QUASIREGULAR VALUES AND RICKMAN'S PICARD THEOREM

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Abstract. We prove a far-reaching generalization of Rickman's Picard theorem for a surprisingly large class of mappings, based on the recently developed theory of quasiregular values. Our results are new even in the planar case.

1. Introduction

Geometric Function Theory (GFT) is largely concerned with generalizations of the theory of holomorphic functions of one complex variable. A widely studied example is the theory of quasiregular maps, which provides such a generalization for spaces of real dimension $n \geq 2$. We recall that, given a domain $\Omega \subset \mathbb{R}^n$ and a constant $K \geq 1$, a *K-quasiregular map* $f\colon \Omega \to \mathbb{R}^n$ is a continuous map in the Sobolev space $W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ which satisfies the distortion inequality

$$|Df(x)|^n \leq K J_f(x) \tag{1.1}$$

for almost every (a.e.) $x \in \Omega$. Here, $|Df(x)|$ is the operator norm of the weak derivative of f at x, and J_f denotes the Jacobian determinant of f.

A significant achievement in the theory of higher-dimensional quasiregular maps is the extension of the classical Picard theorem to n real dimensions. This highly non-trivial result is due to Rickman [43].

Theorem 1.1 (Rickman's Picard Theorem). *For every $K \geq 1$ and $n \geq 2$, there exists a positive integer $q = q(n, K) \in \mathbb{Z}_{>0}$ such that if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is K-quasiregular and $\mathbb{R}^n \setminus f(\mathbb{R}^n)$ has cardinality at most q, then f is constant.*

Rickman's theorem leaves an impression that the global distortion control of quasiregular mappings is necessary for the bound on the number of omitted points. However, in this article, we show that the distortion bound only needs to hold in an asymptotic sense when f is near the omitted points, and can in fact be replaced with an appropriate Sobolev norm estimate elsewhere. Our result is formulated using a recently developed theory of quasiregular values [29]. In particular, supposing that $y_0 \in \mathbb{R}^n$ and that Ω is a domain in $\mathbb{R}^n$ with $n \geq 2$, a map $f\colon \Omega \to \mathbb{R}^n$ in the Sobolev space $W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ has a

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(K,Σ)*-quasiregular value at* y_0 if it satisfies the inequality

$$|Df(x)|^n \le KJ_f(x) + |f(x) - y_0|^n \,\Sigma(x) \tag{1.2}$$

for a.e. $x \in \Omega$, where $K \ge 1$ is a constant as in (1.1) and Σ is a nonnegative function on Ω. Note that results on mappings with quasiregular values typically assume a sufficient degree of L^p-regularity for Σ.

Notably, (1.2) only provides control on the distortion of a mapping f as $f(x)$ equals or asymptotically approaches y_0. Away from y_0, these maps behave similarly to an arbitrary Sobolev map. For instance, a non-constant map f satisfying (1.2) may for instance have a Jacobian that changes sign, an entirely 1-dimensional image, or a bounded image even when f is defined in all of $\mathbb{R}^n$. In addition, a map f satisfying (1.2) needs not be locally quasiregular even in any neighborhood of a point $x_0 \in f^{-1}\{y_0\}$; in fact, it is possible that every neighborhood of such a point meets a region where $J_f < 0$.

In spite of these vast differences, Rickman's Picard theorem still generalizes to the theory of quasiregular values in the following form.

Theorem 1.2. *Let $K \ge 1$ and $\Sigma \in L^{1+\varepsilon}(\mathbb{R}^n) \cap L^{1-\varepsilon}(\mathbb{R}^n)$ for some $\varepsilon > 0$. Then there exists a positive integer $q = q(n,K) \in \mathbb{Z}_{>0}$ such that no continuous map $f \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ has a (K,Σ)-quasiregular value at q distinct points $y_1, \dots, y_q \in \partial f(\mathbb{R}^n)$.*

While the standard Rickman's Picard theorem concerns omitted points $y_i \notin f(\mathbb{R}^n)$, Theorem 1.2 reveals that at this generality, Rickman's Picard Theorem is in fact a result on points y_i in the boundary $\partial f(\mathbb{R}^n)$. Indeed, a version of Theorem 1.2 that instead assumes $y_1, \dots, y_q \notin f(\mathbb{R}^n)$ is immediately shown to be false by any smooth compactly supported map $f \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Regardless of this difference in statements, the standard Rickman's Picard theorem follows almost immediately from the case $\Sigma \equiv 0$ of Theorem 1.2; see Remark 7.4.

The integrability assumptions on Σ in Theorem 1.2 are sharp on the L^p-scale. Indeed, we show in Section 9 that neither $\Sigma \in L^{1+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \cap L^{1-\varepsilon}(\mathbb{R}^n)$ nor $\Sigma \in L^{1+\varepsilon}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ is sufficient for the result. The constructed maps even satisfy (1.2) with $K = 1$. We however expect a logarithmic Orlicz-type sharpening of the integrability assumptions to be possible, though we elect not to pursue log-scale results in this work unless explicitly required by an argument.

1.1. Background on quasiregular maps and the Picard theorem.

The classical Picard theorem states that if $f\colon \mathbb{C} \to \mathbb{C}$ is an entire holomorphic function, then either f is constant or $\mathbb{C} \setminus f(\mathbb{C})$ contains at most one point. The Picard theorem is among the most striking and universally known results in complex analysis, with numerous different proofs discovered over the years; see e.g. [2, 7, 8, 15, 21, 31, 47, 52].

The theory of quasiregular maps originates from the planar setting, with roots in the work of Grötzsch [18] and Ahlfors [1]. More specifically, when $n = 2$, the distortion inequality (1.1) can be rewritten as a linear Beltrami equation

$$f_{\overline{z}} = \mu f_z, \tag{1.3}$$

where $f_z, f_{\overline{z}}$ are the (weak) Wirtinger derivatives of f and $\mu \in L^\infty(\Omega, \mathbb{C})$ satisfies $\|\mu\|_{L^\infty} \leq k < 1$ with $k = (K-1)/(K+1)$. If $K = 1$, then (1.3) reduces to the Cauchy–Riemann system $f_{\overline{z}} = 0$; indeed, a planar map is 1-quasiregular exactly if it is holomorphic. Moreover, homeomorphic solutions of (1.1) or (1.3) are called K-*quasiconformal*, and we also have that a map is 1-quasiconformal exactly if it is a conformal transformation.

In addition to this link to holomorphic maps, planar quasiregular maps satisfy the Stoïlow factorization theorem, which states that a quasiregular map $f\colon \Omega \to \mathbb{C}$ is of the form $f = h \circ g$ where $g\colon \Omega \to \Omega$ is quasiconformal and $h\colon \Omega \to \mathbb{C}$ is holomorphic, see e.g. [4, Chapter 5.5]. The Stoïlow factorization theorem immediately generalizes the topological properties of holomorphic maps to planar quasiregular maps, such as the open mapping theorem, Liouville's theorem, and even the Picard theorem.

The higher-dimensional version of the theory began with the study of n-dimensional quasiconformal mappings by e.g. Šabat [46], Väisälä [49], Gehring [16], and Zorič [53]. Afterwards, the theory of n-dimensional quasiregular mappings was originated by Reshetnyak [38, 41, 40, 39], with significant early contributions by Martio, Rickman, and Väisälä [33, 34, 35]. The theory is by now a central topic in modern analysis, with important connections to partial differential equations, complex dynamics, differential geometry and the calculus of variations; see the textbooks of Väisälä [50], Rickman [45], Reshetnyak [42], and Iwaniec and Martin [27].

Unlike in the planar case, one cannot reduce the topological properties of higher dimensional quasiregular maps to a better understood class of mappings. Indeed, the best known Stoïlow-type theorem in higher dimensions [32] still has a relatively irregular non-injective component. Nevertheless, many topological properties of holomorphic maps have non-trivial extensions to spatial quasiregular mappings as well. For instance, the open mapping theorem generalizes to Reshetnyak's theorem [41, 40], which states that if $f\colon \Omega \to \mathbb{R}^n$ is a non-constant quasiregular map, then f is an open, discrete map with positive local index $i(x, f)$ at every $x \in \Omega$.

Rickman's Picard theorem, stated in Theorem 1.1, is perhaps the most clear demonstration of the similarities between the theory of higher dimensional quasiregular maps and single-variable complex analysis. Consequently, it has become one of the most widely studied results in quasiregular analysis. For instance, a version of Rickman's Picard Theorem has been shown for quasiregular maps $f\colon \mathbb{R}^n \to M$ into an oriented Riemannian n-manifold M by Holopainen and Rickman [23, 24]. A version of the theorem has also been shown by Rajala [37] in the case where f is a mapping of finite distortion, i.e. a mapping satisfying (1.1) with a non-constant K.

When $n = 2$, the Stoïlow factorization approach yields that the constant $q(2, K)$ in Rickman's Picard theorem is equal to 2, and is thus in fact independent on K. It was conjectured for some time that one could also have $q(n, K) = 2$ for all $n \geq 2$ and $K \geq 1$. However, counterexamples by Rickman [44] in the case $n = 3$, and by Drasin and Pankka [10] in the case $n \geq 4$, show that for a fixed $n > 2$ one has $q(n, K) \to \infty$ as $K \to \infty$.

1.2. **The theory of quasiregular values.** Various generalizations of (1.1) and (1.3) occur in the study of complex analysis. For instance, the condition

$$|Df|^2 \leq KJ_f + C\,, \tag{1.4}$$

where $K \geq 1$ and $C \geq 0$ are constants, arises naturally in the theory of elliptic PDEs [17, Chapter 12]. The Hölder regularity of planar domain solutions of (1.4) has been shown by Nirenberg [36], Finn and Serrin [13], and Hartman [20]. Similar ideas also play a key role in the work of Simon [48], where he obtains Hölder estimates for solutions of (1.4) between surfaces, and applies them to the study of equations of mean curvature type.

The theory of quasiregular values stems from another similar generalization of (1.1) and (1.3), namely

$$f_{\overline{z}} = \mu f_z + Af, \tag{1.5}$$

where $\|\mu\|_{L^\infty} < 1$ and $A \in L^{2+\varepsilon}_{\text{loc}}(\Omega, \mathbb{C})$ for some $\varepsilon > 0$. In particular, (1.5) corresponds to the case $n = 2, y_0 = 0$ of the definition (1.2) of quasiregular values. Much of the initial theory on solutions of (1.5) was developed by Vekua [51]. One of the standout applications for (1.5) arose when Astala and Päivärinta used it in their solution to the planar Calderón problem [5]. The solutions of (1.5) play a key part of various other uniqueness theorems as well; we refer to the book of Astala, Iwaniec and Martin [4] for details.

Astala and Päivärinta relied on two results for entire solutions of (1.5), which were essentially modeled on Liouville's theorem and the argument principle; see [5, Proposition 3.3] and [4, Sect. 8.5 and 18.5]. The original key idea behind the planar results is that any solution f of (1.5) is of the form $f = ge^\theta$, where g is quasiregular and $\theta\colon \Omega \to \mathbb{C}$ is a solution of $\theta_{\overline{z}} = \mu\theta_z + A$. Since the existence theory of Beltrami equations and the aforementioned decomposition $f = ge^\theta$ lack higher-dimensional counterparts, this planar approach fails to generalize to the n-dimensional setting. Nevertheless, we have recently in [28, 29, 30] managed to obtain higher-dimensional counterparts to the planar results used by Astala and Päivärinta. The Liouville-type theorem in particular answers the Astala–Iwaniec–Martin uniqueness question from [4, Sect. 8.5]; see [28, Theorem 1.3] and the correction [30].

The higher-dimensional results opened up an entirely new direction of study in GFT, as they led us to introduce the notion of quasiregular values in [29]. The term "quasiregular value" is partially motivated by the single-value versions of various foundational results of quasiregular maps that follow from (1.2). The other main motivation for the term is the fact that K-quasiregularity of a map $f \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$ can be fully characterized by f having a (K, Σ_y)-quasiregular value with $\Sigma_y \in L^{1+\varepsilon}_{\text{loc}}(\Omega)$ at every $y \in \mathbb{R}^n$; see [29, Theorem 1.3].

The following theorem lists the two most notable current results of quasiregular values, which are the single-value versions of Liouville's theorem and Reshetnyak's theorem. They were shown in [28] and [29], respectively, and are key components behind the higher-dimensional versions of the planar results for solutions of (1.5). The addition of Theorem 1.2 to this growing list of results furthers the evidence that quasiregular values have a rich theory comparable to that of quasiregular mappings.

Theorem 1.3 ([28, Theorem 1.2] and [29, Theorem 1.2])**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain, let $\varepsilon > 0$, and let $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ be a continuous map with a (K, Σ)-quasiregular value at $y_0 \in \mathbb{R}^n$, where $K \geq 1$ and $\Sigma\colon \Omega \to [0, \infty)$. Then the following results hold:*

(i) (Liouville's theorem) If $\Omega = \mathbb{R}^n$, $\Sigma \in L^{1+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, and f is bounded, then either $f \equiv y_0$ or $y_0 \notin f(\mathbb{R}^n)$.

(ii) (Reshetnyak's theorem) If $\Sigma \in L^{1+\varepsilon}_{\mathrm{loc}}(\Omega)$ and if f is not the constant function $f \equiv y_0$, then $f^{-1}\{y_0\}$ is discrete, the local index $i(x, f)$ is positive at every $x \in f^{-1}\{y_0\}$, and f maps every neighborhood $U \subset \Omega$ of a point of $f^{-1}\{y_0\}$ to a neighborhood $f(U)$ of y_0.

We note that by [28, Theorem 1.1], solutions $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ of (1.2) always have a continuous representative if $\Sigma \in L^{1+\varepsilon}_{\mathrm{loc}}(\Omega)$ for some $\varepsilon > 0$; see also [9] which explores how much these assumptions can be relaxed for continuity to remain true. Hence, the continuity assumption in our results only amounts to making sure that our chosen representative of the Sobolev map is the continuous one.

1.3. **Other versions of Theorem 1.2.** Besides the standard formulation for quasiregular mappings $f\colon \mathbb{R}^n \to \mathbb{R}^n$, Rickman's Picard theorem is often also equivalently formulated for quasiregular mappings $f\colon \mathbb{R}^n \to \mathbb{S}^n$. In our setting, we similarly obtain a version of Theorem 1.2 for mappings $f\colon \mathbb{R}^n \to \mathbb{S}^n$ with little extra effort, though it requires formulating a spherical version of (1.2). Given $\Omega \subset \mathbb{R}^n$, $K \geq 0$, $y_0 \in \mathbb{S}^n$, and $\Sigma \in L^{1+\varepsilon}_{\mathrm{loc}}(\Omega)$ with $\varepsilon > 0$ and $\Sigma \geq 0$, we say that a continuous mapping $h \in W^{1,n}(\Omega, \mathbb{S}^n)$ has a (K, Σ)-*quasiregular value with respect to the spherical metric at* $w_0 \in \mathbb{S}^n$ if f satisfies

$$|Dh(x)|^n \leq K J_h(x) + \sigma^n(h(x), w_0)\Sigma(x) \tag{1.6}$$

at a.e. $x \in \Omega$, where $\sigma(\cdot, \cdot)$ denotes the spherical distance on $\mathbb{S}^n$, and $|Dh(x)|$ and $J_h(x)$ are defined using the standard Riemannian metric and orientation on $\mathbb{S}^n$. With this definition, the resulting version of Theorem 1.2 is as follows.

Theorem 1.4. *Let $K \geq 1$ and $\Sigma \in L^{1+\varepsilon}(\mathbb{R}^n) \cap L^{1-\varepsilon}(\mathbb{R}^n)$ for some $\varepsilon > 0$. Then there exists a positive integer $q = q(n, K) \in \mathbb{Z}_{>0}$ such that no continuous map $h \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{S}^n)$ has a (K, Σ)-quasiregular value with respect to the spherical metric at q distinct points $w_1, \ldots, w_q \in \partial h(\mathbb{R}^n)$.*

We remark that if we identify $\mathbb{S}^n$ with $\mathbb{R}^n \cup \{\infty\}$ via the stereographic projection, then a map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ has a quasiregular value with respect to the Euclidean metric at $y_0 \in \mathbb{R}^n$ if and only if f has a quasiregular value with respect to the spherical metric at both y_0 and ∞. Hence, (1.6) is in some sense a weaker assumption than (1.2). The comparison between these two definitions is discussed in greater detail in Section 3.

While the assumption $\Sigma \in L^{1+\varepsilon}(\mathbb{R}^n) \cap L^{1-\varepsilon}(\mathbb{R}^n)$ in Theorems 1.2 and 1.4 is sharp, the proof we use does yield us some additional information even under a weaker assumption of $\Sigma \in L^1(\mathbb{R}^n) \cap L^{1+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^n)$. This result is more elegantly stated using spherical quasiregular values.

Theorem 1.5. *Let $K \geq 1$ and $\Sigma \in L^1(\mathbb{R}^n) \cap L^{1+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $\varepsilon > 0$. Then there exists a positive integer $q = q(n, K) \in \mathbb{Z}_{>0}$ with the following property: if a continuous map $h \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{S}^n)$ has a (K, Σ)-quasiregular value with* *respect to the spherical metric at q distinct points $w_1, \dots, w_q \in \partial h(\mathbb{R}^n)$, then $|Dh| \in L^n(\mathbb{R}^n)$.*

1.4. **The planar case.** In the case $n = 2$, similarly to the standard Picard theorem, our main results end up having $q(2, K) = 2$ for maps $f\colon \mathbb{C} \to \mathbb{C}$, and $q(2, K) = 3$ for maps $h\colon \mathbb{C} \to \mathbb{S}^2$. Even this planar version of Theorem 1.2 and Theorem 1.4 is new.

Theorem 1.6. *Let $K \geq 1$ and $\Sigma \in L^{1+\varepsilon}(\mathbb{C}) \cap L^{1-\varepsilon}(\mathbb{C})$ for some $\varepsilon > 0$. Then no continuous map $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{C}, \mathbb{C})$ has a (K, Σ)-quasiregular value at two distinct points $z_1, z_2 \in \partial f(\mathbb{C})$. Similarly, no continuous map $h \in W^{1,2}_{\mathrm{loc}}(\mathbb{C}, \mathbb{S}^2)$ has a (K, Σ)-quasiregular value with respect to the spherical metric at three distinct points $w_1, w_2, w_3 \in \partial h(\mathbb{C})$.*

We prove Theorem 1.6 by reducing it to Theorem 1.2. The version of the argument for quasiregular maps is incredibly simple: If $f\colon \mathbb{C} \to \mathbb{C}$ is a K-quasiregular map omitting two distinct points $z_1, z_2 \in \mathbb{C}$, then the lift $\gamma\colon \mathbb{C} \to \mathbb{C}$ of f in the exponential map $z \mapsto z_1 + e^z$ is a K-quasiregular map that omits the infinitely many values of $\log(z_2 - z_1)$, which is impossible by Rickman's Picard Theorem. Attempting the same idea for maps with quasiregular values using Theorem 1.2 is less straightforward, but we are ultimately able to construct a proof around this fundamental idea through use of the decomposition $f = ge^\theta$ and existing results on quasiregular values; see Section 8 for details.

1.5. **Main ideas of the proof.** While the classical Picard theorem has numerous proofs, only a few of them have been successfully generalized to a proof of the n-dimensional Rickman's Picard Theorem. The original proof by Rickman [43] uses path lifting and conformal modulus techniques in order to estimate spherical averages of the multiplicity function of f. Later, work by Eremenko and Lewis [11, 31] resulted in an alternate proof using Harnack inequalities of $\mathcal{A}$-harmonic maps. Both of these approaches run into significant obstacles in our setting, as solutions of (1.2) currently lack counterparts to e.g. conformal modulus estimates and the natural conformal structure $G_f(x) = J_f^{2/n}(x)[D^T f(x) Df(x)]^{-1}$ of f.

Recently, however, a third method of proof has been discovered by Bonk and Poggi-Corradini [6], which is closer to being applicable in our situation. Motivated by the Ahlfors–Shimizu value distribution theory of holomorphic functions, they study the pull-back $v \circ f$ of a subharmonic logarithmic singularity function $v\colon \mathbb{S}^n \setminus \{x_0\} \to [0, \infty)$, where the spherical n-Laplacian of v is identically 1. They are then able to leverage the preservation of the spherical measure under isometric rotations of $\mathbb{S}^n$ to obtain growth rate estimates for the measure $\mu = f^* \mathrm{vol}_{\mathbb{S}^n}$, from which the result follows via ideas reminiscent of the ones used in Rickman's original argument.

We prove Theorem 1.2 by adopting the structure of the proof of Bonk and Poggi-Corradini, but with key developments to the proof in multiple places where its current form is insufficient for us. Notably, in order to avoid use of the conformal structure G_f, we completely eliminate the use of $\mathcal{A}$-subharmonic theory in our proof, and we instead obtain the required growth estimates by directly using (1.2) and the properties of the logarithmic singularity function. Issues caused by the extra term in (1.2) and the fact that μ is a signed measure are eliminated by the global L^1-integrability of Σ.

The greatest challenges in our setting are tied to replacing the use of [6, Lemma 4.4], which yields that if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a non-constant entire quasiregular map and $r > 0$, then every component of the set $\{|f| > r\}$ is unbounded. In our case, this is not true; instead, we essentially obtain control on the total A_f-measure of any bounded components of $\{|f| > r\}$. One of our primary tools in addressing this problem is to introduce a "pseudosupremum" based on unbounded components of pre-images. Indeed, when the growth estimates for A_f are formulated in terms of this pseudosupremum, they can be combined in a similar manner as in the case of quasiregular maps.

However, the pseudosupremum does not solve the second major challenge surrounding [6, Lemma 4.4], which is the problem of showing that mappings with multiple quasiregular values in $\partial f(\mathbb{R}^n)$ satisfy $A_f(\mathbb{R}^n) = \infty$. We note that Theorem 1.5 is obtained by essentially ignoring this issue and instead assuming a-priori that $A_f(\mathbb{R}^n) = \infty$. For non-constant quasiregular maps $f\colon \mathbb{R}^n \to \mathbb{S}^n \setminus \{x_1, x_2\}$, the fact that $A_f(\mathbb{R}^n) = \infty$ follows easily; see for example [45, Lemma IV.2.7] or [6, p. 631]. In our setting, however, this step becomes nontrivial, involving challenges somewhat similar to the ones encountered in the study of the Astala–Iwaniec–Martin uniqueness question. In particular, the part about excluding the case $A_f(\mathbb{R}^n) < \infty$ is the only part of the proof where the precise integrability assumptions of Theorems 1.2 and 1.4 become relevant.

1.6. **The structure of this paper.** In Section 2, we recall some preliminary information on Sobolev differential forms that is used in our computations. Section 3 is a discussion on the connections between the Euclidean and spherical definitions of quasiregular values. In Section 4, we discuss spherical logarithms of maps with quasiregular values, and prove a boundedness result that is used later in the proof of Theorems 1.2 and 1.4. Section 5 is then dedicated to proving the relevant Caccioppoli-type estimates that are used in the proofs of the main results.

With these preliminaries complete, we then prove Theorem 1.5 in Section 6. The proof of Theorems 1.2 and 1.4 is then completed in Section 7, with an entire section dedicated to dealing with the special case $A_f(\mathbb{R}^n) < \infty$. In Section 8, we prove the sharp planar result given in Theorem 1.6 by using Theorem 1.2. Finally, in Section 9, we provide counterexamples which show the sharpness of the assumptions of Theorem 1.2.

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2. Preliminaries on Sobolev differential forms

Throughout this paper, we use $C(a_1, a_2, \dots, a_m)$ to denote a positive constant that depends on the parameters a_i. The value of $C(a_1, a_2, \dots, a_m)$ may change in each estimate even if the parameters remain the same. We also use the shorthand $A_1 \lesssim_{a_1,a_2,\dots,a_m} A_2$ which stands for $A_1 \le C(a_1, a_2, \dots, a_m)A_2$, where we always list the dependencies of the constant on the $\lesssim$-symbol. The shorthand $A_1 \gtrsim_{a_1,a_2,\dots,a_m} A_2$ is defined similarly. Additionally, if $B = \mathbb{B}^n(x, r) \subset \mathbb{R}^n$ is a Euclidean ball and $c \in (0, \infty)$, then we use cB to denote the ball $\mathbb{B}^n(x, cr)$.

Let U be an open subset of $\mathbb{R}^n$. We use $L^p(\wedge^k T^*U)$, $L^p_{\mathrm{loc}}(\wedge^k T^*U)$, $W^{1,p}(\wedge^k T^*U)$, $W^{1,p}_{\mathrm{loc}}(\wedge^k T^*U)$, and $C^l(\wedge^k T^*U)$ to denote differential k-forms $\omega = \sum_I \omega_I dx_I$ on U for which ω_I are in $L^p(U)$, $L^p_{\mathrm{loc}}(U)$, $W^{1,p}(U)$, $W^{1,p}_{\mathrm{loc}}(U)$, or $C^l(U)$, respectively. We also use the subscript 0 to denote spaces of differential forms or real-valued functions with compact supports; for instance, $C^\infty_0(U)$ denotes the space of compactly supported smooth real-valued functions on U.

Given a differential form $\omega\colon U \to \wedge^k T^*\mathbb{R}^n$, we use $\omega_x \in \wedge^k T^*_x\mathbb{R}^n$ to denote the value of ω at x. We use $|\omega_x|$ for the norm of ω_x, which is the l^2-norm on the coefficients of ω_x with respect to the standard basis; in particular $|\omega|$ is a function $U \to [0, \infty)$. Recall that $|\omega_1 \wedge \omega_2| \le C(n) |\omega_1| |\omega_2|$. If either ω_1 or ω_2 is a wedge product of 1-forms, then one in fact has $|\omega_1 \wedge \omega_2| \le |\omega_1| |\omega_2|$. We also use $\star\omega$ to denote the Hodge star of a differential k-form ω.

If $\omega \in L^1_{\mathrm{loc}}(\wedge^k T^*U)$, then $d\omega \in L^1_{\mathrm{loc}}(\wedge^{k+1} T^*U)$ is a *weak differential* of ω if

$$\int_U d\omega \wedge \eta = (-1)^{k+1} \int_U \omega \wedge d\eta$$

for every $\eta \in C^\infty_0(\wedge^{n-k-1} T^*U)$. We denote the space of $\omega \in L^p_{\mathrm{loc}}(\wedge^k T^*U)$ with a weak differential $d\omega \in L^q_{\mathrm{loc}}(\wedge^{k+1} T^*U)$ by $W^{d,p,q}_{\mathrm{loc}}(\wedge^k T^*U)$, with the abbreviation $W^{d,p}_{\mathrm{loc}}(\wedge^k T^*U) = W^{d,p,p}_{\mathrm{loc}}(\wedge^k T^*U)$. We also define versions of these spaces with global integrability, denoted $W^{d,p,q}(\wedge^k T^*U)$ and $W^{d,p}(\wedge^k T^*U)$, as well as versions for forms with compact supports, denoted $W^{d,p,q}_0(\wedge^k T^*U)$ and $W^{d,p}_0(\wedge^k T^*U)$. Recall that $W^{1,p}_{\mathrm{loc}}(\wedge^k T^*U) \subset W^{d,p}_{\mathrm{loc}}(\wedge^k T^*U)$, along with the similar inclusions $W^{1,p}(\wedge^k T^*U) \subset W^{d,p}(\wedge^k T^*U)$ and $W^{1,p}_0(\wedge^k T^*U) \subset W^{d,p}_0(\wedge^k T^*U)$, where the weak differential of an element of $W^{1,p}_{\mathrm{loc}}(\wedge^k T^*U)$ is obtained component-wise by the rule $d(f dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}) = df \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$.

If $\omega_1 \in W^{1,p}_{\mathrm{loc}}(\wedge^k T^*U)$ and $\omega_2 \in W^{1,q}_{\mathrm{loc}}(\wedge^l T^*U)$ with $p^{-1} + q^{-1} = r^{-1} \le 1$, then standard product rules of Sobolev functions yield that $\omega_1 \wedge \omega_2 \in W^{1,r}_{\mathrm{loc}}(\wedge^{k+l} T^*U)$, and

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2. \tag{2.1}$$

By a convolution approximation argument, we have that (2.1) also holds if one instead assumes that $\omega_1 \in W^{d,p_1,q_1}_{\mathrm{loc}}(\wedge^k T^*U)$ and $\omega_2 \in W^{d,p_2,q_2}_{\mathrm{loc}}(\wedge^l T^*U)$ with $p_1^{-1} + p_2^{-1} = r^{-1} \le 1$ and $\max(p_1^{-1} + q_2^{-1}, p_2^{-1} + q_1^{-1}) = s^{-1} \le 1$, in which case $\omega_1 \wedge \omega_2 \in W^{d,r,s}_{\mathrm{loc}}(\wedge^{k+l} T^*U)$. Moreover, if $\omega \in W^{d,1}_0(\wedge^{n-1} T^*U)$, then a convolution-based argument similarly yields

$$\int_U d\omega = 0. \tag{2.2}$$

We also note that if $\omega \in W^{d,1}_{\mathrm{loc}}(\wedge^k T^*U)$, then $d\omega \in W^{d,1}_{\mathrm{loc}}(\wedge^{k+1} T^*U)$ with $dd\omega = 0$.

If $\omega \in C(\wedge^k T^*V)$, i.e. if the coefficients ω_I of ω are continuous, and if $f \in W^{1,n}_{\mathrm{loc}}(U, \mathbb{R}^n)$, then the pull-back $f^*\omega$ is well-defined and lies in $L^{n/k}_{\mathrm{loc}}(\wedge^k T^*U)$. We recall that in this case, we have the estimate

$$|f^*\omega| \leq (|\omega| \circ f) \, |Df|^k \,. \tag{2.3}$$

Indeed, if $\omega = \varphi dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, then $|f^*\omega| = |(\varphi \circ f) df_{i_1} \wedge \cdots \wedge df_{i_k}| \leq (|\varphi| \circ f) \, |Df|^k$, and the result for general ω then follows by the Pythagorean theorem.

Moreover, if $\omega \in C^1_0(\wedge^k T^*V)$ and $f \in W^{1,n}_{\mathrm{loc}}(U, \mathbb{R}^n)$, the chain rule of C^1_0-functions and $W^{1,n}_{\mathrm{loc}}$-functions then yields that $f^*\omega \in W^{d,n/k,n/(k+1)}_{\mathrm{loc}}(\wedge^k T^*U)$ and $df^*\omega = f^*d\omega$; see e.g. the proof of [28, Lemma 2.2]. If f is additionally continuous, then the assumption $\omega \in C^1_0(\wedge^k T^*V)$ can be weakened to $\omega \in C^1(\wedge^k T^*V)$ by using smooth cutoff functions.

In what follows, we also use a chain rule for $f \in C(U, V) \cap W^{1,n}_{\mathrm{loc}}(U, \mathbb{R}^n)$ and $\omega \in C(\wedge^k T^*V) \cap W^{1,\infty}_{\mathrm{loc}}(\wedge^k T^*V)$. Note that this assumption on ω is equivalent with the coefficients ω_I being locally Lipschitz. At this level of generality, caution is required with the use of chain rules; for instance, the weak differential $d\omega$ is only unique up to a null-set under these assumptions, and changing $d\omega$ in a null-set could change $f^*d\omega$ in a set of positive measure, making $f^*d\omega$ ill-defined. However, these assumptions are still sufficient to obtain weak differentiability of $f^*\omega$. We record the precise statement we use in the following Lemma, which follows in a straightforward manner from the chain rule for Lipschitz and Sobolev maps; see e.g. Ambrosio and Dal Maso [3, Corollary 3.2].

Lemma 2.1. *Let $U, V \subset \mathbb{R}^n$ be open sets, let $f \in C(U, V) \cap W^{1,n}_{\mathrm{loc}}(U, \mathbb{R}^n)$, and let $\omega \in C(\wedge^k T^*V) \cap W^{1,\infty}_{\mathrm{loc}}(\wedge^k T^*V)$ for $k \in \{0, \dots, n-1\}$; i.e., we assume that ω has locally Lipschitz coefficients. Then $f^*\omega \in W^{d,n/k,n/(k+1)}_{\mathrm{loc}}(\wedge^k T^*U)$.*

In particular, combining Lemma 2.1 with (2.1) unlocks the following tool.

Corollary 2.2. *Let $U, V \subset \mathbb{R}^n$ be open sets, let $f \in C(U, V) \cap W^{1,n}_{\mathrm{loc}}(U, \mathbb{R}^n)$, let $\omega_1 \in C(\wedge^k T^*V) \cap W^{1,\infty}_{\mathrm{loc}}(\wedge^k T^*V)$, and let $\omega_2 \in C(\wedge^l T^*V) \cap W^{1,\infty}_{\mathrm{loc}}(\wedge^l T^*V)$, with $k, l \in \mathbb{Z}_{\geq 0}$, $k + l \leq n - 1$. Then*

$$f^*(\omega_1 \wedge \omega_2) \in W^{d, \frac{n}{k+l}, \frac{n}{k+l+1}}_{\mathrm{loc}}(\wedge^{k+l} T^*U),$$

with

$$df^*(\omega_1 \wedge \omega_2) = (df^*\omega_1) \wedge \omega_2 + (-1)^k f^*\omega_1 \wedge (df^*\omega_2).$$

3. Quasiregular values and maps into spheres

Let e_i denote the standard basis vectors of $\mathbb{R}^n$, let $\langle \cdot, \cdot \rangle$ denote the Euclidean inner product on $\mathbb{R}^n$, and let $|\cdot|$ denote the Euclidean norm. The n-dimensional unit sphere $\mathbb{S}^n$ consists of all $w \in \mathbb{R}^{n+1}$ with $|w| = 1$. Recall that on $\mathbb{R}^n$, the inverse $s_n \colon \mathbb{R}^n \to \mathbb{S}^n \setminus \{-e_1\}$ of the stereographic projection is defined by

$$s_n(x) = \frac{1}{1+|x|^2}\Big(1-|x|^2, 2x_1, 2x_2, \dots, 2x_n\Big).$$

The map s_n is then extended to $\mathbb{R}^n \cup \{\infty\}$ by setting $s_n(\infty) = -e_1$.

We recall that the *spherical distance* σ on $\mathbb{S}^n$ is given by $\sigma(w_1, w_2) = \arccos\langle w_1, w_2\rangle$ for $w_1, w_2 \in \mathbb{S}^n$, using the inclusion of $\mathbb{S}^n$ into $\mathbb{R}^{n+1}$. We also define the spherical distance on the space $\mathbb{R}^n \cup \{\infty\}$ by setting $\sigma(x_1, x_2) = \sigma(s_n(x_1), s_n(x_2))$ for $x_1, x_2 \in \mathbb{R}^n \cup \{\infty\}$. Via an elementary computation, one sees for $x_1, x_2 \in \mathbb{R}^n$ that

$$\cos(\sigma(x_1, x_2)) = \langle s_n(x_1), s_n(x_2)\rangle = 1 - \frac{2\,|x_1 - x_2|^2}{(1+|x_1|^2)(1+|x_1|^2)}.$$

In particular,

$$\sin\frac{\sigma(x_1, x_2)}{2} = \frac{|x_1 - x_2|}{\sqrt{(1+|x_1|^2)(1+|x_2|^2)}} \qquad \text{for } x_1, x_2 \in \mathbb{R}^n. \tag{3.1}$$

By letting x_2 tend to infinity in (3.1), we also see that

$$\sin\frac{\sigma(x_1, \infty)}{2} = \frac{1}{\sqrt{1+|x_1|^2}} \qquad \text{for } x_1 \in \mathbb{R}^n. \tag{3.2}$$

We equip $\mathbb{S}^n$ with the standard Riemannian metric that arises from the inclusion to $\mathbb{R}^{n+1}$, and orient $\mathbb{S}^n$ so that its volume form $\operatorname{vol}_{\mathbb{S}^n}$ is given by the restriction of the n-form $\star d(2^{-1}|x|^2) \in C^\infty(\wedge^n T^*\mathbb{R}^{n+1})$. When $\mathbb{S}^n$ is equipped with this metric and volume form, the map $s_n\colon \mathbb{R}^n \to \mathbb{S}^n$ is conformal; more precisely,

$$|Ds_n(x)|^n = J_{s_n}(x) = \frac{2^n}{\left(1+|x|^2\right)^n} \tag{3.3}$$

for every $x \in \mathbb{R}^n$. Moreover, given a set $U \subset \mathbb{R}^n \cup \{\infty\}$, we denote its spherical measure by $\operatorname{vol}_{\mathbb{S}^n}(U)$. By (3.3) , we see that

$$\operatorname{vol}_{\mathbb{S}^n}(U) = \int_U \frac{2^n \operatorname{vol}_n}{\left(1+|x|^2\right)^n}. \tag{3.4}$$

Suppose then that $f \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$. We define a measurable map $h\colon \mathbb{R}^n \to \mathbb{S}^n$ by $h = s_n \circ f$. Since $s_n\colon \mathbb{R}^n \to \mathbb{R}^{n+1}$ is a smooth Lipschitz map, it follows that $h \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^{n+1})$ and $Dh(x) = Ds_n(f(x))Df(x)$ for a.e. $x \in \mathbb{R}^n$. In particular, the image of $Dh(x)$ lies in $T_{h(x)}\mathbb{S}^n$ for a.e. x, and hence Dh can be understood as a measurable map $T\mathbb{R}^n \to T\mathbb{S}^n$. Consequently, we obtain a Jacobian of h by $J_h \operatorname{vol}_n = h^* \operatorname{vol}_{\mathbb{S}^n}$. Since s_n is conformal, we obtain by (3.3) that

$$|Dh|^n = \frac{2^n\,|Df|^n}{\left(1+|f|^2\right)^n} \qquad \text{and} \qquad J_h = \frac{2^n J_f}{\left(1+|f|^2\right)^n} \tag{3.5}$$

a.e. in $\mathbb{R}^n$.

We then prove comparison results for the two definitions of quasiregular values given in (1.2) and (1.6). We begin with a spherical interpretation of Euclidean quasiregular values.

Lemma 3.1. *Let $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ with $\Omega \subset \mathbb{R}^n$. Let $h = s_n \circ f$, let $w_0 = s_n(y_0)$ for some $y_0 \in \mathbb{R}^n$, let $K \in \mathbb{R}$, and let $\Sigma\colon \Omega \to [0, \infty)$ be measurable. Then the following conditions are equivalent up to an extra constant factor $C = C(n, y_0)$ on Σ:*

(1) f has a (K, Σ)-quasiregular value at y_0;
(2) h has a (K, Σ)-quasiregular value with respect to the spherical metric at both w_0 and $s_n(\infty)$;
(3) h satisfies

$$|Dh|^n \leq KJ_h + \sigma^n(h, w_0)\sigma^n(h, s_n(\infty))\Sigma$$

a.e. in Ω.

Proof. We first show the (almost) equivalence of (1) and (3). We multiply (1.2) on both sides by $2^n(1 + |f|^2)^{-n}$ and use (3.5), obtaining that (1.2) is equivalent to

$$|Dh|^n \leq KJ_h + 2^n \frac{|f - y_0|^n}{\left(1 + |f|^2\right)^n}\Sigma.$$

Now, using (3.1) and (3.2), we observe that

$$\begin{aligned}\frac{|f - y_0|}{1 + |f|^2} &= \frac{|f - y_0|}{\sqrt{(1 + |f|^2)(1 + |y_0|^2)}} \cdot \frac{1}{\sqrt{1 + |f|^2}} \cdot \sqrt{1 + |y_0|^2} \\ &= \sin\frac{\sigma(f, y_0)}{2} \cdot \sin\frac{\sigma(f, \infty)}{2} \cdot \left(\sin\frac{\sigma(y_0, \infty)}{2}\right)^{-1}.\end{aligned}$$

Thus, (1.2) is equivalent to

$$|Dh|^n \leq KJ_h + \frac{2^n \sin^n\left(2^{-1}\sigma(f, y_0)\right)\sin^n\left(2^{-1}\sigma(f, \infty)\right)}{\sin^n(2^{-1}\sigma(y_0, \infty))}\Sigma.$$

Since $(2/\pi)t \leq \sin(t) \leq t$ whenever $t \in [0, \pi/2]$, the previous equation is equivalent to the one in part (3), up to a multiplicative constant on Σ.

It remains to show the (almost) equivalence of (2) and (3). Since $\sigma(\cdot, \cdot)$ is bounded from above by π, it is clear from the definition of spherical quasiregular values in (1.6) that (3) implies (2) up to an extra factor of π^n on Σ. For the other diection, we use the fact that for any distinct $w_1, w_2 \in \mathbb{S}^n$, the function $w \mapsto \min(\sigma^{-1}(w, w_1), \sigma^{-1}(w, w_2))$ is continuous and has a maximum value of $2/\sigma(w_1, w_2)$. Thus, if (2) holds, then we have the estimate

$$\begin{aligned}|Dh|^n &\leq KJ_h + \min\left(\sigma^n(h, w_0), \sigma^n(h, s_n(\infty))\right)\Sigma \\ &= KJ_h + \min\left(\sigma^{-n}(h, s_n(\infty)), \sigma^{-n}(h, w_0)\right)\sigma^n(h, w_0)\sigma^n(h, s_n(\infty))\Sigma \\ &\leq KJ_h + C(n, y_0)\sigma^n(h, w_0)\sigma^n(h, s_n(\infty))\Sigma\end{aligned}$$

a.e. on Ω, completing the proof. ☐

Next, we give a Euclidean interpretation of spherical quasiregular values.

Lemma 3.2. *Let $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ with $\Omega \subset \mathbb{R}^n$. Let $h = s_n \circ f$, let $w_0 = s_n(y_0)$ for some $y_0 \in \mathbb{R}^n$, let $K \in \mathbb{R}$, and let $\Sigma\colon \Omega \to [0, \infty)$ be measurable. Then the following conditions are equivalent up to an extra constant factor $C = C(n, y_0)$ on Σ:*

(1) h has a (K, Σ)-quasiregular value at w_0;

(2) f satisfies

$$|Df|^n \leq KJ_f + |f - y_0|^n \left(1 + |f|^2\right)^{\frac{n}{2}} \Sigma$$

a.e. in Ω.

Similarly, the following conditions are equivalent up to an extra constant factor $C = C(n)$ on Σ:

(1') h has a (K, Σ)-quasiregular value at $s_n(\infty)$;

(2') f satisfies

$$|Df|^n \leq KJ_f + \left(1 + |f|^2\right)^{\frac{n}{2}} \Sigma$$

a.e. in Ω.

Proof. For the first equivalence, similarly as in the proof of Lemma 3.1, we may use (3.1), (3.2), and (3.5) to show that condition (2) is equivalent to

$$|Dh|^n \leq KJ_h + \frac{2^n \sin^n\left(2^{-1}\sigma(h, w_0)\right)}{\sin^n(2^{-1}\sigma(w_0, s_n(\infty)))} \Sigma.$$

Since this is equivalent to (1.6) up to a multiplicative constant on Σ, the claim follows. The proof of the second equivalence is analogous, as (3.2) and (3.5) yield that condition (2') is equivalent to

$$|Dh|^n \leq KJ_h + 2^n \sin^n\left(\frac{\sigma(h, s_n(\infty))}{2}\right) \Sigma.$$

$\square$

We end this section by pointing out that the single-point Liouville's theorem and Reshetnyak's theorem for Euclidean quasiregular values imply corresponding results for spherical quasiregular values.

Proposition 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a domain, let $\varepsilon > 0$, and let $h \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{S}^n)$ be a continuous map with a quasiregular value with respect to the spherical metric at $w_0 \in \mathbb{S}^n$, for given choices of $K \geq 1$ and $\Sigma \colon \Omega \to [0, \infty)$. Then the following results hold.*

(i) (Reshetnyak's theorem) If $\Sigma \in L^{1+\varepsilon}_{\text{loc}}(\Omega)$ and if h is not the constant function $h \equiv w_0$, then $h^{-1}\{w_0\}$ is discrete, the local index $i(x, h)$ is positive at every $x \in h^{-1}\{w_0\}$, and h maps every neighborhood $U \subset \Omega$ of a point of $f^{-1}\{w_0\}$ to a neighborhood $h(U)$ of w_0.

(ii) (Liouville's theorem) If $\Omega = \mathbb{R}^n$, $\Sigma \in L^{1+\varepsilon}_{\text{loc}}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, and $\overline{h(\mathbb{R}^n)} \neq \mathbb{S}^n$, then either $h \equiv w_0$ or $w_0 \notin h(\mathbb{R}^n)$.

Proof. Suppose first that the assumptions of (i) hold. If we post-compose h with an isometric rotation $R \colon \mathbb{S}^n \to \mathbb{S}^n$, it follows that $R \circ h$ has a (K, Σ)-quasiregular value with respect to the spherical metric at $R(w_0)$. Thus, we may assume that $w_0 \neq \infty$. Let $y_0 \in \mathbb{R}^n$ be the point for which $s_n(y_0) = w_0$.

We fix an open neighborhood U of w_0 for which $\infty \notin \overline{U}$. Now, in the set $\Omega' = h^{-1}U$, there is a bounded, continuous $f \in W^{1,n}_{\text{loc}}(\Omega', \mathbb{R}^n)$ such that $h = s_n \circ f$. By Lemma 3.2, f has a (K, Σ')-quasiregular value at y_0, where $\Sigma' = C(n, y_0)(1+|f|^2)^{n/2}\Sigma$. Since f is bounded, we have that $\Sigma' \in L^{1+\varepsilon}_{\text{loc}}(\Omega')$. Now, the Euclidean result (Theorem 1.3 (ii)) yields the claim for $h|_{\Omega'}$. Since Ω' is the pre-image of a neighborhood of w_0 under h, this in fact implies the result for h.

Suppose then that the assumptions of (ii) hold. If $w_0 \in \mathbb{S}^n \setminus \overline{h(\mathbb{R}^n)}$, then clearly $w_0 \notin h(\mathbb{R}^n)$ and the claim holds. Otherwise, by post-composing with an isometric rotation, we may this time assume that $s_n(\infty) \in \mathbb{S}^n \setminus \overline{h(\mathbb{R}^n)}$ and that $w_0 = s_n(y_0)$ for some $y_0 \in \mathbb{R}^n$. Consequently, we obtain a bounded, continuous map $f \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ satisfying $h = s_n \circ f$. Lemma 3.2 again yields that f has a (K, Σ')-quasiregular value at the point y_0, where $\Sigma' = C(n, y_0)(1+|f|^2)^{n/2}\Sigma$. Since f is bounded, we have $\Sigma' \in L^1(\mathbb{R}^n) \cap L^{1+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^n)$, and hence the corresponding Euclidean result (Theorem 1.3 (i)) implies that either $w_0 \notin h(\mathbb{R}^n)$ or $h \equiv w_0$. □

4. Omitted quasiregular values and the spherical logarithm

In this section, we cover a key boundedness result for mappings $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n) \cap C(\Omega, \mathbb{R}^n)$ which have a (K, Σ)-quasiregular value at an omitted point $y_0 \in \mathbb{R}^n \setminus f(\Omega)$. This result is used near the end of the proof of Theorems 1.2 and 1.4.

4.1. **The spherical logarithm.**

The main result of this section is formulated and proven in terms of the spherical logarithm, a key tool in studying maps with an omitted quasiregular value. The first application of the spherical logarithm to the theory of quasiregular values was in the solution of the Astala–Iwaniec–Martin -question; see [28, Section 7] and the correction [30].

Definition 4.1. Let $\Omega \subset \mathbb{R}^n$ be an open domain. Suppose that a map $f \in C(\Omega, \mathbb{R}^n) \cap W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ has a (K, Σ)-quasiregular value at $y_0 \in \mathbb{R}^n$ with $K \geq 1$ and $\Sigma\colon \Omega \to [0, \infty)$ measurable. Suppose also that $y_0 \notin f(\mathbb{R}^n)$. Then the *spherical logarithm of f centered at y_0* is the map

(4.1)
$$G = (G_{\mathbb{R}}, G_{\mathbb{S}^{n-1}})\colon \Omega \to \mathbb{R} \times \mathbb{S}^{n-1},\ G(x) = \left(\log|f(x) - y_0|\,, \frac{f(x) - y_0}{|f(x) - y_0|} \right).$$

In particular, the spherical logarithm is of the form $G = \Theta_{y_0} \circ f$, where $\Theta_{y_0}\colon \mathbb{R}^n \setminus \{y_0\} \to \mathbb{R} \times \mathbb{S}^{n-1}$ is the conformal diffeomorphism defined by

$$\Theta_{y_0}(y) = \left(\log|y - y_0|\,, \frac{y - y_0}{|y - y_0|} \right)$$

for $y \in \mathbb{R}^n \setminus \{y_0\}$.

Suppose then that $f \in C(\Omega, \mathbb{R}^n) \cap W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ has a (K, Σ)-quasiregular value at an omitted point $y_0 \in \mathbb{R}^n$, and let G be the spherical logarithm of f centered at y_0. Since both f and Θ_{y_0} are continuous, G is also continuous. If we embed $\mathbb{R} \times \mathbb{S}^{n-1}$ isometrically to $\mathbb{R}^{n+1}$, we see by a chain rule that $G \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^{n+1})$. Moreover, if we equip $\mathbb{R} \times \mathbb{S}^{n-1}$ with the standard orientation, then G has a valid Jacobian J_G defined a.e. in $\mathbb{R}^n$ by

$$J_G \operatorname{vol}_n = dG_{\mathbb{R}} \wedge G^*_{\mathbb{S}^{n-1}} \operatorname{vol}_{\mathbb{S}^{n-1}} = dG_{\mathbb{R}} \wedge G^*_{\mathbb{S}^{n-1}} \star d(2^{-1}|x|^2),$$

where the Hodge star is taken in the Euclidean space $\mathbb{R}^n$ containing $\mathbb{S}^{n-1}$.

The main use of the spherical logarithm is that it transforms the definition (1.2) of quasiregular values into a version without the coefficient $|f - y_0|^n$. The following lemma sums up this property; its proof is a straightforward computation that is covered in [28, Lemma 7.1].

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^n$ be an open domain. Suppose that a map $f \in C(\Omega, \mathbb{R}^n) \cap W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ has a (K, Σ)-quasiregular value at $y_0 \in \mathbb{R}^n$ with $K \geq 1$ and $\Sigma \colon \Omega \to [0, \infty)$ measurable. Suppose that $y_0 \notin f(\mathbb{R}^n)$, and let G be the spherical logarithm of f centered at y_0. Then*

$$|DG| = \frac{|Df|}{|f - y_0|}, \quad J_G = \frac{J_f}{|f - y_0|^n}, \tag{4.2}$$

and therefore

$$|DG|^n \leq K J_G + \Sigma. \tag{4.3}$$

4.2. **The boundedness result.** We then state our boundedness result for the spherical logarithm. The result is closely connected to the solution of the Astala–Iwaniec–Martin question in [28, 30], and its proof is similarly technical.

Proposition 4.3. *Suppose that $G \colon \mathbb{R}^n \to \mathbb{R} \times \mathbb{S}^{n-1}$ is continuous, that $G \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R} \times \mathbb{S}^{n-1})$, and that $|DG| \in L^n(\mathbb{R}^n)$. If G satisfies* (4.3) *with $\Sigma \geq 0$ and $\Sigma \in L^{1-\varepsilon}(\mathbb{R}^n) \cap L^{1+\varepsilon}(\mathbb{R}^n)$ for some $\varepsilon \in (0, 1)$, then the $\mathbb{R}$-component $G_\mathbb{R}$ of G is bounded.*

In particular, Proposition 4.3 has the following immediate corollary.

Corollary 4.4. *Let $f \in C(\mathbb{R}^n, \mathbb{R}^n) \cap W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ have a (K, Σ)-quasiregular value at $y_0 \in \mathbb{R}^n$ with $K \geq 1$ and $\Sigma \in L^{1-\varepsilon}(\mathbb{R}^n) \cap L^{1+\varepsilon}(\mathbb{R}^n)$ with $\varepsilon > 0$. Suppose that $y_0 \notin f(\mathbb{R}^n)$ and $|Df| / |f - y_0| \in L^n(\mathbb{R}^n)$. Then f is bounded and* $\operatorname{dist}(y_0, f(\mathbb{R}^n)) > 0$.

Proof. If G is the spherical logarithm of f centered at y_0, then G satisfies (4.3), and (4.2) yields that $|DG| = |Df| / |f - y_0| \in L^n(\mathbb{R}^n)$. Thus, Proposition 4.3 yields that $G_\mathbb{R} = \log |f - y_0|$ is bounded, and the claim follows. □

The first step in the proof of Proposition 4.3 is a higher integrability result for $|DG|$. The argument is a standard proof based on reverse Hölder inequalities, and has already been recounted in e.g. [29, Lemma 6.1] and [9, Section 2.1] in similar situations. Regardless, we state the result and recall the short proof, as the previous statements do not cover the case where the target of G is $\mathbb{R} \times \mathbb{S}^{n-1}$.

Lemma 4.5. *Suppose that $G \colon \mathbb{R}^n \to \mathbb{R} \times \mathbb{S}^{n-1}$ is continuous, that $G \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R} \times \mathbb{S}^{n-1})$, and that $|DG| \in L^n(\mathbb{R}^n)$. If G satisfies* (4.3) *with $\Sigma \geq 0$ and $\Sigma \in L^1(\mathbb{R}^n) \cap L^{1+\varepsilon}(\mathbb{R}^n)$ for some $\varepsilon > 0$, then there exists $\varepsilon' \in (0, \varepsilon)$ such that*

$$\int_{\mathbb{R}^n} |DG|^{(1+\varepsilon')n} \lesssim_{n,K} \int_{\mathbb{R}^n} \Sigma^{1+\varepsilon'} < \infty.$$

Proof. Let Q be a cube in $\mathbb{R}^n$ with side length r. We select a cutoff function $\eta \in C^\infty(\mathbb{R}^n, [0, 1])$ satisfying $\eta|_Q \equiv 1$, $\operatorname{spt} \eta \subset 2Q$, and $\|\nabla \eta\|_{L^\infty} \lesssim_n r^{-1}$, where we interpret $2Q$ as the cube with the same center as Q but doubled side length. First, (4.3) yields

$$\int_{\mathbb{R}^n} \eta^n |DG|^n \leq K \int_{\mathbb{R}^n} \eta^n J_G + \int_{\mathbb{R}^n} \eta^n \Sigma.$$

We then use a Caccioppoli-type inequality for functions $\mathbb{R}^n \to \mathbb{R} \times M$, where M is an oriented Riemannian $(n-1)$-manifold without boundary; see [28, Lemma 2.3]. That is, if $G_\mathbb{R}$ is the $\mathbb{R}$-coordinate function of G, we obtain

$$\int_{\mathbb{R}^n} \eta^n J_G \leq n \int_{\mathbb{R}^n} \eta^{n-1} |DG|^{n-1} |\nabla \eta| \, |G_\mathbb{R} - c|$$

for every $c \in \mathbb{R}$. By combining these estimates, using Hölder's inequality, dividing by r^n, and applying the assumptions on η, we obtain

$$⨍_Q |DG|^n \lesssim_n K r^{-1} \left(⨍_{2Q} |G_\mathbb{R} - c|^{n^2} \right)^{\frac{1}{n^2}} \left(⨍_{2Q} |DG|^{\frac{n^2}{n+1}} \right)^{\frac{n^2-1}{n^2}} + ⨍_{2Q} \Sigma.$$

We then use the Sobolev-Poincaré inequality on the first integral to obtain

$$r^{-1} \left(⨍_{2Q} |G_\mathbb{R} - c|^{n^2} \right)^{\frac{1}{n^2}} \lesssim_n \left(⨍_{2Q} |DG_\mathbb{R}|^{\frac{n^2}{n+1}} \right)^{\frac{n+1}{n^2}} \leq \left(⨍_{2Q} |DG|^{\frac{n^2}{n+1}} \right)^{\frac{n+1}{n^2}},$$

where $c = (G_\mathbb{R})_{2Q}$. In conclusion, we obtain a reverse Hölder inequality

$$⨍_Q |DG|^n \lesssim_n K \left(⨍_{2Q} |DG|^{\frac{n^2}{n+1}} \right)^{\frac{n+1}{n}} + ⨍_{2Q} \Sigma.$$

As this holds for all cubes Q, we may hence use Gehring's lemma (see e.g. [26, Lemma 3.2]), obtaining that for some $\varepsilon' \in (0, \varepsilon)$ we have the estimate

$$\int_{\mathbb{R}^n} |DG|^{n(1+\varepsilon')} \lesssim_{n,K} \int_{\mathbb{R}^n} \Sigma^{1+\varepsilon'} < \infty.$$

□

The second step is a corresponding lower integrability result. For this, we use the corrected version of [28, Lemma 7.2] proven in [30]. This logarithmic lower integrability result builds upon ideas from [12]. As the proof is relatively complicated, we refer the reader to [30] for details.

Lemma 4.6 ([30, Lemma 7.2 (revised)]). *Suppose that $G \colon \mathbb{R}^n \to \mathbb{R} \times \mathbb{S}^{n-1}$ is continuous and non-constant, that $G \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R} \times \mathbb{S}^{n-1})$, and that $|DG| \in L^n(\mathbb{R}^n)$. If G satisfies (4.3) with $\Sigma \geq 0$ and $\Sigma \in L^1(\mathbb{R}^n) \cap L^{1-\varepsilon}(\mathbb{R}^n)$ for some $\varepsilon \in (0,1)$, then*

$$\int_{\mathbb{R}^n} |DG|^n \log^n \left(1 + \frac{1}{M(|DG|)} \right) < \infty,$$

where M stands for the (centered) Hardy-Littlewood maximal function.

With Lemmas 4.5 and 4.6 recorded, we are ready to prove Proposition 4.3.

Proof of Proposition 4.3. We may assume that G is non-constant. By Lemma 4.5 we have $|DG| \in L^{n+\varepsilon'}(\mathbb{R}^n)$ for some $\varepsilon' > 0$, and by Lemma 4.6, we have $|DG| \log(1 + M^{-1}(|DG|)) \in L^n(\mathbb{R}^n)$.

We fix $x_0 \in \mathbb{R}^n$, with the aim of estimating $|G_\mathbb{R}(x_0) - G_\mathbb{R}(0)|$. We base the proof on a standard chain of balls -argument used in e.g. [19]. In particular, for all $i \in \mathbb{Z}$, we let $r_i = |x_0| \, 2^{-|i|-2}$, and fix balls B_i, where $B_i = \mathbb{B}^n(2^{-|i|-1} x_0, r_i)$ for $i \leq 0$ and $B_i = \mathbb{B}^n((1 - 2^{-|i|-1}) x_0, r_i)$ for $i \geq 0$. See Figure 1 for an illustration.

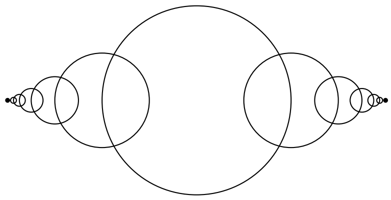

FIGURE 1. The chain of balls B_i from 0 to x_0.

The balls form a chain where the center of B_i is on the boundary of $B_{i-\operatorname{sgn}(i)}$ for $i \neq 0$. Moreover, no point in $\mathbb{R}^n$ is contained in more than two balls B_i, and the overlap of consecutive balls $B_i \cap B_{i-\operatorname{sgn}(i)}$ contains a ball B_i' with radius $r_i' = r_i/2$. By continuity, we also have that the integral averages $(G_\mathbb{R})_{B_i}$ converge to $G_\mathbb{R}(0)$ as $i \to -\infty$, and to $G_\mathbb{R}(x_0)$ as $i \to \infty$.

We thus obtain a telescopic sum estimate

$$|G_\mathbb{R}(x_0) - G_\mathbb{R}(0)| \leq \sum_{i=-\infty}^{\infty} \left| (G_\mathbb{R})_{B_{i+1}} - (G_\mathbb{R})_{B_i} \right|.$$

We show here the estimate for the upper end $i \geq 0$ of the series, as the estimate for the lower end $i < 0$ is analogous. By taking advantage of the ball B_{i+1}' contained in $B_i \cap B_{i+1}$ and by using the Sobolev–Poincaré inequality, we obtain

$$\begin{aligned}
\left| (G_\mathbb{R})_{B_{i+1}} - (G_\mathbb{R})_{B_i} \right| &\leq \left| (G_\mathbb{R})_{B_{i+1}'} - (G_\mathbb{R})_{B_i} \right| + \left| (G_\mathbb{R})_{B_{i+1}'} - (G_\mathbb{R})_{B_{i+1}} \right| \\
&\leq \fint_{B_{i+1}'} |G_\mathbb{R} - (G_\mathbb{R})_{B_i}| + \fint_{B_{i+1}'} \left| G_\mathbb{R} - (G_\mathbb{R})_{B_{i+1}} \right| \\
&\leq 4^n \fint_{B_i} |G_\mathbb{R} - (G_\mathbb{R})_{B_i}| + 2^n \fint_{B_{i+1}} \left| G_\mathbb{R} - (G_\mathbb{R})_{B_{i+1}} \right| \\
&\lesssim_n r_i \fint_{B_i} |DG| + r_{i+1} \fint_{B_{i+1}} |DG|.
\end{aligned}$$

Thus,

$$\sum_{i=0}^{\infty} \left| (G_\mathbb{R})_{B_{i+1}} - (G_\mathbb{R})_{B_i} \right| \lesssim_n \sum_{i=0}^{\infty} r_i^{-(n-1)} \int_{B_i} |DG|.$$

Since r_i is decreasing with respect to i when $i \geq 0$ and tends to zero as $i \to \infty$, there exists an index $i_0 \in \mathbb{Z}_{\geq 0}$ such that $r_i \leq 2$ when $i \geq i_0$, and $r_i > 2$ when $0 \leq i < i_0$. Thus, the end of the series can now be estimated using Hölder's inequality, yielding

$$\begin{aligned}
\sum_{i=i_0}^{\infty} r_i^{-(n-1)} \int_{B_i} |DG| &\lesssim_n \sum_{i=i_0}^{\infty} r_i^{-(n-1)} \left(r_i^n\right)^{\frac{\varepsilon'+n-1}{n+\varepsilon'}} \left(\int_{B_i} |DG|^{n+\varepsilon'} \right)^{\frac{1}{n+\varepsilon'}} \\
&= \sum_{i=i_0}^{\infty} r_i^{\frac{\varepsilon'}{n+\varepsilon'}} \left(\int_{B_i} |DG|^{n+\varepsilon'} \right)^{\frac{1}{n+\varepsilon'}} \leq \|DG\|_{L^{n+\varepsilon'}} \sum_{i=i_0}^{\infty} r_i^{\frac{\varepsilon'}{n+\varepsilon'}} \\
&\lesssim_{n,\varepsilon'} \|DG\|_{L^{n+\varepsilon'}} r_{i_0}^{\frac{\varepsilon'}{n+\varepsilon'}} \leq 2 \|DG\|_{L^{n+\varepsilon'}}.
\end{aligned}$$

In particular, this upper bound for the end of the series is finite by Lemma 4.5. The upper bound is also independent on x_0.

For the beginning part $0 \leq i < i_0$, we use the following elementary inequality: if Φ_1, Φ_2 are positive-valued real functions on an interval $I \subset \mathbb{R}$ with Φ_1 increasing and Φ_2 decreasing, then

$$1 \leq \max\left(\frac{\Phi_1(a)}{\Phi_1(b)}, \frac{\Phi_2(a)}{\Phi_2(b)}\right) \leq \frac{\Phi_1(a)}{\Phi_1(b)} + \frac{\Phi_2(a)}{\Phi_2(b)}$$

for all $a, b \in I$. We use this with $I = (0, \infty)$, $\Phi_1(t) = t^{n-1}$, $\Phi_2(t) = \log(1 + t^{-1})$, $a = M(|DG|)(x)$, and $b = r_i^{-1/2}$ for some $0 \leq i < i_0$. We obtain

$$1 \leq r_i^{\frac{n-1}{2}} M^{n-1}(|DG|) + \frac{\log\left(1 + 1/M(|DG|)\right)}{\log\left(1 + \sqrt{r_i}\right)}.$$

We observe that $\log(1 + \sqrt{r_i}) > \log(\sqrt{r_i}) = \log(r_i)/2$. Moreover, since $0 \leq i < i_0$, we have $r_i > 2$, and consequently $\log(r_i) > 0$. Hence, we conclude that, for $0 \leq i < i_0$, we have

$$1 \leq r_i^{\frac{n-1}{2}} M^{n-1}(|DG|) + \frac{2}{\log(r_i)} \log\left(1 + \frac{1}{M(|DG|)}\right),$$

and in particular,

$$\begin{aligned}\sum_{i=0}^{i_0-1} r_i^{-(n-1)} \int_{B_i} |DG| \leq{}& \sum_{i=0}^{i_0-1} r_i^{-\frac{n-1}{2}} \int_{B_i} |DG|\, M^{n-1}(|DG|) \\ &+ 2\sum_{i=0}^{i_0-1} \frac{1}{r_i^{n-1}\log(r_i)} \int_{B_i} |DG| \log\left(1 + \frac{1}{M(|DG|)}\right).\end{aligned} \tag{4.4}$$

We then utilize the fact that i_0 is the first index for which $r_i \leq 2$, from which it follows that $r_i > 2^{i_0 - i}$ when $0 \leq i < i_0$. Thus, we may estimate the first sum on the right hand side of (4.4) by

$$\sum_{i=0}^{i_0-1} r_i^{-\frac{n-1}{2}} \int_{B_i} |DG|\, M^{n-1}(|DG|) \leq \left(\int_{\mathbb{R}^n} M^n(|DG|)\right) \sum_{j=1}^{\infty} 2^{-\frac{n-1}{2}j},$$

which is again a finite upper bound independent on x_0 due to the Hardy-Littlewood maximal inequality. For the other sum on the right hand side of (4.4), we use both the integral and sum versions of Hölder's inequality, the fact that no point of $\mathbb{R}^n$ is contatined in more than two balls B_i, and the above estimate $r_i > 2^{i_0 - i}$, in order to obtain

$$\begin{aligned}
&\sum_{i=0}^{i_0-1} \frac{1}{r_i^{n-1}\log(r_i)} \int_{B_i} |DG| \log\left(1 + \frac{1}{M(|DG|)}\right)\\
&\quad\lesssim_n \sum_{i=0}^{i_0-1} \frac{1}{\log(r_i)} \left(\int_{B_i} |DG|^n \log^n\left(1 + \frac{1}{M(|DG|)}\right)\right)^{\frac{1}{n}}\\
&\quad\leq \left(\sum_{i=0}^{i_0-1} \frac{1}{\log^{\frac{n}{n-1}}(r_i)}\right)^{\frac{n-1}{n}} \left(\sum_{i=0}^{i_0-1} \int_{B_i} |DG|^n \log^n\left(1 + \frac{1}{M(|DG|)}\right)\right)^{\frac{1}{n}}\\
&\quad\leq \left(\sum_{j=1}^{\infty} \frac{1}{(\log(2)j)^{\frac{n}{n-1}}}\right)^{\frac{n-1}{n}} \left(2\int_{\mathbb{R}^n} |DG|^n \log^n\left(1 + \frac{1}{M(|DG|)}\right)\right)^{\frac{1}{n}}.
\end{aligned}$$

This upper bound is finite by Lemma 4.6. It is also independent of x_0. Thus, combining our estimates, we have an x_0-independent upper bound for the upper end $i \geq 0$ of the telescopic sum of integral averages. An identical argument proves a similar bound for the lower end $i < 0$, completing the proof. □

5. The logarithmic potential and Caccioppoli inequalities

In this section, we prove the Caccioppoli-type inequalities used in the proof. In particular, we prove counterparts to [6, Lemmas 4.2 and 5.4] where we assume (1.2) instead of full quasiregularity. Since our setting still allows for large sets where $J_f(x) = 0$ and $Df(x)$ is non-invertible, we lack a good counterpart for the induced conformal structure $G_f(x) = J_f^{-2/n}(x)[D^T f(x)\, Df(x)]^{-1}$ associated to f. Thus, instead of using $\mathcal{A}$-subharmonic theory as in the original proofs, we rely on more direct computations.

5.1. **The logarithmic potential.** We begin by recalling the logarithmic potential function from [6, Section 3]. We first define a function $S\colon [0,\infty) \to [0,1)$ by

$$S(r) = \frac{\mathrm{vol}_{\mathbb{S}^n}(\mathbb{B}^n(0,r))}{\mathrm{vol}_{\mathbb{S}^n}(\mathbb{R}^n)}. \tag{5.1}$$

By using (3.4), one can see that

$$S(r) = \frac{2^n \,\mathrm{vol}_{\mathbb{S}^{n-1}}(\mathbb{R}^{n-1})}{\mathrm{vol}_{\mathbb{S}^n}(\mathbb{R}^n)} \int_0^r \frac{t^{n-1}\,\mathrm{d}t}{\left(1+t^2\right)^n}.$$

In particular,

$$S'(r) = \frac{C(n) r^{n-1}}{\left(1+r^2\right)^n}, \tag{5.2}$$

and we obtain the following estimates describing the asymptotic behavior of $S(r)$ and $S'(r)$ for large and small r:

$$S(r) \lesssim_n \min(r^n, 1), \qquad S'(r) \lesssim_n \min(r^{n-1}, r^{-(n+1)}). \tag{5.3}$$

Next, we define a function $H\colon [0,\infty) \to [0,\infty)$ by

$$H(r) = \int_0^r \frac{S^{\frac{1}{n-1}}(t)\,\mathrm{d}t}{t}. \tag{5.4}$$

Consequently, we have

$$H'(r) = \frac{S^{\frac{1}{n-1}}(r)}{r}, \tag{5.5}$$

and by applying (5.3), we get the estimates

$$H(r) \lesssim_n \min\left(r^{\frac{n}{n-1}}, 1 + \lvert\log(r)\rvert\right), \qquad H'(r) \lesssim_n \min\left(r^{\frac{1}{n-1}}, r^{-1}\right). \tag{5.6}$$

The logarithmic potential $v_{\mathbb{R}^n}\colon \mathbb{R}^n \to [0,\infty)$ at infinity is then defined on $\mathbb{R}^n$ by

$$v_{\mathbb{R}^n}(x) = H(\lvert x\rvert). \tag{5.7}$$

Since $v_{\mathbb{R}^n}$ is a real-valued radial function, we have

$$\nabla v_{\mathbb{R}^n}(x) = H'(\lvert x\rvert)\frac{x}{\lvert x\rvert} \qquad \text{and} \qquad \lvert\nabla v_{\mathbb{R}^n}(x)\rvert = H'(\lvert x\rvert). \tag{5.8}$$

Moreover, recall that if $v \in C^1(\mathbb{R}^n)$ is a function such that $\lvert\nabla v\rvert^{n-2}\,\nabla v \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, then the *$n$-Laplacian* $\Delta_n v$ of v is defined by

$$\Delta_n v = \nabla \cdot (\lvert\nabla v\rvert^{n-2}\,\nabla v).$$

We note that the n-Laplacian satisfies the identity

$$(\Delta_n v)\,\mathrm{vol}_n = d(\lvert dv\rvert^{n-2} \star dv).$$

We record that the n-Laplacian of $v_{\mathbb{R}^n}$ is in fact exactly the density of the spherical volume; we refer to [6, Lemma 3.1] for the proof.

Lemma 5.1. *We have $v_{\mathbb{R}^n} \in C^1(\mathbb{R}^n)$, $\lvert\nabla v_{\mathbb{R}^n}\rvert^{n-2}\,\nabla v_{\mathbb{R}^n} \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, and*

$$\Delta_n v_{\mathbb{R}^n}(x) = \frac{2^n}{\left(1 + \lvert x\rvert^2\right)^n} = J_{s_n}(x).$$

5.2. **Quasiregular values and superlevel sets.** The study of sublevel and superlevel sets of the form $\{\lvert f - y_0\rvert > L\}$ and $\{\lvert f - y_0\rvert < L\}$ has been perhaps the most fundamental tool in the developement of the prior theory on quasiregular values; see in particular [28, Section 5] and [29, Section 4]. Such sublevel and superlevel sets also play a key role in this paper. Indeed, we prove a counterpart to [6, Lemma 4.4], which essentially yields that superlevel sets $\{\lvert f\rvert > L\}$ of an entire quasiregular function f have no bounded components. As stated in the introduction, superlevel set methods do not fully eliminate the existence of bounded components of $\{\lvert f\rvert > L\}$ in our case, which ends up causing significant complications during the proof. However, we do get a type of control on the total size of the bounded components of $\{\lvert f\rvert > L\}$.

In particular, our main counterpart to [6, Lemma 4.4] is the following general result, which is similar in spirit to [28, Lemma 5.3] and [29, Lemma 4.3].

Lemma 5.2. *Let $y_0 \in \mathbb{R}^n$ and $r > 0$. Suppose that $f \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ is continuous and satisfies an estimate of the form*

$$|Df|^n \leq KJ_f + \tilde{\Sigma}, \tag{5.9}$$

where we assume $K \in \mathbb{R}$, $\tilde{\Sigma} \geq 0$, and $\tilde{\Sigma} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Let U be a bounded component of $f^{-1}(\mathbb{R}^n \setminus \overline{\mathbb{B}^n}(y_0, r))$. Then for any continuous function $\Phi\colon [r, \infty) \to [0, \infty)$, we have

$$\int_U \Phi(|f - y_0|)\,|Df|^n \leq \int_U \Phi(|f - y_0|)\tilde{\Sigma}.$$

Proof. Since U is bounded and since f is continuous, $f(\overline{U})$ is compact, and hence there exists $y_1 \in \mathbb{R}^n \setminus (\overline{\mathbb{B}^n}(y_0, r) \cup f(\overline{U}))$. By the boundedness of U and the continuity of f and Φ, we also have that the functions $\Phi(|f - y_0|)\,|Df|^n$ and $\Phi(|f - y_0|)\tilde{\Sigma}$ are integrable over U. By a Sobolev change of variables, see e.g. [14, Theorem 5.27], we have

$$\int_U \Phi(|f - y_0|)J_f = \int_{\mathbb{R}^n \setminus \overline{\mathbb{B}^n}(y_0, r)} \Phi(|y - y_0|) \deg(f, y, U) \operatorname{vol}_n(y).$$

However, since U is a connected component of $f^{-1}(\mathbb{R}^n \setminus \overline{\mathbb{B}^n}(y_0, r))$, we have $f(\partial U) \subset \partial\mathbb{B}^n(y_0, r)$. Thus, since $\mathbb{R}^n \setminus \overline{\mathbb{B}^n}(y_0, r)$ is connected, we have $\deg(f, y, U) = \deg(f, y_1, U) = 0$ for every $y \in \mathbb{R}^n \setminus \overline{\mathbb{B}^n}(y_0, r)$; see for instance [14, Theorem 2.1 and Theorem 2.3 (3)]. In conclusion,

$$\int_U \Phi(|f - y_0|)J_f = 0.$$

Consequently, the desired estimate follows by multiplying (5.9) by $\Phi(|f - y_0|)$ and by integrating both sides over U. □

5.3. **Pull-backs of the spherical volume.** Let $K \geq 1$ and $\Sigma \in L^1(\mathbb{R}^n)$, and suppose that $f \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ is a continuous map for which $s_n \circ f$ has a (K, Σ)-quasiregular value with respect to the spherical metric at $s_n(\infty)$. Note that by Lemma 3.1, this assumption is true if f has a (K, Σ)-quasiregular value at a point $y_0 \in \mathbb{R}^n$, up to an additional multiplicative constant $C = C(n, y_0)$ on Σ. By Lemma 3.2, the map f satisfies

$$|Df|^n \leq KJ_f + C(n)\big(1 + |f|^2\big)^{\frac{n}{2}}\Sigma \tag{5.10}$$

a.e. in $\mathbb{R}^n$.

We use the notation $\Sigma(E)$ to denote the integral of Σ over a measurable set $E \subset \mathbb{R}^n$. Moreover, we use J_f^+ and J_f^- to denote the positive and negative parts of the Jacobian of f, which are given by $J_f^+(x) = \max(0, J_f(x))$ and $J_f^-(x) = \max(0, -J_f(x))$ for a.e. $x \in \mathbb{R}^n$. In particular, $J_f = J_f^+ - J_f^-$ a.e. in $\mathbb{R}^n$.

We then let A_f^+ and A_f^- denote the positive and negative parts of the pull-back of the spherical volume under f. That is, if $E \subset \mathbb{R}^n$ is Lebesgue measurable, then $A_f^+(E)$ and $A_f^-(E)$ are defined by

$$A_f^+(E) = \int_E \frac{2^n J_f^+}{\big(1 + |f|^2\big)^n} \quad \text{and} \quad A_f^-(E) = \int_E \frac{2^n J_f^-}{\big(1 + |f|^2\big)^n}. \tag{5.11}$$

Both A_f^+ and A_f^- are measures on $\mathbb{R}^n$. We observe the following fact about A_f^-.

Lemma 5.3. *Let $f \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ be a non-constant continuous function that satisfies* (5.10)*, where $K \geq 1$ and $\Sigma \in L^1(\mathbb{R}^n)$ with $\Sigma \geq 0$. Then for every measurable $E \subset \mathbb{R}^n$, we have*

$$A_f^-(E) \lesssim_n \Sigma(E) < \infty.$$

Proof. The equation (5.10) can be rewritten as

$$|Df|^n + KJ_f^- \leq KJ_f^+ + C(n)(1 + |f|^2)^{n/2}\Sigma.$$

Since J_f^+ vanishes when J_f^- is non-zero, we hence obtain

$$J_f^- \lesssim_n K^{-1}(1 + |f|^2)^{n/2}\Sigma. \tag{5.12}$$

In particular, using $K^{-1} \leq 1$, (5.12) yields the estimate

$$A_f^-(E) = \int_E \frac{2^n J_f^-}{\left(1 + |f|^2\right)^n} \lesssim_n \int_E \frac{2^n}{K\left(1 + |f|^2\right)^{\frac{n}{2}}} \Sigma \lesssim_n \Sigma(E).$$

□

In particular, for a map $f \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n) \cap C(\mathbb{R}^n, \mathbb{R}^n)$ satisfying (5.10) with $K \geq 1$ and $\Sigma \in L^1(\mathbb{R}^n)$, the quantity $A_f^+(E) - A_f^-(E)$ is well-defined for every measurable $E \subset \mathbb{R}^n$, since Lemma 5.3 eliminates the possibility of the expression evaluating to $\infty - \infty$. Thus, we may define $A_f(E) = A_f^+(E) - A_f^-(E)$ for every measurable $E \subset \mathbb{R}^n$, obtaining a signed measure A_f on $\mathbb{R}^n$. The measure A_f is the pull-back of the spherical volume under f, and is given by

$$A_f(E) = \int_E \frac{2^n J_f}{\left(1 + |f|^2\right)^n}. \tag{5.13}$$

for every measurable $E \subset \mathbb{R}^n$. We also use $|A_f|$ to denote the total variation measure of A_f, which is naturally given by

$$|A_f|(E) = A_f^+(E) + A_f^-(E) = \int_E \frac{2^n |J_f|}{\left(1 + |f|^2\right)^n}$$

for every measurable $E \subset \mathbb{R}^n$

5.4. **Measure estimates and Caccioppoli-type inequalities.** With the measure A_f defined, we then proceed to obtain various estimates for A_f. We start with a technical Caccioppoli-type estimate that sees multiple uses in the proofs.

Lemma 5.4. *Let $f \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ be a non-constant continuous function that satisfies* (5.10)*, where $K \geq 1$, $\Sigma \geq 0$, and $\Sigma \in L^1(\mathbb{R}^n) \cap L^{1+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $\varepsilon > 0$. Then for every $L \geq 0$ and $\eta \in C_0^\infty(\mathbb{R}^n)$ with $\eta \geq 0$, we have*

$$\int_{\{v_{\mathbb{R}^n} \circ f < L\}} \eta^n \left(|dv_{\mathbb{R}^n}|^n \circ f\right) |Df|^n \lesssim_n K^n L^n \int_{\{v_{\mathbb{R}^n} \circ f < L\}} |d\eta|^n \\ + KL \int_{\{v_{\mathbb{R}^n} \circ f < L\}} \eta^n \, \mathrm{d}|A_f| + \int_{\{v_{\mathbb{R}^n} \circ f < L\}} \eta^n \Sigma.$$

Proof. For brevity, we denote $v_L = \min(v_{\mathbb{R}^n}, L)$, $u = v_{\mathbb{R}^n} \circ f$, and $u_L = \min(u, L) = v_L \circ f$. We may assume $L > 0$, as the case $L = 0$ is trivial due to $\{u < L\}$ being empty in this case.

We first observe that $(|dv_{\mathbb{R}^n}| \circ f)(1+|f|^2)^{1/2} = H'(|f|)(1+|f|^2)^{1/2} \leq C(n)$ by (5.6) and (5.8). We combine this with (5.10) to obtain

$$\begin{aligned}(5.14)\quad &\int_{\{u<L\}} \eta^n \big(|dv_{\mathbb{R}^n}|^n \circ f\big) \, |Df|^n \\ &\leq K \int_{\{u<L\}} \eta^n \big(|dv_{\mathbb{R}^n}|^n \circ f\big) J_f + C(n) \int_{\{u<L\}} \eta^n \big(|dv_{\mathbb{R}^n}|^n \circ f\big) \big(1 + |f|^2\big)^{\frac{n}{2}} \Sigma \\ &\leq K \int_{\{u<L\}} \eta^n f^*(|dv_{\mathbb{R}^n}|^n \operatorname{vol}_n) + C(n) \int_{\{u<L\}} \eta^n \Sigma.\end{aligned}$$

Let then $\mathcal{X}_{\{u<L\}}$ be the characteristic function of $\{u < L\}$. We claim that

$$\mathcal{X}_{\{u<L\}} f^*(|dv_{\mathbb{R}^n}|^n \operatorname{vol}_n) = du_L \wedge f^*(|dv_{\mathbb{R}^n}|^{n-2} \star dv_{\mathbb{R}^n}) \tag{5.15}$$

a.e. in $\mathbb{R}^n$. Indeed, du_L vanishes a.e. in the set $\{u \geq L\}$; see e.g. [22, Corollary 1.21]. In $\{u < L\}$, we may compute as follows:

$$\begin{aligned}du_L \wedge f^*(|dv_{\mathbb{R}^n}|^{n-2} \star dv_{\mathbb{R}^n}) &= f^* dv_{\mathbb{R}^n} \wedge f^*(|dv_{\mathbb{R}^n}|^{n-2} \star dv_{\mathbb{R}^n}) \\ &= f^*(|dv_{\mathbb{R}^n}|^{n-2} \, dv_{\mathbb{R}^n} \wedge \star dv_{\mathbb{R}^n}) = f^*(|dv_{\mathbb{R}^n}|^n \operatorname{vol}_n).\end{aligned}$$

Since the $(n-1)$-form $(v_L - L) \, |dv_{\mathbb{R}^n}|^{n-2} \star dv_{\mathbb{R}^n}$ has Lipschitz coefficients, we have $f^*((v_L - L) \, |dv_{\mathbb{R}^n}|^{n-2} \star dv_{\mathbb{R}^n}) \in W^{d,n/(n-1),1}_{\text{loc}}(\wedge^{n-1} T^* \mathbb{R}^n)$ by Lemma 2.1. Thus, by Corollary 2.2,

$$\begin{aligned}&df^*((v_L - L) \, |dv_{\mathbb{R}^n}|^{n-2} \star dv_{\mathbb{R}^n}) \\ &\quad = du_L \wedge f^*(|dv_{\mathbb{R}^n}|^{n-2} \star dv_{\mathbb{R}^n}) + (u_L - L) df^*(|dv_{\mathbb{R}^n}|^{n-2} \star dv_{\mathbb{R}^n}).\end{aligned}$$

Now, by using (2.2), we may compute that

$$\begin{aligned}&\int_{\mathbb{R}^n} \eta^n du_L \wedge f^*(|dv_{\mathbb{R}^n}|^{n-2} \star dv_{\mathbb{R}^n}) \\ &\quad = \int_{\mathbb{R}^n} \eta^n df^*((v_L - L) \, |dv_{\mathbb{R}^n}|^{n-2} \star dv_{\mathbb{R}^n}) \\ &\qquad\quad - \int_{\mathbb{R}^n} \eta^n (u_L - L) df^*(|dv_{\mathbb{R}^n}|^{n-2} \star dv_{\mathbb{R}^n}) \\ &\quad = - \int_{\mathbb{R}^n} d\eta^n \wedge f^*((v_L - L) \, |dv_{\mathbb{R}^n}|^{n-2} \star dv_{\mathbb{R}^n}) \\ &\qquad\quad - \int_{\mathbb{R}^n} \eta^n (u_L - L) df^*(|dv_{\mathbb{R}^n}|^{n-2} \star dv_{\mathbb{R}^n}).\end{aligned} \tag{5.16}$$

By Lemma 5.1, we have that $|dv_{\mathbb{R}^n}|^{n-2} \star dv_{\mathbb{R}^n}$ is a C^1-smooth form, and consequently

$$df^*(|dv_{\mathbb{R}^n}|^{n-2} \star dv_{\mathbb{R}^n}) = f^* d(|dv_{\mathbb{R}^n}|^{n-2} \star dv_{\mathbb{R}^n}) = f^* s_n^* \operatorname{vol}_{\mathbb{S}^n} \tag{5.17}$$

in the weak sense. On the other hand, by using (2.3), we obtain

$$\left| f^*((v_L - L) \, |dv|^{n-2} \star dv_{\mathbb{R}^n}) \right| \leq |u_L - L| \left(|dv_{\mathbb{R}^n}|^{n-1} \circ f \right) |Df|^{n-1} . \tag{5.18}$$

By combining (5.15), (5.16), (5.17) and (5.18), we obtain the estimate

$$
\begin{aligned}
&K \int_{\{u<L\}} \eta^n f^*(|dv_{\mathbb{R}^n}|^n \operatorname{vol}_n) = K \int_{\mathbb{R}^n} \eta^n du_L \wedge f^*(|dv_{\mathbb{R}^n}|^{n-2} \star dv) \\
&\leq K \int_{\mathbb{R}^n} |d\eta^n| \left| f^*((v_L - L)\, |dv_{\mathbb{R}^n}|^{n-2} \star dv_{\mathbb{R}^n}) \right| + K \int_{\mathbb{R}^n} |u_L - L|\, \eta^n \,|f^* s_n^* \operatorname{vol}_{\mathbb{S}^n}| \\
&\leq Kn \int_{\mathbb{R}^n} |u_L - L|\, |d\eta|\, (\eta(|dv_{\mathbb{R}^n}| \circ f)\, |Df|)^{n-1} + K \int_{\mathbb{R}^n} \eta^n \,|u_L - L| \,\mathrm{d}|A_f|.
\end{aligned}
$$

Moreover, since $u_L - L = 0$ in $\{u \geq L\}$, and since $|u_L - L| \leq L$, we obtain

$$
\begin{aligned}
(5.19)\quad &K \int_{\{u<L\}} \eta^n f^*(|dv_{\mathbb{R}^n}|^n \operatorname{vol}_n) \\
&\leq KLn \int_{\{u<L\}} |d\eta|\, (\eta(|dv_{\mathbb{R}^n}| \circ f)\, |Df|)^{n-1} + KL \int_{\{u<L\}} \eta^n \,\mathrm{d}|A_f|.
\end{aligned}
$$

We recall Young's inequality, which states that $ab \leq a^p/p + b^q/q$ for $a, b \geq 0$ and $p, q \geq 1$ with $p^{-1} + q^{-1} = 1$. We estimate the first term of the right hand side of (5.19) by Young's inequality, resulting in

$$
\begin{aligned}
(5.20)\quad &KLn \int_{\{u<L\}} |d\eta|\, (\eta(|dv_{\mathbb{R}^n}| \circ f)\, |Df|)^{n-1} \\
&\leq K^n L^n n^{n-1} \int_{\{u<L\}} |d\eta|^n + \frac{n-1}{n} \int_{\{u<L\}} \eta^n \left(|dv_{\mathbb{R}^n}|^n \circ f\right) |Df|^n .
\end{aligned}
$$

We note that since $|dv_{\mathbb{R}^n}|$ is bounded by (5.6) and (5.8), $\eta^n(|dv_{\mathbb{R}^n}|^n \circ f)\, |Df|^n$ has finite integral over $\mathbb{R}^n$. We chain (5.14), (5.19), and (5.20) together, and absorb the integral of $\eta^n(|dv_{\mathbb{R}^n}|^n \circ f)\, |Df|^n$ from the right side of (5.20) to the left side of (5.14). The claim follows. □

The most immediate consequence of Lemma 5.4 is the following corollary, which is our counterpart to [6, Lemma 5.4].

Corollary 5.5. *Let $f \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ be a non-constant continuous function that satisfies (5.10), where $K \geq 1$, $\Sigma \geq 0$, and $\Sigma \in L^1(\mathbb{R}^n) \cap L^{1+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $\varepsilon > 0$. Then for every open ball $B \subset \mathbb{R}^n$ and every $L > 0$, we have*

$$
\int_{B \cap \{v_{\mathbb{R}^n} \circ f < L\}} |d(v_{\mathbb{R}^n} \circ f)|^n \lesssim_n K^n L^n + KL|A_f|(2B) + \Sigma(\mathbb{R}^n).
$$

Proof. Let $B = \mathbb{B}^n(x_0, r)$ with $x_0 \in \mathbb{R}^n$ and $r > 0$. We fix a cutoff function $\eta \in C^\infty_0(\mathbb{R}^n, [0, 1])$ satisfying $\eta \equiv 1$ on B, $\operatorname{spt} \eta \subset 2B$, and $\|d\eta\|_{L^\infty} \leq 2r^{-1}$.

Since v is C^1, the chain rule of C^1 and Sobolev functions yields that

$$
|du| = |df^* v| = |f^* dv| \leq (|dv| \circ f)\, |Df| .
$$

Hence,

$$
\int_{B \cap \{u<L\}} |du|^n \leq \int_{\{u<L\}} \eta^n \left(|dv|^n \circ f\right) |Df|^n .
$$

We now use Lemma 5.4, obtaining

$$\begin{aligned}\int_{\{u<L\}} &\eta^n \big(|dv|^n \circ f\big)\, |Df|^n \\ &\lesssim_n K^n L^n \int_{\{u<L\}} |d\eta|^n + KL \int_{\{u<L\}} \eta^n \,\mathrm{d}|A_f| + \int_{\{u<L\}} \eta^n \Sigma \\ &\qquad\qquad \leq K^n L^n \, \|d\eta\|_{L^n}^n + KL|A_f|(2B) + \Sigma(\mathbb{R}^n).\end{aligned}$$

Since

$$\|d\eta\|_{L^n}^n \leq \|d\eta\|_{L^\infty}^n \operatorname{vol}_n(\operatorname{spt}\eta) \leq \big(2r^{-1}\big)^n \operatorname{vol}_n(2B) \leq C(n),$$

the claim follows. □

5.5. **Pseudosupremum of the induced potential.** Besides Corollary 5.5, we also use a counterpart to [6, Lemma 4.2], which in the quasiregular setting yields $A_f(B) \lesssim_{n,K} \sup_{2B}(v_{\mathbb{R}^n} \circ f)^{n-1}$ for every ball $B \subset \mathbb{R}^n$. An estimate based on $\sup_{2B}(v_{\mathbb{R}^n} \circ f)^{n-1}$ is however insufficient for us, since the function $v_{\mathbb{R}^n} \circ f$ need not have the property that every component of $\{v_{\mathbb{R}^n} \circ f > t\}$ is unbounded for every $t > 0$. To compensate for this, we define a pseudosupremum of a continuous function $\varphi\colon \mathbb{R}^n \to [0, \infty)$ as follows.

Definition 5.6. Let $\varphi\colon \mathbb{R}^n \to [0, \infty)$ be continuous, and let $E \subset \mathbb{R}^n$. The *pseudosupremum* $\widetilde{\sup}_E \varphi$ of φ over E is defined by

$$\widetilde{\sup_E}\, \varphi = \sup\{t \in \mathbb{R} : E \text{ meets an unbounded component of } \varphi^{-1}(t, \infty)\}.$$

Remark 5.7. The definition of the pseudosupremum is similar in spirit to that of the classical essential supremum used e.g. in the definition of L^∞-spaces. Recall that if $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ is a measurable function and if $E \subset \mathbb{R}^n$ is measurable, then the essential supremum of φ over E is given by

$$\operatorname*{ess\,sup}_E \varphi = \sup\{t \in \mathbb{R} : E \cap \varphi^{-1}(t, \infty) \text{ has positive measure}\}.$$

For bounded E, we clearly have $0 \leq \widetilde{\sup}_E \varphi \leq \sup_E \varphi < \infty$ for every continuous $\varphi\colon \mathbb{R}^n \to [0, \infty)$. Moreover, if $E_1 \subset E_2$, then $\widetilde{\sup}_{E_1} \varphi \leq \widetilde{\sup}_{E_2} \varphi$. We also note that $(\widetilde{\sup}_E \varphi)^p = \widetilde{\sup}_E(\varphi^p)$ for $p \geq 0$, allowing us to ignore this distinction in our notation.

By combining the pseudosupremum with Lemma 5.2, we obtain a key lemma which, given a ball $B \subset \mathbb{R}^n$ and an entire map f with a quasiregular value, essentially grants us strong control over the L^n-norm of $\nabla(v_{\mathbb{R}^n} \circ f)$ on the set $\{x \in B : v_{\mathbb{R}^n} \circ f(x) \geq \widetilde{\sup}_B(v_{\mathbb{R}^n} \circ f)\}$.

Lemma 5.8. *Let $f \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ be a non-constant continuous function that satisfies* (5.10)*, where $K \geq 1$, $\Sigma \geq 0$, and $\Sigma \in L^1(\mathbb{R}^n) \cap L^{1+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $\varepsilon > 0$. Then for every open ball $B \subset \mathbb{R}^n$, every $\eta \in C_0^\infty(B)$ with $\eta \geq 0$, and every $L > \widetilde{\sup}_B u$, we have*

$$\int_{\{v_{\mathbb{R}^n} \circ f \geq L\}} \eta^n \big(|dv_{\mathbb{R}^n}|^n \circ f\big)\, |Df|^n \lesssim_n \|\eta\|_{L^\infty} \Sigma(\mathbb{R}^n).$$

Proof. We denote $u = v_{\mathbb{R}^n} \circ f$ for brevity, and let $U = \{u > L\}$. By definition, B meets only bounded components of U; denote the union of these components of U that meet B by U_B. Now, recalling that $\operatorname{spt}\eta \subset B$, that $|dv_{\mathbb{R}^n}| \circ f = H'(|f|)$ by (5.8), and that f satisfies (5.10), we may use Lemma 5.2 with $\tilde{\Sigma} = C(n)(1+|f|^2)^{n/2}\Sigma$, $\Phi(t) = [H'(t)]^n$, and $y_0 = 0$ to obtain the estimate

$$\begin{aligned}\int_{\{u>L\}} \eta^n \big(|dv_{\mathbb{R}^n}|^n \circ f\big) \, |Df|^n &\leq \|\eta\|_{L^\infty} \int_{U_B} \big(|dv_{\mathbb{R}^n}|^n \circ f\big) \, |Df|^n \\ &\lesssim_n \|\eta\|_{L^\infty} \int_{U_B} \big(|dv_{\mathbb{R}^n}|^n \circ f\big) \big(1+|f|^2\big)^{\frac{n}{2}} \Sigma.\end{aligned}$$

Since also $J_f = 0$ a.e. in $\{u = L\}$ due to image of this set under f having zero Hausdorff n-measure, we also have $|Df|^n \lesssim_n (1+|f|^2)^{n/2}\Sigma$ a.e. in $\{u = L\}$ by (5.10). Hence, we may improve the previous estimate to

$$\int_{\{u\geq L\}} \eta^n \big(|dv_{\mathbb{R}^n}|^n \circ f\big) \, |Df|^n \lesssim_n \|\eta\|_{L^\infty} \int_{U_{2B}\cup\{u=L\}} \big(|dv_{\mathbb{R}^n}|^n \circ f\big) \big(1+|f|^2\big)^{\frac{n}{2}} \Sigma.$$

Since $(|dv_{\mathbb{R}^n}| \circ f)(1+|f|^2)^{1/2} \leq C(n)$ by (5.6) and (5.8), we obtain the desired estimate

$$\int_{\{u\geq L\}} \eta^n \big(|dv_{\mathbb{R}^n}|^n \circ f\big) \, |Df|^n \lesssim_n \|\eta\|_{L^\infty} \Sigma(U_{2B} \cup \{u = L\}) \leq \|\eta\|_{L^\infty} \Sigma(\mathbb{R}^n).$$

□

With this, we prove our counterpart to [6, Lemma 4.2].

Lemma 5.9. *Let $f \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ be a non-constant continuous function that satisfies* (5.10)*, where $K \geq 1$, $\Sigma \geq 0$, and $\Sigma \in L^1(\mathbb{R}^n) \cap L^{1+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $\varepsilon > 0$. Then for every open ball $B \subset \mathbb{R}^n$, we have*

$$|A_f|(B) \lesssim_n K^{n-1} \widetilde{\sup_{2B}} (v_{\mathbb{R}^n} \circ f)^{n-1} + C(n)\big(\Sigma(\mathbb{R}^n) + [\Sigma(\mathbb{R}^n)]^{\frac{n-1}{n}}\big).$$

Proof. Let B be an open ball with radius $r > 0$. We fix a cutoff function $\eta \in C^\infty_0(\mathbb{R}^n, [0,1])$ satisfying $\eta \equiv 1$ on B, $\operatorname{spt} \eta \subset 2B$, and $\|d\eta(x)\|_{L^\infty} \leq 2r^{-1}$.

We first estimate that

$$|A_f|(B) \leq \int_{\mathbb{R}^n} \eta^n \, \mathrm{d}|A_f| \leq \int_{\mathbb{R}^n} \eta^n \, \mathrm{d}A_f + 2A_f^-(2B)$$

By Lemma 5.3, we have $A_f^-(2B) \lesssim_n \Sigma(B)$, and by Lemma 5.1, we have $d(|dv_{\mathbb{R}^n}|^{n-2} \star dv_{\mathbb{R}^n}) = s_n^* \operatorname{vol}_{\mathbb{S}^n}$. Hence, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \eta^n \, \mathrm{d}A_f &= \int_{\mathbb{R}^n} \eta^n f^* s_n^* \operatorname{vol}_{\mathbb{S}^n} = \int_{\mathbb{R}^n} \eta^n f^* d(|dv_{\mathbb{R}^n}|^{n-2} \star dv_{\mathbb{R}^n}) \\ &= \int_{\mathbb{R}^n} \eta^n df^*(|dv_{\mathbb{R}^n}|^{n-2} \star dv_{\mathbb{R}^n}) \leq n \int_{\mathbb{R}^n} \eta^{n-1} \, |d\eta| \, |f^*(|dv_{\mathbb{R}^n}|^{n-2} \star dv_{\mathbb{R}^n})|,\end{aligned}$$

where d and f^* commute since the form $|dv_{\mathbb{R}^n}|^{n-2} \star dv_{\mathbb{R}^n}$ is C^1-smooth. Furthermore, we may estimate using (2.3) that

$$|f^*(|dv_{\mathbb{R}^n}|^{n-2} \star dv)| \leq |Df|^{n-1} \, (|dv_{\mathbb{R}^n}|^{n-1} \circ f).$$

Consequently by Hölder's inequality,

$$\int_{\mathbb{R}^n} \eta^{n-1} \lvert d\eta \rvert \lvert f^*(\lvert dv_{\mathbb{R}^n} \rvert^{n-2} \star dv_{\mathbb{R}^n}) \rvert \\ \leq \left(\int_{\mathbb{R}^n} \lvert d\eta \rvert^n \right)^{\frac{1}{n}} \left(\int_{\mathbb{R}^n} \eta^n \big(\lvert dv_{\mathbb{R}^n} \rvert^n \circ f \big) \lvert Df \rvert^n \right)^{\frac{n-1}{n}}.$$

Since $\operatorname{spt} \eta \subset 2B$, we have by our estimate $\lvert d\eta \rvert^n \leq 4^n r^{-n}$ that $\lVert d\eta \rVert_{L^n} \leq C(n)$. In conclusion,

$$\lvert A_f \rvert (B) \leq \int_{\mathbb{R}^n} \eta^n \,\mathrm{d} \lvert A_f \rvert \\ \lesssim_n \left(\int_{\mathbb{R}^n} \eta^n \big(\lvert dv_{\mathbb{R}^n} \rvert^n \circ f \big) \lvert Df \rvert^n \right)^{\frac{n-1}{n}} + C(n) \Sigma(\mathbb{R}^n). \tag{5.21}$$

We then proceed to estimate the integral in (5.21). Let $L > \widetilde{\sup}_{2B} (v_{\mathbb{R}^n} \circ f)$. By Lemma 5.8, we obtain

$$\int_{\{v_{\mathbb{R}^n} \circ f \geq L\}} \eta^n \big(\lvert dv_{\mathbb{R}^n} \rvert^n \circ f \big) \lvert Df \rvert^n \lesssim_n \Sigma(\mathbb{R}^n).$$

In the remaining set $\{v_{\mathbb{R}^n} \circ f > L\}$ we use Lemma 5.4, which, recalling that $\lVert d\eta \rVert_{L^n} \leq C(n)$, yields the estimate

$$\int_{\mathbb{R}^n} \eta^n \big(\lvert dv_{\mathbb{R}^n} \rvert^n \circ f \big) \lvert Df \rvert^n \\ \lesssim_n K^n L^n + KL \int_{\mathbb{R}^n} \eta^n \,\mathrm{d} \lvert A_f \rvert + C(n) \Sigma(\mathbb{R}^n). \tag{5.22}$$

Next, chaining together (5.21) and (5.22) and using the elementary inequality $(a+b)^p \lesssim_p a^p + b^p$ for $a, b, p \geq 0$, we obtain

$$\int_{\mathbb{R}^n} \eta^n \,\mathrm{d} \lvert A_f \rvert \lesssim_n K^{n-1} L^{n-1} + (KL)^{\frac{n-1}{n}} \left(\int_{\mathbb{R}^n} \eta^n \,\mathrm{d} \lvert A_f \rvert \right)^{\frac{n-1}{n}} \\ + C(n) \left(\Sigma(\mathbb{R}^n) + [\Sigma(\mathbb{R}^n)]^{\frac{n-1}{n}} \right). \tag{5.23}$$

We then apply Young's inequality to obtain

$$C(n) (KL)^{\frac{n-1}{n}} \left(\int_{\mathbb{R}^n} \eta^n \,\mathrm{d} \lvert A_f \rvert \right)^{\frac{n-1}{n}} \leq \frac{[C(n)]^n K^{n-1} L^{n-1}}{n} + \frac{n-1}{n} \int_{\mathbb{R}^n} \eta^n \,\mathrm{d} \lvert A_f \rvert,$$

where the last integral is finite and can hence be absorbed to the left side of (5.23). In conclusion, we obtain

$$\lvert A_f \rvert (B) \leq \int_{\mathbb{R}^n} \eta^n \,\mathrm{d} \lvert A_f \rvert \lesssim_n K^{n-1} L^{n-1} + C(n) \left(\Sigma(\mathbb{R}^n) + [\Sigma(\mathbb{R}^n)]^{\frac{n-1}{n}} \right).$$

Since $L > \widetilde{\sup}_{2B} (v_{\mathbb{R}^n} \circ f)$ is arbitrary, the claim follows. $\square$

5.6. **Existence of unbounded components.** To finish this section, we show that if $|A_f|(\mathbb{R}^n) = \infty$, then $\widetilde{\sup}_{\mathbb{R}^n}(v_{\mathbb{R}^n} \circ f) = \infty$. The result is a relatively immediate consequence of Lemma 5.9.

Lemma 5.10. *Let $f \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ be a non-constant, unbounded, continuous function that satisfies* (5.10)*, where $K \geq 1$, $\Sigma \geq 0$, and $\Sigma \in L^1(\mathbb{R}^n) \cap L^{1+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $\varepsilon > 0$. Then for every $t > 0$, there exists $s = s(n, K, \Sigma(\mathbb{R}^n), t) > 0$ such that if $|A_f|(B) > s$ for some ball $B \subset \mathbb{R}^n$, then $2B$ meets an unbounded component of $(v_{\mathbb{R}^n} \circ f)^{-1}(t, \infty)$. In particular, if $|A_f|(\mathbb{R}^n) = \infty$, then for every $t > 0$ the set $(v_{\mathbb{R}^n} \circ f)^{-1}(t, \infty)$ has an unbounded component.*

Proof. Let B be a ball, and let $t > 0$ be such that $2B$ meets no unbounded component of $(v_{\mathbb{R}^n} \circ f)^{-1}(t, \infty)$. Then $\widetilde{\sup}_{2B}(v_{\mathbb{R}^n} \circ f) \leq t$, and Lemma 5.9 yields

$$|A_f|(B) \leq C(n)K^{n-1}t + C(n)\big(\Sigma(\mathbb{R}^n) + [\Sigma(\mathbb{R}^n)]^{\frac{n-1}{n}}\big).$$

Hence, we may set $s(n, K, \Sigma(\mathbb{R}^n), t)$ to be bigger than the right hand side of the above estimate, and the claim follows. □

6. The proof of Theorem 1.5

Following the proofs of the Caccioppoli-type estimates in Section 5, we then proceed to show that the Picard theorem for quasiregular values is true when $|A_f|(\mathbb{R}^n) = \infty$, assuming $\Sigma \in L^1(\mathbb{R}^n) \cap L^{1+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^n)$. For this part of the result, we follow the proof of Bonk and Poggi-Corradini from [6] relatively closely, with the main difference being our use of the pseudosupremum $\widetilde{\sup}$ instead of the usual maximum.

We begin by recalling a key tool in the proof that is colloquially referred to as *Rickman's hunting lemma.* For further details including the proof of the lemma, we refer to [6, Lemma 2.1 and p. 627].

Lemma 6.1 (Rickman's Hunting Lemma)**.** *Let μ be a (non-negative) Borel measure on $\mathbb{R}^n$ such that $\mu(\mathbb{R}^n) = \infty$, $\mu(B) < \infty$ for every ball $B \subset \mathbb{R}^n$, and μ has no atoms. Then there exists a constant $D = D(n) > 1$ and a sequence of balls B_j, $j \in \mathbb{Z}_{>0}$ such that $\mu(8B_j) \leq D\mu(B_j)$ and $\lim_{j\to\infty} \mu(B_j) = \infty$.*

We also recall a lemma on conformal capacity that is essentially similar to [6, Lemma 5.3] but phrased in a more abstract way; this more general formulation will become relevant in the next section. Recall that if E, F are compact and mutually disjoint subsets of $\mathbb{R}^n$, then the *(conformal) capacity* of the condenser (E, F) is defined by

$$\mathrm{Cap}(E, F) = \inf\left\{\int_{\mathbb{R}^n} |d\eta|^n : \eta \in C^\infty_0(\mathbb{R}^n), \eta|_E \geq 1, \eta|_F \leq 0\right\}. \tag{6.1}$$

By a standard convolution approximation argument, an equivalent definition is obtained if the assumption $\eta \in C^\infty_0(\mathbb{R}^n)$ in (6.1) is replaced by $\eta \in W^{1,n}_0(\mathbb{R}^n) \cap C(\mathbb{R}^n)$. We call a function $\eta \in W^{1,n}_0(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ with $\eta|_E \geq 1$ and $\eta|_F \leq 0$ *admissible* for the condenser (E, F).

Lemma 6.2. *Let $q \geq 2$. For each $k \in \{1, \dots, q\}$, let E_k and F_k be closed subsets of $\mathbb{R}^n$ such that $E_k \cap F_k = \emptyset$ for every k and $F_l \cup F_k = \mathbb{R}^n$ whenever*

$l \neq k$. Suppose that $B = \mathbb{B}^n(x_0, r)$ meets an unbounded component of E_k for every $k \in \{1, \dots, q\}$. Let $t > 1$, and define

$$E_{k,t} = E_k \cap (\overline{tB} \setminus B), \qquad F_{k,t} = F_k \cap (\overline{tB} \setminus B). \tag{6.2}$$

Then we have

$$\sum_{k=1}^{q} \operatorname{Cap}(E_{k,t}, F_{k,t}) \gtrsim_n q^{\frac{n}{n-1}} \log t.$$

Proof. If $l \neq k$, we observe that since $F_l \cup F_k = \mathbb{R}^n$ and $E_l \cap F_l = \emptyset$, we have $E_l \subset \mathbb{R}^n \setminus F_l \subset F_k$. Consequently, B also meets an unbounded component of F_k for every $k \in \{1, \dots, q\}$. Due to our assumption that $q \geq 2$, we may fix $l \in \{1, \dots, q\} \setminus \{k\}$ and note that B meets an unbounded component of $E_l \subset F_k$. It follows that $(\partial sB) \cap E_k \neq \emptyset \neq (\partial sB) \cap F_k$ for every $s \geq 1$, and we may thus use a capacity estimate given e.g. in [6, Lemma 3.3] to conclude that

$$\operatorname{Cap}(E_{k,t}, F_{k,t}) \gtrsim_n \int_1^t \frac{r \, \mathrm{d}s}{[\mathcal{H}^{n-1}((\partial sB) \setminus (E_k \cup F_k))]^{\frac{1}{n-1}}}.$$

We note that the denominator $\mathcal{H}^{n-1}((\partial sB) \setminus (E_k \cup F_k))$ in the above integral is non-zero for every $s \geq 1$; indeed, $(\partial sB) \setminus (E_k \cup F_k))$ is an open subset of ∂sB, and $(\partial sB) \setminus (E_k \cup F_k))$ is non-empty since ∂sB is connected and E_k and F_k are disjoint closed sets.

We then observe that the sets $\mathbb{R}^n \setminus (E_k \cup F_k)$ are pairwise disjoint, since $(\mathbb{R}^n \setminus (E_k \cup F_k)) \cap (\mathbb{R}^n \setminus (E_l \cup F_l)) \subset \mathbb{R}^n \setminus (F_k \cup F_l) = \emptyset$ whenever $k \neq l$. Thus, the sets $(\partial sB) \setminus (E_k \cup F_k)$ are disjoint for every $s \geq 1$, and Hölder's inequality for sums yields that

$$\begin{aligned} q &= \sum_{k=1}^{q} \left[\mathcal{H}^{n-1}((\partial sB) \setminus (E_k \cup F_k))\right]^{\frac{1}{n}} \frac{1}{[\mathcal{H}^{n-1}((\partial sB) \setminus (E_k \cup F_k))]^{\frac{1}{n}}} \\ &\leq \left[\mathcal{H}^{n-1}(\partial sB)\right]^{\frac{1}{n}} \left(\sum_{k=1}^{q} \frac{1}{[\mathcal{H}^{n-1}((\partial sB) \setminus (E_k \cup F_k))]^{\frac{1}{n-1}}} \right)^{\frac{n-1}{n}}. \end{aligned}$$

Since $[\mathcal{H}^{n-1}(\partial sB)]^{1/n} \lesssim_n (rs)^{(n-1)/n}$, we hence obtain the desired estimate

$$\begin{aligned} \sum_{k=1}^{q} \operatorname{Cap}(E_{k,t}, F_{k,t}) &\gtrsim_n \int_1^t \sum_{k=1}^{q} \frac{r ds}{[\mathcal{H}^{n-1}((\partial sB) \setminus (E_k \cup F_k))]^{\frac{1}{n-1}}} \\ &\gtrsim_n \int_1^t \frac{q^{\frac{n}{n-1}} r \, \mathrm{d}s}{rs} = q^{\frac{n}{n-1}} \log t. \end{aligned}$$

□

Now, we begin the proof of Theorem 1.5. We recall the statement for the convenience of the reader.

Theorem 1.5. *Let $K \geq 1$ and $\Sigma \in L^1(\mathbb{R}^n) \cap L^{1+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $\varepsilon > 0$. Then there exists a positive integer $q = q(n, K) \in \mathbb{Z}_{>0}$ with the following property: if a continuous map $h \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{S}^n)$ has a (K, Σ)-quasiregular value with respect to the spherical metric at q distinct points $w_1, \dots, w_q \in \partial h(\mathbb{R}^n)$, then $|Dh| \in L^n(\mathbb{R}^n)$.*

Proof. Suppose that $h \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{S}^n)$ is continuous and has a (K, Σ)-quasiregular value with respect to the spherical metric at q distinct points $w_1, \dots, w_q \in \partial h(\mathbb{R}^n)$, yet $\|Dh\|_{L^n} = \infty$. Our objective is hence to find an upper bound on q that only depends on n and K. We may assume $q \geq 2$. Since $w_k \in \partial h(\mathbb{R}^n)$, by the single-value Reshetnyak's theorem for spherical quasiregular values given in Proposition 3.3 (i), we conclude that $w_k \notin h(\mathbb{R}^n)$.

For every point w_k, we select a rotation $R_k \colon \mathbb{S}^n \to \mathbb{S}^n$ that takes w_k to $s_n(\infty)$, and denote $h_k = R_k \circ h$. Since R_k is an orientation-preserving isometry of $\mathbb{S}^n$, it follows that h_k has a (K, Σ)-quasiregular value with respect to the spherical metric at $s_n(\infty)$.

Consequently, we obtain maps $f_k \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ satisfying $h_k = s_n \circ f_k$. Notably, for every $k \in \{1, \dots, q\}$ and every measurable $E \subset \mathbb{R}^n$, we have

$$A_{f_k}(E) = \int_E f_k^* s_n^* \operatorname{vol}_{\mathbb{S}^n} = \int_E h_k^* \operatorname{vol}_{\mathbb{S}^n} = \int_E h^* R_k^* \operatorname{vol}_{\mathbb{S}^n} = \int_E h^* \operatorname{vol}_{\mathbb{S}^n}.$$

That is, every A_{f_k} is the same measure; we denote this signed measure by μ, with $|\mu|$ denoting the total variation measure of μ. Since $\|Dh\|_{L^n} = \infty$ and $\Sigma(\mathbb{R}^n) < \infty$, (1.6) yields that

$$|\mu|(\mathbb{R}^n) \geq \mu(\mathbb{R}^n) = \int_{\mathbb{R}^n} J_h \geq \frac{1}{K}\left(\int_{\mathbb{R}^n} |Dh|^n - \pi^n \int_{\mathbb{R}^n} \Sigma\right) = \infty.$$

We also note that since every h_k has a (K, Σ)-quasiregular value with respect to the spherical metric at $s_n(\infty)$, we obtain that every f_k satisfies (5.10) by Lemma 3.2, allowing us to use the results of Section 5 on f_k.

We then let $u_k = v_{\mathbb{R}^n} \circ f_k$ for every $k \in \{1, \dots, q\}$. We note that the sets $s_n(\{\infty\} \cup v_{\mathbb{R}^n}^{-1}(t, \infty))$ form a neighborhood basis of $s_n(\infty)$, where the neighborhoods become smaller as $t > 0$ increases. Hence, there exists $C_0 = C_0(n, w_1, w_2, \dots, w_q) > 0$ such that for every $t \geq C_0$, the sets $u_k^{-1}(t, \infty), k \in \{1, \dots, q\}$ are pairwise disjoint. Moreover, by Lemma 5.10, there exists $A_0 = A_0(n, K, \Sigma(\mathbb{R}^n), w_1, w_2, \dots, w_q)$ such that if $B \subset \mathbb{R}^n$ is a ball with $\mu(B) > A_0$, then $2B$ meets an unbounded component of each of the sets $u_k^{-1}(3C_0, \infty)$.

Since $|\mu|(\mathbb{R}^n) = \infty$, we may also use Rickman's Hunting Lemma 6.1 to obtain a sequence (B_j) of balls in $\mathbb{R}^n$ for which $\lim_{j\to\infty} |\mu|(B_j) = \infty$ and $|\mu|(8B_j) \lesssim_n |\mu|(B_j)$. Then there exists $j_0 > 0$ such that $|\mu|(B_j) > A_0$ whenever $j \geq j_0$. For all such j and for every $k \in \{1, \dots, q\}$, we define

$$L_{j,k} = \widetilde{\sup_{2B_j}}\, u_k.$$

We also define

$$E_k^j = u_k^{-1}[2L_{j,k}/3, \infty), \qquad F_k^j = u_k^{-1}[0, L_{j,k}/3],$$

and

$$E_{k,2}^j = E_k^j \cap (\overline{4B_j} \setminus 2B_j), \qquad F_{k,2}^j = E_k^j \cap (\overline{4B_j} \setminus 2B_j).$$

We claim that for every $j \geq j_0$, the sets E_k^j and F_k^j with $k \in \{1, \dots q\}$ satisfy the assumptions of Lemma 6.2. Indeed, it is clear from the definition that $E_k^j \cap F_k^j = \emptyset$ for every k. Since $\mu(B_j) > A_0$, $2B_j$ meets an unbounded component of $u_k^{-1}(3C_0, \infty)$, and hence $L_{j,k} \geq 3C_0 > 0$ for every k. Thus, the sets $\mathbb{R}^n \setminus F_k^j = u_k^{-1}(L_{j,k}/3, \infty)$ are pairwise disjoint, and consequently

$F_k^j \cup F_l^j = \mathbb{R}^n$ whenever $k \neq l$. Since $0 < L_{j,k} = \widetilde{\sup}_{2B_j} u_k$, we also have that $2B_j$ meets an unbounded component of every $u_k^{-1}(2L_{j,k}/3, \infty)$, and consequently $2B_j$ also meets an unbounded component of every E_k^j. Thus, the assumptions of Lemma 6.2 are satisfied, and it follows that for every $j \geq j_0$, we have

$$\sum_{k=1}^{q} \operatorname{Cap}(E_{k,2}^j, F_{k,2}^j) \gtrsim_n q^{\frac{n}{n-1}}. \tag{6.3}$$

We are now ready to estimate q. Let $j \geq j_0$. By using Lemma 5.9 on f_k, we obtain

$$|\mu|(B_j) \lesssim_n K^{n-1} L_{j,k}^{n-1} + C(n, \Sigma(\mathbb{R}^n)). \tag{6.4}$$

for every $k \in \{1, \dots, q\}$. Since $\lim_{j\to\infty} |\mu|(B_j) = \infty$ by our use of Rickman's Hunting Lemma, we conclude that

$$\lim_{j\to\infty} \min_k L_{j,k} = \infty. \tag{6.5}$$

We may fix a function $\psi_j \in C_0^\infty(8B_j)$ for which $\|\nabla \psi_j\|_{L^n} \leq C(n)$ and $\psi_j \equiv 1$ on a neighborhood of $\overline{4B_j}$. Now, the function

$$\eta_j = \left(\frac{3\min(u_k, L_{j,k})}{L_{j,k}} - 1 \right) \psi$$

is admissible for the condenser $(E_{k,2}^j, F_{k,2}^j)$. It follows that

$$\operatorname{Cap}(E_{k,2}^j, F_{k,2}^j) \leq \int_{\mathbb{R}^n} |\nabla \eta_j|^n \lesssim_n \|\nabla \psi_j\|_{L^n}^n + \int_{4B_j \cap \{u_k < L_{j,k}\}} \frac{|\nabla u_k|^n}{L_{j,k}^n}$$

We apply Corollary 5.5 to the last integral and use $\|\nabla \psi_j\|_{L^n} \lesssim_n 1 \leq K^n$ to obtain

$$\operatorname{Cap}(E_{k,2}^1, E_{k,2}^2) \lesssim_n 2K^n + \frac{K|\mu|(8B_j)}{L_{j,k}^{n-1}} + \frac{C(n, \Sigma(\mathbb{R}^n))}{L_{j,k}^n}.$$

By (6.3), there always exists an index $k = k(j,h) \in \{1, \dots, q\}$ such that $\operatorname{Cap}(E_{k,2}^1, E_{k,2}^2) \geq C(n) q^{1/(n-1)}$. Hence, for this specific choice of k, we have

$$q^{\frac{1}{n-1}} \lesssim_n 2K^n + \frac{K|\mu|(8B_j)}{L_{j,k}^{n-1}} + \frac{C(n, \Sigma(\mathbb{R}^n))}{L_{j,k}^n}.$$

We then apply the estimate $|\mu|(8B_k) \lesssim_n |\mu|(B_k)$ and (6.4) to obtain

$$q^{\frac{1}{n-1}} \lesssim_n 3K^n + \frac{KC(n, \Sigma(\mathbb{R}^n))}{L_{j,k}^{n-1}} + \frac{C(n, \Sigma(\mathbb{R}^n))}{L_{j,k}^n} \tag{6.6}$$

for our specific choice of $k = k(j,h)$. Finally, let $j \to \infty$ in (6.6). It follows from (6.5) that the terms involving $L_{j,k}$ vanish at the limit, and we obtain the desired

$$q \leq C(n) K^{n(n-1)},$$

concluding the proof. □

7. The proof of Theorems 1.2 and 1.4

In order to prove Theorems 1.2 and 1.4, what remains is essentially to show that, under the assumptions of Theorem 1.2, we have $|A_f|(\mathbb{R}^n) = \infty$. As stated in the introduction, this is a small step in the quasiregular version of the proof [6, p.631], but grows into a significantly more complex undertaking in our setting, involving e.g. the boundedness result shown in Proposition 4.3 and Corollary 4.4.

7.1. **The two cases.** The starting point of our argument is that if one does not have $A_f(\mathbb{R}^n) = \infty$, then one essentially obtains an L^n-integrability condition for $\nabla \log |f|$. This general idea of obtaining L^n-regularity for $\nabla \log |f|$ when the behavior of f differs from that of a quasiregular map is frequent in the proofs of other results on quasiregular values [28, 29]. This underlying dichotomy is summarized in the following result.

Proposition 7.1. *Let $K \geq 1$ and $\Sigma \in L^1(\mathbb{R}^n) \cap L^{1+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $\varepsilon > 0$. Suppose that $f \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ is an unbounded, continuous function such that f has a (K,Σ)-quasiregular value at 0 and $0 \notin f(\mathbb{R}^n)$. Then*

$$|A_f|(\mathbb{R}^n) = \infty \qquad \text{or} \qquad \int_{\mathbb{R}^n} \frac{|Df|^n}{|f|^n} < \infty.$$

We divide the proof into two cases. For this division, note that if $s \in [0,\infty]$ and $\{|f| > s\}$ has an unbounded component, then $\{|f| > s'\}$ has an unbounded component for all $s' < s$. Similarly, if $s \in [0,\infty]$ and $\{|f| < s\}$ has an unbounded component, then $\{|f| < s'\}$ has an unbounded component for all $s' > s$. The first case is when there exists an overlap between the region where $\{|f| > s\}$ has an unbounded component and the region where $\{|f| < s\}$ has an unbounded component; see Figure 2 for an illustration.

Lemma 7.2. *Let $K \geq 1$ and $\Sigma \in L^1(\mathbb{R}^n) \cap L^{1+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $\varepsilon > 0$. Suppose that $f \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ is an unbounded, continuous function such that f has a (K,Σ)-quasiregular value at 0 and $0 \notin f(\mathbb{R}^n)$. If there exist $0 < s_1 < s_2 < \infty$ for which $\{|f| > s_2\}$ and $\{|f| < s_1\}$ both have an unbounded component, then $|A_f|(\mathbb{R}^n) = \infty$.*

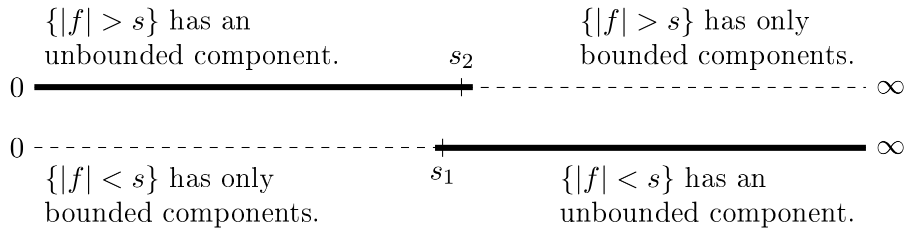

FIGURE 2. The case covered in Lemma 7.2, where both $\{|f| > s_2\}$ and $\{|f| < s_1\}$ have an unbounded component.

Proof. The argument is reminiscent of the proof that $|A_f|(\mathbb{R}^n) = \infty$ in the quasiregular case. We begin by observing that, since f has a (K,Σ)-quasiregular value at 0, we can use Lemmas 3.1 and 3.2 to conclude that f

satisfies (5.10). Since f is also unbounded, we may hence use the results of Section 5 on f. We divide the proof into two main cases.

We fix a ball B that meets the unbounded components of both $\{|f| > s_2\}$ and $\{|f| < s_1\}$. We choose values c_1, c_2, c_3, c_4 such that $s_1 < c_1 < c_2 < c_3 < c_4 < s_2$. We let $E_1 = \{|f| \geq c_4\}$, $F_1 = \{|f| \leq c_3\}$, $E_2 = \{|f| \leq c_1\}$, and $F_2 = \{|f| \geq c_2\}$. Since we have $\{|f| > s_2\} \subset E_1$ and $\{|f| < s_1\} \subset E_2$, B meets an unbounded component of E_1 and E_2. Moreover, $E_1 \cap F_1 = \emptyset = E_2 \cap F_2$ and $F_1 \cup F_2 = \mathbb{R}^n$. Consequently the sets E_i and F_i satisfy the conditions of Lemma 6.2 with $q = 2$. Hence, if $t > 1$, and $E_{i,t}$, $F_{i,t}$ are as in (6.2), we get

$$\operatorname{Cap}(E_{1,t}, F_{1,t}) + \operatorname{Cap}(E_{2,t}, F_{2,t}) \gtrsim_n \log t.$$

Thus, for each $t > 1$, we have $\operatorname{Cap}(E_{1,t}, F_{1,t}) \gtrsim_n \log t$ or $\operatorname{Cap}(E_{2,t}, F_{2,t}) \gtrsim_n \log t$.

Consider first the case where one can find arbitrarily large values of t such that $\operatorname{Cap}(E_{1,t}, F_{1,t}) \gtrsim_n \log t$. We let $u = v \circ f$ where v is as in (5.7), and select a function $\psi \in C_0^\infty(2tB, [0,1])$ with $\|\nabla \psi\|_{L^n} \leq C(n)$ and $\psi \equiv 1$ in a neighborhood of tB. Similarly to the beginning of the proof of Theorem 1.5, we obtain that

$$\eta = \left(\frac{\min(u, H(c_4)) - H(c_3)}{H(c_4) - H(c_3)} \right) \psi$$

is admissible for the condenser $(E_{1,t}, F_{1,t})$, where H is as in (5.4). We then use Corollary 5.5 to obtain that

$$\begin{aligned}
\log t \lesssim_n \operatorname{Cap}(E_{1,t}, F_{1,t}) &\leq \int_{\mathbb{R}^n} |\nabla \eta|^n \\
&\lesssim_n \frac{1}{(H(c_4) - H(c_3))^n} \left(H^n(c_3) \|\nabla \psi\|_{L^n}^n + \int_{2tB \cap \{u < H(c_4)\}} |\nabla u|^n \right) \\
&\lesssim_n \frac{H^n(c_3) C(n) + K^n H^n(c_4) + K H^n(c_4) |A_f|(4tB) + \Sigma(\mathbb{R}^n)}{(H(c_4) - H(c_3))^n} \\
&\leq C(n, K, c_3, c_4, \Sigma(\mathbb{R}^n)) + C(n, K, c_3, c_4) |A_f|(4tB).
\end{aligned}$$

Letting $t \to \infty$, we conclude that $|A_f|(\mathbb{R}^n) = \infty$.

In the other case where $\operatorname{Cap}(E_{2,t}, F_{2,t}) \gtrsim_n \log t$ for arbitrarily large t, we repeat the above proof with the function

$$\eta = \left(\frac{H(c_2) - \min(u, H(c_2))}{H(c_2) - H(c_1)} \right) \psi.$$

Indeed, this η is admissible for the condenser $(E_{2,t}, F_{2,t})$, and provides an analogous upper bound for $\log t$ in terms of $|A_f|(4tB)$ by a similar proof. □

The other case in the proof of Proposition 7.1 is when there is no overlap between the region where $\{|f| > s\}$ has an unbounded component and the region where $\{|f| < s\}$ has an unbounded component, or alternatively when this overlap is merely a single endpoint. See Figure 3 for an illustration.

Lemma 7.3. *Let $K \geq 1$ and $\Sigma \in L^1(\mathbb{R}^n) \cap L^{1+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $\varepsilon > 0$. Suppose that $f \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ is an unbounded, continuous function such that f has a (K, Σ)-quasiregular value at 0 and $0 \notin f(\mathbb{R}^n)$. If there exists an $s_0 \in [0, \infty]$ for which $\{|f| > s\}$ has only bounded components whenever*

$s > s_0$, and $\{|f| < s\}$ has only bounded components whenever $s < s_0$, then $|f|^{-1}\,|Df| \in L^n(\mathbb{R}^n)$.

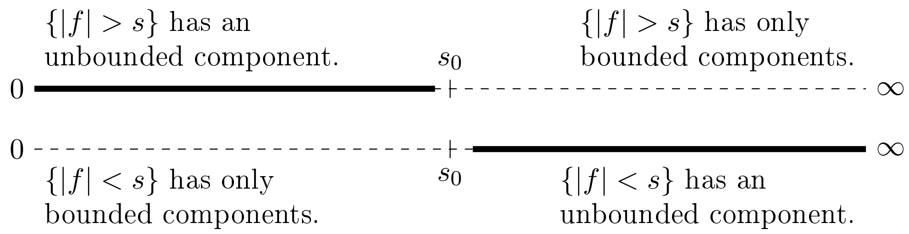

FIGURE 3. The case covered in Lemma 7.3, where $\{|f| > s\}$ has only bounded components for $s > s_0$, and $\{|f| < s\}$ has only bounded components for $s < s_0$. Note that if the endpoints of the regions coincide, this case also applies, with s_0 as the shared endpoint.

Proof. Let $s > s_0$. Since f has a (K, Σ)-quasiregular value at 0 and since $\{|f| > s\}$ has only bounded components, we may use Lemma 5.2 with $\Psi(t) = t^{-n}$ and $\tilde{\Sigma} = |f|^n\,\Sigma$ to conclude that

$$\int_{\{|f|>s\}} \frac{|Df|^n}{|f|^n} \leq C(n) \int_{\{|f|>s\}} \Sigma \leq C(n)\Sigma(\mathbb{R}^n).$$

Monotone convergence consequently yields that

$$\int_{\{|f|>s_0\}} \frac{|Df|^n}{|f|^n} \lesssim_n \Sigma(\mathbb{R}^n) < \infty.$$

We then consider the map $\tilde{f} = \iota \circ f$, where $\iota\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ is the conformal inversion across the unit $(n-1)$-sphere. Then since we have $0 \notin f(\mathbb{R}^n)$, we obtain that $\tilde{f} \in C(\mathbb{R}^n, \mathbb{R}^n) \cap W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ and $0 \notin \tilde{f}(\mathbb{R}^n)$. By the conformality of ι and the fact that $|\iota(y)| = |y|^{-1}$ and $|D\iota(y)| = |y|^{-2}$, we obtain that

$$\frac{|D\tilde{f}|}{|\tilde{f}|} = \frac{(|D\iota| \circ f)\,|Df|}{|f|^{-1}} = \frac{|Df|}{|f|}.$$

It also follows that the map $\tilde{f}$ also has a (K, Σ)-quasiregular value at 0, since

$$|D\tilde{f}|^n = \frac{|Df|^n}{|f|^{2n}} \leq \frac{KJ_f}{|f|^{2n}} + \frac{\Sigma}{|f|^n} = KJ_{\tilde{f}} + |\tilde{f}|^n\Sigma.$$

Furthermore, for every $\tilde{s} > s_0^{-1}$, we have that $\{|\tilde{f}| > \tilde{s}\} = \{|f| < s^{-1}\}$ has no unbounded components. Hence, similarly as before, we may use Lemma 5.2 to obtain that

$$\int_{\{|f|<s_0\}} \frac{|Df|^n}{|f|^n} = \int_{\{|\tilde{f}|>s_0^{-1}\}} \frac{|D\tilde{f}|}{|\tilde{f}|} \leq C(n)\Sigma(\mathbb{R}^n) < \infty.$$

In conclusion,

$$\int_{\{|f|\neq s_0\}} \frac{|Df|^n}{|f|^n} \lesssim_n \Sigma(\mathbb{R}^n) < \infty.$$

It remains to show that if $0 < s_0 < \infty$, then the integral of $|f|^{-n} |Df|^n$ over $\{|f| = s_0\}$ is finite. If $s_0 \in \{0, \infty\}$, then this set is empty. Otherwise, for a.e. $x \in \{|f| = s_0\}$, we may estimate as follows:

$$\frac{|Df(x)|^n}{|f(x)|^n} = \frac{|Df(x)|^n}{s_0^n} \leq \frac{K J_f(x)}{s_0^n} + \Sigma(x).$$

Here, Σ has finite integral over $\mathbb{R}^n$, and $J_f = 0$ a.e. in $\{|f| = s_0\}$ due to the set having an image with zero Hausdorff n-measure. The proof of the lemma is hence complete. □

The proof of Proposition 7.1 is hence complete, since if the assumptions of Proposition 7.1 are satisfied, then the assumptions of either Lemma 7.2 or Lemma 7.3 are satisfied.

7.2. **Completing the proofs.** It remains to complete the proofs of Theorems 1.2 and 1.4. We start with Theorem 1.4, where we recall the statement for the convenience of the reader.

Theorem 1.4. *Let $K \geq 1$ and $\Sigma \in L^{1+\varepsilon}(\mathbb{R}^n) \cap L^{1-\varepsilon}(\mathbb{R}^n)$ for some $\varepsilon > 0$. Then there exists a positive integer $q = q(n, K) \in \mathbb{Z}_{>0}$ such that no continuous map $h \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{S}^n)$ has a (K, Σ)-quasiregular value with respect to the spherical metric at q distinct points $w_1, \ldots, w_q \in \partial h(\mathbb{R}^n)$.*

Proof. Suppose that $h \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{S}^n)$ has a (K, Σ)-quasiregular value with respect to the spherical metric at q distinct points $w_1, \ldots, w_q \in \partial f(\mathbb{R}^n)$, with $q \geq 2$. By Theorem 1.5, we must have either $q \leq q_0(n, K)$, or $|Dh| \in L^n(\mathbb{R}^n)$. Suppose then that we are in the latter case, with the aim of deriving a contradiction. From this point onwards, we may ignore all of the spherical quasiregular values w_i except the first two, w_1 and w_2.

By the single-point Reshetnyak's theorem given in Proposition 3.3 (i), we have $w_1, w_2 \notin h(\mathbb{R}^n)$. By post-composing h with an isometric spherical rotation, we may assume that $w_2 = s_n(\infty)$. In this case, we have an unbounded continuous map $\tilde{f} \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ such that $h = s_n \circ \tilde{f}$. We let $y_1 \in \mathbb{R}^n$ be the point for which $s_n(y_1) = w_1$. It follows that $\tilde{f}$ is unbounded, that $y_1 \in \partial \tilde{f}(\mathbb{R}^n)$, and that $\tilde{f}$ has a $(K, C(n)\Sigma)$-quasiregular value at y_1 by Lemma 3.1. The fact that $|Dh| \in L^n(\mathbb{R}^n)$ also yields that

$$|A_{\tilde{f}}|(\mathbb{R}^n) = \int_{\mathbb{R}^n} |J_h| \leq \int_{\mathbb{R}^n} |Dh|^n < \infty.$$

We then consider the map $f = \tilde{f} - y_1$. It follows that f is a continuous, unbounded map in $W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$, that $0 \notin f(\mathbb{R}^n)$, and that f has a $(K, C(n)\Sigma)$-quasiregular value at 0. Moreover, since $J_f = J_{\tilde{f}}$ and $1 + |f|^2 \gtrsim_{n,y_1} 1 + |f - y_1|^2$, we obtain

$$A_f(\mathbb{R}^n) = \int_{\mathbb{R}^n} \frac{2^n J_f}{(1 + |f|^2)^n} \lesssim_{n,y_1} \int_{\mathbb{R}^n} \frac{2^n |J_f|}{(1 + |f - y_1|^2)^n} = |A_{\tilde{f}}|(\mathbb{R}^n) < \infty.$$

Thus, we may apply Proposition 7.1 on f, and conclude that $|f|^{-1} |Df| \in L^n(\mathbb{R}^n)$.

However, since f omits 0 and $\Sigma \in L^{1+\varepsilon}(\mathbb{R}^n) \cap L^{1-\varepsilon}(\mathbb{R}^n)$, we may now apply Corollary 4.4 to conclude that f is bounded. This is a contradiction, since f is unbounded. The proof is hence complete. □

Theorem 1.2 is then an immediate corollary of Theorem 1.4. We recall the statement and give a short proof.

Theorem 1.2. *Let $K \geq 1$ and $\Sigma \in L^{1+\varepsilon}(\mathbb{R}^n) \cap L^{1-\varepsilon}(\mathbb{R}^n)$ for some $\varepsilon > 0$. Then there exists a positive integer $q = q(n, K) \in \mathbb{Z}_{>0}$ such that no continuous map $f \in W^{1,n}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ has a (K, Σ)-quasiregular value at q distinct points $y_1, \dots, y_q \in \partial f(\mathbb{R}^n)$.*

Proof. Suppose that $f \in W^{1,n}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ is continuous and has a (K, Σ)-quasiregular value at q distinct points $y_1, \dots, y_q \in \partial f(\mathbb{R}^n)$. Let $h = s_n \circ f$. Then by Lemma 3.1, h has a $(K, \tilde{\Sigma})$-quasiregular value with respect to the spherical metric at each of the points $s_n(y_1), \dots, s_n(y_q) \in \partial h(\mathbb{R}^n)$, where $\tilde{\Sigma} = C(n, y_1, \dots, y_q)\Sigma$. Now, Theorem 1.4 yields an upper bound on q dependent only on n and K, completing the proof. □

Remark 7.4. With Theorems 1.2 and 1.4 proven, we conclude this section by briefly pointing out how the standard Rickman's Picard Theorem follows almost immediately from the case $\Sigma \equiv 0$ of our main results. Besides Theorem 1.2, the only other result of quasiregular theory used in the argument is either Liouville's theorem or Reshetnyak's Theorem; the single-value versions from Theorem 1.3 can also be used for this.

Both arguments begin in the same manner. Suppose towards contradiction that $f \colon \mathbb{R}^n \to \mathbb{R}^n$ is an entire non-constant K-quasiregular map that omits $q+1$ distinct points $y_1, \dots y_{q+1} \notin f(\mathbb{R}^n)$, where $q = q(n, K)$ is as in Theorem 1.2. We note that f has a $(K, 0)$-quasiregular value at every $y \in \mathbb{R}^n$. Hence, by Theorem 1.2, we obtain that $\partial f(\mathbb{R}^n)$ contains at most q points. Since f omits more than q different points, the set $\mathbb{R}^n \setminus \overline{f(\mathbb{R}^n)}$ must be non-empty.

For the argument based on Reshetnyak's theorem, we argue as follows. Since f is non-constant, we have by Reshetnyak's theorem that $f\mathbb{R}^n$ is an open set. Since $\partial f(\mathbb{R}^n)$ separates two non-empty open subsets of $\mathbb{R}^n$, it is not finite; see e.g. [25, Theorem IV 4] for a formal justification. This is a contradiction, since $\partial f(\mathbb{R}^n)$ consists of at most q points. The proof is thus complete.

For the argument based on Liouville's theorem, we instead use the non-emptiness of $\mathbb{R}^n \setminus \overline{f(\mathbb{R}^n)}$ to fix a point $y_0 \in \mathbb{R}^n \setminus \overline{f(\mathbb{R}^n)}$ and a Möbius transformation $T \colon \mathbb{R}^n \cup \{\infty\} \to \mathbb{R}^n \cup \{\infty\}$ satisfying $T(y_0) = \infty$. Now, since f is a quasiregular map that omits a neighborhood of y_0, the map $\tilde{f} = T \circ f$ is a quasiregular map that omits a neighborhood of ∞, and thus $\tilde{f}$ is bounded. Hence, Liouville's theorem implies that $\tilde{f}$ is constant, resulting in a contradiction and completing the proof.

8. The planar case

In this section, we prove Theorem 1.6. The result is derived directly from Theorem 1.2 with the use of a trick.

Before beginning the proof, we recall a corollary of the single-value Reshetnyak's theorem from [29]. The corollary generalizes a version of the argument principle used by Astala and Päivärinta [5, Proposition 3.3 b)]. Recall that if $f \in W^{1,n}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n) \cap C(\mathbb{R}^n, \mathbb{R}^n)$ has a (K, Σ)-quasiregular value at a point

$y_0 \in \mathbb{R}^n$ with $K \geq 1$ and $\Sigma \in L^{1+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $\varepsilon > 0$, then $f^{-1}\{y_0\}$ is discrete by the single-value Reshetnyak's theorem. This is sufficient to ensure the existence of a local index $i(x, f)$ for every $x \in f^{-1}\{y_0\}$, as detailed e.g. in [14, Theorem 2.8].

Lemma 8.1 ([29, Corollary 1.6]). *Let $f_1, f_2 \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n) \cap C(\mathbb{R}^n, \mathbb{R}^n)$ be such that both f_i have a (K_i, Σ_i)-quasiregular value at $y_0 \in \mathbb{R}^n$, with $K_i \geq 1$ and $\Sigma_i \in L^{1+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $\varepsilon > 0$. Suppose that*

$$\liminf_{x\to\infty} |f_2(x) - y_0| \neq 0 \quad \text{and} \quad \liminf_{x\to\infty} |f_1(x) - f_2(x)| = 0.$$

Then

$$\sum_{x \in f_1^{-1}\{y_0\}} i(x, f_1) = \sum_{x \in f_2^{-1}\{y_0\}} i(x, f_2).$$

Note that the sum of local indices in Lemma 8.1 acts as a replacement of the global degree.

We also recall a version of the main structure theorem for planar maps with a quasiregular value. A proof for the result can essentially be found embedded in [4, Proof of Theorem 8.5.1]. We regardless go over the key ideas of the argument.

Lemma 8.2. *Suppose that $f\colon \mathbb{C} \to \mathbb{C}$ has a (K, Σ)-quasiregular value at $z_0 \in \mathbb{C}$, where $K \geq 1$ and $\Sigma \in L^{1+\varepsilon}(\mathbb{C}) \cap L^{1-\varepsilon}(\mathbb{C})$ for some $\varepsilon > 0$. Then f is of the form*

$$f(z) = z_0 + g(z)e^{\theta(z)},$$

where $g\colon \mathbb{C} \to \mathbb{C}$ is an entire quasiregular map, and $\theta \in C(\mathbb{C}, \mathbb{C})$ with $\lim_{z\to\infty} \theta(z) = 0$.

Proof. We first rewrite (1.2) in the form of a Beltrami equation. Indeed, recalling that $|Df| = |f_z| + |f_{\overline{z}}|$ and $J_f = |f_z|^2 - |f_{\overline{z}}|^2$, we have

$$|f_z|^2 + |f_{\overline{z}}|^2 \leq |Df|^2 \leq K(|f_z|^2 - |f_{\overline{z}}|^2) + |f - z_0|^2 \Sigma.$$

Rearranging, we have

$$|f_{\overline{z}}|^2 \leq \frac{K-1}{K+1} |f_z|^2 + |f - z_0|^2 \frac{\Sigma}{K+1}.$$

Due to the elementary inequality $\sqrt{a^2 + b^2} \leq |a| + |b|$, we hence have

$$|f_{\overline{z}}| \leq k\, |f_z| + \sigma\, |f - z_0|\,, \tag{8.1}$$

where

$$k = \sqrt{\frac{K-1}{K+1}} \in [0, 1) \quad \text{and} \quad \sigma = \sqrt{\frac{\Sigma}{K+1}} \in L^{2+2\varepsilon}(\mathbb{C}) \cap L^{2-2\varepsilon}(\mathbb{C}).$$

Moreover, (8.1) can be rewritten as a Beltrami-type equation

$$f_{\overline{z}} = \mu f_z + A(f - z_0), \tag{8.2}$$

where $\|\mu\|_{L^\infty} \leq k < 1$ and $A \in L^{2+2\varepsilon}(\mathbb{C}, \mathbb{C}) \cap L^{2-2\varepsilon}(\mathbb{C}, \mathbb{C})$.

To prove the structure theorem, one first studies the auxiliary equation

$$\theta_{\overline{z}} = \mu \theta_z + A. \tag{8.3}$$

By standard existence theory of Beltrami-type equations as discussed in e.g. [4], one can find a solution for (8.3) by $\theta = \mathcal{C}(I - \mu\mathcal{S})^{-1}A$, where $\mathcal{C}$

is the Cauchy transform and $\mathcal{S}$ is the Beurling transform. In particular, since $A \in L^{2+2\varepsilon}(\mathbb{C},\mathbb{C}) \cap L^{2-2\varepsilon}(\mathbb{C},\mathbb{C})$, the map θ ends up being a bounded, continuous map with $\lim_{z\to\infty} \theta = 0$: see e.g. [4, Theorem 4.3.11 and Section 5.4].

Then, with the solution θ of (8.3), one defines $g = (f - z_0)e^{-\theta}$, in which case $f = z_0 + ge^{\theta}$. Using (8.2) and (8.3), one computes directly that $g_{\overline{z}} = \mu g_z$. Hence, g is an entire quasiregular map, completing the argument. □

We then prove a lemma on the preservation of quasiregular values under complex logarithms.

Lemma 8.3. *Let $\Omega \subset \mathbb{C}$ be a domain, let $\gamma \in W^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{C}) \cap C(\Omega,\mathbb{C})$, and let $f = e^{\gamma}$. If f has a (K,Σ)-quasiregular value at both 0 and $z_0 = e^{w_0} \in \mathbb{C}\setminus\{0\}$, where $K \geq 1$ and $\Sigma \colon \Omega \to [0,\infty)$, then γ has a $(K, 4\Sigma)$-quasiregular value at every point of the form $w_0 + 2\pi i k$, $k \in \mathbb{Z}$.*

Proof. We have

$$|D\gamma|^2 = \frac{|Df|^2}{|f|^2} \leq K\frac{J_f}{|f|^2} + \frac{\min(|f|^2, |f - z_0|^2)}{|f|^2}\Sigma$$
$$= KJ_\gamma + \min\left(1, \left|1 - e^{w_0 - \gamma}\right|^2\right)\Sigma.$$

Now, fix $k \in \mathbb{Z}$, and suppose first that $|\gamma(z) - w_0 - 2\pi i k| \leq 2^{-1}$. Then

$$\begin{aligned}\left|1 - e^{w_0 - \gamma(z)}\right| &= \left|e^{w_0 + 2\pi i k - \gamma(z)} - 1\right| \\ &\leq |w_0 + 2\pi i k - \gamma(z)| \left(\sum_{j=1}^{\infty} \frac{|w_0 + k2\pi i - \gamma(z)|^{j-1}}{j!}\right) \\ &\leq |w_0 + 2\pi i k - \gamma(z)| \left(\sum_{j=1}^{\infty} \frac{1}{2^{j-1} j!}\right) \leq 2\,|\gamma(z) - w_0 - 2\pi i k|\,.\end{aligned}$$

If on the other hand we have $|\gamma(z) - w_0 - 2\pi i k| \geq 2^{-1}$, then we obtain $1 \leq 4\,|\gamma(z) - w_0 - 2\pi i k|^2$. In either case, we have

$$\min(1, |1 - e^{-\gamma - \theta}|^2) \leq 4\,|\gamma - w_0 - 2\pi i k|^2\,.$$

Thus,

$$|D\gamma|^2 \leq KJ_\gamma + |\gamma - w_0 - 2\pi i k|^2\, 4\Sigma.$$

□

We are now ready to prove Theorem 1.6. We again first recall the statement for the convenience of the reader.

Theorem 1.6. *Let $K \geq 1$ and $\Sigma \in L^{1+\varepsilon}(\mathbb{C}) \cap L^{1-\varepsilon}(\mathbb{C})$ for some $\varepsilon > 0$. Then no continuous map $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{C},\mathbb{C})$ has a (K,Σ)-quasiregular value at two distinct points $z_1, z_2 \in \partial f(\mathbb{C})$. Similarly, no continuous map $h \in W^{1,2}_{\mathrm{loc}}(\mathbb{C},\mathbb{S}^2)$ has a (K,Σ)-quasiregular value with respect to the spherical metric at three distinct points $w_1, w_2, w_3 \in \partial h(\mathbb{C})$.*

Proof. We first reduce the case $h\colon \mathbb{C} \to \mathbb{S}^2$ to the case $f\colon \mathbb{C} \to \mathbb{C}$. Suppose that $h \in W^{1,2}_{\mathrm{loc}}(\mathbb{C}, \mathbb{S}^2)$ has a (K, Σ)-quasiregular value with respect to the spherical metric at three distinct points $w_1, w_2, w_3 \in \partial h(\mathbb{C})$. By post-composing h with an isometric rotation, we may assume that $w_3 = s_2(\infty)$. The single-point Reshetnyak's theorem given in Proposition 3.3 (i) then again yields that $s_2(\infty) \notin h(\mathbb{R}^n)$; indeed, otherwise $h(\mathbb{R}^n)$ would be a neighborhood of $s_2(\infty)$ by the openness part, contradicting $s_2(\infty) = w_3 \in \partial h(\mathbb{C})$. Thus, if we define $f\colon \mathbb{C} \to \mathbb{C}$ by $s_2 \circ f = h$, we have by Lemma 3.1 that f has a $(K, C(h)\Sigma)$-quasiregular value at two distinct points $z_1, z_2 \in \partial f(\mathbb{C})$, where $s_2(z_1) = w_1$ and $s_2(z_2) = w_2$.

Suppose then towards contradiction that $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{C}, \mathbb{C})$ has a (K, Σ)-quasiregular value at two distinct points $z_1, z_2 \in \partial f(\mathbb{C})$. For convenience, we may assume $z_1 = 0$ and $z_2 = 1$ by replacing f with the map $(f - z_1)/(z_2 - z_1)$, an operation which only introduces a multiplicative constant $C(z_1, z_2)$ to Σ. As before, by the single-point Reshetnyak's theorem, we also have that $0, 1 \notin f(\mathbb{C})$.

Since $\Sigma \in L^{1+\varepsilon}(\mathbb{C}) \cap L^{1-\varepsilon}(\mathbb{C})$, we may use Lemma 8.2 to write $f(z) = g(z)e^{\theta(z)}$, where $g\colon \mathbb{C} \to \mathbb{C}$ is an entire quasiregular map and $\theta \in C(\mathbb{C}, \mathbb{C})$ with $\lim_{z\to\infty} \theta(z) = 0$. Since $f(z) \neq 0$ and $e^{\theta(z)} \neq 0$ for all $z \in \mathbb{C}$, we conclude that g omits 0. Hence, we may lift g in the exponential map to find an entire quasiregular map $\gamma\colon \mathbb{C} \to \mathbb{C}$ such that $g = e^\gamma$. In particular,

$$f(z) = e^{\gamma(z)+\theta(z)}.$$

We first observe that γ is non-constant. Indeed, suppose towards contradiction that $\gamma \equiv c$. Then we have $\lim_{z\to\infty} f(z) = e^c$. However, this is impossible, since it follows from $\lim_{z\to\infty} \theta(z) = 0$ that $(\partial f(\mathbb{C})) \setminus f(\mathbb{C}) \subset \{e^c\}$, yet $(\partial f(\mathbb{C})) \setminus f(\mathbb{C})$ must at least contain the two distinct points 0 and 1. Hence, we conclude that γ is non-constant; in particular, by the Picard theorem for entire quasiregular maps, γ omits at most a single point in $\mathbb{C}$.

By Lemma 8.3, $\gamma + \theta$ has a $(K, 4\Sigma)$-quasiregular value at each of the points $2\pi i k, k \in \mathbb{Z}$. Since $\Sigma \in L^{1+\varepsilon}(\mathbb{C}) \cap L^{1-\varepsilon}(\mathbb{C})$, Theorem 1.2 provides a constant $q = q(n, K)$ such that $2\pi i k \in \partial[(\gamma + \theta)(\mathbb{C})]$ for at most q different values of k. Since γ omits at most one point of $\mathbb{C}$, we may fix $k_0 \in \mathbb{Z}$ for which $2\pi i k_0 \in \gamma(\mathbb{C})$ and $2\pi i k_0 \notin \partial[(\gamma + \theta)(\mathbb{C})]$. Since $1 \notin f(\mathbb{C})$, we also have $2\pi i k_0 \notin (\gamma + \theta)(\mathbb{C})$, and hence there exists a radius $r_0 > 0$ such that $(\gamma + \theta)(\mathbb{C}) \cap \mathbb{B}^2(k_0 2\pi i, r_0) = \emptyset$.

Now, for the final step of the argument, we apply Lemma 8.1. Indeed, we have

$$\liminf_{z\to\infty} |(\gamma + \theta)(z) - 2\pi i k_0| \geq r_0 > 0 \quad \text{and} \quad \lim_{z\to\infty} |(\gamma + \theta)(z) - \gamma(z)| = 0.$$

Moreover, $\gamma + \theta$ has a $(K, 4\Sigma)$-quasiregular value at $2\pi i k_0$, and γ is a non-constant quasiregular map. Hence, we conclude that

$$0 = \sum_{z \in (\gamma+\theta)^{-1}\{2\pi i k_0\}} i(z, \gamma + \theta) = \sum_{z \in \gamma^{-1}\{2\pi i k_0\}} i(z, \gamma) > 0,$$

which is a contradiction. The proof is thus complete. $\square$

9. Counterexamples

In this chapter, we discuss the sharpness of the assumptions of Theorem 1.2. In particular, we show that the assumption $\Sigma \in L^1(\mathbb{R}^n) \cup L^{1+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^n)$ in Theorem 1.5 is not sufficient to obtain the conclusions of Theorem 1.2.

Example 9.1. In our first example, we construct for every $q \in \mathbb{Z}_{>0}$ a continuous map $f \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ such that f has q distinct $(1, \Sigma)$-quasiregular values, where $\Sigma \in L^1(\mathbb{R}^n) \cap L^{1-\varepsilon}(\mathbb{R}^n) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ for every $\varepsilon \in (0, 1)$. See Figure 4 for a rough illustration of the example in the case $n = 2$.

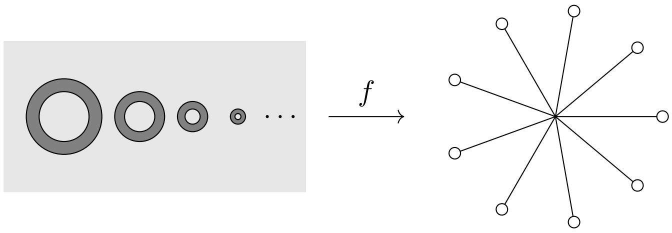

Figure 4. Rough illustration of the map f of Example 9.1 in the case $n = 2$. The map f takes each of the infinitely many shaded annuli on the domain side to one of the open-ended stalks on the target side, stopping partway through. In the lighter shaded part of $\mathbb{R}^2$ the map f is locally constant, with the unbounded component mapped to the center of the stalks. The tips of the stalks are quasiregular values of f and are contained in $\partial f(\mathbb{R}^2)$.

We begin by selecting q distinct points $\{y_1, \dots, y_q\} \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$. Let $d_0 > 0$ be the minimum distance from a point y_k to a line $\{ty_l, t \in \mathbb{R}\}$, where $k \neq l$.

We consider the function $\theta\colon (0, 2^{-1}) \to [0, \infty)$ given by

$$\theta(r) = \log^{\frac{n-1-\delta}{n}} \frac{1}{r},$$

where $\delta \in (0, n-1)$. Note that θ is decreasing. We also define a function $\Theta\colon \mathbb{B}^n(0, 2^{-1}) \setminus \{0\} \to [0, \infty)$ by

$$\Theta(x) = \theta(|x|).$$

Then we have

$$\int_{\mathbb{B}^n(0,2^{-1})} |\nabla\Theta|^n \lesssim_{n,\delta} \int_{\mathbb{B}^n(0,2^{-1})} \frac{1}{|x|^n \log^{1+\delta} |x|^{-1}} < \infty.$$

Thus, $\nabla\Theta \in L^n(\mathbb{B}^n(0, 2^{-1}))$, and consequently, by Hölder's inequality, $\nabla\Theta \in L^{(1-\varepsilon)n}(\mathbb{B}^n(0, 2^{-1}))$ for every $\varepsilon \in (0, 1)$. Note also that

$$\lim_{x \to 0} \Theta(x) = \lim_{r \to 0} \theta(r) = \infty.$$

Thus, we may select radii $2^{-1} = R_1 > R_2 > \dots$ for which we have $\theta(R_{i+1}) - \theta(R_i) = i$ for all $i \in \mathbb{Z}_{\geq 0}$.

We then pick a discrete set of points $\{x_i : i \in \mathbb{Z}_{>0}\} \subset \mathbb{R}^n$ such that the closures of the balls $B_i = \mathbb{B}^n(x_i, R_i)$ are pairwise disjoint. We also denote $B_i' = \mathbb{B}^n(x_i, R_{i+1})$, and $k_i = (i \mod q) \in \{1, \dots, q\}$. We then define a function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ as follows: in the set $\mathbb{R}^n \setminus \bigcup_i B_i$, we define $f \equiv 0$, in the sets $B_i \setminus B_i'$, we define

$$f(x) = (1 - e^{\Theta(x-x_i)-\theta(R_i)})y_{k_i},$$

and in the sets B_i', we define $f(x) \equiv (1-e^{-i})y_{k_i}$.

By our construction, we observe that $f \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$, f is continuous, and $y_j \in \partial f(\mathbb{R}^n)$ for every $j \in \{1, \dots, q\}$. We also have $J_f \equiv 0$ everywhere since the image of f is a 1-dimensional tree, and $|Df| \equiv 0$ in $\mathbb{R}^n \setminus \bigcup_i B_i$ and in every B_i'. Hence, we may select $\Sigma \equiv 0$ in these sets, and have $|Df| \leq J_f + |f - y_j|^n \, \Sigma$ for every $j \in \{1, \dots, q\}$.

It remains to consider the regions $B_i \setminus B_i'$. In these regions, we have

$$\frac{|Df|}{|f - y_{k_i}|} = \frac{|y_{k_i}| \, e^{\Theta(x-x_i)-\theta(R_i)} \, |\nabla \Theta(x - x_i)|}{|y_{k_i}| \, e^{\Theta(x-x_i)-\theta(R_i)}} = |\nabla \Theta(x - x_i)| \, .$$

Moreover, whenever $j \neq k_i$, we may use $e^{\Theta(x-x_i)-\theta(R_i)} \leq 1$, $|y_j| = 1$, and $|f - y_j| \geq d_0$ to obtain

$$\frac{|Df|}{|f - y_j|} = \frac{|y_j| \, e^{\Theta(x-x_i)-\theta(R_i)} \, |\nabla \Theta(x - x_i)|}{|f - y_j|} \leq d_0^{-1} \, |\nabla \Theta(x - x_i)| \, .$$

Thus, we may select $\Sigma = \max(1, d_0^{-n}) \, |\nabla \Theta(x - x_i)|^n$. Now, since the regions $B_i \setminus B_i'$ are translates of the concentric annuli $\mathbb{B}^n(0, R_i) \setminus \mathbb{B}^n(0, R_{i+1})$ by x_i, and since $|\nabla \Theta| \in L^p(\mathbb{B}^n(0, R_1))$ for all $p \in (0, n]$, we obtain that $\Sigma \in L^1(\mathbb{R}^n) \cap L^{1-\varepsilon}(\mathbb{R}^n)$ for every $\varepsilon \in (0,1)$. Moreover, since $\{x_i\}$ is discrete and since Σ is bounded on every $B_i \setminus B_i'$, we get that $\Sigma \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$.

Finally, the following example shows the necessity of the global lower integrability assumption in Theorem 1.2.

Example 9.2. In this example, we construct for every $q \in \mathbb{Z}_{>0}$ a continuous map $f \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ with q distinct $(1, \Sigma)$-quasiregular values in $\partial f \mathbb{R}^n$, where $\Sigma \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Our strategy is similar to the one used in Example 9.1, but we use increasingly large annuli instead of increasingly small ones.

We let $\{y_1, \dots, y_q\} \in \mathbb{S}^{n-1}$ and $d_0 > 0$ be as in the previous example. This time, we consider the map $\theta\colon (2, \infty) \to [0, \infty)$ given by

$$\theta(r) = \log^{\frac{n-1-\delta}{n}} r,$$

where $\delta \in (0, n-1)$. We define $\Theta\colon \mathbb{R}^n \setminus \mathbb{B}^n(0,2) \to [0, \infty)$ by $\Theta(x) = \theta(|x|)$. Similarly to last time, we have

$$\int_{\mathbb{R}^n \setminus \mathbb{B}^n(0,2)} |\nabla \Theta|^n \lesssim_{n,\delta} \int_{\mathbb{R}^n \setminus \mathbb{B}^n(0,2)} \frac{1}{|x|^n \log^{1+\delta} |x|} < \infty.$$

Moreover, we have $\lim_{r \to \infty} \theta(r) = \infty$ and $|\nabla \Theta| \in L^\infty(\mathbb{R}^n \setminus \mathbb{B}^n(0,2))$.

We again split $\mathbb{R}^n \setminus \mathbb{B}^n(0,2)$ into sub-annuli by fixing radii $2 = R_1 < R_2 < \dots$ satisfying $\theta(R_{i+1}) - \theta(R_i) = i$. We select points $\{x_i\}$ so that the closures of the balls $B_i = \mathbb{B}^n(x_i, R_{i+1})$ are pairwise disjoint; note that this time $\{x_i\}$ is automatically discrete and in fact extremely sparse, as we have $|x_i - x_j| \geq R_i + R_j \geq 4$ whenever $i \neq j$. We also again denote $B_i' = \mathbb{B}^n(x_i, R_i)$ and $k_i = (i \mod q) \in \{1, \dots, q\}$.

We then define $f\colon \mathbb{R}^n \to \mathbb{R}^n$ so that in the set $\mathbb{R}^n \setminus \bigcup_i B_i$ we have $f \equiv 0$, in the sets $B_i \setminus B_i'$ we have

$$f(x) = (1 - e^{\Theta(x-x_i)-\theta(R_{i+1})})y_{k_i},$$

and in the sets B_i' we have $f(x) = (1 - e^{-i})y_{k_i}$. We again get that f is continuous, that $y_j \in \partial f(\mathbb{R}^n)$ for all $j \in \{1, \dots, q\}$, that $J_f \equiv 0$, and moreover that $f \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$. In order for all y_j to be $(\Sigma, 1)$-quasiregular values of f, we can again pick $\Sigma \equiv 0$ in $\mathbb{R}^n \setminus \bigcup_i B_i$ and in the sets B_i'. Moreover, in the sets $B_i \setminus B_i'$, a similar argument as in the last example shows that we may pick $\Sigma = \max(1, d_0^{-n}) \left|\nabla\Theta(x - x_i)\right|^n$, in which case $\Sigma \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

References

[1] L. Ahlfors. Zur Theorie der überlagerungsflächen. *Acta Math.*, 65(1):157–194, 1935.

[2] L. V. Ahlfors. *Conformal invariants: topics in geometric function theory.* McGraw-Hill Series in Higher Mathematics. McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1973.

[3] L. Ambrosio and G. Dal Maso. A general chain rule for distributional derivatives. *Proc. Amer. Math. Soc.*, 108(3):691–702, 1990.

[4] K. Astala, T. Iwaniec, and G. Martin. *Elliptic partial differential equations and quasiconformal mappings in the plane.* Princeton university press, 2009.

[5] K. Astala and L. Päivärinta. Calderón's inverse conductivity problem in the plane. *Ann. of Math. (2)*, 163(1):265–299, 2006.

[6] M. Bonk and P. Poggi-Corradini. The Rickman-Picard theorem. *Ann. Acad. Sci. Fenn. Math.*, 44:615–633, 2019.

[7] E. Borel. Sur les zéros des fonctions entières. *Acta Math.*, 20(1):357–396, 1897.

[8] B. Davis. Picard's theorem and Brownian motion. *Trans. Amer. Math. Soc.*, 213:353–362, 1975.

[9] A. Doležalová, I. Kangasniemi, and J. Onninen. Mappings of generalized finite distortion and continuity. *J. London Math. Soc.*, 109(1), 2024.

[10] D. Drasin and P. Pankka. Sharpness of Rickman's Picard theorem in all dimensions. *Acta Math.*, 214(2):209–306, 2015.

[11] A. Eremenko and J. Lewis. Uniform limits of certain A-harmonic functions with applications to quasiregular mappings. *Ann. Acad. Sci. Fenn. Math.*, 16:361–375, 1991.

[12] D. Faraco and X. Zhong. A short proof of the self-imroving regularity of quasiregular mappings. *Proc. Amer. Math. Soc.*, 134(1):187–192, 2005.

[13] R. Finn and J. Serrin. On the Hölder continuity of quasi-conformal and elliptic mappings. *Trans. Amer. Math. Soc.*, 89:1–15, 1958.

[14] I. Fonseca and W. Gangbo. *Degree theory in analysis and applications*, volume 2 of *Oxford Lecture Series in Mathematics and its Applications.* The Clarendon Press, Oxford University Press, New York, 1995. Oxford Science Publications.

[15] W. H. J. Fuchs. *Topics in the theory of functions of one complex variable.* Van Nostrand Mathematical Studies, No. 12. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1967. Manuscript prepared with the collaboration of Alan Schumitsky.

[16] F. W. Gehring. Rings and quasiconformal mappings in space. *Trans. Amer. Math. Soc.*, 103(3):353–393, 1962.

[17] D. Gilbarg and N. Trudinger. *Elliptic partial differential equations of second order.* Springer, 2001.

[18] H. Grötzsch. Über die Verzerrung bei schlichten nichtkonformen Abbildungen und über eine damit zusammenhängende Erweiterung des Picardischen Satzes. *Ber. Vehr. Sächs. Akad. Wiss. Leipzig*, 80:503–507, 1928.

[19] P. Hajłasz and P. Koskela. Sobolev met Poincaré. *Mem. Amer. Math. Soc.*, 145(688):x+101, 2000.

[20] P. Hartman. Hölder continuity and non-linear elliptic partial differential equations. *Duke Math. J.*, 25:57–65, 1958.

[21] W. K. Hayman. *Meromorphic functions.* Oxford Mathematical Monographs. Clarendon Press, Oxford, 1964.

[22] J. Heinonen, T. Kilpeläinen, and O. Martio. *Nonlinear potential theory of degenerate elliptic equations.* Dover, 2006.

[23] I. Holopainen and S. Rickman. A Picard type theorem for quasiregular mappings of $\mathbb{R}^n$ into n-manifolds with many ends. *Rev. Math. Iberoam.*, 8(2):131–148, 1992.

[24] I. Holopainen and S. Rickman. Ricci curvature, Harnack functions, and Picard-type theorems for quasiregular mappings. In C. A. Cazacu, O. Lehto, and T. Rassias, editors, *Analysis and Topology*, pages 315–326. World Scientific, 1998.

[25] W. Hurewicz and H. Wallman. *Dimension theory (PMS-4).* Princeton University Press, 1948.

[26] T. Iwaniec. The Gehring lemma. In P. Duren, J. Heinonen, B. Osgood, and B. Palka, editors, *Quasiconformal mappings and analysis: a collection of papers honoring F.W. Gehring.* Springer, 1998.

[27] T. Iwaniec and G. Martin. *Geometric function theory and non-linear analysis.* Clarendon Press, 2001.

[28] I. Kangasniemi and J. Onninen. On the heterogeneous distortion inequality. *Math. Ann.*, 384:1275–1308, 2022.

[29] I. Kangasniemi and J. Onninen. A single-point Reshetnyak's theorem. 2022. Pre-print, https://arxiv.org/abs/2202.06917.

[30] I. Kangasniemi and J. Onninen. Correction to "On the heterogeneous distortion inequality". *Math. Ann.*, 2023.

[31] J. L. Lewis. Picard's theorem and Rickman's theorem by the way of Harnack's inequality. *Proc. Amer. Math. Soc.*, 122(1):199–206, 1994.

[32] G. Martin and K. Peltonen. Stoïlow factorization for quasiregular mappings in all dimensions. *Proc. Amer. Math. Soc.*, 138(1):147–151, 2010.

[33] O. Martio, S. Rickman, and J. Väisälä. Definitions for quasiregular mappings. *Ann. Acad. Sci. Fenn. Ser. AI. Math.*, 448:1–40, 1969.

[34] O. Martio, S. Rickman, and J. Väisälä. Distortion and singularities of quasiregular mappings. *Ann. Acad. Sci. Fenn. Ser. AI. Math.*, 465:1–13, 1970.

[35] O. Martio, S. Rickman, and J. Väisälä. Topological and metric properties of quasiregular mappings. *Ann. Acad. Sci. Fenn. Ser. AI. Math.*, 488:1–31, 1971.

[36] L. Nirenberg. On nonlinear elliptic partial differential equations and Hölder continuity. *Comm. Pure Appl. Math.*, 6:103–156; addendum, 395, 1953.

[37] K. Rajala. Mappings of finite distortion: the Rickman-Picard theorem for mappings of finite lower order. *J. Anal. Math.*, 94(1):235–248, 2004.

[38] Y. G. Reshetnyak. Bounds on moduli of continuity for certain mappings. *Sibirsk. Mat. Zh.*, 7:1106–1114, 1966. (Russian).

[39] Y. G. Reshetnyak. The Liouville theorem with mininal regularity conditions. *Sibirsk. Mat. Zh.*, 8:835–840, 1967. (Russian).

[40] Y. G. Reshetnyak. On the condition of the boundedness of index for mappings with bounded distortion. *Sibirsk. Mat. Zh.*, 9:368–374, 1967. (Russian).

[41] Y. G. Reshetnyak. Space mappings with bounded distortion. *Sibirsk. Mat. Zh.*, 8:629–659, 1967. (Russian).

[42] Y. G. Reshetnyak. *Space mappings with bounded distortion*, volume 73 of *Translations of Mathematical Monographs.* American Mathematical Society, Providence, RI, 1989.

[43] S. Rickman. On the number of omitted values of entire quasiregular mappings. *J. Anal. Math.*, 37:100–117, 1980.

[44] S. Rickman. The analogue of Picard's theorem for quasiregular mappings in dimension three. *Acta Math.*, 154(3):195–242, 1985.

[45] S. Rickman. *Quasiregular mappings*, volume 26. Springer-Verlag, 1993.

[46] B. V. Šabat. On the theory of quasiconformal mappings in space. *Dokl. Akad. Nauk SSSR*, 130:1210–1213, 1960.

[47] S. L. Segal. *Nine introductions in complex analysis*, volume 80 of *Notas de Matemática [Mathematical Notes].* North-Holland Publishing Co., Amsterdam-New York, 1981.

[48] L. Simon. A Hölder estimate for quasiconformal maps between surfaces in euclidean space. *Acta Math.*, 139:19–51, 1977.

[49] J. Väisälä. On quasiconformal mappings in space. *Ann. Acad. Sci. Fenn. Ser. AI*, 298:1–36, 1961.

[50] J. Väisälä. *Lectures on n-dimensional quasiconformal mappings.* Springer, 1971.

[51] I. N. Vekua. *Generalized analytic functions.* Addison-Wesley Publ. Co., 1962.

[52] G. Y. Zhang. Curves, domains and Picard's theorem. *Bull. London Math. Soc.*, 34(2):205–211, 2002.

[53] V. A. Zorič. On a correspondence of boundaries for q-quasiconformal mappings of a sphere. *Dokl. Akad. Nauk SSSR*, 145:31–34, 1962.

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