WEAK TOPOLOGY ON CAT(0) SPACES

ALEXANDER LYTCHAK AND ANTON PETRUNIN

Abstract. We analyze weak convergence on CAT(0) spaces and the existence and properties of corresponding weak topologies.

1. Introduction

Weak convergence and coarse topologies in CAT(0) spaces have appeared in relation to very different problems and settings in the last years, see [Jos94, Mon06, KP08, Bac13, Kel14, Str16, BDL17, GN20] and the survey [Bac18] for an overview. On the other hand, some related fundamental questions have remained open. This note aims to close some of these gaps.

Definition 1.1. *A bounded sequence (x_n) in a* CAT(0) *space X* converges weakly *to a point x if for any compact geodesic c starting at x, the closest-point projections $Proj_c(x_n)$ of x_n to c converge to x.*

This notion of convergence (also known as Δ-convergence), introduced in [Jos94], generalizes weak convergence in Hilbert spaces. It can be defined in many other natural ways and is suitable for questions concerning the existence of fixed points and gradient flows, see [Bac18]. The weak convergence generalizes verbatim to convergence of nets and satisfies natural compactness and separation properties.

We begin by resolving the question asked by William Kirk and Bancha Panyanak in [KP08, Question 1] and discussed, for instance, in [Bac14, Bac18, Kel14, DST16]. The question concerns the existence of a weak topology inducing the weak convergence. Somewhat surprisingly, the answer is different for sequences and for general nets. In the case of sequences, the answer is always affirmative and the proof is general nonsense, not involving geometry:

Theorem 1.2. *Let X be a* CAT(0) *space. There exists a unique topology $\mathcal{T}_\Delta$ on X with the following two properties:*

- *A sequence (x_n) converges in X with respect to $\mathcal{T}_\Delta$ to a point x if and only if the sequence is bounded and converges to x weakly.*

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- *The topology $\mathcal{T}_\Delta$ is sequential.*

Recall, that a topology $\mathcal{T}$ is called *sequential* if a subset is $\mathcal{T}$-closed whenever it contains any $\mathcal{T}$-limit point of any sequence of its elements.

This topology $\mathcal{T}_\Delta$, which we want to call the *weak topology*, has the following additional properties; see Proposition 3.1, Corollary 5.3: $\mathcal{T}_\Delta$ is sequentially Hausdorff; any metrically closed, bounded, convex subset of X is $\mathcal{T}_\Delta$-closed, $\mathcal{T}_\Delta$-sequentially compact and $\mathcal{T}_\Delta$-compact. However,

Proposition 1.3. *There exists a bounded, separable, two-dimensional* CAT(0) *simplicial complex X such that $\mathcal{T}_\Delta$ is not Hausdorff.*

Together with Proposition 1.3, the next theorem implies that, in general, there is no topology on a CAT(0) space which induces the weak convergence of *nets*:

Theorem 1.4. *Let X be a* CAT(0) *space and let $\mathcal{T}_\Delta$ be the weak topology defined in Theorem 1.2. For a topology $\mathcal{T}$ on X the following two conditions are equivalent:*

- *A bounded net (x_α) converges to a point $x \in X$ weakly if and only if (x_α) converges to x with respect to $\mathcal{T}$.*
- *The restriction of $\mathcal{T}$ to any closed ball in X is Hausdorff and coincides with $\mathcal{T}_\Delta$.*

We discuss $\mathcal{T}_\Delta$ in some examples and relate this topology to another coarse topology, the *coconvex* topology introduced by Nicolas Monod in [Mon06]. This coconvex topology $\mathcal{T}_{co}$ on a CAT(0) space X is defined as the coarsest topology $\mathcal{T}$ on X for which all metrically closed, convex subsets are $\mathcal{T}$-closed.

Every metrically closed, bounded convex subset of X is $\mathcal{T}_{co}$-compact and $\mathcal{T}_{co}$-sequentially compact, see Section 6. The weak topology $\mathcal{T}_\Delta$ is finer than the coconvex topology $\mathcal{T}_{co}$ (Proposition 3.1); these topologies can be different even for bounded CAT(0) spaces X (Lemma 4.1). The topologies $\mathcal{T}_{co}$ and $\mathcal{T}_\Delta$ coincide on all bounded subsets of X if and only if the topology $\mathcal{T}_{co}$ is sequential and sequentially Hausdorff on bounded convex subsets. Whenever $\mathcal{T}_{co}$ is Hausdorff on bounded subsets, the topologies $\mathcal{T}_{co}$ and $\mathcal{T}_\Delta$ coincide on bounded subsets.

Whenever the CAT(0) space X is locally compact, the metric topology $\mathcal{T}_{metric}$ coincides with $\mathcal{T}_\Delta$. On the other hand, for smooth 3-dimensional Riemannian CAT(-1) manifolds or symmetric spaces of higher rank, the coconvex topology $\mathcal{T}_{co}$ can be non-Hausdorff and not first countable, as we will observe in Section 6. The failure of the Hausdorff property for symmetric spaces has been expected in [Mon06], a first explicitly confirmed failure of the Hausdorff property for some

CAT(0) space seems to be the example of the Euclidean cone over a Hilbert space provided by Martin Kell in [Kel14].

On the other hand, $\mathcal{T}_{co}$ is Hausdorff (and therefore coincides with $\mathcal{T}_\Delta$ and induces the weak convergence of nets) in some geometric cases:

Proposition 1.5. *The topology $\mathcal{T}_{co}$ is Hausdorff in the following cases:*

(1) *X is homeomorphic to the plane.*
(2) *X is a Riemannian manifold with pinched negative curvature.*
(3) *X is a finite-dimensional cubical complex.*

The answer we provide to the second point above is a direct consequence of the construction of convex hulls in manifolds with pinched negative curvature due to Michael Anderson. While the main construction of [And83] works without changes in infinite dimensions, it seems not to be sufficient to answer another question from [Mon06]:

Problem 1.6. *Is the coconvex topology $\mathcal{T}_{co}$ Hausdorff on the infinite-dimensional complex projective space $X = CH^\infty$?*

Despite Lemma 4.1 and the examples by Alano Ancona [Anc94], see Example 6.4 below, we do not know the answer to the following:

Problem 1.7. *Find an example of bounded* CAT(0) *spaces for which $\mathcal{T}_\Delta$ is Hausdorff but different from $\mathcal{T}_{co}$.*

A natural question is whether for the class of non-locally compact CAT(0) spaces appearing in most applications, as in [Mon06, Str16, BDL17, Cla13, CR13], the weak topology is Hausdorff, at least when restricted to bounded subsets. Most of the examples are subsumed by or related to the example in the following question (we refer to [Mon06] for the definition and properties of the spaces of L^2-maps):

Problem 1.8. *Let Ω be a probability space and X a locally compact* CAT(0) *space. What are the separation properties of the weak and the coconvex topologies on the space of L^2-maps $\mathcal{L}^2(\Omega, X)$?*

Also, the following question seems to be very natural in view of the somewhat cumbersome formulation of Theorem 1.4:

Problem 1.9. *If the restriction of the weak topology $\mathcal{T}_\Delta$ on any bounded subset is Hausdorff, does it have to be a Hausdorff topology on X?*

The paper arose in an attempt to better understand the behavior of convex subsets and convex hulls in CAT(0) spaces. The non-Hausdorff properties of $\mathcal{T}_{co}$ should be related to Gromov's question:

Problem 1.10. *Is the closed convex hull of a compact subset of any* CAT(0) *space compact?*

The paper is structured as follows. In Section 3 we recall some basic properties of the weak convergence and provide a rather straightforward proof of Theorem 1.2. In Section 4 we provide the example verifying Proposition 1.3. In Section 5 we prove Theorem 1.4. The main technical point in the proof is a CAT(0)-version of the theorem of Eberlein–Smulian in functional analysis, saying that a bounded subset is weakly closed if and only if it is weakly sequentially closed (Proposition 5.1). In the final Section 6, we discuss the relations with the coconvex topology.

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2. Preliminaries

2.1. **CAT(0).** We assume familiarity with the geometry of CAT(0) spaces and refer to [BH99], [AKP19a], [AKP19b]. All CAT(0) spaces here are by definition complete and geodesic.

By $d(x,y) = d_X(x,y)$ we denote the distance in a metric space X. By $B_r(x)$ we denote the closed ball of radius r around the point x.

Any bounded subset A in a CAT(0) space X has a unique circumcenter $x \in X$ such that for some $r = r(A) \in \mathbb{R}$, the *circumradius* of A, we have $A \subset B_r(x)$ but $A \not\subset B_r(y)$, for any other point $y \neq x$ [BH99].

2.2. **General topology.** We refer to [Eng89] for a detailed explanation of the notions below.

A *directed set* I is a partially ordered set, such that for any pair $\alpha_1, \alpha_2 \in I$ there exist α with $\alpha \geq \alpha_1$ and $\alpha \geq \alpha_2$.

A net (x_α) in a set X is given by a map $\alpha \to x_\alpha$ from a directed set I to X. We will mostly suppress the directed set I since it will not play any special role.

In a topological space X, a net (x_α) converges to a point x if for any neighborhood U of x there exists some α_0 such that, for all $\alpha \geq \alpha_0$, the elements x_α are contained in U.

In a topological space, convergence of nets can be used as the right generalization of convergence of sequences from the theory of metric spaces. For instance, a topological space is Hausdorff if and only if any net can converge to at most one point in X. A topological space

is compact if and only if any net in X has a converging subnet. The closure of a subset $A \subset X$ consists of all limit points x of all nets (x_α) with $x_\alpha \in A$.

Replacing in the above statement general nets by sequences, we obtain the following properties of spaces which will appear below.

A topological space X is called *sequentially Hausdorff* if any sequence in X has at most one limit point. Any Hausdorff space is sequentially Hausdorff but the opposite does not hold.

A topological space X is called *sequentially compact* if any sequence in X has a convergent subsequence. A compact space does not need to be sequentially compact and a sequentially compact space does not need to be compact.

2.3. **Basics on weak convergence.** Let X be a CAT(0) space. We stick to the definition of weak convergence given in the introduction and refer to [Bac18] for other descriptions and for the explanations and references of the following properties frequently used below:

Any bounded net in X has at most one weak limit point in X. Any subnet of a weakly converging net converges weakly to the same limit point. Any bounded sequence (net) has a weakly converging subsequence (subnet).

3. Weak convergence of sequences

In this section we provide the rather straightforward:

Proof of Theorem 1.2. Define the topology $\mathcal{T}_\Delta$ as follows. We say that a subset $A \subset X$ is $\mathcal{T}_\Delta$-closed, if, for any bounded sequence $x_n \in A$ weakly converging to a point $x \in X$, we have $x \in A$.

By definition, the empty set and the whole set are $\mathcal{T}_\Delta$-closed. Moreover, any intersection of $\mathcal{T}_\Delta$-closed subsets A_α is $\mathcal{T}_\Delta$-closed.

Finally, if $A_1, ..., A_m$ are $\mathcal{T}_\Delta$-closed and (x_n) is a bounded sequence in $A_1 \cup .... \cup A_m$ weakly converging to x, then we find a subsequence of (x_n) contained in one of the A_i. This subsequence also weakly converges to x, therefore $x \in A_i$. Hence $A_1 \cup ... \cup A_m$ is $\mathcal{T}_\Delta$-closed.

Altogether, this shows that the family of all $\mathcal{T}_\Delta$-closed sets is the family of closed sets of a topology, which we will denote by $\mathcal{T}_\Delta$.

We claim that a sequence (x_n) in X converges to a point x with respect to $\mathcal{T}_\Delta$ if and only if (x_n) is bounded and converges to x weakly.

Firstly, let (x_n) be bounded and weakly converge to x. If (x_n) does not $\mathcal{T}_\Delta$-converge to x, we would find a $\mathcal{T}_\Delta$-open subset U containing x and a subsequence (x_{m_n}) contained in the complement $A := X \setminus U$.

However, (x_{m_n}) also converges to x weakly, hence, by the definition of $\mathcal{T}_\Delta$-closed subsets, we infer $x \in A$, a contradiction.

On the other hand, let a sequence (x_n) converge in the $\mathcal{T}_\Delta$-topology to x. If (x_n) is not bounded, we could find a subsequence (x_{m_n}) such that $d(x_1, x_{m_n}) \to \infty$. Then the countable set $\{x_{m_n}\}$ is $\mathcal{T}_\Delta$-closed. Hence, (x_{m_n}) does not $\mathcal{T}_\Delta$-converge to x. Therefore, (x_n) must be bounded.

Assume that x_n does not converge weakly to x. Then we find a subsequence (x_{m_n}) of (x_n) which converges weakly to some point $y \neq x$. Moreover, deleting finitely many elements from the sequence, we may assume that x_{m_n} is not equal to x for all n. Then the union A of all x_{m_n} and the point y is $\mathcal{T}_\Delta$-closed. Thus, the complement of A is a $\mathcal{T}_\Delta$-open neighborhood of x, which does not contain all but finitely many elements of the sequence (x_n). This contradiction proves that (x_n) weakly converges to x and finishes the proof of the claim.

The claim and the definition of $\mathcal{T}_\Delta$ imply that a subset A of X is $\mathcal{T}_\Delta$-closed if every $\mathcal{T}_\Delta$-limit $x \in X$ of a sequence of points in A is contained in A. This means that $\mathcal{T}_\Delta$ is sequential.

We have verified the required properties of $\mathcal{T}_\Delta$. Let $\mathcal{T}$ be another sequential topology on X, for which a sequence (x_n) converges to x if and only if (x_n) is bounded and weakly converges to x. Then, for $\mathcal{T}$ and $\mathcal{T}_\Delta$ the convergence of sequences coincide. Since both topologies are sequential, this implies that the properties of being closed with respect to $\mathcal{T}$ and $\mathcal{T}_\Delta$ coincide. Hence, $\mathcal{T} = \mathcal{T}_\Delta$. □

Basic properties of the weak topology $\mathcal{T}_\Delta$ are direct consequence of the definition and the corresponding properties of weak convergence:

Proposition 3.1. *The weak topology $\mathcal{T}_\Delta$ on a* CAT(0) *space X is finer than the coconvex topology and coarser than the metric topology:*

$$\mathcal{T}_{co} \subset \mathcal{T}_\Delta \subset \mathcal{T}_{metric}.$$

The topology $\mathcal{T}_\Delta$ is sequentially Hausdorff. Any metrically closed, bounded, convex subset $C \subset X$ is $\mathcal{T}_\Delta$-sequentially compact.

The less trivial statement that any closed, bounded, convex subset is $\mathcal{T}_\Delta$-compact will be derived later in Corollary 5.3.

We finish the section with two simple examples. The first example is a direct consequence of the definition and the theorem of Hopf–Rinow:

Example 3.2. Assume that the CAT(0) space X is locally compact. Then $\mathcal{T}_\Delta$ coincides with the metric topology.

The second example is a special case of the fact that the weak convergence as defined above corresponds to the usual weak convergence in the case of Hilbert spaces, [Bac18] and Theorem of Eberlien-Smulian,

[Whi67], in the case of Hilbert spaces, saying that a subset is compact in the weak topology if and only if it is sequentially compact.

Example 3.3. For a Hilbert space X, the topology $\mathcal{T}_\Delta$ coincides with the weak topology of the Hilbert space and with $\mathcal{T}_{co}$.

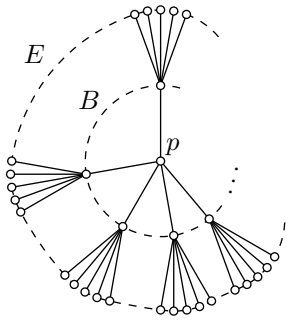
4. Example

We are going to show that $\mathcal{T}_\Delta$ can be non-Hausdorff:

Proof of Proposition 1.3. Let Y_1 be a countable family of intervals $[0, \frac{\pi}{4}]$ glued together at the common boundary point 0. Fix an endpoint b among the countably many endpoints of the tree Y_1. Choose a countable family of isometric copies of Y_1 and glue all of them together by identifying the chosen "endpoints" b with each other.

The arising space Y is a tree with a special point p (the point at which all subtrees isometric to Y_1 are glued together). Point p is the unique circumcenter of the simplicial tree Y. The tree has countably many branches at p and every point at distance $\frac{\pi}{4}$ from p. There are no other branching points in Y; all edges of the tree Y have length $\frac{\pi}{4}$.

We denote by E the set of endpoints of the tree Y and by B the set of the branching points different from p (thus the $\frac{\pi}{4}$-sphere around p). Any pair of different points of B lie at distance $\frac{\pi}{2}$ from each other. Any pair of different points in E either are at distance π and have p as the midpoint or are at distance $\frac{\pi}{2}$ and have a point from B as their midpoint.



Let $\hat{X}$ denote the Euclidean cone over Y. We identify Y with the unit sphere around the tip o in $\hat{X}$. For a point $y \in Y$ and a number $\lambda \geq 0$, we denote by $\lambda \cdot y$ the point in the cone $\hat{X}$ on the radial ray in the direction of y at distance λ from the vertex o.

For any edge I of Y with endpoints y_1, y_2 consider the triangle S_I defined by the points $o, 2 \cdot y_1, 2 \cdot y_2$ in $\hat{X}$. The union of all such triangles is a closed convex subset X of $\hat{X}$. This subset X is bounded and contains the unit ball $B_1(o)$. Moreover, X is a 2-dimensional simplicial complex with countably many simplices.

We are going to verify that the points o and $\frac{1}{2} \cdot p$ are not separated in the weak topology $\mathcal{T}_\Delta$ on space X.

Firstly, for any pair of different points in $E \subset Y \subset X$ the unique geodesic in X connecting them either has its midpoint in o (if the points

are at distance π in Y) or it has its midpoints in $\frac{1}{\sqrt{2}} \cdot b$ for the unique midpoint $b \in B$ of the corresponding geodesic in Y.

Given any sequence (x_n) of elements in $E \subset Y \subset X$ with pairwise distance π in Y, we see that the convex hull of $\{x_n\}$ is the union of the geodesic segments $[o, x_n]$, thus a tree with a unique vertex in o. In this case, the sequence (x_n) converges weakly to o.

Given any sequence x_n of pairwise different elements in E with pairwise distance $\frac{\pi}{2}$ in Y, the convex hull of $\{x_n\}$ is again a tree with a unique vertex $\frac{1}{\sqrt{2}} \cdot b$, the common midpoint of any pair of different points in the sequence (x_n). Thus, (x_n) weakly converges to $\frac{1}{\sqrt{2}} \cdot b$.

Similarly, for any sequence of different point $b_n \in B \subset Y \subset X$, the sequence (b_n) weakly converges in X to the point $\frac{1}{\sqrt{2}} \cdot p$. Thus, by rescaling, the sequence $\frac{1}{\sqrt{2}} \cdot b_n$ converges weakly to $\frac{1}{2} \cdot p$.

Assume that o and $\frac{1}{2} \cdot p$ can be separated in $\mathcal{T}_\Delta$. Thus, we find $\mathcal{T}_\Delta$-closed subsets C_1 and C_2 such that $o \notin C_1$, $\frac{1}{2} \cdot p \notin C_2$ and $C_1 \cup C_2 = X$.

By above, C_1 cannot contain infinitely many points of E, which have in Y pairwise distance π.

Thus, for all but finitely many branch-points $b \in B \subset Y$ all points in E at distance $\frac{\pi}{4}$ from b are contained in C_2. By above, for any such b we must have $\frac{1}{\sqrt{2}} \cdot b \in B$. Since we have infinitely many such points, we conclude $\frac{1}{2} \cdot p \in C_2$, in contradiction to our assumption.

Thus, we have verified that $(X, \mathcal{T}_\Delta)$ is not Hausdorff. □

The provided example implies that $\mathcal{T}_{co}$ and $\mathcal{T}_\Delta$ may be different:

Lemma 4.1. *The weak topology $\mathcal{T}_\Delta$ and coconvex topology $\mathcal{T}_{co}$ do not coincide on the bounded* CAT(0) *space X constructed above.*

Proof. Consider the set

$$A = E \cup \tfrac{1}{\sqrt{2}} \cdot B \cup \{\tfrac{1}{2} \cdot p\} \cup \{o\}$$

which has appeared above. As explained in the proof of Proposition 1.3 above, the set A is $\mathcal{T}_\Delta$-closed.

We are going to prove that $\frac{1}{4} \cdot p$ is contained in the $\mathcal{T}_{co}$ closure of A. Assuming the contrary, we find finitely many convex, metrically closed subsets $C_1, ..., C_n$ in X which cover A and do not contain $\frac{1}{4} \cdot p$.

For any $b \in B$, consider the set E^b of points in E which are at distance $\frac{\pi}{4}$ from b. Then a counting argument implies that at least one of the sets C_i contains at least 2 points in any of the sets E^{b_1}, E^{b_2}, for different $b_1, b_2 \in B$. Then this convex set C_i contains the origin o (as the midpoint of a point in E^{b_1} and E^{b_2}), the points $\frac{1}{\sqrt{2}} \cdot b_i$ and therefore

their midpoint $\frac{1}{2}\cdot p$. Hence, C_i also contains the whole geodesic $[o, \frac{1}{2}\cdot p]$ and, therefore, $\frac{1}{4}\cdot p \in C_i$, in contradiction to our assumption.

Thus, the set A is not $\mathcal{T}_{co}$-closed, finishing the proof. □

5. Compactness

The following result can be seen as an analog of the theorem of Eberlein–Smulian in functional analysis. Unlike Theorem 1.2, here the CAT(0) geometry plays an important role several times:

Proposition 5.1. *Let (x_α) be a bounded net in a* CAT(0) *space X weakly converging to a point x. Then there exists a sequence $x_{\alpha_1}, x_{\alpha_2}, \dots$ of elements of the net weakly converging to x.*

Proof. Replacing the net by a subnet we may assume that the net $r_\alpha := d(x_\alpha, x)$ of real numbers converges to some $r \geq 0$. If $r = 0$, we find some x_{α_i} such that $\lim_{i\to\infty} r_{\alpha_i} = 0$. Thus, the sequence x_{α_i} converges to x in the metric topology, and, therefore, also weakly. Thus, we may assume $r > 0$ and, after rescaling, $r = 1$.

We choose inductively $\alpha_k \in I$, for $k = 1, 2, ...$, starting with an arbitrary α_1. Let the elements $\alpha_1 \leq ... \leq \alpha_k$ in I be already chosen.

For any non-empty subset $S \subset \{1, ..., k\}$, denote by m_S the unique circumcenter m_S of the finite set $\{x_{\alpha_i}, i \in S\}$. Since the net (x_α) converges weakly to x and (r_α) converges to 1, we find some $\alpha_{k+1} \geq \alpha_k$ with the two following properties, for any $\alpha \geq \alpha_{k+1}$:

1) $|r_\alpha - 1| \leq 2^{-k-1}$.

2) For all nonempty $S \subset \{1, ..., k\}$ the projection $Proj_c(x_\alpha)$ of x_α onto the geodesic $c = [xm_S]$ has distance at most 2^{-k-1} from x.

Note that any subsequence of the sequence (x_{α_i}) has also the properties (1) and (2). We claim that the so-defined sequence (x_{α_i}) converges to x weakly. The proof of the claim relies only on the strict convexity of the squared distance functions and is rather straightforward. For the convenience of the reader, we present the somewhat lengthy details.

Assuming the contrary and replacing the sequence by a subsequence we may assume that the sequence converges weakly to a point $z \neq x$. Set $\delta := d(z, x)$. Choosing yet another subsequence we may assume that $s_{\alpha_i} := d(x_{\alpha_i}, z)$ converge to some $s \geq 0$, for $i \to \infty$.

We set $\varepsilon \leq \frac{\delta^2}{10}$ and find some i_0 such that $(1 - 2^{-i_0-1})^2 > 1 - \varepsilon$ and such that, for all $i \geq i_0$,

$$|r_{\alpha_i}^2 - 1| < \varepsilon \ ; \ |s_{\alpha_i}^2 - s^2| < \varepsilon.$$

Using the weak convergence of (x_{α_i}) to z and CAT(0) comparison, we may assume in addition, that for all $i \geq i_0$

$$d^2(x_{\alpha_i}, x) - d^2(x_{\alpha_i}, z)^2 \geq \delta^2 - \varepsilon = 9\varepsilon.$$

For $j = 1, 2...$ we consider the point $p_j := x_{\alpha_{i_0+j}}$. By above, the circumradius t of the countable set $\{p_j\}$ satisfies

$$t^2 < 1 - \tfrac{1}{2}\delta^2 = 1 - 5\varepsilon\,.$$

Denote by $0 \leq t_k \leq t$ the circumradius of the set $\{p_1, ..., p_k\}$. We claim that there exists some positive $\rho > 0$, such that $t_{k+1}^2 - t_k^2 > \rho$ for all $k \geq 1$. Since the sequence (t_k) is bounded above by t, this would provide a contradiction and finish the proof.

In order to prove the claim, consider the circumcenter m_k of the subset $p_1, ..., p_k$. Thus, m_k is the point at which the 2-convex function,

$$f(y) := \max_{1 \leq i \leq k} d^2(y, p_i)$$

assumes its unique minimum t_k^2. By the 2-convexity, we deduce

$$f(m_{k+1}) \geq t_k^2 + d^2(m_k, m_{k+1}).$$

On the other hand, $f(m_{k+1}) \leq t_{k+1}^2$, hence

$$t_{k+1}^2 \geq t_k^2 + d^2(m_k, m_{k+1}).$$

By construction of the sequence x_{α_i}, we have

$$d^2(p_{k+1}, m_k) \geq (1 - 2^{-i_0-1})^2 > 1 - \varepsilon.$$

Thus, by the triangle inequality and the fact

$$d^2(p_{k+1}, m_{k+1}) \leq t_{k+1}^2 \leq t^2 \leq 1 - 5\varepsilon$$

we obtain some positive lower bound $\rho > 0$ on $d^2(m_k, m_{k+1})$. This finishes the proof of the claim and of the proposition. □

As a consequence, we derive:

Lemma 5.2. *If a bounded net (x_α) in X converges to the point x weakly then (x_α) converges to x with respect to the $\mathcal{T}_\Delta$-topology.*

Proof. Assume the contrary. Then, replacing the net by a subnet, we find a $\mathcal{T}_\Delta$-open neighborhood U of x which does not contain any x_α. Using Proposition 5.1 we find a sequence $x_{\alpha_1}, ..., x_{\alpha_k}, ...$ of elements of the net converging weakly to x. Then x is contained in the $\mathcal{T}_\Delta$-closed set $X \setminus U$ which contains all elements of the sequence. This contradicts the definition of $\mathcal{T}_\Delta$-closed sets. □

Since any bounded net has weakly convergent subnets, we infer:

Corollary 5.3. *Every bounded $\mathcal{T}_\Delta$-closed subset A of X is $\mathcal{T}_\Delta$-compact.*

Now we provide:

Proof of Theorem 1.4. Let $\mathcal{T}$ be a topology on X, such that a bounded net (x_α) weakly converges to x if and only if this net $\mathcal{T}$-converges to x. Since any net has at most one weak limit point and since the Hausdorff property can be recognized by the uniqueness of limit points of nets, we deduce that any bounded subset of X is Hausdorff with respect to $\mathcal{T}$.

Let A be a bounded subset of X. By definition, A is $\mathcal{T}$-closed if and only if it contains all weak limit points of any net (x_α) of elements in A. From Proposition 5.1, this happens if and only if A contains all weak limit points of any *sequence* of elements in A. Thus, if and only if A is $\mathcal{T}_\Delta$-closed. We infer, that $\mathcal{T}$ coincides with $\mathcal{T}_\Delta$ on bounded subsets.

Assume, on the other hand, that the weak topology $\mathcal{T}_\Delta$ is Hausdorff on any ball in X. We claim that a bounded net (x_α) converges weakly to x if and only if (x_α) converges to x with respect to $\mathcal{T}_\Delta$.

Due to Lemma 5.2, the only if conclusion always holds. On the other hand, assume that (x_α) converges to x with respect to $\mathcal{T}_\Delta$ but does not weakly converge to x. Replacing (x_α) by a subnet we may assume that (x_α) weakly converges to another point y. Due to Lemma 5.2, this implies that the net (x_α) converges to the point y with respect to the topology $\mathcal{T}_\Delta$. But this contradicts the assumption that $\mathcal{T}_\Delta$ is Hausdorff on the bounded ball which contains the net (x_α).

This proves the "if"-direction and finishes the proof of the theorem. □

Remark 5.4. Using the considerations above, it is not difficult to prove another form of Theorem 1.4. Namely, the topology $\mathcal{T}_\Delta$ is Hausdorff on any bounded subset of X (and thus weak convergence of bounded nets is equivalent to the $\mathcal{T}_\Delta$-convergence) if and only if the topology $\mathcal{T}_\Delta$ is *Frechet–Urysohn* on any bounded set. Recall, that a topology $\mathcal{T}$ is called Frechet–Urysohn, if the closure of any set A in this topology is the set of all $\mathcal{T}$-limit points in X of all *sequences* contained in A.

6. Coconvex topology

The coconvex topology $\mathcal{T}_{co}$ is coarser than $\mathcal{T}_\Delta$, Proposition 3.1. Thus, convergence of sequences (nets) with respect to $\mathcal{T}_\Delta$ implies convergence with respect to $\mathcal{T}_{co}$. This immediately implies that any bounded, $\mathcal{T}_{co}$-closed set is $\mathcal{T}_{co}$-compact and $\mathcal{T}_{co}$-sequentially compact.

Proposition 6.1. *The topologies $\mathcal{T}_{co}$ and $\mathcal{T}_\Delta$ coincide on all bounded subsets of a* CAT(0) *space X if and only if the topology $\mathcal{T}_{co}$ is sequential and sequentially Hausdorff on every closed ball $B_r(x)$ in X. This happens if $B_r(x)$ is $\mathcal{T}_{co}$-Hausdorff.*

Proof. We may replace X by a ball $B_r(x)$ and assume that X is bounded. The only if statement follows from Proposition 3.1.

On the other hand, assume that $\mathcal{T}_{co}$ is sequential and sequentially Hausdorff on the bounded CAT(0) space X. Due to Proposition 3.1, the identity map $Id\colon (X, \mathcal{T}_\Delta) \to (X, \mathcal{T}_{co})$ is continuous. In order to prove that the inverse $Id\colon (X, \mathcal{T}_{co}) \to (X, \mathcal{T}_\Delta)$ is continuous, consider a $\mathcal{T}_\Delta$-closed subset A and assume that A is not $\mathcal{T}_{co}$-closed. Since $\mathcal{T}_{co}$ is sequential, we find a sequence (x_n) in A which $\mathcal{T}_{co}$-converges to a point $x \in X \setminus A$. Using that X is $\mathcal{T}_\Delta$-sequentially compact, we may replace (x_n) by a subsequence and assume that (x_n) converges to some point y in X with respect to $\mathcal{T}_\Delta$. Then, by Proposition 3.1, the sequence converges to y also with respect to $\mathcal{T}_{co}$. The assumption that $\mathcal{T}_{co}$ is sequentially Hausdorff gives us $x = y$. Since A is $\mathcal{T}_\Delta$-closed, we deduce $x = y \in A$. This contradiction implies that $Id\colon (X, \mathcal{T}_{co}) \to (X, \mathcal{T}_\Delta)$ is continuous, hence $\mathcal{T}_\Delta = \mathcal{T}_{co}$.

Finally, if $\mathcal{T}_{co}$ is Hausdorff on the bounded CAT(0) space X, then the compactness of $\mathcal{T}_\Delta$ and the continuity of the identity map $Id\colon (X, \mathcal{T}_\Delta) \to (X, \mathcal{T}_{co})$ imply $\mathcal{T}_\Delta = \mathcal{T}_{co}$. □

In the proof of Proposition 1.5 below, we assume more knowledge of non-positive curvature than in the rest of this paper. We refer to [LN19] for properties of geodesically complete CAT(0) spaces, to [Sch19] for properties of cubical complexes, and to [And83] and [Bor92] for manifolds of pinched negative curvature.

Proof of Proposition 1.5. Assume first that X is homeomorphic to the plane $\mathbb{R}^2$. Then each geodesic $\gamma\colon [a, b] \to X$ extends to a geodesic $\hat{\gamma}\colon \mathbb{R} \to X$, [BH99]. Moreover, by Jordan's theorem, $\hat{\gamma}$ divides X into two connected components both having $\hat{\gamma}$ as their boundaries. The closures of the connected components are convex. Thus, the open components are $\mathcal{T}_{co}$ open.

In order to prove that $\mathcal{T}_{co}$ is Hausdorff it suffices to find, for any pair of points x, y, some geodesic $\gamma\colon \mathbb{R} \to X$, such that x and y are in different components of $X \setminus \gamma$. We connect x and y by a geodesic η and take the midpoint m of η. We find two points $p^\pm$ sufficiently close to m which lie in different components of $X \setminus \hat{\eta}$ for some extension of η to an infinite geodesic. Then consider a geodesic $\gamma\colon \mathbb{R} \to X$ which contains p^+ and p^-. The geodesic γ intersect η between x and y. We

infer that x and y lie in different components of $X \setminus \gamma$. This finishes the proof if X is homeomorphic to a plane.

Assume now that X is a finite-dimensional cubical complex and choose $x, y \in X$. Taking a sufficiently fine cubical subdivision, we may assume that the diameter of all cubes is much smaller than the distance between x and y. Then the geodesic between x and y intersects at least one *hyperwall* in X. Any such hyperwall is a convex subsets dividing X into two connected and convex components. As above, we deduce that x and y are separated in $\mathcal{T}_{co}$.

Finally, let X be a Riemannian manifold with pinched negative curvature and let $x, y \in X$ be arbitrary different points. Fix $r > d(x, y)$ and set $B = B_r(x)$. By Anderson's construction, [And83], see also [Bor92, Theorem 2.1], we find finitely many closed convex subsets C_i in $X \setminus B$, such that $V := X \setminus \cup_{i=1}^m C_i$ is bounded. Then V is a $\mathcal{T}_{co}$-open set containing B and contained in some larger closed ball B'.

On the compact ball B' the topology $\mathcal{T}_{co}$ coincides with the metric topology, [Mon06, Lemma 17], hence it is Hausdorff. Thus, we find $\mathcal{T}_{co}$-open subsets U_1 and U_2 in X containing x and y, respectively, such that $U_1 \cap U_2 \cap B'$ is empty. Then $U_1 \cap U$ and $U_2 \cap U$ are disjoint $\mathcal{T}_{co}$-neighborhoods of x and y in X. □

Remark 6.2. As the proof and the reference to [Bor92] shows, in condition (2) one can replace the pinching by the assumption that the quotient of the minimal and maximal curvature in the ball $B_r(x_0)$ around some chosen point x_0 is at most $2^{\lambda r}$ for some $\lambda \in \mathbb{R}$. Moreover, a closer look at the proof shows that under the assumptions (1) or (2), the topology $\mathcal{T}_{co}$ coincides with the metric topology on all of X.

We discuss finally two examples showing that the coconvex topology can be quite strange even for rather regular spaces. Below we denote for a locally compact CAT(0) space X by X^∞ its boundary at infinity with its cone topology, [BH99]. Recall that X^∞ is compact.

Lemma 6.3. *Let X be a locally compact* CAT(0) *space. Assume that X is not bounded and for any closed convex subset A of X different from X, the boundary at infinity A^∞ is nowhere dense in X^∞. Then the coconvex topology $\mathcal{T}_{co}$ on X is non-Hausdorff and not first-countable.*

Proof. Since X^∞ is a compact space, it is not a countable union of nowhere dense subsets, by Baire's theorem.

Therefore, by our assumption, X is not a finite union of closed convex subsets different from X. Thus, any finite intersection of non-empty $\mathcal{T}_{co}$-open subsets is non-empty. In particular, $\mathcal{T}_{co}$ is not Hausdorff.

Assume now that $\mathcal{T}_{co}$ is first-countable on X, fix an arbitrary $x \in X$ and a $\mathcal{T}_{co}$-fundamental system of its open neighborhoods $U_1, ..., U_n, ...$ By definition of $\mathcal{T}_{co}$, we may assume that each U_i is the complement of a finite union of closed convex subsets K_i^j, not containing the point x. Hence, the union of the boundaries at infinity

$$\cup_{i,j}(K_i^j)^\infty \subset X^\infty$$

is not all of X^∞. Consider an arbitrary point $z \in X^\infty$ not contained in this union and a ray γ in X with endpoint $z \in X^\infty$, such that x is not on γ. Then $X \setminus \gamma$ is a $\mathcal{T}_{co}$-open neighborhood of x which does not contain any of the set U_i. This contradiction shows that $\mathcal{T}_{co}$ is not first-countable. □

The first example directly follows from Lemma 6.3 above and [Anc94, Theorem B, Corollary C]:

Example 6.4. There exists a smooth 3-dimensional CAT(−1) Riemannian manifold X for which the coconvex topology $\mathcal{T}_{co}$ is not Hausdorff and not first-countable.

In the final example, we use some facts about geometry of spherical buildings arising as the boundary at infinity of symmetric space with their corresponding *Tits-metric*, see [KL06], [KL97], [KLP18]. The following result might be known to specialists, accordingly to Nicolas Monod it was known to Bruce Kleiner many years ago.

Proposition 6.5. *Let X be an irreducible, non-positively curved symmetric space of rank at least two. Then X satisfies the assumptions, and, therefore, the conclusions of Lemma 6.3.*

Proof. Assume the contrary and consider any closed convex subset A of X such that the boundary at infinity A^∞ of A has non-empty interior in the $(n-1)$-dimensional sphere X^∞; here n is the dimension of X.

Thus, in the cone topology, A^∞ has dimension $n-1$. Therefore, there are no totally geodesic symmetric spaces $Y \subsetneq X$ with $A^\infty \subset Y^\infty$. On the other hand, if $A^\infty = X^\infty$ then $A = X$. Thus, we may assume $A^\infty \neq X^\infty$. Applying [KL06, Theorem 3.1], we deduce that A^∞ is not a *sub-building* of the *spherical building* X^∞.

Since A^∞ contains an open subset in the cone topology, we find a non-empty subset O of A^∞, open in the cone topology and consisting of *regular* points only. If, for some $p \in O$, we find an antipode $q \in A^\infty$ (with respect to the *Tits-distance*) then A^∞ contains a spherical apartment (the boundary of a maximal flat in X), as the convex hull in the Tits-metric of q and a Tits-ball around p. By [BL06, Theorem 1.1], this would imply that A^∞ is a sub-building, in contradiction to the

statements above. Thus, for no $p \in O$ and $q \in A^\infty$ the Tits-distance between p and q equals π.

We are going to construct a pair of antipodes $p \in O$ and $q \in A^\infty$ and achieve a contradiction. We start with an arbitrary point $p \in O$.

Let G be the isometry group of X (and of X^∞) and denote by Δ the spherical Coxeter chamber X^∞/G of the spherical building X^∞. Let $\mathcal{P} : X^\infty \to \Delta$ be the canonical projection. Denote by $I : \Delta \to \Delta$ the isometry of the Coxeter chamber induced by the action of $-Id$ on any apartment of X^∞. The map I is an involution, which is the identity map if and only if the Coxeter group W of X^∞ has a non-trivial center (note, that this is the case for all Weyl groups, which are not of type A_m, E_6 or D_{2m+1}, see [Hum72, p.71]).

Consider the orbit $L := G \cdot p = \mathcal{P}^{-1}(\mathcal{P}(p)) \subset X^\infty$. Any element $p' \in L$ is contained in a unique Coxeter chamber $\Delta_{p'}$. Consider the set L_p^{op} of all elements p' in L which are in an *opposite* Coxeter to p, thus such that the Coxeter chamber $\Delta_{p'}$ through p' contains an antipode of p. Then L_p^{op} is open and dense in the manifold $G \cdot p$, see, for instance, [KLP18]. Thus, we find an element $p' \in O \cap L_p^{op}$.

If the isometry $I : \Delta \to \Delta$ is the identity (see the discussion above), then p' is an antipode of p and we are done. If I is not the identity, then looking at an apartment through p and p' we deduce that the Tits-geodesic between p and p' contains a point q which is projected by $\mathcal{P}$ onto $I(p)$. Then L contains all antipodes of q. By convexity, $q \in A^\infty$. As above, the set $L_q^{op} \cap O$ of elements in O contained in a chamber opposite to q is not empty. For any such element $p' \in L_q^{op} \cap O$, the distance between q and p' is π.

Thus, in both cases we have found a pair of antipodes $p \in O$ and $q \in A^\infty$, finishing the proof. □

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