



Congruence modules in higher codimension and zeta lines in Galois cohomology

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This article builds on recent work of the first three authors where a notion of congruence modules in higher codimension is introduced. The main results are a criterion for detecting regularity of local rings in terms of congruence modules, and a more refined version of a result tracking the change of congruence modules under deformation. Number theoretic applications include the construction of canonical lines in certain Galois cohomology groups arising from adjoint motives of Hilbert modular forms.

congruence module | Hecke module | modularity lifting | Wiles defect

1. Introduction

Let p be a prime number, $\mathcal{O}$ the ring of integers of a finite extension of $\mathbb{Q}_p$, and R a complete noetherian local $\mathcal{O}$ -algebra with an $\mathcal{O}$ -algebra morphism $\lambda: R \rightarrow \mathcal{O}$ such that the local ring $R_{\mathfrak{p}}$, where $\mathfrak{p} := \text{Ker}(\lambda)$, is regular. Let c denote the codimension of λ , namely, the height of $\mathfrak{p}$. We denote the category of such pairs (R, λ) by $C_{\mathcal{O}}(c)$.

In ref. 1, we develop an analog of the Wiles–Lenstra–Diamond numerical criterion in arbitrary codimension, with the original criterion (see refs. 2 and 3) corresponding to the codimension 0 case. This gives a criterion for a finitely generated R -module M to have a free direct summand and for R to be a complete intersection ring. This involves two invariants associated to λ : the torsion $\Phi_{\lambda}(R)$ of the cotangent space of λ , and the congruence module $\Psi_{\lambda}(M)$ of M . When $c = 0$ one has $\mathfrak{p}/\mathfrak{p}^2 = \Phi_{\lambda}(R)$.

Wiles uses the criterion in his work on the modularity of elliptic curves over $\mathbb{Q}$ to go from modularity lifting theorems in the minimal case to those in the nonminimal case. In ref. 1, the numerical criterion in higher codimension is used along the same lines to prove integral modularity lifting results for nonminimal lifts in situations of positive defect which arise in considering Galois representations over imaginary quadratic fields. In this work, we explore the meaning of the invariants $\Phi_{\lambda}(R)$ and $\Psi_{\lambda}(M)$ in certain number theoretic situations, relating them to the index of zeta elements in global Galois cohomology groups.

To begin with, we focus on one of the key ideas of ref. 1, namely the definition and properties of congruence modules, and congruence ideals, associated to an augmentation λ in arbitrary codimension. This is the content of Section 2. The highlights are a characterization, Theorem 2.6, of regularity of rings (A, λ) in our category $C_{\mathcal{O}}(c)$ in terms of vanishing of the invariants $\Phi_{\lambda}(A)$ and $\Psi_{\lambda}(A)$; a more transparent description, in Section 2.10, of the connection between duality and our congruence modules than in the earlier paper, and a refinement, Theorem 2.39, of a result about deformation invariance of Wiles defect.

Section 3 focuses on number theoretic applications, and concerns the congruence ideal attached to an augmentation of Hida's ordinary Hecke algebra $\mathbb{T}^{\text{ord}}$, or ordinary deformation ring R^{ord} , arising from cohomological Hilbert modular forms f over totally real fields F . The functorial properties of the congruence ideal are used to relate it to lines (that is to say, free $\mathcal{O}$ -modules of rank one) in the Galois cohomology with coefficients in $\text{ad } \rho_f$; see Theorem 3.9. The index of their image under global-to-local restriction maps to singular local Galois cohomology $H_{\text{ord}/f}^1(G_p, \text{ad } \rho_f)$ is related to the classical congruence module of f , and to special values of adjoint L -function $L^{\text{alg}}(1, \text{ad } \rho_f)$, by work of Hida (4) and its generalization by Dimitrov (5). This is connected to the “zeta elements” of (6, Theorem 1.1). The terminology is due to Kato (7), who used it for elements of Galois cohomology he constructs in a related context, arising from Beilinson–Flach classes.

In ref. 6, exact sequences of Selmer groups are used while the proof of Theorem 3.9 uses in addition congruence modules in higher codimension. This allows us to bypass local complete intersection results on the corresponding Hecke ring used in ref. 6, and

Significance

In 1994, Wiles deduced Fermat's Last theorem, a problem that had resisted solution for over 350 y, from his proof of a more structural and central conjecture in number theory called the modularity conjecture for elliptic curves defined over the rational numbers. A key ingredient of Wiles' proof of modularity is a numerical criterion whose verification depends on subtle methods to study congruences between modular forms, initiated by Hida and Ribet. Recent work of the first three authors takes forward Wiles' numerical criterion into more complicated contemporaneous situations, involving modularity of other geometric objects of relevance to modern number theory. Our results in this paper further develop these ideas and obtain applications, in particular to Galois cohomology.

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the argument does not require any $R = \mathbb{T}$ theorems. This connection arises from the following circumstance.

Let $\mathbf{t} := t_1, \dots, t_c$ be indeterminates, $\Lambda_c := \mathcal{O}[[\mathbf{t}]]$ the power-series ring, and $\Lambda_c \rightarrow \mathcal{O}$ the natural augmentation. Fix (A, λ) in $C_{\mathcal{O}}(c)$ equipped with a finite flat map $\iota: \Lambda_c \rightarrow A$ of $\mathcal{O}$ -algebras over $\mathcal{O}$, so that composite map $\Lambda_c \xrightarrow{\iota} A \xrightarrow{\lambda} \mathcal{O}$ is the augmentation. For $A_0 = A/(\mathbf{t})$ one has a map $\text{Ext}_A^c(\mathcal{O}, A) (= A_0[\ker \lambda]) \rightarrow \bigwedge_{\mathcal{O}}^c \text{Hom}(\mathfrak{p}/\mathfrak{p}^2, \mathcal{O})$, whose cokernel is $\Psi_{\lambda}(A)$. Abstractly both domain and range are simply $\mathcal{O}$'s. In applications when A is a nearly ordinary deformation ring and λ arises from the classical form f , the range is a Selmer group as $\text{Hom}(\mathfrak{p}/\mathfrak{p}^2, \mathcal{O}) = H_{\mathcal{L}}^1(G_{F,S}, \text{ad } \rho_f)$, with local conditions $\mathcal{L} = (\mathcal{L}_v)$ such that $\mathcal{L}_v \subset H^1(G_v, \text{ad } \rho_f)$ for $v \in S$ and $\mathcal{L}_v$ is the unramified subspace $H_{\text{unr}}^1(G_v, \text{ad } \rho_f)$ for v not in S . When A is a nearly ordinary Hecke algebra $\text{Hom}(\mathfrak{p}/\mathfrak{p}^2, \mathcal{O})$ is a subspace of $H^1(G_{F,S}, \text{ad } \rho_f)$. This gives a “pure thought” construction of canonical lines in Galois cohomology; see Theorem 3.9.

2. Higher Congruence Modules and Wiles Defects

We begin by recalling the setup of ref. 1. This section complements the material presented in (1, Part 1). The highlights are Theorems 2.6 and 2.39, and Proposition 2.25. Along the way, we provide a different perspective and proofs of some key results from ref. 1.

2.1. Let $\mathcal{O}$ be a complete discrete valuation ring, with valuation $\text{ord}(-)$ and uniformizer ϖ . Throughout we fix a complete local $\mathcal{O}$ -algebra A and a finitely generated A -module M . Given a map $\lambda: A \rightarrow \mathcal{O}$ of $\mathcal{O}$ -algebras, set

$$\mathfrak{p}_{\lambda} := \text{Ker } \lambda \quad \text{and} \quad c := \text{height } \mathfrak{p}_{\lambda}.$$

For any finitely generated A -module M , set $F_{\lambda}^i(M) := \text{Ext}_A^i(\mathcal{O}, M)^{\text{tf}}$, the torsion-free quotient of the $\mathcal{O}$ -module $\text{Ext}_A^i(\mathcal{O}, M)$. Here $\mathcal{O}$ is viewed as an A -module via λ . The congruence module of M at λ is the $\mathcal{O}$ -module

$$\Psi_{\lambda}(M) := \text{coker} \left(F_{\lambda}^c(M) \xrightarrow{F_{\lambda}^c(\lambda \otimes M)} F_{\lambda}^c(M/\mathfrak{p}_{\lambda}M) \right).$$

We have also to consider $\mathcal{O}$ -module $\Phi_{\lambda}(A) := \text{tors}(\mathfrak{p}_{\lambda}/\mathfrak{p}_{\lambda}^2)$, the torsion part of the cotangent module $\mathfrak{p}_{\lambda}/\mathfrak{p}_{\lambda}^2$ of λ .

We say an A -module M has a certain property at λ if the $A_{\mathfrak{p}_{\lambda}}$ -module $M_{\mathfrak{p}_{\lambda}}$ has the stated property. For instance, A is regular at λ means the local ring $A_{\mathfrak{p}_{\lambda}}$ is regular. The starting point is the result below; see (1, Theorem 2.5 and Lemma 2.6).

Theorem 2.2. *With $\lambda: A \rightarrow \mathcal{O}$ as above, the following conditions are equivalent:*

1. *The local ring A is regular at λ .*
2. *The rank of the $\mathcal{O}$ -module $\mathfrak{p}_{\lambda}/\mathfrak{p}_{\lambda}^2$ is $\text{height } \mathfrak{p}_{\lambda}$.*
3. *The $\mathcal{O}$ -module $\Psi_{\lambda}(A)$ is torsion.*
4. *The $\mathcal{O}$ -module $\Psi_{\lambda}(M)$ is torsion for each finitely generated A -module M .*

Moreover, when these conditions hold the $\mathcal{O}$ -module $\Psi_{\lambda}(A)$ is cyclic. □

Condition (2) is that the embedding dimension of the ring $A_{\mathfrak{p}_{\lambda}}$ equals its Krull dimension, so $(1) \Leftrightarrow (2)$ is one definition of regularity; see (8, Definition 2.2.1). The key input in proving $(1) \Leftrightarrow (3)$ is the following result due to Lescot (9); see also ref. 10.

2.3. A noetherian local ring R is regular if and only if the map $\text{Ext}_R(k, R) \rightarrow \text{Ext}_R(k, k)$ induced by the canonical surjection $R \rightarrow k$ to the residue field of R , is nonzero. When this is the case, the map above is nonzero in (upper) degree $\dim R$.

The result below is implicit in the proof of $(1) \Rightarrow (4)$ in Theorem 2.2, in ref. 1. We make it explicit, for it is used also in proving Lemma 2.5 and Theorem 2.6 below; the proof is standard.

Lemma 2.4. *Let $\varepsilon: R \rightarrow S$ be a surjective map of noetherian rings where the ideal $\text{Ker}(\varepsilon)$ is generated by a regular sequence, and set $c := \dim R - \dim S$. For any R -module M the map $\text{Ext}_R^c(S, \varepsilon \otimes_R M): \text{Ext}_R^c(S, M) \rightarrow \text{Ext}_R^c(S, S \otimes_R M)$ is bijective.* □

We denote $C_{\mathcal{O}}$ the category whose objects are pairs (A, λ) satisfying the equivalent conditions in Theorem 2.2. A morphism $\varphi: (A, \lambda) \rightarrow (A', \lambda')$ in this category is a map of $\mathcal{O}$ -algebras $\varphi: A \rightarrow A'$ over $\mathcal{O}$; that is to say, with $\lambda' \circ \varphi = \lambda$. We write $C_{\mathcal{O}}(c)$ for the subcategory of $C_{\mathcal{O}}$ consists of pairs (A, λ) such that $\text{height } \mathfrak{p}_{\lambda} = c$.

Lemma 2.5. *For (A, λ) in $C_{\mathcal{O}}(c)$ and finitely generated A -module M the map $F_{\lambda}^c(\lambda \otimes_A M)$ is one-to-one.*

Proof: Set $R := A_{\mathfrak{p}_{\lambda}}$ and let $\varepsilon: R \rightarrow E$ be the map obtained by localizing λ at $\mathfrak{p}_{\lambda}$; here E is the residue field of R , which is also the field of fractions of $\mathcal{O}$. Since injectivity of a map of torsion-free $\mathcal{O}$ -modules can be detected after passing to the field of fractions, it suffices to check that the map $F_{\lambda}^c(\lambda \otimes_A M)_{\mathfrak{p}_{\lambda}} \cong \text{Ext}_R^c(E, \varepsilon \otimes_R M)$ is one-to-one. Since R is regular the ideal $\text{Ker}(\varepsilon)$ is generated by a regular sequence of length c ; see (8, Proposition 2.2.4). It remains to apply Lemma 2.4. □

In the work of Hida (4) and Ribet (11) congruence modules (for codimension $c = 0$) are attached to augmentations $\mathbb{T} \rightarrow \mathcal{O}$ of Hecke algebras $\mathbb{T}$ that act faithfully on certain localized Betti cohomology groups $H^1(X_1(N), \mathcal{O})_{\mathfrak{m}}$. They measure the complexity of $\mathbb{T}$ and their vanishing is equivalent to $\mathbb{T}$ being smooth, namely just $\mathcal{O}$. Analogously we show in the result below that for rings $A \in C_{\mathcal{O}}$ the vanishing of either the congruence module $\Psi_{\lambda}(A)$ or $\Phi_{\lambda}(A)$, the torsion part of the cotangent module, at any augmentation $\lambda: A \rightarrow \mathcal{O}$ implies A is smooth.

Unlike most results in (1, Part 1), the following theorem does not make assumptions on the depth of the ring A .

Theorem 2.6. For (A, λ) in $C_{\mathcal{O}}$, the local ring A is regular if and only if $\Phi_{\lambda}(A) = 0$, if and only if $\Psi_{\lambda}(A) = 0$.

Proof: We first verify that A is regular if and only if $\Phi_{\lambda}(A) = 0$. As A is a complete $\mathcal{O}$ -algebra, one has $A \cong P/I$ where $P := \mathcal{O}[[t_1, \dots, t_n]]$, a ring of formal power series over $\mathcal{O}$, the ideal $I \subseteq (\varpi)(\mathfrak{t}) + (\mathfrak{t})^2$, and $\lambda: A \rightarrow \mathcal{O}$ is quotient by $(\mathfrak{t})$.

Let $\mathbf{f} := f_1, \dots, f_m$ be a minimal generating set for I . The cotangent module $\mathfrak{p}_{\lambda}/\mathfrak{p}_{\lambda}^2$ depends only on n and the linear part of the f_i , in the following sense: By our assumption on I , each f_i has a unique expression of the form

$$f_i := \sum_{j=1}^n u_{ij} t_j + g_i \quad \text{with } u_{ij} \in (\varpi)\mathcal{O} \text{ and } g_i \in (\mathfrak{t})^2. \quad [2.7]$$

Then one has a presentation $\mathcal{O}^m \xrightarrow{(u_{ij})} \mathcal{O}^n \rightarrow \mathfrak{p}_{\lambda}/\mathfrak{p}_{\lambda}^2 \rightarrow 0$. The torsion part of $\mathfrak{p}_{\lambda}/\mathfrak{p}_{\lambda}^2$ is zero if and only if $(u_{ij}) = 0$, that is to say, $(\mathbf{f}) \subseteq (\mathfrak{t})^2$. Since A is regular at λ this condition is equivalent to $\mathbf{f} = 0$, as desired.

Next, we verify the claim that A is regular if and only if $\Psi_{\lambda}(A) = 0$.

When A is regular, $\text{Ker}(\lambda: A \rightarrow \mathcal{O})$ is generated by a regular sequence of length $c := \text{height}(\text{Ker } \lambda)$; see (8, Proposition 2.2.4). Thus Lemma 2.4 yields that the map $\text{Ext}_A^c(\mathcal{O}, \lambda)$ is one-to-one so $\Psi_{\lambda}(A) = 0$.

Assume $\Psi_{\lambda}(A) = 0$. To verify that A is regular it suffices to verify that the map

$$\text{Ext}_A(k, \varepsilon): \text{Ext}_A(k, A) \rightarrow \text{Ext}_A(k, k),$$

induced by the quotient map $\varepsilon: A \rightarrow k$, is nonzero, for then Lescot's result 2.3 gives the desired conclusion.

Let M be a finitely generated A -module. The exact sequence

$$0 \rightarrow \mathcal{O} \xrightarrow{\varpi} \mathcal{O} \rightarrow k \rightarrow 0 \quad [2.8]$$

of A -modules induces exact sequences of k -modules $0 \rightarrow k \otimes_{\mathcal{O}} \text{Ext}_A^i(\mathcal{O}, M) \xrightarrow{\bar{\partial}^{i+1}(M)} \text{Ext}_A^i(k, M) \rightarrow \text{Ext}_A^{i+1}(\mathcal{O}, M)[\varpi] \rightarrow 0$. For what follows the relevant point is that the maps $\bar{\partial}^i(M)$ are inclusions. Set $c := \text{height}(\mathfrak{p}_{\lambda})$ and consider the following commutative diagram of k -vector spaces:

$$\begin{array}{ccccc} k \otimes_{\mathcal{O}} \text{Ext}_A^c(\mathcal{O}, A) & \longrightarrow & k \otimes_{\mathcal{O}} \text{Ext}_A^c(\mathcal{O}, \mathcal{O}) & & \\ \bar{\partial}(A) \downarrow & \searrow & \downarrow \bar{\partial}(\mathcal{O}) & & \\ \text{Ext}_A^{c+1}(k, A) & \longrightarrow & \text{Ext}_A^{c+1}(k, \mathcal{O}) & \hookrightarrow & \text{Ext}_A^{c+1}(k, k) \\ & \searrow & \text{Ext}_A^{c+1}(k, \varepsilon) \nearrow & & \end{array}$$

The map in the top row is induced by $\lambda: A \rightarrow \mathcal{O}$ and the ones in the lower row are induced by $A \rightarrow \mathcal{O} \rightarrow k$. That the map in the lower right is one-to-one follows by considering the long exact sequence in cohomology that arises by applying $\text{Hom}_A(k, -)$ to the exact sequence Eq. 2.8. It is easy to verify that the hypothesis $\Psi_{\lambda}(A) = 0$ implies the map in the top row is nonzero, and hence so is the diagonal map. It then follows from the commutative diagram that the map $\text{Ext}_A^{c+1}(k, \varepsilon)$ is nonzero. $\square$

Remark 2.9: Consider the ordinary Hida Hecke algebra $\mathbb{T}^{\text{ord}}$ of tame level N . It is finite flat over $\Lambda := \mathcal{O}[[t]]$, with t the weight variable, and $\mathbb{T}^{\text{ord}}/(t) = \mathbb{T}$ is a classical Hecke algebra, acting faithfully on $H^1(X_1(Np), \mathcal{O})_{\mathfrak{m}}$. Consider an augmentation $\lambda: \mathbb{T}^{\text{ord}} \rightarrow \mathbb{T} \rightarrow \mathcal{O}$ arising from a $(p$ -stabilized) newform $f \in S_1(\Gamma_1(Np^r))$. The vanishing of the congruence module of $\mathbb{T}$ for the augmentation $\mathbb{T} \rightarrow \mathcal{O}$ implies $\mathbb{T} = \mathcal{O}$ and $\mathbb{T}^{\text{ord}} = \Lambda$ while the vanishing of the congruence module for $\mathbb{T}^{\text{ord}} \rightarrow \mathcal{O}$ implies that $\mathbb{T}^{\text{ord}}$ is smooth, while $\mathbb{T}$ may not be smooth. In other words $\Psi_{\lambda}(\mathbb{T}^{\text{ord}}) = 0$ implies $\mathbb{T}^{\text{ord}} = \mathcal{O}[[x]]$, furthermore x can be taken to be the weight variable t if and only if the classical congruence module $\Psi_{\lambda}(\mathbb{T}) = 0$ also vanishes.

Next, we describe a pairing associated with the definition of congruence modules. This too appears in ref. 1, but does not play a major role in the development there. The presentation below is more transparent.

2.10. For any finitely generated A -module M , one has natural isomorphisms

$$\text{Ext}_A^c(\mathcal{O}, M/\mathfrak{p}_{\lambda}M)^{\text{tf}} \cong \text{Ext}_A^c(\mathcal{O}, \mathcal{O})^{\text{tf}} \otimes_{\mathcal{O}} (M/\mathfrak{p}_{\lambda}M)^{\text{tf}} \cong \text{Hom}_{\mathcal{O}}(\text{Hom}_{\mathcal{O}}(M, \mathcal{O}), \text{Ext}_A^c(\mathcal{O}, \mathcal{O})^{\text{tf}}).$$

Thus, the map $F_{\lambda}^c(\lambda \otimes M): \text{Ext}_A^c(\mathcal{O}, M) \rightarrow \text{Ext}_A^c(\mathcal{O}, M/\mathfrak{p}_{\lambda}M)^{\text{tf}}$, whose cokernel is the congruence module of M , is adjoint to

$$\langle -, - \rangle_M: \text{Ext}_A^c(\mathcal{O}, M)^{\text{tf}} \otimes_{\mathcal{O}} \text{Hom}_A(M, \mathcal{O}) \longrightarrow \text{Ext}_A^c(\mathcal{O}, \mathcal{O})^{\text{tf}}.$$

The congruence ideal of M , with respect to the augmentation λ , is the image of this pairing: $\eta_{\lambda}(M) := \text{Image}(\langle -, - \rangle_M)$. It is easy to check that the free $\mathcal{O}$ -modules $F_{\lambda}^c(M)$ and $\text{Hom}_A(M, \mathcal{O})$ have the same rank and that $F_{\lambda}^c(\mathcal{O})$ has rank one, so

$$\text{length}_{\mathcal{O}}(\mathcal{O}/\eta_{\lambda}(M)) \leq \text{length}_{\mathcal{O}} \Psi_{\lambda}(M) \leq \text{rank}_{\lambda}(M) \cdot \text{length}_{\mathcal{O}}(\mathcal{O}/\eta_{\lambda}(M)).$$

Here $\text{rank}_{\lambda}(M)$ denotes the rank of M at λ , that is to say, the rank of the A_{λ} -module $M_{\mathfrak{p}_{\lambda}}$. In particular, when this rank equals 1, the length of the congruence module can be computed from the pairing.

The pairing above is induced—by passage to torsion-free quotients—by the natural pairing given by composition of morphisms:

$$\langle -, - \rangle_M: \text{Ext}_A^c(\mathcal{O}, M) \otimes_{\mathcal{O}} \text{Hom}_A(M, \mathcal{O}) \longrightarrow \text{Ext}_A^c(\mathcal{O}, \mathcal{O}). \quad [2.11]$$

Namely, $\text{Ext}_A^c(\mathcal{O}, M)$ can be realized as $\text{Hom}_{\mathcal{D}}(\mathcal{O}, M[c])$, the morphisms in the derived category of A from $\mathcal{O}$ to $M[c]$, and given such a morphism f and a map $g: M \rightarrow \mathcal{O}$, the pairing above is

$$\langle f, g \rangle := g \circ f: \mathcal{O} \rightarrow \mathcal{O}[c].$$

In the Yoneda interpretation of $\text{Ext}_A^c(\mathcal{O}, M)$ as equivalence classes of exact sequences $0 \rightarrow M \rightarrow X_{c-1} \rightarrow \cdots \rightarrow X_0 \rightarrow \mathcal{O} \rightarrow 0$, the pairing is given by taking push-out along $g: M \rightarrow \mathcal{O}$.

2.11.1. Cohen–Macaulay modules. When M is Cohen–Macaulay of dimension $c + 1$, local duality yields an identification

$$\text{Hom}_A(\mathcal{O}, M) \cong \text{Ext}_A^c(\mathcal{O}, M^\vee) \quad \text{where } M^\vee \cong H_{c+1}(\text{RHom}_A(M, \omega_A)).$$

Here ω_A is the dualizing complex of A , normalized as in ref. 12; see ref. 1, §4 for details. With this identification, the pairing Eq. 2.11 takes the form

$$\langle -, - \rangle: \text{Ext}_A^c(\mathcal{O}, M) \otimes_{\mathcal{O}} \text{Ext}_A^c(\mathcal{O}, M^\vee) \longrightarrow \text{Ext}_A^c(\mathcal{O}, \mathcal{O}). \quad [2.12]$$

See ref. 1, Proposition 4.7. The pairing Eq. 2.11 can be described concretely when $c \leq 1$. One simplification that occurs then is that $\text{Ext}_A^1(\mathcal{O}, \mathcal{O})$ is already torsion-free, as is explained below. To that end, consider the exact sequence

$$0 \longrightarrow \mathfrak{p}_\lambda \longrightarrow A \longrightarrow \mathcal{O} \longrightarrow 0. \quad [2.13]$$

Applying $\text{Hom}_A(-, M)$ yields the exact sequence $M \cong \text{Hom}_A(A, M) \rightarrow \text{Hom}_A(\mathfrak{p}_\lambda, M) \rightarrow \text{Ext}_A^1(\mathcal{O}, M) \rightarrow \text{Ext}_A^1(A, M) = 0$. This justifies the following result.

Lemma 2.14. *For any $c \geq 0$, and any A -module M , there is a natural isomorphism of $\mathcal{O}$ -modules*

$$\text{Ext}_A^1(\mathcal{O}, M) \cong \text{coker}(M \rightarrow \text{Hom}(\mathfrak{p}_\lambda, M)). \quad \square$$

The isomorphism assigns an A -linear map $f: \mathfrak{p}_\lambda \rightarrow M$ to the exact sequence obtained by push-out of the exact sequence in Eq. 2.13 along f . The natural map $\mathcal{O} \rightarrow \text{Hom}_A(\mathfrak{p}_\lambda, \mathcal{O})$ is zero, so for $M := \mathcal{O}$ the isomorphism in Lemma 2.14 becomes

$$\text{Ext}_A^1(\mathcal{O}, \mathcal{O}) \cong \text{Hom}_A(\mathfrak{p}_\lambda, \mathcal{O}) \cong \text{Hom}_{\mathcal{O}}(\mathfrak{p}_\lambda/\mathfrak{p}_\lambda^2, \mathcal{O}) \quad [2.15]$$

which is already torsion-free. Now we return to the pairing Eq. 2.11.

2.16. The Case $c = 0$. Since $\text{Hom}_A(\mathcal{O}, M) = M[\mathfrak{p}_\lambda]$, the $\mathfrak{p}_\lambda$ -torsion submodule of M , the pairing Eq. 2.11 becomes

$$M[\mathfrak{p}_\lambda] \otimes_{\mathcal{O}} \text{Hom}_A(M, \mathcal{O}) \longrightarrow \mathcal{O} \quad \text{where } m \otimes f \mapsto f(m).$$

When $\text{depth}_A M \geq 1$, one has $M^\vee \cong \text{Hom}_{\mathcal{O}}(M, \mathcal{O})$ this pairing is equivalent to the one given by the composition

$$M[\mathfrak{p}_\lambda] \otimes_{\mathcal{O}} M^\vee[\mathfrak{p}_\lambda] \longrightarrow M \otimes_{\mathcal{O}} M^\vee \longrightarrow \mathcal{O},$$

where the map on the right is the obvious one.

2.17. The Case $c = 1$. With this description, for $c = 1$ the pairing Eq. 2.11 is induced by the obvious pairing given by composition:

$$\text{Hom}_A(\mathfrak{p}, M) \otimes_A \text{Hom}_A(M, \mathcal{O}) \longrightarrow \text{Hom}_A(\mathfrak{p}, \mathcal{O}) \cong \text{Hom}_{\mathcal{O}}(\mathfrak{p}/\mathfrak{p}^2, \mathcal{O}).$$

Since $\text{Ext}_A^1(\mathcal{O}, \mathcal{O})$ is torsion-free, as in the case $c = 0$ the ideal $\eta_\lambda(M)$ is just the image of the pairing above.

2.18. Structure of $F_A^*(\mathcal{O})$. A key input in the development of the commutative algebraic properties of the congruence module is a structure theorem for $F_A^*(\mathcal{O})$. The Yoneda product gives $\text{Ext}_A^*(\mathcal{O}, \mathcal{O})$ the structure of a graded $\mathcal{O}$ -algebra, and this is inherited by its torsion-free quotient, $F_A^*(\mathcal{O})$. The remarkable fact (1, Theorem 6.8) is that although the Ext-algebra itself can be highly noncommutative, and infinite, $F_A^*(\mathcal{O})$ is just an exterior algebra generated by its degree one component $F_A^1(\mathcal{O}) \cong \text{Hom}_{\mathcal{O}}(\mathfrak{p}_\lambda/\mathfrak{p}_\lambda^2, \mathcal{O})$; see Eq. 2.15. As explained in ref. 1, Introduction, this may be seen as an integral version of a result, due to Serre, on the structure of the Ext algebra of a regular local ring. The proof of this structure theorem for $F_A^*(\mathcal{O})$ uses ideas from the theory of differential graded algebras. An important takeaway is that there is a natural isomorphism of $\mathcal{O}$ -modules

$$\bigwedge^c \text{Hom}_{\mathcal{O}}(\mathfrak{p}_\lambda/\mathfrak{p}_\lambda^2, \mathcal{O}) \xrightarrow{\cong} F_A^c(\mathcal{O}). \quad [2.19]$$

The naturality of this isomorphism leads to an invariance of domain property for congruence modules; see ref. 1, Theorem 7.4.

Theorem 2.20. *Given a surjective map $\varphi: (A, \lambda) \rightarrow (A', \lambda')$ in $C_{\mathcal{O}}(c)$, and an A' -module M' with $\text{depth}_{A'} M' \geq c$, there is a natural isomorphism of $\mathcal{O}$ -modules $\Psi_{\lambda'}(M') \cong \Psi_\lambda(M')$.* $\square$

Since φ is surjective, $\mathfrak{p}_\lambda \cdot A' = \text{Ker } \lambda'$. The hypotheses in the statement above imply that $\varphi_{\mathfrak{p}_\lambda}: A_{\mathfrak{p}_\lambda} \rightarrow A'_{\mathfrak{p}_\lambda}$ is surjective map of regular local rings of dimension c and hence an isomorphism.

2.21. Freeness Criterion. Fix (A, λ) in $C_{\mathcal{O}}$ and a finitely generated A -module M . For any A -module X one has a map

$$\mathrm{Ext}_A^c(\mathcal{O}, X) \otimes_{\mathcal{O}} (M/\mathfrak{p}_{\lambda}M) \cong \mathrm{Ext}_A^c(\mathcal{O}, X) \otimes_A M \longrightarrow \mathrm{Ext}_A^c(\mathcal{O}, X \otimes_A M),$$

where the one on the right is a Künneth map. This is functorial in X , and one gets the commutative diagram below:

$$\begin{array}{ccc} \mathrm{Ext}_A^c(\mathcal{O}, A)^{\mathrm{tf}} \otimes_{\mathcal{O}} (M/\mathfrak{p}_{\lambda}M)^{\mathrm{tf}} & \longrightarrow & \mathrm{Ext}_A^c(\mathcal{O}, \mathcal{O})^{\mathrm{tf}} \otimes_{\mathcal{O}} (M/\mathfrak{p}_{\lambda}M)^{\mathrm{tf}} \\ \downarrow & & \downarrow \\ \mathrm{Ext}_A^c(\mathcal{O}, M)^{\mathrm{tf}} & \longrightarrow & \mathrm{Ext}_A^c(\mathcal{O}, M/\mathfrak{p}_{\lambda}M)^{\mathrm{tf}} \end{array}$$

The horizontal maps are one-to-one, by Lemma 2.5. Moreover, the one on the right is an isomorphism, as can be verified easily. It follows that the map on the left is one-to-one. This justifies the following result.

Lemma 2.22. *The diagram above induces a natural surjective map of $\mathcal{O}$ -modules $a_{\lambda}(M): \Psi_{\lambda}(A)^{\mu} \rightarrow \Psi_{\lambda}(M)$, where $\mu := \mathrm{rank}_{\lambda}(M)$. In particular there is an equality $\mathrm{length}_{\mathcal{O}} \Psi_{\lambda}(M) = \mu \cdot \mathrm{length}_{\mathcal{O}} \Psi_{\lambda}(A) - \mathrm{length}_{\mathcal{O}} \mathrm{Ker}(a_{\lambda}(M))$.*

When A is Gorenstein and M is maximal Cohen–Macaulay, $\mathrm{Ker}(a_{\lambda}(M))$ can be interpreted as a “stable” cohomology module of the pair (A, M) . This identification leads to the criterion below for detecting free summands of M ; see ref. 1, Theorem 9.2.

Theorem 2.23. *When A Gorenstein and M is maximal Cohen–Macaulay, $\mathrm{length}_{\mathcal{O}} \Psi_{\lambda}(M) = \mu \cdot \mathrm{length}_{\mathcal{O}} \Psi_{\lambda}(A)$ if and only if $M \cong A^{\mu} \oplus W$ and $W_{\mathfrak{p}_{\lambda}} = 0$, as A -modules. In this case, when $\mu \neq 0$ the A -module M is faithful. $\square$*

2.24. Isomorphism Criteria. The preceding results leads to criteria for detecting isomorphisms between rings.

Proposition 2.25. *Let $\varphi: A \rightarrow B$ be a surjective map of complete local $\mathcal{O}$ -algebra. Assume there exists an augmentation $\lambda: B \rightarrow \mathcal{O}$ such that $(A, \lambda\varphi)$ and (B, λ) are in $C_{\mathcal{O}}(c)$ for some $c \geq 0$, and either of the following conditions hold:*

1. *The ring A is Gorenstein, B is Cohen–Macaulay, and $\mathrm{length}_{\mathcal{O}} \Psi_{\lambda\varphi}(A) = \mathrm{length}_{\mathcal{O}} \Psi_{\lambda}(B)$;*
2. *The ring B is complete intersection and $\mathrm{length}_{\mathcal{O}} \Phi_{\lambda\varphi}(A) = \mathrm{length}_{\mathcal{O}} \Phi_{\lambda}(B)$.*

Then the map φ is an isomorphism.

Proof: (1) The hypotheses imply that φ is an isomorphism at $\lambda\varphi$ so $\mathrm{rank}_{\lambda\varphi} B = 1$. Thus Theorem 2.23 implies that B is a faithful A -module, so $\mathrm{Ker} \varphi = (0)$. As to (2), it follows by a simple argument using the Jacobi–Zariski sequence arising from maps $A \rightarrow B \rightarrow \mathcal{O}$ and Nakayama’s Lemma; see ref. 1, Lemma 5.10 for details. $\square$

The isomorphism Eq. 2.19 is also a critical input in tracking the behavior of congruence modules under deformations.

2.26. Deformations. Fix (A, λ) in $C_{\mathcal{O}}(c)$ and elements $\mathbf{f} := f_1, \dots, f_n$ in $\mathfrak{p}_{\lambda}$ such that their residue classes in the $\mathcal{O}$ -module $\mathfrak{p}_{\lambda}/\mathfrak{p}_{\lambda}^2$ form a linearly independent set. Set $\bar{A} := A/\mathbf{f}A$. The augmentation $\lambda: A \rightarrow \mathcal{O}$ factors through $\bar{A}$ so we an augmentation $\bar{\lambda}: \bar{A} \rightarrow \mathcal{O}$. The hypotheses on $\mathbf{f}$ is equivalent to saying that the pair $(\bar{A}, \bar{\lambda})$ is in $C_{\mathcal{O}}(c - n)$; see ref. 1, §8. A straightforward computation yields an equality

$$\mathrm{length}_{\mathcal{O}} \Phi_{\bar{\lambda}}(\bar{A}) = \mathrm{length}_{\mathcal{O}} \Phi_{\lambda}(A) + \sum_i \mathrm{ord}(f_i), \quad [2.27]$$

where $\mathrm{ord}(f_i)$ is the order of f_i in $\mathfrak{p}_{\lambda}/\mathfrak{p}_{\lambda}^2$, defined by $(\varpi^{\mathrm{ord}(f_i)})\mathcal{O} = \{\alpha(f_i) | \alpha \in \mathrm{Hom}_{\mathcal{O}}(\mathfrak{p}_{\lambda}/\mathfrak{p}_{\lambda}^2, \mathcal{O})\}$; see ref. 1, §8.5.

Theorem 2.28. *In the context above, let M be a finitely generated A -module with $\mathrm{depth}_A M \geq c + 1$ and set $\bar{M} := M/\mathbf{f}M$. If $\mathbf{f}$ is M -regular, then $\mathrm{length}_{\mathcal{O}} \Psi_{\bar{\lambda}}(\bar{M}) = \mathrm{length}_{\mathcal{O}} \Psi_{\lambda}(M) + (\mathrm{rank}_{\lambda} M) \sum_i \mathrm{ord}(f_i)$.*

Sketch of proof: It is enough to consider the case when $n = 1$. One first reduces to the case when f is not a zerodivisor on A as well; this uses the invariance of domain property for congruence modules, Theorem 2.20. See ref. 1, Proof of Theorem 8.2 for details. The essence of the argument is captured $c = 1$, so we sketch the proof in that context. Since f is in $\mathfrak{p}_{\lambda}$, and it is not a zerodivisor on A nor on M one gets the isomorphism on the right:

$$\mathrm{coker}(M \rightarrow \mathrm{Hom}(\mathfrak{p}_{\lambda}, M)) \xrightarrow{\cong} \mathrm{Ext}_A^1(\mathcal{O}, M) \xrightarrow{\cong} \mathrm{Hom}_{A_0}(\mathcal{O}, M_0).$$

The one on the left is from Lemma 2.14. It is straightforward to check that the composite map is induced by the assignment $\alpha \mapsto -\alpha(f) \bmod \mathbf{f}M$ for $\alpha \in \mathrm{Hom}_A(\mathfrak{p}_{\lambda}, M)$. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_A(\mathfrak{p}_{\lambda}, M) \otimes_A \mathrm{Hom}_A(M, \mathcal{O}) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}}(\mathfrak{p}_{\lambda}/\mathfrak{p}_{\lambda}^2, \mathcal{O}) \\ \alpha \mapsto \alpha(f) \downarrow \cong & & \downarrow \cong \\ \bar{M}[\mathfrak{p}_{\lambda}] \otimes_A \mathrm{Hom}_{\bar{A}}(\bar{M}, \mathcal{O}) & \longrightarrow & \mathcal{O} \end{array}$$

The image of the vertical map on the right is precisely the order ideal of f , that is to say, $(\varpi^{\mathrm{ord}(f)})$. Since $\Psi_{\lambda}(M)$ and $\Psi_{\bar{\lambda}}(\bar{M})$ are cokernel of the maps adjoint to the top and bottom row, respectively, the desired equality follows.

The case $c \geq 2$ is tackled by a reduction to the $c = 1$ case, using (8, Lemma 1.2.4), and the isomorphism Eq. 2.19. $\square$

2.29. Wiles Defect. Fix a pair (A, λ) in $C_{\mathcal{O}}$ and a finitely generated A -module M . Since A is regular at $\mathfrak{p}_\lambda$, and in particular a domain, the $A_{\mathfrak{p}_\lambda}$ -module $M_{\mathfrak{p}_\lambda}$ has a rank. The Wiles defect of M at λ is the integer

$$\delta_\lambda(M) := \text{rank}_\lambda(M) \cdot \text{length}_{\mathcal{O}} \Phi_\lambda(A) - \text{length}_{\mathcal{O}} \Psi_\lambda(M).$$

In particular the Wiles defect of A at λ is $\text{length}_{\mathcal{O}} \Phi_\lambda(A) - \text{length}_{\mathcal{O}} \Psi_\lambda(A)$. We refer to ref. 1, Introduction for a discussion on precedents to this definition. Theorem 2.28 and Eq. 2.27 give the following result, which is (1, Theorem 8.2):

Theorem 2.30. *One has $\delta_\lambda(M/fM) = \delta_\lambda(M)$ for M, f as in Theorem 2.28.* □

Also, Theorem 2.20 implies the following (which is ref. 1, Theorem 7.4):

Lemma 2.31. *If $\varphi: (A', \lambda') \rightarrow (A, \lambda)$ is a surjective map in $C_{\mathcal{O}}(c)$, then $\delta_{\lambda'}(M) \geq \delta_\lambda(M)$, with equality if and only if $\Phi_{\lambda'}(A') \cong \Phi_\lambda(A)$ holds.* □

With $a_\lambda(M)$ as in Eq. 2.32, one gets a “defect formula”:

$$\delta_\lambda(M) = \text{rank}_\lambda(M) \cdot \delta_\lambda(A) + \text{length}_{\mathcal{O}} \text{Ker}(a_\lambda(M)). \tag{2.32}$$

In particular $\delta_\lambda(M) \geq 0$ for all M if and only if $\delta_\lambda(A) \geq 0$.
The result below extends to modules; see ref. 1, Theorem 9.6.

Theorem 2.33. *For $(A, \lambda) \in C_{\mathcal{O}}(c)$ with $\text{depth } A > c$ one has $\delta_\lambda(A) \geq 0$; equality holds if and only if A is complete intersection.*

In ref. 1 this result was proved by reduction to the case $c = 0$, using Theorem 2.28. Here is an alternative argument, under the slightly more restrictive case where A is Cohen–Macaulay (so $\dim A = c + 1$), that argues by “going up” to a regular ring.

Proof: First, we verify that $\delta_\lambda(A) = 0$ when $A \in C_{\mathcal{O}}(c)$ is complete intersection, that is to say, isomorphic to

$$\mathcal{O}[[t_1, \dots, t_n]]/(f_1, \dots, f_m)$$

for some regular sequence $f := f_1, \dots, f_m$ in (t) . Since A is in $C_{\mathcal{O}}(c)$ it follows that $n - m = c$ and that f satisfies the hypothesis of Theorem 2.28, so we get the first equality in: $\delta_\lambda(A) = \delta_{\lambda\epsilon}(\mathcal{O}[[t]]) = 0$. The second one is by Theorem 2.6.

Next, we verify that when A is Cohen–Macaulay $\delta_\lambda(A) \geq 0$, and that if equality holds A is complete intersection. Since $\dim A = c + 1$ one can find a surjection $\epsilon: C \rightarrow A$ where C is a complete intersection in $C_{\mathcal{O}}(c)$ and ϵ induces an isomorphism $\Phi_{\lambda\epsilon}(C) \cong \Phi_\lambda(A)$; see (1, Theorem 5.6). Thus $\delta_\lambda(A) = \delta_{\lambda\epsilon}(A) \geq 0$, where the equality is by Lemma 2.31 and the inequality follows from Eq. 2.32 and the fact that $\delta_{\lambda\epsilon}(C) = 0$ as C is complete intersection. Moreover we get that if $\delta_{\lambda\epsilon}(A) = 0$, then $\text{Ker}(a_{\lambda\epsilon}(A)) = 0$, so A is faithful as a C -module, hence ϵ is an isomorphism. □

2.34. Λ -Structures. Motivated by number theory, we consider a setting where the algebra A in $C_{\mathcal{O}}$ has additional structure, and give a variant of the computation of change of congruence modules in Section 2 on going modulo regular sequences.

Let $t := t_1, \dots, t_c$ be indeterminates, $\Lambda_c := \mathcal{O}[[t]]$ the power-series ring, and $\Lambda_c \rightarrow \mathcal{O}$ the natural augmentation. Fix (A, λ) in $C_{\mathcal{O}}(c)$ equipped with a finite flat map $\iota: \Lambda_c \rightarrow A$ of $\mathcal{O}$ -algebras over $\mathcal{O}$, so that composite map

$$\Lambda_c \xrightarrow{\iota} A \xrightarrow{\lambda} \mathcal{O},$$

is the augmentation (that is, so that ι is a morphism in $C_{\mathcal{O}}(c)$). Since ι is flat the sequence $\iota(t_1), \dots, \iota(t_c)$ is A -regular. We assume also that the residue classes of t in $\mathfrak{p}_\lambda/\mathfrak{p}_\lambda^2$ form a linearly independent set. Thus, setting $A_0 := A/tA$, the map λ factors through A_0 , yielding an augmentation $\lambda_0: A_0 \rightarrow \mathcal{O}$, and (A_0, λ_0) is in $C_{\mathcal{O}}(0)$. One gets a commutative diagram of $\mathcal{O}$ -algebras

$$\begin{array}{ccc} \Lambda_c & \xrightarrow{\iota} & A \\ \epsilon \downarrow & & \downarrow \alpha \\ \mathcal{O} & \longrightarrow & A_0 \xrightarrow{\lambda_0} \mathcal{O} \end{array} \tag{2.35}$$

all augmented to $\mathcal{O}$, via λ_0 . We wish to track the change in cotangent modules and congruence modules along α , and we do that by using the diagram above, to reducing the problem to one about the map ϵ , where it is trivial, and the map ι , where it is easier to handle.

In the rest of this discussion we write $\mathfrak{p}$ and $\mathfrak{p}_0$ instead of $\mathfrak{p}_\lambda$ and $\mathfrak{p}_{\lambda_0}$, respectively.

We first discuss the change in cotangent modules in passing from λ to λ_0 . Since A_0 is regular at λ_0 , the $\mathcal{O}$ -module $D_2(\mathcal{O}/A_0, \mathcal{O})$, the second André–Quillen homology of the map $A_0 \rightarrow \mathcal{O}$, is torsion. Moreover one has

$$D_1(A_0/A, \mathcal{O}) \cong D_1(\mathcal{O}/\Lambda_c, \mathcal{O}) \cong \mathfrak{m}/\mathfrak{m}^2 \quad \text{where } \mathfrak{m} := (t)\Lambda_c.$$

In particular, this is a free $\mathcal{O}$ -module, of rank c . Thus the Jacobi–Zariski sequence arising from the maps $A \rightarrow A_0 \rightarrow \mathcal{O}$ yields an exact sequence of $\mathcal{O}$ -modules

$$0 \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\iota} \mathfrak{p}/\mathfrak{p}^2 \longrightarrow \mathfrak{p}_0/\mathfrak{p}_0^2 \longrightarrow 0,$$

where we use ι also to denote the map induced on cotangent modules by the ring map ι . One gets that $\Phi_\lambda(A)$ is a submodule of $\Phi_{\lambda_0}(A_0)$, and an exact sequence

$$0 \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\iota} (\mathfrak{p}/\mathfrak{p}^2)^{\text{tf}} \longrightarrow \Phi_{\lambda_0}(A_0)/\Phi_\lambda(A) \longrightarrow 0.$$

From a number theory perspective, it is more natural to consider the exact sequence obtained by applying $(-)^* := \text{Hom}_{\mathcal{O}}(-, \mathcal{O})$, namely the sequence

$$0 \longrightarrow (\mathfrak{p}/\mathfrak{p}^2)^* \xrightarrow{\iota^*} (\mathfrak{m}/\mathfrak{m}^2)^* \longrightarrow \text{Ext}_{\mathcal{O}}^1(\Phi_{\lambda_0}(A_0)/\Phi_\lambda(A), \mathcal{O}) \longrightarrow 0. \quad [2.36]$$

On the subcategory of torsion $\mathcal{O}$ -modules one has an isomorphism of functors $\text{Ext}_{\mathcal{O}}^1(-, \mathcal{O}) \cong \text{Hom}_{\mathcal{O}}(-, E/\mathcal{O})$, where E is the field of fractions of $\mathcal{O}$. Since the functor on the right preserves lengths, the computations above yield

$$\text{length}_{\mathcal{O}} \Phi_{\lambda_0}(A_0) - \text{length}_{\mathcal{O}} \Phi_\lambda(A) = \text{length}_{\mathcal{O}}(\Phi_{\lambda_0}(A_0)/\Phi_\lambda(A)) = \text{length}_{\mathcal{O}} \text{coker}(\iota^*) = \text{length}_{\mathcal{O}} \text{coker}(\wedge^c \iota^*). \quad [2.37]$$

The third equality holds because ι^* is a map between free $\mathcal{O}$ -modules of rank c .

Now we move on to the congruence modules. Given commutative diagram Eq. 2.35 of algebras over $\mathcal{O}$ and the functorial properties of $F_{\lambda}^c(\mathcal{O})$ one gets a commutative diagram of $\mathcal{O}$ -modules

$$\begin{array}{ccc} F_{\lambda_\ell}^c(\mathcal{O}) & \longleftarrow & F_{\lambda}^c(\mathcal{O}) \\ \downarrow \cong & & \downarrow \\ \mathcal{O} = F_{\text{id}}^0(\mathcal{O}) & = & F_{\lambda_0}^0(\mathcal{O}) = \mathcal{O} \end{array}$$

The isomorphism in the lower row is clear from the definitions; the vertical isomorphism is by a direct computation. The identity map on $\mathcal{O}$ is a canonical generator for $F_B^0(\mathcal{O}) = \text{Hom}_B(\mathcal{O}, \mathcal{O})$, for any B in $\text{C}\mathcal{O}$; this is why we write equalities in the last row. Using the commutative diagram above and the functoriality of the map Eq. 2.19, one gets a commutative diagram

$$\begin{array}{ccc} \wedge^c(\mathfrak{p}/\mathfrak{p}^2)^* & \xrightarrow{\cong} & F_{\lambda}^c(\mathcal{O}) \\ \wedge^c \iota^* \downarrow & & \downarrow \\ \wedge^c(\mathfrak{m}/\mathfrak{m}^2)^* & \xrightarrow{\cong} & F_{\lambda_0}^0(\mathcal{O}) \end{array}$$

Consider a finitely generated A -module M such that $\mathfrak{t}$ is also regular on M and $\text{depth}_A M \geq c + 1$. Setting $M_0 := M/(\mathfrak{t})M$ and using the identifications above, one gets a commutative diagram like so:

$$\begin{array}{ccc} \text{Ext}_A^c(\mathcal{O}, M) \otimes_{\mathcal{O}} \text{Hom}_A(M, \mathcal{O}) & \longrightarrow & \wedge^c(\mathfrak{p}/\mathfrak{p}^2)^* \\ \downarrow \cong & & \downarrow \cong \\ \text{Ext}_{A_0}^0(\mathcal{O}, M_0) \otimes_{\mathcal{O}} \text{Hom}_{A_0}(M_0, \mathcal{O}) & \longrightarrow & \wedge^c(\mathfrak{m}/\mathfrak{m}^2)^* \end{array} \quad [2.38]$$

All these lead to the following structural refinement of Theorem 2.28.

Theorem 2.39. *Viewing $\eta_\lambda(M)$ and $\eta_{\lambda_0}(M_0)$ as submodules of $\wedge^c(\mathfrak{p}/\mathfrak{p}^2)^*$ and $\wedge^c(\mathfrak{m}/\mathfrak{m}^2)^*$, respectively, there is an equality*

$$\eta_{\lambda_0}(M_0) = (\wedge^c \iota^*)(\eta_\lambda(M)).$$

Moreover, with $\mu := \text{rank}_\lambda M$ there are equalities

$$\text{length}_{\mathcal{O}} \Psi_{\lambda_0}(M_0) - \text{length}_{\mathcal{O}} \Psi_\lambda(M) = \mu \cdot \text{length}_{\mathcal{O}} \text{coker}(\wedge^c \iota^*) = \mu \cdot (\text{length}_{\mathcal{O}} \Phi_{\lambda_0}(A_0) - \text{length}_{\mathcal{O}} \Phi_\lambda(A)).$$

Proof: The first part is immediate from the commutative diagram Eq. 2.38. The second part follows, given also Eq. 2.37. $\square$

3. Zeta Lines and Congruence Modules

We focus on number theory applications of the results in Section 2, notably the exact sequence Eq. 2.36 and Theorem 2.39. The main result is Theorem 3.9. We begin with Proposition 3.2, which is a simple consequence of the Poitou-Tate exact sequence and is used to prove Proposition 3.13.

Let F be a number field, S a finite set of places of F , and $G_{F,S}$ the Galois group of F_S/F , the maximal extension of F unramified outside the places above S in an algebraic closure of F . Fix a prime number p , a finite extension $E/\mathbb{Q}_p$, and let $\mathcal{O}$ denote the ring of integers of E . Let A be a $\mathcal{O}$ -module, which is finitely or cofinitely generated, with an action of $G_{F,S}$. The Pontryagin dual and the twisted Pontryagin dual of A , respectively, are the $G_{F,S}$ -modules $A^\vee := \text{Hom}_{\mathcal{O}}(A, E/\mathcal{O})$ and $A' = A^\vee(1) = \text{Hom}_{\mathcal{O}}(A, E/\mathcal{O}(1))$. A Selmer datum for S and A is a collection $\mathcal{L} = \{\mathcal{L}_v\}_v$, where $\mathcal{L}_v$ is an $\mathcal{O}$ -submodule of $H^1(G_v, A)$ for each $v \in S$. The corresponding Selmer group, $H_{\mathcal{L}}^1(F, A)$ is the kernel of the map $H^1(G_{F,S}, A) \rightarrow \prod_{v \in S} H^1(G_v, A)/\mathcal{L}_v$. Local Tate-duality induces the perfect pairing $H^1(G_v, A) \times H^1(G_v, A') \rightarrow H^2(G_v, E/\mathcal{O}(1)) \cong E/\mathcal{O}$.

The dual Selmer datum $\mathcal{L}^\perp$ (for S and A') is defined with $\mathcal{L}_v^\perp \subset H^1(G_v, A')$ the annihilator of $\mathcal{L}_v$ under this pairing. The dual Selmer group of A is $H_{\mathcal{L}^\perp}^1(F, A')$. For $i = 1, 2$, set $\text{III}_S^i(F, A) := \text{Ker}(H^i(G_{F,S}, A) \rightarrow \prod_{v \in S} H^i(G_v, A))$. The result below is standard; see ref. 13, 8.7.9.

Lemma 3.1. *One has an exact sequence*

$$0 \rightarrow H_{\mathcal{L}}^1(F, A) \rightarrow H^1(G_{F,S}, A) \rightarrow \prod_{v \in S} \frac{H^1(G_v, A)}{\mathcal{L}_v} \rightarrow H_{\mathcal{L}^\perp}^1(F, A')^\vee \rightarrow \text{III}_S^2(F, A) \rightarrow 0.$$

If A is finite, then

$$\frac{\#H_{\mathcal{L}}^1(F, A)}{\#H_{\mathcal{L}^\perp}^1(F, A')} = \frac{\#H^0(F, A)}{\#H^0(F, A')} \cdot \prod_{v \in S} \frac{\#\mathcal{L}_v}{\#H^0(F_v, A)}.$$

□

Given Selmer datum $\mathcal{L}$ and $\tilde{\mathcal{L}}$ for S and A , we write $\mathcal{L} \subseteq \tilde{\mathcal{L}}$ if $\mathcal{L}_v \subseteq \tilde{\mathcal{L}}_v$ for all v .

Proposition 3.2. *Assume A is compact or cocompact and that $\mathcal{L} \subseteq \tilde{\mathcal{L}}$ are Selmer data. Then one has a natural exact sequence*

$$0 \rightarrow \frac{H_{\tilde{\mathcal{L}}}^1(S, A)}{H_{\mathcal{L}}^1(S, A)} \rightarrow \prod_{v \in S} \frac{\tilde{\mathcal{L}}_v}{\mathcal{L}_v} \rightarrow \left(\frac{H_{\tilde{\mathcal{L}}^\perp}^1(S, A')}{H_{\mathcal{L}^\perp}^1(S, A')} \right)^\vee \rightarrow 0.$$

Proof: From the definitions one gets that natural maps $H_{\mathcal{L}}^1(S, A) \rightarrow H_{\tilde{\mathcal{L}}}^1(S, A)$ and $H_{\tilde{\mathcal{L}}^\perp}^1(S, A') \rightarrow H_{\mathcal{L}^\perp}^1(S, A')$ are injective. Setting $K := \text{Ker}(H_{\mathcal{L}^\perp}^1(F, A')^\vee \rightarrow \text{III}_S^2(F, A))$ and similarly $\tilde{K}$ with $\tilde{\mathcal{L}}$ in place of $\mathcal{L}$, the naturality of the exact sequence in Lemma 3.1 yields the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \frac{H^1(G_{F,S}, A)}{H_{\mathcal{L}}^1(F, A)} & \longrightarrow & \prod_{v \in S} \frac{H^1(G_v, A)}{\mathcal{L}_v} & \longrightarrow & K \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \frac{H^1(G_{F,S}, A)}{H_{\tilde{\mathcal{L}}}^1(F, A)} & \longrightarrow & \prod_{v \in S} \frac{H^1(G_v, A)}{\tilde{\mathcal{L}}_v} & \longrightarrow & \tilde{K} \longrightarrow 0 \end{array}$$

The Snake Lemma yields the exact sequence

$$0 \rightarrow \frac{H_{\tilde{\mathcal{L}}}^1(F, A)}{H_{\mathcal{L}}^1(F, A)} \rightarrow \prod_{v \in S} \frac{\tilde{\mathcal{L}}_v}{\mathcal{L}_v} \rightarrow K \rightarrow \tilde{K} \rightarrow 0.$$

By applying the Snake Lemma to the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & H_{\mathcal{L}^\perp}^1(F, A')^\vee & \longrightarrow & \text{III}_S^2(F, A) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \tilde{K} & \longrightarrow & H_{\tilde{\mathcal{L}}^\perp}^1(F, A')^\vee & \longrightarrow & \text{III}_S^2(F, A) \longrightarrow 0 \end{array}$$

and using Pontryagin duality, we obtain isomorphisms

$$\text{Ker}(K \rightarrow \tilde{K}) \cong \text{Ker}(H_{\mathcal{L}^\perp}^1(F, A')^\vee \rightarrow H_{\tilde{\mathcal{L}}^\perp}^1(F, A')^\vee) \cong \left(\frac{H_{\mathcal{L}^\perp}^1(F, A')}{H_{\tilde{\mathcal{L}}^\perp}^1(F, A')} \right)^\vee,$$

concluding the proof of the proposition. □

3.3. Nearly Ordinary Hilbert Modular Forms. We specialize to a situation which corresponds to that of ref. 6. Thus, F is a totally real number field, p an odd prime, and f a nearly ordinary at p , holomorphic and cohomological cuspidal Hilbert modular newform for $\text{GL}_2(\mathbb{A}_F)$. Our coefficients A as in the section above arise from $\text{ad } \rho_f$ (and $\text{ad}(\rho_f \otimes_{\mathcal{O}} E/\mathcal{O})$) the trace 0 submodule of the adjoint of (an) integral Galois representation

$$\rho_f: G_{F,S} \rightarrow \text{GL}_2(\mathcal{O})$$

associated to f and an embedding of $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. We assume the residual representation $\bar{\rho}_f: G_{F,S} \rightarrow \text{GL}_2(k)$ is irreducible (which implies that it is absolutely irreducible as ρ_f is totally odd and $p > 2$), and thus there is a unique integral representation ρ_f associated to f (by a well-known result of Carayol).

We apply the results of the previous section to study the Galois cohomology of the adjoint representation of ρ_f with several local conditions at places dividing p . For each $v|p$, we fix a decomposition subgroup D_v at v . We call $I_v \subset D_v$ the inertia subgroup, F_v the completion of F at v , and d_v its degree over $\mathbb{Q}_p$. By nearly ordinarity of f , for each $v|p$ there exists $g_v \in \text{GL}_2(\mathcal{O})$ such that the restriction to the decomposition subgroup D_v at v of $g_v \rho_f g_v^{-1}$ is upper triangular. We also assume that it is v -distinguished (that is,

the characters appearing on the diagonal are distinct modulo the uniformizer ϖ of $\mathcal{O}$) and indecomposable. We then consider the following summands:

$$\mathcal{F}_v^+ := \{g_v \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} g_v^{-1}\} \subset \mathcal{F}_v^0 := \{g_v \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} g_v^{-1}\} \subset \text{ad } \rho_f.$$

We denote $Gr_v^0 := \mathcal{F}_v^0 / \mathcal{F}_v^+$ and fix an isomorphism of D_v -modules $Gr_v^0 \cong \mathcal{O}$. Let B be a $\mathcal{O}$ -module. The ordinary condition $H_{\text{ord}}^1(F_v, \text{ad } \rho_f \otimes B)$ at v is given by the image of $H^1(F_v, \mathcal{F}_v^0 \otimes B)$ in $H^1(F_v, \text{ad } \rho_f \otimes B)$, and (in the terminology of Wiles (3)) the Selmer condition $H_{\text{Sel}}^1(F_v, \text{ad } \rho_f \otimes B)$ at v is given by the image of

$$\text{Ker}(H^1(F_v, \mathcal{F}_v^+ \otimes B) \longrightarrow H^1(I_v, Gr_v^0 \otimes B))$$

in $H_{\text{ord}}^1(F_v, \text{ad } \rho_f \otimes B)$. Since the representation ρ_f is v -distinguished, we get an exact sequence:

$$0 \rightarrow H_{\text{Sel}}^1(F_v, \text{ad } \rho_f \otimes B) \rightarrow H_{\text{ord}}^1(F_v, \text{ad } \rho_f \otimes B) \rightarrow H^1(I_v, B)^{\frac{D_v}{I_v}} \rightarrow 0.$$

The orthogonal of the finite Selmer condition $H_{\text{Sel}}^1(F_v, \text{ad } \rho_f \otimes B(1))$ and of the ordinary condition $H_{\text{ord}}^1(F_v, \text{ad } \rho_f \otimes B(1))$ are respectively given by the images of $H^1(F_v, \mathcal{F}_v^0 \otimes B(1))$ and of $H^1(F_v, \mathcal{F}_v^+ \otimes B(1))$ in $H^1(F_v, \text{ad } \rho_f \otimes B(1))$.

Remark 3.4: Assume that the action of D_v on $\mathcal{F}_v^+$ is distinct from the cyclotomic character and that the Hodge-Tate weights of $\text{ad } \rho_f$ belong to an interval of length $\leq p-2$. Then an extension

$$0 \longrightarrow \text{ad } \rho_f \otimes \mathcal{O} / \varpi^n \mathcal{O} \longrightarrow E \longrightarrow \mathcal{O} / \varpi^n \mathcal{O} \longrightarrow 0$$

is crystalline if and only if it is the reduction of an ordinary crystalline representation. Since the finite Bloch-Kato condition classifies the extensions which are reduction of crystalline extension, the finite Selmer condition is nothing else but the finite Bloch-Kato condition in that case. In other words, one has

$$\begin{aligned} H_{\text{Sel}}^1(F_v, \text{ad } \rho_f \otimes B) &= H_f^1(F_v, \text{ad } \rho_f \otimes B) \\ H_{\text{Sel}}^1(F_v, \text{ad } \rho_f \otimes B(1)) &= H_f^1(F_v, \text{ad } \rho_f \otimes B(1)). \end{aligned}$$

Next, we interpret some higher congruence modules in terms of Galois cohomology (local and global) and by applying Theorem 2.39 in the situation described below.

3.5. Congruence Modules and Galois Cohomology. We recall some ingredients of the setup of ref. 6; the notation is borrowed from it.

Let $\kappa := (\sum_{\sigma} k_{\sigma} \cdot \sigma, \sum_{\sigma} l_{\sigma} \cdot \sigma) \in \mathbb{Z}[\Sigma_F]^2$ be the weight of the cohomological Hilbert modular cusp form f . We have $k_{\sigma} \geq 2$ for all $\sigma \in \Sigma_F$ and $w = k_{\sigma} + 2l_{\sigma}$ is independent of σ . For such a weight and a O_F -algebra S with F' the normal closure of F , we consider the algebraic representation of $\text{GL}_2(O_F)$:

$$L(\kappa, S) := \bigotimes_{\sigma} \text{Sym}^{k_{\sigma}-2}(S^2) \otimes \det^{l_{\sigma}}.$$

For each neat open compact subgroup $K \subset \text{GL}_2(\mathbb{A}_f \otimes F)$, this defines a local system $L(\kappa, \mathbb{C})$ on the Hilbert modular variety

$$X(K) := \text{GL}_2(F) \backslash \text{GL}_2(\mathbb{A} \otimes F) / KK_{\infty}^0 Z(F \otimes \mathbb{R}),$$

where Z is the center of GL_2 and K_{∞}^0 is the special orthogonal subgroup of $\text{GL}_2(\mathbb{R} \otimes F)$. In particular, we have an action of the group of connected components of the maximal compact $K_{\infty} / K_{\infty}^0$ on the cohomology of $X(K)$. Since its group of characters can be naturally identified with $\{\pm 1\}^{\Sigma_F}$, for each $\epsilon \in \{\pm 1\}^{\Sigma_F}$, we can define the ϵ -part of the cohomology of $X(K)$.

Let $\mathfrak{n} \subset O_F$ be the tame conductor of f . It is a nonzero integral ideal of O_F prime to p . Let $K_{11}^p(\mathfrak{n}) \subset \text{GL}_2(\widehat{\mathbb{Z}}^p \otimes O_F)$ be the subgroup of matrices which are upper unipotent modulo $\mathfrak{n}$ and where we have written $\widehat{\mathbb{Z}}^p$ for the prime-to- p part of the profinite completion of $\mathbb{Z}$. We will assume that $K_{11}^p(\mathfrak{n})$ is neat. Let ω be the central character of the cuspidal representation attached to f . It is an idèle class character of conductor dividing $\mathfrak{n}p^{\infty}$ and infinity type $|\cdot|^w$.

For each positive integer n , we denote $K_0(p^n)$ the subgroup of $\text{GL}_2(O_F \otimes \mathbb{Z}_p)$ of matrices which are upper triangular modulo p^n and by $K_1(p^n)$ its subgroup of those such that the diagonal entries are congruent modulo p^n . We identify $K_0(p^n)/K_1(p^n)$ with $(O_F/p^n O_F)^{\times}$ via the map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a^{-1}d$.

Let $h_{\kappa}^{\text{ord}}(\mathfrak{n}p^n, \omega)$ be the nearly ordinary Hecke algebra of level $K_{11}^p(\mathfrak{n})K_1(p^n)$, weight κ and central character ω . We then consider the universal nearly ordinary Hecke algebra of weight κ and tame level K^p and action of the center given by ω .

$$\mathbf{h}^{\text{ord}} = \mathbf{h}_{\kappa}^{\text{ord}}(\mathfrak{n}) := \varprojlim_n h_{\kappa}^{\text{ord}}(\mathfrak{n}p^n).$$

The Hecke ring $b_k^{\text{ord}}(\mathfrak{n}p^n)$ has a natural structure of $\mathcal{O}[(O_F/p^n O_F)^\times]$ -algebra, inducing a structure of Λ_F -algebra on $\mathbf{h}^{\text{ord}}$ with

$$\Lambda_F := \mathcal{O}[\![O_{F,p}^1]\!] \cong \mathcal{O}[\![t_1, \dots, t_d]\!]$$

and $O_{F,p}^1 \cong \mathbb{Z}_p^d$ the subgroup of $O_{F,p}^\times$ of local units congruent to 1 modulo p . Let $\mathfrak{m}$ be the kernel of the map $\Lambda_F \rightarrow \mathcal{O}$ corresponding to the trivial character of $O_{F,p}^1$.

The Hecke eigensystem attached to our nearly ordinary Hilbert modular form f gives us an homomorphism:

$$\lambda_f: \mathbf{h}^{\text{ord}} \rightarrow b_k^{\text{ord}}(\mathfrak{n}p^r) \rightarrow \mathcal{O}.$$

with r the smallest integer so that f is $K_1(p^r)$ -invariant.

We denote $\mathbb{T}^{\text{ord}}$ (resp. $\mathbb{T}_0$) the localization of $\mathbf{h}^{\text{ord}}$ (resp. $b_k^{\text{ord}}(\mathfrak{n}p^r)$) at its maximal ideal $\mathfrak{m}_f$ containing $\ker \lambda_f$. By work of Hida (14, Corollary 4.3) one gets that $\mathbb{T}^{\text{ord}}$ is free of finite rank over Λ_F . Moreover, we have a canonical isomorphism

$$\mathbb{T}^{\text{ord}} \otimes_{\Lambda_F} \mathcal{O} \cong \mathbb{T}_0.$$

However, we will not need these facts. We now construct a $\mathbb{T}^{\text{ord}}$ -module which is free over Λ_F and interpolates the nearly ordinary cohomology of the Hilbert modular variety localized at the maximal ideal associated to f . For any p -adically complete $\mathcal{O}$ -algebra A and $n \geq r$, let

$$C_n(\kappa, A) := \text{Ind}_{K_1(p^n)}^{K_1(p^r)} L(\kappa, A)$$

and write $\mathcal{C}(\kappa, A)$ for the direct limit of the $C_n(\kappa, A)$ for the obvious transition maps, and $\mathcal{C}(\kappa, \mathcal{O})$ for the inverse limit of the $\mathcal{C}(\kappa, \mathcal{O}/p^m \mathcal{O})$ as m varies. It is clearly a $\Lambda_F[K_1(p^r)]$ -module.

Let $\eta_{\lambda_f}(M)$ be the congruence ideal of M with respect to λ_f introduced in Section 2.10. For any ideal $\mathfrak{b} \subset O_F$, we denote $K_{11}(\mathfrak{b})$ the open compact subgroup of $\text{GL}_2(O_F \otimes \hat{\mathbb{Z}})$ of upper triangular matrices which are unipotent modulo $\mathfrak{b}$.

Proposition 3.6. *Assume the image of $\bar{\rho}_f$ is not solvable. Then for any ϵ in $\{\pm 1\}^{\Sigma_F}$, the $\mathbb{T}^{\text{ord}}$ -module*

$$\mathbf{M}^\epsilon := H^d(X(K_{11}(\mathfrak{n}p^r), \mathcal{C}(\kappa, \mathcal{O}))_{\mathfrak{m}_f}^\epsilon$$

is free of finite rank over Λ_F , and $\mathbf{M}^\epsilon / \mathfrak{m} \mathbf{M}^\epsilon = \mathbf{M}_0^\epsilon := H^d(X(K_{11}(\mathfrak{n}p^r), \mathcal{L}(\kappa, \mathcal{O}))_{\mathfrak{m}_f}^\epsilon$. Moreover $\eta_{\lambda_f}(\mathbf{M}_0^\epsilon) = (\xi_f^\epsilon)$, where

$$\xi_f^\epsilon := \frac{\Gamma(\text{ad } \rho_f, 1) L^{S_f}(\text{ad } \rho_f, 1)}{\Omega_f^\epsilon \Omega_f^{-\epsilon}},$$

where S_f is the set of finite places where ρ_f is ramified and $(\Omega_f^\epsilon)_{\epsilon \in \{\pm 1\}^{\Sigma_F}}$ are the canonical complex periods attached to the Hilbert modular form f in (5, §7.1).*

Proof: This is a classical exercise in Hida theory since the localization at $\mathfrak{m}_f$ captures a direct factor of the nearly ordinary part of the cohomology. The fact that the module is free over Λ_F follows from a control theorem and the vanishing Theorem 7.1.1 of Caraiani and Tamiozzo (15). The last part of the proposition follows from a computation of Dimitrov in sections 7.2 and 7.3 of ref. 5, and in particular its equations (50) and (51). $\square$

We use the inclusion $\eta_{\lambda_f}(\mathbf{M}^\epsilon) \subset F_{\lambda_f}^d(\mathcal{O}) = \bigwedge_{\mathcal{O}}^d(\mathfrak{p}/\mathfrak{p}^2)^*$ to define the zeta $\mathcal{O}$ -module associated to f .

3.7. Construction of Zeta Lines. Given an $\mathcal{O}$ -module A we set $A^* := \text{Hom}_{\mathcal{O}}(A, \mathcal{O})$. Let R^{ord} (resp. R_0) be the universal deformation ring of $\bar{\rho}_f$ with fixed determinant equal to $\det \rho_f$ and with nearly ordinary conditions (resp. with ordinary condition of weight κ) at places dividing p and the unramified condition at finite places away from those dividing $\mathfrak{n}p\infty$.

We have a canonical surjective map $R^{\text{ord}} \rightarrow \mathbb{T}^{\text{ord}}$. Set $\mathfrak{p} := \text{Ker}(\mathbb{T}^{\text{ord}} \rightarrow \mathcal{O})$ and $\mathfrak{p}_R := \text{Ker}(R^{\text{ord}} \rightarrow \mathcal{O})$. One has isomorphisms

$$H_{\text{full,ord}}^1(F, \text{ad } \rho_f \otimes E/\mathcal{O})^\vee \cong \mathfrak{p}_R/\mathfrak{p}_R^2 \quad \text{and} \quad H_{\text{full,ord}}^1(F, \text{ad } \rho_f) \cong (\mathfrak{p}_R/\mathfrak{p}_R^2)^*;$$

see for example refs. 16 or 17, Section 2.5. The subscript full means that no local conditions are required at places dividing $\mathfrak{n}$. From the natural surjection

$$H_{\text{full,ord}}^1(F, \text{ad } \rho_f \otimes E/\mathcal{O})^\vee \cong \mathfrak{p}_R/\mathfrak{p}_R^2 \twoheadrightarrow \mathfrak{p}/\mathfrak{p}^2.$$

one gets maps

$$F_{\lambda_f}^d(\mathcal{O}) = \bigwedge_{\mathcal{O}}^d(\mathfrak{p}/\mathfrak{p}^2)^* \longrightarrow \bigwedge_{\mathcal{O}}^d(\mathfrak{p}_R/\mathfrak{p}_R^2)^* \cong \bigwedge_{\mathcal{O}}^d H_{\text{full,ord}}^1(F, \text{ad } \rho_f).$$

*In (5), the ϵ -parts of the cohomology and the periods are indexed by the subsets $J \subset \Sigma_F$ corresponding to the character ϵ_J .

Definition 3.8: The image of the submodule $\eta_{\lambda_f}(\mathbf{M}^\epsilon) \subset \bigwedge^d (\mathfrak{p}/\mathfrak{p}^2)^*$ under the composition of maps above is a cyclic $\mathcal{O}$ -submodule; we write it as (z_f^ϵ) and call it the zeta line. Thus z_f^ϵ is well defined only up to multiplication by a unit in $\mathcal{O}$.

For each v , the quotient map $\mathcal{F}_v^0 \rightarrow Gr_v^0$ induces the map

$$H^1(F_v, \mathcal{F}_v^0) \longrightarrow H^1(F_v, Gr_v^0) \longrightarrow H^1(I_v, Gr_v^0)^{D_v/I_v} \cong \mathcal{O}^{d_v}.$$

Since $\sum_{v|p} d_v = [F : \mathbb{Q}] = d$ these induce the map

$$\bigwedge^d \left(\prod_{v|p} H^1(F_v, \mathcal{F}_v^0) \right) \longrightarrow \bigotimes_{v|p} \bigwedge_{\mathcal{O}}^{d_v} H^1(I_v, Gr_v^0)^{D_v/I_v} \cong \mathcal{O}.$$

Precomposing this with the d th exterior power of the restriction map $\text{res}_p: H_{\text{full,ord}}^1(F, \text{ad } \rho_f) \rightarrow \prod_{v|p} H^1(F_v, \mathcal{F}_v^0)$ yields

$$\bigwedge^d \text{res}_p: \bigwedge^d H_{\text{full,ord}}^1(F, \text{ad } \rho_f) \longrightarrow \bigotimes_{v|p} \bigwedge_{\mathcal{O}}^{d_v} H^1(I_v, Gr_v^0)^{D_v/I_v}.$$

The following theorem is the main result of this section.

Theorem 3.9. Assume that the residual representation $\bar{\rho}_f$ has nonsolvable image and choose $\epsilon \in \{\pm 1\}^{\Sigma_F}$. Then

$$\left(\bigwedge^d \text{res}_p \right) (z_f^\epsilon) = (\xi_f^\epsilon) \quad \text{where as before} \quad \xi_f^\epsilon := \frac{\Gamma(\text{ad } \rho_f, 1) L^{S_f}(\text{ad } \rho_f, 1)}{\Omega_f^\epsilon \Omega_f^{-\epsilon}}.$$

Proof: Let R_0 be the universal deformation ring of $\bar{\rho}_f$ with fixed determinant equal to $\det \rho_f$ and with ordinary condition of weight κ at places dividing p and the unramified condition at finite places away from those dividing $n p \infty$, and $R_0 \rightarrow \mathbb{T}_0$ the canonical surjection. The restriction of the universal deformations to the decomposition subgroups at places dividing p gives an homomorphism $\Lambda_F \rightarrow R^{\text{ord}}$ making the map $R^{\text{ord}} \rightarrow \mathbb{T}^{\text{ord}}$ an Λ_F -algebra homomorphism and a canonical isomorphism $R^{\text{ord}} \otimes_{\Lambda_F} \mathcal{O} \cong R_0^{\text{ord}}$. Setting $\mathfrak{p}_0 := \text{Ker}(\mathbb{T}_0^{\text{ord}} \rightarrow \mathcal{O}) = \mathfrak{p}/\mathfrak{m}$ and $\mathfrak{p}_{R,0} := \text{Ker}(R_0^{\text{ord}} \rightarrow \mathcal{O}) = \mathfrak{p}_R/\mathfrak{m}$, the isomorphism and surjective maps induces the following commutative diagram of Kähler differentials.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{m}/\mathfrak{m}^2 & \longrightarrow & \mathfrak{p}/\mathfrak{p}^2 & \longrightarrow & \mathfrak{p}_0/\mathfrak{p}_0^2 \longrightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathfrak{m}/\mathfrak{m}^2 & \longrightarrow & \mathfrak{p}_R/\mathfrak{p}_R^2 & \longrightarrow & \mathfrak{p}_{R,0}/\mathfrak{p}_{R,0}^2 \longrightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ 0 & \rightarrow & \bigoplus_{v|p} (H^1(I_v, \frac{E}{\mathcal{O}})^{\frac{D_v}{I_v}})^\vee & \rightarrow & H_{\text{full,ord}}^1(F, \text{ad } \rho_f \otimes \frac{E}{\mathcal{O}})^\vee & \rightarrow & H_{\text{full,Sel}}^1(F, \text{ad } \rho_f \otimes \frac{E}{\mathcal{O}})^\vee \rightarrow 0 \end{array}$$

Here $H_{\text{full,Sel}}^1(F, \text{ad } \rho_f \otimes \frac{E}{\mathcal{O}})$ means no condition at primes dividing n , and the Selmer condition at places of F above p . The exactness on the left in the top row follows from Hida's theorem that $\mathbb{T}^{\text{ord}}$ is unramified over the weight space Λ_F at the augmentation λ_f arising from the holomorphic cohomological newform f . The vertical arrows are surjective and that the $\mathcal{O}$ -module $\mathfrak{p}_0/\mathfrak{p}_0^2 \cong \Omega_{\mathbb{T}_0/\mathcal{O}} \otimes_{\lambda_f} \mathcal{O}$ is torsion, and therefore both $\mathfrak{m}/\mathfrak{m}^2$ and $(\mathfrak{p}/\mathfrak{p}^2)^{\text{tf}}$ are free of rank d over $\mathcal{O}$. The diagram above yields the commutative diagram

$$\begin{array}{ccc} \bigwedge^d (\mathfrak{m}/\mathfrak{m}^2)^* & \longleftarrow & \bigwedge^d (\mathfrak{p}/\mathfrak{p}^2)^* \\ \parallel & & \downarrow \\ \bigwedge^d (\mathfrak{m}/\mathfrak{m}^2)^* & \longleftarrow & \bigwedge^d (\mathfrak{p}_R/\mathfrak{p}_R^2)^* \\ \parallel & & \parallel \\ \bigotimes_{v|p} \bigwedge^{d_v} H^1(I_v, \mathcal{O})^{\frac{D_v}{I_v}} & \longleftarrow & \bigwedge^d H_{\text{full,ord}}^1(F, \text{ad } \rho_f) \end{array}$$

Given this diagram, Proposition 3.6, and Theorem 2.39, it follows that the image of z_f^ϵ under the local restriction map at p is ξ_f^ϵ , fixing an isomorphism with $\mathcal{O}$. $\square$

Remark 3.10: As explained in ref. 6, (z_f^ϵ) is the bottom class of an Euler system of rank d . We have shown that (z_f^ϵ) is related to the L -value ξ_f . It would be interesting to extend our method to construct the other classes using higher congruence modules for the base change of f to abelian extensions of F , for we would be able to construct the p -adic L -function $L_p^{S_f}(\text{ad } \rho_f, s)$.

Remark 3.11: By Eq. 2.36, $\text{length}_{\mathcal{O}} \text{coker}((\mathfrak{p}/\mathfrak{p}^2)^* \rightarrow (\mathfrak{m}/\mathfrak{m}^2)^*) = \text{length}_{\mathcal{O}}(\Phi_{\lambda_f}(\mathbb{T})) - \text{length}_{\mathcal{O}}(\Phi_{\lambda_f}(\mathbb{T}^{\text{ord}}))$. Thus the classical Selmer group $\Phi_{\lambda}(R)$ “factors” into a part coming from $\Phi_{\lambda}(R^{\text{ord}})$ and a part coming from the cokernel of $(\mathfrak{p}/\mathfrak{p}^2)^* \rightarrow (\mathfrak{m}/\mathfrak{m}^2)^*$.

3.12. The Cotorson in Galois Cohomology. In this section, we assume that ρ_f is a minimal deformation of $\bar{\rho}_f$ as in ref. 5, §4.2. We replace the rings R^{ord} and R_0 by their minimal deformation analogues. Then it is known that the maps $R_{\min}^{\text{ord}} \rightarrow \mathbb{T}^{\text{ord}}$ and $R_{0,\min} \rightarrow \mathbb{T}_0$ are isomorphisms of complete intersection rings thanks to the works of Fujiwara (18) and Dimitrov (5). Note that we can and do replace the big image assumption of Dimitrov for $\bar{\rho}_f$ by the much weaker one of ref. 15 of being nonsolvable, which implies the Taylor-Wiles hypothesis that ρ_f restricted to $G_{F(\mu_p)}$ is irreducible. In addition, we make the following hypothesis justified by the Remark 3.4. For all $v|p$, we assume that the action of D_v on $\mathcal{F}_v^+$ is distinct from the cyclotomic character. By our minimality assumption, the local conditions at places away from p are the finite Bloch-Kato conditions for all the Galois cohomology groups considered in this section, so that we can now make the following identifications:

- $\text{Hom}(\mathfrak{p}/\mathfrak{p}^2, \mathcal{O}) = H_{\text{ord}}^1(F, \text{ad } \rho_f)$
- $\text{Hom}(\mathfrak{m}/\mathfrak{m}^2, \mathcal{O}) = H_{\text{ord}/f}^1(F_p, \text{ad } \rho_f) = \bigoplus_{v|p} H^1(I_v, \mathcal{O})^{\frac{D_v}{\tau_v}}$
- $H_f^1(F, \text{ad } \rho_f \otimes_{\mathcal{O}} E/\mathcal{O})^{\vee} = \mathfrak{p}_0/\mathfrak{p}_0^2 := \Phi_{\lambda_f}$ since $\mathbb{T}_0$ is finite over $\mathcal{O}$,
- $\text{cotor}_s(H_{\mathcal{L}^{\text{ord}}}^1(\mathbb{Q}, \text{ad } \rho_f \otimes_{\mathcal{O}} E/\mathcal{O})) = \text{tors}(\mathfrak{p}/\mathfrak{p}^2) = \Phi_{\lambda_f}^{\text{ord}} := \Phi_{\lambda_f}(\mathbb{T}^{\text{ord}})$.

We abbreviate $\eta_{\lambda_f}(\mathbb{T}_0)$ and $\eta_{\lambda_f}(\mathbb{T}^{\text{ord}})$ to η_{λ_f} and $\eta_{\lambda_f}^{\text{ord}}$, respectively, and view them as ideals of $\mathcal{O}$. Since we have assumed that the Hecke rings are complete intersection we have $\eta_{\lambda_f} = \text{Fitt}_{\mathcal{O}}(\Phi_{\lambda_f}) \subset \eta_{\lambda_f}^{\text{ord}} = \text{Fitt}_{\mathcal{O}}(\Phi_{\lambda_f}^{\text{ord}})$.

Here is an interpretation of the invariants $\Phi_{\lambda_f}^{\text{ord}}$ and $\Psi_{\lambda_f}^{\text{ord}}$.

Proposition 3.13. *With the minimality assumptions as above, the following statements hold.*

1. *There are isomorphisms $\Phi_{\lambda_f}^{\text{ord}} \cong \text{cotor}_s(H_{\text{ord}}^1(F, \text{ad } \rho_f \otimes_{\mathcal{O}} E/\mathcal{O}))$, and an equality*

$$\text{length}_{\mathcal{O}}(\Phi_{\lambda_f}^{\text{ord}}) = \text{length}_{\mathcal{O}}(H_{\text{ord}^{\perp}}^1(F, \text{ad } \rho_f \otimes_{\mathcal{O}} E/\mathcal{O}(1))).$$

2. *There is an isomorphism $\Psi_{\lambda_f}^{\text{ord}} \cong (\bigwedge^d H_{\text{ord}}^1(F, \text{ad } \rho_f))/(z_f^{\varepsilon})$.*

Proof: From Eq. 2.36 we get the exact sequence $0 \rightarrow (\mathfrak{p}/\mathfrak{p}^2)^* \rightarrow (\mathfrak{m}/\mathfrak{m}^2)^* \rightarrow (\Phi_{\lambda_f}/\Phi_{\lambda_f}^{\text{ord}})^{\vee} \rightarrow 0$. Comparing with the following Poitou-Tate duality exact sequence

$$0 \rightarrow H_{\text{ord}}^1(F, \text{ad } \rho_f) \rightarrow H_{\text{ord}/f}^1(F_p, \text{ad } \rho_f) \rightarrow \left(\frac{H_f^1(F, \text{ad } \rho_f \otimes_{\mathcal{O}} E/\mathcal{O}(1))}{H_{\text{ord}^{\perp}}^1(F, \text{ad } \rho_f \otimes_{\mathcal{O}} E/\mathcal{O}(1))} \right)^{\vee} \rightarrow 0$$

from Proposition 3.2 and the identifications recalled above, we get

$$\frac{\Phi_{\lambda_f}}{\Phi_{\lambda_f}^{\text{ord}}} \cong \frac{H_f^1(F, \text{ad } \rho_f \otimes_{\mathcal{O}} E/\mathcal{O}(1))}{H_{\text{ord}^{\perp}}^1(F, \text{ad } \rho_f \otimes_{\mathcal{O}} E/\mathcal{O}(1))}.$$

On the other hand, we have a canonical isomorphism $\Phi_{\lambda_f} \cong H_f^1(F, \text{ad } \rho_f \otimes_{\mathcal{O}} E/\mathcal{O})^{\vee}$. It remains to note that by the balanced properties of Bloch-Kato Selmer groups and Lemma 3.1, the $\mathcal{O}$ -module above has finite length, equal to $\text{length}_{\mathcal{O}}(H_f^1(F, \text{ad } \rho_f \otimes_{\mathcal{O}} E/\mathcal{O}(1)))$.

(2): From Theorem 2.39 one gets an exact sequence $0 \rightarrow \bigwedge^d(\mathfrak{p}/\mathfrak{p}^2)^* \rightarrow \bigwedge^d(\mathfrak{m}/\mathfrak{m}^2)^* \rightarrow \Psi_{\lambda_f}/\Psi_{\lambda_f}^{\text{ord}} \rightarrow 0$, and therefore

$$0 \rightarrow \frac{\bigwedge^d(\mathfrak{p}/\mathfrak{p}^2)^*}{(z_f^{\varepsilon})} \rightarrow \frac{\bigwedge^d(\mathfrak{m}/\mathfrak{m}^2)^*}{(\bigwedge^d \text{res}_p)(z_f^{\varepsilon})} \rightarrow \frac{\Psi_{\lambda_f}}{\Psi_{\lambda_f}^{\text{ord}}} \rightarrow 0.$$

The desired isomorphism follows since after identification of $\bigwedge^d(\mathfrak{m}/\mathfrak{m}^2)^*$ with $\mathcal{O}$, we have $\bigwedge^d \text{res}_p(z_f^{\varepsilon}) = (\xi_f^{\varepsilon})$ which is the same as the ideal η_{λ_f} by construction. $\square$

The next result is immediate from Proposition 3.13 and Theorem 2.6. It is an analog of the well-known fact that the p -part of the class group of $\mathbb{Q}(\zeta_p)^+$ being trivial (Vandiver's conjecture is that this is always the case) is equivalent to the group of cyclotomic units having index prime to p inside the global units of $\mathbb{Q}(\zeta_p)$.

Corollary 3.14. *With previous assumptions, $\mathbb{T}^{\text{ord}}$ is regular $\iff H_{\text{ord}^{\perp}}^1(F, \text{ad } \bar{\rho}_f(1)) = 0 \iff (z_f^{\varepsilon}) = \bigwedge^d H_{\text{ord}}^1(F, \text{ad } \rho_f)$.* $\square$

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