

Tail Adversarial Stability for Regularly Varying Linear Processes and their Extensions

Shuyang Bai[†](bsy9142@uga.edu) and Ting Zhang (tingzhang@uga.edu)

University of Georgia

July 24, 2023

Abstract

The notion of tail adversarial stability has been proven useful in obtaining limit theorems for tail dependent time series. Its implication and advantage over the classical strong mixing framework has been examined for max-linear processes, but not yet studied for additive linear processes. In this article, we fill this gap by verifying the tail adversarial stability condition for regularly varying additive linear processes. We in addition consider extensions of the result to a stochastic volatility generalization and to a max-linear counterpart. We also address the invariance of tail adversarial stability under monotone transforms. Some implications for limit theorems in statistical context are also discussed.

Keywords: time series, linear processes, moving average, power laws, regularly varying distributions, tail dependence.

MSC Classification: 62M10, 62G32.

1 Introduction

Compared with the conventional notion of correlation-based dependence that mainly concerns co-movements around the mean, tail dependence or extremal dependence refers to the dependence in the joint extremes that mainly concerns the co-occurrence of tail events. In bivariate or finite-dimensional distributions, the concept of tail dependence and its quantification have been extensively explored in the literature; see for example Sibuya (1960), de Haan and Resnick (1977), Joe (1993), Ledford and Tawn (1996), Coles et al. (1999), Embrechts et al. (2002), Draisma et al. (2004), Poon et al. (2004), McNeil et al. (2005), Zhang (2008), Balla et al. (2014), Hoga (2018) and references therein. In the time series setting, Leadbetter et al. (1983), Smith and Weissman (1994) and Ferro and Segers (2003) considered the use of an extremal index to describe the degree of tail dependence. Zhang (2005) proposed to extend the bivariate tail dependence index of Sibuya (1960) to the time series setting, and Linton and Whang (2007) considered a variant that uses a quantile as the tail threshold; see also the extremogram of Davis and Mikosch (2009) and Hill (2009), the tail autocorrelation of Zhang (2022), as well as the notion of tail process in Basrak

[†]Corresponding author.

and Segers (2009) which characterizes the local tail dependence structure in a stationary regularly varying time series. We also refer to a recent review paper Zhang (2021b) and monograph Kulik and Soulier (2020) for additional discussions and references.

Although various tail dependence measures have been proposed, most of them focused on summarizing the degree of tail dependence in observed data and few is useful for developing limit theorems of tail dependent time series. This is similar to the correlation-based dependence case, in which the correlation coefficient is prominent in summarizing the underlying degree of dependence, but generally does not lead to sufficient conditions for limit theorems to be developed. To achieve the latter, a popular approach is due to the influential work of Rosenblatt (1956) which introduced the notion of strong mixing along with a “big block-small block” argument that have led to the development of various limit theorems under the strong mixing condition. Although not being originally developed for handling dependence in the tail, the strong mixing framework has been applied to the tail setting as a major tool for developing limit theorems; see for example Smith and Weissman (1994), Drees (2003), Ferro and Segers (2003), Chernozhukov (2005), Davis and Mikosch (2009), Chernozhukov and Fernández-Val (2011), Davis et al. (2012), Mikosch and Zhao (2014), Kulik and Soulier (2020), and references therein. To handle tail statistics from time series data, however, the strong mixing condition often has to be used together with additional anti-clustering conditions that control more specifically the degree of dependence in the tail; see for example condition (9.67) of Chernozhukov (2005), condition (3.3) of Davis and Mikosch (2009), Assumption 4 of Chernozhukov and Fernández-Val (2011), and condition (2.4) of Mikosch and Zhao (2014), among others. Such conditions may lead to more restrictions on the underlying dependence than the strong mixing condition itself.

Recently, Zhang (2021a) introduced the notion of tail adversarial stability, which provides an alternative framework for developing asymptotic theories of analysis of tail dependent time series. Compared with the traditional strong mixing framework that involves a supremum distance between two sigma algebras, the tail adversarial stability framework of Zhang (2021a) relies on the tail adversarial effect of coupled innovations expressed as a conditional probability, which is much more tractable. It has been shown in Zhang (2021a) and Zhang (2022) that the tail adversarial stability measure can be easily calculated for the max-linear (or say moving-maximum) processes (see, e.g., Hsing (1986) and Hall et al. (2002)), and can lead to cleaner and weaker conditions than the traditional strong mixing framework. Besides the max-linear process and its variants covered in the recent review of Zhang (2021b), the additive autoregressive and moving-average (ARMA) model with heavy-tailed innovations has also been a popular choice for modeling time series with tail dependence. The practical meaning of tail adversarial stability for such additively

structured processes, however, has not been addressed so far. This article aims to fill this gap by verifying the tail adversarial stability condition for regularly varying (additive) linear processes, or say moving-average processes. Note that the causal ARMA processes are covered due to their moving-average representations. In particular, we develop probabilistic bounds that are uniform across all lags at which the underlying process is decoupled (Lemma 3.4 and Corollary 3.8). Based on these bounds, we then obtain a bound of the tail adversarial stability measure in terms of the coefficients of the linear process (Theorem 3.9). We hence are able to verify the tail adversarial stability condition under relatively mild assumptions (Corollary 3.10). Parallel results for their stochastic volatility extensions are presented in Theorem 4.3 and Corollary 4.4. We also revisit the the class of max-linear processes, which extends the additive linear process to its maximal counterpart.

The remaining of the article is organized as follows. Section 2 reviews the tail adversarial stability measure. Section 3 provides an explicit calculation of the tail adversarial stability measure for regularly varying linear processes. Section 4 concerns an extension to stochastic volatility type models that are driven by regularly varying linear processes. Section 5 revisits the max-linear process when the innovations are general regularly varying random variables. Section 6 addresses the invariance of tail adversarial stability under monotone transforms. Section 7 discusses some implications of our results in statistical context. Section 8 concludes the paper.

2 Tail Adversarial Stability

The notion of tail adversarial stability is formulated for a stationary process $X = (X_i)$ of the following form

$$X_i = g(e_i, e_{i-1}, \dots, e_1, e_0, e_{-1}, \dots), \quad i = 0, 1, 2, \dots, \quad (1)$$

where g is a measurable function, $(e_j)_{j \in \mathbb{Z}}$ is a sequence of independent and identically distributed (i.i.d.) random variables, or more generally, random elements such as random vectors. The framework (1) covers a wide range of common linear and nonlinear time series models. Let e_0^* be an innovation that has the same distribution as e_0 but independent of $(e_j)_{j \in \mathbb{Z}}$. Then

$$X_i^* = g(e_i, e_{i-1}, \dots, e_1, e_0^*, e_{-1}, \dots), \quad i \geq 0$$

represents the coupled observation at time i whose innovation at time zero is replaced by an i.i.d. copy. The difference between X_i and the coupled version X_i^* , measured through, e.g., the L^p norm $\|X_i - X_i^*\|_p$, leads to the *physical or functional dependence measure* formulated by Wu (2005). It has since been applied extensively for establishing various asymptotic results for

analysis of dependent data; see for example [Wu \(2005\)](#), [Wu \(2007\)](#), [Liu and Wu \(2010\)](#), [Zhou and Wu \(2010\)](#), [Zhang and Wu \(2011\)](#), [Zhang \(2013\)](#), and [Zhang \(2015\)](#), among others.

Let $\mathcal{U}_X = \inf\{x \in \mathbb{R} : \mathbb{P}(X_0 \leq x) = 1\} \in (-\infty, +\infty]$ be the upper end point of the marginal distribution of X_0 . Now the *tail adversarial stability* (TAS) measure, first introduced in [Zhang \(2021a\)](#) (although in a triangular array setting), is given by

$$\theta_y(i) = \theta_y^{(X)}(i) = \sup_{z \geq y} \mathbb{P}(X_i^* \leq z \mid X_i > z), \quad y < \mathcal{U}_X, \quad (2)$$

which quantifies the adversarial effect of a perturbation of the innovation e_0 has on whether the observed data at time i is an upper-tail observation. Note that if X_i does not depend on e_0 , then $X_i^* = X_i$ and $\theta_y(i) = 0$, meaning that e_0 will not have any tail adversarial effect on X_i . Let

$$\Theta_{y,q} = \Theta_{y,q}^{(X)} = \sum_{i=0}^{\infty} \{\theta_y(i)\}^{1/q}, \quad q > 0, \quad (3)$$

which measures the cumulative tail adversarial effect of e_0 on all future observations. Then the process (X_i) is said to be *tail adversarial q -stable* or $(X_i) \in \text{TAS}_q$, if

$$\lim_{y \uparrow \mathcal{U}_X} \Theta_{y,q} < \infty. \quad (4)$$

[Zhang \(2021a\)](#) obtained the consistency and central limit theorem for high quantile regression estimators when the underlying process is TAS_2 . [Zhang \(2022\)](#) established the consistency and central limit theorem for sample tail autocorrelations when $(X_i) \in \text{TAS}_q$ for some $q > 4$. We anticipate that more statistical theories for analysis of tail-dependent time series can be developed under the TAS framework.

In this article, we shall mostly consider the case where $\mathbb{P}(X_0 > z) > 0$ for all $z > 0$, that is, the distribution of X_i is unbounded on the positive side so that $\mathcal{U}_X = \infty$, for which it suffices to consider $y > 0$ in [\(2\)](#).

3 Regularly Varying Linear Processes

3.1 Basic Setup

Let $(a_j)_{j \geq 0}$ be a sequence of real coefficients, we consider the additive linear process

$$X_i = \sum_{j=0}^{\infty} a_j \epsilon_{i-j}, \quad i \in \mathbb{Z}, \quad (5)$$

where $(\epsilon_j)_{j \in \mathbb{Z}}$ is a sequence of i.i.d. innovation random variables. It is also known as a (possibly infinite-order) moving-average process, which includes finite-order ARMA processes as special

cases due to their moving-average representations. The process (5) is covered by the form in (1) when we set the function g to be linear which is measurable (see, e.g., Samorodnitsky (2016, Example 2.1.9)), and the random elements $e_i = \epsilon_i$, $i \in \mathbb{Z}$.

We assume that $|\epsilon_0|$ is regularly varying with index $-\nu$, that is,

$$\mathbb{P}(|\epsilon_0| > x) = x^{-\nu} \ell(x), \quad (6)$$

where the function $\ell(\cdot)$ is slowly varying at ∞ , namely, $\ell(\cdot)$ is a positive function such that $\lim_{z \rightarrow \infty} \ell(\lambda z)/\ell(z) = 1$ for any $\lambda > 0$. See Bingham et al. (1989) for more details about regularly and slowly varying functions. In addition, we assume that the distribution of ϵ_0 satisfies a tail balance condition:

$$\lim_{x \rightarrow \infty} \frac{\mathbb{P}(\epsilon_0 > x)}{\mathbb{P}(|\epsilon_0| > x)} = p \quad (7)$$

for some $p \in [0, 1]$. This is equivalent to assuming that $\lim_{x \rightarrow \infty} \mathbb{P}(\epsilon_0 < -x)/\mathbb{P}(|\epsilon_0| > x) = 1 - p$. A distribution of ϵ_0 satisfying both (6) and (7) is often known as being *balanced regularly varying*, where p is the *tail balance parameter*. Note that if ϵ_0 comes from a positive distribution such as the Fréchet distribution, then for any $x > 0$, $\mathbb{P}(\epsilon_0 > x) = \mathbb{P}(|\epsilon_0| > x)$ and hence it has $p = 1$.

On the other hand, for the coefficient sequence (a_j) in (5), we assume that for some $\varepsilon > 0$,

$$\sum_{j=0}^{\infty} |a_j|^{\frac{\nu}{\nu+2}-\varepsilon} < \infty. \quad (8)$$

Since the exponent $\frac{\nu}{\nu+2} - \varepsilon < \min(1, \nu)$, the random series in (5) converges almost surely and hence the resulting linear process is well defined; see for example Samorodnitsky (2016, Corollary 4.2.12). It is worth noting that as $\nu \rightarrow \infty$ and $\varepsilon \rightarrow 0$, the condition (8) approaches $\sum_{j=0}^{\infty} |a_j| < \infty$, a well-known condition of short memory for linear processes. The condition (8) is imposed for establishing Lemma 3.4, an important uniform estimate for verifying the TAS condition for general regularly varying linear processes. On the other hand, we shall mention in Remark 3.7 below that when (ϵ_j) are ν -stable innovations (here $\nu \in (0, 2)$), the restriction (8) can be relaxed. We also assume without loss of generality that $a_j \neq 0$ for infinitely many $j \geq 1$. The case when $a_j \neq 0$ for finitely many $j \geq 1$ degenerates to an m -dependent process, for which $\Theta_{y,q} \leq m + 1$ and the process trivially belongs to TAS_q for any $q > 0$.

Given the assumptions made above, it is known (e.g., Samorodnitsky (2016, Corollary 4.2.12)) that each X_i is also balanced regularly varying with index $-\nu$ and

$$\lim_{x \rightarrow \infty} \frac{\mathbb{P}(X_0 > x)}{\mathbb{P}(|\epsilon_0| > x)} = \sum_{j \geq 0} \{p(a_j)_+^\nu + (1-p)(a_j)_-^\nu\} \quad (9)$$

as $x \rightarrow \infty$, where $(\cdot)_+$ and $(\cdot)_-$ stand for positive and negative part respectively.

We make an additional mild assumption: the density f_ϵ of ϵ_0 exists and satisfies for any $\delta \in (0, \nu + 1)$, there exists a constant $c_0 > 0$, such that

$$f_\epsilon(x) \leq c_0 \min(1, |x|^{-\nu-1+\delta}). \quad (10)$$

Remark 3.1. *In view of Karamata's Theorem ([Bingham et al. \(1989\)](#), Proposition 1.5.8)) and Potter's bound ([Bingham et al. \(1989\)](#), Theorem 1.5.6)), all the assumptions made so far on ϵ_0 are satisfied if f_ϵ is bounded, and either both $f_\epsilon(x)$ and $f_\epsilon(-x)$ are regularly varying with index $-\nu - 1$ as $x \rightarrow \infty$ with $\lim_{x \rightarrow \infty} f_\epsilon(x)/f_\epsilon(-x)$ existent and positive, or $f_\epsilon(x)$ is regularly varying with index $-\nu - 1$ on one side and is of smaller order on the other side (which corresponds to $p = 0$ or 1 in [\(7\)](#)). These conditions cover a broad family of power-law distributions such as Pareto, Fréchet, Student-t, F-distributions (with the numerator degree of freedom ≥ 1) and non-Gaussian stable distributions (including Cauchy).*

3.2 Preparations

Throughout the article we use c to denote a generic positive constant whose value may change from one expression to another. In this section, we collect some important auxiliary results we need for the rest of the article.

The following lemma collects some variants of the Potter's bound useful for handling regularly varying tails. Recall a random variable $Z \geq 0$ is said to be regularly varying with index $-\nu$, $\nu > 0$, if $\lim_{z \rightarrow \infty} \mathbb{P}(Z > \lambda z)/\mathbb{P}(Z > z) = \lambda^{-\nu}$ for any $\lambda > 0$.

Lemma 3.2. *Suppose random variable $Z \geq 0$ is regularly varying with index $-\nu$, $\nu > 0$. Given any fixed $\varepsilon > 0$ (and in addition $\varepsilon < \nu$ for [\(11\)](#) below), $z_0 > 0$ and $x_0 > 0$, there exists a constant $c > 0$, such that*

$$\mathbb{P}(Z > z) \leq cz^{-\nu+\varepsilon}, \quad z > 0, \quad (11)$$

$$\mathbb{P}(Z > z) \geq cz^{-\nu-\varepsilon}, \quad z > z_0, \quad (12)$$

and

$$\frac{\mathbb{P}(xZ > z)}{\mathbb{P}(Z > z)} \leq cx^{\nu-\varepsilon}, \quad z > z_0, \quad x \in [0, x_0]. \quad (13)$$

Proof. The lemma follows readily from [Kulik and Soulier \(2020\)](#), Propositions 1.4.1 and 1.4.2). $\square$

We also need the following fact on the (truncated) moments of regularly varying random variables. Below and throughout, we write $E[Z; A] = \mathbb{E}[Z1_A]$ for random variable Z , event A and indicator 1_A .

Lemma 3.3. Suppose random variable $Z \geq 0$ is regularly varying with index $-\nu$, $\nu > 0$. If $\beta \in (0, \nu)$, then $\mathbb{E}[Z^\beta] < \infty$; if $\beta > \nu$, then

$$\lim_{z \rightarrow \infty} \frac{\mathbb{E}[Z^\beta; Z \leq z]}{z^\beta \mathbb{P}(Z > z)} = \frac{\nu}{\beta - \nu} > 0;$$

In addition,

$$\lim_{z \rightarrow \infty} \frac{\mathbb{E}[Z^\nu; Z \leq z]}{z^{\nu+\varepsilon} \mathbb{P}(Z > z)} = 0$$

for any $\varepsilon > 0$.

Proof. The first two claims directly follows from [Kulik and Soulier \(2020\)](#), Proposition 1.4.6). By [Kulik and Soulier \(2020\)](#), Proposition 1.4.6) again, $\mathbb{E}[Z^\nu; Z \leq z]$ is slowly varying as $z \rightarrow \infty$, and so is $z^\nu \mathbb{P}(Z > z)$. Hence the last conclusion follows from [Bingham et al. \(1989\)](#), Proposition 1.3.6). $\square$

Following Section [2](#), let ϵ_0^* be a random variable with the same distribution as ϵ_0 but independent of $(\epsilon_j)_{j \in \mathbb{Z}}$. Then the coupled version of X_i is

$$X_i^* = X_i - a_i \epsilon_0 + a_i \epsilon_0^* = a_0 \epsilon_i + \cdots + a_{i-1} \epsilon_1 + a_i \epsilon_0^* + a_{i+1} \epsilon_{-1} + \cdots \quad (14)$$

Introduce

$$Y_i = X_i - a_i \epsilon_0 = \sum_{j \geq 0, j \neq i} a_j \epsilon_{i-j}. \quad (15)$$

and hence

$$X_i = Y_i + a_i \epsilon_0, \quad X_i^* = Y_i + a_i \epsilon_0^*.$$

Below we develop a uniform estimate for the densities of $\{Y_i\}$ which will be the key for establishing the main results. The condition [\(8\)](#) plays an important role in an infinite-order induction argument.

Lemma 3.4. Fix any $\delta \in (0, \nu + 1)$. Suppose that $a_j \neq 0$ for infinitely many $j \geq 0$. Under the assumptions [\(6\)](#), [\(7\)](#), [\(8\)](#) and [\(10\)](#). The density f_i of each Y_i , $i \geq 0$, exists, and we have for all $x \in \mathbb{R}$ and $i \geq 0$,

$$f_i(x) \leq c \min(1, |x|^{-\nu-1+\delta}) \quad (16)$$

for some positive constant $c > 0$ that does not depend on i or x .

Remark 3.5. Observe that [\(16\)](#) is equivalent to imposing both the uniform boundedness $\sup_i \sup_x f_i(x) < \infty$ and the uniform power decay $\sup_i f_i(x) = O(|x|^{-\nu-1+\delta})$ as $|x| \rightarrow \infty$ for any $\delta \in (0, \nu + 1)$.

Proof of Lemma 3.4. We shall assume that $a_j \neq 0$ for every $j \geq 0$. Otherwise, if $a_i = 0$, $i \geq 0$, then $Y_i = X_i$ with f_i being the marginal density of X_0 . The proof below with a slight modification readily covers this case. We also assume that $|a_j| < 1$. Otherwise, apply a proper scaling.

We first prove the existence and the uniform boundedness of f_i . Suppose first $i \geq 1$. Use $P_{i,1}$ to denote the distribution of $Z_i := \sum_{j \geq 1, j \neq i} a_j \epsilon_{i-j}$. In view of Fubini's theorem, the density of $Y_i = a_0 \epsilon_i + Z_i$ exists and can be identified with the convolution

$$f_i(x) := \int_{\mathbb{R}} |a_0|^{-1} f_\epsilon(a_0^{-1}(x-y)) P_{i,1}(dy).$$

Hence by (10), writing $\|g\|_\infty = \sup_{x \in \mathbb{R}} |g(x)|$ for a function g , we have for $i \geq 1$,

$$\|f_i\|_\infty \leq \int |a_0|^{-1} \|f_\epsilon\|_\infty P_{i,1}(dy) = |a_0|^{-1} \|f_\epsilon\|_\infty \leq |a_0|^{-1} c_0 < \infty.$$

The existence and boundedness for f_0 can be obtained similarly by replacing the role of $a_0 \epsilon_i$ by $a_1 \epsilon_{i-1}$ in the argument above. Hence the uniform boundedness required in (16) follows.

Now we turn to the uniform power decay in (16). Recall δ in (10) can be specified arbitrarily small. Since $\min(1, |x|^{-\nu-1+\delta})$ is non-decreasing with respect to $\delta \in (0, \nu+1)$, it suffices to prove (16) for any sufficiently (to be specified later) small $\delta \in (0, \nu)$. We only prove the uniform power decay on the positive side $x > 0$, and the case $x < 0$ follows similarly. Set

$$h_\kappa(x) = c \min(1, |x|^{-\kappa}), \quad \kappa := \nu + 1 - \delta > 1,$$

where $c > 0$ is a constant such that (see (10))

$$f_\epsilon(x) \leq h_\kappa(x). \tag{17}$$

Now for a function $g \geq 0$, we define

$$\mathcal{M}^\kappa g = \sup_{t>0} t^\kappa g(t).$$

To prove the uniform power decay in (16) on the positive side, it suffices to show

$$\sup_{i \geq 0} \mathcal{M}^\kappa f_i < \infty. \tag{18}$$

Suppose first $i \geq 2$.

Below we apply an infinite-order induction argument similar to the proof of Barbe and McCormick (2009, Lemma 6.6.3). For a fixed constant $\rho > 0$ to be specified later, let

$$d_j = |a_j|^\rho \in (0, 1), \quad j \geq 1.$$

Let $g_{i,n}$ be the density of $\sum_{0 \leq j \leq n, j \neq i} a_j \epsilon_{i-j}$, $n \geq 0$, the truncated approximation of Y_i . Note that $g_{i,0}(\cdot) = |a_0|^{-1} f_\epsilon(a_0^{-1} \cdot)$ since we have supposed $i \geq 2$.

If $n \neq i$ and $n \geq 1$, we decompose the convolution $g_{i,n} = g_{i,n-1} * (|a_n|^{-1} f_\epsilon(a_n^{-1} \cdot))$ as

$$g_{i,n}(x) = \int_{-\infty}^{d_n x} g_{i,n-1}(x-y) |a_n|^{-1} f_\epsilon(a_n^{-1} y) dy + \int_{-\infty}^{(1-d_n)x} g_{i,n-1}(y) |a_n|^{-1} f_\epsilon(a_n^{-1}(x-y)) dy. \quad (19)$$

Note that

$$\sup_{y < d_n x} g_{i,n-1}(x-y) \leq \left(\sup_{t > (1-d_n)x} t^{-\kappa} \right) \left(\sup_{t > (1-d_n)x} t^\kappa g_{i,n-1}(t) \right) \leq (1-d_n)^{-\kappa} x^{-\kappa} \mathcal{M}^\kappa g_{i,n-1}.$$

Therefore, since $\int_{\mathbb{R}} |a_n|^{-1} f_\epsilon(a_n^{-1} y) dy = 1$, we have for all $x > 0$ that

$$\int_{-\infty}^{d_n x} g_{i,n-1}(x-y) |a_n|^{-1} f_\epsilon(a_n^{-1} y) dy \leq (1-d_n)^{-\kappa} x^{-\kappa} \mathcal{M}^\kappa g_{i,n-1}.$$

On the other hand, by the bound (17) and the symmetry of h_κ , we have $\sup_{y < (1-d_n)x} f_\epsilon(a_n^{-1}(x-y)) \leq \sup_{y < (1-d_n)x} h_\kappa(|a_n|^{-1}(x-y)) \leq c_0 |a_n|^{-1} d_n x)^{-\kappa}$. Hence using $\int_{\mathbb{R}} g_{i,n-1}(y) dy = 1$, we have for all $x > 0$ that

$$\int_{-\infty}^{(1-d_n)x} g_{i,n-1}(y) |a_n|^{-1} f_\epsilon(a_n^{-1}(x-y)) dy \leq c |a_n|^{\kappa-1} d_n^{-\kappa} x^{-\kappa}.$$

Applying the two displayed bounds above to (19), we conclude that

$$\mathcal{M}^\kappa g_{i,n} \leq (1-d_n)^{-\kappa} \mathcal{M}^\kappa g_{i,n-1} + c |a_n|^{\kappa-1} d_n^{-\kappa}. \quad (20)$$

If $n = i \geq 2$, then $g_{i,n} = g_{i,n-1}$, and the bound above trivially follows from monotonicity.

Define for $1 \leq j+1 \leq n$ that

$$B_{j,n} = \prod_{j+1 \leq \ell \leq n} (1-d_\ell)^{-\kappa}.$$

Set also $B_{n,n} = 1$. Now by an induction based on the recursive bound (20), it can be verified that for all $n \geq 1$,

$$\mathcal{M}^\kappa g_{i,n} \leq B_{0,n} \mathcal{M}^\kappa g_{i,0} + c \left(\sum_{j=1}^n B_{j,n} |a_j|^{\kappa-1} d_j^{-\kappa} \right).$$

Note that $B_{j,n}$ increases as j decreases or as n increases. In view of the monotonicity, we have

$$\mathcal{M}^\kappa g_{i,n} \leq B \mathcal{M}^\kappa g_{i,0} + c B \left(\sum_{j=1}^{\infty} |a_j|^{\kappa-1} d_j^{-\kappa} \right), \quad (21)$$

where $B = \lim_n B_{0,n}$. Now set $\rho = \kappa(1 - \rho) - 1$ which implies

$$\rho = \frac{\kappa - 1}{\kappa + 1} = \frac{\nu - \delta}{\nu + 2 - \delta}.$$

By assumption (8) with $\delta > 0$ chosen sufficiently small so that $\rho \geq \frac{\nu}{\nu+2} - \varepsilon$, we have

$$\sum_{j=0}^{\infty} d_j = \sum_{j=0}^{\infty} |a_j|^\rho < \infty, \quad \text{and} \quad \sum_{j=1}^{\infty} |a_j|^{\kappa-1} d_j^{-\kappa} = \sum_{j=1}^{\infty} |a_j|^{\kappa(1-\rho)-1} < \infty.$$

Note that the summability $\sum_{j=0}^{\infty} d_j < \infty$ implies $B \in (0, \infty)$. Recall $g_{i,0}(\cdot) = |a_0|^{-1} f_\epsilon(a_0^{-1} \cdot)$, and hence by (17) and (21) we have

$$\sup_{n \geq 1, i \geq 2} \mathcal{M}^\kappa g_{i,n} < \infty. \quad (22)$$

Let $P_{i,2,n}$ denote the distribution of $Z_{i,2,n} := \sum_{2 \leq j \leq n, j \neq i} a_j \epsilon_{i-j}$, $n \geq 2$. The a.s. convergence of $Z_{i,2,n}$ to $Z_{i,2} := \sum_{j \geq 2, j \neq i} a_j \epsilon_{i-j}$ implies the weak convergence of $P_{i,2,n} \Rightarrow P_{i,2}$ as $n \rightarrow \infty$, where $P_{i,2}$ is the distribution of $Z_{i,2}$. On the other hand, the function $g_{i,1}$ (recall $i \geq 2$), as a convolution between two bounded integrable functions $|a_0|^{-1} f_\epsilon(a_0^{-1} \cdot)$ and $|a_1|^{-1} f_\epsilon(a_1^{-1} \cdot)$, is bounded and continuous (e.g., Bogachev (2007), Corollary 3.9.6). Hence for any $x \in \mathbb{R}$ and $n \geq 1$, we have for all $x > 0$ that

$$g_{i,n}(x) = \int_{\mathbb{R}} g_{i,1}(x-y) P_{i,2,n}(dy) \rightarrow \int_{\mathbb{R}} g_{i,1}(x-y) P_{i,2}(dy) = f_i(x) \quad (23)$$

as $n \rightarrow \infty$. So combining (23) with (22), we conclude that

$$\sup_{i \geq 2} \mathcal{M}^\kappa f_i < \infty. \quad (24)$$

By a similar argument which uses some other indices to replace the roles of $i = 0, 1$ above, we can also show that $\mathcal{M}^\kappa f_i < \infty$ for $i = 0, 1$. This combined with (24) concludes (18). $\square$

Remark 3.6. *It is possible to further improve the tail decay in the bound (16). Under additional assumptions including certain smooth (regular) variation (cf., Bingham et al., 1989, Section 1.8)) conditions on the distribution of ϵ_0 , the remarkable work of Barbe and McCormick (2009) developed uniform asymptotic expansions for the marginal distribution of regularly varying linear series. In particular, their Theorem 2.5.1 implies a sharp uniform bound: there exists $y_0 > 0$, such that for all $y > y_0$,*

$$\sup_{i \geq 0} f_i(y) \leq c|y|^{-\nu-1} \ell(y), \quad i \geq 0,$$

with the slowly varying $\ell(y)$ as in (6). On the other hand, the slightly weaker tail bound in (16) is verified under less stringent assumptions compared to Barbe and McCormick (2009), which suffices for our purposes.

Remark 3.7. The summability assumption (8) is imposed for establishing the uniform power decay estimate in (16). On the other hand, the restriction (8) is likely only an artifact of the current proof. Indeed, consider the case where the innovations (ϵ_i) are standard symmetric ν -stable ($S\nu S$), $\nu \in (0, 2)$, specified by the characteristic function $\mathbb{E}[e^{i\theta\epsilon_0}] = e^{-|\theta|^\nu}$. See, e.g., [Nolan (2020)] for more details. This special case satisfies the balanced regularly variation assumptions (6) and (7). In this case, for the linear process (5) to be well-defined, it suffices to assume $\sum_{i=0}^{\infty} |a_i|^{\nu \wedge 1} < \infty$ (e.g., [Kokoszka and Taqqu (1995)]). In addition, it follows from the sum-stability property of $S\nu S$ distributions that $Y_i \stackrel{d}{=} \left(\sum_{j \geq 0, j \neq i} |a_j|^\nu\right)^{1/\nu} \epsilon_0$, where $\inf_{i \geq 0} \left(\sum_{j \geq 0, j \neq i} |a_j|^\nu\right)^{1/\nu} > 0$. This implies a uniform bound

$$f_i(y) \leq c \min(1, |y|^{-\nu-1})$$

for all $i \geq 0$ (cf., e.g., Sections 1.4 and 1.5 of [Nolan (2020)]). The discussion above is generalizable to non-symmetric stable distributions which for simplicity is omitted.

Corollary 3.8. Fix any $\delta \in (0, \nu + 1)$. Under the same assumptions as Lemma 3.4, for all $-|z|/2 \leq v \leq u \leq |z|/2$, $z \in \mathbb{R}$ and $i \geq 0$, we have

$$\mathbb{P}(Y_i \in [z - u, z - v]) \leq c(u - v) \min(1, |z|^{-\nu-1+\delta}).$$

for some constants $c > 0$.

Proof. Write

$$\mathbb{P}(Y_i \in [z - u, z - v]) = \int_{z-u}^{z-v} f_i(y) dy.$$

By Lemma 3.4, we have the constant bound $f_i(y) \leq c$ and the power-law bound $f_i(y) \leq c|y|^{-\nu-1+\delta}$ for all $i \geq 0$. The constant bound yields

$$\mathbb{P}(Y_i \in [z - u, z - v]) \leq c(u - v).$$

The power-law bound combined with the restriction on u and v yields

$$\mathbb{P}(Y_i \in [z - u, z - v]) \leq c \int_{z-u}^{z-v} |y|^{-\nu-1+\delta} dy \leq c \int_{z-u}^{z-v} |z/2|^{-\nu-1+\delta} dy \leq c|z|^{-\nu-1+\delta}(u - v).$$

Combining the bounds concludes the proof. $\square$

3.3 Verification of TAS Condition

We shall provide an explicit bound of the TAS measure (2) for the linear process (5) in Theorem 3.9 below. The bound enables an immediate verification of TAS_q in Corollary 3.10 below.

Below is the main result.

Theorem 3.9. Suppose (X_i) is a linear process as in (5) with i.i.d. innovations (ϵ_i) satisfying (6), (7) and (10). Assume the coefficients $a_i \neq 0$ for infinitely many $j \geq 0$, the summability condition (8) holds, and the right-hand side of (9) is nonzero. Fix an arbitrary $\eta \in (0, \nu)$ when $\nu \leq 1$. There exist constants $c > 0$ and $y_0 > 0$, such that for all $y > y_0$ and $i \geq 0$, the TAS measure $\theta_y(i)$ in (2) satisfies

$$\theta_y(i) \leq \begin{cases} c|a_i|, & \nu > 1, \\ c|a_i|^\eta, & \nu \leq 1, \end{cases}$$

In fact, Theorem 3.9 follows from Theorem 4.3 below in the special case where each $S_i = 1$ in (28). For convenience, we include a separate and more transparent proof for this special case below.

Proof. Assume without loss of generality that every $a_j \neq 0$ and let $z > 0$. Below $y_0 > 0$ is a constant which does not depend on $i \geq 0$, whose value may be increased if necessary each time when mentioned. Write

$$\begin{aligned} \mathbb{P}(X_i^* \leq z \mid X_i > z) &= \frac{\mathbb{P}(z - a_i \epsilon_0 < Y_i \leq z - a_i \epsilon_0^*)}{\mathbb{P}(X_i > z)} \\ &\leq \frac{\mathbb{P}(z - a_i \epsilon_0 < Y_i \leq z - a_i \epsilon_0^*, -z/2 \leq a_i \epsilon_0^* < a_i \epsilon_0 \leq z/2)}{\mathbb{P}(X_0 > z)} + \frac{P(|a_i \epsilon_0| > z/2)}{\mathbb{P}(X_0 > z)} \\ &=: A_i(z) + B_i(z). \end{aligned}$$

- Suppose $\nu > 1$.

Recall that $\mathbb{P}(X_0 > z)$ is regularly varying with index $-\nu$ in view of (9). Hence by Potter's bound (12), for any chosen $\delta \in (0, 1)$, there exists $y_0 > 0$ and constant $c > 0$ such that

$$\mathbb{P}(X_0 > z) \geq cz^{-\nu-1+\delta}$$

for all $z > y_0$. Then by independence and Corollary 3.8 with the same $\delta > 0$, we have for all $z > y_0$,

$$\begin{aligned} A_i(z) &= \int_{-z/2 \leq a_i v < a_i u \leq z/2} \frac{\mathbb{P}(z - a_i u \leq Y_i \leq z - a_i v)}{\mathbb{P}(X_0 > z)} f_\epsilon(u) f_\epsilon(v) du dv \\ &\leq c|a_i| \int_{|a_i u|, |a_i v| \leq z/2} |u - v| f_\epsilon(u) f_\epsilon(v) du dv. \end{aligned}$$

We can bound the integral above as

$$\int_{|a_i u|, |a_i v| \leq z/2} |u - v| f_\epsilon(u) f_\epsilon(v) du dv \leq \mathbb{E}|\epsilon_0 - \epsilon_0^*| \leq 2\mathbb{E}|\epsilon_0|. \quad (25)$$

Hence if $\nu > 1$ under which $\mathbb{E}|\epsilon_0| < \infty$, we have

$$A_i(z) \leq c|a_i|$$

for $z > y_0$. On the other hand, it follows from (9), the restriction $\nu > 1$ and Potter's bound (13) that for all $z > y_0$,

$$B_i(z) \leq c|a_i|.$$

- Suppose $\nu \in (0, 1]$.

By the Potter's bound (12), for any δ chosen such that $0 < \delta < \eta < \nu$, we have

$$\mathbb{P}(X_0 > z) \geq cz^{-\nu-\eta+\delta}$$

for all $z > y_0$. Then similarly as above, applying Corollary 3.8 with the same δ , we have for $z > y_0$ that

$$A_i(z) \leq c|a_i|z^{\eta-1} \int_{|a_i u|, |a_i v| \leq z/2} |u-v| f_\epsilon(u) f_\epsilon(v) du dv.$$

For all $z > y_0$, the integral above is bounded by

$$2\mathbb{E}[|\epsilon_0|; |\epsilon_0| \leq z/(2|a_i|)] \leq c \left(\frac{z}{|a_i|} \right)^{1-\eta}. \quad (26)$$

The last inequality follows from $\sup_j |a_j| < \infty$, Lemma 3.3 and Potter's bound (11). Then for $z > y_0$,

$$A_i(z) \leq c|a_i|^\eta.$$

On the other hand, it follows from (9) and Potter's bound (13) that for all $z > y_0$,

$$B_i(z) \leq c \frac{\mathbb{P}(|a_i \epsilon_0| > z/2)}{\mathbb{P}(|\epsilon_0| > z)} \leq c|a_i|^\eta.$$

The conclusion follows. □

Under the conditions of the theorem above, we have for any $\eta \in (0, \nu)$ arbitrarily close to ν ,

$$\Theta_{y,q} = \sum_{i=0}^{\infty} \theta_y(i)^{1/q} \leq \begin{cases} c \sum_{i=0}^{\infty} |a_i|^{1/q} & \nu > 1, \\ c \sum_{i=0}^{\infty} |a_i|^{\eta/q} & \nu \leq 1. \end{cases}$$

Combining this with (8), we arrive at the following sufficient condition for the TAS_q condition.

Corollary 3.10. Suppose the assumptions of Theorem 3.9 holds. Then TAS_q condition (4) holds if for some $\varepsilon > 0$,

$$\sum_{i \geq 0} |a_i|^{\vartheta(\nu, q, \varepsilon)} < \infty;$$

where

$$\vartheta(\nu, q, \varepsilon) = \begin{cases} \frac{1}{q}, & \text{when } \nu > 1 \text{ and } \frac{\nu}{\nu+2} > \frac{1}{q}; \\ \frac{\nu}{\nu+2} - \varepsilon, & \text{when } \nu > 1 \text{ and } \frac{\nu}{\nu+2} \leq \frac{1}{q}; \\ \frac{\nu}{q} - \varepsilon, & \text{when } \nu \in (0, 1] \text{ and } \nu < q - 2; \\ \frac{\nu}{\nu+2} - \varepsilon, & \text{when } \nu \in (0, 1] \text{ and } \nu \geq q - 2. \end{cases}$$

Remark 3.11. As mentioned in Remark 3.7, the restriction (8) can be relaxed when the innovations (ϵ_i) are $S\nu S$ random variables, $\nu \in (0, 2)$. In this case, it follows a similar line of argument as the proof of Theorem 3.9 and the properties of stable distributions that the TAS_q condition holds if $\sum_{i \geq 0} |a_i|^{\vartheta(\nu, q, \varepsilon)} < \infty$ but with $\vartheta(\nu, q, \varepsilon)$ in Corollary 3.10 above replaced by

$$\vartheta(\nu, q, \varepsilon) = \begin{cases} \frac{1}{q}, & \text{when } \nu > 1; \\ 1/q - \varepsilon, & \text{when } \nu = 1; \\ \frac{\nu}{q}, & \text{when } \nu < 1; \end{cases}$$

We conjecture that for a large class of regularly varying linear processes, the uniform estimate in Lemma 3.4 holds under less stringent conditions than (8), and that TAS_q holds under conditions close to the one mentioned above for the $S\nu S$ case.

4 A Stochastic Volatility Extension

4.1 Model Setup

Consider the following model of stochastic volatility type. Let $X_i = \sum_{j=0}^{\infty} a_j \epsilon_{i-j}$ be the linear process with innovations ϵ_j balanced regularly varying with index $-\nu < 0$, which satisfies all the assumptions in Section 3.1. Here we allow the right-hand side of (9) to be zero (i.e., left tail of X_i dominates instead). In particular, we have

$$\lim_{x \rightarrow \infty} \frac{\mathbb{P}(X_0 > x)}{\mathbb{P}(|\epsilon_0| > x)} = A_1, \quad \lim_{x \rightarrow \infty} \frac{\mathbb{P}(X_0 < -x)}{\mathbb{P}(|\epsilon_0| > x)} = A_2 \quad (27)$$

where $A_1 = \sum_{j=0}^{\infty} (p(a_j)_+^{\nu} + (1-p)(a_j)_-^{\nu})$ and $A_2 = \sum_{j=0}^{\infty} (p(a_j)_-^{\nu} + (1-p)(a_j)_+^{\nu})$, where either $A_1 > 0$ or $A_2 > 0$ since $A_1 + A_2 = \sum_{j \geq 0} |a_j|^{\nu} > 0$.

Let (S_i) be i.i.d. random variables independent of (ϵ_j) . Then consider the model of stochastic volatility type:

$$R_i = S_i X_i. \quad (28)$$

Note that this follows the causal process form [\(1\)](#) with $e_i := (S_i, \epsilon_i)$.

We introduce for notational simplicity $(S, X) \stackrel{d}{=} (S_0, X_0)$ and set $R = SX$. Write also $(\epsilon, S) \stackrel{d}{=} (\epsilon_0, S_0)$. Below Z_+ and Z_- denote the positive and negative parts of random variable Z respectively. We make the following assumption.

Assumption 4.1. *Assume either of the following cases holds.*

(I) *(S has lighter tail than X) For some $\beta > \nu$ that*

$$E|S|^\beta < \infty, \quad (29)$$

and

$$\mathbb{P}(A_1 S_+ + A_2 S_- > 0) > 0; \quad (30)$$

(II) *(S has heavier tail than X) S is balanced regularly varying with index $-\beta$ and tail balance parameter $q := \lim_{x \rightarrow \infty} \mathbb{P}(S > x) / \mathbb{P}(|S| > x) \in [0, 1]$, where $\beta \in (0, \nu)$, and*

$$\mathbb{P}(qX_+ + (1 - q)X_- > 0) > 0; \quad (31)$$

(III) *(S has comparable tail as X) S is balanced regularly varying with index $-\beta$ and tail balance parameter q as above, $\beta = \nu$, and*

$$\text{if } A_2 = 0, q > 0; \text{ if } A_1 = 0, q < 1. \quad (32)$$

Throughout the paper, we write $a_i \sim b_i$ if $a_i/b_i \rightarrow 1$ as $i \rightarrow \infty$.

Remark 4.2. *Under Case (I), by Breiman's Lemma (e.g., [\(Kulik and Soulier, 2020\)](#), Lemma 1.4.3)) and [\(9\)](#) (i.e., [\(Samorodnitsky, 2016\)](#), Corollary 4.2.12)), one has as $z \rightarrow \infty$ that*

$$\mathbb{P}(R > z) = \mathbb{P}(S_+ X_+ > z) + \mathbb{P}(S_- X_- > z) \sim (A_1 \mathbb{E} S_+^\nu + A_2 \mathbb{E} S_-^\nu) \mathbb{P}(|\epsilon_0| > z), \quad (33)$$

where $A_1 \mathbb{E} S_+^\nu + A_2 \mathbb{E} S_-^\nu > 0$ under the assumption [\(30\)](#) and hence $\mathbb{P}(R > z)$ is regularly varying with index $-\nu$ as $z \rightarrow \infty$.

Under (II) when $\beta < \nu$, since $E|X|^{\beta+\gamma} < \infty$ for $\gamma \in (0, \nu - \beta)$ (Lemma [3.3](#)), by Breiman's Lemma similarly as above, we have as $z \rightarrow \infty$ that

$$\mathbb{P}(R > z) = \mathbb{P}(S_+ X_+ > z) + \mathbb{P}(S_- X_- > z) \sim (q \mathbb{E} X_+^\beta + (1 - q) \mathbb{E} X_-^\beta) \mathbb{P}(|S| > z), \quad (34)$$

where $q\mathbb{E}X_+^\beta + (1-q)\mathbb{E}X_-^\beta > 0$ under the assumption (31) and hence $\mathbb{P}(R > z)$ is regularly varying with index $-\beta$ as $z \rightarrow \infty$.

Under (III) when $\beta = \nu$, by (Embrechts and Goldie, 1980, COROLLARY of Theorem 3), the tail $\mathbb{P}(|R| > z)$ is a regularly varying with index $-\nu$ as $z \rightarrow \infty$. However, the same result cannot conclude regular variation of $\mathbb{P}(R > z) = \mathbb{P}(S_+X_+ > z) + \mathbb{P}(S_-X_- > z)$ as $z \rightarrow \infty$ in all the possible cases. For example, when $A_1, A_2 > 0, q = 0$, while $\mathbb{P}(S_+ > z) = o(\mathbb{P}(S_- > z))$, it could happen that $\mathbb{P}(S_+ > z)$ is neither regularly varying nor of smaller order than $\mathbb{P}(X_+ > z)$ as $z \rightarrow \infty$. In this case, (Embrechts and Goldie, 1980, COROLLARY of Theorem 3) is not applicable to conclude the regular variation of $\mathbb{P}(S_+X_+ > z)$, although the regular variation of $\mathbb{P}(S_-X_- > z)$ follows.

Note that the condition (32) excludes the special cases $A_2 = q = 0$ or $A_1 = 1 - q = 0$. These two cases introduce some technical difficulty to the current proof. On the other hand, these two special cases possibly allow $R_- = S_+X_- + S_-X_+$ to have a heavier tail than $R_+ = S_+X_+ + S_-X_-$, which is less relevant since the focus is on the right tail of R .

4.2 Verification of TAS Condition

Let R_i^* be as R_i except that e_0 is replaced by an identically distributed copy $e_0^* = (S_i^*, \epsilon_i^*)$ independent of (e_i) . Define as before

$$\theta_y(i) := \sup_{z \geq y} \mathbb{P}(R_i^* \leq z \mid R_i > z) \quad (35)$$

and then

$$\Theta_{y,q} := \sum_{i=0}^{\infty} \theta_y(i)^{1/q}$$

It turns out that the same conclusion as Theorem 3.9 holds for the stochastic volatility extension.

Theorem 4.3. Suppose (R_i) is of the form (28) with (X_i) specified as a linear process in (5) with the coefficient $a_i \neq 0$ for infinitely many $i \geq 0$, satisfying (6), (7), (8) and (10). Suppose also that Assumption 4.1 holds. Fix an arbitrary $\eta \in (0, \nu)$ when $\nu \leq 1$. There exist constants $c > 0$ and $y_0 > 0$, such that for all large $y \geq y_0$ and $i \geq 0$, the TAS measure $\theta_y(i)$ in (35) satisfies

$$\theta_y(i) \leq \begin{cases} c|a_i| & \nu > 1, \\ c|a_i|^\eta & \nu \leq 1. \end{cases}$$

Proof. Assume without loss of generality $S_i \neq 0$ a.s. (otherwise condition on $\{S_i \neq 0\}$) and every $a_j \neq 0, j \geq 0$. Recall $(S, X) \stackrel{d}{=} (S_i, X_i)$ and $R = SX$, and write P_S for the distribution of S .

Suppose $z > 0$. Below $y_0 > 0$ is a constant which does not depend on i , whose value may be increased if necessary each time when mentioned. We have

$$\begin{aligned} \mathbb{P}(R_i^* \leq z \mid R_i > z) &= \frac{\mathbb{P}(z - a_i S_i \epsilon_0 < S_i Y_i \leq z - a_i S_i \epsilon_0^*)}{\mathbb{P}(R_i > z)} \\ &\leq \frac{\mathbb{P}(z - a_i S_i \epsilon_0 < S_i Y_i \leq z - a_i S_i \epsilon_0^*, \quad -z/2 \leq a_i S_i \epsilon_0^* < a_i S_i \epsilon_0 \leq z/2)}{\mathbb{P}(R > z)} + \frac{\mathbb{P}(|a_i S \epsilon| > z/2)}{\mathbb{P}(R > z)} \\ &= A_i(z) + B_i(z). \end{aligned} \quad (36)$$

• Suppose $\nu > 1$.

For some $\delta \in (0, 1)$ to be chosen later, the numerator of $A_i(z)$ above can be bounded using Corollary 3.8 as

$$\begin{aligned} &\int_{|s| \in (0, \infty)} P_S(ds) \int_{-z/2 \leq a_i s v < a_i s u \leq z/2} \mathbb{P}(s Y_i \in (z - a_i s u, z - a_i s v]) f_\epsilon(u) f_\epsilon(v) du dv \\ &\leq \int_{|s| \in (0, z]} P_S(ds) \int_{|a_i s v|, |a_i s u| \leq z/2} c |a_i| |u - v| |z/s|^{-\nu-1+\delta} f_\epsilon(u) f_\epsilon(v) du dv \\ &+ \int_{|s| \in (z, \infty)} P_S(ds) \int_{|a_i s v|, |a_i s u| \leq z/2} c |a_i| |u - v| f_\epsilon(u) f_\epsilon(v) du dv \\ &\leq c |a_i| z^{-\nu-1+\delta} \mathbb{E}[|S|^{\nu+1-\delta}; |S| \leq z] + c |a_i| \mathbb{P}(|S| > z), \end{aligned} \quad (37)$$

where in the last inequality above we have applied

$$\int_{|a_i s u|, |a_i s v| \leq z/2} |u - v| f_\epsilon(u) f_\epsilon(v) du dv \leq \mathbb{E}|\epsilon_0 - \epsilon_0^*| \leq 2\mathbb{E}|\epsilon_0|. \quad (38)$$

We consider the Cases (I)~(III) in Assumption 4.1 separately.

Case (I).

Suppose that $\delta \in (0, 1)$ is chosen sufficiently close to 1 so that $\nu + 1 - \delta \in (0, \beta)$ (recall $\nu < \beta$). Then $\mathbb{E}|S|^{\nu+1-\delta} < \infty$ and hence $\mathbb{P}(|S| > z) \leq c z^{-\nu-1+\delta}$ by Markov inequality. Note that $\mathbb{P}(R > z) \geq c z^{-\nu-1+\delta}$ when $z > y_0$, which is a consequence of regular variation of $\mathbb{P}(R > z) \sim c \mathbb{P}(|\epsilon| > z)$ of index $-\nu$ as $z \rightarrow \infty$ as described in (33) and Potter's bound (12). Combining these facts to (37) we conclude that for $z > y_0$,

$$A_i(z) \leq c |a_i|.$$

Next, observe that $\mathbb{P}(|S \epsilon| > z) \sim \mathbb{E}|S|^\nu \mathbb{P}(|\epsilon| > z)$ as $z \rightarrow \infty$ by Breiman's Lemma. This implies that for all $z > 0$ and $i \geq 0$, we have $\mathbb{P}(|a_i S \epsilon| > z/2) \leq c \mathbb{P}(|a_i \epsilon| > z/2)$ for some large enough constant $c > 0$. Combining this with the aforementioned fact $\mathbb{P}(R > z) \sim c \mathbb{P}(|\epsilon| > z)$ as $z \rightarrow \infty$, we have for $z > y_0$ that

$$B_i(z) \leq c \frac{\mathbb{P}(|a_i \epsilon| > z/2)}{\mathbb{P}(|\epsilon| > z)} \leq c |a_i|, \quad (39)$$

where in the last inequality we have applied Potter's bound (13) and fact $\sup_{j \geq 0} |a_j| < \infty$, as well as the restriction $\nu > 1$. So putting these together we have for all $z > y_0$,

$$\mathbb{P}(R_i^* \leq z \mid R_i > z) \leq c|a_i|. \quad (40)$$

Case (II).

In this case, the tail of S is regularly varying with index $-\beta > -\nu$, and the choice $\delta \in (0, 1)$ always ensures $\beta < \nu + 1 - \delta$. By Lemma 3.3, the truncated moment in the first term of (37) satisfies for $z > y_0$ that

$$\mathbb{E}[|S|^{\nu+1-\delta}; |S| \leq z] \leq cz^{\nu+1-\delta} \mathbb{P}(|S| > z). \quad (41)$$

By (34), we have $\mathbb{P}(R > z) \sim c\mathbb{P}(|S| > z)$ as $z \rightarrow \infty$. Combining these facts, one has for $z > y_0$ that

$$A_i(z) \leq c|a_i|.$$

On the other hand, by Breiman's Lemma, $\mathbb{P}(|S\epsilon| > z) \sim \mathbb{E}|\epsilon|^\beta \mathbb{P}(|S| > z)$ as $z \rightarrow \infty$. Hence arguing similarly as Case (I) above, we have for $z > y_0$,

$$B_i(z) \leq c \frac{\mathbb{P}(|a_i S| > z/2)}{\mathbb{P}(|S| > z)} \leq c|a_i|,$$

So (40) holds in this case as well.

Case (III).

Now $\beta = \nu$. We claim that there exists a constant $c > 0$ such that

$$\mathbb{P}(R > z) \geq c\mathbb{P}(|S\epsilon| > z) \quad (42)$$

for all $z > 0$. We prove this below. First we consider the case both $A_1, A_2 > 0$. In view of (27), for some small enough constant $c > 0$, we have $\mathbb{P}(X_+ > z) \geq c\mathbb{P}(|\epsilon| > z)$ and $\mathbb{P}(X_- > z) \geq c\mathbb{P}(|\epsilon| > z)$ for all $z > 0$. This by independence implies that $\mathbb{P}(S_+ X_+ > z) \geq c\mathbb{P}(S_+ |\epsilon| > z)$ and $\mathbb{P}(S_- X_- > z) \geq c\mathbb{P}(S_- |\epsilon| > z)$. So

$$\mathbb{P}(R > z) = \mathbb{P}(S_+ X_+ > z) + \mathbb{P}(S_- X_- > z) \geq c\mathbb{P}((S_+ + S_-)|\epsilon| > z) = c\mathbb{P}(|S\epsilon| > z).$$

Now consider the case $A_2 = 0$ and $q > 0$. The other case where $A_1 = 0$ and $q < 1$ is similar and will be omitted. Since $A_2 = 0$ implies $A_1 > 0$ and $q > 0$, we have $\mathbb{P}(X_+ > z) \geq c\mathbb{P}(|\epsilon| > z)$ and $\mathbb{P}(S_+ > z) \geq c\mathbb{P}(|S| > z)$ for some small enough constant $c > 0$. Hence by independence, we have $\mathbb{P}(S_+ X_+ > z) \geq c\mathbb{P}(S_+ |\epsilon| > z)$ and $\mathbb{P}(S_+ |\epsilon| > z) \geq c\mathbb{P}(|S\epsilon| > z)$. Therefore, for all $z > 0$,

$$\mathbb{P}(R > z) \geq \mathbb{P}(S_+ X_+ > z) \geq c\mathbb{P}(|S\epsilon| > z).$$

Hence (42) is concluded.

Now in view of (37), (41) and (42), for $z > y_0$,

$$A_i(z) \leq c|a_i| \frac{\mathbb{P}(|S| > z)}{\mathbb{P}(|S\epsilon| > z)} \leq c|a_i|, \quad (43)$$

where the last inequality follows from $\mathbb{P}(|S\epsilon| > z) \geq \mathbb{P}(|\epsilon| \geq 1)\mathbb{P}(|S| > z)$. In addition, again by (42), for $z > 0$, we have

$$B_i(z) \leq c \frac{\mathbb{P}(|a_i S \epsilon| > z/2)}{\mathbb{P}(|S\epsilon| > z)}. \quad (44)$$

By (Embrechts and Goldie, 1980, COROLLARY of Theorem 3), $\mathbb{P}(|S\epsilon| > z)$ is regularly varying with index $-\nu = -\beta < -1$ as $z \rightarrow \infty$. So by Potter's bound (13), when $z > y_0$,

$$B_i(z) \leq c|a_i|.$$

So (40) holds in Case (III) as well.

- Suppose $\nu \leq 1$.

Start as the case $\nu > 1$ until the step before (37). Note that now (38) may not be applicable since $\mathbb{E}|\epsilon|$ is possibly infinite. Instead, applying $|u - v| \leq |u| + |v|$, we bound the last two lines above (37) by

$$\begin{aligned} E_i(z) + F_i(z) &:= \\ c|a_i|z^{-\nu-1+\delta} \mathbb{E} \left[|\epsilon||S|^{\nu+1-\delta}; |a_i \epsilon S| \leq z, |S| \leq z \right] &+ c|a_i| \mathbb{E} [|\epsilon|; |a_i \epsilon S| \leq z, |S| > z], \end{aligned} \quad (45)$$

where we fix $\delta \in (0, \nu)$ to be specified later.

By Lemma 3.3, Potter's bound (11) and the fact $\sup_{j \geq 0} |a_j| < \infty$, with fixed $\eta \in (0, \nu)$, we have for all $|s| \in (0, z]$, $i \geq 0$ and $z > 0$ that

$$\mathbb{E} [|\epsilon|; |\epsilon| \leq z/(s|a_i|)] \leq c(zs^{-1}|a_i|^{-1})^{1-\eta}.$$

Hence by independence and integrating out the randomness of ϵ , for $z > 0$,

$$E_i(z) \leq c|a_i|^\eta z^{-\nu+\delta-\eta} \mathbb{E} \left[|S|^{\nu-\delta+\eta}; |S| \leq z \right]. \quad (46)$$

On the other hand, by independence, Lemma 3.3 and Potter's bound (11), we have for $z > 0$,

$$F_i(z) \leq c|a_i| \mathbb{E} [|\epsilon|; |\epsilon| \leq |a_i|^{-1}] \mathbb{P}(|S| > z) \leq c|a_i|^\eta \mathbb{P}(|S| > z). \quad (47)$$

Case (I).

Now $\nu < \beta$. In this case, choose $\eta \in (\delta, \nu)$ but sufficiently close to δ , so that $\nu + \eta - \delta \leq \beta$. Then $\mathbb{E}|S|^{\nu+\eta-\delta} < \infty$. The bound (46) simplifies to

$$E_i(z) \leq c|a_i|^\eta z^{-\nu+\delta-\eta}.$$

Note, on the other hand, that $\mathbb{P}(R > z) \geq cz^{-\nu+\delta-\eta}$ for $z > y_0$ due to the regular variation of $\mathbb{P}(R > z)$ with index $-\nu$ in (33) and Potter's bound (12) since $\delta - \eta < 0$. Combining these above with (47) and Markov inequality $\mathbb{P}(|S| > z) \leq cz^{-\beta} \leq cz^{-\nu+\delta-\eta}$ when $z > y_0 > 1$, we have for $z > y_0$

$$A_i(z) \leq c|a_i|^\eta.$$

It follows from a similar argument as (39) using Potter's bound (13) that for $z > y_0$,

$$B_i(z) \leq c|a_i|^\eta.$$

Hence for $z > y_0$,

$$\mathbb{P}(R_i^* \leq z \mid R_i > z) \leq c|a_i|^\eta. \quad (48)$$

The conclusion follows by noting that δ and η can be chosen arbitrarily close to ν .

Case (II).

Now $\beta < \nu$. Choosing again $0 < \delta < \eta < \nu$, the bound (46) in view of Lemma 3.3 becomes for $z > y_0$,

$$E_i(z) \leq c|a_i|^\eta \mathbb{P}(|S| > z).$$

This time $\mathbb{P}(R > z) \sim c\mathbb{P}(|S| > z)$ and $\mathbb{P}(|S\epsilon| > z) \sim c\mathbb{P}(|S| > z)$ as $z \rightarrow \infty$ in view of (34) and Breiman's Lemma respectively. Combining these with (47), we can deduce the bound $c|a_i|^\eta$ for $A_i(z)$ when $z > y_0$. The same bound for $B_i(z)$ follows similarly as the case $\nu > 1$. So (48) holds.

Case (III).

For this case we work with a bound different from (46). Decompose the expectation in $E_i(z)$ in (45) into $|\epsilon| \leq 1$ and $|\epsilon| > 1$ parts. Then drop the restriction $|a_i\epsilon S| \leq z$ in the part with $\epsilon \leq 1$. Drop the restriction $|S| \leq z$ and apply the inequality $|\epsilon| \leq |\epsilon|^{\nu+1-\delta}$ in the part with $|\epsilon| > 1$. We then have the bound

$$E_i(z) \leq c|a_i|z^{-\nu-1+\delta}\mathbb{E}[|S|^{\nu+1-\delta}; |S| \leq z] + c|a_i|z^{-\nu-1+\delta}\mathbb{E}[|\epsilon S|^{\nu+1-\delta}; |\epsilon S| \leq z/|a_i|].$$

Both $\mathbb{P}(|S| > z)$ and $\mathbb{P}(|\epsilon S| > z)$ (Embrechts and Goldie (1980)) are regularly varying with index $-\nu = -\beta > -\nu - 1 + \delta$ as $z \rightarrow \infty$. So by Lemma 3.3 and Potter's bound (13), with a fixed $\eta \in (0, \delta)$, we have for any $z > y_0$ that

$$\begin{aligned} E_i(z) &\leq c|a_i|\mathbb{P}(|S| > z) + c|a_i|^{\delta-\nu}\mathbb{P}(|\epsilon S| > z/|a_i|) \\ &\leq c|a_i|\mathbb{P}(|S| > z) + c|a_i|^\eta\mathbb{P}(|\epsilon S| > z). \end{aligned} \quad (49)$$

We have also $\mathbb{P}(|S| > z)\mathbb{P}(|\epsilon| \geq 1) \leq \mathbb{P}(|S\epsilon| > z)$, and $\mathbb{P}(R > z) \geq c\mathbb{P}(|S\epsilon| > z)$ as in (42), both of which hold for all $z > 0$. Combining these with (49) and (47) (choose $\eta \in (0, \min(\delta, \nu))$), the bound $c|a_i|^\eta$ holds for both $A_i(z)$ and $B_i(z)$ when $z > y_0$. Hence for $z > y_0$,

$$\mathbb{P}(R_i^* \leq z \mid R_i > z) \leq c|a_i|^\eta.$$

The conclusion follows since δ and η can be chosen arbitrarily close to ν . $\square$

Corollary 4.4. *Suppose the assumptions of Theorem 4.3 holds. Then the conclusion of Corollary 3.10 continues to hold for the stochastic volatility extension (28).*

Remark 4.5. *Remark 3.11 on the possibility of relaxing the restriction (8) also applies to the stochastic volatility type model (R_i).*

Remark 4.6. *The causal representation (1) covers a wide class of nonlinear time series models beyond the stochastic volatility type models considered in this section, including GARCH, autoregression with random coefficients, nonlinear autoregression, bilinear models, etc. See, for instance, Section 3 of Liu and Lin (2009). The verification of the TAS condition for these models requires nontrivial extensions and is left for future works.*

5 The Max-Linear Extension: A Revisit

In this section, we revisit the max-linear extension that replaces the additive structure in (5) by its maximal counterpart. Davis and Resnick (1989) presented a max-ARMA process that extends the usual additive ARMA process to its extreme-value counterpart. Hall et al. (2002) considered the class of infinite-order moving-maximum processes, and showed that they are dense in the class of stationary processes whose finite-dimensional distributions are extreme-value of a given type. As commented in Zhang (2021b), the additive structure in traditional time series models cannot describe the extremal clusters and tail dependence satisfactorily in many applications, and it seems desirable to consider their non-additive extensions such as the max-linear process. Zhang (2021a) studied the implication of the TAS_q condition on the moving-maximum process of Hall et al. (2002) when the innovation distribution is Fréchet, and we shall here extend their results to the case when the innovations are from a general non-negative regularly varying distribution. In particular, let $(\epsilon_j)_{j \in \mathbb{Z}}$ be i.i.d. non-negative random variables with regularly varying tail:

$$\mathbb{P}(\epsilon_0 > x) = x^{-\nu} \ell(x) \tag{50}$$

for some function ℓ slowly varying at $+\infty$ and $\nu > 0$. Let $\{a_j\}_{j \geq 0}$ be non-negative coefficients such that

$$\sum_{j=0}^{\infty} a_j^{\nu'} < \infty, \quad (51)$$

for some $\nu' \in (0, \nu)$. Then as shown in [Hsing \(1986\)](#), the moving-maximum process

$$X_i = \bigvee_{j=0}^{\infty} a_j \epsilon_{i-j} \quad (52)$$

is a.s. finite and

$$\lim_{x \rightarrow \infty} \frac{\mathbb{P}(X_0 > x)}{\mathbb{P}(\epsilon_0 > x)} = \sum_{j=0}^{\infty} a_j^{\nu}. \quad (53)$$

Following [Section 2](#), let ϵ_0^* be an i.i.d. copy of ϵ_0 which is independent of (ϵ_j) . Define X_i^* as X_i except that ϵ_0 is replaced by ϵ_0^* . Introduce

$$Y_i = \bigvee_{j \geq 0, j \neq i}^{\infty} a_j \epsilon_{i-j}.$$

Then $X_i = Y_i \vee (a_i \epsilon_0)$ and $X_i^* = Y_i \vee (a_i \epsilon_0^*)$. So by [\(53\)](#) and Potter's bound [\(13\)](#), there exists $y_0 > 0$ which does not depend on i , such that for $z \geq y_0$,

$$\begin{aligned} \mathbb{P}(X_i^* \leq z | X_i > z) &= \frac{\mathbb{P}(Y_i \leq z) \mathbb{P}(a_i \epsilon_0^* \leq z) \mathbb{P}(a_i \epsilon_0 > z)}{\mathbb{P}(X_0 > z)} \leq \frac{\mathbb{P}(a_i \epsilon_0 > z)}{\mathbb{P}(X_0 > z)} \\ &\leq c a_i^{\eta}, \end{aligned} \quad (54)$$

where $\eta > 0$ can be fixed arbitrarily close to ν . Hence we have proved the following.

Corollary 5.1. *TAS_q condition [\(4\)](#) holds for the moving-maximum process [\(52\)](#) if $\sum_{i \geq 0} a_i^{\eta/q} < \infty$ for some $\eta \in (0, \nu)$.*

Remark 5.2. *It is possible to slightly improve [\(54\)](#) for certain slowly varying function $\ell(x)$ in [\(50\)](#). For example in [Zhang \(2021a\)](#), the bound [\(54\)](#) can be strengthened to $c a_i^{\nu}$ for Fréchet distribution which leads to the sufficient condition $\sum_{i \geq 0} a_i^{\nu/q} < \infty$ for TAS_q. Similar improvements can also be considered for Theorems [3.9](#) and [4.3](#). We do not pursue such a refinement here since it does not lead to a substantial statistical consequence.*

6 Extensions via Monotone Transforms

Recall a process $X = (X_i)$ given by [\(1\)](#) satisfies the TAS_q condition if $\lim_{y \uparrow \mathcal{U}_X} \Theta_{y,q}^{(X)} < \infty$ (cf. [\(4\)](#)). Under the TAS_q condition, there exists a real

$$x_q^* = \inf\{y < \mathcal{U}_X : \Theta_{y,q}^{(X)} < \infty\} < \mathcal{U}_X.$$

Note that $\Theta_{y,q}^{(X)} < \infty$ for any $y > x_q^*$. The following proposition provides sufficient conditions for TAS_q to carry over through monotonic transforms.

Proposition 6.1. *Suppose a stationary process $X = (X_i)$ is given by the model (1), whose marginal distribution has lower and upper end points $\mathcal{L}_X = \sup\{x \in \mathbb{R} : \mathbb{P}(X_0 \geq x) = 0\}$ and $\mathcal{U}_X = \inf\{x \in \mathbb{R} : \mathbb{P}(X_0 \leq x) = 1\}$ respectively. Suppose X satisfies the TAS_q condition, $q > 0$.*

Let $K : [\mathcal{L}_X, \mathcal{U}_X] \mapsto [-\infty, \infty]$ be a non-decreasing function. Suppose the transformed stationary process $Y = (Y_i) = (K(X_i))$ has marginal upper end point $\mathcal{U}_Y$ satisfying $\mathcal{U}_Y = K(\mathcal{U}_X)$. Then Y satisfies the TAS_q condition under either of the following conditions:

- (a) *The function K is strictly increasing on $(x_0, \mathcal{U}_X]$ for some $x_0 < \mathcal{U}_X$;*
- (b) *We have $x_1 := \inf\{x \in [\mathcal{L}_X, \mathcal{U}_X] : K(x) = \mathcal{U}_Y\} > x_q^*$, and there exists $x_0 < x_1$ such that $P(X_0 = x) = 0$ for all $x \in (x_0, x_1)$.*

Remark 6.2. *Condition (a) says K is ultimately strictly increasing. In Condition (b), note that $x_1 = \mathcal{U}_X$ if $K(x) < \mathcal{U}_Y$ for all $x < \mathcal{U}_X$, under which $x_1 > x_q^*$ always holds if X satisfies TAS_q . The second assumption in Condition (b) imposes ultimate continuity of the marginal distribution X_0 .*

The assumption $\mathcal{U}_Y = K(\mathcal{U}_X)$ is made without loss of generality. In general, it is possible that $K(\mathcal{U}_X) > \mathcal{U}_Y$. But since $\mathbb{P}(Y_0 > \mathcal{U}_Y) = 0$, one may modify the definition of K by a truncation as $K1_{\{K \leq \mathcal{U}_Y\}} + \mathcal{U}_Y 1_{\{K > \mathcal{U}_Y\}}$ without changing Y almost surely.

In the case where K is only defined on the open interval $(\mathcal{L}_X, \mathcal{U}_X)$ (similarly for other half-open-type intervals), one may without loss of generality extend the domain of K to $[\mathcal{L}_X, \mathcal{U}_X]$ by setting $K(\mathcal{L}_X) = \lim_{u \downarrow \mathcal{L}_X} K(x)$ and $K(\mathcal{U}_X) = \lim_{u \uparrow \mathcal{U}_X} K(x)$.

Proof of Proposition 6.1. We follow the notation in Section 2.

(a) Let K^{-1} denote the inverse of K when the latter is restricted to $(x_0, \mathcal{U}_X]$. With $K(x_0, \mathcal{U}_X]$ denoting the image of $(x_0, \mathcal{U}_X]$ under K , observe that $\inf K(x_0, \mathcal{U}_X] < \mathcal{U}_Y$. So with $\inf K(x_0, \mathcal{U}_X] < y < \mathcal{U}_Y$, one has

$$\begin{aligned} \sup_{z \geq y} \mathbb{P}(K(X_i^*) \leq z \mid K(X_i) > z) &\leq \sup_{z \in K(x_0, \mathcal{U}_X]} \mathbb{P}(X_i^* \leq K^{-1}(z) \mid X_i > K^{-1}(z)) \\ &\leq \sup_{u \geq K^{-1}(x_0)} \mathbb{P}(X_i^* \leq u \mid X_i > u) = \theta_{K^{-1}(x_0)}^{(X)}(i). \end{aligned}$$

The conclusion follows if, without loss of generality, x_0 is chosen sufficiently close to $\mathcal{U}_X$ so that $K^{-1}(x_0) > x_q^*$.

(b) For $z \in [\mathcal{L}_Y, \mathcal{U}_Y]$, we define $I_z = \{x \in [\mathcal{L}_X, \mathcal{U}_X] : K(x) \leq z\}$ and $J_z = \{x \in [\mathcal{L}_X, \mathcal{U}_X] : K(x) > z\}$, both of which are intervals due to the monotonicity of K . Set $b(z) = \sup I_z = \inf J_z$, which is non-decreasing in z . Assume without loss of generality $x_0 \in (x_q^*, x_1)$.

We claim that as $z \uparrow \mathcal{U}_Y$, we have $b(z) \uparrow x_1$. Indeed, otherwise, there exists $x'_1 < x_1$ such that $b(z) \leq x'_1$ for any $z < \mathcal{U}_Y$. Hence $K(x) < \mathcal{U}_Y$ implies $x \leq x'_1$, which contradicts with the definition of x_1 .

Now with the claim above, we can choose $y < \mathcal{U}_Y$ sufficiently close to $\mathcal{U}_Y$ so that $x_1 > b(y) > x_0 > x_q^*$. Then applying the assumptions, we have

$$\begin{aligned} \sup_{z \geq y} \mathbb{P}(K(X_i^*) \leq z \mid K(X_i) > z) &= \sup_{z \geq y} \mathbb{P}(X_i^* \in I_z \mid X_i \in J_z) = \sup_{z \geq y} \mathbb{P}(X_i^* \leq b(z) \mid X_i > b(z)) \\ &\leq \sup_{u \geq b(y)} \mathbb{P}(X_i^* \leq u \mid X_i > u), \end{aligned}$$

and the conclusion follows.

Example 6.3. Consider a linear process (X_i) as in (5), which satisfies the TAS_q condition, $q > 0$ (cf. Corollary 3.10 and Remark 3.11). Based on the assumptions made, the marginal distribution of X_0 is typically continuous (cf. the Proof of Lemma 3.4), and we shall assume so.

To model integer-valued tail-dependent data, one may consider $(Y_i) = (K(X_i)) = (\lfloor X_i \rfloor)$, where $K(x) = \lfloor x \rfloor$ is the floor function (i.e., greatest integer not exceeding x). Based on Proposition 6.1, in particular, applying Condition (b) (note that $x_1 = \mathcal{U}_X = \infty$ in this case), the integer-valued process (Y_i) also satisfies TAS_q .

The same consideration applies to the stochastic volatility extension in Section 4 and the max-linear process in Section 5.

□

7 Application: Limit Theorems in Statistical Context

In this section, we provide implications of the developed results on some limit theorems of tail quantities with statistical motivations.

7.1 High Quantile Regression

We first consider the high quantile regression problem studied in Zhang (2021a). Suppose we observe the n -th row of a triangular array which consists of response variables $U_{1,n}, \dots, U_{n,n} \in \mathbb{R}$

associated with a set of explanatory variables $W_{1,n}, \dots, W_{n,n} \in \mathbb{R}^p$ according to the quantile regression model (Koenker and Bassett, 1978)

$$U_{i,n} = W_{i,n}^\top \beta_n + X_{i,n},$$

where $^\top$ denotes the transpose, $\beta_n \in \mathbb{R}^p$ is the regression coefficient for the $(1 - \alpha_n)$ -th quantile, and $X_{i,n} = U_{i,n} - W_{i,n}^\top \beta_n$ is the auxiliary variable satisfying $\mathbb{P}(X_{i,n} \leq 0) = \mathbb{P}(U_{i,n} \leq W_{i,n}^\top \beta_n) = 1 - \alpha_n$. The quantile regression coefficient β_n can then be estimated by the high quantile regression estimator

$$\hat{\beta}_n = \operatorname{argmin}_{\eta \in \mathbb{R}^p} \sum_{i=1}^n \phi_{1-\alpha_n}(U_{i,n} - W_{i,n}^\top \eta), \quad (55)$$

where $\phi_{1-\alpha_n}(u) = (1 - \alpha_n)u^+ + \alpha_n(-u)^+$ is the check function with $u^+ = \max(u, 0)$. Compared with the traditional quantile regression (Koenker and Bassett, 1978, Koenker, 2005), the high quantile regression in (55) requires the quantile level $1 - \alpha_n$ to approach the unit as the sample size increases to capture the tail phenomena. Assuming that the auxiliary process $(X_{i,n}) \in \text{TAS}_2$ (a triangular array variant), under some mild conditions on the smoothness of the marginal distribution and the design matrix, Zhang (2021a) obtained the consistency and the central limit theorem for the high quantile regression estimator (55); see Theorems 1 and 2 of Zhang (2021a).

In many applications, one is interested in estimating a high quantile of a given stationary tail dependent time series, which relates to the situation when $W_{i,n} \equiv 1$. In this case, one observes a stationary time series $(U_{i,n}) = (U_i)$ whose marginal distribution is denoted by $F(u) = \mathbb{P}(U_i \leq u)$, and (55) can still be used to obtain an estimator for the $(1 - \alpha_n)$ -th quantile β_n . We make the following assumption.

- (Q) There exists an $\alpha \in (0, 1)$ such that $F(\cdot)$ is continuously differentiable with uniformly bounded and strictly positive derivative $f(\cdot)$ in its upper tail $\{F^{-1}(1 - \alpha), F^{-1}(1)\}$ with $|F^{-1}(1) - F^{-1}(1 - \alpha)| > 0$.

Assumption (Q) mostly concerns the smoothness of the underlying distribution $F(\cdot)$ in the tail part and is satisfied by many commonly used distributions. Let $X_{i,n} = U_i - \beta_n$ be the associated auxiliary variable, the following theorem provides the consistency and central limit theorem of $\hat{\beta}_n$, which follows from Theorems 1 and 2 of Zhang (2021a), with some of the conditions simplified for the current intercept case.

Theorem 7.1 (Zhang, 2021a). *Assume (Q), $(U_i) \in \text{TAS}_2$, $\alpha_n \rightarrow 0$ and $n\alpha_n \rightarrow \infty$. If*

$$\psi_n = (n\alpha_n)^{1/2} \frac{f_n(0)}{1 - F_n(0)} \rightarrow \infty$$

and

$$\max_{1 \leq i \leq n} \sup_{|\eta| \leq c} \left| \frac{f_n(\psi_n^{-1}\eta) - f_n(0)}{f_n(0)} \right| \rightarrow 0$$

for any $c < \infty$, then

$$\hat{\beta}_n - \beta_n = O_p(\psi_n^{-1}).$$

If in addition the limit

$$\rho_k = \lim_{n \rightarrow \infty} \text{cor}(1_{\{X_{0,n} > 0\}}, 1_{\{X_{k,n} > 0\}})$$

exists for each $k \in \mathbb{Z}$ and $\sum_{k \in \mathbb{Z}} \rho_k > 0$, then

$$\psi_n(\hat{\beta}_n - \beta_n) \rightarrow N\left(0, \sum_{k \in \mathbb{Z}} \rho_k\right).$$

Assumptions concerning $F(\cdot)$ and $f(\cdot)$ in the above theorem can be verified for a number of distribution functions, including the uniform, exponential, normal and Pareto distributions; see for example the discussions in [Zhang \(2021a\)](#). We shall in the following provide a discussion on the tail adversarial stability condition that $(U_i) \in \text{TAS}_2$.

For the linear process [\(5\)](#) with $S\nu S$ innovations, $\nu \in (0, 2)$, by the discussion in [Section 3](#), one can show that the TAS_2 condition needed for high quantile regression inference as in [Zhang \(2021a\)](#) is satisfied if the coefficients

$$a_i \sim ci^{-\zeta}, \quad i \rightarrow \infty,$$

for some $\zeta > \max(2, 2/\nu)$. For more general linear regularly varying process with index $-\nu$, $\nu > 0$, satisfying the assumptions of [Theorem 3.9](#), we need $\zeta > \max(2, 1 + 2/\nu)$ under the power decay condition for a_i above. By [Corollary 4.4](#), this will continue to hold for the stochastic volatility extension given in [\(28\)](#) as well. As a comparison, [Chernozhukov \(2005\)](#) studied high quantile regression under the strong mixing framework of [Rosenblatt \(1956\)](#) and used an additional condition to control the joint probability of nearby tail events. Such a condition can essentially be interpreted as a negligibility condition on tail dependence, and is generally not expected to hold for processes exhibiting nonnegligible tail dependence. Therefore, the TAS framework seems to provide a convenient framework for studying high quantile regression of tail dependent time series data.

7.2 Tail Autocorrelation Analysis

We in this section consider the problem of tail autocorrelation analysis, which extends the traditional autocorrelation analysis to the tail setting. For this, let $x_n \rightarrow \infty$ be an extremal threshold,

then the degree of tail dependence at lag k can be quantified by the conditional probability $\mathbb{P}(X_{1+k} > x_n \mid X_1 > x_n)$ as proposed in [Zhang \(2005\)](#); see also [Linton and Whang \(2007\)](#) when the threshold x_n is represented using quantiles. The tail autocorrelation at lag k is then defined as

$$\tau_{x_n}(k) = \frac{\mathbb{P}(X_{1+k} > x_n \mid X_1 > x_n) - \mathbb{P}(X_{1+k} > x_n)}{1 - \mathbb{P}(X_1 > x_n)},$$

which standardizes the conditional probability $\mathbb{P}(X_{1+k} > x_n \mid X_1 > x_n)$ in the form of a correlation coefficient. [Zhang \(2022\)](#) established the consistency and a two-phase central limit theorem of sample tail autocorrelations under the tail adversarial stability framework, where it was assumed that

(Z1) the underlying process $(X_i) \in \text{TAS}_q$ for some $q > 4$; and

(Z2) the extremal threshold satisfies $\bar{F}(x_n) \rightarrow 0$ and $n\bar{F}(x_n) \rightarrow \infty$,

with $\bar{F}(x_n) = \mathbb{P}(X_1 > x_n)$ being the marginal survival function. By Corollary [3.10](#) with $q > 4$, condition (Z1) holds for the regularly varying linear process [\(5\)](#) if

$$\sum_{i=0}^{\infty} |a_i|^\iota < \infty$$

for some $\iota < \min(\nu/4, 1/4)$. Condition (Z2) is very mild as $\bar{F}(x_n) \rightarrow 0$ only requires the threshold x_n to be in the tail and $n\bar{F}(x_n) \rightarrow \infty$ essentially requires the amount of the data in the tail goes to infinity so that we can have the consistency without assuming any parametric assumption on the tail.

On the other hand, [Davis and Mikosch \(2009\)](#) considered adopting the strong mixing framework and provided a central limit theorem for sample tail autocorrelations in their Corollary 3.4 that aligns with the Phase I result of [Zhang \(2022\)](#). Let $\mathcal{F}_{i,j} = \sigma(X_k, i \leq k \leq j)$ be the σ -field generated by $X_i, \dots, X_j$ for $i \leq j$, it was assumed in [Davis and Mikosch \(2009\)](#) that

(DM1) the underlying process (X_i) is α -mixing and the strong mixing coefficient

$$\alpha(i) = \sup_{A \in \mathcal{F}_{-\infty}^k, B \in \mathcal{F}_{k+i}^\infty} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|$$

satisfies

$$\lim_{n \rightarrow \infty} m_n \sum_{i=r_n}^{\infty} \alpha(i) = 0$$

for some $m_n, r_n \rightarrow \infty$ with $\lim_{n \rightarrow \infty} m_n \mathbb{P}(|X_1| > x_n) = 1$, $m_n/n \rightarrow 0$ and $r_n/m_n \rightarrow 0$;

(DM2) for all $\varpi > 0$,

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} m_n \sum_{i=k}^{r_n} \mathbb{P}(|X_i| > \varpi x_n, |X_0| > \varpi x_n) = 0;$$

and

(DM3) $n\alpha_{r_n}/m_n \rightarrow 0$ and $m_n = o(n^{1/3})$, where $m_n = o(n^{1/3})$ can be replaced by

$$\frac{m_n^4}{n} \sum_{i=r_n}^{m_n} \alpha(i) \rightarrow 0 \text{ and } \frac{m_n r_n^3}{n} \rightarrow 0.$$

We shall here verify conditions (DM1)–(DM3) for the regularly varying linear process (5). Davis and Mikosch (2009) considered the special case of a finite-order ARMA model, for which the coefficient a_i in its linear representation follows a geometric decay. In this case, the strong mixing coefficient also follows a geometric decay, which largely simplified the verification of conditions (DM1)–(DM3). As before, we assume that

$$a_i \sim ci^{-\zeta}, \quad i \rightarrow \infty,$$

for some $\zeta > 0$. For simplicity of illustration, we also assume that the regularly varying innovations in (5) satisfies

$$\nu = 1$$

and $\ell(x) \rightarrow 1$ as $x \rightarrow \infty$ in (6). On the other hand, obtaining a sharp estimate of the strong mixing coefficient is highly nontrivial. The best estimate we can find in literature is Lemma 15.3.1 of Kulik and Soulier (2020) adapted from the results of Pham and Tran (1985). Specifically, assuming that the index $\zeta > 3$, the strong mixing coefficient has the bound

$$\alpha(n) = O\{n^{-(\zeta-1)(1-\varepsilon)/(2-\varepsilon)+1}\}, \quad (56)$$

where $\varepsilon \in (0, 1)$ is a constant that can be taken arbitrarily small. Note that $m_n \sim 1/\bar{F}(x_n)$, condition (DM1) is satisfied if $\bar{F}(x_n) \rightarrow 0$, $n\bar{F}(x_n) \rightarrow \infty$, $r_n \rightarrow \infty$, and

$$r_n \bar{F}(x_n) + r_n^{2-(\zeta-1)(1-\varepsilon)/(2-\varepsilon)} / \bar{F}(x_n) \rightarrow 0. \quad (57)$$

In addition, by a similar argument used in (15.3.33) of Kulik and Soulier (2020), condition (DM2) is satisfied if

$$\sum_{i=0}^{\infty} i |a_i|^{\zeta} < \infty$$

for some $\varsigma \in (0, 1)$. Since $a_i \sim ci^{-\varsigma}$, there exists a compatible $r_n \rightarrow \infty$ such that (DM1) and (DM2) are satisfied if $\varsigma - 1 > 1$ and

$$2 - (\varsigma - 1)(1 - \varepsilon)/(2 - \varepsilon) < -1.$$

By choosing $\varepsilon > 0$ arbitrarily small, the above indicates that $\varsigma > 7$. In contrast, condition (Z1) from the tail adversarial stability framework only requires that $\varsigma > 4$. It is remarkable that the strong mixing framework requires an additional condition (DM3), which typically leads to more restrictive conditions on how extremal the tail can be. For example, the condition $m_n = o(n^{1/3})$ in (DM3) requires that $n\{\bar{F}(x_n)\}^3 \rightarrow \infty$, while in comparison condition (Z2) from the tail adversarial stability framework only requires that $n\bar{F}(x_n) \rightarrow \infty$. Note that the condition $m_n = o(n^{1/3})$ in (DM3) can be replaced by its alternative $(m_n^4/n) \sum_{i=r_n}^{m_n} \alpha(i) \rightarrow 0$ and $m_n r_n^3/n \rightarrow 0$, for which by (56) and Karamata's theorem it suffices to have

$$\frac{r_n^{2-(\varsigma-1)(1-\varepsilon)/(2-\varepsilon)}}{n\{\bar{F}(x_n)\}^4} \rightarrow 0 \text{ and } \frac{r_n^3}{n\bar{F}(x_n)} \rightarrow 0.$$

This, together with (57), make it difficult to work out the actual condition as it depends on the nontrivial interplay between how extremal the tail can be and how fast the linear coefficients decay to zero. Since $r_n \bar{F}(x_n) \rightarrow 0$ and $r_n^{(\varsigma-1)(1-\varepsilon)/(2-\varepsilon)-2} \bar{F}(x_n) \rightarrow \infty$ by (57), it is then necessary, though probably not sufficient, to have

$$n\{\bar{F}(x_n)\}^{6-(\varsigma-1)(1-\varepsilon)/(2-\varepsilon)} \rightarrow \infty \text{ and } n\{\bar{F}(x_n)\}^{1+3/\{(\varsigma-1)(1-\varepsilon)/(2-\varepsilon)-2\}} \rightarrow \infty,$$

which is still stronger than condition (Z2) from the tail adversarial stability framework. We also remark that the condition $n\alpha_{r_n}/m_n \sim n\bar{F}(x_n)\alpha_{r_n} \rightarrow 0$ in (DM3) prevents $\bar{F}(x_n)$ from going to zero too slowly, while condition (Z2) only requires that $\bar{F}(x_n) \rightarrow 0$. Therefore, in addition to being more tractable, the tail adversarial stability framework can also lead to cleaner and weaker conditions on not only how strong the tail dependence can be but also how extremal the tail can be.

7.3 Tail Empirical Distribution

We in this section consider estimating the tail probability $T(x_n) := \bar{F}(x_n) = \mathbb{P}(X_i > x_n)$ by its empirical version

$$\hat{T}(x_n) = n^{-1} \sum_{i=1}^n 1_{\{X_i > x_n\}}$$

when the threshold $x_n \rightarrow \infty$ satisfying also $\mathbb{E}[n\hat{T}(x_n)] = n\bar{F}(x_n) \rightarrow \infty$. For simplicity we assume $T(x) \sim cx^{-\nu}$ as $x \rightarrow \infty$, and hence the aforementioned condition becomes $x_n \ll n^{1/\nu}$ as $n \rightarrow \infty$

(recall we write $a_n \ll b_n$ if $a_n = o(b_n)$). To understand the convergence rate and the associated asymptotic distribution, it requires a limit theorem on the difference

$$\widehat{T}(x_n) - T(x_n) = n^{-1} \sum_{i=1}^n [1_{\{X_i > x_n\}} - T(x_n)]. \quad (58)$$

For this, by the proof of Theorem 2 in [Zhang \(2021a\)](#), one can show that the central limit theorem

$$\left[\frac{n}{T(x_n)\{1 - T(x_n)\}} \right]^{1/2} \{\widehat{T}(x_n) - T(x_n)\} \rightarrow_d N(0, \sigma^2)$$

as $n \rightarrow \infty$ holds for some $\sigma^2 > 0$ if the process $(X_i) \in \text{TAS}_2$ along with some other mild regularity conditions. [Rootzén \(2009\)](#) applied the β -mixing condition and obtained a weak convergence result for the tail empirical process (introducing an additional parameter into [\(58\)](#)) which implies the above central limit theorem; see also [\(Kulik and Soulier, 2020, Chapter 9\)](#). We leave a full development of functional limit theorem for tail empirical process under the tail adversarial stability framework as a future work, and restrict the discussion on the marginal central limit theorem.

We shall make a comparison between the TAS framework and the β -mixing framework described in [\(Kulik and Soulier, 2020, Section 9.2.3\)](#) for heavy-tailed linear processes [\(5\)](#). Assume as before that the linear process coefficients satisfy for some $\zeta > 0$ that

$$a_i \sim ci^{-\zeta}, \quad i \rightarrow \infty.$$

Below for simplicity, we focus only on the implications on the exponent ζ and the threshold x_n and omit some additional technical assumptions involved. For the TAS framework, as in [Section 7.1](#), for a linear process with $S\nu S$ innovations, $\nu \in (0, 2)$, the process is TAS_2 if $\zeta > \max(2, 2/\nu)$; for the more general regularly varying linear processes satisfying the assumptions of [Theorem 3.9](#), we need the stronger restriction $\zeta > \max(2, 1 + 2/\nu)$. On the other hand, to establish the central limit theorem under the β -mixing framework as described in [\(Kulik and Soulier, 2020, Proposition 9.2.5\)](#), one needs the conditions denoted as $\mathcal{R}(r_n, x_n)$, $\beta(r_n, \ell_n)$ and $\mathcal{S}(r_n, x_n)$, where r_n and ℓ_n are two sequences tending to infinity such that $\ell_n \ll r_n \ll n$ as $n \rightarrow \infty$. First, in view of [\(Kulik and Soulier, 2020, Section 15.3\)](#), the β -mixing condition is satisfied if $\zeta > 2 + 1/\nu$ with a β -mixing coefficient estimate $\beta_n = O(n^{1-(\zeta-1)(\nu-\varepsilon)/(1+\nu-\varepsilon)}) = o(1)$ as $n \rightarrow \infty$, where $\varepsilon > 0$ can be chosen arbitrarily small. Now the conditions $\mathcal{R}(r_n, x_n)$ and $\beta(r_n, \ell_n)$ respectively require:

$$r_n^{1/\nu} \ll x_n \ll n^{1/\nu} \quad \text{and} \quad n\ell_n^{1-(\zeta-1)(\nu-\varepsilon)/(1+\nu-\varepsilon)} \ll r_n. \quad (59)$$

According to [\(Kulik and Soulier, 2020, Section 15.13\)](#), the condition $\mathcal{S}(r_n, x_n)$ holds when $\zeta > \max(2/\nu, 1)$. As a summary, the β -mixing framework minimally requires $\zeta > \max(2/\nu, 2 + 1/\nu)$,

which is more stringent than the TAS requirement for $S\nu S$ innovations when $\nu \in (1/2, 2)$, and more stringent than the TAS requirement for general regularly varying innovations when $\nu > 1$. The β -mixing framework also introduces a lower boundary rate for the threshold x_n which is not present in the TAS framework: (59) and $\ell_n \ll r_n$ together imply that $x_n \gg n^{(1+\nu)/((\zeta-1)\nu^2)}$.

We also consider the moving-maximum process (52). Assume for simplicity that the innovations are ν -Fréchet and again the coefficients

$$a_i \sim ci^{-\zeta}, \quad i \rightarrow \infty.$$

In view of Section 5, the process is TAS_2 if $\zeta > 2/\nu$. On the other hand, by (Kulik and Soulier, 2020, Theorems 13.4 and 13.5), the β -mixing condition holds if $\zeta > 3/\nu$ (more stringent than the TAS requirement) with a beta mixing coefficient estimate $\beta_n = O(n^{3-\zeta\nu}) = o(1)$ as $n \rightarrow \infty$, and the condition $\mathcal{S}(r_n, x_n)$ mentioned above also follows. The conditions $\mathcal{R}(r_n, x_n)$ and $\beta(r_n, \ell_n)$ mentioned above respectively require

$$r_n^{1/\nu} \ll x_n \ll n^{1/\nu} \quad \text{and} \quad n\ell_n^{3-\zeta\nu} \ll r_n,$$

which as above imply a lower rate restriction for the threshold: $x_n \gg n^{1/(\nu(\zeta\nu-2))}$, which is not present in the TAS case.

8 Conclusion

Although various tail dependence measures have been proposed to summarize the degree of the underlying tail dependence, few is useful for developing limit theorems of tail dependent time series. Because of this limitation on available tools, the existing literature to date still largely relies on the strong mixing condition of (Rosenblatt (1956)) to obtain limit theorems of tail dependent time series. However, the strong mixing condition of (Rosenblatt (1956)) was not originally developed to handle dependence in the tail, and as a result additional conditions that control more specifically the degree of dependence in the tail are often needed together with the strong mixing condition. Such conditions can lead to either additional restrictions on the strong mixing coefficient that cannot be easily made explicit or conditions that cannot be fully captured by the strong mixing coefficient. In addition, the supreme over two sigma algebras makes it generally a difficult task to derive a sharp estimate of the strong mixing coefficient. Recently, (Zhang (2021a)) proposed an alternative framework based on a new notion of tail adversarial stability, which has been shown to be useful in obtaining nontrivial limit theorems of tail dependent time series. The advantage over the classical strong mixing framework was illustrated in (Zhang (2021a)) for the

moving-maximum process of [Hall et al. \(2002\)](#). This article studies the tail adversarial stability for the class of regularly varying additive linear processes, which has also been adopted in modeling extremal clusters and tail dependence in time series. It can be seen from our main results in [Section 3](#) that the tail adversarial stability condition can be translated into mild conditions on the linear coefficients, which can be weaker than those under the strong mixing framework; see for example the discussion in [Section 7](#). Extensions to the stochastic volatility model and the max-linear processes are also considered.

Declarations

Ethical approval

Not applicable.

Availability of supporting data

Not applicable.

Competing interests

No, I declare that the authors have no competing interests as defined by Springer, or other interests that might be perceived to influence the results and/or discussion reported in this paper.

Funding

Ting Zhang is supported by the NSF CAREER Award DMS-2131821.

Authors' contributions

The authors contributed equally to this work.

Acknowledgments

We thank the anonymous reviewers for their useful suggestions which greatly improved the manuscript. In particular, some of the materials in [Sections 6](#) and [7](#) were suggested by the reviewers.

References

- BALLA, E., ERGEN, I. AND MIGUEIS, M. (2014). Tail dependence and indicators of systemic risk for large us depositories. *Journal of Financial Stability*, **15**, 195–209.
- BARBE, P. AND MCCORMICK, W. P. (2009). *Asymptotic expansions for infinite weighted convolutions of heavy tail distributions and applications*. American Mathematical Soc.
- BASRAK, B. AND SEGERS, J. (2009). Regularly varying multivariate time series. *Stochastic Processes and their Applications*, **119**, 1055–1080.
- BINGHAM, N., GOLDIE, C. AND TEUGELS, J. (1989). *Regular Variation*. Encyclopedia of Mathematics and Its Applications. Cambridge University Press.
- BOGACHEV, V. I. (2007). *Measure Theory*, vol. 1. Springer.
- CHERNOZHUKOV, V. (2005). Extremal quantile regression. *The Annals of Statistics*, **33**, 806–839.
- CHERNOZHUKOV, V. AND FERNÁNDEZ-VAL, I. (2011). Inference for extremal conditional quantile models, with an application to market and birthweight risks. *The Review of Economic Studies*, **78**, 559–589.
- COLES, S., HEFFERNAN, J. AND TAWN, J. (1999). Dependence measures for extreme value analyses. *Extremes*, **2**, 339–365.
- DAVIS, R. A. AND MIKOSCH, T. (2009). The extremogram: A correlogram for extreme events. *Bernoulli*, **15**, 977–1009.
- DAVIS, R. A., MIKOSCH, T. AND CRIBBEN, I. (2012). Towards estimating extremal serial dependence via the bootstrapped extremogram. *Journal of Econometrics*, **170**, 142–152.
- DAVIS, R. A. AND RESNICK, S. I. (1989). Basic properties and prediction of max-ARMA processes. *Advances in Applied Probability*, **21**, 781–803.
- DE HAAN, L. AND RESNICK, S. I. (1977). Limit theory for multivariate sample extremes. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, **40**, 317–337.
- DRAISMA, G., DREES, H., FERREIRA, A. AND DE HAAN, L. (2004). Bivariate tail estimation: dependence in asymptotic independence. *Bernoulli*, **10**, 251–280.
- DREES, H. (2003). Extreme quantile estimation for dependent data, with applications to finance. *Bernoulli*, **9**, 617–657.
- EMBRECHTS, P. AND GOLDIE, C. M. (1980). On closure and factorization properties of subexponential and related distributions. *Journal of the Australian Mathematical Society*, **29**, 243–256.
- EMBRECHTS, P., MCNEIL, A. AND STRAUMANN, D. (2002). Correlation and dependence in risk management: Properties and pitfalls. In M. A. H. Dempster, editor, *Risk Management: Value at Risk and Beyond*. Cambridge University Press, pp. 176–223.

- FERRO, C. A. T. AND SEGERS, J. (2003). Inference for clusters of extreme values. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, **65**, 545–556.
- HALL, P., PENG, L. AND YAO, Q. (2002). Moving-maximum models for extrema of time series. *Journal of Statistical Planning and Inference*, **103**, 51–63.
- HILL, J. B. (2009). On functional central limit theorems for dependent, heterogeneous arrays with applications to tail index and tail dependence estimation. *Journal of Statistical Planning and Inference*, **139**, 2091–2110.
- HOGA, Y. (2018). A structural break test for extremal dependence in β -mixing random vectors. *Biometrika*, **105**, 627–643.
- HSING, T. (1986). Extreme value theory for suprema of random variables with regularly varying tail probabilities. *Stochastic processes and their applications*, **22**, 51–57.
- JOE, H. (1993). Parametric families of multivariate distributions with given margins. *Journal of Multivariate Analysis*, **46**, 262–282.
- KOENKER, R. (2005). *Quantile Regression*. Cambridge University Press, Cambridge.
- KOENKER, R. AND BASSETT, G. J. (1978). Regression quantiles. *Econometrica*, **46**, 33–50.
- KOKOSZKA, P. S. AND TAQQU, M. S. (1995). Fractional arima with stable innovations. *Stochastic processes and their applications*, **60**, 19–47.
- KULIK, R. AND SOULIER, P. (2020). *Heavy-Tailed Time Series*. Springer.
- LEADBETTER, M. R., LINDGREN, G. AND ROOTZÉN, H. (1983). *Extremes and Related Properties of Random Sequences and Processes*. Springer, Berlin.
- LEDFORD, A. W. AND TAWN, J. A. (1996). Statistics for near independence in multivariate extreme values. *Biometrika*, **83**, 169–187.
- LINTON, O. B. AND WHANG, Y.-J. (2007). The quantilogram: With an application to evaluating directional predictability. *Journal of Econometrics*, **141**, 250–282.
- LIU, W. AND LIN, Z. (2009). Strong approximation for a class of stationary processes. *Stochastic Processes and their Applications*, **119**, 249–280.
- LIU, W. AND WU, W. B. (2010). Asymptotics of spectral density estimates. *Econometric Theory*, **26**, 1218–1245.
- MCNEIL, A. J., FREY, R. AND EMBRECHTS, P. (2005). *Quantitative Risk Management—Concept, Techniques and Tools*. Princeton University Press, Princeton, NJ.
- MIKOSCH, T. AND ZHAO, Y. (2014). A fourier analysis of extreme events. *Bernoulli*, **20**, 803–845.
- NOLAN, J. P. (2020). *Univariate stable distributions*. Springer.
- PHAM, T. D. AND TRAN, L. T. (1985). Some mixing properties of time series models. *Stochastic processes and their applications*, **19**, 297–303.

- POON, S.-H., ROCKINGER, M. AND TAWN, J. (2004). Extreme value dependence in financial markets: diagnostics, models, and financial implications. *The Review of Financial Studies*, **17**, 581–610.
- ROOTZÉN, H. (2009). Weak convergence of the tail empirical process for dependent sequences. *Stochastic Processes and their Applications*, **119**, 468–490.
- ROSENBLATT, M. (1956). A central limit theorem and a strong mixing condition. *Proceedings of the National Academy of Sciences of the United States of America*, **42**, 43–47.
- SAMORODNITSKY, G. (2016). *Stochastic Processes and Long Range Dependence*, vol. 26. Springer.
- SIBUYA, M. (1960). Bivariate extreme statistics, I. *Annals of the Institute of Statistical Mathematics*, **11**, 195–210.
- SMITH, R. L. AND WEISSMAN, I. (1994). Estimating the extremal index. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, **56**, 515–528.
- WU, W. B. (2005). Nonlinear system theory: another look at dependence. *Proceedings of the National Academy of Sciences of the United States of America*, **102**, 14150–14154.
- WU, W. B. (2007). Strong invariance principles for dependent random variables. *The Annals of Probability*, **35**, 2294–2320.
- ZHANG, T. (2013). Clustering high-dimensional time series based on parallelism. *Journal of the American Statistical Association*, **108**, 577–588.
- ZHANG, T. (2015). Semiparametric model building for regression models with time-varying parameters. *Journal of Econometrics*, **187**, 189–200.
- ZHANG, T. (2021a). High-quantile regression for tail-dependent time series. *Biometrika*, **108**, 113–126.
- ZHANG, T. (2022). Asymptotics of sample tail autocorrelations for tail dependent time series: phase transition and visualization. *Biometrika*, **109**, 521–534.
- ZHANG, T. AND WU, W. B. (2011). Testing parametric assumptions of trends of a nonstationary time series. *Biometrika*, **98**, 599–614.
- ZHANG, Z. (2005). A new class of tail-dependent time series models and its applications in financial time series. *Advances in Econometrics*, **20**, 323–358.
- ZHANG, Z. (2008). Quotient correlation: A sample based alternative to Pearson’s correlation. *The Annals of Statistics*, **36**, 1007–1030.
- ZHANG, Z. (2021b). On studying extreme values and systematic risks with nonlinear time series models and tail dependence measures. *Statistical Theory and Related Fields*, **5**, 1–25.
- ZHOU, Z. AND WU, W. B. (2010). Simultaneous inference of linear models with time varying coefficients. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, **72**,

513-531.