

Linear orders in NIP structures

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Abstract

We show that every unstable NIP theory admits a \forall -definable linear quasi-order, over a finite set of parameters. In particular, if the theory is ω -categorical, then it interprets an infinite linear order. This partially answers a longstanding open question.

1 Introduction

A first order structure is *NIP* if every family of uniformly definable sets has finite VC-dimension. We like to think of NIP structures as being *geometric*, indeed the classical examples comprise algebraically closed fields (the domain of algebraic geometry), real closed fields (semi-algebraic geometry), the field \mathbb{Q}_p of p -adic numbers and algebraically closed valued fields (non-archimedean geometry). This class contains that of stable structures, for which we now have an extremely rich theory (see [She90], [Pil96]).

In his paper [She71], Shelah introduced NIP theories and proved that any unstable NIP theory is SOP, that is admits a definable partial order with infinite chains. A longstanding open question asks whether this can be strengthened to an interpretable infinite linear order¹. In this paper, we give a positive answer to a weaker form of this question: we find a \forall -definable equivalence relation such that the quotient by it is infinite and linearly ordered: see Theorem 5.8. In the case of ω -categorical theories, we obtain a *bona fide* interpretable linear order.

In fact, we show slightly more. Following [GH15], we define the op-dimension of a type as a variation on the dp-rank which only sees order-like dimensions. This dimension precisely gives the number of independent linear orders that one can define on a type. In the last section, we then define stable dimension as a counterpart to op-dimension, but this is not used elsewhere in the paper.

There is an important difference between our result and Shelah's theorem, giving a partial order. The existence of a partial order is a non-structure result. Since partial orders can be arbitrarily complicated, it gives no positive information on the models of the theory. A linear order however is a much more constrained object. In fact, we hope that this theorem could open up a new perspective on NIP theories. It shows that NIP is a more structured world than was thought before and makes it reasonable to expect classification statements and analyses similar to those for stable (or superstable) theories, where linear orders would be explicitly present. Indeed we would like the linear orders to have similar role in NIP theories (or subclasses of it) as for instance strongly minimal sets play in the study of ω -stable structures. Isomorphism types of

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¹We are not aware of any occurrence of this question in print. It seems to have been raised independently by several people, including at least Shelah and Hrushovski, from whom we first heard it.

linear orders could replace dimensions of regular types (or rather complement them, since an NIP theory can have stable components). Of course, this still seems far away. A natural special case to initiate this program is the case of ω -categorical structures. This will be studied in future works, starting with [Sim18] which deals with ω -categorical structures of thorn rank 1 and completely classifies the primitive ones.

2 Preliminaries

Throughout this paper, T is a complete first order theory in a language L . We let \mathcal{U} be a monster model, which is $\bar{\kappa}$ -saturated and $\bar{\kappa}$ -strongly homogeneous for some large enough $\bar{\kappa}$. All sets of parameters considered have size smaller than $\bar{\kappa}$.

We use the notation ϕ^0 to mean $\neg\phi$ and ϕ^1 to mean ϕ .

Letters such as a, b, c usually denote finite tuples of variables, whereas A, B, C denote small subsets of \mathcal{U} .

The concatenation of two sequences I and J will be denoted $I + J$ or $I \smallfrown J$. The first notation is used for indiscernible sequences and the second one for concatenation of tuples or families of any kind.

Assumption: Throughout this paper, we assume that T is NIP.

2.1 Invariant types

By an A -invariant type, we mean a global type p which is invariant under automorphisms fixing A pointwise. If $p(x)$ and $q(y)$ are both A -invariant, we can define the type $p(x) \otimes q(y)$ whose restriction to any set $B \supseteq A$ is $\text{tp}(a, b/B)$, where $b \models q|B$ and $a \models p|Bb$. It is also an A -invariant type. A Morley sequence of p over A is a sequence $I = (a_i : i \in \mathcal{I})$ such that for each $i \in \mathcal{I}$, $a_i \models p|Aa_{<i}$. A Morley sequence of p over A is indiscernible over A and all Morley sequences of p over A indexed by the same order have the same type over A .

Two invariant types $p(x)$ and $q(y)$ commute if $p(x) \otimes q(y) = q(y) \otimes p(x)$.

2.2 Indiscernible sequences

We set here some terminology concerning indiscernible sequences that we copy from [Sim20].

Sequences $(I_i : i < \alpha)$ are *mutually indiscernible* if each I_i is indiscernible over $I_{\neq i}$.

The EM-type of an indiscernible sequence I is the set $\{p_n : n < \omega\}$, where $p_n = \text{tp}(a_1, \dots, a_n)$ for some/any elements $a_1 <_I \dots <_I a_n$ of I .

If I is an indiscernible sequence, we let $\text{op}(I)$ denote the sequence I indexed in the opposite order. If I is an endless indiscernible sequence and T is NIP, let $\lim(I)$ denote the limit type of I : the global I -invariant type defined by $\phi(x) \in \lim(I)$ if $\phi(I)$ is cofinal in I . Observe that if $\text{op}(I_1)$ is a Morley sequence of $\lim(I)$ over I , then $I + I_1$ is indiscernible.

A cut $\mathfrak{c} = (I_0, I_1)$ of I is a pair of subsequences of I such that I_0 is an initial segment of I and I_1 the complementary final segment, i.e., $I = I_0 + I_1$. If J is a sequence such that $I_0 + J + I_1$ is indiscernible, we say that J fills the cut \mathfrak{c} . To such a cut, we can associate two limit types: $\lim(I_0)$ and $\lim(\text{op}(I_1))$ (which are defined respectively if I_0 and $\text{op}(I_1)$ have no last element). The cut (I_0, I_1) is *Dedekind* if both I_0 and $\text{op}(I_1)$ have infinite cofinalities, in particular are not empty.

We now recall the important theorem about shrinking of indiscernibles and introduce a notation related to it (see e.g. [Sim15, Chapter 3]).

Definition 2.1. A finite convex equivalence relation on \mathcal{I} is an equivalence relation \sim on \mathcal{I} which has finitely many classes, all of which are convex subsets of \mathcal{I} .

Proposition 2.2 (Shrinking of indiscernibles). *Let $(a_t)_{t \in \mathcal{I}}$ be an indiscernible sequence. Let d be any tuple and $\phi(y_0, \dots, y_{n-1}; d)$ a formula. There is a finite convex equivalence relation \sim_ϕ on \mathcal{I} such that given:*

- $t_0 < \dots < t_{n-1}$ in \mathcal{I} ;
- $s_0 < \dots < s_{n-1}$ in \mathcal{I} with $t_k \sim_\phi s_k$ for all k ;

we have $\phi(a_{t_0}, \dots, a_{t_{n-1}}; d) \leftrightarrow \phi(a_{s_0}, \dots, a_{s_{n-1}}; d)$.

Furthermore, there is a coarsest such equivalence relation.

Given $A, I = (a_t)_{t \in \mathcal{I}}, \phi(y_0, \dots, y_{n-1}; d)$ as above, we let $\mathbb{T}(I, \phi)$ denote the number of equivalence classes in the coarsest \sim_ϕ given by the proposition. By compactness, the number $\mathbb{T}(I, \phi)$ is bounded by an integer depending only on $\phi(y_0, \dots, y_{n-1}; z)$.

If $I \subseteq J$ are indiscernible sequences and A is any set of parameters, we write $I \trianglelefteq_A J$ if for every $\phi(y_0, \dots, y_{n-1}; d) \in L(A)$, we have $\mathbb{T}(I, \phi) = \mathbb{T}(J, \phi)$. Intuitively, formulas with parameters in A do not alternate more on J than they do on I .

Note the following special cases:

- If I is indiscernible over A , then $I \trianglelefteq_A J$ simply means that J is A -indiscernible and contains I .
- If I is without endpoints, $I \trianglelefteq_A I_0 + I + I_1$ is equivalent to the statement that I_0 is a Morley sequence in $\lim(\text{op}(I))$ over IA and $\text{op}(I_1)$ is a Morley sequence in $\lim(I)$ over AI_0 .

The following will be used repeatedly without mention.

Lemma 2.3. *If $I = (a_i : i \in \mathcal{I})$ is indiscernible, where the indexing order \mathcal{I} is dense without endpoints, then given any $\mathcal{I} \subseteq \mathcal{J}$ and any set A of parameters, we can find $J = (a_i : i \in \mathcal{J})$ extending I such that $I \trianglelefteq_A J$.*

Proof. We give two arguments for this. For the first one, let M be a model containing I and A . Assume that I is a sequence of k -tuples. Expand M by adding a k -ary predicate $P(x)$ to name the sequence I and a $2k$ -ary predicate \leq_P for the order on that sequence. Let M_P be the resulting structure. Let N_P be a $|J|^+$ -saturated elementary extension of M_P . Then $P(x)$ names a sequence K , ordered by \leq_P which extends I . We then have, in the original language, $I \trianglelefteq_A K$ since the fact that a formula has a certain number of alternations is first order expressible in L_P . By saturation, we can find a subsequence J of K with the right order type.

We can also build J explicitly as follows: for every cut $\mathfrak{c} = (I_0, I_1)$ of I , by density of I either I_0 has no last element or I_1 has no first element. Assume for example the latter. Let $J_{\mathfrak{c}}$ be a Morley sequence of $\lim(\text{op}(I_1))$ over everything constructed so far which is indexed by the subsequence of \mathcal{J} which lies in the cut of \mathcal{I} corresponding to \mathfrak{c} . Doing this iteratively for all cuts of I and adding all those sequences to I , we obtain J as required. \square

2.3 Dp-rank

The dp-rank will not be used in this paper, but we will define variations of it and hence it seems useful to recall its definition and some of its properties.

Definition 2.4. Let π be a partial type over a set A , and let κ be a (finite or infinite) cardinal. We say $\text{dp-rk}(\pi, A) < \kappa$ if for every family $(I_t : t < \kappa)$ of mutually indiscernible sequences over A and $b \models \pi$, there is $t < \kappa$ such that I_t is indiscernible over Ab .

If $b \in \mathcal{U}$, then $\text{dp-rk}(b/A)$ stands for $\text{dp-rk}(\text{tp}(b/A), A)$.

A theory T is NIP if and only if we have $\text{dp-rk}(\pi, A) < |T|^+$ for every finitary type π ([Sim15, Observation 4.13]).

The term *rank* used for dp-rank is misleading as the dp-rank is a cardinal and not an ordinal. (Strictly speaking, it is not a cardinal, since we only defined $\text{dp-rk}(\pi, A) < \kappa$ and not $\text{dp-rk}(\pi, A) = \kappa$. This is due to a problem at limit cardinals: we can have say $\text{dp-rk}(\pi, A) < \aleph_0$ and yet $\text{dp-rk}(\pi, A) \geq n$ for each $n < \omega$. Some authors write this as $\text{dp-rk}(\pi, A) = \aleph_0^-$.) The reason for it is historical: Shelah gave a more general definition of dp-ranks in [She14], which were indeed ordinals. The definition we use now was extracted from that paper in [Usv09] and the name stayed. In the following sections, we will define two variations on the dp-rank, one from [GH15] which sees only the order component, and a new one which sees only the stable component. We will call them *dimensions* instead of *ranks*.

Some properties of dp-rank in NIP theories (of unequal difficulties):

- If $A \subseteq B$ and π is over A , then $\text{dp-rk}(\pi, A) = \text{dp-rk}(\pi, B)$.
- Given π a partial type over A and let κ be any cardinal. Then we have $\text{dp-rk}(\pi, A) < \kappa$ if and only if for any family $(I_t : t \in X)$ of sequences, mutually indiscernible over A and any $b \models \pi$, there is $X_0 \subseteq X$ of size $< \kappa$ such that $(I_t : t \in X \setminus X_0)$ are mutually indiscernible over Ab .
- (Additivity) Let $a, b \in \mathcal{U}$, A a small set of parameters and κ_1, κ_2 be two cardinals such that $\text{dp-rk}(b/A) < \kappa_1$ and $\text{dp-rk}(a/Ab) < \kappa_2$, then

$$\text{dp-rk}(a, b/A) < \kappa_1 + \kappa_2 - 1.$$

The first bullet is relatively straightforward. The second one is from [KOU13], as well as the third, which follows from it. Proofs can be also found in [Sim15, Section 4].

3 Indiscernible sequences stable over a set

Recall that we assume T to be NIP.

In [Sim13] we introduced the idea that there are two minimal ways in which an indiscernible sequence I can fail to be indiscernible over a tuple a : either some formula $\phi(x; a)$ changes truth value at one cut of the sequence I , or there is an element $b \in I$ such that some formula takes a different truth value on b , but removing b from I yields an indiscernible sequence over a . Distal theories are exactly those for which the second behavior never happens. The following fact from [Sim13] says that this second behavior cannot happen on a large subset of I .

Fact 3.1 ([Sim13], Theorem 3.30, Corollary 3.32). *Let $I_1 + I_2 + I_3$ be indiscernible, with I_1, I_3 infinite without endpoints. Write $I_2 = (b_i : i \in \mathcal{I})$. Assume that $I_1 + I_3$ is indiscernible over A , then:*

1. if $\phi(x; a) \in L(A)$, then $\{i \in \mathcal{I} : \models \phi(b_i; a)\}$ is finite or co-finite in \mathcal{I} ;
2. there is $\mathcal{J} \subseteq \mathcal{I}$ of size $\leq |A| + |T|$ such that $I_1 + (a_i : i \in \mathcal{I} \setminus \mathcal{J}) + I_3$ is indiscernible over A .

Proposition 3.2. *Let I be an endless densely ordered indiscernible sequence and A a set of parameters. The following are equivalent:*

1. there are infinite endless sequences I_0, I_1 such that $I_0 + I + I_1$ is indiscernible and $I_0 + I_1$ is indiscernible over A ;
2. if I_0 is a Morley sequence in $\lim(\text{op}(I)/IA)$ and $\text{op}(I_1)$ a Morley sequence in $\lim(I/II_0A)$ (equivalently, $I \trianglelefteq_A I_0 + I + I_1$), then $I_0 + I_1$ is indiscernible over A ;
3. if $I \trianglelefteq_A J$, then $J \setminus I$ is indiscernible over A ;
4. for every formula $\phi(x_1, \dots, x_n) \in L(A)$, there is a finite set $I_\phi \subseteq I$ and a truth value \mathbf{t} such that for every $a_1 < \dots < a_n$ in $I \setminus I_\phi$, $\models \phi^{\mathbf{t}}(a_1, \dots, a_n)$.

Proof. The implications (4) \rightarrow (3) \rightarrow (2) \rightarrow (1) are straightforward: (4) implies (3) as limit types are unaffected by removing finitely many points from a sequence, (2) is a special case of (3) and (1) follows directly from (2). The implication (1) \rightarrow (4) is proved as in [Sim13, Corollary 3.32]. \square

Definition 3.3. If the conditions of the proposition are satisfied, we say that I is stable over A .

If I is any infinite indiscernible sequence (not necessarily densely ordered), we say that it is stable over A if condition (1) above is satisfied. This implies that (4) also holds. If I has no endpoints, then (2) also holds.

Note that if $I \subseteq J$ are two endless indiscernible sequences, I cointial and cofinal in J , then I is stable over some set A if and only if J is stable over A . This follows from (2) above since the limit types of I and J (from both sides) are the same.

Also, it follows from Fact 3.1 that if I is stable over A , then there is $I' \subseteq I$, $|I'| \leq |T| + |A|$ such that $I \setminus I'$ is indiscernible over A .

Finally, we note that this definition first appeared in [GH15] under the name *almost indiscernible sequence*. We use a different terminology, mainly because the next definition of mutually stable sequences does not coincide with almost mutually indiscernible sequences from [GH15].

Proposition 3.4. *Let $(I_i : i < \alpha)$ be a family of endless densely ordered indiscernible sequences and A a set of parameters. The following are equivalent:*

1. there are infinite endless sequences J_i^0, J_i^1 , $i < \alpha$, such that the sequences $J_i^0 + I_i + J_i^1$ are indiscernible and $J_i^0 + J_i^1$ are mutually indiscernible over A ;
2. if for each $i < \alpha$, J_i^0 realizes $\lim(\text{op}(I_i)/I_{<\alpha}AJ_{<i}^0J_{<i}^1)$ and $\text{op}(J_i^1)$ realizes $\lim(I/I_{<\alpha}AJ_{\leq i}^0J_{<i}^1)$, then the sequences $(J_i^0 + J_i^1 : i < \alpha)$ are mutually indiscernible over A ;
3. if we construct inductively $I_i \trianglelefteq_{AI_{<\alpha}J_{<i}} J_i$, then the sequences $(J_i \setminus I_i : i < \alpha)$ are mutually indiscernible over A ;

4. for every formula $\phi(x_1^1, \dots, x_{n_1}^1; x_1^2, \dots, x_{n_2}^2; \dots; x_1^k, \dots, x_{n_k}^k) \in L(A)$ and indices $i_1 > \dots > i_k$ there is a truth value \mathbf{t} , a finite set $I^1 \subseteq I_{i_1}$ such that for any $a_1^1 < \dots < a_{n_1}^1 \in I_{i_1} \setminus I^1$, there is a finite set $I^2 \subseteq I_{i_2}$ such that for any $a_1^2 < \dots < a_{n_2}^2 \in I_{i_2} \setminus I^2$, there is a finite set $I^3 \subseteq I_{i_3}$ such that \dots for any $a_1^k < \dots < a_{n_k}^k \in I_{i_k} \setminus I^k$, we have $\models \phi^{\mathbf{t}}(a_1^1, \dots, a_{n_1}^1; a_1^2, \dots, a_{n_2}^2; \dots; a_1^k, \dots, a_{n_k}^k)$.

Proof. The implications (4) \rightarrow (3) \rightarrow (2) \rightarrow (1) are as above. We show (1) \rightarrow (4). Assume (1) and take the sequences J_i^0, J_i^1 to be countable. Assume that (4) fails, as witnessed by some formula ϕ and indices $i_1 > \dots > i_k$. For simplicity of notations, assume $k = 2$, $(i_1, i_2) = (1, 0)$ and $\phi = \phi(x^1, x^0)$ (we changed the variable name from x^2 to x^0 to improve readability). The general case is similar. Let \mathbf{t} be the truth value of $\phi(a^1, a^0)$ for some/any $a^1 \in J_1^0, a^0 \in J_0^0$. Then we can find an infinite $I^1 \subseteq I_1$ such that for all $a \in I^1$, there is an infinite $I_a^0 \subseteq I_0$ such that for all $b \in I_a^0$, we have

$$\neg \phi^{\mathbf{t}}(a, b).$$

By compactness, we can increase the sequences I_1 and I_0 and assume that I^1 and each I_a^0 have size $\geq |T|^+$. As the sequence $J_1^0 + J_1^1$ is indiscernible over $J_0^0 + J_0^1$ and the latter is countable, by Fact 3.1, there is a subset $I_*^1 \subseteq I_1$ of size $\leq |T|$ such that $J_1^0 + (I_1 \setminus I_*^1) + J_1^1$ is indiscernible over $J_0^0 + J_0^1$. Let $a \in (I^1 \setminus I_*^1)$. Then $J_1^0 + (a) + J_1^1$ is indiscernible over $J_0^0 + J_0^1$, and hence those two sequences are mutually indiscernible. Next, we can similarly find $I_*^0 \subseteq I_0$ of size $\leq |T|$ such that $J_0^0 + (I_0 \setminus I_*^0) + J_0^1$ is indiscernible over $J_1^0 + (a) + J_1^1$. So there is some $b \in I_a^0 \setminus I_*^0$. Then the sequences $J_1^0 + (a) + J_1^1$ and $J_0^0 + (b) + J_0^1$ are mutually indiscernible. But this contradicts the construction of I_a^0 . \square

When the conditions in the last proposition are satisfied, we say that the family $(I_i : i < \alpha)$ is *mutually stable* over A . Condition (1) shows that this notion does not depend on the ordering of the family. As previously, we extend this definition to arbitrary indexing orders using condition (1). This implies that condition (4) holds and, if the sequences (I_i) are endless, condition (2) holds.

The following lemmas will be used repeatedly.

Lemma 3.5. *Let I_1, \dots, I_m be infinite sequences, mutually indiscernible over some A . Assume that the sequence $I_1 + \dots + I_m$ is indiscernible and stable over A . Then $I_1 + \dots + I_m$ is indiscernible over A .*

Proof. Fix a formula $\phi(x_1, \dots, x_n)$ over A . By Proposition 3.2(4), there is a finite set $I_\phi \subseteq I_1 + \dots + I_m$ and a truth value \mathbf{t} such that for every $a_1 < \dots < a_n$ in $I_1 + \dots + I_m \setminus I_\phi$, $\models \phi^{\mathbf{t}}(a_1, \dots, a_n)$. Now take an arbitrary $a_1, \dots, a_n \in I_1 + \dots + I_m$. Since the sequences I_1, \dots, I_m are infinite and mutually indiscernible over A , we can find $a'_1, \dots, a'_n \in I_1 + \dots + I_m$ having the same type as a_1, \dots, a_n over A and disjoint from I_ϕ . We know that $\phi^{\mathbf{t}}(a'_1, \dots, a'_n)$ holds, so also $\phi^{\mathbf{t}}(a_1, \dots, a_n)$ is true. Hence the sequence $I_1 + \dots + I_m$ is indiscernible over A . \square

Lemma 3.6. *Let $(J_i : i < \alpha)$ be sequences, mutually stable over some A , where each J_i can be written as $J_i = I_i^1 + \dots + I_i^k$ and the sequences $(I_i^j : i < \alpha, j \leq k)$ are infinite and mutually indiscernible over A . Then the sequences $(J_i : i < \alpha)$ are mutually indiscernible over A .*

Proof. The proof is the same as that of the previous lemma using Proposition 3.4(4) instead of Proposition 3.2(4). \square

4 Op-dimension

Op-dimension was introduced by Guingona and Hill in [GH15]. It measures the number of independent orderings that one can define on an infinite subset of a (partial) type. We give a self-contained exposition, slightly different from, but equivalent to, the one in [GH15] (in the case of finite cardinals, see the remarks below).

Definition 4.1. Let A be any set of parameters, π a partial type over A and κ a cardinal. We say that $\text{opD}(\pi, A) < \kappa$ if we cannot find:

- $a \models \pi$;
- a family $(I_i : i < \kappa)$ of sequences mutually indiscernible over A , where $I_i = (b_j^i : j \in \mathcal{I}_i)$;
- for each $i < \kappa$, a formula $\phi_i(x; y_i) \in L(A)$ (with $|x| = |a|$ and $|y_i| = |b_j^i|$), such that $\{j \in \mathcal{I}_i : \models \phi_i(a; b_j^i)\}$ is infinite and co-infinite in \mathcal{I}_i .

It follows at once from the definition that $\text{opD}(\pi, A) \leq \text{dp-rk}(\pi, A)$. In particular, if T is NIP and π is a partial type in finitely many variables, then $\text{opD}(\pi, A) < |T|^+$.

Lemma 4.2. If $A \subseteq B$ and $\pi(x)$ is a partial type over A , then $\text{opD}(\pi, A) = \text{opD}(\pi, B)$.

Proof. If $a, (I_i : i < \kappa)$ is a witness to $\text{opD}(\pi, B) \geq \kappa$, then $a, (I'_i : i < \kappa)$ witnesses $\text{opD}(\pi, A) \geq \kappa$, where I'_i is obtained from I_i by concatenating a fixed enumeration of B to the end of every element of it.

Conversely, if $(I_i : i < \kappa)$ are A -mutually indiscernible and witness $\text{opD}(\pi, A) \geq \kappa$, then there is $B' \equiv_A B$ such that those sequences are mutually indiscernible over B' . We then have $\text{opD}(\pi, B) = \text{opD}(\pi, B') \geq \kappa$. Hence $\text{opD}(\pi, B) \geq \text{opD}(\pi, A)$. \square

Hence it makes sense to write $\text{opD}(\pi)$ to stand for $\text{opD}(\pi, A)$ for any A over which π is defined.

This definition of op-dimension does not appear in this form in [GH15], but it is easily seen by Ramsey compactness to be equivalent to the one using IRD-patterns given as Lemma 1.23 in that paper. (The only difference is that we allow it to take infinite cardinal values, whereas the definition as stated in [GH15] only allows integers or ∞ .)

Proposition 4.3. Let $\pi(x)$ be a partial type and κ a cardinal. The following are equivalent:

1. $\text{opD}(\pi) < \kappa$;
2. for every A over which π is defined, for every family $(I_i : i < \kappa)$ of endless sequences mutually stable over A and $a \models \pi$, there is $i < \kappa$ such that I_i is stable over Aa ;
3. for every A over which π is defined, for every family $(I_i : i < \lambda)$ of endless sequences mutually stable over A and $a \models \pi$, there is $X \subseteq \lambda$, $|X| < \kappa$ such that $(I_i : i \in \lambda \setminus X)$ are mutually stable over Aa .

Proof. (1) \Rightarrow (2): Assume we are given a family $(I_i : i < \kappa)$ of endless sequences mutually stable over A such that no I_i is stable over Aa . Using the second bullet after Proposition 2.2, build inductively dense endless sequences J_i , so that

$$I_i \trianglelefteq J_i^0 + I_i + J_i^1,$$

over everything constructed so far. Then the sequences $(J_i^0 + J_i^1 : i < \kappa)$ are mutually indiscernible over A , but no $J_i^0 + J_i^1$ is indiscernible over Aa . Since the sequences $(J_i^0, J_i^1 :$

$i < \kappa$) are mutually indiscernible over Aa , we can find a formula $\phi(x; y) \in L(AJ_i^0 J_i^1)$ such that $\phi(a; b)$ holds for all b in some end segment of J_i^0 , whereas $\neg\phi(a; b')$ holds for all b' in some initial segment of J_i^1 . Adding the parameters of ϕ to the base and trimming the sequences gives a witness of $\text{opD}(\pi) \geq \kappa$.

(2) \Rightarrow (3): Assume that $\text{opD}(\pi, A) < \kappa$. Let $(I_i : i < \lambda)$ be mutually stable over A and let $a \models \pi$. Using the second bullet after Proposition 2.2, build inductively dense endless sequences $J_i^k, i < \lambda, k < 4$, so that

$$I_i \leq J_i^0 + J_i^1 + I_i + J_i^2 + J_i^3$$

over everything constructed so far. Let $A' = AJ_{<\lambda}^0 J_{<\lambda}^3$. The sequences $(J_i^1 + J_i^2 : i < \lambda)$ are mutually stable (indeed mutually indiscernible) over A' . Let $X \subseteq \lambda$ be the set of indices i for which $J_i^1 + J_i^2$ is not indiscernible over $A'a$. Since the sequences $(J_i^1, J_i^2 : i < \lambda)$ are mutually indiscernible over $A'a$, Lemma 3.5 implies that for $i \in X$, $J_i^1 + J_i^2$ is not stable over $A'a$. Hence by (2), $|X| < \kappa$. Now since $(J_i^0 + J_i^1, J_i^2 + J_i^3 : i < \lambda)$ are mutually indiscernible over Aa , each sequence $J_i^0 + J_i^1 + J_i^2 + J_i^3$ for $i \notin X$ is indiscernible over $AaJ_{\neq i}$. Hence the sequences $(J_i^0 + J_i^1, J_i^2 + J_i^3 : i \in \lambda \setminus X)$ are mutually indiscernible over Aa and the sequences $(I_i : i \in \lambda \setminus X)$ are mutually stable over Aa .

(3) \Rightarrow (1) is immediate. \square

The following is [GH15, Theorem 2.2]. The proof we give is very similar to the one by Guingona and Hill, except that we use mutually stable sequences instead of almost mutually indiscernible, which is a stronger notion.

Proposition 4.4. *Let A be any set of parameters and a, b two tuples. If $\text{opD}(a/A) < \kappa_1$ and $\text{opD}(b/Aa) < \kappa_2$, then $\text{opD}(a, b/A) < \kappa_1 + \kappa_2 - 1$.*

Proof. Let $(I_i : i < \lambda)$ be mutually stable over A . We can find $X_1 \subseteq \lambda, |X_1| < \kappa_1$ such that the sequences $(I_i : i \in \lambda \setminus X_1)$ are mutually stable over Aa . We then find $X_2 \subseteq \lambda, |X_2| < \kappa_2$ such that the sequences $(I_i : i \in \lambda \setminus (X_1 \cup X_2))$ are mutually stable over Aab . This proves $\text{opD}(a, b/A) < \kappa_1 + \kappa_2 - 1$. \square

5 Constructing linear orders

Assumption: All indiscernible sequences in this section are assumed to be indexed by \mathbb{Q} . This will be recalled at times. The density of the indexing order is only for convenience and could be removed in most places, however having no endpoints is often essential for the arguments to go through, as we want to be able to extend sequences on both sides by realizing limit types.

Definition 5.1. A quintuple $\mathbf{u} = (\pi(x), I, J, \phi; A)$ is *good* if:

- $I = (a_i : i \in \mathbb{Q})$ and $J = (b_j : j \in \mathbb{Q})$ are sequences of tuples, A is a small set of parameters and $\pi(x)$ is a partial type over AIJ ;
- the sequence $I + J$ is indiscernible over A ;
- $\phi = \phi(x; y) \in L(A)$ with x the same variable as that of $\pi(x)$ and $|y| = |a_i| = |b_j|$;
- there is $a \models \pi(x)$ such that for each $i \in \mathbb{Q}$, we have $\models \phi(a; a_i)$ and for each $j \in \mathbb{Q}$, we have $\models \neg\phi(a; b_j)$.

We will sometimes omit A from the notation if it is irrelevant.

We write $(a, I, J, \phi; A)$ for $(\text{tp}(a/AIJ), I, J, \phi; A)$.

If $\mathbf{u} = (a, I, J, \phi; A)$ and $\mathbf{v} = (a', I', J', \phi'; A')$ are good, write $\mathbf{u} \leq \mathbf{v}$ to mean:

- $\phi = \phi'$;
- $A \subseteq A'$;
- $\text{tp}(a/A) = \text{tp}(a'/A)$;
- $I + J$ and $I' + J'$ have the same EM-type over A .

Definition 5.2. Let $\mathbf{u} = (a, I, J, \phi; A)$ be good and set $p(\mathbf{u}) = \text{tp}(a/AIJ)$. We define the following binary relations on realizations of $p(\mathbf{u})$:

- $(a, b) \in E(\mathbf{u})$ if for any sequence K such that $I + K + J$ is A -indiscernible, if $\phi(a; d)$ (resp. $\neg\phi(a; d)$) holds for all $d \in K$, then $\phi(b; d)$ (resp. $\neg\phi(b; d)$) holds for almost all $d \in K$ (all but finitely many).
- $(a, b) \in R(\mathbf{u})$ if we cannot find a sequence K such that both $(a, I + K, J, \phi; A)$ and $(b, I, K + J, \phi; A)$ are good.

Lemma 5.3. Let \mathbf{u} be good, then $E(\mathbf{u})$ and $R(\mathbf{u})$ are \forall -definable relations on $p(\mathbf{u})$. The relation $E(\mathbf{u})$ is an equivalence relation and $R(\mathbf{u})$ is reflexive, transitive and $E(\mathbf{u})$ -equivariant. Furthermore $(a, b) \in E(\mathbf{u})$ if and only if both (a, b) and (b, a) are in $R(\mathbf{u})$.

Proof. Only transitivity of $R(\mathbf{u})$ is not immediate from the definition. Let $a, b, c \models p(\mathbf{u})$ and assume that $(a, c) \notin R(\mathbf{u})$ as witnessed by K : that is $(a, I + K, J, \phi; A)$ and $(c, I, K + J, \phi; A)$ are good. Replacing K by a Morley sequence of its limit type, we may assume that K is indiscernible over Ab . If $\phi(b; d)$ holds for $d \in K$, then $(b, I + K, J, \phi; A)$ is good. This implies that $(b, c) \notin R(\mathbf{u})$. Similarly, if $\neg\phi(b; d)$ holds for $d \in K$, then $(b, I, K + J, \phi; A)$ is good and $(a, b) \notin R(\mathbf{u})$. \square

Definition 5.4. We say that \mathbf{u} is linear if it is good and any two realizations of $p(\mathbf{u})$ are $R(\mathbf{u})$ -comparable.

In other words, $\mathbf{u} = (a, I, J, \phi; A)$ is linear if it is good and we cannot find $a, b \models p(\mathbf{u})$ and K, L two endless indiscernible sequences such that both $(a, I, K + J, \phi; A)$ and $(b, I + K, J, \phi; A)$ are good as well as $(a, I + L, J, \phi; A)$ and $(b, I, L + J, \phi; A)$.

If \mathbf{u} is linear, then $R(\mathbf{u})$ induces a linear order on the quotient of $p(\mathbf{u})$ by $E(\mathbf{u})$. It is usually easier to consider the type-definable relation $\neg R(\mathbf{u})$ which induces a strict linear order on that quotient.

Lemma 5.5. Let $\mathbf{u}_\alpha = (a, I_\alpha^0, J_\alpha^0, \phi_\alpha; A)$, $\alpha < \eta$, be good, where the sequences $(I_\alpha^0 + J_\alpha^0 : \alpha < \eta)$ are mutually indiscernible over A . There is μ with $\eta \leq \mu \leq \text{opD}(a/A)$ and a family $(I_\alpha, J_\alpha, \phi_\alpha)_{\alpha < \mu}$ such that:

- ₀ each $(a, I_\alpha, J_\alpha, \phi_\alpha; A)$ is good;
- ₁ for $\alpha < \eta$, $\mathbf{u}_\alpha \trianglelefteq (a, I_\alpha, J_\alpha, \phi_\alpha; A)$;
- ₂ the sequences $(I_\alpha + J_\alpha : \alpha < \mu)$ are mutually indiscernible over A ;
- ₃ whenever $K_\alpha^0, K_\alpha^1, \alpha < \mu$ are sequences and $A' \supseteq A$ are such that the sequences $(I_\alpha + J_\alpha : \alpha < \mu)$ are mutually indiscernible over A' and $(a, I_\alpha + K_\alpha^0, K_\alpha^1 + J_\alpha, \phi_\alpha; A)$ are good, then the sequences in the family $(I_\alpha + K_\alpha^0 : \alpha < \mu) \cap (K_\alpha^1 + J_\alpha : \alpha < \mu)$ are mutually stable over $A'a$;
- ₄ if L is indiscernible over $AI_{<\mu}J_{<\mu}$ and the sequences $(I_\alpha + J_\alpha : \alpha < \mu)$ are mutually indiscernible over AL , then L is stable over $AI_{<\mu}J_{<\mu}a$;
- ₅ the sequences in the family $(I_\alpha : \alpha < \mu) \cap (J_\alpha : \alpha < \mu)$, are mutually indiscernible over Aa .

Proof. Call a family $(\psi_\alpha : \alpha < \mu)$ *good* if each $\psi_\alpha = \psi_\alpha(x; y_\alpha) \in L(A)$, with $|x| = |a|$ and there are sequences $(I_\alpha + J_\alpha : \alpha < \mu)$ mutually indiscernible over A such that each $(a, I_\alpha, J_\alpha, \psi_\alpha; A)$ is good and for $\alpha < \eta$, $I_\alpha + J_\alpha$ has same EM-type as $I_\alpha^0 + J_\alpha^0$ over A . Any good family has length $\mu \leq \text{opD}(a/A) < |T|^+$. Also by compactness, an increasing union of good families is again good. Hence there is a maximal good family $(\phi_\alpha : \alpha < \mu)$ extending $(\psi_\alpha : \alpha < \eta)$. Let its goodness be witnessed by $(I_\alpha + J_\alpha : \alpha < \mu)$. Properties \bullet_0 and \bullet_2 are immediate by construction. We will prove that this family satisfies \bullet_3 and \bullet_4 . We can then enforce \bullet_5 by building $I_\alpha \sqsubseteq I_\alpha + I'_\alpha$ and $J_\alpha \sqsubseteq J'_\alpha + J_\alpha$ over everything and replacing each (I_α, J_α) by (I'_α, J'_α) . The new family still witnesses goodness and therefore also satisfies $\bullet_{1 \rightarrow 4}$.

Assume that \bullet_4 does not hold. So there is some L such that the sequences $(I_i + J_i : i < \mu)$ along with L are mutually indiscernible over A , but L is not stable over $AI_{<\mu}J_{<\mu}a$. Without loss, L has no endpoints. Let $L \sqsubseteq_{AI_{<\mu}J_{<\mu}a} L_0 + L + L_1$. Then L_0, L_1 are mutually indiscernible over $AI_{<\mu}J_{<\mu}a$ but $L_0 + L_1$ is not indiscernible over that same set. There is a formula $\phi_*(a, y; \vec{d})$, $\vec{d} \in AI_{<\mu}J_{<\mu}L_0L_1$, which holds on an end segment of L_0 , whereas its negation holds on an initial segment of L_1 . Take L'_0 an end segment of L_0 , L'_1 an initial segment of L_1 such that L'_0, L'_1 contain no element from \vec{d} . Take also end segments I'_α of I_α and initial segments J'_α of J_α for $\alpha < \mu$ such that those also do not contain any element from \vec{d} . Finally let L''_0, L''_1 be the sequences L'_0, L'_1 respectively, with \vec{d} concatenated to every element of the sequence. Then the sequences $I'_\alpha + J'_\alpha$, $\alpha < \mu$ and $L''_0 + L''_1$ are mutually indiscernible over A . Each $(a, I'_\alpha, J'_\alpha, \phi_\alpha; A)$ is good as is $(a, L''_0, L''_1, \phi'_*; A)$, where $\phi'_*(x; y\vec{z}) = \phi_*(x, y; \vec{z})$. Also the EM-type over A of each $I'_\alpha + J'_\alpha$ is the same as that of $I_\alpha + J_\alpha$. This contradicts maximality of the initial family.

From \bullet_4 , we can deduce two seemingly stronger statements:

- \bullet'_4 If $A \subseteq A'$, L is indiscernible over $A'I_{<\mu}J_{<\mu}$ and the sequences $(I_\alpha + J_\alpha : \alpha < \mu)$ are mutually indiscernible over $A'L$, then L is stable over $A'I_{<\mu}J_{<\mu}a$.
- \bullet''_4 If $A \subseteq A'$, $(L_i : i < \beta)$ is a family of sequences mutually indiscernible over $A'I_{<\mu}J_{<\mu}$ such that $(I_\alpha + J_\alpha : \alpha < \mu)$ are mutually indiscernible over $A'L_{<\beta}$, then $(L_i : i < \beta)$ are mutually stable over $A'I_{<\mu}J_{<\mu}a$.

To see that \bullet'_4 follows, consider the sequence L' obtained from L by concatenating A' to each of its elements. Then \bullet_4 applied to L' gives \bullet'_4 . To deduce \bullet''_4 , let $(L_i : i < \beta)$ be given as above. Construct inductively on $i < \beta$, $L_i \sqsubseteq M_i + L_i + N_i$ over everything built so far, including all of the L_i 's. Then by \bullet'_4 , $M_i + N_i$ is stable over $A'I_{<\mu}J_{<\mu}M_{\neq i}N_{\neq i}a$. It follows from Lemma 3.5 that $M_i + N_i$ is indiscernible over that set. Thus the sequences $(M_i + N_i : i < \beta)$ are mutually indiscernible over $A'I_{<\mu}J_{<\mu}a$ as required.

We now show that \bullet_3 holds. So let K_α^0, K_α^1 , $\alpha < \mu$, and $A' \supseteq A$ be given such that $(I_\alpha + J_\alpha : \alpha < \mu)$ are mutually indiscernible over A' and $(a, I_\alpha + K_\alpha^0, K_\alpha^1 + J_\alpha, \phi_\alpha; A)$ are good. Build inductively on $\alpha < \mu$,

$$I_\alpha + K_\alpha^0 \sqsubseteq M_\alpha^0 + (I_\alpha + K_\alpha^0) + N_\alpha^0 + I'_\alpha$$

and

$$K_\alpha^1 + J_\alpha \sqsubseteq J'_\alpha + N_\alpha^1 + (K_\alpha^1 + J_\alpha) + M_\alpha^1,$$

each \sqsubseteq being understood to hold over everything built so far (including a). Then the tuples $(a, I'_\alpha, J'_\alpha, \phi_\alpha; A')$ are good. Furthermore as the sequences $(I_\alpha + K_\alpha^0 + K_\alpha^1 + J_\alpha : \alpha < \mu)$ are mutually stable over A' (as witnessed by $I_\alpha + J_\alpha$), the sequences

$$(M_\alpha^0 + N_\alpha^0 + I'_\alpha + J'_\alpha + N_\alpha^1 + M_\alpha^1 : \alpha < \mu)$$

are mutually indiscernible over A' (property (3) of Proposition 3.4). The family $(I'_\alpha + J'_\alpha)_\alpha$ also witnesses maximality of the family $(\phi_\alpha)_{\alpha < \mu}$, hence we can apply \bullet_4'' to it. We deduce that the sequences $(M_\alpha^0 + N_\alpha^0 : \alpha < \mu) \frown (N_\alpha^1 + M_\alpha^1 : \alpha < \mu)$ are mutually stable over $A' I'_{<\mu} J'_{<\mu} a$. By Lemma 3.6, they are mutually indiscernible over that set. Therefore $(I_\alpha + K_\alpha^0 : \alpha < \mu) \frown (K_\alpha^1 + J_\alpha : \alpha < \mu)$ are mutually stable over $A' a$. \square

Remark 5.6. If $\text{opD}(a/A) \geq \eta$, then there is a family $(I_\alpha, J_\alpha, \phi_\alpha)_{\alpha < \mu}$ satisfying \bullet_0 and $\bullet_{2 \rightarrow 4}$ with $\mu \geq \eta$.

We now come to the main technical point of the construction.

Proposition 5.7. *Let $\mathbf{u}_\alpha = (a, I_\alpha, J_\alpha, \phi_\alpha; A)$, $\alpha < \mu$ be a family satisfying \bullet_0 and $\bullet_{2 \rightarrow 5}$ of Lemma 5.5. Then there is $A' \supseteq A$ such that each $\mathbf{v}_\alpha := (a, I_\alpha, J_\alpha, \phi_\alpha; A')$ is linear.*

Proof. During the proof we will often replace a sequence K say by a sequence K' so that $K \trianglelefteq K + K'$ over everything that we have built so far. Thus the new sequence K' is indiscernible over all parameters considered. This will never affect previous assumptions: for instance if K was indiscernible over some B , then K' has the same type as K over it. If some $(b, K, L, \psi; B)$ was good, then so is $(b, K', L, \psi; B)$. If K was stable over some B , then K' is also stable over B . In particular, note that by \bullet_5 , doing this to the original sequences I_α, J_α does not change their type over Aa (and over each other), hence at the cost of applying automorphisms, we can assume that they do not change during the construction.

Let $A_\mu = A I_{<\mu} J_{<\mu}$ and set $p_0(x) = \text{tp}(a/A_\mu)$.

Step 0: Note that if some $(b, I, J, \psi; B)$ is linear, then so is $(b, I, J, \psi; B')$ for any $B' \supseteq B$ for which this quintuple is good. To prove the proposition it suffices then to find $A' \supseteq A$ such that the sequences $(I_\alpha + J_\alpha : \alpha < \mu)$ are mutually indiscernible over A' and $(a, I_0, J_0, \phi_0; A')$ is linear. Indeed, having done this, properties \bullet_0 and $\bullet_{2 \rightarrow 4}$ still hold for A' replacing A (for \bullet_4 , this is given by \bullet_4') and we can enforce \bullet_5 as in the beginning of the proof of Lemma 5.5. We can then inductively increase A' to make each $(a, I_\alpha, J_\alpha, \phi_\alpha; A')$ linear one after the other.

Step 1: Set $\phi = \phi_0$ and $\mathbf{u} = (a, I_0, J_0, \phi; A)$. Let n be larger than the VC-dimension of the formula $\phi(x; y)$. We show that one cannot find tuples $a_k \models p_0$, $k < n$ and sequences K_α^k , $\alpha < \mu$, $k < n - 1$ such that:

- \boxtimes_0 the sequences $(I_\alpha + K_\alpha^0 + \cdots + K_\alpha^{n-2} + J_\alpha : \alpha < \mu)$ are mutually indiscernible over A ;
- \boxtimes_1 each $(a_k, I_\alpha + K_\alpha^{<k}, K_\alpha^{\geq k} + J_\alpha, \phi_\alpha; A)$ is good;
- \boxtimes_2 for each $k \neq k' < n$, the tuples a_k and $a_{k'}$ are $R(\mathbf{u})$ -incomparable.

Assume for a contradiction that we are given such tuples and sequences. We show that for any σ permutation of n , we can find sequences $K_{0,\sigma}^k$, $k < n - 1$ such that:

- $\boxtimes_{3,\sigma}$ the sequence $I_0 + K_{0,\sigma}^0 + \cdots + K_{0,\sigma}^{n-2} + J_0$ is indiscernible over A ;
- $\boxtimes_{4,\sigma}$ each $(a_k, I_0 + K_{0,\sigma}^{<\sigma(k)}, K_{0,\sigma}^{\geq \sigma(k)} + J_0, \phi)$ is good.

Why is this enough? Fix any σ a permutation of n , $i < n$ and let e be an element of $K_{0,\sigma}^i$ (or J_0 if $i = n - 1$). Then we have $\models \phi(a_j, e) \iff \sigma(j) > i$. This shows that the set $\{a_0, \dots, a_{n-1}\}$ is shattered by ϕ and contradicts the choice of n .

We now turn to the construction of $K_{0,\sigma}^k$. For σ the identity, we can take $K_{0,\sigma}^k = K_0^k$. Assume that we have built those sequences for some value of σ . Let $i < n-1$ and set $\tau = (i, i+1) \circ \sigma$. We show how to build the sequences $K_{0,\tau}^k$. Since the sequences $(I_\alpha + J_\alpha : \alpha < \mu)$ are mutually indiscernible over A , the sequences $I_\alpha + K_\alpha^0 + \dots + K_\alpha^{n-2} + J_\alpha$, $0 < \alpha < \mu$ and $I_0 + K_{0,\sigma}^0 + \dots + K_{0,\sigma}^{n-2} + J_0$ are mutually stable over A . Replacing each of the sequences $I_\alpha, J_\alpha, K_\alpha^k$ and $K_{0,\sigma}^k$ by a Morley sequence of their limit types and applying Lemma 3.6, we obtain:

⊠_{5, σ} the sequences $I_\alpha + K_\alpha^0 + \dots + K_\alpha^{n-2} + J_\alpha$, $0 < \alpha < \mu$ and $I_0 + K_{0,\sigma}^0 + \dots + K_{0,\sigma}^{n-2} + J_0$ are mutually indiscernible over A .

Set $u = \sigma^{-1}(i)$, $v = \sigma^{-1}(i+1)$ and let $b = a_u$ and $c = a_v$. Assume $u < v$ (the case $u > v$ is similar). The main thing to prove is:

⊠ The two sequences $I_0 + K_{0,\sigma}^{<i}$ and $K_{0,\sigma}^{\geq i} + J_0$ are mutually indiscernible over Abc .

To prove this, let us first consider the situation over the base Ab . For $0 < \alpha < \mu$, the tuple $(b, I_\alpha + K_\alpha^{<u}, K_\alpha^{\geq u} + J_\alpha, \phi_\alpha; A)$ is good, and so is $(b, I_0 + K_{0,\sigma}^{<i}, K_{0,\sigma}^{\geq i} + J_0, \phi; A)$. Hence by \bullet_3 , the sequences $I_\alpha + K_\alpha^{<u}, K_\alpha^{\geq u} + J_\alpha, I_0 + K_{0,\sigma}^{<i}$ and $K_{0,\sigma}^{\geq i} + J_0$, where α ranges in $0 < \alpha < \mu$, are mutually stable over Ab . Similarly, the sequences $I_\alpha + K_\alpha^{<v}, K_\alpha^{\geq v} + J_\alpha, I_0 + K_{0,\sigma}^{<i+1}$ and $K_{0,\sigma}^{\geq i+1} + J_0$ are mutually stable over Ac . Replacing all the sequences K_α^j and $K_{0,\sigma}^j$ by Morley sequences of their limit types over everything and applying Lemma 3.6, we can replace “mutually stable” by “mutually indiscernible” in the two previous sentences and obtain:

⊠_{6, σ} The sequences $I_\alpha + K_\alpha^{<u}, K_\alpha^{\geq u} + J_\alpha, I_0 + K_{0,\sigma}^{<i}$ and $K_{0,\sigma}^{\geq i} + J_0$, where α ranges over $0 < \alpha < \mu$, are mutually indiscernible over Ab .

⊠_{7, σ} The sequences $I_\alpha + K_\alpha^{<v}, K_\alpha^{\geq v} + J_\alpha, I_0 + K_{0,\sigma}^{<i+1}$ and $K_{0,\sigma}^{\geq i+1} + J_0$ where α ranges over $0 < \alpha < \mu$, are mutually indiscernible over Ac

By ⊠_{6, σ} , the family

$$(I_\alpha + K_\alpha^{<u}, K_\alpha^u : 0 < \alpha < \mu) \frown (I_0 + K_{0,\sigma}^{<i}, K_{0,\sigma}^i)$$

has the same type as

$$(I_\alpha, J_\alpha : 0 < \alpha < \mu) \frown (I_0, J_0)$$

over Ab . In particular, each of $(b, I_\alpha + K_\alpha^{<u}, K_\alpha^u, \phi_\alpha; A)$, $\alpha > 0$, and $(b, I_0 + K_{0,\sigma}^{<i}, K_{0,\sigma}^i, \phi; A)$ is good and together they have the same type as $(b, I_\alpha, J_\alpha, \phi_\alpha; A)$, $\alpha > 0$, and $(b, I_0, J_0, \phi; A)$. In particular, they satisfy properties $\bullet_{2 \rightarrow 5}$.

The sequences $I_\alpha + K_\alpha^{<u} + K_\alpha^u, I_0 + K_{0,\sigma}^{<i} + K_{0,\sigma}^i$ are mutually indiscernible over $AK_{0,\sigma}^{>i}J_0c$ (we are using $u < v$ here). Then applying \bullet_3 (where the K 's there are empty), we get that the sequences $(I_\alpha + K_\alpha^{<u}, K_\alpha^u : 0 < \alpha < \mu)$ along with $I_0 + K_{0,\sigma}^{<i}$ and $K_{0,\sigma}^i$ are mutually indiscernible over $AK_{0,\sigma}^{>i}J_0bc$. In particular:

$$I_0 + K_{0,\sigma}^{<i} \text{ is indiscernible over } AK_{0,\sigma}^{>i}J_0bc.$$

By a symmetric reasoning, interchanging the roles of b and c , we get that $(K_\alpha^{v-1}, K_\alpha^{\geq v} + J_0 : 0 < \alpha < \mu)$ along with $K_{0,\sigma}^i$ and $K_{0,\sigma}^{>i} + J_0$ are mutually indiscernible over $AI_0K_{0,\sigma}^{<i}bc$, and in particular:

$K_{0,\sigma}^{>i} + J_0$ is indiscernible over $AI_0K_{0,\sigma}^{<i}bc$.

Thus \boxplus follows from those two statements.

We can now finish the construction. By \boxplus , $(I_0 + K_{0,\sigma}^{<i}, K_{0,\sigma}^{>i} + J_0)$ and (I_0, J_0) have the same type over Abc . As b, c are not $R(\mathbf{u})$ -comparable, there is a sequence $K_{0,\tau}^i$ such that each of $(b, I_0 + K_{0,\sigma}^{<i} + K_{0,\tau}^i, K_{0,\sigma}^{>i} + J_0, \phi)$ and $(c, I_0 + K_{0,\sigma}^{<i}, K_{0,\tau}^i + K_{0,\sigma}^{>i} + J_0, \phi)$ are good. Set $K_{0,\tau}^j = K_{0,\sigma}^j$ for $j \neq i$. We claim that $\boxtimes_{3,\tau}$ and $\boxtimes_{4,\tau}$ are satisfied. The construction immediately gives $\boxtimes_{3,\tau}$ along with $\boxtimes_{4,\tau}$ where k there is either u or v . Take now k different from u and v . Assume that $\sigma(k) < i$ (the case $\sigma(k) > i + 1$ is similar). Then $\boxtimes_{4,\tau}$ will follow for this value of k if we know that $\neg\phi(a_k, b)$ holds for all $b \in K_{0,\tau}^i$. By the argument that lead to $\boxtimes_{6,\sigma}$, taking a_k instead of b , we have that the sequence $K_{0,\sigma}^{i-1} + J_0$ is indiscernible over Ab . We also know that $\neg\phi(a_k, b)$ holds for b in that sequence. It follows that $K_{0,\sigma}^{i-1} + K_{0,\tau}^i + J_0$ is stable over Ab and by Lemma 3.5 it is indiscernible over it. Hence $\boxtimes_{4,\tau}$ follows.

Step 2: Let n be maximal such that there are a_k , $k < n$, and sequences K_α^k , $\alpha < \mu$, $k < n - 1$, such that \boxtimes_{0-2} above hold and let such elements and sequences be given. If $n = 1$, set $b = a$, $A' = AI_{>0}J_{>0}$ and $I'_0 = I_0$. If $n > 1$, set $b = a_{n-1}$, $A' = AI_{>0}K_{>0}^{<n-1}J_{>0}I_0K_0^{<n-2}a_{<n-1}$ and $I'_0 = K_0^{n-2}$. Let $\mathbf{v} = (b, I'_0, J_0, \phi; A')$. We show that \mathbf{v} is linear. Assume not, then we can find some $c \equiv_{A'I'_0J_0} b$ such that b and c are $R(\mathbf{v})$ -incomparable. This means that there are sequences L_0, L_1 such that all of $(b, I'_0, L_0 + J_0, \phi; A')$, $(c, I'_0 + L_0, J_0, \phi; A')$, $(b, I'_0 + L_1, J_0, \phi; A')$ and $(c, I'_0, L_1 + J_0, \phi; A')$ are good. Then c also satisfies p_0 . Note that $I_0 + L_0 + J_0$ is indiscernible over A : if $n = 1$ this is clear, if $n > 1$, then $I'_0 + L_0 + J_0$ is indiscernible over A' , in particular over AI_0 and the results follows from the fact that $I_0 + I'_0 + J_0$ is indiscernible over A . The same is true for $I_0 + L_1 + J_0$. Therefore c is $R(\mathbf{u})$ -incomparable with b , as witnessed by L_0, L_1 . Since c has the same type as b over $A'I_0J_0$, it is also $R(\mathbf{u})$ -incomparable to all the a_k 's, $k < n - 1$.

Construct, for $0 < \alpha < \mu$, sequences K_α^{n-1} which are Morley sequences of $\lim(\text{op}(J_\alpha))$ over everything built so far and each other.

We prove by induction on k that for $k < n - 1$:

(P_k) $(K_\alpha^{\geq k} + J_\alpha : \alpha < \mu)$, are mutually indiscernible over $AI_{<\mu}K_{<\mu}^{<k}a_{\leq k}$.

First, by the argument of $\boxtimes_{6,\sigma}$, for each k we have:

\boxtimes'_6 The sequences $(I_\alpha + K_\alpha^{<k}, K_\alpha^{\geq k} + J_\alpha : \alpha < \mu)$ are mutually indiscernible over Aa_k .

Taking $k = 0$, we obtain (P_0) . Assume (P_{k-1}) and we show (P_k) . The argument is the same as that used to show \boxplus above. The tuples $(a_k, K_\alpha^{k-1}, K_\alpha^{\geq k} + J_\alpha, \phi_\alpha; A)$, $\alpha < \mu$ have the same type all together as $(a, I_\alpha, J_\alpha, \phi_\alpha; A)$ and hence satisfy $\bullet_{2 \rightarrow 5}$. By induction hypothesis, the sequences $K_\alpha^{k-1} + K_\alpha^{\geq k} + J_\alpha$, $\alpha < \mu$, are mutually indiscernible over $Aa_{\leq k-1}$. By \bullet_3 , the sequences $(I_\alpha + K_\alpha^{<k}, K_\alpha^{\geq k} + J_\alpha : \alpha < \mu)$ are mutually stable over $Aa_{\leq k-1}a_k$. By Lemma 3.6, those sequences are mutually indiscernible over $Aa_{\leq k}$ and (P_k) holds.

It follows that the sequences $K_\alpha^{n-2} + K_\alpha^{n-1} + J_\alpha$, $0 < \alpha < \mu$ and $K_0^{n-1} + L_0 + J_0$ are mutually stable over $AI_{<\mu}K_{<\mu}^{<n-2}a_{<n}$ (if $n = 1$, take I_α instead of K_α^{n-2} and remove $I_{<\mu}$ from the base.) Replacing those sequences by Morley sequences of their limit types if necessary and applying Lemma 3.6, we obtain:

\boxplus' The sequences $K_\alpha^{n-2} + K_\alpha^{n-1} + J_\alpha$, $0 < \alpha < \mu$ and $K_0^{n-1} + L_0 + J_0$ are mutually indiscernible over $AI_{<\mu}K_{<\mu}^{<n-2}a_{<n}$.

Decompose in an arbitrary non-trivial way L_0 as $L_0 = L' + L''$. Then by \boxplus' , there is a point $d \models p(\mathbf{v})$ such that $(d, I_\alpha + K_\alpha^{\leq n}, J_\alpha, \phi; A)$ is good for all $0 < \alpha < \mu$, as is $(d, I'_0 + L', L'' + J_0, \phi; A)$. Then neither (d, b) nor (c, d) are in $R(\mathbf{u})$, as witnessed by L' and L'' respectively. There is an infinite subsequence $L'_1 \subseteq L_1$ such that either $(d, I'_0 + L'_1, J_0, \phi; A)$ or $(d, I'_0, L'_1 + J_0, \phi; A)$ is good. In the first case, (d, c) is not in $R(\mathbf{u})$ and d is $R(\mathbf{u})$ -incomparable with c . In this case, set $(a'_{n-1}, a_n) = (c, d)$ and $K_0^{n-1} = L'_1$. In the second case, (b, d) is not in $R(\mathbf{u})$. Thus d is $R(\mathbf{u})$ -incomparable with b . In that case, set $(a'_{n-1}, a_n) = (b, d)$ and $K_0^{n-1} = L'$. Since d satisfies $p(\mathbf{v})$ it is also $R(\mathbf{u})$ -incomparable with each $a_k, k < n - 1$. In both cases, the sequence $(a_0, \dots, a_{n-2}, a'_{n-1}, a_n)$ along with $K_{<\mu}^{\leq n}$ contradicts the maximality of n .

Step 3: We have shown that $\mathbf{v} = (b, I'_0, J_0, \phi; A')$ is linear. If $n = 1$, then we are done. Otherwise, for $0 < \alpha < \mu$, build inductively sequences $K_\alpha^{n-2} \trianglelefteq K_\alpha^{n-2} + I'_\alpha$ and $J_\alpha \trianglelefteq J'_\alpha + J_\alpha$, over everything constructed so far. Let also $J'_0 = J_0$. Then the tuple $(b, (I'_\alpha, J'_\alpha)_{\alpha < \mu}, A)$ has the same type as $(a, (I_\alpha, J_\alpha)_{\alpha < \mu}, A)$. Let σ be an automorphism sending the first tuple to the second. Then we can take $\sigma(A')$ as the A' we need to finish the proof. \square

In the following theorem, by a *linear quasi-order*, we mean a reflexive and transitive relation for which any two points are comparable.

Theorem 5.8. *Let T be NIP, $p(x)$ any type with $\text{opD}(p) \geq \mu$. Then there is an extension $q \supseteq p$ over some set A , relations $R_\alpha(x, y)$, \forall -definable over A , such that each R_α defines a linear quasi-order \leq_α with infinite chains on the set of realizations of $q(x)$. Furthermore, those orders are dense and independent in the sense that if $a_\alpha <_\alpha b_\alpha$ are given for $\alpha < \mu$, then there is $c \models q$ such that $a_\alpha <_\alpha c <_\alpha b_\alpha$ for all $\alpha < \mu$.*

Proof. Let $a \models p$. By the assumption that $\text{opD}(p) \geq \mu$, we can find some $\mathbf{u}_\alpha = (a, I_\alpha, J_\alpha, \phi_\alpha; A)$ which are good and such that the sequences $I_\alpha + J_\alpha$ are mutually indiscernible over A . Using then Lemma 5.5 and Proposition 5.7 and replacing A by A' there, we can assume that all the \mathbf{u}_α are linear. Let then $q = \text{tp}(a / AI_{<\mu}J_{<\mu})$ and $R_\alpha = R(\mathbf{u}_\alpha)$.

To see that the independence condition holds, let $a_\alpha, b_\alpha \models q$ be given with $\neg R_\alpha(b_\alpha, a_\alpha)$ (that is $a_\alpha <_\alpha b_\alpha$). For each $\alpha < \mu$, let K_α be such that both $(a_\alpha, I_\alpha, K_\alpha + J_\alpha, \phi; A)$ and $(b_\alpha, I_\alpha + K_\alpha, J_\alpha, \phi; A)$ are good. The sequences $(I_\alpha + J_\alpha : \alpha < \mu)$ are mutually indiscernible over A , hence the sequences $(I_\alpha + K_\alpha + J_\alpha : \alpha < \mu)$ are mutually stable over A . Replacing the sequences K_α by Morley sequences of their limit types, we can assume that the sequences $(I_\alpha + K_\alpha + J_\alpha : \alpha < \mu)$ are mutually indiscernible over A . Decompose each K_α into two infinite pieces as $K_\alpha = K_\alpha^0 + K_\alpha^1$. By mutual indiscernibility, there is $c \models q$ such that all $(c, I_\alpha + K_\alpha^0, K_\alpha^1 + J_\alpha, \phi; A)$ are good. We then have $a_\alpha <_\alpha c <_\alpha b_\alpha$. This proves both density and independence.

Note that if we carry out this construction without b_α , we get c such that $a_\alpha <_\alpha c$ for each α . This proves that for each order $<_\alpha$ there are two realizations of q which are strictly comparable. Then by density, each order $<_\alpha$ has infinite chains on realizations of $q(x)$. \square

In the case where μ in the above theorem is finite, then we can modify this result to have A be finite, at the cost of weakening independence. This boils down to a simple compactness argument. However, we first need to change slightly the notion of linearity. The reason is that with the previous definitions, we considered sequences K with a certain type over AIJ , hence the base contains I and J and is thus always infinite. In order to be able to have the base be finite, we need to remove I and J from it. To this end, we introduce the following definitions.

Definition 5.9. Let $\mathbf{u} = (\pi(x), I, J, \phi; A)$ be good. We define the following binary relations on realizations of $\pi(x)$:

- $(a, b) \in E_0(\mathbf{u})$ if for any infinite sequence L of same EM-type as I over A , if $\phi(a; e)$ holds for all $e \in L$, then $\phi(b; e)$ holds for almost all $e \in L$ and symmetrically if $\neg\phi(a; e)$ holds for all $e \in L$, then $\neg\phi(b; e)$ holds for almost all $e \in L$;
- $(a, b) \in R_0(\mathbf{u})$ if we cannot find an infinite sequence L of same EM-type as I over A such that both $\phi(a; e)$ and $\neg\phi(b; e)$ holds for all $e \in L$.

We have as previously that $E_0(\mathbf{u})$ is a \forall -definable equivalence relation on realizations of $\pi(x)$, that $R_0(\mathbf{u})$ is a \forall -definable reflexive, transitive, $E_0(\mathbf{u})$ -equivariant, relation. Furthermore we have $E_0(a, b) \iff R_0(a, b) \wedge R_0(b, a)$.

We say that \mathbf{u} is linear_0 if any two realizations of $p(\mathbf{u})$ are $R_0(\mathbf{u})$ -comparable.

The following properties follow at once from the definitions.

Lemma 5.10. Let $\mathbf{u} = (a, I, J, \phi; A)$ be good and build $\mathbf{u}' = (a, I', J', \phi; AIJ)$. where $I \trianglelefteq_{AIJ} a$, $I + I'$ and $J \trianglelefteq_{AIJ} a$, $J' + J$.

Then:

1. \mathbf{u}' is good;
2. $E(\mathbf{u}) = E_0(\mathbf{u}')$ and $R(\mathbf{u}) = R_0(\mathbf{u}')$ on realizations of $p(\mathbf{u}')$;
3. if \mathbf{u} is linear, then \mathbf{u}' is linear_0 .

The following lemma is true with linear instead of linear_0 , but is truly useful only in the latter case.

Lemma 5.11. If $\mathbf{u} = (\pi(x), I, J, \phi; A)$ is linear_0 then there is a finite $A_0 \subseteq A$ and a formula $\theta(x) \in \pi(x)$ such that $(\theta(x), I, J, \phi; A_0)$ is linear_0 .

Proof. Saying that $(\pi(x), I, J, \phi; A)$ is linear_0 is saying that it is good and that one cannot find two infinite sequences $L = (c_i : i < \omega)$ and $K = (d_i : i < \omega)$, both having the same EM-type as I over A such that for $i < \omega$,

$$\models \phi(a; c_i) \wedge \neg\phi(a'; c_i) \wedge \neg\phi(a; d_i) \wedge \phi(a'; d_i).$$

By compactness, one can find a finite $A_0 \subseteq A$ and formula $\theta(x) \in \pi(x)$ such that this also holds with A replaced by A_0 and $\pi(x)$ replaced by $\theta(x)$. Then $(\theta(x), I, J, \phi; A_0)$ is also good and is linear_0 . \square

Note that the goodness hypothesis is not needed for the compactness argument, but is necessary to ensure that the order we construct is not trivial.

We now state our main theorem. We only state the result with finite bases for one order, to simplify the statement. The n -order version is below, stated under the ω -categorical assumption.

Theorem 5.12. Let T be NIP, unstable. Then there is a finite set A , a formula $\theta(x)$ over A and a relation $R(x, y)$ \forall -definable over A which defines a dense linear quasi-order on $\theta(x)$ with an infinite chain.

Proof. As T is unstable, the op-dimension of $x = x$ is at least 1 and we can find some good quintuple $\mathbf{u} = (a, I, J, \phi; A)$. By Proposition 5.7, up to increasing A , we can take \mathbf{u} to be linear. By Lemma 5.10, increasing A some more, we can assume that \mathbf{u} is linear_0 . Then by Lemma 5.11, we get some $\mathbf{u}_0 = (\theta(x), I, J, \phi; A_0)$ which is linear_0 , with A_0 finite. Then $R = R_0(\mathbf{u}_0)$ is as required. \square

Theorem 5.13. *If the theory T is ω -categorical, NIP, $\text{opD}(x = x) \geq n > 0$, then there is a finite set A_* , an A_* -definable set X and n A_* -definable linear quasi-orders \leq_1, \dots, \leq_n on p , such that the structure $(X; \leq_1, \dots, \leq_n)$ contains an isomorphic copy of every finite structure $(X_0; \leq_1, \dots, \leq_n)$ equipped with n linear orders.*

Proof. This is similar to Theorem 5.12 except that we start with n good quintuples $\mathbf{u}_k = (a, I_k, J_k, \phi; A)$, where $(I_k + J_k : k < n)$ are mutually indiscernible over A . By Proposition 5.7, we can assume that they are linear. By the same argument as in the proof of Theorem 5.8, the order $R(\mathbf{u}_k)$ are independent.

As in Lemma 5.10, we can successively construct $I_k \trianglelefteq I_k + I'_k$ and $J_k \trianglelefteq J'_k + J_k$, each time over everything constructed so far. Then replacing I_k by I'_k , J_k by J'_k and A by $AI_{<n}J_{<n}$ we can assume that the quintuples are linear₀. Note that after having done this substitution, the sequence $(I_k + J_k : k < n)$ are still mutually indiscernible over A . Lemma 5.11 then gives us linear₀ quintuples $\mathbf{u}_k^0 = (\theta_k(x), I_k, J_k, \phi; A_k)$, with A_k finite. Let $A_* = \bigcup_{k < n} A_k$ and $\theta(x) = \bigwedge_{k < n} \theta_k(x)$. Then each $\mathbf{u}_k^1 := (\theta(x), I_k, J_k, \phi; A_*)$ is linear₀ and we define the order \leq_k to be given by the relation $R_0(\mathbf{u}_k^1)$.

Note that if $(b, a) \notin R(\mathbf{u}_k)$, then also $(b, a) \notin R_0(\mathbf{u}_k^1)$ witnessed by the same sequences, and thus $a <_k b$. The statement about universality therefore follows from the independence of the orders $R(\mathbf{u}_k)$. \square

5.1 Theories with no interpretable linear order

Having found a linear order, the natural next step would be to understand the induced structure on it. When the order is interpretable, this becomes an instance of the classical problem of studying NIP ordered structures. The dp-minimal case in particular has received some attention (see e.g. [Goo10], [Sim11]), though most results assume an ordered-group structure. The ω -categorical case is considered in [Sim18]. However, we expect that more often than not, the order we constructed will be strictly \forall -definable. It seems likely that one could actually take advantage of it as the non-definability limits the possibilities for the induced structure. We give an example of that here and leave further studies for later. We show that if the theory does not interpret any infinite linear order, then in some precise sense, the induced structure on the \forall -definable quotient is weakly o-minimal.

We work in a general context not relying on the previous notations. Let D be a definable set over some A and $S(x, y)$ a \wedge -definable relation over A such that $S(x, y) \rightarrow D(x) \wedge D(y)$ and $S(x, y)$ is a strict linear quasi-order with infinite chains on D that is:

- S is transitive and anti-reflexive;
- $E(x, y) := \neg S(x, y) \wedge \neg S(y, x)$ is (a \forall -definable) equivalence relation such that S is E -equivariant;
- S induces an infinite strict linear order on the quotient D/E .

Assume also that S is type-definable by a countable conjunction of formulas. (Note that we can always ensure this in our construction since we end up with a finite A and we can work in a reduct to a countable sublanguage.)

We can then write $S(x, y) = \bigwedge_{i < \omega} S_i(x, y)$ such that:

- $S_0(x, y) \rightarrow D(x) \wedge D(y)$;
- $\neg(\exists x, y) S_0(x, y) \wedge S_0(y, x)$;
- $(\forall x, y) S_{i+1}(x, y) \rightarrow S_i(x, y)$;
- $(\forall x, y) S_{i+1}(x, y) \wedge S_{i+1}(y, z) \rightarrow S_i(x, z)$.

Note that for all $i < \omega$, we have:

$$(\forall x, y) [(S(x, y) \wedge S_i(y, z)) \rightarrow S(x, z)].$$

Why? Assume that $\neg S(x, z)$ and $S(x, y)$ hold. Then as S is linear, we must have $S(z, y)$, hence $S_0(z, y)$ holds. By the second bullet above, this implies $\neg S_0(y, z)$ and hence $\neg S_i(y, z)$.

In particular:

$$(\Box) \quad (\forall x, y) [(S(x, y) \wedge S_i(y, z)) \rightarrow S_i(x, z)].$$

Now for $i < \omega$, define $x \leq_i y$ as:

$$a \leq_i b \iff (\forall c \in D) S_i(b, c) \rightarrow S_i(a, c).$$

Then \leq_i is a transitive, reflexive relation and by (\Box) , for $a, b \in D$ we have

$$S(a, b) \rightarrow a \leq_i b \rightarrow \neg S(b, a).$$

We also define $a <_i b \iff (a \leq_i b) \wedge \neg(b \leq_i a)$. This is a transitive, irreflexive relation.

Proposition 5.14. *Assume that there is a definable subset $X \subseteq D$ whose projection on D/E is not a finite union of convex sets. Then there is an infinite interpretable linear order.*

Proof. Let B be such that X is defined over B . The assumption implies that for all $n < \omega$, we can find $a_1, b_1, \dots, a_n, b_n \in D$ such that:

- if a'_i is E -equivalent to a_i , then $a'_i \notin X$;
- $b_i \in X$ for all $i \leq n$;
- we have $S(a_1, b_1) \wedge S(b_1, a_2) \wedge S(a_2, b_2) \wedge \dots \wedge S(a_n, b_n)$.

Consider the definable set $F = \{x \in D : (\forall y \in X) x <_0 y \vee y <_0 x\}$, that is the set of points strictly \leq_0 -comparable to all points in X . Note that if $x \in D$ is not E -equivalent to any point in X , then $x \in F$, since it will even be S -comparable to all points in X . For $a \in F$, let $X(a) = \{x \in X : a <_0 x\}$.

Claim: The sets $X(a)$, $a \in F$ are linearly ordered by inclusion.

Proof: Assume that $a, b \in F$ are such that $X(a) \not\subseteq X(b)$ and let $x \in X(a) \setminus X(b)$. We then have $a <_0 x$ and $\neg(b <_0 x)$. Since b is in F , we must have $x <_0 b$. If there is $y \in X(b) \setminus X(a)$, then we have $a <_0 x <_0 b <_0 y <_0 a$ and $a <_0 a$ by transitivity of $<_0$, which is absurd. We conclude that $X(b) \subseteq X(a)$.

On F we can define the equivalence relation

$$E_X(a, b) \iff X(a) = X(b).$$

By the assumptions on X , the quotient F/E_X is infinite. It is also linearly ordered by $a \leq b \iff X(a) \subseteq X(b)$, which finishes the proof. \square

6 Stable dimension

This section is independent of the rest of the paper. We define the natural counterpart to op-dimension. We only show basic properties and leave its in depth study for later. Throughout this section, we assume that T is NIP.

Definition 6.1. Let A be a set of parameters and $\pi(x)$ a partial type over A . We say that $\text{st-dim}(\pi, A) < \kappa$ if we cannot find the following:

- a tuple $a \models \pi$;
- infinite sequences $(I_i : i < \kappa)$ and $(J_i : i < \kappa)$ such that $(I_i + J_i : i < \kappa)$ are mutually indiscernible over Aa ;
- tuples $(b_i : i < \kappa)$ such that $(I_i + b_i + J_i : i < \kappa)$ are mutually indiscernible over A , but for each $i < \kappa$, $I_i + b_i + J_i$ is not indiscernible over Aa .

Note that if we have such a witness to $\text{st-dim}(\pi, A) < \kappa$, then we can build one where the sequences have any given order type, by replacing them by Morley sequences of one of their limit types. In particular, we can ask for them to be indexed by \mathbb{Q} .

The following follows at once from the definitions.

Lemma 6.2. *If $\text{dp-rk}(\pi, A) < \kappa$, then $\text{st-dim}(\pi, A) < \kappa$. In particular, if T is NIP, then $\text{st-dim}(\pi, A) < |T|^+$.*

The base change lemma is slightly harder to prove than for dp-rank. We start with a basic lemma about NIP.

Lemma 6.3. *For $i < \alpha$, let $I_i = (a_{i,j} : j < \beta)$ be an indiscernible sequence. Assume that the sequences $(I_i : i < \alpha)$ and mutually indiscernible over \emptyset and the sequence of columns $((a_{i,j} : i < \alpha) : j < \beta)$ is indiscernible over A . Then the sequences $(I_i : i < \alpha)$ are mutually indiscernible over A .*

Proof. To simplify notations, let us assume that $\alpha = 2$. The general case is similar. Start by increasing the sequence of columns to one indexed by \mathbb{Q} of same EM-type over A . If the sequences I_0, I_1 are not mutually indiscernible over A , then there is a formula $\phi(x_1, \dots, x_n; y_1, \dots, y_m) \in L(A)$ and four increasing tuples of elements of \mathbb{Q} : $\bar{i} := (i_1, \dots, i_n)$, $\bar{i}' := (i'_1, \dots, i'_n)$, $\bar{j} := (j_1, \dots, j_m)$ and $\bar{j}' := (j'_1, \dots, j'_m)$ such that $\models \phi(a_{0,\bar{i}}, a_{1,\bar{j}}) \wedge \neg \phi(a_{0,\bar{i}'}, a_{1,\bar{j}'})$.

Now construct two sequences $\bar{i}_0 < \bar{i}_1 < \dots$ and $\bar{j}_0 < \bar{j}_1 < \dots$ of increasing tuples of elements of \mathbb{Q} so that for k even (\bar{i}_k, \bar{j}_k) has same order-type as (\bar{i}, \bar{j}) and for k odd, (\bar{i}_k, \bar{j}_k) has same order-type as (\bar{i}', \bar{j}') . Then the sequence of $n + m$ -tuples $(a_{0,\bar{i}_k} \hat{a}_{1,\bar{j}_k} : k < \omega)$ is indiscernible and the formula ϕ alternates infinitely on it. This contradicts NIP. \square

Lemma 6.4. *If $A \subseteq B$ and $\pi(x)$ is a partial type over A , then $\text{st-dim}(\pi, A) = \text{st-dim}(\pi, B)$.*

Proof. If $(a, I_i, J_i, b_i : i < \kappa)$ is a witness to $\text{st-dim}(\pi, B) \geq \kappa$, then $(a, I'_i, J'_i, b'_i : i < \kappa)$ witnesses $\text{st-dim}(\pi, A) \geq \kappa$, where I'_i is obtained from I_i by concatenating a fixed enumeration of B to the end of every element of it, and same for J'_i and b'_i .

Conversely, assume that $(a, I_i, J_i, b_i : i < \kappa)$ is a witness to $\text{st-dim}(\pi, A) \geq \kappa$. By Ramsey and compactness, we can find sequences I'_i, J'_i of same EM-type as I_i, J_i over Aa such that $(I'_i + J'_i : i < \kappa)$ are mutually indiscernible over Ba . Replacing I_i, J_i by I'_i, J'_i , we can assume that $(I_i + J_i : i < \kappa)$ are mutually indiscernible over Ba . Without loss, all sequences are indexed by \mathbb{Q} . Increase each sequence to $I_i + J_i + K_{i,0} + K_{i,1} + \dots$, preserving mutual indiscernibility over Ba . Then for each $n < \omega$, the family of pairs $((I_i + J_i + K_{i,<n}, K_{i,\geq n}) : i < \kappa)$ has the same type as $((I_i, J_i) : i < \kappa)$ over Ba . For each such n , let $b_{i,n}$ be such that

$$I_i, b_i, J_i \equiv_{Aa} I_i + J_i + K_{i,<n}, b_{i,n}, K_{i,\geq n}.$$

Thinking of the sequences $(I_i + J_i + K_{i,<n} : i < \kappa)$ as rows of an array, the sequence of columns is indiscernible. Applying [Sim13, Lemma 2.8] to this sequence of columns, there is an automorphism σ fixing $A, I_i, J_i, K_{i,<\omega}$ such that for each n , the sequence of columns of the array $(I_i + J_i + K_{i,<n} + \sigma(b_{i,n}) + K_{i,\geq n} : i < \kappa)$ is indiscernible over B . By Lemma 6.3, for each n , the sequences $(I_i + J_i + K_{i,<n} + \sigma(b_{i,n}) + K_{i,\geq n} : i < \kappa)$ are mutually indiscernible over B . Given a finite set $X \subseteq \kappa$, a finite set Δ of formulas and a finite $B_0 \subseteq B$, by shrinking of indiscernibles in NIP (e.g. [Sim15, Proposition 3.32]), there is $n = n(\Delta)$ such that the sequences $(K_{i,<n} + K_{i,\geq n} : i \in X)$ are mutually Δ -indiscernible over $B_0\sigma(a)$. By compactness, we can find $(a', I'_i, J'_i, b'_i : i < \kappa)$ so that

$$a', I'_i, b'_i, J'_i \equiv_A a, I_i, b_i, J_i$$

and $(I'_i + b'_i + J'_i : i < \kappa)$ are mutually indiscernible over B . This witnesses $\text{st-dim}(\pi, B) \geq \kappa$. \square

We can now define $\text{st-dim}(\pi)$ as being equal to $\text{st-dim}(\pi, A)$ for some/any A over which π is defined. As usual, we define $\text{st-dim}(a/A)$ as $\text{st-dim}(\text{tp}(a/A))$.

Proposition 6.5. *Let $\pi(x)$ be any partial type and κ a cardinal. Then the following are equivalent:*

1. $\text{st-dim}(\pi) < \kappa$;
2. *given A over which π is defined, $a \models \pi$ and dense endless sequences $(I_i : i < \kappa)$ which are mutually indiscernible over A and mutually stable over Aa , there is $i < \kappa$ such that I_i is indiscernible over Aa ;*
3. *given A over which π is defined, $a \models \pi$ and dense endless sequences $(I_i : i < \lambda)$ which are mutually indiscernible over A and mutually stable over Aa , we can find $X \subseteq \lambda$, $|X| < \kappa$ such that the sequences $(I_i : i \in \lambda \setminus X)$ are mutually indiscernible over Aa .*

Proof. (1) \Rightarrow (2): Let A, a and $(I_i : i < \kappa)$ be as in (2). Construct sequences J_i^0, J_i^1 indexed by \mathbb{Z} so that $I_i \trianglelefteq_{Aa} J_i^0 + I_i + J_i^1$. By the assumption of mutual stability, the sequences $(J_i^0 + J_i^1 : i < \kappa)$ are mutually indiscernible over Aa . Assume that the conclusion of (2) fails. Then for each $i < \kappa$, there is an integer n_i and a subsequence \bar{a}_i of I_i of size n_i such that $J_i^0 + \bar{a}_i + J_i^1$ is not indiscernible over Aa . If $J_i^0 = (b_i : i \in \mathbb{Z})$, define $J_i^0 = (b_{kn_i} \cap \dots \cap b_{kn_i+n_i-1} : k \in \mathbb{Z})$. Define J_i^1 similarly. Having done this for all $i < \kappa$, we see that the family $(J_i^0 + (\bar{a}_i) + J_i^1 : i < \kappa)$ witnesses $\text{st-dim}(\pi) \geq \kappa$.

(2) \Rightarrow (3): The argument is similar as for the analogous result for dp-rank from [KOU13] (also presented in [Sim15, Proposition 4.17]). The case where κ is infinite is rather straightforward: If the conclusion of (3) fails, we can construct inductively a sequence $(\delta_t : t < \kappa)$ of elements of λ and a sequence $(\Delta_t : t < \kappa)$ of finite subsets of λ such that:

- the sequence I_{δ_t} is not indiscernible over $Aa \cup \bigcup \{I_i : i \in \Delta_t\}$;
- the sets $\Delta'_t := \Delta_t \cup \{\delta_t\}$, $t < \kappa$, are pairwise disjoint.

For $t < \kappa$, let J_t be the sequence of columns of the array whose rows are the sequences $(I_i : i \in \Delta'_t)$. Since by hypothesis those rows are not mutually indiscernible over Aa , Lemma 6.3 implies that the sequence J_t is not indiscernible over Aa . Furthermore, the sequences $(J_t : t < \kappa)$ are mutually stable over A : take the same sequences witnessing mutual stability of the I_t 's (in the sense of Proposition 3.4 (1)). This shows that (2) fails.

For $\kappa = n + 1$ finite, we prove the result by induction on λ . If $\lambda \leq n$, then we can take $X = \lambda$. Assume that $\lambda = n + k + 1$ is finite. Construct inductively endless sequences J_i^0, J_i^1 so that $I_i \leq J_i^0 + I_i + J_i^1$, each one over everything built so far. Let $B = A \cup \bigcup \{J_i^0, J_i^1 : i < \lambda\}$. Then the sequences $(I_i : i < \lambda)$ are mutually indiscernible over B and mutually stable over Ba . By (2), there is $i(*) < \lambda$ such that $I_{i(*)}$ is indiscernible over Ba .

Claim: The sequences $(I_i : i \neq i(*))$ are mutually stable over $AI_{i(*)}a$.

Proof: It suffices to show that the sequences $(J_i^0 + J_i^1 : i \neq i(*))$ are mutually indiscernible over $AI_{i(*)}a$. If not, then there is some formula $\phi(\bar{u}; \bar{v}; \bar{a})$ witnessing it, where \bar{u} is a tuple of elements from $(J_i^0 + J_i^1 : i \neq i(*))$, \bar{v} a tuple of elements from $I_{i(*)}$ and \bar{a} a tuple of elements from Aa . Now by mutual stability, $J_{i(*)}^0 + J_{i(*)}^1$ is indiscernible over $(J_i^0 + J_i^1 : i \neq i(*)) \cup Aa$. By indiscernibility, \bar{u} can be taken to be any tuple of the right order type from $I_{i(*)}$. Therefore by Fact 3.1, if we take \bar{u}' in $J_{i(*)}^0$ of the same order type, we also have $\phi(\bar{u}; \bar{v}; \bar{a})$. But this contradicts the fact that $(J_i^0 + J_i^1 : i \neq i(*))$ are mutually indiscernible over $AaJ_{i(*)}^0J_{i(*)}^1$.

By induction hypothesis, working over the base set $AI_{i(*)}$, there is $X_0 \subseteq X \setminus \{i(*)\}$ of size at most n such that the sequences $(I_i : i \notin X_0 \cup \{i(*)\})$ are mutually indiscernible over $AI_{i(*)}a$. If $I_{i(*)}$ is indiscernible over $Aa \cup (I_i : i \notin X_0 \cup \{i(*)\})$, then we can take $X = X_0$. Assume that this is not the case and we reach a contradiction as in the previous claim. As the sequences $(I_i : i \notin X_0 \cup \{i(*)\})$ are mutually indiscernible over $AI_{i(*)}a$ and the J_i^0 's are built over all the I_i 's and Aa , the parameters from $(I_i : i \notin X_0 \cup \{i(*)\})$ needed to witness that $I_{i(*)}$ is not indiscernible can be taken in $(J_i^0 : i \notin X_0 \cup \{i(*)\})$ instead. But then $I_{i(*)}$ is not indiscernible over Ba : contradiction.

Finally, the case of infinite λ can be deduced easily from the finite case as in [Sim15, Proposition 4.17].

(3) \Rightarrow (1) is clear. □

Proposition 6.6. *Let A be any set of parameters and a, b two tuples. If $\text{st-dim}(a/A) < \kappa_1$ and $\text{st-dim}(b/Aa) < \kappa_2$, then $\text{st-dim}(a, b/A) < \kappa_1 + \kappa_2 - 1$.*

Proof. Let $(I_i : i < \lambda)$ be mutually indiscernible over A and mutually stable over Aab . We can find $X_1 \subseteq \lambda$, $|X_1| < \kappa_1$ such that the sequences $(I_i : i \in \lambda \setminus X_1)$ are mutually indiscernible over Aa . Next, we find $X_2 \subseteq \lambda$, $|X_2| < \kappa_2$ such that the sequences $(I_i : i \in \lambda \setminus (X_1 \cup X_2))$ are mutually indiscernible over Aab . This shows that $\text{st-dim}(a, b/A) < \kappa_1 + \kappa_2 - 1$. □

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