NIP ω-categorical structures: the rank 1 case

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Abstract

We classify primitive, rank 1, ω-categorical structures having polynomially many types over finite sets. We show that there are only finitely many such structures with a fixed number of 4 types and that they are built out of finitely many linear or circular orders interacting in a restricted number of ways. As an example of application, we deduce the classification of primitive structures homogeneous in a language consisting of n linear orders as well as all reducts of such structures.

1 Introduction

Since the work of Lachlan on finite homogeneous structures, interactions between homogeneous structures and model theory have been very fruitful in both directions. Lachlan [Lac84] realized that the property of stability and the toolbox that comes with it were relevant in the finite case. Geometric stability theory had its birth in Zilber's work on totally categorical structures [Zil] and this in turn lead to a fairly detailed understanding of the ω-stable ω-categorical structures ([CHL85], [Hru89]). Following a suggestion of Lachlan, this analysis was then generalized to smoothly approximable structures. This was done first by Kantor, Liebeck, Macpherson [KLM89] in the primitive case using classification of finite simple groups, and then by Cherlin and Hrushovski [CH03] in the general case by model-theoretic methods. In that latter work, many features of simple theories first appeared. The present paper fits in this line of research and begins the study of yet another class of ω-categorical structures defined by a model theoretic condition.

To define this class, let us restrict first to the case of structures homogeneous in a finite relational language (which we also call *finitely homogeneous*). If M is such a structure, then given any finite $A \subseteq M$, the number of 1-types over A (that is, the number of orbits under the stabilizer of A) is finite. For a given n, we let $f_M(n)$ be the maximal number of 1-types over a set $A \subseteq M$ of size n. For instance, if $M = (\mathbb{Q}, \leq)$, then $f_M(n) = 2n+1$. If $M = (G, R)$ is a model of the random graph, then $f_M(n) = 2^n + n$. A well-known theorem of Sauer and Shelah implies that this function has either polynomial or exponential growth. Following the model-theoretic terminology, we call a finitely

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homogeneous structure M *NIP* if the function f_M has polynomial growth. (*NIP* stands for the negation of the independence property. We like to think of those structures as being *geometric* in some sense.) For instance, dense linear orders are NIP, whereas the random graph is not. Intuitively, NIP structures have no random-like behavior. Another important example of NIP structure is the Fraïssé limit of finite trees (where a tree $(T, \leq, \wedge)$ is a partial order such that the predecessors of a point form a chain, and $a \wedge b$ is the infimum of $\{a, b\}$).

Within structures homogeneous in a finite relational language, there is another characterization of NIP obtained by counting orbits on unordered k-tuples, or equivalently finite substructures of size k up to isomorphism. If M is homogeneous in a finite relational language (or more generally an ω-categorical relational structure), define $\pi_M(k)$ as being the number of substructures of M of size k. Cameron showed in [Cam81] that this function is always non-decreasing and in [Cam76] he classified the case where π_M is constant equal to 1. Macpherson [Mac85] showed that if M is primitive, then π_M is either constant equal to 1 or grows at least exponentially. A number of structures for which the growth is no faster than exponential are given by Cameron in [Cam87]: they are all order-like or tree-like structures. Cameron also remarks there that those seem to be essentially the only examples of such structures known at the time. In [Mac87], Macpherson shows that for structures homogeneous in a finite relational language, there is a gap in the possible growth rates of the function π_M. Using the aforementioned Sauer–Shelah theorem, we can state a stronger version of his result: if M is NIP, then $\pi_M(k) = O(2^{ck \ln k})$ for some $c > 0$ (see the remark after Fact 2.13). If M has IP (is not NIP), then $\pi_M(k) \geq 2^{p(k)}$ for some polynomial $p(X)$ of degree at least 2. Hence finitely homogeneous structures with π_M of exponential growth are a subclass of NIP homogeneous structures. See e.g. [Mac11, Section 6.3] for many more results on this function.

We conjecture that NIP finitely homogeneous structures can be reasonably well classified, and in particular that there are only countably many up to bi-interpretability. We will give some precise conjectures at the end of this paper. What we have in mind is that those structures are all built out of linear orders, possibly branching into trees. However, we are for now not capable of saying much in the general case, and introduce another condition, which should be thought of as forbidding trees in the structure: we ask that there is a rank function on definable sets satisfying certain axioms. This limits the size of a nested sequence of definable equivalence relations. In model theory, this condition is called *rosiness*. It is always satisfied by binary structures, so one may want to think of this work as studying binary NIP homogeneous structures, though our actual hypothesis are *a priori* more general. We will actually relax the homogeneity assumption to ω-categoricity. Similarly, NIP, which we defined by counting types, becomes a condition on formulas. Under those hypotheses, we conjecture that the results on ω-categorical ω-stable and quasi-finite structures essentially go through *mutatis mutandis*. In particular, we should have coordinatization by rank 1 sets and quasi-finite axiomatization. We deal here only with the rank 1 primitive case, for which we give a complete classification, up to bi-definability. The general finite rank case is studied in a subsequent work [OS21] with Alf Onshuus.

As a rather straightforward application, we classify primitive homogenous multi-orders (also called finite-dimensional permutation structures): that is primitive structures homogeneous in a language consisting of n linear orders. For $n = 2$, this was solved by Cameron [Cam02] and for $n = 3$ by Braunfeld [Bra18], where the general case is conjectured. We show that for any n, there is only one primitive homogeneous multi-order, where no two orders are equal or reverse of each other: the Fraïssé limit of finite sets with n orders. We also classify all reducts of such structures, generalizing the work of Linman and Pinsker [LP15] on the case $n = 2$.

Looking at it from the point of view of model theory, one can see this work as a development of the study of (rosy) NIP structures along the lines of stable theories. We hope that it will eventually lead to new insights into general NIP structures. At any rate, the results demonstrate that there is a richer theory of NIP than one suspected only a few years ago and that this world is much more structured and closer to stability than was expected. It does not seem completely unreasonable to hope for classification results for some subclasses of NIP in the spirit of Shelah's classification for superstable theories, where cardinal dimensions will be complemented by isomorphism types of linear orders (which are shown to exist in [Sim22]). But we are not quite there yet.

1.1 Summary of results

We are concerned with structures M such that:

($\star$) M is an ω-categorical, rank 1, primitive, unstable NIP structure,

where "rank 1" means that, in M^{eq}, there is no infinite definable set E and uniformly definable family $(X_t)_{t \in E}$ of infinite subsets of M which is k-inconsistent for some k: that is for any k values $t_1, \dots, t_k \in E$, we have $X_{t_1} \cap \cdots \cap X_{t_k} = \emptyset$. Those hypotheses will be fully enforced only in Section 6. In sections before that, we study ω-categorical linear and circular orders under a weakening of the rank 1 assumption, but make no use of NIP. Results there might be of some use in the classification of other classes of ordered homogeneous structures. We then give a fairly explicit description of structures satisfying ($\star$) up to bi-definability. They all admit an interpretable finite cover composed of a disjoint union of linear and circular order, independent of each other.

Here are some examples of structures that satisfy the hypotheses.

Example 1.1.

- *A dense linear order or any of its 3 non-trivial reducts: a betweenness relation, circular order or separation relation.*
- *The Fraïssé limit of finite sets equipped with n orders.*
- *The class of structures equipped with two linear orders $\leq_1, \leq_2$ and a binary relation R that satisfies $a' \leq_1 a \; R \; b \leq_2 b' \Rightarrow a' \; R \; b'$ and $\neg a \; R \; a$ is a*

Fraïssé class. Its Fraïssé limit satisfies $(\star)$. This kind of structure will be studied in Section 3.1.

- *The class of finite sets equipped with a circular order C and an equivalence relation E all of whose classes have exactly two elements is a Fraïssé class. The quotient by E of the Fraïssé limit of this class satisfies $(\star)$. It does not admit a circular order definable over $\operatorname{acl}^{eq}(\emptyset)$ but does have one definable over any one parameter.*

As a consequence of the classification we obtain the following theorems (the terminology will be explained later).

Theorem 1.2. *Given an integer n, there are, up to bi-definability, finitely many ω-categorical primitive NIP structures M of rank 1 having at most n 4-types.*

Theorem 1.3. *If M is an ω-categorical, primitive, rank 1, NIP, unstable structure, then:*

1. *over $\emptyset$, there is an interpretable set W, which is a finite union of circular orders and admits a finite-to-one map to M;*
2. *up to inter-definability, M is homogeneous in a finite relational language and finitely axiomatizable;*
3. *after naming a finite set of points, M admits elimination of quantifiers in a binary language and has a definable linear order;*
4. *M is distal of finite op-dimension;*
5. *M has trivial geometry: $\operatorname{acl}(A) = A$ for every $A \subseteq M$, equivalently the stabilizer of any finite $A \subseteq M$ in the automorphism group of M has no finite orbit on $M \setminus A$.*

Statement 1 follows from the construction in Section 6. Statements 2 and 3 are proved in Section 6.6, along with distality. Statement 5 also follows from the discussion there. Finiteness of op-dimension is Proposition 6.19.

As regards homogeneous multi-orders, we prove the following.

Theorem 1.4. *Let $(M; \leq_1, \ldots, \leq_n)$ be homogeneous, primitive and such that each $\leq_i$ defines a linear order on M. Assume that no two of those orders are equal or reverse of each other. Then M is the Fraïssé limit of finite sets with n orders.*

The proof of this last theorem requires only a small part of the paper, namely Sections 2, 3 and 7. The imprimitive case is classified in [BS20], joint with Samuel Braunfeld.

1.2 Overview of the proof

Let M be an L-structure that satisfies $(\star)$. The starting point for this work is the result proved in [Sim22] that any NIP ω-categorical unstable structure interprets a linear order. In fact more is true: Guingona and Hill introduce in [GH15] the notion of op-dimension, which tells us the maximal number of independent orders that a structure (or type) can have. The main theorem of [Sim22] says—in the ω-categorical case—that if M is NIP of op-dimension at least n, then we can find some infinite definable set X on which we can interpret n linear orders. By transitivity of M, the family of conjugates of X covers M.

In Section 3, we show that any extra structure on a rank 1 linear order must be dense with respect to the order and that different definable orders can interact only in a few prescribed ways. This is extended to circular orders in Section 4. (Those sections make no use of NIP.) This allows us to glue the orders coming from various conjugates of X together. Each order might then wrap around itself, yielding a circular order. We construct in this way a 0-definable finite family W of linear and circular orders. We also show that this W is a finite cover of M, that is it admits a finite-to-one map onto M. This is all done in Section 6.

We then have to analyze the additional structure on W. Using op-dimension, we show that any additional structure must come from stable formulas. By rank 1, those formulas cannot fork. Using finiteness of the number of non-forking extensions, those formulas can be defined from *local* equivalence relations with finitely many classes. Here *local* means that the equivalence relation is only defined locally, on bounded intervals of the orders, and may not glue as an equivalence relation on the whole structure. Such relations are studied in Section 5, in which a purely topological discussion shows that they must come from connected finite covers of circular orders.

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2 Preliminaries

2.1 Model theoretic terminology

We will use standard model-theoretic notation and terminology. Lowercase letters such as a, b, c, x, y, z will usually denote finite tuples of elements or variables: $a \in M$ means $a \in M^{|a|}$. We will sometimes write say $\bar{a}, \bar{x}$ if we want to emphasize this.

For the sake of completeness, we recall some basic definitions. More details can be found in any introductory book on model theory, for instance [Mar02] or [TZ12]. We first give the general definitions that make sense in arbitrary structures, and then give equivalent formulations in terms of automorphism groups, that are only valid in the ω-categorical case, over finite set of parameters.

We work in a structure M, in a countable language L. Let $B \subseteq M$ be any set. A subset $X \subseteq M^k$ is *definable over B*, or *B-definable* if it is the solution set of a first-order formula $\phi(x;b)$, where b is a tuple of parameters from B. A set is *definable* if it is definable over some B. We write *0-definable* to mean $\emptyset$-definable. The notation $M \models \phi(a;b)$ and $a \models \phi(x;b)$ both mean that $\phi(a;b)$ is true in M. Since we will work throughout in a fixed structure M, we will usually not indicate it and simply write $\models \phi(a;b)$ instead of $M \models \phi(a;b)$. If $\pi(x)$ is a set of formulas all with variable x, we write $a \models \pi(x)$ to mean $a \models \phi(x)$ for all $\phi(x) \in \pi(x)$.

The *type* (or *complete type*) of a tuple $a \in M^k$ over B, denoted $\mathrm{tp}(a/B)$, is the set of formulas $\phi(x;b)$, $|x| = |a|$, with parameters b in B that hold of a. If $B = \emptyset$, we may omit it. If $p = \mathrm{tp}(a/B)$, we usually write $p \vdash \phi(x;b)$ to mean $\phi(x;b) \in p$. The set of types in k variables over B is denoted $S_k(B)$. We sometimes omit k if it is clear from the context, or irrelevant. We write $a \equiv_B a'$ to mean that a and a' have the same type over B. We will often abuse terminology by saying that a formula $\phi(x)$ is a complete type if it implies a complete type, or in other words can be uniquely extended to a complete type. Similarly, we will say that a definable set X is a complete type if it is the set of realization of a complete type, or equivalently is defined by a complete type.

The *definable closure* of a set A, denoted $\mathrm{dcl}(A)$, is the set of elements $c \in M$ for which there exists a formula $\phi(x;a)$, a a tuple from A, of which c is the only solution in M. Similarly the *algebraic closure* of a set A, denoted $\mathrm{acl}(A)$, is the set of elements $c \in M$ for which there exists a formula $\phi(x;a)$, a a tuple from A, satisfied by c and which has only finitely many solutions in M.

It is often important to consider not only definable subsets of M (or M^k), but also quotients of definable subsets by definable equivalence relations. A convenient way to do this is to introduce a multisorted structure M^{eq} in which such quotients are represented by definable sets. More precisely, M^{eq} has a sort M_E for every 0-definable equivalence relation E on some M^n. The sort M_E is interpreted as the quotient of M by E. The sort $M_=$ is identified with M and equipped with the same structure as M. Furthermore, for each E as above, we equip M^{eq} with the canonical projection map π_E from M^n to M_E. One can then show that, for any $A \subseteq M$, a subset of M_E is A-definable in M^{eq} if and only if its pre-image under π_E is A-definable in the original M. In particular, the original M and the copy of M inside M^{eq} have the same definable sets.

We recall that a countable structure M is *ω-categorical* if any of the following equivalent conditions is satisfied:

– For any $n < \omega$, there are finitely many types of the form $\mathrm{tp}(a/\emptyset)$, with $a \in M^n$.

– For any finite $A \subseteq M$, there are finitely many 1-types $\mathrm{tp}(a/A)$, with $a \in M$ a singleton.

– For any $n < \omega$, the action of $\mathrm{Aut}(M)$ on M^n has finitely many orbits.

– Any countable N elementarily equivalent to M is isomorphic to it.

Assume from now on that M is ω-categorical. Then one can define most model-theoretic notions using the automorphism group alone (at least over finite parameter sets). Let $A \subseteq M$ be finite. A subset $X \subseteq M^n$ is A-definable if and only if it is (setwise) invariant under the group $\mathrm{Aut}(M)_A$ of automorphisms fixing A pointwise. In particular, X is 0-definable if and only if it is $\mathrm{Aut}(M)$-invariant.

Still assuming that A is finite, two tuples a and a' have the same type over A, denoted $a \equiv_A a'$, if and only if there is an automorphism of M fixing A pointwise and sending a to a'. Thus types over A are in natural bijection with orbits of $\mathrm{Aut}(M)_A$.

An element c of M is in the definable closure of A if and only if it is fixed by $\mathrm{Aut}(M)_A$. Similarly, c is in the algebraic closure of A if and only if its orbit under $\mathrm{Aut}(M)_A$ is finite.

We will often consider the algebraic closure evaluated in M^{eq}: $\mathrm{acl}^{eq}(a)$. This can be though of as containing a name for each equivalence class of a under a 0-definable equivalence relation with finitely many classes. In particular, a subset $X \subseteq M^n$ is definable over $\mathrm{acl}^{eq}(\emptyset)$ if and only if it has finitely many conjugates under the automorphism group of M. If a and b are in M, then $a \in \mathrm{acl}(b)$ if and only if $a \in \mathrm{acl}^{eq}(b)$, since the definable subsets of M^k are the same seen in M or in M^{eq}. The *strong type* of a over A is the type of a over $\mathrm{acl}^{eq}(A)$: two elements have the same strong type over A if they are equivalent for every A-definable equivalence relation with finitely many classes.

Let $A \subseteq M$ be any set of parameters and let $X \subseteq M^k$ be an A-definable set. We say that X is *transitive* over A if any two elements of X have the same type over A. Note that since X is A-definable, any element of M^k having the same type as a member of X is itself in X. Thus an A-definable set X is transitive over A if and only if $\mathrm{Aut}(M)_A$ acts transitively on it. This is just another way of saying that X is a complete type over A. Similarly, we say that the A-definable set X is *primitive* over A if the action of $\mathrm{Aut}(M)_A$ on X is primitive, or equivalently X does not admit any non-trivial A-definable equivalence relation. If $A = \emptyset$, then we will usually omit "over A".

We say that two structures M and N are *inter-definable* if they have the same universe and the same 0-definable sets (in all cartesian powers). Hence M and N are essentially the same structure, but in possibly different languages. We say that M and N are *bi-definable* if M is inter-definable with a structure isomorphic to N (or equivalently N is inter-definable with a structure isomorphic to M).

Assumption: Throughout this paper, we work in an ω-categorical structure M in a language L. That assumption will in general not be recalled, and is implicitly assumed in all statements.

2.2 Homogeneous structures

We will call a countable structure M in a relational language L *homogeneous* if for any finite $A \subseteq M$ and $\sigma\colon A \to M$ a partial isomorphism (that is, $\sigma\colon A \to \sigma(A)$ is an isomorphism, where A and $\sigma(A)$ are equipped with the induced structure from M), there is an automorphism $\tilde{\sigma}\colon M \to M$ that extends σ. This is also sometimes called *ultrahomogeneous*.

We call a structure M *finitely homogeneous* if it is homogeneous and its language is finite and relational. A structure M is *finitely homogenizable* if it is inter-definable with a finitely homogeneous structure. Note that any finitely homogenenizable structure is ω-categorical: since the language is finite relational, the number of isomorphism types of substructures of a fixed size $n < \omega$ is finite, hence by homogeneity, the action of $\mathrm{Aut}(M)$ on M^n has finitely many orbits.

It is easy to see that a structure M is finitely homogenizable if and only if there is $k < \omega$ such that the following two conditions hold:

- There are finitely many types of k-tuples of elements of M.
- For any $n < \omega$, any two n-types $p(\bar{x})$ and $q(\bar{x})$ of tuples of elements of M are equal if and only if they have the same restriction to any set of k-variables.

2.3 Linear orders and their reducts

There is only one countable homogeneous linear order: $(\mathbb{Q}, \leq)$. It is also the only ω-categorical linear order with transitive automorphism group. Its reducts follow from Cameron's result on highly homogeneous permutations groups [Cam76]: there are five of them. Apart from the trivial reduct to pure equality, there are three unstable proper reducts:

- the generic betweenness relation $(\mathbb{Q}; B(x,y,z))$, where
$$B(x,y,z) \leftrightarrow (x \leq y \leq z) \vee (z \leq y \leq x);$$
- the generic circular order $(\mathbb{Q}; C(x,y,z))$, where
$$C(x,y,z) \leftrightarrow (x \leq y \leq z) \vee (z \leq x \leq y) \vee (y \leq z \leq x);$$
- the generic separation relation $(\mathbb{Q}; S(x,y,z,t))$, where
$$S(x,y,z,t) \leftrightarrow (C(x,y,z) \wedge C(y,z,t) \wedge C(z,t,x) \wedge C(t,x,y)) \vee \\ (C(t,z,y) \wedge C(z,y,x) \wedge C(y,x,t) \wedge C(x,t,z)).$$

The automorphism group of the betweenness relation is generated by the automorphisms of the linear order along with a bijection that reverses the order, for instance $x \mapsto -x$. Similarly, the automorphism group of the separation relation is generated from that of the circular order along with an order-reversing bijection.

Depending on the context, order will mean either linear order or circular order; by default linear. Linear and circular orders will play an essential role in this paper, but the betweenness and separation relations will not explicitly appear. They will be accounted for in the analysis by having every order come with a dual in order-reversing bijection with it. Thus the betweenness relation for example will be present in our classification as the quotient of two linear orders in order-reversing bijection.

2.4 Rank

We define rank as in [CH03, Section 2.2.1], restricting to the ω-categorical context. This notion of rank also coincides with what is now called thorn-rank, which is defined for any structure: see [Ons06, Definition 4.1, Remark 4.1.9]. We start by giving the definition as it appears in [CH03], then discuss it briefly before giving an equivalent definition. Hopefully, this will help the reader gain some intuition of this notion.

Definition 2.1. Given a definable set $D \subseteq M^{eq}$ and ordinal α, we define inductively $\mathrm{rk}(D) \geq \alpha$:

- $\mathrm{rk}(D) \geq 1$ if and only if D is infinite;
- $\mathrm{rk}(D) \geq \alpha + 1$ if there is in M^{eq} a definable set D_1, a definable finite-to-one map $\pi\colon D_1 \to D$ and a map $f\colon D_1 \to D_2$ such that D_2 is infinite and $\mathrm{rk}(f^{-1}(d)) \geq \alpha$ for all $d \in D_2$.
- for limit λ, $\mathrm{rk}(D) \geq \lambda$ if $\mathrm{rk}(D) \geq \alpha$ for all $\alpha < \lambda$.

The rank of a definable set D is either an ordinal or ∞ in the case where $\mathrm{rk}(D) \geq \alpha$ for all α. We say that a structure M is *ranked* if $rk(M) < \infty$. The rank of a type $\mathrm{tp}(a/b)$, denoted $\mathrm{rk}(a/b)$, is the minimal rank of a b-definable set containing a (in M^{eq}).

To get some intuition on this definition, start by considering the case where $D_1 = D$ and π is the identity. Then the definition implies that if there is $f\colon D \to D_2$ with D_2 infinite (hence of rank ≥ 1) and all fibers of rank n, then the rank of D is at least $n+1$. In fact, we will see that in structures where the rank is finite, we have the stronger property that given any definable $f\colon D \to D_2$, if $\mathrm{rk}(D_2) = n$ and all fibers of f have rank m, then D has rank $n+m$. This is a familiar property shared by many notions of dimension. It would be tempting to define the rank as the minimal notion of dimension satisfying this property (along with the first condition above on infinite sets). In fact we see that the definition we gave is slightly more complicated. We add the possibility to replace D by a finite cover D_1 of it. This does not change the rank ($\mathrm{rk}(D_1) = \mathrm{rk}(D)$, as we will see below), but might be required to find the map f. Here is an example illustrating why this can be necessary.

Let M be an infinite set with no structure and let D be the set of subsets of M of size 2. This is naturally a definable set in M^{eq}: it is a quotient of $D_1 :=$

$\{(a,b) \in M^2 : a \neq b\} \subseteq M^2$. The projection map $\pi \colon D_1 \to D$ has all fibers of size 2. It is not too hard to see that there is no definable map $f : D \to D_2$ to some infinite definable set D_2 in M with infinite fibers (intuitively, we cannot choose an element from each subset in a definable way). However, there is such a map $f \colon D_1 \to M$, for instance $f \colon (a,b) \mapsto a$ and hence D_1 has rank 2 as it should be.

Definition 2.1 is rather impractical given the quantification on the finite cover D_1 and the fact the it takes place in M^{eq}. We now give a second definition which might be more palatable, and prove right after that the two definitions are equivalent.

By a *uniformly definable* family $(X_t : t \in E)$, we mean that E is a definable set and there is a formula $\phi(x;y)$ such that for $t \in E$, X_t is the set defined by $\phi(x;t)$. For the purposes of the following definition, we will say that a family $(X_t : t \in E)$ is *weakly k-inconsistent* (k a natural number) if any *k pairwise distinct* members of the family have trivial intersection.

Definition 2.2. This definition takes place in M (not M^{eq}). Given a definable set $D \subseteq M^k$ and ordinal α, we define inductively $\widetilde{\operatorname{rk}}(D) \geq \alpha$:

- $\widetilde{\operatorname{rk}}(D) \geq 1$ if and only if D is infinite;
- $\widetilde{\operatorname{rk}}(D) \geq \alpha + 1$ if there is a uniformly definable family $(X_t : t \in E)$ of subsets of D which is weakly k-inconsistent for some k, contains infinitely many pairwise distinct sets, and such that $\widetilde{\operatorname{rk}}(X_t) \geq \alpha$ for each $t \in E$;
- for limit λ, $\widetilde{\operatorname{rk}}(D) \geq \lambda$ if $\widetilde{\operatorname{rk}}(D) \geq \alpha$ for all $\alpha < \lambda$.

Note that since M^{eq} does not add new definable subsets of M^k, this definition in fact does not change if we allow E and the family $(X_t : t \in E)$ to be definable in M^{eq}.

Proposition 2.3. *For any structure M and any definable set $D \subseteq M^k$, we have* $\operatorname{rk}(D) = \widetilde{\operatorname{rk}}(D)$.

Proof. We first show that for any α, if the definable map $\pi \colon D' \to D$ is finite-to-one and onto, then $\widetilde{\operatorname{rk}}(D') \geq \alpha \iff \widetilde{\operatorname{rk}}(D) \geq \alpha$. This can be shown by an easy induction on α: In one direction, given a weakly k-inconsistent family of subsets of D, we can pull it back by π to obtain a weakly k-inconsistent family of subsets of D'. In the other direction, let l be the maximal size of a fiber of π (which exists by ω-categoricity). Then if $(X_t : t \in E)$ is a weakly k-inconsistent family of subsets of D', it is not hard to see that the family of images $(\pi(X_t) : t \in E)$ is a weakly kl-inconsistent family of subsets of D.

We now show by induction on α that $\operatorname{rk}(D) \geq \alpha \iff \widetilde{\operatorname{rk}}(D) \geq \alpha$. For $\alpha = 1$, this follows from the first point in both definitions. Assume we know it for some α and that $\operatorname{rk}(D) \geq \alpha + 1$ as witnessed by D_1, D_2, π and f. Here, D_1 and D_2 are definable subsets of M^{eq}. For $d \in D_2$, let $X_d = \pi(f^{-1}(t)) \subseteq D$. Then the family $(X_d : d \in D_2)$ is uniformly definable in M^{eq}. By construction, there is a natural number k such that this family is k-inconsistent. By definition of M^{eq}, there is some $E \subseteq M^l$ and a definable surjection $g\colon E \to D_2$ in M^{eq}. For $t \in E$, let $X'_t = X_{g(t)}$. Then the family $(X'_t : t \in E)$ is uniformly definable in M^{eq}, hence also in M since M^{eq} does not add any definable subsets in powers of M. It is also weakly k-inconsistent. By induction hypothesis $\widetilde{\mathrm{rk}}(f^{-1}(d)) \geq \alpha$ for each $d \in D_2$ hence also $\widetilde{\mathrm{rk}}(X'_t) \geq \alpha$ for each $t \in E$ by the first paragraph of this proof. Therefore $\widetilde{\mathrm{rk}}(D) \geq \alpha + 1$.

Assume now that $\widetilde{\mathrm{rk}}(D) \geq \alpha + 1$ as witnessed by the weakly k-inconsistent family $(X_t : t \in E)$. Let $\sim$ be the equivalence relation on E defined by $t \sim t'$ if $X_t = X_{t'}$. Let $D_2 = E/\sim$ seen as a subset of M^{eq}. For $t \in D_2$, we define X_t in the natural way. Let $D_1 = \{(a,t) \in D \times D_2 : a \in X_t\}$. Then the canonical projection $\pi\colon D_1 \to D$ is k-to-one. Let $f : D_1 \to D_2$ be the other canonical projection. For any $t \in D_2$, $\widetilde{\mathrm{rk}}(f^{-1}(t)) \geq \alpha$ as it admits a one-to-one projection to X_t. By induction, $\mathrm{rk}(f^{-1}(t)) \geq \alpha$. Hence by definition, $\mathrm{rk}(D) \geq \alpha + 1$. □

If $M_=$ is an infinite set with no structure, then it has rank 1 since every infinite definable subset of $M_=$ is cofinite, hence there are no infinite weakly k-inconsistent families of definable subsets of $M_=$. Another example of a rank 1 structure is a dense linear order $(M, \leq)$. Here is a sketch of a proof of this: If $(X_t : t \in E)$ is a uniformly definable weakly k-inconsistent family of intervals of M, then the set of left endpoints of those intervals cannot contain an interval, hence it has to be finite. Similarly for the set of right end points. It follows that the family contains only finitely many sets. A same argument can be made for a family of unions of n intervals. By quantifier elimination, every definable set is a union of finitely many intervals, hence we are done.

We state some basic properties of the rank. See [CH03, Section 2.2.1] for proofs of (1)-(5); (6) and (7) are simple consequences of (5).

Proposition 2.4. *1.* $\mathrm{rk}(a/b) = 0$ *if and only if* $a \in \mathrm{acl}^{eq}(b)$.

2. $\mathrm{rk}(D_1 \cup D_2) = \max(\mathrm{rk}(D_1), \mathrm{rk}(D_2))$.

3. *If* $B_1 \subseteq B_2$, *then* $\mathrm{rk}(a/B_1) \geq \mathrm{rk}(a/B_2)$

4. *If D is definable over B, then there is $a \in D$ such that* $\mathrm{rk}(a/B) = \mathrm{rk}(D)$.

5. *If* $\mathrm{rk}(a/bc)$ *and* $\mathrm{rk}(b/c)$ *are finite, then so is* $\mathrm{rk}(ab/c)$ *and we have*
$$\mathrm{rk}(ab/c) = \mathrm{rk}(a/bc) + \mathrm{rk}(b/c).$$
In particular, if $a' \in \mathrm{acl}^{eq}(ab)$, *then by point 1 above,* $\mathrm{rk}(a'/ab) = 0$, *hence* $\mathrm{rk}(a'/b) \leq \mathrm{rk}(aa'/b) = \mathrm{rk}(a/b)$.

6. *In a finite rank structure we have*
$$\mathrm{rk}(ab) = \mathrm{rk}(a/b) + \mathrm{rk}(b) = \mathrm{rk}(b/a) + \mathrm{rk}(a).$$

7. *For tuples a, b, a' in a finite rank structure, we have $a' \in \mathrm{acl}(ab)$ if and only if* $\mathrm{rk}(aa'/b) = \mathrm{rk}(a/b)$. *If this holds, we have in particular* $\mathrm{rk}(a'/b) \leq \mathrm{rk}(a/b)$.

From point 5, we deduce that if M has finite rank, then any finite tuple of elements of M, or indeed M^{eq}, has finite rank.

The operator acl always defines a closure relation in the sense that $\mathrm{acl}(\mathrm{acl}(A)) = \mathrm{acl}(A)$ for all A and $\mathrm{acl}(A) \subseteq \mathrm{acl}(B)$ whenever $A \subseteq B$. Assuming that $\mathrm{rk}(M) = 1$, then it furthermore satisfies the exchange property: for all $A \subseteq M$ and two singletons $a, b \in M$, we have:

$$b \in \mathrm{acl}(Aa) \setminus \mathrm{acl}(A) \iff a \in \mathrm{acl}(Ab) \setminus \mathrm{acl}(A).$$

A set M equipped with a closure operator is a *pregeometry* if the closure operator satisfies the exchange property. Thus if M has rank one, then algebraic closure defines a pregeometry on M.

We can then define independent sets and bases as one does for vector spaces, with rk playing the role of dimension. We will only make mild use of this fact.

Lemma 2.5. *Let $p(x) \in S(A)$ be a type of finite rank n and let $B \supseteq A$, then there is a tuple d realizing p such that $\mathrm{rk}(d/B) = n$.*

Proof. Let $\pi(x)$ be the conjunction of all formulas of the form $\neg\psi(x, b)$, with $b \in B$ such that for some formula $\phi(x, a)$ in $p(x)$, $\mathrm{rk}(\phi(x, a) \wedge \psi(x, b)) < n$. Then $\pi(x)$ contains $p(x)$ and by point 2 above, it is finitely consistent. Now take d realizing $\pi(x)$. □

Lemma 2.6. *Let A be a set of parameters; let D be an A-definable set and E an A-definable equivalence relation on D. Let $D_1 \subseteq D$ be an E-class which is not algebraic over A (in other words, the corresponding element of M^{eq} is not in $\mathrm{acl}^{eq}(A)$). Then $\mathrm{rk}(D_1) < \mathrm{rk}(D)$.*

Proof. Seeing D/E as a subset of M^{eq}, let $d_1 \in D/E$ be the element of M^{eq} corresponding to D_1. Let $F \subseteq D/E$ be the definable set of elements having the same type as d_1 over A. Consider the definable family $(D_t : t \in F)$, where $D_t \subseteq D$ is the E-class coded by t. Then for each $t \in F$, $\mathrm{rk}(D_t) = \mathrm{rk}(D_1)$ since $\mathrm{tp}(t/A) = \mathrm{tp}(d_1/A)$. As d_1 is not algebraic over A, F is infinite. Furthermore the D_t's are pairwise disjoint. It follows form the definition of the rank that $\mathrm{rk}(D_1) = \mathrm{rk}(D_t) < \mathrm{rk}(D)$. □

Still following [CH03], we define rank independence.

Definition 2.7. (M has finite rank.) Say that two tuples a and b are *independent over E* and write $a \perp_E b$ if

$$\mathrm{rk}(ab/E) = \mathrm{rk}(a/E) + \mathrm{rk}(b/E).$$

This is a symmetric notion in a and b and it satisfies transitivity: a and bc are independent over E if and only if a and b are independent over Ec and a and c are independent over E.

Finally, we show the following useful property.

Lemma 2.8. *(M has finite rank.) Two tuples a and b are independent over e if and only if* $\mathrm{rk}(a/be) = \mathrm{rk}(a/e)$*. Furthermore if this holds then* $\mathrm{acl}^{eq}(ae) \cap \mathrm{acl}^{eq}(be) = \mathrm{acl}^{eq}(e)$.

Proof. We have in general $\mathrm{rk}(ab/e) = \mathrm{rk}(a/be) + \mathrm{rk}(b/e)$. The first statement then follows at once form the definition of independence.

Let now $d \in \mathrm{acl}^{eq}(ae) \cap \mathrm{acl}^{eq}(be)$ be a tuple. Assume that $d \notin \mathrm{acl}^{eq}(e)$, so that $\mathrm{rk}(d/e) \geq 1$. We have:

$$\begin{aligned}
\mathrm{rk}(a/bde) &\leq \mathrm{rk}(a/de) \\
&= \mathrm{rk}(ad/e) - \mathrm{rk}(d/e) && \text{using Proposition 2.4(6)} \\
&= \mathrm{rk}(a/e) - \mathrm{rk}(d/e) && \text{by Proposition 2.4(1) and } d \in \mathrm{acl}^{eq}(ae) \\
&\leq \mathrm{rk}(a/e) - 1 && \text{as } d \notin \mathrm{acl}^{eq}(e) \\
&< \mathrm{rk}(a/e).
\end{aligned}$$

On the other hand, since $d \in \mathrm{acl}^{eq}(be)$, $\mathrm{rk}(abde) = \mathrm{rk}(abe)$ and $\mathrm{rk}(bde) = \mathrm{rk}(be)$. Applying Proposition 2.4(6) on both sides of

$$\mathrm{rk}(abde) = \mathrm{rk}(abe)$$

we get

$$\mathrm{rk}(a/bde) + \mathrm{rk}(bde) = \mathrm{rk}(a/be) + \mathrm{rk}(be)$$

and since $\mathrm{rk}(bde) = \mathrm{rk}(be)$,

$$\mathrm{rk}(a/bde) = \mathrm{rk}(a/be).$$

Finally, $\mathrm{rk}(a/be) = \mathrm{rk}(a/e)$ since $a \mathop{\smile\hskip-0.9em\lower0.3ex\hbox{$\mid$}}_e b$.

From the series of inequalities above we have $\mathrm{rk}(a/bde) < \mathrm{rk}(a/e)$, but we have just shown $\mathrm{rk}(a/bde) = \mathrm{rk}(a/e)$; contradiction. □

2.5 Stability

Recall that a formula $\phi(x; y)$ is *stable* (in some structure M) if for some integer k, we cannot find tuples $(a_i : i < k)$ and $(b_j : j < k)$ such that

$$\phi(a_i; b_j) \iff i \leq j.$$

We say that the structure M is *stable* if all formulas are stable. Stability is preserved under elementary equivalence and we say that a complete theory T is stable if some/any model of T is stable.

We are concerned in this paper with unstable structures, but stable formulas will appear briefly at the end of the analysis in Section 6.5. There, we will need the following fact, which the reader not well acquainted with stability theory can take as a black box.

Fact 2.9. *Let M be a ranked ω-categorical structure and let $\phi(x;y)$ be a stable formula. Let $p \in S(A)$ be a type over some set $A \subseteq M$ and let $B \subseteq M$ be any set, then the set*

$$\left\{ \mathrm{tp}_{\phi}(a/B) : a \models p, a \underset{A}{\downarrow} B \right\}$$

is finite.

Proof. (Assuming knowledge of stability theory: see for instance [Pil96, Chapter 1].) First, note that by [Ons06, Theorem 5.1.1], forking and thorn-forking are the same for stable formulas. Hence if $a \models p$, $a \downarrow_A B$, then the partial type $p \cup \mathrm{tp}_{\phi}(a/B)$ does not fork over A. Since ϕ is stable, there are only finitely many non-forking extensions of p to a ϕ-type over B. □

2.5.1 Strongly minimal sets

We check that Theorem 1.2 holds in the stable case and for that assume familiarity with stability theory. None of this will be used later.

A structure M is *strongly minimal* if for any N elementarily equivalent to M, any definable (over parameters) subset of N is either finite or cofinite. If M is ω-categorical, then it is enough to check the condition for $N = M$. The classification of strongly minimal primitive ω-categorical structures was established by Zilber [Zil] using model-theoretic methods. The paper [CHL85] gives an exposition of this result, as well as a shorter proof attributed to Cherlin and Mills, using the classification of finite simple groups. The results are expressed in terms of the geometry coming from algebraic closure. We have explained in Section 2.4 how any rank 1 structure is equipped with a pregeometry whose closure operation is given by algebraic closure. If M is primitive, this pregeometry is in fact a *geometry*, meaning that $\mathrm{acl}(a) = \{a\}$ for any element $a \in M$. By the acl-geometry of M, we mean the set M equipped with the closure operator acl.

Fact 2.10. *If M is strongly minimal, primitive and ω-categorical, then either:*

1. *M is a pure set;*
2. *the* acl-*geometry on M is that of an infinite-dimensional affine space over a finite field;*
3. *the* acl-*geometry on M is that of an infinite-dimensional projective space over a finite field.*

Cases 2 and 3 do not completely determine M up to bi-definability, but they do determine it up to finitely many possibilities corresponding to automorphism groups G with $AGL_{\omega}(F_q) \subseteq G \subseteq A\Gamma L_{\omega}(F_q)$ in the affine case and $PGL_{\omega}(F_q) \subseteq G \subseteq P\Gamma L_{\omega}(F_q)$ in the projective case.

Proposition 2.11. *A rank 1, primitive, stable, ω-categorical structure M is strongly minimal. For a given $n < \omega$, there are, up to bi-definability, finitely many such structures having at most n 4-types.*

Proof. If M is stable of finite rank, then rank-independence is the same thing as forking-indepedence: see [Ons06, Theorem 5.1.1]. Thus if M is stable of rank 1, it is superstable of U-rank 1. If M is furthermore primitive, then $x = x$ is a complete strong type over $\emptyset$ and therefore for any definable set $D \subseteq M$, either D or its complement forks over $\emptyset$. Hence by U-rank 1, either D or its complement is finite. Therefore a stable, rank 1, primitive, ω-categorical structure is strongly minimal.

Fact 2.10 and the remark following it describe the possibilities. We can assume that M is not a pure set. Assume first that M is affine over a field F_q, $q = p^n$. Then if we fix a point a as the origin, making M linear, and take b, c colinear, we have $c = \lambda \cdot b$ for some $\lambda \in F_q$, defined in the worst case up to an element of $Gal(F_q/F_p)$. That Galois group has size n and therefore the number of orbits goes to infinity with q. Hence so does the number of 3-types. The projective case is similar, except that we need to name two points to serve as 0 and ∞ and obtain that the number of 4-types goes to infinity with q. □

2.6 NIP and op-dimension

We recall some basic facts about NIP theories and refer the reader to [Sim15] for more details.

Definition 2.12. A formula $\phi(x; y)$ is *NIP* in M if for some integer k, we cannot find tuples $(a_i : i < k)$ and $(b_J : J \in \mathfrak{P}(k))$ in M with:

$$M \models \phi(a_i; b_J) \iff i \in J.$$

If a formula $\phi(x; y)$ is NIP, then it stays so in any structure N elementarily equivalent to M. We say that the theory T is NIP if for some/any model of T, all formulas are NIP.

By a result of Shelah, if all formulas $\phi(x; y)$ with $|x| = 1$ are NIP, then the theory is NIP. Stable theories are NIP and so is for example the theory of dense linear orders.

The NIP condition can be characterized by counting ϕ-types over finite sets. See [Sim15, Chapter 6]. In the finitely homogeneous case, this becomes a particularly natural condition.

Fact 2.13. *A structure M homogeneous in a finite relational language is NIP if and only if there is a polynomial $P(X)$ such that the number of types over any finite set A is bounded by $P(|A|)$.*

Note in particular, that if M is NIP and homogeneous in a finite relational language, then the size of $S_n(\emptyset)$ is bounded by $P(1) \cdot P(2) \cdots P(n-1)$, where $P(X)$ is the polynomial given by the previous fact. Hence $|S_n(\emptyset)| = O(2^{cn \ln(n)})$ for some $c > 0$.

We now give a short account of [Sim22] which establishes that NIP unstable theories interpret linear orders. First, we define op-dimension as in [GH15], which will allow us to determine how many independent orders we can hope to find.

Definition 2.14. An *ird-pattern* of length κ for the partial type $\pi(x)$ is given by:

- a family $(\phi_\alpha(x; y_\alpha) : \alpha < \kappa)$ of formulas;
- an array $(b_{\alpha,k} : \alpha < \kappa, k < \omega)$ of tuples, with $|b_{\alpha,k}| = |y_\alpha|$;

such that for any $\eta \colon \kappa \to \omega$, there is $a_\eta \models \pi(x)$ such that for any $\alpha < \kappa$ and $k < \omega$, we have

$$\models \phi_\alpha(a_\eta; b_{\alpha,k}) \iff \eta(\alpha) < k.$$

Remark 2.15. This definition is from [She90, III.7.1]. The letters *ird* stand for *independent orders*.

Definition 2.16. We say that T has *op-dimension* less than κ, and write $\mathrm{opD}(T) < \kappa$ if, in a saturated model of T, there is no ird-pattern of length κ for the partial type $x = x$.

If a structure is NIP, then it has op-dimension less than $|T|^+$ (otherwise, we can assume $\phi_\alpha = \phi$ is constant and then ϕ has IP: we can take $\{b_{\alpha,0} : \alpha < \omega\}$ as the a_i's in Definition 2.12). Conversely, if for some cardinal κ, we have $\mathrm{opD}(T) < \kappa$, then T is NIP. (If $\phi(x; y)$ has IP, we can find by compactness an ird-pattern of any length with $\phi_\alpha = \phi$.)

We also define the op-dimension of a partial type $p(\bar{x})$: $\mathrm{opD}(p) < \kappa$ if, in a saturated model of T, there is no ird-pattern of length κ for $p(\bar{x})$. We let $\mathrm{opD}(\bar{a}) = \mathrm{opD}(\mathrm{tp}(\bar{a}/\emptyset))$ and $\mathrm{opD}(\bar{a}/A) = \mathrm{opD}(\mathrm{tp}(\bar{a}/A))$.

We say that $\mathrm{opD}(p) = n$ if we have $\mathrm{opD}(p) < n+1$, but not $\mathrm{opD}(p) < n$ (and the same for T instead of p).

Fact 2.17. *Op-dimension is sub-additive:* $\mathrm{opD}(\bar{a}\bar{b}/A) \leq \mathrm{opD}(\bar{b}/A\bar{a}) + \mathrm{opD}(\bar{a}/A)$. *In particular, if* $\bar{b} \subseteq \mathrm{acl}^{eq}(A\bar{a})$*, then* $\mathrm{opD}(\bar{a}\bar{b}/A) = \mathrm{opD}(\bar{a}/A)$.

Proof. The first statement is [GH15, Theorem 2.2]. See also [Sim22, Section 4]. The "in particular" part follows from the fact that if $\bar{b}$ is algebraic over $\bar{a}$, then $\mathrm{opD}(\bar{b}/\bar{a}) = 0$ which is clear from the definition since the points a_η there have to be pairwise distinct. □

By a linear quasi-order, we mean a reflexive, transitive relation $\leq$ for which any two elements are related. If $\leq$ is a linear quasi-order, then the associated strict order $<$ is defined by

$$a < b \iff a \leq b \wedge \neg(b \leq a).$$

Furthermore, the relation $aEb \iff (a \leq b) \wedge (b \leq a)$ is an equivalence relation and $\leq$ induces a linear order on the quotient.

The main result of [Sim22] in the ω-categorical case is the following.

Fact 2.18 ([Sim22], Theorem 6.14). *If the theory T is ω-categorical, NIP,* $\mathrm{opD}(T) \geq n > 0$*, then there is a finite set A, a set D definable and transitive over A and n A-definable linear quasi-orders $\leq_1, \ldots, \leq_n$ on D, such that the structure $(D; \leq_1, \ldots, \leq_n)$ contains an isomorphic copy of every finite structure $(X_0; \leq_1, \ldots, \leq_n)$ equipped with n linear orders.*

Note that by transitivity, for each i, the quotient of D by the equivalence relation associated with $\leq_i$ is infinite and, using ω-categoricity, $\leq_i$ induces on it a dense linear order without endpoints.

2.6.1 Distality

Distality was introduced in [Sim13]. It is meant to capture the notion of a purely unstable NIP structure. We give here the equivalent definition from [CS15].

Definition 2.19. A structure M is called *distal* if for every formula $\phi(x;y)$, there is a formula $\psi(x;z)$ such that for any finite set $A \subseteq M$ and tuple $a \in M^{|x|}$, there is $d \in A^{|z|}$ such that $\psi(a;d)$ holds and for any instance $\phi(x;b) \in \mathrm{tp}(a/A)$, we have the implication

$$M \models (\forall x)\psi(x;d) \to \phi(x;b).$$

Assume that M is homogeneous in a finite relational language. Then if M is distal, there is an integer k such that for any finite set A and singleton $a \in M$, there is $A_0 \subseteq A$ of size $\leq k$ such that $\mathrm{tp}(a/A_0) \vdash \mathrm{tp}(a/A)$. (That is, if $\mathrm{tp}(a'/A_0) = \mathrm{tp}(a/A_0)$, then $\mathrm{tp}(a'/A) = \mathrm{tp}(a/A)$.) In fact, the converse is also true, as can be seen by induction on $|x|$ in the definition above, but we will not need this.

For instance, DLO is distal, and we can take $k = 2$. We will see in Theorem 8.3 that a distal finitely homogeneous structure is always finitely axiomatizable.

3 Linear orders

We will consider definable linear orders $(V, \leq)$, meaning that the underlying set V is parameter-definable and so is the order relation $\leq$. We will often abuse notation by denoting the pair $(V, \leq)$ by V, sometimes by $\leq$. If we have two definable orders $(V_0, \leq_0)$, $(V_1, \leq_1)$, it may happen that the underlying sets V_0, V_1 are equal. This will, however, be irrelevant for most of what we say and it might be more convenient to think of V_0 and V_1 as two disjoint copies of the same set. In any case, V_0 will mean the set equipped with the order $\leq_0$ and V_1 the set equipped with the order $\leq_1$. The reverse of the order $(V, \leq)$ is $(V, \geq)$.

Orders are always equipped with the order topology, and product of orders with the product topology. Hence, in the situation above, $V_0 \times V_1$ is equipped with the product topology coming from $\leq_0$ on the first coordinate and $\leq_1$ on the second, regardless of whether the underlying sets V_0 and V_1 are equal or not.

Lemma 3.1. *Let $(V, \leq)$ be a 0-definable infinite linear order, which is a complete type over $\emptyset$. Then the order $\leq$ is dense and for any $a \in V$, $\mathrm{acl}(a) \cap V = \{a\}$.*

Proof. If $\leq$ is not dense, then some point $a \in V$ has an immediate successor. Since V is a complete type over $\emptyset$, all points have a successor and hence the order is discrete. By ω-categoricity, V is finite.

If $b \in \mathrm{acl}(a) \cap V$, say $b > a$, then again as V is a complete type, there is $b_1 > b$, $b_1 \in \mathrm{acl}(b)$ and iteratively $b_{k+1} > b_k$, $b_{k+1} \in \mathrm{acl}(b_k)$. This gives infinitely many elements in $\mathrm{acl}(a)$, contradicting ω-categoricity. □

A *convex equivalence relation* on an order $(V, \leq)$ is an equivalence relation E with convex classes: that is:

$$a \leq b \leq c \wedge a\ E\ c \Longrightarrow a\ E\ b.$$

Such a relation is non-trivial if it has more than one class and is not equality.

Definition 3.2. Let $(V, \leq)$ be an A-definable linear order.

- We say that $(V, \leq)$ has *topological rank 1* if it does not admit any definable (over parameters) convex equivalence relation E with infinitely many infinite classes.
- We say that $(V, \leq)$ is *weakly transitive* over A if it is a dense order and any A-definable subset of V is either empty or dense in V.
- We say that $(V, \leq)$ is *minimal* over A if it is weakly transitive over A, and has topological rank 1. If $A = \emptyset$, then we omit it.

The name *topological rank 1* comes from the fact that a rank 1 structure, in the sense of Section 2.4, cannot have a definable equivalence relation with infinitely many infinite classes. Here, we forbid such equivalence relations that have convex classes. We will not define topological rank in general.

Note that if $(V, \leq)$ is transitive over A, in the sense that it is a complete type over A, then it is weakly transitive over A. As an example, consider the structure $(\mathbb{Q}; \leq, P)$, where $\leq$ is the usual order on $\mathbb{Q}$ and $P(x)$ is a unary predicate that is dense co-dense in $\mathbb{Q}$. Then the order $(\mathbb{Q}; \leq)$ is weakly transitive (over $\emptyset$), but is not a complete type.

Lemma 3.3. *1. If $(V, \leq)$ is weakly transitive over A, then it has neither first nor last element.*

2. A definable subset of a topological rank 1 linear order has itself topological rank 1.

3. If $(V, \leq)$ is an A-definable dense order without first or last element, and $W \subseteq V$ is a dense A-definable subset of V, then V has topological rank 1 (resp. is weakly transitive over A, resp. is minimal over A) if and only if W has the same property

Proof. 1. If V has a first (resp. last) element a, then $\{a\}$ is an A-definable subset of V that is neither empty nor dense.

2. Let $(V, \leq)$ have topological rank 1 and $W \subseteq V$ be definable (over some parameters). Let E be a definable convex equivalence relation on W with infinitely many infinite classes. Define a relation $\overline{E}$ on V by: $\overline{E}(a, b)$ holds if all the points of W in the interval $a \leq x \leq b$ are in one E-equivalence class. Then $\overline{E}$ is a definable convex equivalence relation on V with infinitely many infinite classes. This contradicts V having topological rank 1.

3. If E is a definable convex equivalence relation on V, then its restriction $E|_W$ to W is also a definable convex equivalence relation. Furthermore if E has infinitely many infinite classes on V, each of those classes has infinite intersection with W by density, hence $E|_W$ shows that W does not have topological rank 1. Along with the first point, this shows that V has topological rank 1 if and only if W has topological rank 1.

Assume that V is not weakly transitive. Let $X \subseteq V$ be A-definable, non-empty and not dense in V. If X is finite, let a be its first element. Then the set $\{x \in W : x < a\}$ is A-definable, non-empty (as V has no first element and W is dense in V) and not dense in W (as V has no last element). This shows that W is not weakly transitive over A. If now X is infinite, let $\overline{X}$ be its topological closure in V. Then $W \cap \overline{X}$ is A-definable, non-empty and not dense in W. Thus W is not weakly transitive over A. Conversely, it is immediate that if W is not weakly transitive over A then neither is V. □

Lemma 3.4. *Let $(V, \leq)$ be a definable dense order of topological rank 1. Then any definable closed (or open) subset of V is a finite union of convex sets.*

Proof. Let $X \subseteq V$ be a definable closed subset. Consider the equivalence relation E_X which holds of a pair (a, b) in V^2 if either $a = b$ or there is no element of X in the interval $a \leq x \leq b$. This is a convex equivalence relation. Moreover, any E_X-class is either of the form $\{a\}$, $a \in X$, or of the form $a < x < b$, with $a, b \in X \cup \{\pm\infty\}$. Since $(V, \leq)$ is dense, classes of the second type are infinite. By topological rank 1, there can be only finitely many such classes. This implies that the complement of X is a finite union of convex sets. Then so is X. □

Note that with $(V, \leq)$ as above, if $A \subseteq V$ is the closure of a complete type X over some $\bar{c}$, then A is a convex set. Indeed, if say $A = A_0 \cup \cdots \cup A_{n-1}$ where the A_i's are disjoint convex sets and n is minimal, then A and each A_i is definable over $\bar{c}$. Thus for each $i < n$, $X \cap A_i$ is definable over $\bar{c}$ and non-empty. By completeness of X, $n = 1$.

3.1 Intertwinings

Let $(V, \leq)$ be an A-definable dense order with no first or last element. By a *cut* in V we mean an initial segment of it which is neither empty nor the whole of V and has no last element. We let $\overline{V}$ be the set of definable (over any parameters) cuts of V. Let $\phi(x; y)$ be a formula without parameters. The set $C_\phi := \{b : \phi(V; b)$ is a cut of $V\}$ is definable over A. The set of cuts of V definable by a formula of the form $\phi(x;b)$ can be identified with the quotient of C_ϕ by the equivalence relation $b \sim b' \iff (\forall x \in V)(\phi(x;b) \leftrightarrow \phi(x;b'))$. Hence the set of cuts in V that can be defined by an instance of $\phi(x;y)$ is naturally an A-definable set in M^{eq}. If Φ is a finite set of formulas as above, write $C_\Phi = \bigcup_{\phi\in\Phi} C_\phi$. This is also an A-definable set in M^{eq}. Now $\overline{V} = \bigcup_\Phi C_\Phi$, where Φ runs over all finite set of formulas of the form $\phi(x;y)$, is naturally a directed union of A-definable sets. (It would more rigorous to describe it as a direct limit of A-definable sets, but we will do without introducing such formalism.) In all arguments using $\overline{V}$, one can replace $\overline{V}$ with a big enough definable subset of it of the form C_Φ.

If there is a finite set Φ of formulas such that every definable cut of V can be defined by some $\phi(x;b)$ for $\phi(x;y) \in \Phi$, then $\overline{V}$ can be indentified with C_Φ. In this case, $\overline{V}$ is a bona fide definable set in M^{eq}. In that case, we will say that $\overline{V}$ *is definable*. We will see in Corollary 6.2 that this is true in a rank 1 structure. Hence the reader will not loose much by assuming that this is the case when reading this paper.

A function $f\colon X \to \overline{V}$ is said to be definable over some $B \supseteq A$ if there is a B-definable binary relation $F \subseteq X \times V$ such that for all $a \in X$, the fiber $F_a := \{x \in V : (a,x) \in F\} \subseteq V$ is equal to $f(a)$. This is consistent with the view of $\overline{V}$ as a union of definable sets: a function $f\colon X \to \overline{V}$ is B-definable if and only if it takes values inside a fixed definable subset C_Φ of $\overline{V}$ and is B-definable in the usual sense.

We identify V with a (definable) subset of $\overline{V}$ by $a \mapsto \{x \in V : x < a\}$. The order $\leq$ naturally extends to $\overline{V}$, where it coincides with inclusion. Note that V is dense in $\overline{V}$. Note also that if $a \in V$ and $c \in \overline{V}$, then $a \in c$ is equivalent to $a < c$, where $<$ is meant in $\overline{V}$ with the identification just discussed. We will use both notations.

Lemma 3.5. *Let $(V,\leq)$ be definable and minimal over some A. Any A-definable non-empty subset of $\overline{V}$ is dense in $\overline{V}$.*

Proof. Let $X \subseteq \overline{V}$ be A-definable. We define a relation E_X on V by:

$$a \; E_X \; b \iff (a = b) \vee (\forall x \in \overline{V})(a \leq x \leq b \to x \notin X).$$

Then E_X is a convex A-definable equivalence relation on V and by topological rank 1, it has only finitely many infinite classes. Assume it has an infinite class, then that class is A-definable and by weak transitivity, it is the whole of V. This implies that X is empty. If there is no infinite class, then by density of V, all classes have one element, which implies that X is dense in $\overline{V}$. ☐

Lemma 3.6. *Let $(V,\leq)$ be definable and minimal over some A and let $W \subseteq \overline{V}$ be an A-definable subset of $\overline{V}$ containing V. Then W is minimal over A.*

Proof. We know that V is dense in $\overline{V}$, hence also in W. The result then follows from Lemma 3.3. ☐

Lemma 3.7. *Given a finite tuple $\bar{a}$ and an $\bar{a}$-definable dense order V, $\mathrm{dcl}^{eq}(\bar{a}) \cap \overline{V}$ is finite.*

Proof. Formally, the conclusion says that there is some number $k < \omega$ such that $\operatorname{dcl}^{eq}(\bar{a}) \cap V_0$ has size at most k for all $\bar{a}$-definable $V_0 \subseteq \overline{V}$. Let V_0 be such a set and let $m_1 < \cdots < m_n$ be in $\operatorname{dcl}^{eq}(\bar{a}) \cap V_0$. By density of V in $\overline{V}$, we can find $b_0, b_1, \ldots, b_{n-1} \in V$ with $m_1 < b_1 < m_2 < \cdots < b_{n-1} < m_n$. Each b_i has a different type over $\bar{a}$, and hence there are at least $n-1$ different types of elements of V over $\bar{a}$. Hence $\operatorname{dcl}^{eq}(\bar{a}) \cap V_0$ has size bounded by the number of 1-types over $\bar{a}$ of elements of V, which is finite by ω-categoricity. □

Definition 3.8. Let $(V, \leq_V)$ and $(W, \leq_W)$ be orders, definable and weakly transitive over A. We say that they are *intertwined* over A if there is an A-definable non-decreasing map $f \colon V \to \overline{W}$. If A is clear from the context, we omit it.

Note that this is the same thing as saying that there is an A-definable binary relation $R \subseteq V \times W$ such that

$$(a\, R\, b) \wedge (a' \leq_V a) \wedge (b \leq_W b') \Longrightarrow a'\, R\, b'.$$

Indeed, the relation R is defined from f by

$$x\, R\, y \iff f(x) \leq_{\overline{W}} y \iff \neg F(x, y),$$

where F is associated to f as above. Observe also that by weak transitivity, no element of $\overline{W}$ is definable over A, hence the image of f has to be cofinal and coinitial in $\overline{W}$.

Lemma 3.9. *For any fixed A, intertwining is an equivalence relation on orders that are definable and weakly transitive over A.*

Proof. Any order is intertwined with itself via the identity function. If R as above is an intertwining relation from V to W, then R' defined by $x\, R'\, y \iff \neg y\, R\, x$ is an intertwining relation from W to V. Finally if R is an intertwining relation from V to W and S an intertwining relation from W to Z, then T defined by $x\ T\ y \iff (\exists z)(x\ R\ z \wedge z\ S\ y)$ intertwines V and Z. □

Working over some base A, let V and W be two weakly transitive orders and $f \colon V \to \overline{W}$ an intertwining map. If W has topological rank 1, then the image of f must be dense in $\overline{W}$ (otherwise we can define an equivalence relation as in the proof of Lemma 3.4; it cannot have finitely many classes as W is weakly transitive). If V has topological rank 1, then f is injective: $f(x) = f(y)$ is a convex equivalence relation on V; it cannot have finitely many infinite classes by weak transitivity and cannot have infinitely many by topological rank 1. Hence all classes are singletons and f is injective. We conclude that if both V and W have topological rank 1, an intertwining gives an increasing injection of V into a dense subset of $\overline{W}$. Furthermore, the map f extends to an increasing bijection $\tilde{f} \colon \overline{V} \to \overline{W}$ defined as follows: if $c \in \overline{V}$ is a cut in V, seen as a subset of V, we let $\tilde{f}(c) = \{y \in W : y < f(x) \text{ for some } x \in c\}$. Since f is increasing and c has no last element, $\tilde{f}(c)$ also has no last element and is a definable cut in W. One sees at once that $\tilde{f}$ extends f and is increasing. Also if $d \in \overline{W}$ is a definable cut in W, then $c := \{x \in V : f(x) < d\}$ is a definable cut in V and $\tilde{f}(c) = d$. Hence $\tilde{f}\colon \overline{V} \to \overline{W}$ is a bijection. It follows that we can—and will—think of V and W as having a common definable completion, or equivalently as being dense in each other's completion.

Lemma 3.10. *Working over A, if V and W are minimal linear orders which are intertwined, then there is a unique A-definable intertwining map $f\colon V \to \overline{W}$.*

Proof. Assume that we are given two increasing maps $f, g\colon V \to \overline{W}$, both definable over A. Keeping only the parameters needed to define V, W, f and g, we may assume that A is finite. The two maps f and g extend uniquely to increasing bijections from $\overline{V}$ to $\overline{W}$, still denoted by f and g. If for some $a \in V$, $f(a) < g(a)$, then we have $a < f^{-1}(g(a))$ and hence $g(a) < g(f^{-1}(g(a)))$. Continuing in this way we find

$$a < f^{-1}(g(a)) < f^{-1}(g(f^{-1}(g(a)))) < \cdots,$$

which gives infinitely many elements in $\operatorname{dcl}(Aa) \cap \overline{V}$, contradicting Lemma 3.7. $\square$

It will follow from Lemma 3.18 that even over a larger set of parameters, there cannot be another intertwining map from an interval of V to an interval of W.

The following lemma will be useful later and can also be seen as a warm-up for the next proposition as the proof will have a similar flavor.

Lemma 3.11. *Let $(V, \leq)$ be definable and minimal over some A. There is a subset $B \supseteq A$ and a B-definable bounded convex subset $W \subseteq V$ such that W is minimal over B.*

Proof. Let $a \in V$ be any point. Assume that there is an element in $\operatorname{dcl}(Aa) \cap \overline{V}$ which is larger than a. Take $m \in \operatorname{dcl}(Aa) \cap \overline{V}$ to be minimal such. Then the convex subset $(a, m) := \{x \in V : a < x < m\}$ is Aa-definable. It has topological rank 1 since V does and it is weakly transitive over Aa by minimality of m (no cut of it is definable over Aa). Hence it is minimal over Aa and we have want we want by setting $B = Aa$ and $W = (a, m)$.

Assume now that $\operatorname{dcl}(Aa) \cap \overline{V}$ contains no element greater than a. Let $b \in V$, $b > a$. If $\operatorname{dcl}(Aab) \cap \overline{V}$ contains no element strictly between a and b, then the interval $(a, b) \subseteq V$ is definable and minimal over Aab and we can take $B = Aab$, $W = (a, b)$. If $\operatorname{dcl}(Aab) \cap \overline{V}$ contains an element strictly between a and b, let m be minimal such. Then the convex subset (a, m) is definable and minimal over Aab. $\square$

We now study definable subsets of cartesian powers of a minimal order.

Proposition 3.12. *Working over some A, let $(V, \leq)$ be a minimal definable linear order. Let $p(x_0, \ldots, x_{n-1}) \in S(A)$ be a type in V^n such that $p \vdash x_0 < x_1 < \ldots < x_{n-1}$. Then given open intervals $I_0 < \cdots < I_{n-1}$ of V, we can find $a_i \in I_i$ such that $(a_0, \ldots, a_{n-1}) \models p$.*

Proof. For simplicity of notation, assume $A = \emptyset$. The strategy of the proof is as follows: we first ignore the type p and produce by induction on $l < \omega$, types $r_l \in S_l(\emptyset)$ which satisfy the conclusion of the proposition. We then show how the existence of r_{2n} implies that p itself has the required density property by sandwiching elements of a realization of p between elements of a realization of r_{2n}.

For any finite tuple $\bar{d}$, let $m(\bar{d})$ denote the maximal element of $\mathrm{dcl}^{eq}(\bar{d}) \cap \overline{V}$. Note that for a fixed tuple of variables $\bar{y}$, the relation $\phi(x;\bar{y}) := x > m(\bar{y})$ is invariant under $\mathrm{Aut}(M)$, and therefore definable.

We construct an increasing sequence of types $r_l(x_0, \ldots, x_{l-1}) \in S(\emptyset)$, $l > 0$, of elements of V^l. For $l = 1$, let $a_0 \in V$ by any element and set $r_1 = \mathrm{tp}(a_0)$ and $m_0 = m(a_0) \in \overline{V}$. Pick any point $a_1 > m_0$ and let $r_2 = \mathrm{tp}(a_0, a_1)$. We continue in this way: having constructed $r_l = \mathrm{tp}(a_0, \ldots, a_{l-1})$, let $m_{l-1} = m(a_{\leq l})$[1]. Pick any $a_l > m_{l-1}$ and set $r_{l+1} = \mathrm{tp}(a_0, \ldots, a_l)$. We note that

$$r_{l+1}(x_0, \ldots, x_l) \vdash x_l > m(x_0, \ldots, x_{l-1}).$$

This being done, let $I_0 < \cdots < I_{l-1}$ be open intervals of V. We claim that we can find $(b_0, \ldots, b_{l-1}) \models r_l$ such that $m(b_{\leq k})$ lies in I_k for each k. We do this by induction. Assume that $b_{<k}$ have been selected and set $m = m(b_{<k})$ (if $k = 0$, take $m = -\infty$). Define the relation E_k on $V_{>m}$ by $v \; E_k \; w$ if either $v = w$, or for no s with $\mathrm{tp}(b_{<k}, s) = r_{k+1}$ do we have $v \leq m(b_{<k}s) \leq w$. This is an equivalence relation with convex classes. By the topological rank 1 assumption, it must have finitely many infinite classes. The infima and suprema of those classes are elements of $\overline{V}$ definable over $b_{<k}$. However, by definition, no cut above $m(b_{<k})$ is definable over $b_{<k}$. Hence all classes of E_k are finite and by density of the order, all classes have one element. It follows that we can find b_k with $\mathrm{tp}(b_{\leq k}) = r_{k+1}$ and $m(b_{\leq k})$ lying in I_k.

Let now $p(x_0, \ldots, x_{n-1})$ be as in the statement of the lemma and $\bar{a} \models p$. Let $r = r_{2n}$. Then by the previous paragraph, we can find $\bar{b} \models r$ such that for each k, $m(b_{\leq 2k}) < a_k < m(b_{\leq 2k+1})$. Pick open intervals $I_0 < \cdots < I_{n-1}$ of V. For each k, let $J_{2k} < J_{2k+1}$ be two subintervals of I_k. Applying the previous paragraph again, we can find $\bar{b}' \models r$ such that for each i, $m(b'_{\leq i}) \in J_i$. Since $\bar{b}$ and $\bar{b}'$ have the same type, there is $\sigma \in \mathrm{Aut}(M)$ with $\sigma(\bar{b}) = \bar{b}'$. Let $\bar{a}' = \sigma(\bar{a})$. We then have $m(b'_{\leq 2k}) < a'_k < m(b'_{\leq 2k+1})$ for each k. By the choice of $\bar{b}'$, this implies $a'_k \in I_k$ as required. ☐

Remark 3.13. Let $(V, \leq)$ be definable and minimal over A. Let $p(x_0, \ldots, x_{n-1}) \in S(A)$ be a type in $\overline{V}^n$ such that $p \vdash x_0 < \cdots < x_{n-1}$. Then there is some A-definable $W \subseteq \overline{V}$ containing V such that p lies in W^n. By Lemma 3.6, W is also minimal over A and we can apply the previous proposition with W instead of V. This shows that Proposition 3.12 can be applied to types in $\overline{V}^n$ instead of V^n.

[1]Where $a_{\leq l} := a_0, \ldots, a_l$

Corollary 3.14. *Let $(V, \leq)$ be a minimal definable linear order over some A. Let $X \subseteq V^n$ be an A-definable subset, then the topological closure of X is a boolean combination of sets of the form $x_i \leq x_j$.*

Proof. We can write $X = \bigcup_{i<n} Y_i$, where the Y_i's are pairwise disjoint and each Y_i is A-definable and defines a complete type over A. Since the closure of X is the union of the closures of the Y_i's, it is enough to prove the statement for each Y_i. We may therefore assume that X defines a complete type over A. Let $(a_0, \ldots, a_{n-1}) \in X$. For some permutation σ of $\{0, \ldots, n-1\}$, we have $a_{\sigma(0)} \leq \ldots \leq a_{\sigma(n-1)}$. If the coordinates of $\bar{a}$ are pairwise distinct, then the previous proposition implies that X is dense in the set defined by $x_{\sigma(0)} \leq \ldots \leq x_{\sigma(n-1)}$. In general, X is dense in the intersection of that set with the set defined by the conjunction of the equations $x_{\sigma(i)} = x_{\sigma(i+1)}$ that hold in $\bar{a}$. □

In the end of this section, we give a more concrete description of intertwined orders and show that there is only one transitive structure composed of n intertwined orders, up to isomorphism and permutation of the orders. (See Proposition 3.16 for a precise statement.)

Proposition 3.15. *Consider the language $L_n = \{\leq, P_0, \ldots, P_{n-1}, f_1, \ldots, f_{n-1}\}$, where the P_i's are unary predicates and the f_i's unary functions. Let the theory T_n say that:*

- *$\leq$ defines a dense linear order without endpoints;*
- *the P_i's partition the universe and are dense (with respect to $\leq$);*
- *the function f_i is the identity outside of P_0; its restriction to P_0 is a bijection between P_0 and P_i;*
- *for all $x \in P_0$, we have $x < f_1(x) < f_2(x) < \cdots < f_{n-1}(x)$;*
- *given any open intervals $I_0 < I_1 < \cdots < I_n$, there is $x \in I_0$ such that $f_i(x) \in I_i$ for each $1 \leq i < n$.*

Then the theory T_n is complete, ω-categorical and has elimination of quantifiers.

Proof. This can be shown by a straightforward back-and-forth argument. Alternatively, one can see that T_n is the Fraïssé limit of the class of finite L_n structures satisfying:

- $\leq$ defines a linear order;
- the P_i's partition the universe;
- the function f_i is the identity outside of P_0 and for $x \in P_0$, we have $P_i(f_i(x))$ and $x < f_1(x) < f_2(x) < \cdots < f_{n-1}(x)$.

It follows that T_n has elimination of quantifiers. Hence it is complete and ω-categorical (because the structure generated by a set of size m has size at most nm). □

Let now $(V; \leq_0, \ldots, \leq_{n-1})$ be a structure equipped with n distinct linear orders. Assume that each order $V_i := (V, \leq_i)$ has topological rank 1 and that any two V_i, V_j are intertwined. Further assume that the structure V is transitive (that is, there is a unique 1-type over $\emptyset$). For each $i < n$, there is by Lemma 3.10 a unique increasing 0-definable map $f_i \colon V_i \to \overline{V_0}$. Inside V, we interpret an L_n-structure V_* as follows: the universe of V_* is the union of n disjoint copies of V, which we think of as representing the orders V_0 to V_{n-1}. The unary predicate P_i names the i-th copy of V, which we identify with the image $f_i(V_i)$ inside $\overline{V_0}$. The order $\leq$ on V_* is then given by the order on $\overline{V_0}$ using those identifications. Finally, the function f_i sends a point $x \in P_0(V_*)$ to the corresponding point in $P_i(V_*)$: remember, that both are just copies of V, so f_i is just the canonical identification of one copy of V with the other. Define also f_0 as being the identity function on V_*.

Since we assumed that V has a unique 1-type over $\emptyset$, then for some permutation σ of $\{0, \ldots, n-1\}$, we have that for all $x \in P_0(V_*)$,

$$f_{\sigma(0)}(x) < f_{\sigma(1)}(x) < \cdots < f_{\sigma(n)}(x).$$

If σ is the identity, then V_* is a model of T_n as defined above. Otherwise, we obtain a model of T_n by applying the same construction to the structure $(V; \leq_{\sigma^{-1}(0)}, \ldots, \leq_{\sigma^{-1}(n-1)})$. Note that there is a unique σ with this property.

Conversely, given a (countable) model $(V_*; \leq, P_0, \ldots, P_{n-1}, f_1, \ldots, f_{n-1})$ of T_n, we can construct a structure $(V^{(n)}; \leq_0, \ldots, \leq_{n-1})$ by taking as universe $V^{(n)} = P_0(V_*)$, interpreting $\leq_0$ as $\leq$ and $\leq_i$, $i > 0$ by:

$$x \leq_i y \iff f_i(x) \leq f_i(y).$$

Note that by ω-categoricity of T_n, the structure $V^{(n)}$ is uniquely defined up to isomorphism. For each $i \leq n$, let $V_i^{(n)}$ be the definable linear order $(V^{(n)}; \leq_i)$. It might seem that by going to $V^{(n)}$, we have lost the intertwining between the orders, but in fact this is not the case. Indeed, the orders $V_0^{(n)}$ and $V_i^{(n)}$, $i < n$ are intertwined in $V^{(n)}$: let $x \in V^{(n)}$ and consider the set

$$g_i(x) := \{y \in V^{(n)} : (\forall z <_i y) z <_0 x\}.$$

Then $g_i(x)$ is a cut of $V_i^{(n)}$ and we leave it to the reader to check that g_i does define an intertwining from $V_0^{(n)}$ to $V_i^{(n)}$.

If we apply the first construction above to $V^{(n)}$, then we recover the V_* we started with. The following proposition now follows from this discussion.

Proposition 3.16. *Let $(V; \leq_0, \ldots, \leq_{n-1})$ be a transitive (countable) structure equipped with n distinct linear orders. Assume that each order $V_i := (V, \leq_i)$ has topological rank 1, any two V_i, V_j are intertwined. Then for some unique permutation σ of $\{0, \ldots, n-1\}$, $(V; \leq_{\sigma(0)}, \ldots, \leq_{\sigma(n-1)})$ is isomorphic to the structure $(V^{(n)}; \leq_0, \ldots, \leq_{n-1})$ defined above. In particular, there are exactly $n!$ such structures up to isomorphism.*

3.2 Independent orders

Definition 3.17. Let V and W be two orders, definable over some A. We say that V and W are *independent* if there does not exist:

- a set of parameters $B \supseteq A$,
- B-definable infinite subsets $X \subseteq V$ and $Y \subseteq W$, both weakly transitive over B, which we equip with the induced orders from V and W respectively,
- a B-definable intertwining from X to either Y or the reverse of Y.

Note that independence is a symmetric relation.

Lemma 3.18. *Let $(V, \leq)$ be definable and minimal over some A. Let $B \supseteq A$ and $I, J \subseteq V$ be two infinite B-definable disjoint convex subsets, weakly transitive over B, then $(I, \leq)$ and $(J, \leq)$ are independent.*

Proof. Without loss of generality, B is finite. Assume first that there is an intertwining map f from I to J, definable over B. Then f extends to an increasing bijection from $\overline{I} \to \overline{J}$, which we still denote by f (see the paragraph after Lemma 3.9). Assume for definiteness that $I < J$. Let c_1, c_2 be the infimum and supremum of I respectively, seen as elements of $\overline{V}$. Define similarly d_1, d_2 for J. Hence we have $c_1 < c_2 < d_1 < d_2$. By Proposition 3.12 (and Remark 3.13), we can find $(c_1', c_2', d_1', d_2') \equiv_A (c_1, c_2, d_1, d_2)$ such that

$$c_1 < c_1' < c_2' < c_2 < d_1' < d_1 < d_2 < d_2'.$$

Let $\sigma \in \mathrm{Aut}(M)_A$ send (c_1, c_2, d_1, d_2) to (c_1', c_2', d_1', d_2'). Let I', J' be the images of I, J respectively under σ and set $g = \sigma \circ f \circ \sigma^{-1}$, so that g is an increasing map from $\overline{I'}$ to $\overline{J'}$.

Let $a \in I \setminus I'$, say $c_1 < a < c_1'$. Then f sends a to a point in $J \subset J'$. So g^{-1} is defined on $f(a)$ and sends it to a point in $\overline{I'}$, hence $a < g^{-1}(f(a))$. Applying f, we obtain $f(a) < f(g^{-1}(f(a)))$, thus $g^{-1}(f(a)) < g^{-1}(f(g^{-1}(f(a))))$. Iterating, we find infinitely many elements in $\mathrm{dcl}^{eq}(aB\sigma(B)) \cap \overline{V}$:

$$a < g^{-1} \circ f(a) < g^{-1} \circ f \circ g^{-1} \circ f(a) < g^{-1} \circ f \circ g^{-1} \circ f \circ g^{-1} \circ f(a) < \cdots .$$

This contradicts Lemma 3.7. The same argument shows that I is not intertwined over B with the reverse of J.

Now take $B' \supseteq B$ and $X \subseteq I$, $Y \subseteq J$ two infinite subsets, definable and weakly transitive over B'. Since V is minimal, the closure of X is a finite union of convex subsets. By weak transitivity, it is just one convex subset I'. Similarly the closure of Y is a convex subset J'. An intertwining between X and Y induces naturally an intertwining between I' and J'. Using the previous paragraph we see that there is no such intertwining. We conclude that I and J are independent. ☐

Corollary 3.19. *Let $(V, \leq)$ be definable and minimal over some A. Let $B \supseteq A$ and $I, J \subseteq V$ be two infinite B-definable convex subsets. Assume that I and J are intertwined, then $I = J$ and the intertwining map is the identity.*

Proof. Let h be the intertwining map from I to $\overline{J}$. If h is not the identity, then we can find some convex subset $I' \subseteq I$ such that the convex hull of $h(I')$ is disjoint from I (since h is increasing). Taking I' smaller if necessary, we can assume that it is weakly transtive over parameters C defining it. Then $h(I')$ is also weakly transitive over C, and I' and $h(I')$ contradict the previous lemma. □

Corollary 3.20. *Let $(V, \leq)$ be definable and minimal over some A. Then $(V, \leq)$ is not intertwined with its reverse $(V, \geq)$.*

Proof. If there is an A-definable decreasing map $f\colon V \to \overline{V}$, then we can find an open interval $I \subseteq V$ such that the convex hull of the image $f(I)$ is disjoint from I. Let J be the intersection of the convex hull of $f(I)$ with V. Then I and J contradict the previous lemma. □

Lemma 3.21. *Let V_0, V_1 be two linear orders definable and minimal over some A. Assume that they are not independent. Then there is either an A-definable intertwining from V_0 to V_1 or an A-definable intertwining from V_0 to the reverse of V_1.*

Proof. Without loss, assume that A is finite and let $B \supseteq A$ be also finite. Assume that we have some B-definable $X_0 \subseteq V_0$ and $X_1 \subseteq V_1$ both weakly transitive over B and a B-definable increasing map $f\colon X_0 \to \overline{X_1}$ (if there is a decreasing map from X_0 to $\overline{X_1}$, replace V_1 by its reverse). Restricting X_0, we may assume that it is transitive over B. Let $a \in X_0$. By topological rank 1, both X_0 and X_1 are dense in their convex hulls and f extends to an increasing map $\overline{X_0} \to \overline{X_1}$. Assume that $f(a) \notin \mathrm{dcl}(Aa)$. Then, there is $\sigma \in \mathrm{Aut}(M)_{Aa}$ such that $\sigma(f(a)) \neq f(a)$. Let $f' = \sigma(f)$ so that

$$f'(a) = \sigma(f)(\sigma(a)) = \sigma(f(a)) \neq f(a).$$

Then there is an open interval I of V_0 containing a such that f' is defined on I and induces an increasing map $f'|_I\colon I \to \overline{V_1}$ (indeed, we can take $I = \sigma(\overline{X_0})$). Reducing I, we can assume that $f(I)$ and $f'(I)$ are disjoint. But then $f' \circ f^{-1}$ gives an intertwining map from $f(I)$ to $f'(I)$, which contradicts Lemma 3.18.

It follows that $f(a) \in \mathrm{dcl}(Aa)$. Let g be the A-definable map sending a to $f(a)$ and for simplicity assume V_0 is transitive over A (otherwise, replace it by the locus of $\mathrm{tp}(a/A)$). As X_0 is transitive over B, g coincides with f on X_0 and therefore is increasing on it. Let $\tau \in \mathrm{Aut}(M)_A$ and let $X'_0 = \tau(X_0)$. Then g is also increasing on X'_0. Assume that the convex hulls of X_0 and X'_0 in V_0 have an open interval Z in their intersection. We can construct two increasing maps from Z to $\overline{V}_1$: one induced by $g|_{X_0}$ and one induced by $g|_{X'_0}$. By Lemma 3.10, those two maps coincide. As V_0 is transitive over A, the sets $\tau(X_0)$, for τ ranging in $\mathrm{Aut}(M)_A$ cover V_0. In particular, they cover the convex hull of X_0 in V_0. It follows that g is increasing on the convex hull of X_0. Therefore as V_0 is transitive over A, g is locally increasing on V_0: for each $a \in V_0$, there is an open convex subset of V_0 containing a on which g is increasing. Let C_a denote the maximal such set. The the sets C_a form an A-definable partition of V_0 into infinite convex sets. As V_0 has topological rank 1, $C_a = V_0$ for all a and g is increasing on V_0. It follows that g intertwines V_0 and V_1. □

Lemma 3.22. *Working over some A, let $(V_0, \leq_0)$, $(V_1, \leq_1)$ be two minimal independent definable orders. Let $f_0 \colon V_0 \to \overline{V_0}$ and $f_1 \colon V_0 \to \overline{V_1}$ be two A-definable functions. Then the set*

$$\{(f_0(x), f_1(x)) : x \in V_0\}$$

is dense in $\overline{V_0} \times \overline{V_1}$.

Proof. First, the images of f_0 and f_1 are definable subsets of $\overline{V_0}$ and $\overline{V_1}$ respectively. By Lemma 3.6, we can replace V_0 by $V_0 \cup f_0(V_0)$ and V_1 by $V_1 \cup f_1(V_0)$ and assume that f_0 and f_1 take values in V_0 and V_1 respectively.

Let $V \subseteq V_0$ be A-definable and transitive over A. Then by minimality, V is dense in V_0 and it is enough to prove that $\{(f_0(x), f_1(x)) : x \in V\}$ is dense in $\overline{V_0} \times \overline{V_1}$. Next, notice that since V_0 and V_1 are minimal over A and f_0, f_1, V are A-definable, $f_0(V)$ is dense in $\overline{V_0}$ and $f_1(V)$ is dense in $\overline{V_1}$. Fix $a \in V$ and consider the set

$$X_a = \{f_0(x) : x \in V, f_1(x) <_1 f_1(a)\}.$$

This set is non-empty by the previous sentence. Let also Y_a be the closure of X_a. Then by Lemma 3.4, Y_a is a finite union of convex sets.

The infima and suprema of those convex sets are either $\pm\infty$ or elements of $\overline{V_0}$. Let $W \subseteq \overline{V_0}$ be an A-definable subset containing V_0 along with all those elements. By Lemma 3.6, W is minimal over A. Assume that Y_a contains a bounded interval

$$c \leq_0 x \leq_0 d, \quad c, d \in W,$$

and this interval is maximal in Y_a. By Proposition 3.12 applied to W, there is an automorphism σ such that $c <_0 \sigma(c) <_0 d <_0 \sigma(d)$. But then, we have neither $Y_{\sigma(a)} \subseteq Y_a$, nor $Y_a \subseteq Y_{\sigma(a)}$ and this is impossible by the definition of X_a. We can do the same thing if Y_a contains two disjoint unbounded intervals. We conclude that Y_a is either an initial segment, an end segment, or the whole of $\overline{V_0}$.

Assume that Y_a is an initial segment and define $g(a) \in W$ to be its supremum. Then as V is a complete type, $Y_{a'}$ is an initial segment for each $a' \in V$. Let $h \colon f_1(V) \to V_0$ send a point $b = f_1(a')$ to $g(a')$. This is well defined as $X_{a'}$ and hence $g(a')$ depends only on $f_1(a')$. Note that h is non-decreasing and therefore intertwines $f_1(V)$ and V_0. This contradicts independence. Similarly, if Y_a is an end segment, we obtain an intertwining from $f_1(V)$ to the reverse of V_0. We therefore conclude that Y_a is equal to $\overline{V_0}$. We also have symmetrically that $\{f_0(x) : x \in V, f_1(x) >_1 f_1(a)\}$ is dense in $\overline{V_0}$ for all $a \in V$.

Assume now towards a contradiction that for some bounded open interval $I \subset V_0$, the set

$$H(I) := \{f_1(x) : x \in V_0, f_0(x) \in I\}$$

is not dense in $\overline{V_1}$ (where we have identified I with its convex closure in $\overline{V_0}$). Let $J \subset V_0$ be any bounded interval. By Proposition 3.12, there is an automorphism σ over A such that $\sigma(J) \subseteq I$. Then $H(\sigma(J)) \subseteq H(I)$ is not dense in $\overline{V_1}$. Therefore, also $H(J)$ is not dense in $\overline{V_1}$.

By what we know so far, $H(I)$ is cofinal and coinitial in $\overline{V_1}$ (since for any $d \in f_1(V)$, the sets $\{f_0(x) : x \in V, f_1(x) <_1 d\}$ and $\{f_0(x) : x \in V, f_1(x) >_1 d\}$

are dense in $\overline{V_0}$ and $f_1(V)$ is dense in V_1). Let $C(I) = V_1 \setminus \overline{H(I)}$. Then $C(I)$ is a non-empty finite union of bounded open intervals. Let $\tilde{C}(I)$ be its convex hull. If $I \subseteq J$, then $H(I) \subseteq H(J)$, so $C(I) \supseteq C(J)$ and $\tilde{C}(I) \supseteq \tilde{C}(J)$. As any two intervals are contained in a third one, any two intervals of the form $\tilde{C}(J)$ intersect, where J is any open interval of V_0. Let $a \in V_1$ to the left of $\tilde{C}(I)$ and $b \in V_1$ to the right of it of same type as a. Then there is an automorphism σ over A sending a to b. Then $\sigma(\tilde{C}(I)) = \tilde{C}(\sigma(I))$ is disjoint from $\tilde{C}(I)$. This is a contradiction. □

Having described in Corollary 3.14 the closed definable subsets of minimal orders, and hence of products of intertwined orders, we now complete the picture with the case of pairwise independent orders.

Proposition 3.23. *Working over some set A, let $V_0, \dots, V_{n-1}$ be pairwise independent minimal definable orders. Then any A-definable closed set $X \subseteq V_0^{k_0} \times \cdots \times V_{n-1}^{k_{n-1}}$ is a finite union of products of the form $D_0 \times \ldots \times D_{n-1}$, where each D_i is an A-definable closed subset of $V_i^{k_i}$.*

Proof. We assume for simplicity that $A = \emptyset$. Let $p \in S(\emptyset)$ be a type on $\overline{V_0}^{k_0} \times \cdots \times \overline{V_{n-1}}^{k_{n-1}}$. Let D be the closure in $\overline{V_0}^{k_0} \times \cdots \times \overline{V_{n-1}}^{k_{n-1}}$ of its set of realizations. Say that p has property ⊠ if there are closed 0-definable sets $D_i \subseteq V_i^{k_i}$ such that

$$D \cap (V_0^{k_0} \times \cdots \times V_{n-1}^{k_{n-1}}) = D_0 \times \cdots \times D_{n-1}.$$

We prove the following two statements by induction on n:

(A_n) Let $V_0, \dots, V_{n-1}$ be pairwise independent minimal orders. If $f_i \colon V_0 \to \overline{V}_i$, $i < n$, are 0-definable functions, then $\{(f_0(x), f_1(x), \dots, f_{n-1}(x)) : x \in V_0\}$ is dense in $\overline{V_0} \times \cdots \times \overline{V_{n-1}}$.

(B_n) Let $V_0, \dots, V_{n-1}$ be pairwise independent minimal orders. Then any type $p \in S(\emptyset)$ on $\overline{V_0}^{k_0} \times \cdots \times \overline{V_{n-1}}^{k_{n-1}}$ has property ⊠.

The statement of the theorem then follows from (B_n) since by ω-categoricity, any definable set is a finite union of types.

Property (A_1) follows from minimality and (B_1) holds trivially. We will show that (A_n) and (B_n) together imply (A_{n+1}) and then that (A_{n+1}) implies (B_{n+1}).

$(A_n) + (B_n) \Rightarrow (A_{n+1})$: The property (A_2) is Lemma 3.22, so we can assume $n > 1$. We follow closely the proof of Lemma 3.22. Fix $a \in V_0$ and define

$$X_a = \{(f_0(x), f_1(x), \dots, f_{n-1}(x)) : x \in V_0, f_n(x) < f_n(a)\} \subseteq V_0 \times \cdots \times V_{n-1}.$$

For each $i < n$, let $Y_i \subseteq V_i$ be a complete type over a. Note that $\overline{Y}_i$ is convex in $\overline{V}_i$ (it is a finite union of convex sets by minimality and then is convex since it defines a complete type over a). Set

$$\hat{Y} = \prod_{i<n} \overline{Y_i} \subseteq \prod_{i<n} \overline{V_i}.$$

Working over the parameter a, the Y_i's are pairwise independent minimal orders. The property (A_n) then implies that $X_a \cap \hat{Y}$ is either dense in $\hat{Y}$ or empty. It now follows that the closure $\overline{X_a}$ of X_a in $\prod_{i<n} \overline{V_i}$ is a union of finitely many rectangles of the form $\prod_{i<n} I_i$, where each $I_i \subseteq \overline{V_i}$ is a convex set. The tuple of endpoints (or endcuts) of those convex sets is an element of some $\overline{V_1}^{k_1} \times \cdots \times \overline{V_{n-1}}^{k_{n-1}}$. Let p be the type of that tuple over $\emptyset$. Applying (B_n), we see that p has property ⊠. In addition, it follows from the definition of X_a that for any a' having the same type as a, one of $\overline{X_a}$ and $\overline{X_{a'}}$ is included in the other. Since property ⊠ allows us to move the endpoints of the convex sets defining X_a freely, this is only possible if $\overline{X_a}$ is either the full product $\prod_{i<n} \overline{V_i}$, or is a rectangle, unbounded on all but at most one coordinate. However, by (B_2), we know that $\overline{X_a}$ must have full projection on each coordinate. Hence the only possibility is that $\overline{X_a} = \prod_{i<n} \overline{V_i}$.

We end as in Lemma 3.22. Density of X_a in the product implies that for any product $\hat{I} = \prod_{i<n} I_i$ of open intervals, the set

$$s(\hat{I}) := \{f_n(x) : (f_0(x), f_1(x), \ldots, f_{n-1}(x)) \in \hat{I}\}$$

is coinitial in V_n. By applying the same argument to the reverse order, we get that it is also cofinal. Furthermore, by minimality, the closure of $s(\hat{I})$ is a finite union of convex sets. Hence, given any $\hat{I}$, there is a unique minimal bounded convex set $c(\hat{I}) \subseteq V_n$ such that $s(\hat{I})$ is dense in $V_n \setminus c(\hat{I})$. If $\hat{I} \subseteq \hat{I}'$, then $c(\hat{I}) \supseteq c(\hat{I}')$. As V_n is weakly transitive, the intersection of all $c(\hat{I})$ is empty. Since the family of $\hat{I}$'s is upward-directed under inclusion, $c(\hat{I}_*)$ must be empty for some $\hat{I}_*$. But then by (B_n), for any $\hat{I}'$, one can find $\hat{I}'_* \subseteq I_*$ which is a conjugate of $\hat{I}'$. Hence $c(\hat{I}')$ is also empty and $s(\hat{I}')$ is dense in V_n. Since this holds for any $\hat{I}'$, (A_{n+1}) follows.

$(A_{n+1}) \Rightarrow (B_{n+1})$: As in the proof of Proposition 3.12, to show that all types have property ⊠, it is enough to find, for all $k < \omega$, one type in $\overline{V}_0^k \times \cdots \times \overline{V}_n^k$ having property ⊠ and for which no two coordinates are equal. To this end, take $b_0 \in V_0$. For each $i \leq n$, let $m_i(b_0)$ denote the largest element of $\overline{V_i}$ definable from b_0. Set $a_{0,i} = m_i(b_0)$. Then by (A_{n+1}) applied to the functions m_i, we see that $p_1 := \text{tp}(a_{0,i} : i \leq n)$ has property ⊠: its set of realizations is dense in $\overline{V_0} \times \cdots \times \overline{V_n}$.

Assume that b_l, $a_{l,i}$ have been constructed for $l < k$, $i \leq n$, with $a_{l,i} = m_i(b_{\leq l})$. For $i \leq n$, let X_i be a complete type over $b_{<k}$ of elements in V_i, greater than $a_{k-1,i}$. So X_i is dense in $\{x \in V_i : x > a_{k-1,i}\}$. Work over $b_{<k}$ and consider the sets $X_0, \ldots, X_n$ equipped with the induced orders. They are pairwise independent. Pick any $b_k \in X_0$ and define $a_{k,i} = m_i(b_{\leq k})$, $i \leq n$. Then again by (A_{n+1}), the set of realizations of $\text{tp}(a_{k,i} : i \leq n)$ is dense in $\overline{X_0} \times \cdots \times \overline{X_n}$. It

follows inductively that the resulting type $p_k := \mathrm{tp}(a_{l,i} : l \leq k, i \leq n)$ satisfies $\boxtimes$. ☐

Definition 3.24. Let $(V, \leq)$ be a parameter-definable linear order. For $k < \omega$, a *sector* of V^k is a subset of V^k defined by a formula $\phi(x_0, \ldots, x_{k-1})$ which is a finite boolean combination of relations of the form $x_i = x_j$ and $x_i \leq x_j$, $i, j < k$.

If $V_0, \ldots, V_{n-1}$ are pairwise independent parameter-definable orders, a *sector* of $V_0^{k_0} \times \cdots \times V_{n-1}^{k_{n-1}}$ is a finite union of products $D_0 \times \ldots \times D_{n-1}$, where each D_i is a sector of $V_i^{k_i}$.

Corollary 3.25. *Working over some set A, let $V_0, \ldots, V_{n-1}$ be pairwise independent minimal definable orders. Let $X \subseteq V_0^{k_0} \times \cdots \times V_{n-1}^{k_{n-1}}$ be A-definable. Then the topological closure of X is a sector of $V_0^{k_0} \times \cdots \times V_{n-1}^{k_{n-1}}$.*

Proof. This follows at once from Proposition 3.23 applied to the topological closure of X (which is also A-definable) along with Corollary 3.14. ☐

We say that a betweenness relation has topological rank 1 if one (or equivalently both) of its associated linear orders has topological rank 1.

Corollary 3.26. *Let V be a definable transitive set and let $B_1, \ldots, B_n$ be distinct 0-definable betweenness relations on V of topological rank 1. Then for any subset $I \subseteq n$, we can find $a_I, b_I, c_I \in V$ such that $B_i(a_I, b_I, c_I)$ holds if and only if $i \in I$.*

We now extend Proposition 3.16 to a structure equipped with any n minimal linear orders.

Proposition 3.27. *Let $(M; \leq_1, \ldots, \leq_n)$ be countable, ω-categorical, transitive over $\emptyset$. Assume that each $M_i := (M, \leq_i)$ is a linear order of topological rank 1 and that no two of them are equal or reverse of each other. Then for each $i \neq j \leq n$, exactly one of the following holds:*

- *$\leq_i$ and $\leq_j$ are independent;*
- *$\leq_i$ is intertwined with $\leq_j$ and if $f_{ij} \colon M_i \to \overline{M_j}$ is the unique 0-definable increasing map, we have $f_{ij}(x) <_j x$ for all x;*
- *$\leq_i$ is intertwined with $\leq_j$ and we have $f_{ij}(x) >_j x$ for all x;*
- *$\leq_i$ is intertwined with the reverse of $\leq_j$ and if $f_{ij} \colon M_i \to \overline{M_j}$ is the unique 0-definable decreasing map, we have $f_{ij}(x) <_j x$ for all x;*
- *$\leq_i$ is intertwined with the reverse of $\leq_j$ and we have $f_{ij}(x) >_j x$ for all x.*

Furthermore, the data of which of those cases holds for each pair $i \neq j$ completely determines the isomorphism type of M.

Proof. The argument is similar to that of Proposition 3.16, which we present a little bit differently. First note that by Corollary 3.20, by replacing some orders $\leq_i$ with their reverses, we can assume that the last two cases never occur. Let then E be the equivalence relation on $\{1, \dots, n\}$ which holds for i, j if M_i and M_j are intertwined. Let $s_1, \dots, s_k$ be representatives of the E-classes and for each $i \leq n$, let $t(i)$ be such that $i \; E \; s_{t(i)}$. Define also $\iota_i \colon M_i \to \overline{M_{s_{t(i)}}}$ to be the unique increasing 0-definable map intertwining M_i and $M_{s_{t(i)}}$.

For $l \leq k$, define $V_l \subseteq \overline{M_{s_l}}$ as the union

$$V_l = \bigcup_{i E s_l} \iota_i(M_i),$$

and let $\preceq_l$ be its canonical linear order. Then $(V_l, \preceq_l)$ is a minimal, 0-definable order. Define

$$\Gamma = \{(\iota_1(x), \dots, \iota_n(x)) : x \in M\} \subseteq \prod_{i \leq n} V_{t(i)}.$$

Now by the previous proposition, Γ is dense in a product $D_1 \times \cdots \times D_k$ of closed subsets of the V_i's. By Corollary 3.14, D_k is dense in a set defined by a boolean combination of inequalities on variables $x_i \leq x_j$. Those inequalities are determined by inequalities $\iota_i(x) \lesseqgtr_{s_{t(i)}} x$ that are true in M and are part of the data that we are given. We conclude by a direct back-and-forth argument as in Proposition 3.15. □

We will need the following corollary later on.

Corollary 3.28. *Let $(M; \leq_1, \dots, \leq_n, \dots)$ be as in the previous proposition with possibly additional structure. Then there is a finite set of parameters $\bar{d}$ and some $\bar{d}$-definable subset $X \subseteq M$ such that X is transitive over $\bar{d}$ and the orders $\leq_1, \dots, \leq_n$ are pairwise independent when restricted to X.*

Proof. We construct X as a subset of an intersection of intervals for the various orders. We will use the notation $M_i = (M, \leq_i)$ as in the previous proposition.

For each $i \leq n$, we pick $a_i <_i b_i$ in M. We can make this choice in such a way that the intervals $a_i <_i x <_i b_i$ and $a_j <_j x <_j b_j$ are independent when equipped with $<_i$ and $<_j$ respectively: if M_i and M_j are independent, then any choice of points will do and if they are intertwined, pick two intervals so that their convex closures in the common completion $\overline{M_i} = \overline{M_j}$ are disjoint (similarly if they are intertwined up to reversal). Having done this, define let $\bar{d} = (a_i, b_i)_{i \leq n}$ and let X_0 be the set defined by the conjunction of the formulas $a_i <_i x <_i b_i$ for $i \leq n$. The set X_0 need not be transitive over $\bar{d}$, so let $X \subseteq X_0$ be any $\bar{d}$-definable infinite subset of X_0 which is transitive over $\bar{d}$. Then then n orders are pairwise independent on X as required. □

3.3 Weakly minimal orders

We have defined intertwining only for weakly transitive orders. This was required to ensure that being intertwined is a symmetric relation. This will be too restrictive later on, especially because the property of being weakly transitive is not invariant under adding parameters to the base. We therefore need a more general notion that allows us to talk about orders being intertwined even if they are not weakly transitive. To keep the nice properties of this relation (in particular, symmetry and uniqueness of the intertwining), we instead assume that the orders are *weakly minimal*, as defined below. To motivate this definition, note that if $(V, \leq)$ is A-definable and has topological rank 1, then $\mathrm{dcl}^{eq}(A) \cap \overline{V}$ is finite of size n say. Enumerate its elements as $s_1 < \cdots < s_n$. We use the interval notation $(a, b) := \{x \in V : a < x < b\}$ for $a, b \in \overline{V}$. Then the $n + 1$ convex subsets $(-\infty, s_1)$, (s_i, s_{i+1}) and $(s_n, +\infty)$ are each A-definable and minimal over A (and furthermore those are the *only* A-definable infinite convex subsets of V minimal over A since for any A-definable convex subset W of V the infimum and supremum of W are definable over A). Any two of those minimal convex subsets are either independent or intertwined over A. The following definition simply says that the latter never happens.

Definition 3.29. Let A be a set of parameters. An A-definable order $(V, \leq)$ is *weakly minimal over A* if:
- it is densely ordered with neither smallest no largest element;
- it has topological rank 1;
- any two distinct A-definable minimal convex subsets of it are independent.

Lemma 3.30. *Let $(V, \leq)$ be an A-definable linear order and $B \supseteq A$ a larger set of parameters. Then V is weakly minimal over A if and only if it is weakly minimal over B.*

Proof. Assume that V is weakly minimal over A. Let $W_1, W_2 \subseteq V$ be two B-definable distinct convex subsets of V that are minimal over B. Let W_1^*, W_2^* the A-definable convex subsets of V that are minimal over A and contain W_1 and W_2 respectively. If $W_1^* = W_2^*$, then W_1, W_2 are independent by Lemma 3.18. If $W_1^* \neq W_2^*$, then W_1^* and W_2^* are independent and hence so are W_1 and W_2 by definition of independence.

Conversely, assume that V is not weakly minimal over A. Let W_1^*, W_2^* be two distinct A-definable convex subsets of V that are minimal over A and intertwined. Let $f : \overline{W_1^*} \to \overline{W_2^*}$ be the A-definable increasing bijection. Let W_1 be an infinite convex subset of W_1^* which is definable and minimal over B. Let W_2 be equal to $f(\overline{W_1^*}) \cap W_2^*$. Then W_2 is definable and minimal over B and W_1 and W_2 are intertwined. Hence V is not weakly minimal over B. □

It follows from the lemma that we can drop "over A" in the definition of weakly minimal: given a parameter-definable order $(V, \leq)$, we can say whether it is weakly minimal or not without mentioning a set of parameters over which it is defined since the definition does not depend on the set of parameters actually chosen. (If V is definable over both A and A', let $B = A \cup A'$ and apply the previous lemma to the pair (A, B) and to the pair (A', B).)

Definition 3.31. Let V, W be two linear orders, definable over some A and weakly minimal. We say that V and W are *intertwined over* A if there is an A-definable increasing bijection $f : \overline{V} \to \overline{W}$.

Proposition 3.32. *If V and W are weakly minimal A-definable orders which are intertwined over A, then there is a unique intertwining between V and W.*

Proof. Assume that V and W are intertwined and let $f : \overline{V} \to \overline{W}$ be the A-definable increasing bijection. Enumerate $\mathrm{dcl}^{eq}(A) \cap \overline{V}$ as $s_1 < \cdots < s_n$ and enumerate $\mathrm{dcl}^{eq}(A) \cap \overline{W}$ as $t_1 < \cdots < t_m$. Formally set s_0 and t_0 to be equal to $-\infty$ and s_{n+1}, t_{m+1} to be equal to $+\infty$. Let $i < n+1$. The subset (s_i, s_{i+1}) of V is definable and minimal over A. Since f is definable over A, both $f(s_i)$ and $f(s_{i+1})$ are in $\mathrm{dcl}^{eq}(A) \cap \overline{W}$ and the subset $(f(s_i), f(s_{i+1})$ is an A-definable convex subset of W. (Here, we have increased f formally to map $\pm\infty$ to $\pm\infty$.) Furthermore, $(f(s_i), f(s_{i+1})$ is minimal over A: if not, there would be some $j < m+1$ such that $f(s_i) < t_j < f(s_{i+1})$, but then $f^{-1}(t_j)$ would be an element of $\mathrm{dcl}^{eq}(A) \cap \overline{V}$ lying strictly between s_i and s_{i+1}. Therefore for some j, $f(s_i) = t_j$ and $f(s_{i+1}) = t_{j+1}$. Since f is increasing, this implies that $n = m$ and $f(s_i) = f(t_i)$ for all $i < n+1$. Now, for each $i < n+1$, there is by Lemma 3.10, a unique intertwining between (s_i, s_{i+1}) and (t_i, t_{i+1}). This shows that f is completely determined and there is a unique intertwining between V and W. □

It follows from the proposition that we can drop "over A" in the definition of being intertwined: if V and W are definable both over A and over B, then they are intertwined over A if and only if they are intertwined over B and the (unique) intertwining is definable over any set of parameters over which V and W are defined.

Note that if V and W are weakly minimal, but not intertwined, they are not necessarily independent: It could be that some proper convex subset of V is intertwined with a convex subset of W, but that intertwining does not extend to the whole of V. For instance let $(V_1, \leq_1), (V_2, \leq_2)$ be two definable independent minimal orders, definable over $\emptyset$ and assume for simplicity that V_1, V_2 are two disjoint subsets of the model. Set V be the linear order $V_1 + V_2$, that is V has underlying set $V_1 \cup V_2$ equipped with the linear ordering obtained by making the elements of V_1 smaller than those of V_2 and keeping the existing orders on V_1 and V_2 respectively. Then V is weakly minimal and is neither independent nor intertwined with V_1.

Proposition 3.23 and Corollary 3.25 go through for weakly minimal orders instead of minimal orders, except that we have to extend our notion of sector to allow for definable convex subsets.

Definition 3.33. let $A \subseteq M^{eq}$ and let $(V, \leq)$ be an A-definable linear order. For $k < \omega$, an *A-sector* of V^k is a subset of V^k defined by a formula $\phi(x_0, \ldots, x_{k-1})$ which is a finite boolean combination of relations of the form:

- $x_i = x_j$, for $i, j < k$;
- $x_i \leq x_j$, for $i, j < k$;

- $x_i = a$, for $i < k$ and $a \in \mathrm{dcl}^{eq}(A) \cap V$;
- $x_i \leq a$, for $i < k$ and $a \in \mathrm{dcl}^{eq}(A) \cap \overline{V}$.

In other words, an A-sector is a subset of V^k which is quantifier-free definable from the order along with unary predicates for A-definable cuts of V.

Definition 3.34. let $A \subseteq M^{eq}$ and let $V_0, \dots, V_{m-1}$ be pairwise independent A-definable linear orders. For $n < \omega$, an *A-sector* of $V_0^{k_0} \times \cdots \times V_{m-1}^{k_{m-1}}$ is a finite union of sets of the form $D_0 \times \cdots \times D_{m-1}$, where each D_i is an A-sector of $V_i^{k_i}$.

Proposition 3.35. *Let $V_0, \dots, V_{n-1}$ be pairwise independent weakly minimal A-definable orders. Let $X \subseteq V_0^{k_0} \times \cdots \times V_{n-1}^{k_{n-1}}$ be an A-definable set. Then the closure of X is an A-sector of $V_0^{k_0} \times \cdots \times V_{n-1}^{k_{n-1}}$.*

Proof. Each V_i decomposes over A as a finite union of points in $\mathrm{dcl}(A) \cap V_i$ and minimal convex subsets with end-cuts in $\mathrm{dcl}^{eq}(A) \cap \overline{V_i}$. Note that all of those are A-sectors of V_i. Any two of those minimal convex subsets are independent. The product $V_0^{k_0} \times \cdots \times V_{n-1}^{k_{n-1}}$ then decomposes over A as a finite union of products of powers of A-definable minimal orders and A-definable points, such that any two distinct minimal orders are independent. It is then enough to prove the proposition for one such product, but that is given by Corollary 3.25. ☐

Let V be a weakly minimal linear order. We write $W \subseteq_{\mathrm{wmc}} V$ to say that $W \subseteq V$ is a convex subset of V which is weakly minimal. Since V is weakly minimal, this is equivalent to asking that W is a convex subset of V that has no first nor last element.

4 Circular orders

Most of the results above generalize to circular orders, though some extra arguments are required.

Let (V, C) be a circular order. We will abuse notation by writing say $a < b < c < d$ to mean that a, b, c, d are pairwise distinct and (a, b, c, d) lie in this order on V: that is $C(a, b, c) \wedge C(b, c, d) \wedge C(c, d, a) \wedge C(d, a, b)$. So $a < b$ only means that $a \neq b$ and $a < b < c$ is equivalent to $a \neq b \neq c \wedge C(a, b, c)$. Hopefully, this will not lead to confusion. For any $a < b$ on V, the set defined by $a < x < b$ is called an *open interval* of V. Any interval has a canonical linear order on it coming from the circular order on V. The notations are consistent in the sense that if $I \subseteq V$ is an open interval, and $c, d, e \in I$, then we have $c < d < e$ in the sense of the circular order if and only if we have $c < d < e$ in the sense of the induced linear order on I.

For $a \in V$, we let $V_{a\rightarrow} = V \setminus \{a\}$, equipped with the linear order inherited from C. We say that V has *topological rank 1* if it does not admit a parameter-definable convex equivalence relation with infinitely many infinite classes. Then V has topological rank 1 if and only if some/any $V_{a\to}$ has topological rank 1.

Let V be circularly ordered. A subset $I \subseteq V$ is *convex* if for any $a \neq b \in I$, one of the two intervals $a < x < b$ and $b < x < a$ is included in I. A convex set I is *bounded* if its complement is infinite. Note that if V is dense, then any open interval is bounded. A bounded convex set $I \subseteq V$ has a well defined linear order induced by the circular order on V. If I and J are two bounded convex subsets of V with no last element (in their induced linear orders), we say that I and J *define the same cut* in V if one is an end segment of the other.

We define the completion $\overline{V}$ of V as the set of definable bounded convex subsets of V quotiented by the equivalence relation of defining the same cut. As for linear orders, this is naturally a countable union of interpretable sets (or rather a direct limit). In fact, given $a \in V$, $\overline{V}$ can be canonically identified with $\overline{V_{a\to}} \cup \{a\}$: the element $a \in V$ is identified with the class of an open interval $b < x < a$ and any cut of $V_{a\to}$ is a bounded convex subset of V and is identified with its class in $\overline{V}$. As in the case of linear orders, $\overline{V}$ is naturally equipped with a circular order, and there is a canonical embedding of V in $\overline{V}$ which sends V to a dense subset of $\overline{V}$.

We say that the A-definable circular order V is *weakly transitive* over A if it is densely ordered and no element in $\overline{V}$ is algebraic over A.

Lemma 4.1. *If, over some A, (V, C) is definable and weakly transitive of topological rank 1, then any A-definable subset of V is dense in V.*

Proof. By topological rank 1, any closed A-definable subset of V is a finite union of convex sets. The cuts defining these convex sets are algebraic over A, but there can be no such cut by weak transitivity. □

If V and W are two A-definable weakly transitive circular orders, we say that they are *intertwined* over A if there is an A-definable order-preserving injective map $f : V \to \overline{W}$. As for linear orders, this is an equivalence relation. It is no longer true that such a map has to be unique, however, we will see that there can be at most finitely many.

Definition 4.2. A *self-intertwining* of a circular order (V, C) is an intertwining map $f : V \to \overline{V}$ which is not the identity.

Let (V, C) be a 0-definable circular order of topological rank 1 and fix some $a \in V$. Then we can write $V = F \cup V_1 \cup \cdots \cup V_n$, where $F = \mathrm{dcl}(a) \cap V$ and the V_i's are convex subsets of V, definable and weakly transitive over a, with $V_1 < V_2 < \cdots < V_n$. (In other words, the V_i's are the infinite convex subsets of V defined as the set of points between two consecutive elements of $\mathrm{dcl}^{eq}(a) \cap \overline{V}$. Note that F only contains those elements of $\mathrm{dcl}^{eq}(a) \cap \overline{V}$ that actually lie in V.) The V_i's are then minimal over A and by Lemma 3.10, for any $i, j \leq n$, there is at most one intertwining map $f_{ij} : V_i \to \overline{V_j}$. If it exists, f_{ij} has dense image.

Let now $f : V \to \overline{V}$ be a self-intertwining map defined over some set A. Let $W \subseteq V$ be a minimal Aa-definable infinite convex subset of V. Say that

W is a subset of V_i and assume that f sends W to some $\overline{V_j}$. Composing by f_{ij}^{-1}, we get an intertwining between two subsets of V_i. By Corollary 3.19, this must be the identity. It follows that f coincides with f_{ij} on W. By continuity of f, f must coincide with f_{ij} on the whole of V_i. So f sends V_i to some $\overline{V_j}$ via f_{ij}. Assume that for some i, f sends V_i to $\overline{V_{i+k}}$. Then as f preserves the order, it must send V_{i+1} to $\overline{V_{i+k+1}}$ (addition modulo n) and iteratively send any V_j to $\overline{V_{j+k}}$. The number k completely determines f, as does therefore the image of a. The possibilities for k form a subgroup in $\mathbb{Z}/n\mathbb{Z}$. Hence the set of self-intertwinings along with the identity map, equipped with composition, is isomorphic to $\mathbb{Z}/\delta\mathbb{Z}$ for some integer δ.

Definition 4.3. A circular order V is *minimal* if it is weakly transitive, of topological rank 1 and admits no self-intertwining.

Lemma 4.4. *Let V be a circular order and let X_a, $a \in D$, be a uniformly definable family of non-empty subsets of V which is directed: for any $a, a' \in D$ there is $a'' \in D$ such that $X_{a''} \subseteq X_a \cap X_{a'}$. Then there is some $c \in \overline{V}$ such that for any $a \in D$ and any neighborhood I of c in V, $I \cap X_a \neq \emptyset$.*

Proof. We fix a point $d \in V$ and work in the linear order $V_{d\to}$. Let $c \in \overline{V}$ be equal to $\inf_{a\in D}(\sup X_a)$. (If $c = \pm\infty$, then set $c = d$.) Then c has the required property. □

Proposition 4.5. *Working over some A, let V be a minimal definable circular order. Then for any type $p(x_1, \ldots, x_n) \vdash x_1 < \cdots < x_n$ over A, and any open intervals $I_1 < \cdots < I_n$ of V, we can find $a_i \in I_i$ with $(a_1, \ldots, a_n) \models p$.*

Proof. For simplicity, assume $A = \emptyset$. Fix $a < b$ in V and let $q(x, y) = \mathrm{tp}(a, b)$. Call a convex subset I of V *small* if there are no $a' < b'$ in I with $\mathrm{tp}(a', b') = q$ (where the order $<$ is the canonical one on I). Assume that there is some small interval. Then by weak transitivity, for any point c of V, there is a small open interval containing c. For any $c \in V$, let $s(c)$ be the maximal cut in $V_{c\to}$ so that $(c, s(c))$ is small. We have

$$c < d < s(c) \Longrightarrow c < d < s(c) \leq s(d).$$

Note that if $c < d < s(c) = s(d)$, then $s(c) = s(e)$ for any e, $c < e < d$. Hence the preimage of a cut by s is a convex set. If the preimage of some cut is infinite, then this is true for infinitely many cuts in $\overline{V}$ by weak transitivity. But then the relation $s(x) = s(y)$ is a convex equivalence relation with infinitely many infinite classes, contradicting topological rank 1. It follows that s is injective. Hence $s\colon V \to \overline{V}$ is a self-intertwining, which contradicts minimality. We have established that no interval is small.

Given $a \in V$, let $m(a)$ denote the maximal cut in $V_{a\to}$ definable over a (and $m(a) = a$ if there is no such cut). Chose $a_* \leq m(a_*) < b_*$ in V and set $q = \mathrm{tp}(a_*, b_*)$. Now pick $I_0 < I_1 < \cdots < I_n$ open intervals of V. By the previous paragraph, we can find some pair $(a, b) \models q$ such that $a, b \in I_0$. The interval $x > m(a)$ in $V_{a\to}$ is a linear order which is weakly transitive over a. Let $p_{1n}(x_1, x_n)$ be the restriction of p to the variables (x_1, x_n). Applying the previous paragraph to p_{1n}, we see that there is a realization of p_{1n} in $\{x \in V_{a\to} : x > m(a)\}$. Such a realization extends to a realization of p lying in that same interval. By Lemma 3.12, we can find $a_1 \in I_1, \ldots, a_n \in I_n$ with $\operatorname{tp}(a_1, \ldots, a_n) = p$. □

Lemma 4.6. *Let V be a minimal circular order and $I, J \subseteq V$ two disjoint open intervals, then I and J are independent (as linear orders).*

Proof. Assume that some two disjoint intervals I, J of V are intertwined (over some set of parameters). Then by the previous proposition, we can find $I' \subset I$ and $J' \supset J$ disjoint such that the pair (I', J') is a conjugate of (I, J). In particular I' and J' are intertwined and we conclude as in Lemma 3.18. □

Definition 4.7. Two circular orders V and W are *independent* if any open interval of V is independent (as a linear order) from any open interval of W.

Lemma 4.8. *Working over some A, let V be a weakly transitive definable circular order of topological rank 1 and W a weakly transitive definable linear order of topological rank 1. Then an open interval of V is independent with any open interval of W.*

Proof. Assume that $I \subseteq V$ and $J \subseteq W$ are two open intervals definable and weakly transitive over some B, which are intertwined. Let $a \in I$. If I_0 is an open interval containing a intertwined with some interval J_0 of W, then by uniqueness of intertwinings (and Lemma 3.18), the intertwining maps $I_0 \to \overline{W}$ and $I \to \overline{W}$ must coincide on $I_0 \cap I$. It follows that the element of $\overline{W}$ to which a is mapped lies in $\operatorname{dcl}(Aa)$: say it is equal to $g(a)$ for some A-definable function g. Then $g\colon V \to \overline{W}$ is locally increasing, which is impossible. □

Proposition 4.9. *Working over some A, let $V_0, \ldots, V_{n-1}$ be minimal definable circular orders, pairwise independent. Then any A-definable closed set $X \subseteq V_0^{k_0} \times \cdots \times V_{n-1}^{k_{n-1}}$ is a finite union of products of the form $D_0 \times \ldots \times D_{n-1}$, where each D_i is an A-definable closed subset of $V_i^{k_i}$.*

Proof. Assume $A = \emptyset$. We show the following two statements by induction on n. Note that (B_n) implies what we want since by ω-categoricity, any definable set is a finite union of types.

(A_n) Let $p(\bar{x}_i : i < n)$ be a type in some product $V_0^{l_0} \times \cdots \times V_{n-1}^{l_{n-1}}$, then given any intervals $I_i \subseteq V_i$, we can find $(\bar{a}_i : i < n) \models p$ with $\bar{a}_i \in I_i$ for each $i < n$.

(B_n) For any type p over $\emptyset$ on $V_0^{k_0} \times \cdots \times V_{n-1}^{k_{n-1}}$, the closure X of the set of realizations of p is equal to the product of its projections to each factor $V_i^{k_i}$.

(B_n): Assume we know (A_n) and we show that (B_n) follows.

Let X be given as in (B_n) and for $i < n$, let D_i be the projection of X to $V_i^{k_i}$. For each $i < n$, let $T_i \subseteq V_i$ be an open interval and set $T = T_0^{k_0} \times \cdots \times T_{n-1}^{k_{n-1}}$. Since we can choose T to contain any given finite set, it is enough to show the result for $X \cap T$ instead of X.

Let $\bar{e}$ be any tuple of parameters containing at least two points from each V_i, $i < n$. For each $i < n$, let $a_i, b_i \in \operatorname{dcl}(\bar{e}) \cap \overline{V_i}$ be such that the complement of the convex set $a_i \leq x \leq b_i$ in V_i is infinite and weakly transitive over $\bar{e}$. By (A_n), we may choose $\bar{e}$ so that each convex set $a_i \leq x \leq b_i$ is disjoint from T_i. Then over $\bar{e}$, the T_i's are intervals in some weakly transitive $\bar{e}$-definable linear orders, which are pairwise independent. Therefore by Proposition 3.23, the restriction of X to T is the product of its projections to each factor, as required.

(A_n): Assume that we know (B_{n-1}) and we prove (A_n).

Let $V = V_0$ and $W = \prod_{0<i<n} V_i$. Given a point $d \in \prod_{0<i<n} V_i$, a neighborhood of d will mean a product $\prod_{0<i<n} J_i$, where each J_i is an open interval containing d_i.

Let $c \in V$. Say that a subset $J = \prod_{0<i<n} J_i \subseteq W$ is *good for c* if for any open interval $I \subset V$ containing c, there is $(\bar{b}_i)_{i<n} \models p$, with $\bar{b}_0 \in I$ and $\bar{b}_i \in J_i$, $i > 0$. We claim that there are bounded convex sets $J_i \subset V_i$, $i < n$ such that $\prod_{0<i<n} J_i$ is good for c. To see this, take for each $i < n$, $K_{i,1}, \ldots, K_{i,t}$ disjoint open intervals of V_i, with $t > |\bar{b}_i|$ and set $J_{i,s} = V_i \setminus K_{i,s}$: a bounded convex subset of V_i. By Proposition 4.5, for any neighborhood I of c, there is $(\bar{b}_i)_{i<n} \models p$ with $\bar{b}_0 \in I$ and $\bar{b}_i \in V_i$, $0 < i$. For each $0 < i < n$, there must be some $s(i)$ such that no coordinate of $\bar{b}_i$ lies in $K_{i,s(i)}$. As the family of possible I is directed downwards, there is a choice of $s(i)$ which works for all I. Let $J_i = J_{i,s(i)}$, $i < n$. Then the set $\prod_{0<i<n} J_i$ is good for c.

For any $J \subseteq W$ a product of bounded convex sets, let $X(J) \subseteq V$ be the set of elements $c \in V$ for which J is good. Note that $X(J)$ is closed in the order topology and hence is a finite union of closed intervals. For $d \in W$, the family $\{X(J) : J \text{ neighborhood of } d\}$ is directed. By Lemma 4.4, there is some $c \in \overline{V}$ which lies in the closures of each such $X(J)$. We then have the following property: for any neighborhoods I of c and J of d, there is $(\bar{b}_i)_{i<n} \models p$, with $\bar{b}_0 \in I$ and $\bar{b}_i \in J_i$, $i > 0$. Take a set of parameters $\bar{e}$ containing two points from each V_i and such that neither c nor d lies in $\operatorname{acl}(\bar{e})$. Then, over $\bar{e}$, there are intervals $J_i \subseteq V_i$ that are weakly transitive and with $c \in J_0$ and $d_i \in J_i$. By assumption, the J_i's are pairwise independent. Therefore by Lemma 3.23, given any subintervals $J'_i \subseteq J_i$, we can find a realization of p in $\prod_{i<n} J'_i$.

Given $d \in W$, let $Z(d)$ be the set of points $c \in V$ such that any neighborhood d is good for c. By the previous paragraph, there is d such that $Z(d)$ has non-empty interior. Then by Proposition 4.5, for any open interval I_* of V, there is $d_* \in W$ such that $Z(d_*) \supseteq I_*$. Let $Z_*(I_*)$ denote the set of such points d_*. For $i < n$ let $\pi_i \colon \prod_{0<j<n} V_j \to V_i$ be the canonical projection. Fix $c \in V$ and for each $0 < i < n$ consider the family $\{\pi_i(Z_*(I)) : I \text{ open interval disjoint from } c\}$. Since the map Z_* is decreasing, that family is directed and by Lemma 4.4 there is $e_i \in \overline{V_i}$ in the closure of all of its elements. Now (B_{n-1})

implies that the closure of $Z_*(I)$ can be written as a union of definable subsets of the form $D_1 \times \ldots \times D_{n-1}$, where each D_i is a closed definable subset of V_i. Restricting it, we can assume that it is equal to one such set. We then have that $e = (e_1, \ldots, e_{n-1})$ is in the closure of each $Z_*(I)$, I open interval disjoint from c.

We have thus obtained the following property: for any neighborhood J of e in W and any open interval I of V not containing c, there are $\bar{a} \in I$ and $\bar{b} \in J$ such that $(\bar{a}, \bar{b}) \models p$. Since any open interval contains a subinterval not containing c, we can remove the requirement that I does not contain c. By (A_{n-1}), the locus of e is dense in $\overline{W}$: any product of open intervals in W contains a conjugate of e. This shows that for any open $I \subseteq V$ and $J_i \subseteq V_i$, we can find $(\bar{b}_i)_{i<n} \models p$, with $\bar{b}_0 \in I$ and $\bar{b}_i \in J_i$, $0 < i$, as required. □

Lemma 4.10. *Working over some A, let $V_0, \ldots, V_{n-1}$ be pairwise independent, minimal definable circular orders. Let $V_n, \ldots V_{m-1}$ be pairwise independent minimal definable linear orders. Let $p(\bar{x}_i : i < m)$ be a type over A in some product $V_0^{l_0} \times \cdots \times V_{m-1}^{l_{m-1}}$, then given any open intervals $I_i \subseteq V_i$, $i < n$ and initial segments $I_i \subseteq V_i$, $n \leq i < m$, we can find $(\bar{a}_i : i < m) \models p$ with $\bar{a}_i \in I_i^{l_i}$ for each $i < m$.*

Proof. We can assume that $A = \emptyset$ and that if $\bar{a} \models p$, then no two coordinates of the tuple $\bar{a}$ are equal. Let us first assume that all the orders are circular, that is $n = m$. Let $X \subseteq V_0^{l_0} \times \cdots \times V_{n-1}^{l_{n-1}}$ be the closure of the set of realizations of the type p. By Proposition 4.9, $X = D_0 \times \cdots \times D_{n-1}$, where $D_k \subseteq V_k^{l_k}$ is a closed 0-definable subset. Fix some intervals $I_i \subseteq V_i$, $i < n$. By Proposition 4.5, for each $k < n$, $D_k \cap I_k^{l_k}$ has non-empty interior. It follows that p has a realization $(\bar{a}_i : i < n)$, where each $\bar{a}_k$ lies in I_k.

Now for the general case, let $p_c(\bar{x}_i : i < n)$ and $p_l(\bar{x}_i : n \leq i < m)$ denote the restrictions of p to the corresponding variables. Fix intervals $I_i \subseteq V_i$, $i < n$ in the circular orders. Then by the previous paragraph applied to the restriction of p to the first n tuples of variables, we can find $(\bar{a}_i : i < m) \models p$ with $\bar{a}_i \in I_i$ for each $i < n$ (find a realization of p_c, then extend it to a realization of p).

Let B be a finite set of parameters over which the intervals I_i are definable. For each $n \leq i < m$, let $J_i \subseteq V_i$ be the minimal B-definable initial segment of V_i. Hence J_i is a minimal linear order over B. Restricting the I_i's if necessary, we can assume that they are also minimal over B. If there is a realization $(\bar{a}_i : i < m)$ of p such that each $\bar{a}_i \in I_i$ for $i < n$ and $\bar{a}_i \in J_i$ for $n \leq i < m$, then the result follows at once from Corollary 3.25 applied to the linear orders I_i and J_i.

Assume now that there is no realization $(\bar{a}_i : i < m)$ of p such that $\bar{a}_i \in I_i$ for $i < n$ and $\bar{a}_i \in J_i$ for $n \leq i < m$. Let $\tilde{I} = \prod_{i<n} I_i^{l_i}$. By Propositions 4.9 and 4.5, there are finitely many automorphisms $\sigma_1, \ldots, \sigma_k$ such that $\bigcup_{t \leq k} \sigma_t(\tilde{I})$ covers $\prod_{i<n} V_i^{l_i}$. Let $J'_i = \bigcap_{t \leq k} \sigma_t(J_i)$, so that each J'_i is a non-empty initial segment of V_i. Then we see that there is no realization $(\bar{a}_i : n \leq i < m)$ of p_l with $\bar{a}_i \in J'_i$ for each i. This contradicts Corollary 3.25 applied to the independent linear orders V_i, $n \leq i < m$. □

Theorem 4.11. *Working over some A, let $V_0, \ldots, V_{n-1}$ be pairwise independent, minimal definable circular orders. Let $V_n, \ldots V_{m-1}$ be pairwise independent minimal definable linear orders. Then any A-definable closed subset $D \subseteq V_0^{k_0} \times \cdots \times V_{m-1}^{k_{m-1}}$ is a finite union of products of the form $D_0 \times \cdots \times D_{m-1}$, where each D_i is an A-definable closed subset of $V_i^{k_i}$.*

Proof. The proof is very similar to that of (B_n) in Proposition 4.9, using Lemma 4.10. Assume $A = \emptyset$.

Let D be given as in the statement and assume that it is the closure of a complete type. For $i < m$, let D_i be the projection of D to $V_i^{k_i}$. For each $i < m$, let $T_i \subseteq V_i$ be a bounded interval and set $T = T_0^{k_0} \times \cdots \times T_{m-1}^{k_{m-1}}$. It is enough to show the result for $D \cap T$ instead of D.

Let $\bar{e}$ be any tuple of parameters containing at least two points from each V_i, $i < m$. For each $i < n$, let $a_i, b_i \in \mathrm{dcl}(\bar{e}) \cap \overline{V_i}$ be such that the complement of the interval $a_i \leq x \leq b_i$ in V_i is infinite and weakly transitive over $\bar{e}$. For each $n \leq i < m$, let $d_i \in \mathrm{dcl}(\bar{e}) \cap \overline{V_i}$ such that the end-segment $x > d_i$ is weakly transitive over $\bar{e}$.

By Lemma 4.10, we may choose $\bar{e}$ so that each interval $a_i \leq x \leq b_i$ is disjoint from T_i for $i < n$, and for $n \leq i < m$, we have $d_i < T_i$. Then over $\bar{e}$, the T_i's are intervals in some weakly transitive $\bar{e}$-definable linear orders, which are pairwise independent. Therefore by Proposition 3.23, the restriction of X to T is the product of its projections to each factor, as required. □

Definition 4.12. Let $A \subseteq M^{eq}$ and let (V, C) be an A-definable circular order. For $n < \omega$, an *A-sector* of V^n is a subset of V^n defined by a formula $\phi(x_0, \ldots, x_{n-1})$ which is a finite boolean combination of relations of the form:

- $x_i = x_j$, for $i, j < n$;
- $C(x_i, x_j, x_k)$, for $i, j, k < n$;
- $C(a, x_i, x_j)$, for $i, j < n$ and $a \in \mathrm{acl}^{eq}(A) \cap \overline{V}$;
- $x_i \in W$, for $i < n$ and W an $\mathrm{acl}^{eq}(A)$-definable convex subset of V.

Definition 4.13. Let $A \subseteq M^{eq}$ and let $V_0, \ldots, V_{m-1}$ be pairwise independent A-definable linear or circular orders. For $n < \omega$, a *sector* (resp. *A-sector*) of $V_0^{k_0} \times \cdots \times V_{m-1}^{k_{m-1}}$ is a set of the form $D_0 \times \cdots \times D_{m-1}$, where each D_i is a sector (resp. A-sector) of $V_i^{k_i}$.

Corollary 4.14. *Let $V_0, \ldots, V_{n-1}$ be pairwise independent, minimal 0-definable circular orders. Let $V_n, \ldots V_{m-1}$ be pairwise independent minimal 0-definable linear orders. Let $X \subseteq V_0^{k_0} \times \cdots \times V_{m-1}^{k_{m-1}}$ be definable over some parameters A. Then the topological closure of X is an A-sector of $V_0^{k_0} \times \cdots \times V_{m-1}^{k_{m-1}}$.*

Proof. Each V_i breaks over A into finitely many A-definable points and A-definable convex subsets, each weakly transitive over A. Any two such convex subsets are independent by Lemmas 3.18, 4.6 and 4.8. Applying Theorem 4.11 to the closure of X, it is enough to prove the statement for one linear or circular order. The case of a linear order is Corollary 3.14 and the circular case follows similarly from Proposition 4.5. $\square$

Corollary 4.15. *Let $V_0, \dots, V_{m-1}$ be pairwise independent, minimal 0-definable linear or circular orders. Let $\bar{a}$ be a finite tuple of d pairwise distinct elements of $\bigcup_{i<m} \overline{V_i}$. Then* $\mathrm{opD}(\bar{a}) \geq d$.

Proof. Let $p(\bar{x}) = \mathrm{tp}(\bar{a}/\emptyset)$. Then by the previous corollary, $p(\bar{x})$ is dense in a sector S of $\prod_{i<m} V_i$. Since $p(\bar{x})$ implies that all coordinates of $\bar{x}$ are distinct, the sector S must have non-empty interior: it contains a product $\prod_{i<m} I_i$, where $I_i \subseteq V_i$ is a bounded open interval of V_i. Those intervals are definable over some set A.

We can now construct an ird-pattern of length d as follows: write $\bar{a} = (a_\alpha : \alpha < d)$. For each $\alpha < d$ let $i(\alpha) < m$ be such that $a_\alpha \in V_{i(\alpha)}$. Pick a sequence of points $(b_{\alpha,k} : k < \omega)$ of elements of $I_{i(\alpha)}$, increasing along the order on $I_{i(\alpha)}$. Let $\phi_\alpha(\bar{x}; b_{\alpha,k})$ be a formula wih extra parameters from A saying that the α-th element of $\bar{x}$ lies in the interval I_i above the point $b_{\alpha,k}$. Then by density, for any $\eta : d \to \omega$, we can find a tuple $\bar{a}_\eta \models p$ such that for any $\alpha < d$ and $k < \omega$, we have

$$\phi_\alpha(a_\eta; b_{\alpha,k}) \iff \eta(\alpha) < k.$$

This shows that $\mathrm{opD}(p) \geq d$. $\square$

With the same argument as for Proposition 3.27, we can show the following classification result.

Corollary 4.16. *Let $(M; C_1, \dots, C_m, \leq_1, \dots, \leq_n)$ be countable, ω-categorical, transitive, equipped with m circular orders and n linear orders, each minimal. Write $M_i = (M, \leq_i)$. Then the isomorphism type of M is completely determined up to automorphism by the following information:*

- *For any $i, j \leq m$, whether C_i and C_j are equal, equal up to reversal, intertwined, intertwined up to reversal, or independent.*

- *For any $i, j \leq n$, whether $\leq_i$ and $\leq_j$ are equal, equal up to reversal, intertwined, intertwined up to reversal, or independent.*

- *For any $i < j \leq n$ such that $\leq_i$ and $\leq_j$ are intertwined (possibly up to reversal) but not equal, if $f_{ij} \colon M_i \to \overline{M_j}$ is the intertwining map, whether we have $f_{ij}(x) <_j x$ or $x <_j f_{ij}(x)$ for some/any $x \in M$.*

5 Local equivalence relations and local formulas

We now aim at describing a certain kind of definable sets on products of minimal orders, which we call *local*. This will only be used at the end of the analysis to show the finiteness result of Theorem 1.2. We advise the reader to skip this section at first and come back to it when it is called for.

We start by giving examples of local definable sets.

EXAMPLE 5.1. *All structures are assumed to be countable.*

1. *Let $(V, \leq)$ be a dense linear order without endpoints and let E be an equivalence relation on V with finitely many classes, each of which is dense co-dense. In the structure $(V; \leq, E)$, the order $(V, \leq)$ is weakly transitive and rank 1. The isomorphism type of this structure is determined by the number of classes. One could further expand this structure by adding any structure on the finite quotient V/E. We will see that those are the only weakly transitive, rank 1 and op-dimension 1 expansions of a linear order.*

2. *Let (V, C) be a dense circular order. We may similarly expand it by adding an equivalence relation E with finitely many classes, each of which is dense co-dense. Again, the isomorphism type of the expansion is determined by the number of classes and one can expand the resulting structure by putting any structure on the quotient V/E.*

3. *Take (V, C) a dense (countable) circular order. Let $\pi\colon W \to V$ be a connected k-fold cover of V: that is W is itself a circular order, the map π is locally an isomorphism and is k-to-one. Up to isomorphism, there is a unique such structure. Now let $s\colon V \to W$ be a section of π which is generic in the sense that on any small interval of V, s takes values in the k sheets of the cover above that interval. Again, those conditions determine the isomorphism type of $(W, V; \pi, s)$.*

   The induced structure on V can be described in various ways. If $k > 1$, let $R(x, y)$ be the binary relation which holds for two points a, b if π is injective on the interval $s(a) < x < s(b)$. Note that the circular order on V is definable from R and in fact the whole structure is bi-interpretable with $(V; R)$. Those structures $(V; R)$ are sometimes named $S(k)$ in the literature. We will call them finite covers *of V (in general a finite cover of a structure M is a structure N equipped with a finite-to-one projection map onto M).*

   Another way to encode the structure on V which will be more natural to us is as a local equivalence relation. *Define a 4-ary predicate*

$$E(s, t; x, y) \equiv (s < x = y < t) \vee (s < x < y < t \wedge R(x, y)) \vee \\ \vee (s < y < x < t \wedge R(y, x)).$$

Then for any $a \neq b$, the relation $E(a,b;x,y)$ is an equivalence relation on the interval $a < x < b$. It is in this form that those structures will appear in our analysis.

4. *We can combine examples (2) and (3). Fix some integers $(k_1, \dots, k_m)$. Let (V,C) be a dense circular order, equipped with an equivalence relation E with m dense co-dense classes. On the i-th class, we have a k_i-fold cover coded by a local equivalence relation E_i as in (3). The isomorphism type of the structure $(V; C, E_1, \dots, E_m)$ is determined by the tuple $(k_1, \dots, k_m)$.*

   As we will see eventually, those are, up to bi-definability, the only minimal, rank 1, op-dimension 1, expansions of circular orders.

5. *Let (V,C) be a dense circular order equipped with two equivalence relations E and F such that F has two dense classes, each E-class consists of exactly one element from each, and the structure is generic such. Let M be the quotient of V by E. Then M satisfies $(\star)$ and is a proper expansion of the last structure in Example 1.1 (obtained from M by forgetting about F). We then have an equivalence relation with two classes on the set W_* of pairs $(a,b) \in V^2$, with $a \; E \; b$ (given by the F-class of the first coordinate for instance). This is another example of a local equivalence relation. In this case it is a* bona fide *equivalence relation, although not on the structure M itself, but on the finite cover W_*.*

Let $(V_k^* : k < m_*)$ be a finite family of 0-definable minimal linear and circular orders so that any two are independent. Let $\bar{c} = (c_i)_{i<n_*}$ enumerate a relatively algebraically closed subset of $\bigcup V_k^*$ such that $\bar{c} \in \operatorname{acl}(c_i)$ for each i. For $i < n_*$, let $k(i) < m_*$ be such that $c_i \in V_{k(i)}^*$ and set $V_i = V_{k(i)}^*$. Reordering $\bar{c}$ if necessary, assume that for some $n_c \leq n_*$, V_i is circular for $i < n_c$ and linear otherwise. Let $p_0 = \operatorname{tp}(\bar{c})$ and let $W_* \subseteq \prod_{i<n_*} V_i$ be the set of realizations of p_0.

Note that as p_0 is a complete type, if $S \subseteq \prod_{i<n_*} V_i$ is a sector, then either $W_* \subseteq S$ or $W_* \cap S = \emptyset$.

For each $i < n_*$, let $W_i \subseteq V_i$ be the projection of W_* on V_i: it is a dense subset of V_i by minimality and is transitive since p_0 is a complete type. If $i \neq j$ and $V_i = V_j$ are linear, then $W_i \neq W_j$ since algebraic closure must be trivial on W_i by Lemma 3.1. However, if $V_i = V_j$ is circular, then we could have either $W_i \neq W_j$ or $W_i = W_j$.

By the L_0-structure, we mean the structure having one sort for each V_k^* equipped with its linear or circular order and a unary predicate for W_* as a subset of $\prod_{i<n_*} V_{k(i)}^*$.

5.1 Small cells, paths and simple connectedness

A bounded interval of a linear or circular order is an interval of the form $a < x < b$, with $a < b$.

A *small cell* of W_* is a non-empty set of the form $W_* \cap \prod_{i<n_*} I_i$ such that each $I_i \subseteq V_i$ is a bounded interval, and such that for any $i \neq j$ such that $V_i = V_j$,

I_i and I_j are disjoint. Note that any sector of $\prod_{i<n_*} V_i$ intersecting $\prod_{i<n_*} I_i$ actually contains $\prod_{i<n_*} I_i$ since the disjointness condition ensures that order relations between the variables are the same for any two points in that product. It follows from Corollary 4.14 applied to $A = \emptyset$ that the intersection $W_* \cap \prod_{i<n_*} I_i$ is dense in $\prod_{i<n_*} I_i$ (as it is assumed to be non-empty).

A *minimal small cell* of W_* over $\bar{a}$ is a small cell $W_* \cap \prod_{i<n_*} I_i$ such that each I_i is definable and minimal over $\bar{a}$.

In what follows, we equip W_* with the induced topology coming from the product of the order topologies on each V_i.

Lemma 5.2. *Let $X \subseteq W_*$ be a non-empty definable open set and let $C_{\bar{a}} \subseteq W_*$ be a small cell defined by a formula $\phi(\bar{x};\bar{a})$. Then there is $\bar{a}' \equiv \bar{a}$, such that the small cell $C_{\bar{a}'}$ defined by $\phi(\bar{x};\bar{a}')$ is included in X.*

Proof. Let $d \in X$. Then as all elements of W_* have the same type, there is $\bar{b} \equiv \bar{a}$ such that $d \in C_{\bar{b}}$. By Corollary 4.14, the set of realizations of $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$ is dense in a sector, hence we can move the endpoints of the intervals defining $C_{\bar{b}}$ freely, as long as the order type of those endpoints is preserved. Therefore we can find $\bar{b}' \equiv \bar{b}$ such that $C_{\bar{b}'}$ contains d and is small enough to be included in X. □

Lemma 5.3. *Let $C_{\bar{a}}$ and $D_{\bar{b}}$ be two small cells, defined by possibly different formulas. Then there is $\bar{a}' \equiv \bar{a}$ such that $C_{\bar{a}'} \supseteq D_{\bar{b}}$.*

Proof. By the previous lemma, there is $\bar{b}' \equiv \bar{b}$ such that $C_{\bar{a}} \supseteq D_{\bar{b}'}$. Now take $\bar{a}'$ such that $\mathrm{tp}(\bar{a}'\bar{b}) = \mathrm{tp}(\bar{a}\bar{b}')$. □

In the following definition, we slightly abuse terminology: when we write $E_{\bar{a}}$, we are assuming given not just the definable set $E_{\bar{a}}$, but also the formula $\phi(x, y; \bar{z})$ so that $\phi(x, y; \bar{a})$ defines $E_{\bar{a}}$. (In fact, the definition does not actually depend on the formula chosen, but we will not need this.)

Definition 5.4. Let $C_{\bar{a}}$ be a minimal small cell defined over $\bar{a}$ and let $E_{\bar{a}}$ be an $\bar{a}$-definable equivalence relation on $C_{\bar{a}}$. We say that $E_{\bar{a}}$ is a *local equivalence relation* if for any $\bar{a}' \equiv \bar{a}$ such that $C_{\bar{a}'} \subseteq C_{\bar{a}}$, $E_{\bar{a}'}$ and $E_{\bar{a}}$ coincide on $C_{\bar{a}'}$.

Lemma 5.5. *Let $E_{\bar{a}}$ be a local equivalence relation defined on the minimal small cell $C_{\bar{a}}$ and take $\bar{a}' \equiv \bar{a}$. Let $D_{\bar{c}} \subseteq C_{\bar{a}} \cap C_{\bar{a}'}$ be a small cell defined possibly by a different formula. Then $E_{\bar{a}}$ and $E_{\bar{a}'}$ coincide on $D_{\bar{c}}$.*

Proof. Let $\bar{d}$ be a finite tuple of points from $D_{\bar{c}}$. By Lemma 5.2, there is $\bar{a}'' \equiv \bar{a}$ such that $C_{\bar{a}''} \subseteq D_{\bar{c}}$. Note that a minimal small cell is by definition a product of pairwise independent minimal linear orders. It then follows from Corollary 3.25 that we can find a realization of $\mathrm{tp}(\bar{d}/\bar{c})$ inside any open subset of $D_{\bar{c}}$ and in particular we can find such a realization $\bar{d}'$ inside $C_{\bar{a}''}$. Let $\bar{a}_*$ be such that $\bar{d}'\bar{a}'' \equiv_{\bar{c}} \bar{d}\bar{a}_*$. Then $C_{\bar{a}_*}$ is included in $D_{\bar{c}}$ and contains $\bar{d}$. Hence for any finite subset of $D_{\bar{c}}$ we have found a small cell included in $D_{\bar{c}}$ and containing that finite set. The result therefore follows by the definition of a local equivalence relation. □

Observe that a non-empty intersection of two small cells need not be a small cell: the intersection of two intervals in a circular order may be two disjoint intervals. This is where the topological complexity comes in and is the reason why a local equivalence relation does not always give a *bona fide* equivalence relation on the whole of W_*.

Fix a local equivalence relation $E_{\bar{a}}$ and let $\mathcal{E}$ be the family $\{E_{\bar{a}'} : \bar{a}' \equiv \bar{a}\}$. We will also refer to $\mathcal{E}$ as a local equivalence relation. For any small cell C, we can find by Lemma 5.3 $E \in \mathcal{E}$ whose domain contains C. Then by the previous lemma, $E|_C$ does not depend on the choice of $E \in \mathcal{E}$. We will denote that equivalence relation by $\mathcal{E}(C)$ and its set of classes by $C/\mathcal{E}$.

We say that the local equivalence relation $E_{\bar{a}}$ (or $\mathcal{E}$) is *finite* if $E_{\bar{a}}$ has finitely many classes on its domain $C_{\bar{a}}$.

Lemma 5.6. *Let $\mathcal{E}$ be a finite local equivalence relation. For any small cell C, any $\mathcal{E}(C)$-class is dense in C. Furthermore, $\mathcal{E}(C)$ has finitely many classes and that number does not depend on C.*

Proof. It follows at once from Corollary 3.25 that any $E_{\bar{a}}$-class is dense in the minimal small cell $C_{\bar{a}}$. Since any small cell C is included in a conjugate of $C_{\bar{a}}$, the result follows. □

In what follows, $\mathcal{E}$ is a finite local equivalence relation.

If C_0, C_1 are small cells such that $C_0 \cap C_1$ is also a small cell, then we have a natural bijection $f : C_0/\mathcal{E} \to C_1/\mathcal{E}$ given by identifying both $C_0/\mathcal{E}$ and $C_1/\mathcal{E}$ with $C_0 \cap C_1/\mathcal{E}$.

Definition 5.7. A *path* is a family $\mathfrak{p} = (C_i)_{i<n}$ such that each C_i is a small cell and each $C_i \cap C_{i+1}$ is a small cell.

Given a path $\mathfrak{p} = (C_i)_{i<n}$, we can define a map $f_{\mathfrak{p}} : C_0/\mathcal{E} \to C_{n-1}/\mathcal{E}$ given by composing the natural bijections $f_i : C_i/\mathcal{E} \to C_{i+1}/\mathcal{E}$ defined above.

Definition 5.8. Say that a path $\mathfrak{p}' = (C'_i)_{i<n'}$ refines a path $\mathfrak{p} = (C_i)_{i<n}$ if there exist indices

$$0 = i_0 < \cdots < i_{n-1} < i_n = n'$$

such that $i_k \le i < i_{k+1}$ implies $C'_i \subseteq C_k$.

Proposition 5.9.
1. *If a path $\mathfrak{p} = (C_i)_{i<n}$ satisfies that all the C_i's lie in some given small cell C, then $f_{\mathfrak{p}} : C_0/\mathcal{E} \to C_{n-1}/\mathcal{E}$ is given by the identification of $C_0/\mathcal{E}$ and $C_{n-1}/\mathcal{E}$ to $C/\mathcal{E}$.*
2. *If a path $\mathfrak{p}'$ refines $\mathfrak{p}$, then $f_{\mathfrak{p}'}$ is equal to $f_{\mathfrak{p}}$, modulo the canonical identifications of the domain and range given by inclusion maps.*

Proof. The proof of (1) is immediate by induction on n.

To prove (2), let $0 = i_0 < \cdots < i_{n-1} < i_n = n'$ be as in Definition 5.8. The map from $C'_0/\mathcal{E}$ to $C'_{i_1-1}/\mathcal{E}$ obtained following $\mathfrak{p}'$ is given by the identification of both to $C_0/\mathcal{E}$. Then since $C'_{i_1-1} \cap C'_{i_1} \subseteq C_0 \cap C_1$, the map $C'_{i_1-1}/\mathcal{E} \to C'_{i_1}/\mathcal{E}$ is the same—up to canonical identification of domain and range—as the one $C_0/\mathcal{E} \to C_1/\mathcal{E}$. Going on in this way proves the result. □

If $X \subseteq W_*$ then a *path in* X is a path $\mathfrak{p} = (C_i : i < n)$, where each C_i is included in X.

Definition 5.10. 1. An open definable set $X \subseteq W_*$ is *path-connected* if for any two points $a, b \in X$, there is a path $\mathfrak{p} = (C_i : i < n)$ in X with $a \in C_0$ and $b \in C_{n-1}$.

2. An open set $X \subseteq W_*$ is *simply connected* if it is path-connected and for any two paths $\mathfrak{p} = (C_i : i < n)$ and $\mathfrak{p}' = (C'_i : i < n')$ in X with $C_0 = C'_0$, $C_{n-1} = C'_{n'-1}$, the maps $f_\mathfrak{p}$ and $f_{\mathfrak{p}'}$ are equal.

Let $X \subseteq W_*$ be a simply connected open set. Let $a, b \in X$ and take a path $\mathfrak{p}$ in X from some small cell C_a containing a to a small cell C_b containing b. This induces a bijection $f_\mathfrak{p} \colon C_a/\mathcal{E} \to C_b/\mathcal{E}$. Say that a and b are $\mathcal{E}(X)$-related if $f_\mathfrak{p}$ maps the $\mathcal{E}(C_a)$ class of a to the $\mathcal{E}(C_b)$-class of b. This notion does not depend on the choice of $\mathfrak{p}$ by definition. It also does not depend on the choice of C_a and C_b, since if we make a different choice, say C'_a and C'_b, related by a path $\mathfrak{p}'$, then we can find $C''_a \subseteq C_a \cap C'_a$ and $C''_b \subseteq C_b \cap C'_b$ and any map $f_{\mathfrak{p}''} \colon C''_a/\mathcal{E} \to C''_b/\mathcal{E}$ coming from a path must coincide (modulo canonical identifications) with $f_\mathfrak{p}$ and $f_{\mathfrak{p}'}$.

We therefore see that $\mathcal{E}(X)$ is an equivalence relation on X. Furthermore, it follows by construction that if $Y \subseteq X$ are both simply connected, then $\mathcal{E}(X)$ and $\mathcal{E}(Y)$ coincide on Y. Also if C is a small cell, then by Proposition 5.9 (1), this definition of $\mathcal{E}(C)$ coincides with the previous one. Finally, note that if X is definable, then so is $\mathcal{E}(X)$ since it is automorphism-invariant.

Lemma 5.11. *1. If X is simply connected, then any $\mathcal{E}(X)$-class is dense in X.*

2. If X and Y are simply connected, then the equivalence relations $\mathcal{E}(X)$ and $\mathcal{E}(Y)$ have the same number of classes.

Proof. 1. Let X be simply connected and let $C_0, C_1 \subseteq X$ be small cells. Then there is a path $\mathfrak{p}$ in X from some $C'_0 \subseteq C_0$ to some $C'_1 \subseteq C_1$. This path induces a bijection $f_\mathfrak{p} \colon C'_0/\mathcal{E} \to C'_1/\mathcal{E}$ which in turns induces a bijection $C_0/\mathcal{E} \to C_1/\mathcal{E}$ via the canonical identifications induced by the inclusion maps. Therefore C_0 and C_1 intersect the same $\mathcal{E}(X)$-classes, hence every class is dense in X.

2. By Lemma 5.6, any two cells have the same number of $\mathcal{E}$-classes. Furthermore, each of $\mathcal{E}(X)$ and $\mathcal{E}(Y)$ has the same number of classes as $\mathcal{E}(C)$ for some/any small cell C contained in them, since every class is dense. ☐

Lemma 5.12. *Let X be an open subset of W_*. Assume that we have a family $\mathcal{F}$ of definable (over parameters) open subsets of X such that:*

1. for any finite collection $\{C_1, \dots, C_k\}$ of small cells included in X, there is a finite set $F \subseteq \mathcal{F}$ whose union contains all the C_i's;

2. for any non-empty finite set $F \subseteq \mathcal{F}$ the intersection of all the sets in F is non-empty and simply connected.

Then X is simply connected.

Proof. To see that X is connected, let $a, b \in X$. We can find two sets $X_a, X_b \in \mathcal{F}$ that contain a and b respectively. By assumption $X_a \cap X_b$ is non-empty. Pick a point c in it. Then since both X_a and X_b are connected, there are paths from a to c and from c to b, which we can compose to obtain a path from a to b.

Let $\mathfrak{p} = (C_i : i < n)$ and $\mathfrak{p}' = (C'_i : i < n')$ be two paths with $C_0 = C'_0$, $C_{n-1} = C'_{n'-1}$. Let F be the finite set promised by condition 1 for the family $\{C_0, \ldots, C_{n-1}, C'_0, \ldots, C'_{n'-1}\}$. Refining the two paths, we may assume that each C_i and C'_i lies in a unique member of the family. Let F_∞ be the intersection of all the sets in F. By hypothesis F_∞ is simply connected, so $\mathcal{E}(F_\infty)$ is well defined. Then we see that the transition maps from $C_i/\mathcal{E} \to C_{i+1}/\mathcal{E}$ coincide with the identification of both domain and range with $F_\infty/\mathcal{E}$, and same for the primed family. Hence the two maps $f_{\mathfrak{p}}$ and $f_{\mathfrak{p}'}$ are also defined in this way and therefore coincide. □

Lemma 5.13. *Assume that all the orders V_i are linear and that the map k is injective: no two coordinates of a $\bar{c} \in W_*$ lie in the same order. Then W_* is simply connected.*

Proof. Any finite union of small cells of W_* is included in one small cell (any finite union of bounded intervals of a linear order is included in one bounded interval and the same holds for products). Hence Proposition 5.9 (1) directly implies that W_* is simply connected. □

Lemma 5.14. *Assume that all the orders V_i are linear. Then W_* is simply connected.*

Proof. We prove the result by induction on the number of pairs (i, j) for which $V_i = V_j$. If there is no such pair, then the previous lemma applies.

Assume now that say $V_0 = V_1 = \ldots = V_{k-1}$ and $V_i \neq V_0$ for $i \geq k$. Without loss of generality, assume that $p_0(\bar{x}) \vdash x_0 < \cdots < x_{k-1}$. Consider the family $\mathcal{F}$ of non-empty sets of the form

$$W_* \cap J_0 \times \prod_{0<i<k} J_1 \times \prod_{k \leq i} V_i,$$

where J_0 is an initial segment of V_0 and J_1 the complementary end segment. Any finite intersection of those sets is a non-empty set of the form

$$W_* \cap K_0 \times \prod_{0<i<k} K_1 \times \prod_{k \leq i} V_i,$$

where K_0 is an initial segment and K_1 some end segment of V_0. In such a set, the first coordinate lives in the linear order K_0, and all the others are in orders independent from it. By induction, that set is simply connected and we conclude by Lemma 5.12. □

Proposition 5.15. *For each $i < n_*$, let $I_i \subseteq V_i$ be an open bounded interval of V_i, if V_i is circular, or either an interval or the whole of V_i if V_i is linear. Then $X := W_* \cap \prod_{i<n_*} I_i$ is empty or simply connected.*

Proof. This follows at once from Lemma 5.14 applied to $W_* \cap \prod_{i<n_*} I_i$ instead of W_*. □

5.2 Classification of finite local equivalence relations

Let $\mathcal{E}$ be a finite local equivalence relation. Fix an L_0-formula $\psi(x;\bar{y})$ and an L_0-type $q(\bar{t})$ such that for any $\bar{a} \models q$, $C_{\bar{a}} := \psi(M;\bar{a})$ is a small cell. Define the relation $E(\bar{t};\bar{x},\bar{y})$ which holds for $\bar{x},\bar{y} \in W_*$ and $\bar{t} \models q$ if $\bar{x},\bar{y}$ are in $C_{\bar{t}}$ and are $\mathcal{E}(C_{\bar{t}})$-equivalent. Let $L_{\mathcal{E}}$ be the language $L_0 \cup \{E\}$ and our goal now is to describe the possibilities for the isomorphism type of the expansion of the L_0 structure to $L_{\mathcal{E}}$.

Let $\mathcal{C}$ be the set of indices $k < m_*$ for which V_k^* is circular.

For each $k \in \mathcal{C}$, let three distinct points $\alpha_k < \beta_k < \gamma_k \in V_k^*$ be given. Define three intervals $C_{k,0} := \alpha_k < x < \gamma_k$, $C_{k,1} := \beta_k < x < \alpha_k$ and $C_{k,2} := \gamma_k < x < \beta_k$ of V_k^*. The indices $0, 1, 2$ in $C_{k,0}, \ldots$ are considered as elements of the cyclic group $\mathbb{Z}_3$. Let also $A = \{\alpha_k, \beta_k, \gamma_k : k \in \mathcal{C}\}$.

Note that any two of $C_{k,0}, C_{k,1}, C_{k,2}$ intersect in a non-empty bounded interval of V_k^*.

Recall that $n_c \leq n_*$ was defined so that for $i < n_*$, V_i is circular if and only if $i < n_c$. Given a tuple $\bar{t} = (t_k : k < n_c)$ of elements of $\mathbb{Z}_3$, let

$$C_{\bar{t}} = W_* \cap \prod_{i<n_c} C_{k(i),t_k} \times \prod_{n_c \leq i < n_*} V_i.$$

A *big cell* of W_* is a set of the form $C_{\bar{t}}$, with $\bar{t} \in \mathbb{Z}_3^{n_c}$ as above. By Lemma 5.14, each big cell is simply connected. Furthermore, the intersection

$$C(\bar{t},\bar{s}) := C_{\bar{t}} \cap C_{\bar{s}}$$

of two big cells is a non-empty product of intervals and linear orders and hence is also simply connected. It follows that $\mathcal{E}(C_{\bar{t}})$ is a well defined equivalence relation on each big cell $C_{\bar{t}}$ and $\mathcal{E}(C(\bar{t},\bar{s}))$ is a well defined equivalence relation on each $C(\bar{t},\bar{s})$. The latter induces a bijection between $C_{\bar{t}}/\mathcal{E}$ and $C_{\bar{s}}/\mathcal{E}$, which we will denote by $f_{\bar{t},\bar{s}}$.

Let M and M' be two $L_{\mathcal{E}}$ structures with isomorphic L_0-reducts. Fix an isomorphism $\sigma : M \to M'$ between the L_0-reducts. Let $\alpha_k, \beta_k, \gamma_k$ in M be points defining big cells and let $\alpha'_k, \beta'_k, \gamma'_k$ be their images under σ. Assume that the number of $\mathcal{E}$-classes are the same in M and M' and that for each $\bar{t} \in \mathbb{Z}_3^{n_c}$, we have an identification of the classes in $C_{\bar{t}}$ and $C'_{\bar{t}}$ so that the maps $f_{\bar{t},\bar{s}}$, for $\bar{t},\bar{s} \in \mathbb{Z}_3^{n_c}$ are the same in M and M' (modulo this identification). Then M and M' are isomorphic as $L_{\mathcal{E}}$-structures. Indeed, we can construct an isomorphism by a straightforward back-and-forth: $\mathcal{E}$-classes on each $C_{\bar{t}}$ are dense subsets and the identification ensures that the local equivalence relations coincide.

5.3 Local formulas

Say that two small cells C_0, C_1 of W_* are *strongly disjoint* if for any $i, j < n_*$ so that $V_i = V_j$, the projections $\pi_i(C_0)$ and $\pi_j(C_1)$ to V_i and V_j are disjoint.

Definition 5.16. A (parameter-)definable set $R(x_1, \dots, x_k) \subseteq W_*^k$ is *local* if there is a finite local equivalence relation $\mathcal{E}_R$ on W_* such that given strongly disjoint small cells $C_1, \dots, C_k$ and two tuples $(a_1, \dots, a_k), (a'_1, \dots, a'_k) \in C_1 \times \cdots \times C_k$,

$$\bigwedge (a_i, a'_i) \in \mathcal{E}_R(C_i) \Longrightarrow (R(a_1, \dots, a_k) \leftrightarrow R(a'_1, \dots, a'_k)).$$

We say that a formula is *local* if it defines a local definable set.

Remark 5.17. There is a slight clash of terminology with *local equivalence relation*. A local equivalence relation is not the same thing as a local formula defining an equivalence relation, but we will never consider such objects so hopefully this should not lead to confusion.

Proposition 5.18. *Let $R(x_1, \dots, x_k)$ be a local definable set. Let $\bar{a} = (a_1, \dots, a_k), \bar{b} = (b_1, \dots, b_k) \in W_*^k$ be two tuples of pairwise distinct elements. Assume that $\bar{a}$ and $\bar{b}$ have the same L_0-type and that for each $i \leq k$, there is a big cell C of W_* containing both a_i and b_i with $(a_i, b_i) \in \mathcal{E}_R(C)$. Then we have*

$$R(a_1, \dots, a_k) \leftrightarrow R(b_1, \dots, b_k).$$

Proof. (Sketch) For any two k-tuples $\bar{c}$ and $\bar{d}$ of elements of W_*, write $\bar{c} \to \bar{d}$ if for each $i \leq k$, there is a big cell C_i of W_* and a small cell $C'_i \subseteq C_i$ that contains c_i and d_i and such that $(c_i, d_i) \in \mathcal{E}_R(C'_i)$ and the C'_i's are strongly disjoint. To prove the proposition, it is sufficient to find a sequence $\bar{a} = \bar{a}^0 \to \bar{a}^1 \to \cdots \to \bar{a}^m = \bar{b}$. The fact that the L_0-types of $\bar{a}$ and $\bar{b}$ are the same implies that the relative order of the elements in the tuple are the same. Thus we can always find such a path from $\bar{a}$ to $\bar{b}$ by moving the points one by one. □

It follows that a local definable set R is definable over the parameters A used to define the big cells along with parameters defining the equivalence relations $\mathcal{E}_R$ on each big cell and a name for each $\mathcal{E}_R$-equivalence class inside each big cell. Also, for a fixed $L_{\mathcal{E}}$-structure, there are only finitely many local definable sets of each arity.

Proposition 5.19. *For a given L_0-structures and some $n < \omega$ there are finitely many possibilities for the $L_{\mathcal{E}}$-structure, where $\mathcal{E}$ has n classes (in some/any small cell). Furthermore, for a given $L_{\mathcal{E}}$-structure, there are finitely many local definable sets of a given arity.*

5.4 Monodromy

The previous discussion gives us all we need to prove the main theorem of this paper. However, it is natural to push the analysis a little bit further and show that the data contained in the set of maps $f_{\bar{t},\bar{s}}$ can be encoded by an action of the fundamental group of the space on a finite set. We explain this here. This subsection will not be used in the rest of the paper.

Lemma 5.20. *For each $i < n_*$, let $I_i \subseteq V_i$ be either an open bounded interval of V_i or the whole of V_i. Assume that for each $k < m_*$ such that V_k^* is circular, there is exactly one value of i for which $V_i = V_k^*$ and $I_i \neq V_i$. Then $X := W_* \cap \prod_{i<n_*} I_i$ is empty or simply connected.*

Proof. We first explain what this corresponds to in a standard topological framework. Let $\tilde{V}_k^*$, $k < m_*$, be 1-dimensional manifolds, which are thus homeomorphic to either $\mathbb{R}$ or the circle $\mathbb{S}_1$. Let $\tilde{V}_i$, $i < n_*$ be each equal to one of the $\tilde{V}_k^*$ and let $\tilde{U} \subseteq \prod_{i<n_*} \tilde{V}_i$ be the set of tuples with distinct coordinates. Let $\tilde{W}_*$ be a connected component of $\tilde{U}$. Choose open intervals $\tilde{I}_i \subseteq \tilde{V}_i$ satisfying the same condition as in the statement of the lemma. Then the set $\tilde{X} = \tilde{W}_* \cap \prod_{i<n_*} \tilde{I}_i$ is simply connected. In fact this space is contractible. This is not hard to see: First, we can assume that $m_* = 1$, since the space decomposes as a product of spaces each involving one $\tilde{V}_k^*$ and a product of contractible spaces is contractible. Let us assume for example that $\tilde{V}_0^*$ is circular. At least one coordinate, say $i = 0$ is constrained inside a proper interval $\tilde{I}_0$. Fix any element $\bar{a} \in \tilde{X}$. Then we can send any other element $\bar{a}'$ to $\bar{a}$, by sending a_0' to a_0 via a shortest path (and moving the other coordinates with it so that no two cross). We then move only the other coordinates in the circle minus $\{a_0\}$, and this reduces to the linear case which is clear.

Now, we just have to translate this topological intuition into an argument in our context. The reader who is already convinced will not lose anything by skipping the rest of this proof. Assume that X is not empty. As above, we can assume that $m_* = 1$: all points live in the same order V_0^*, since coordinates in different V_k^* are completely independent of each other. If $n_* = 1$, then this follows from Proposition 5.9 (1): any finite set of bounded intervals is included in one bounded interval, so any two paths are included in one common bounded interval and thus define the same functions $f_{\mathfrak{p}}$.

Assume that V_0^* is linear, and we prove the result by induction on n_*. Without loss $p_0(\bar{x}) \vdash x_0 < \cdots < x_{n_*-1}$. Consider the family $\mathcal{F}$ of non-empty sets of the form $X \cap J_0 \times \prod_{i<n_*} J_1$, where J_0 is an initial segment of V_0^* and J_1 the complementary end segment. Any finite intersection of those sets is a non-empty set of the form $X \cap L_0 \times \prod_{i<n_*} L_1$, where L_0 is an initial segment and L_1 some end segment of V_0^*. In such a set, the first coordinate lives in the linear order L_0 and the others in L_1 which is independent from it. By induction, that set is simply connected and we conclude by Lemma 5.12.

Assume next that V_0^* is circular. Without loss, I_0 is a proper interval and $I_i = V_i$ for $i > 0$. We may also assume that $p_0(\bar{x}) \vdash x_0 < x_1 < \cdots < x_{n_*-1}$. Fix some $I_* \subset I_0$ a proper subinterval that has no endpoint in common with I_0 and let J_* be the complement of I_*. Define F to be $W_* \cap I_* \times \prod_{0<i<n_*} J_* \subseteq X$. By the linear case, F is simply connected.

Identify $\{0, \ldots, n_* - 1\}$ with $\mathbb{Z}/n_*\mathbb{Z}$. Let $\mathcal{S}$ be the set of pairs $(t, k) \in \mathbb{Z}/n_*\mathbb{Z}^2$ such that the sequence $(t, t+1, \ldots, t+k)$ contains 0. For $(t, k) \in \mathcal{S}$, let $G_{t,k} \subseteq X$ be the set of tuples $\bar{a} \in X$ for which $a_t, \ldots, a_{t+k}$ lie in I_0 in that order and no other a_i is in I_0. Again using the linear case, any such set is simply connected. Note also that two distinct $G_{t,k}$ are disjoint. For $(t, k) \in \mathcal{S}$, $G_{t,k} \cap F$

has the form $\prod_{i<n_*} I_i$, where the I_i's are intervals, any two of which are either equal or disjoint. From the linear case, it follows that $G_{t,k} \cap F$ is simply connected. Enumerate the elements of $\mathcal{S}$ arbitrarily as $s_1, \ldots, s_v$. For $r \leq v$, let $F_r = F \cup \bigcup_{i<r} G_{s_i}$. By induction using the remarks above and Lemma 5.12 with the two element family $\{F_{r-1}, G_{s_r}\}$, we see that each F_r is simply connected. Since $F_v = X$, we are done. □

Let $\bar{t} \in \mathbb{Z}_3^{\mathcal{C}}$ and take $\bar{e}_0, \bar{e}_1 \in \mathbb{Z}_3^{\mathcal{C}}$ having each exactly one non-zero coordinate, with $\bar{e}_0 \neq \pm\bar{e}_1$. Then the big cells $C_{\bar{t}}$, $C_{\bar{t}+\bar{e}_0}$, $C_{\bar{t}+\bar{e}_1}$, $C_{\bar{t}+\bar{e}_0+\bar{e}_1}$ are included in a common simply connected set. It follows that we have the commutation relation:

$$(\square) \quad f_{\bar{t}+\bar{e}_0,\bar{t}+\bar{e}_0+\bar{e}_1} \circ f_{\bar{t},\bar{t}+\bar{e}_0} = f_{\bar{t}+\bar{e}_1,\bar{t}+\bar{e}_0+\bar{e}_1} \circ f_{\bar{t},\bar{t}+\bar{e}_1}.$$

Denote by $\bar{0} \in \mathbb{Z}_3^{\mathcal{C}}$ the tuple all of whose coordinates are 0 and let $X = C_{\bar{0}}/\mathcal{E}$. We may identify each $C_{\bar{t}}/\mathcal{E}$ with X by following a path of bijections between $C_{\bar{0}}$ and $C_{\bar{t}}$ that never *wraps around*. More formally, order $\mathbb{Z}_3$ by identifying it with $\{0, 1, 2\}$. If $C_{\bar{t}_0}, \ldots, C_{\bar{t}_n}$ and $C_{\bar{s}_0}, \ldots, C_{\bar{s}_n}$ are two sequences of cells with

$$\bar{t}_0 \leq \bar{t}_1 \leq \ldots \leq \bar{t}_n, \bar{s}_0 \leq \bar{s}_1 \leq \ldots \leq \bar{s}_n, \text{ and } \bar{t}_0 = \bar{s}_0, \bar{t}_n = \bar{s}_n$$

and both

$$f_{\bar{t}_{n-1},\bar{t}_n} \circ \cdots \circ f_{\bar{t}_0,\bar{t}_1} \text{ and } f_{\bar{s}_{n-1},\bar{s}_n} \circ \cdots \circ f_{\bar{s}_0,\bar{s}_1}$$

well defined, then those two compositions are equal by iterations of (□). We identify $C_{\bar{t}}/\mathcal{E}$ with $X = C_{\bar{0}}/\mathcal{E}$ by following any sequence of adjacent big cells from $C_{\bar{0}}$ to $C_{\bar{t}}$ as above.

For any $i \in \mathcal{C}$, let $\bar{e}_i \in \mathbb{Z}_3^{\mathcal{C}}$ be the element with coordinates 0 everywhere except for 2 at the i-th place. Now to describe $\mathcal{E}$, it is enough to describe the maps $f_{\bar{t},\bar{t}+\bar{e}_i}$ when the i-th coordinate of $\bar{t}$ is equal to 0. (All other maps $f_{\bar{t},\bar{s}}$ are the identity on X by our identification.) In fact, we can further simplify by noticing that such an $f_{\bar{t},\bar{t}+\bar{e}_i}$ is equal to $f_{\bar{0},\bar{e}_i}$: let g be a composition of maps $f_{\bar{t},\bar{s}}$, which do not wrap around (that is change a coordinate from 2 to 0 or vise-versa), such that $t_i = s_i = 2$ so that the i-th coordinate is not changed and $g \circ f_{\bar{t},\bar{t}+\bar{e}_i}$ maps $C_{\bar{t}}/\mathcal{E}$ to $C_{\bar{e}_i}/\mathcal{E}$. Let h be the same composition as g, but with all i-th coordinate being equal to 0 instead of 2. Then h sends $C_{\bar{t}}/\mathcal{E}$ to $C_{\bar{0}}/\mathcal{E}$ and $f_{\bar{0},\bar{e}_i} \circ h$ also sends $C_{\bar{t}}/\mathcal{E}$ to $C_{\bar{e}_i}/\mathcal{E}$. As neither g nor h wraps around, g and h induce the identity map on X. Furthermore, by successive applications of (□), one sees that

$$g \circ f_{\bar{t},\bar{t}+\bar{e}_i} = f_{\bar{0},\bar{e}_i} \circ h.$$

Hence, seen as maps from X to X, we have $f_{\bar{t},\bar{t}+\bar{e}_i} = f_{\bar{0},\bar{e}_i}$

For each index i, set $h_i = f_{\bar{0},\bar{e}_i}$, seen as a map from X to X. Using (□) and following the standard argument that the fundamental group of a torus is $\mathbb{Z}^2$, one obtains that h_i and h_j commute for all i, j. (Deform the path corresponding to $h_i \circ h_j$ to that corresponding to $h_j \circ h_i$ by successive applications of (□).)

We have thus associated to the local equivalence relation $\mathcal{E}$ a family of pairwise commuting maps $h_i \colon X \to X$, or equivalently, an action of $\mathbb{Z}^{\mathcal{C}}$ on X. We will call this the *monodromy action* of $\mathcal{E}$. Given a decomposition of W_* into big cells, this action is well defined only up to conjugation by a permutation of X. Furthermore, it follows from the analysis above that another choice of big cells would lead to the same family of maps up to conjugation. The monodromy action determines the $L_{\mathcal{E}}$-structure up to isomorphism (given the L_0-reduct) since we can from it reconstruct a set of maps $f_{\bar{t},\bar{s}}$.

6 Classification of rank 1 structures

In this section, we prove our main theorems. In Subsection 6.1 we prove some important technical statements that follow from having a ranked structure: first a kind of *geometric triviality* property saying that any binary function into a minimal order is essentially unary, and second we show that minimal linear orders can intersect (up to intertwining) only in very restricted ways. For instance it is impossible for two minimal orders to have proper initial segments that are intertwined, while the remaining final segments are independent. (Think of two branches in a tree for instance: they start out equal and then diverge. This situation will be ruled out by showing that if it happens then we can increase this pair of orders to a whole tree and trees cannot exist in ranked structures).

In Subsection 6.2 we show how starting with a definable family of minimal linear orders, we can glue them together to construct one (or in fact finitely many) linear or circular orders that are algebraic over $\emptyset$, hence canonical. Intuitively, we start with an order in the family and extend it as much as possible by adjoining other orders in the family that have a convex subset in common. This can never lead to orders *diverging* by the result mentioned above. Hence either we can keep going obtaining a longer and longer linear order, or the construction wraps up on itself, and we obtain a circle. This picture is complicated by the fact that the orders might not actually intersect, but be intertwined. We solve this by first *thickening* every order in our family by adding to it all (convex subsets of) orders that are intertwined with it. Having done that, we do not have to worry about intertwinings any more and the rest of the argument is completely elementary, although rather tedious.

Subsection 6.3 is in some sense the core of the paper. Here we start with an ω-categorical rank 1 primitive unstable NIP structure M and construct a canonical family W composed of finitely many linear and circular orders. Two orders in W are either in order-reversing bijection or independent. Those orders are obtained by applying the procedure described above to all definable families of minimal linear orders. We show that this does indeed lead to only finitely many orders by rank arguments. The structure W is definable in M^{eq} and admits a definable finite-to-one map onto M. From now on, we switch our focus from M to W and see M as a quotient of W. It remains to classify the possible structures on W. In Subsection 6.4 we introduce the basic structure on W that is given by its construction which we call the *skeletal structure*. In Subsection 6.5 we show that any additional structure on W must come from local definable sets. Finally, in Subsection 6.6 we use this analysis to prove our main theorems.

6.1 Preliminary statements

The following proposition shows that certain binary functions are essentially unary.

Proposition 6.1. *Assume that M has finite rank and is NIP. Let a, b be two finite tuples and set $p(x,y) = \mathrm{tp}(a,b)$. Assume that either $a \mathop{\smile\!\!\!\!\!\mid} b$ or $\mathrm{rk}(a) = 1$. Let also V be a 0-definable linear or circular order of topological rank 1 and let $f\colon p(M) \to \overline{V}$ be a 0-definable function. Then $f(a,b) \in \mathrm{acl}^{eq}(a) \cup \mathrm{acl}^{eq}(b)$.*

Proof. If $a \in \mathrm{acl}(b)$, there is nothing to show. If $\mathrm{rk}(a) = 1$ and $a \notin \mathrm{acl}(b)$, then $\mathrm{rk}(a/b) \geq 1 = \mathrm{rk}(a)$, so by Lemma 2.8 $a \mathop{\smile\!\!\!\!\!\mid} b$. Hence we can assume $a \mathop{\smile\!\!\!\!\!\mid} b$. Let $a_1, \ldots, a_n \in M$ be realizations of $p(x,b)$ so that for each k, $\mathrm{rk}(a_i/ba_1 \ldots a_{i-1}) = \mathrm{rk}(a)(= \mathrm{rk}(a_i))$ (this exists by Lemma 2.5). For $i \leq n$, set $c_i = f(a_i, b)$. If c_i is algebraic either over b or over a_i, then since $\mathrm{tp}(a_i b) = \mathrm{tp}(ab)$, it follows that $f(a,b)$ is algebraic either over b or over a and we are done. Assume that this is not the case.

Claim: $c_i \notin \mathrm{acl}^{eq}(ba_1 \ldots a_{i-1})$.

Proof: Assume that $c_i \in \mathrm{acl}^{eq}(ba_1 \ldots a_{i-1})$. Then $c_i \in \mathrm{acl}^{eq}(ba_i) \cap \mathrm{acl}^{eq}(ba_1 \ldots a_{i-1})$. Since $\mathrm{rk}(a_i/ba_1 \ldots a_{i-1}) = \mathrm{rk}(a_i) = \mathrm{rk}(a_i/b)$, by Lemma 2.8, we have

$$a_i \mathop{\smile\!\!\!\!\!\mid}_b a_1 \ldots a_{i-1},$$

and Lemma 2.8 implies $c_i \in \mathrm{acl}^{eq}(b)$. Contradiction.

As a consequence of the claim, the c_i's are pairwise distinct (since for $j < i$, $c_j \in \mathrm{acl}^{eq}(ba_1 \ldots a_{i-1})$).

Set $\bar{a} = (a_1, \ldots, a_n)$. By Proposition 2.4 (5),

$$\mathrm{rk}(\bar{a}b) = \mathrm{rk}(b) + \mathrm{rk}(a_1/b) + \cdots + \mathrm{rk}(a_n/ba_1 \ldots a_{n-1}) = \mathrm{rk}(b) + n\,\mathrm{rk}(a).$$

Hence for any i,

$$\begin{aligned}\mathrm{rk}(\bar{a}/a_i b) = \mathrm{rk}(\bar{a}b) - \mathrm{rk}(a_i b) &= \mathrm{rk}(b) + n\,\mathrm{rk}(a) - (\mathrm{rk}(a_i/b) + \mathrm{rk}(b)) \\ &= (n-1)\,\mathrm{rk}(a).\end{aligned}$$

We also have $\mathrm{rk}(\bar{a}/a_i) = \mathrm{rk}(\bar{a}) - \mathrm{rk}(a_i) = (n-1)\,\mathrm{rk}(a)$ so that we have $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_{a_i} b$. As $c_i \in \mathrm{acl}^{eq}(a_i b) \setminus \mathrm{acl}^{eq}(a_i)$ we deduce from Lemma 2.8 again that $c_i \notin \mathrm{acl}^{eq}(\bar{a})$.

Let Z_i^0 be the subset of V defined by $\mathrm{tp}(c_i/\mathrm{acl}^{eq}(\bar{a}))$ and let Z_i be the topological closure of Z_i^0 in V. Note that Z_i^0, and hence Z_i, is infinite as c_i is not algebraic over $\bar{a}$. Then by Lemma 3.4 and the remark after it, Z_i is a convex subset of V. The set Z_i with the induced order is minimal over $\bar{a}$ as it has topological rank 1 and is the closure of a transitive set. For $i, j \leq n$, the subsets Z_i

and Z_j are either equal or disjoint (since each is the closure of a complete type over $\bar{a}$). If they are disjoint, then they are independent by Lemma 3.18. Taking a larger value of n and restricting to a subtuple of $(c_1, \ldots, c_n)$, we may assume that they are either all equal or pairwise disjoint.

Assume that the Z_i's are pairwise disjoint and that V is linear. Let X be the topological closure of the set of realizations of $q := \mathrm{tp}(c_1, \ldots, c_n/\bar{a})$ in $Z_1 \times \cdots \times Z_n$. By Proposition 3.23, X is a finite union of products of closed subsets of each V_i. Since X is $\bar{a}$-definable and each Z_i is minimal over $\bar{a}$, the only non-empty $\bar{a}$-definable closed subset of Z_i is Z_i itself, hence it must be that $X = \prod_{i \leq n} Z_i$. It follows that for any subset $I \subseteq \omega$, we can find $(c'_1, \ldots, c'_n) \models q$ with

$$c'_i < c_i \iff i \in I.$$

Take b' so that

$$\mathrm{tp}(c'_1, \ldots, c'_n, b'/\bar{a}) = \mathrm{tp}(c_1, \ldots, c_n, b/\bar{a}).$$

We then have

$$f(a_i, b') < c_i \iff i \in I.$$

As n was arbitrary, the formula

$$\phi(xx'; y) \equiv f(x, y) < x'$$

has the independence property. The argument in the circular case is similar using Proposition 4.9 instead of Proposition 3.23.

Assume finally that the Z_i's are all equal to some Z. In the linear case, the argument is exactly the same as in the previous paragraph, using Proposition 3.12 to find the c'_i's. In the circular case we use Proposition 4.5 instead. □

Recall from the beginning of Section 3.1 that we say that $\overline{V}$ is definable if there is a finite set of formulas Φ such that every cut of V can be defined by an instance of some $\phi(x; y) \in \Phi$.

Corollary 6.2. *Assume that M has rank 1 and is NIP. Let $(V, \leq)$ be a minimal 0-definable linear order. Let $\overline{V}(a)$ denote $\mathrm{acl}^{eq}(a) \cap \overline{V} = \mathrm{dcl}^{eq}(a) \cap \overline{V}$[2]. Then:*

1. *for any $a_0, \ldots, a_{n-1} \in M$, we have $\overline{V}(a_0, \ldots, a_{n-1}) = \bigcup_{i<n} \overline{V}(a_i)$;*
2. *$\overline{V}$ is definable, minimal and has rank 1.*

Proof. (1) Let $c \in \overline{V}(a_0, \ldots, a_{n-1})$ and set $p = \mathrm{tp}(a_0, a_1 \ldots a_{n-1})$. Then by definition of $\overline{V}(a_0, \ldots, a_{n-1})$, there is some 0-definable function f defined on realizations of p, such that $f(a_0, a_1 \ldots a_{n-1}) = c$. By Proposition 6.1, $c \in \mathrm{dcl}^{eq}(a_0) \cup \mathrm{dcl}^{eq}(a_1 \ldots a_{n-1})$. We conclude by induction on n.

(2) Let a be a singleton, then $\overline{V}(a)$ is finite, by Lemma 3.7. Hence there is a finite set of formulas Φ such that every element c of $\overline{V}(a)$ is definable by $\phi(x; a)$ for some $\phi(x; y) \in \Phi$. Since M has rank 1 and c is definable over a, $\mathrm{rk}(c) \leq \mathrm{rk}(a) \leq 1$. By (1), every element of $\overline{V}$ is of this form, hence $\overline{V}$ is definable and has rank 1. Finally, $\overline{V}$ is minimal since $V \subseteq \overline{V}$ is minimal and dense inside it (Lemma 3.3 (3)). □

[2] Algebraic closure and definable closure are equal on a linear order.

In what follows, we will consider a definable family $(V_a, \leq_a)$, $a \in D$ of linear orders, by which we mean that D is a definable set and there are formulas $\phi(x;t)$ and $\psi(x,y;t)$ such that for any $a \in D$, the formula $\psi(x,y;a)$ defines a linear order denoted $\leq_a$ on $V_a := \phi(M;a)$.

Recall from Section 3.3 that $I \subseteq_{\mathrm{wmc}} V$ means that I is a weakly minimal definable convex subset of V.

Proposition 6.3. *Let D be a 0-definable set and let $(V_u, \leq_u)$, $u \in D$, be a definable family of linearly ordered sets, with V_u minimal over u. Assume that D is ranked and let $a, b \in D$. Let $I \subseteq_{wmc} V_a$ be intertwined with some $J \subseteq_{wmc} V_b$. Let $h : I \to \overline{J}$ be the intertwining map and take $t \in I$. Then the following two statements hold:*

- *Either $\{x \in V_a : x <_a t\}$ is intertwined with a convex subset of V_b or $\{x \in V_b : x <_b h(t)\}$ is intertwined with a convex subset of V_a.*
- *Either $\{x \in V_a : x >_a t\}$ is intertwined with a convex subset of V_b or $\{x \in V_b : x >_b h(t)\}$ is intertwined with a convex subset of V_a.*

Proof. By reversing the orders, it is enough to prove the second statement. We will drop the indices in the linear orders $\leq_a, \leq_b, \ldots$ when they are implied by the context.

For any $c, d \in D$ and $u \in \overline{V_c}$, consider the definable set $C[c,d,u] \subseteq V_c$ defined as:

$$v \in C[c,d,u] \text{ if } \{x \in V_c : u < x < v\} \text{ is intertwined with some } W \subseteq_{\mathrm{wmc}} V_d.$$

If $C[c,d,u]$ is non-empty then it is an initial segment of $\{x \in V_c : u < x\}$. In that case define

$$f_{c,u}(d) = \sup C[c,d,u] \in \overline{V_c} \cup \{+\infty\}.$$

If $C[c,d,u]$ is empty, then $f_{c,u}(d)$ is undefined.

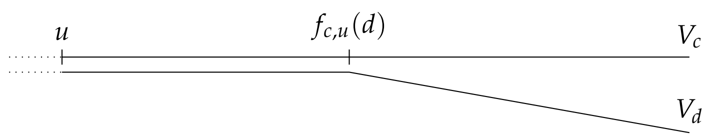

Figure 1: Definition of $f_{c,u}(d)$. Intertwined intervals are represented as parallel lines.

Claim A: If $f_{c,u}(d)$ is defined and different from $+\infty$, then $f_{c,u}(d) \in \mathrm{dcl}^{eq}(cd)$.

Proof: Working over cd, the two linear orders V_c and V_d are weakly minimal and can be each written as a disjoint union of cd-definable points and cd-definable minimal convex subsets (as explained at the beginning of Section 3.3). Any two of those cd-definable minimal orders are either intertwined or independent. Hence $f_{c,u}(d)$, if defined and different from $+\infty$, is the supremum of one of those minimal orders. Therefore $f_{c,u}(d) \in \mathrm{dcl}^{eq}(cd)$.

Note also:

$$\odot \qquad \text{If } u < u' < f_{c,u}(d), \text{ then } f_{c,u}(d) = f_{c,u'}(d).$$

Claim B: Let $c, d, e \in D$ and $u \in V_c$ such that $C[c, d, u] \subseteq V_c$ is non-empty and bounded above. Let $I \subseteq_{\text{wmc}} V_d$ be intertwined with $C[c, d, u]$. Take $v, w \in V_d$ so that

$$v < \inf I < \sup I < w.$$

Assume that $\{x \in V_d : v < x < w\}$ is intertwined with a convex subset of V_e. Then $C[c, e, u] = C[c, d, u]$.

Proof: Write $C = C[c, d, u]$, hence C is intertwined with $I \subseteq_{\text{wmc}} V_d$. By assumption $C[d, e, v] \supset I$. Let $J \subseteq_{\text{wmc}} V_e$ be intertwined with $C[d, e, v]$. By transitivity of intertwining, C is also intertwined with a convex subset J_0 of J. Hence $C \subseteq C[c, e, u]$ by definition of $C[c, e, u]$. Assume that equality does not hold. Then for some $u' > f_{c,u}(d)$ in V_c the subset $C' := \{x \in V_c : u < x < u'\}$ is intertwined with a convex subset J' of V_e. We have $J_0 \subseteq J'$ and J' extends J_0 to the right. Taking a smaller u' if necessary, we may assume that $J' \subseteq J$. But then by transitivity of intertwining, C' is intertwined with a convex subset of V_d. Therefore $f_{c,u}(d) \geq u'$; contradiction.

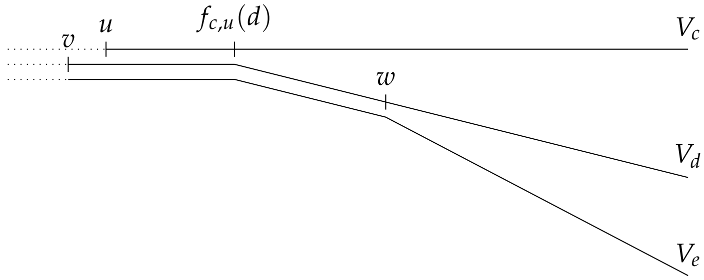

Figure 2: Claim B.

Coming back to the original a, b and t given by the statement of the proposition, the set $C[a, b, t]$ contains all elements of I that are greater than t, hence it is non-empty. If $f_{a,t}(b) = +\infty$, then $\{x \in V_a : x > t\}$ is intertwined with a convex subset of V_b and we are done. Assume this is not the case, so $f_{a,t}(b) \in \overline{V_a}$. Similarly, if $f_{b,h(t)}(a) = +\infty$, then we are done, so we may assume that $f_{b,h(t)}(a) \in \overline{V_b}$.

By Claim A, $f_{a,t}(b)$ is equal to one of the finitely many ab-definable elements of $\overline{V_a}$. List those elements as $s_1 < \ldots < s_n$ and say that $f_{a,t}(b) = s_k$. It follows that the convex subset $s_{k-1} < x < s_k$ of V_a is intertwined with a convex subset of V_b (if $k = 1$, set $s_{k-1} = -\infty$). By minimality of V_a over a and Proposition 3.12 (along with Remark 3.13), for any $u \in V_a$, $u > t$, we can find $b' \equiv_a b$ such that the k-th ab'-definable element of $\overline{V_a}$ is greater than u and, if $k > 1$, the $k-1$-th is smaller than t. Since $\text{tp}(b'/a) = \text{tp}(b/a)$, the convex subset of V_a between t and u is intertwined with a convex subset of $V_{b'}$. Hence $f_{a,t}(b') > u$. This shows that the image of $f_{a,t}$ is dense in the final segment $x > t$ of V_a.

We now construct inductively $(b_i : i < \omega)$ in D and points $(u_i : i < \omega)$, $u_i \in V_{b_i}$. We start by setting $(b_{-1}, u_{-1}) = (a, t)$. Set also $b_0 = b$ and take $u_0 \in J$ to be smaller than $h(t)$. Note that $f_{b_0,u_0}(b_{-1}) > b_0$ since b_0 lies in J which is intertwined with an interval of $V_{b_{-1}}$. We will use the notation $V_k := V_{b_k}$. Having constructed (b_k, u_k), let $(u'_k, w_k) \in V_k \times \overline{V_k}$ be such that

$$(1) \qquad u'_k < u_k < f_{b_k,u_k}(b_{k-1}) < w_k$$

and

$$\mathrm{tp}(u'_k, w_k, b_k) = \mathrm{tp}(u_k, f_{b_k,u_k}(b_{k-1}), b_k).$$

This is possible by Lemma 3.12. Next, let b_{k+1} be such that

$$\mathrm{tp}(u'_k, w_k, b_k, b_{k+1}) = \mathrm{tp}(u_k, f_{b_k,u_k}(b_{k-1}), b_k, b_{k-1}).$$

We then have

$$f_{b_k,u'_k}(b_{k+1}) = w_k,$$

and then by (1) and $\odot$,

$$f_{b_k,u_k}(b_{k+1}) = w_k > f_{b_k,u_k}(b_{k-1}).$$

The interval $\{x \in V_k : u'_k < x < w_k\}$ is intertwined with a convex subset J_{k+1} of V_{k+1} via a function h_k. Pick $u_{k+1} \in J_{k+1}$ smaller than $h(u_k)$.

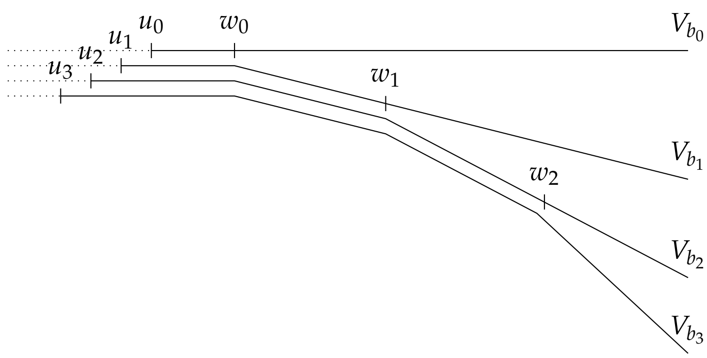

Figure 3: Construction of (b_k, u_k).

Claim C: For $l < k+1$, we have $f_{b_l,u_l}(b_{k+1}) = w_l$.

Proof: We show this by decreasing induction on l. We already know this for $l = k$. Assume we know it for $l+1$. By construction, $C[b_l, b_{l+1}, u_l]$ has supremum w_l and is intertwined with some J in V_{l+1} such that $u_{l+1} < \inf J < \sup J < w_{l+1}$. Furthermore, by induction hypothesis the interval $\{x \in V_{l+1} : u_{l+1} < x < w_{l+1}\}$ is intertwined with a convex subset of V_{k+1}. Hence by Claim B, $C[b_l, b_{k+1}, u_l] = C[b_l, b_{l+1}, u_l]$ hence $f_{b_l,u_l}(b_{k+1}) = f_{b_l,u_l}(b_{l+1}) = w_l$.

For $c \in D$ and $u \in V_c$, define the equivalence relation $E_{c,u}(x,y)$ on D by $f_{c,u}(x) = f_{c,u}(y)$. From Claim C, we deduce that for $k, k' > l$, b_k and $b_{k'}$ are E_{b_l,u_l}-equivalent. In particular the E_{b_l,u_l}-class of b_{l+1} is infinite. At each stage l of the construction, we have infinitely many choices for w_l (since the only condition on it is that it is larger than $f_{b_k,u_k}(b_{k-1})$), hence for the E_{b_l,u_l}-class of b_{l+1}. We can make a choice that is not algebraic over all the elements considered so far. Let D_1 be the E_{b_0,u_0}-class of b_1. As that class is not algebraic over b_0u_0, we have $\mathrm{rk}(D_1) < \mathrm{rk}(D)$ by Lemma 2.6. Next, D_1 is split into infinitely many E_{b_1,u_1}-classes. The E_{b_1,u_1}-class of b_2, say D_2, is not algebraic over $b_0u_0b_1u_1$, hence $\mathrm{rk}(D_2) < \mathrm{rk}(D_1)$. Continuing in this way, we obtain an infinite sequence of definable sets of decreasing ranks, which is absurd. □

6.2 Gluing definable orders

Let $(V_a, \leq_a)$, $a \in D$, be a 0-definable family of linearly ordered sets, with V_a minimal over a. Assume that D is ranked. Our goal in this section is to glue the orders V_a together as much as possible along definable intertwinings between subintervals so as to construct a 0-definable family of pairwise independent orders.

Here is a useful example the reader might want to keep in mind while reading this section.

EXAMPLE 6.4. *Let the structure M be equipped with a linear order $\leq$, an equivalence relation E_0 and no additional structure (a similar construction would work for a circular order). There are infinitely many E_0-classes and each one is dense. Let D be the set of pairs $(a_0, a_1) \in M^2$ such that $a_0 < a_1$. For $\bar{a} = (a_0, a_1) \in D$, define*

$$V_{\bar{a}} = \{x \in M \colon a_0 < x < a_1 \land x \; E_0 \; a_0\}.$$

The set $V_{\bar{a}}$ equipped with the induced order is minimal over $\bar{a}$. It is a dense subset of a convex subset of the original order $(M, \leq)$. In this situation, the construction presented in this section will essentially reconstruct the order $(M, \leq)$ from its pieces $(V_{\bar{a}})_{\bar{a} \in D}$.

For a slightly more complicated example which shows the need for the equivalence relation E in the theorem below, let N be a structure equipped with an equivalence relation E and such that each E-class is a copy of the structure M above and there are no extra relations between the classes. Here the family $(V_a)_{a \in D}$ given by the theorem is the union of the families constructed as above in each class. The theorem will recover E and the linear order on each E-class.

Theorem 6.5. *Let D be a 0-definable subset of M^{eq} which is ranked and let $(V_a, \leq_a)_{a \in D}$ be a definable family of linearly ordered sets, with V_a minimal over a. Then there is a 0-definable set W in M^{eq} and a 0-definable equivalence relation E on W such that:*

- *for $e \in W/E$, let $W[e] \subseteq W$ be the E-class corresponding to e, then $W[e]$ admits either a linear or circular e-definable order;*
- *for $e \in W/E$, $W[e]$ equipped with that order is minimal over e;*
- *for any $e \neq e' \in W/E$, the orders $W[e]$ and $W[e']$ are either independent or in definable order-reversing bijection;*

- *for any $a \in D$, there is (a necessarily unique) $e \in W/E$ such that V_a admits an order-preserving injection into $W[e]$.*

Note that the conclusion becomes stronger if we replace the given family $(V_a, \leq_a)_{a \in D}$ by a larger family $(V'_a, \leq_a)_{a \in D'}$ for some $D' \supseteq D$, with $V'_a = V_a$ for $a \in D$. Having noticed this, we start by increasing the family so that the following property holds:

($\triangle$) For any $a \in D$, there is $\tilde{a} \in D$ such that V_a is in definable order-reversing bijection with $V_{\tilde{a}}$.

To achieve this, we replace D by $D' = D \times \{*_1, *_2\}$, where $*_1, *_2$ are two elements of $\mathrm{dcl}^{eq}(\emptyset)$[3] and let $V_{(a,*_1)}$ be V_a and $V_{(a,*_2)}$ be the reverse of V_a (which is also minimal over a, hence over $(a, *_2)$).

The next stage of the construction involves *thickening* the V_a's by replacing each one by a large enough definable subset of $\overline{V_a}$, which will be called W_a. For instance in Example 6.4, $W_{\bar{a}}$ would be the convex hull of $V_{\bar{a}}$ inside M.

To this end, we first define an equivalence relation $\sim$ on pairs (a, t), $a \in D, t \in V_a$. The intuition is that if $a, b \in D$, $t \in V_a$ and $u \in V_b$, then (a, t) and (b, u) are equivalent if they should be glued together. In Example 6.4, we will have $(\bar{a}, t) \sim (\bar{b}, u)$ if and only if $t = u$.

Given $a, b \in D$, $t \in V_a$ and $u \in V_b$, we say that $(a, t) \sim (b, u)$ if there exist parameter-definable subsets $t \in U_a \subseteq_{\mathrm{wmc}} V_a$ and $u \in U_b \subseteq_{\mathrm{wmc}} V_b$ which are intertwined and such that the unique intertwining $f : \overline{U_a} \to \overline{U_b}$ sends t to u.

Lemma 6.6. *The relation $\sim$ is an equivalence relation on the set of pairs (a, t), $a \in D, t \in V_a$.*

Proof. Reflexivity and symmetry are clear. Let us show transitivity. Let $a, b, c \in D$, $t \in V_a$, $u \in V_b$, $w \in V_c$ and assume $(a, t) \sim (b, u)$ and $(b, u) \sim (c, w)$. Let A be a set of parameters large enough to define all the relevant convex subsets of V_a, V_b and V_c. Working over A, there are $U_a \subseteq_{\mathrm{wmc}} V_a$, $U_b \subseteq_{\mathrm{wmc}} V_b$ containing t and u respectively which are intertwined by $f : \overline{U_a} \to \overline{U_b}$, with $f(t) = u$. Similarly, there are $U'_b \subseteq_{\mathrm{wmc}} V_b$ and $U_c \subseteq_{\mathrm{wmc}} V_c$ intertwined by $g : \overline{U'_b} \to \overline{U_c}$ with $g(u) = w$. Let $U''_b = U_b \cap U'_b$. Then U''_b is an A-definable convex subset of V_b which is weakly minimal and contains u. Let U''_c be the convex subset of V_c intertwined with U''_b via g, that is:

$$U''_c = g(\overline{U''_b}) \cap V_c.$$

And define in the same way:

$$U''_a = f^{-1}(\overline{U''_b}) \cap V_a.$$

[3] Whereas $\mathrm{dcl}(\emptyset)$ can very well be empty, $\mathrm{dcl}^{eq}(\emptyset)$ is always infinite as it contains the quotient of each M^k by the equivalence relation with a unique class.

Then U''_a and U''_c are weakly minimal and both are intertwined with U''_b. By transitivity of intertwining, U''_a and U''_c are intertwined: in fact the intertwining map is $(g|_{U''_b}) \circ (f|_{U''_a})$. That maps sends t to w, hence we have $(a,t) \sim (c,w)$. $\square$

Note that for $t, u \in V_a$ distinct, we have $(a,t) \nsim (a,u)$ since by Lemma 3.18 there can be no intertwining between two disjoint convex subsets of V_a.

Let $[a,t]_\sim$ denote the $\sim$-class of (a,t) and let W be the set of $\sim$-classes. For $a \in D$, define W_a as:

$$W_a = \{[b,t]_\sim \colon \text{some weakly minimal convex subset of } V_b \text{ containing } t \\ \text{is intertwined with a weakly minimal convex subset of } V_a\}.$$

In this definition, the weakly minimal convex subsets of V_a and V_b are allowed to be defined over any set of parameters. In fact, if they exist, they can always be taken to be defined over ab, but we will not use that fact.

Some observations:

$\boxplus_0$ W_a is well defined: the definition does not depend on the representative of $[b,t]_\sim$.

To see this, assume that $(b,t) \sim (c,u)$ and that some weakly minimal convex subset $t \in U_b \subseteq V_b$ is intertwined with some $U_a \subseteq_{\text{wmc}} V_a$. By definition of $\sim$, there are weakly minimal subsets $t \in U'_b \subseteq_{\text{wmc}} V_b$, $u \in U'_c \subseteq_{\text{wmc}} V_c$ which are intertwined via $f : \overline{U'_b} \to \overline{U'_c}$ sending t to u. Define $U''_b = U_b \cap U'_b$ and let U''_c be the convex subset of U'_c corresponding to U''_b, that is $U''_c = f(\overline{U''_b}) \cap V_c$. Then U''_c is weakly minimal, contains u and is intertwined with a convex subset of V_a.

Note that W_a is invariant under automorphisms fixing a and hence since M is ω-categorical, it is definable over a.

$\boxplus_1$ For $a \in D$ and $t \in V_a$, $[a,t]_\sim \in W_a$.

Indeed, in the definition of W_a we can take V_a itself as weakly minimal convex subset of V_a. Note that W_a is in general larger than $\{[a,t]_\sim : t \in V_a\}$ since we are not assuming in the definition that the intertwining sends t to a point in V_a (it could be sent to a point in $\overline{V_a} \setminus V_a$).

$\boxplus_2$ There is an a-definable injective map $\iota_a \colon W_a \to \overline{V_a}$.

To construct this map, let $[b,t]_\sim = [c,u]_\sim \in W_a$. By definition of $\sim$, there are $t \in U_b \subseteq_{\text{wmc}} V_b$ and $u \in U_c \subseteq_{\text{wmc}} V_c$ which are intertwined by some $f : \overline{U_b} \to \overline{U_c}$ sending t to u. By definition of W_a and $\boxplus_0$, up to replacing U_b and U_c by smaller neighborhoods if necessary, U_b and U_c are each intertwined with convex subsets of V_a, via maps $g : \overline{U_b} \to \overline{T_b}$ and $h : \overline{U_c} \to \overline{T_c}$, where T_b and T_c are weakly minimal convex subsets of V_a. Then T_b and T_c are intertwined by $h \circ f \circ g^{-1}$. By Corollary 3.19, we must have $T_b = T_c$ and

$h \circ f \circ g^{-1}$ is the identity map. Since f maps t to u, $g(t) = h(u)$. We then define $\iota_a([b,t]_\sim) = g(t) \in \overline{V_a}$. We have proved that ι_a is well defined. It is a-definable since it is invariant under automorphisms fixing a and we are working in an ω-categorical structure.

To see that ι_a is injective, assume that $\iota_a([b,t]_\sim) = \iota_a([d,v]_\sim)$. Then there are $t \in U_b \subseteq_{\mathrm{wmc}} V_b$ and $v \in U_d \subseteq_{\mathrm{wmc}} V_d$ intertwined with convex subsets of V_a via maps $f : \overline{U_b} \to \overline{V_a}$ and $g : \overline{U_d} \to \overline{V_a}$ with $f(t) = g(v)$. Restricting U_b and U_d, we may assume that f and g have the same image $\overline{U_a}$, where $U_a \subseteq_{\mathrm{wmc}} V_a$. But then $g^{-1} \circ f$ intertwines U_b and U_d and sends t to v. Hence $[b,t]_\sim = [d,v]_\sim$.

It follows from $\boxplus_2$ and $\boxplus_1$ that W_a is in definable bijection with an a-definable dense subset of $\overline{V_a}$. We equip W_a with the order $\leq_a$ inherited from this bijection. The linear order $(W_a, \leq_a)$ is then minimal over a (by Lemma 3.3 applied to a suitable definable subset of $\overline{V_a}$).

Recall that W is the set of $\sim$ classes and that each W_a is a subset of W.

$\boxplus_3$ Let $a, b \in D$ and let $U_a \subseteq_{\mathrm{wmc}} W_a$ and $U_b \subseteq_{\mathrm{wmc}} W_b$. If U_a and U_b are intertwined, then $U_a = U_b$ and the orders induced on that set by W_a and W_b coincide.

Let $[c,t]_\sim \in U_a$. Then by definition of W_a, there is $t \in U_c \subseteq_{\mathrm{wmc}} V_c$ such that U_c is intertwined with a convex subset $U'_a \subseteq \overline{V_a}$. By construction, ι_a sends $[c,t]_\sim \in U_a$ to a point in U'_a. Hence U_c is intertwined with a convex subset of U_a. By transitivity of intertwining, U_c is intertwined with a convex subset of U_b (hence with a convex subset of V_b). So $[c,t]_\sim \in U_b$ by construction. Let h be the intertwining map from U_a to $\overline{U_b}$. There are convex neighborhoods of $[c,t]_\sim$ and $h([c,t]_\sim)$ in $\overline{U_b}$ that are intertwined with neighborhoods of t in U_c (by definition of $[c,t]_\sim$). Hence some convex neighborhoods of $[c,t]_\sim$ and $h([c,t]_\sim)$ in $\overline{U_b}$ are intertwined. By Corollary 3.19 applied to W_a, that intertwining must be the identity and thus h sends $[c,t]_\sim$ to itself.

By symmetry of the roles of a and b, this shows that $U_a = U_b$ and the intertwining map is the identity.

We know that for every $x \in W$, there is a (non-unique) subset $x \in W_a \subseteq W$ which is linearly ordered. We can think of W as a kind of "1-dimensional manifold" and W_a as a neighborhood of x "homeomorphic" to a line. We have to check now that the W_a's do indeed function like neighborhoods, in particular, that their intersections are well behaved and that the orders coincide on them. This will be achieved by Lemma 6.11. We will then be able to mimic the classical proof that a 1-dimension manifold is a disjoint union of lines and circles.

Definition 6.7. Let $V_0, V_1 \subseteq W$ be two parameter-definable subsets equipped with parameter-definable linear orders $\leq_0$ and $\leq_1$ respectively. We say that $(V_0, \leq_0)$ and $(V_1, \leq_1)$ are *in good position* (or that the pair (V_0, V_1) is in good position) if the following two conditions hold:

- for any $t \in V_0 \cap V_1$, there is $\eta \in \{0,1\}$ such that $\{x \in V_\eta : x \leq_\eta t\}$ is a convex subset of $V_{1-\eta}$ and the two orders $\leq_0$ and $\leq_1$ coincide on it.

- for any $t \in V_0 \cap V_1$, there is $\eta \in \{0,1\}$ such that $\{x \in V_\eta : x \geq_\eta t\}$ is a convex subset of $V_{1-\eta}$ and the two orders $\leq_0$ and $\leq_1$ coincide on it.

Lemma 6.8. *Two parameter-definable $V_0, V_1 \subseteq W$ equipped with linear orders as in the previous definition are in good position if and only if one of the following holds:*

1. *$V_0 \cap V_1 = \emptyset$;*

2. *for some $\eta \in \{0,1\}$, V_η is a convex subset of $V_{1-\eta}$ and the two orders coincide on V_η;*

3. *for some $\eta \in \{0,1\}$, $V_0 \cap V_1$ is an initial segment of V_η and a final segment of $V_{1-\eta}$ and the two orders coincide on it;*

4. *$V_0 \cap V_1$ can be written as a disjoint union $I \sqcup J$, where I is an initial segment of V_0 and a final segment of V_1, J is a final segment of V_0 and an initial segment of V_1 and the two orders coincide on each of I and J.*

Proof. It is clear that any one of those conditions imply that V_0 and V_1 are in good position.

Assume V_0 and V_1 are in good position. If $V_0 \cap V_1$ is empty, then item (1) holds. Assume it is not and let $t \in V_0 \cap V_1$. Without loss of generality, the initial segment $\{x \in V_0 : x \leq t\}$ is a convex subset of V_1. Let $I \subseteq V_0$ be the maximal initial segment of V_0 which is a convex subset of V_1 and on which the two orders $\leq_0$ and $\leq_1$ coincide. This is a definable subset of V_0 which contains t. If $I = V_0$, the item (2) holds and we are done. If $\{x \in V_0 : x >_0 t\}$ is a convex subset of V_1 with the same induced order, then the whole of V_0 is such and $I = V_0$. This has been ruled out so it must be that $\{x \in V_1 : x >_1 t\}$ is a convex subset of V_0 with the same induced order. This set is therefore equal to $\{x \in I : x >_0 t\}$ by definition of I, and we have that I is an initial segment of V_0 and a final segment of V_1. If $I = V_0 \cap V_1$, then item (3) holds and we are done. If this is not the case, we show that (4) holds. To this end, take $s \in V_0 \cap V_1 \setminus I$. Then $\{x \in V_1 : x \geq_1 s\}$ contains I as a proper subset, hence cannot be a convex subset of V_0 with the same induced order since I is an initial subset of V_0. Therefore it must be that $\{x \in V_0 : x \geq_0 s\}$ is a convex subset of V_1 with the same induced order. Define J as the maximal final segment of V_0 which is a convex subset of V_1 and on which the two orders coincide. By the same reasoning as above, J is an initial segment of V_1.

It remains to show that $V_0 \cap V_1 = I \cup J$. If not, let $u \in V_0 \cap V_1 \setminus (I \cup J)$. Then by the same reasoning as for s, we cannot have that $\{x \in V_1 : x \geq_1 u\}$ is a convex subset of V_0 (since I is an initial segment of V_0 which is properly contained in it), but by the same argument with J instead of I, $\{x \in V_0 : x \geq_0 u\}$ cannot be a convex subset of V_1. This contradicts the definition of good position. $\square$

Note that cases (2) and (3) are not mutually exclusive: if for instance V_0 is a proper initial segment of V_1, then they both hold.

If item (3) above holds, we will say that V_η is a *simple continuation* of $V_{1-\eta}$. In this case there is a natural definable linear order on $V_0 \cup V_1$ which coincides with $\leq_0$ and $\leq_1$ on V_0 and V_1 respectively and for which every point of $V_\eta \setminus V_{1-\eta}$ is above every point of $V_{1-\eta}$.

If item (4) holds, we will say that V_0 and V_1 are in *circular position*. Note that in this case, there is a natural definable circular order C on $V_0 \cup V_1$ for which V_0 and V_1 are convex subsets and $\leq_0$ and $\leq_1$ are the linear orders induced by C (this completely characterizes C).

At this point we strongly encourage readers to take a moment to convince themselves that from a family of linear orders pairwise in good position, one can glue those orders together and obtain a family of pairwise disjoint linear and circular orders. To achieve this, start say from a linear order V from the family. If some W in the family intersects V non-trivially, add it to it to obtain a longer linear order, or a circular order if case (4) holds. Keep going in this way adding all orders that have non-empty intersection with what you have so far. From the definition of good position, one can see that at any stage of this process, we have either a linear or a circular order (we prove this is details in what follows).

Lemma 6.9. *Let $V_0, V_1, V_2 \subseteq W$ be parameter-definable and linearly ordered. Assume that any two of them are in good position and that (V_0, V_1) is in circular position. Then either V_2 is disjoint from $V_0 \cup V_1$, or it is a convex subset of $V_0 \cup V_1$ when the latter is equipped with its canonical circular order.*

Proof. This is rather straightforward by going through all the cases. The reader is encouraged to make drawings to follow the arguments. Assume that V_2 is not disjoint from $V_0 \cup V_1$. Write $V_0 \cap V_1 = I \cup J$ as in the definition of circular position, so that I is an initial segment of V_0 and J a final segment of it.

Consider first the case where V_2 and V_0 are in circular position. Then we can write $V_0 \cup V_2 = I' \cup J'$ as in the definition of circular position, so that I' is an initial segment of V_0 and J' a final segment of it. As I and I' are two initial segments of V_0, one is included in the other. Assume that $I \subseteq I'$; the other case is similar. Write $I' = I \cup K$. It follows that $J = K \cup J'$. Then $V_0 = I \cup K \cup J'$, $V_1 = K \cup J' \cup I$ and $V_2 = J' \cup I \cup K$, where we are writing each union in the order in which the sets appear in the linear order in question. Then V_1, V_2 are in circular position as witnessed by K and $J' \cup I$ and V_2 is a convex subset of the circular order $V_0 \cup V_1$ (in fact it covers it entirely). Note that in this case the three pairs (V_0, V_1), (V_1, V_2) and (V_2, V_0) are in circular position and define the same circular order on the union.

The case where V_2 and V_1 are in circular position is similar, exchanging the roles of V_0 and V_1 in the arguments above.

Assume that V_0 is included in V_2 and is a convex subset of it. Then we have $I < J$ in V_2, since this holds in V_0. However, we know that $J < I$ holds in V_1. Given that V_1 and V_2 are in good position, this can only happen if V_1 and V_2 are in circular position, since in all other cases in the definition of good position, the intersection of the two orders is ordered in the same way in both orders. We have already covered this case.

Assume that V_2 is included in V_0 and is a convex subset of it. If V_2 is included in either I or J, then V_2 is a convex subset of the circular order $V_0 \cup V_1$ as required (since I and J are such). Otherwise, since V_2 is convex in V_0, it contains a final segment I' of I and an initial segment J' of J. We then have $I' < J'$ in V_2, but $J' < I'$ in V_1. We conclude as in the previous case.

The cases where one of V_1 or V_2 is included in the other as a convex subset are similar by exchanging the roles of V_0 and V_1 in the two paragraphs above.

Assume that V_2 is a simple continuation of V_0. Let $J' = V_0 \cap V_2$, then J' is a final segment of V_0 and an initial segment of V_2. As J and J' are final segments of V_0, one is included in the other. Assume that $J \subsetneq J'$. Write $J' = K \cup J$. Hence $K \subseteq I$. We then have $K < J$ in V_2, but $J < K$ in V_1 and we conclude as we have done twice before. If $J' = J$, then it is an initial segment of both V_2 and V_1. As V_1 and V_2 are in good position, this can only happen if one is included in the other and those cases have been covered. Assume next that $J' \subsetneq J$ and write $J = K \cup J'$. Then $K \cup J'$ is an initial segment of V_1 and J' is an initial segment of V_2. It thus cannot be the case that V_1 is a simple continuation of V_2. Given that we have already covered the cases where those two sets are in circular position, or one is included in the other, we can assume that V_2 is a simple continuation of V_1. So $V_2 \cap V_1$ is an end segment of V_1. As this intersection contains J' and $J' < I$ in V_1, it also contains I. But then $I \subseteq V_0 \cap V_2$ and we have $I < J'$ in V_0 and $J' < I$ in V_2, contradiction to V_2 being a simple continuation of V_0.

The case when V_0 is a simple continuation of V_2 is handled in the same way. With it, we have covered all cases. □

For further reference, we note the following which follows from the proof above.

Lemma 6.10. *Let $V_0, V_1, V_2 \subseteq W$ be parameter-definable and linearly ordered. Assume that any two of them are in good position, that (V_0, V_1) is in circular position and that $V_1 \subseteq V_2$, with the two orders coinciding on V_1. Then V_0 and V_2 are in circular position and $V_0 \cup V_1 = V_0 \cup V_2$.*

Lemma 6.11. *Let $a, b \in D$. Then W_a and W_b are in good position.*

Proof. This follows from Proposition 6.3 along with $\boxplus_3$: If $x \in W_a \cap W_b$, say $x = [c, t]_\sim$, then taking a small enough convex subset $t \in K \subseteq V_c$, there are two convex subsets $x \in I \subseteq W_a$ and $x \in J \subseteq W_b$ which are each intertwined with K. Hence I and J are intertwined by transitivity of intertwining. We now apply Proposition 6.3. By $\boxplus_3$, we can replace in the conclusion "intertwined" by "equal", which precisely gives us that W_a and W_b are in good position. □

We now move to the construction of the E-classes given by the theorem. We start by defining inductively families $(W^k_t)_{t \in D_k}$ of linear orders obtained by gluing finitely many of the W_a's together.

Start by letting $D_1 = D$ and $W^1_a = W_a$ for all $a \in D$.

Having defined $(W^k_a)_{a \in D_k}$, let $D_{k+1} = \{(a, b) : W^k_b$ is a simple continuation of $W^k_a\}$ and for $(a, b) \in D_{k+1}$, $W^{k+1}_{(a,b)} = W^k_a \cup W^k_b$. This set is equipped with its natural linear order extending those of W^k_a and W^k_b.

Lemma 6.12. *For any k and any $a \in D_k$, there is $\tilde{a} \in D_k$ such that $W^k_{\tilde{a}}$ is in order-reversing bijection with W^k_a.*

Proof. We prove the result by induction on k. For $k = 1$, take $a \in D$. Let $\tilde{a}$ be given by property ($\triangle$). Then $\boxplus_3$ implies that W_a and $W_{\tilde{a}}$ are in order-reversing bijection. Assume we know the result for k and let $d = (a, b) \in D_{k+1}$. Take $\tilde{a}, \tilde{b}$ given by the induction hypothesis. Since W^k_b is a simple continuation of W^k_a, $W^k_{\tilde{a}}$ is a simple continuation of $W^k_{\tilde{b}}$ and we can take $\tilde{d} = (\tilde{b}, \tilde{a})$. □

Lemma 6.13. *For any k, if $(a, b), (c, d) \in D_{k+1}$, then $W^k_{(a,b)}$ and $W^k_{(c,d)}$ are in good position.*

Proof. We prove the result by induction on k. For $k = 1$, it follows from Lemma 6.11.

Assume that $s \in W^{k+1}_{(a,b)} \cap W^{k+1}_{(c,d)}$. We will only prove the first half of the definition of good position since the second half is proved in exactly the same way reversing the orders. Up to exchanging the roles of (a, b) and (c, d), one of the three following cases applies:

(a) $s \in W^k_a \cap W^k_c$.

By induction, W^k_a and W^k_c are in good position. Without loss of generality, the initial segment I defined by $x \leq s$ in W^k_a is a convex subset of W^k_c. But I is also the initial segment defined by $x \leq s$ in $W^{k+1}_{(a,b)}$ and hence is a convex subset of $W^{k+1}_{(c,d)}$.

(b) $s \in W^k_a \cap W^k_d$.

If the initial segment defined by $x \leq s$ in W^k_a is a convex subset of W^k_d, we are done. Otherwise, as W^k_a and W^k_d are in good position, it must be that the initial segment $x \leq s$ in W^k_d is a convex subset of W^k_a. But this initial segment contains a point t in W^k_c. The interval $[t, s]$ is the same in W^k_a and in W^k_d, hence also in $W^k_{(a,b)}$ and in $W^k_{(c,d)}$ and we conclude by applying Case (a) to t instead of s.

(c) $s \in W^k_b \cap W^k_d$.

We do a similar reduction as in the previous case. Assume without loss of generality that the initial segment $x \leq s$ in W^k_b is a convex subset of W^k_d. That subset contains a point t in W^k_a and the interval $[t, s]$ is the same in $W^k_{(a,b)}$ and in $W^k_{(c,d)}$. We then conclude by applying case (b) to t instead of s.

□

Note that $W^k_a = W^{k+1}_{(a,a)}$ for any $a \in D_k$.

For $s, t \in W$, write $s \to_k t$ if there is $a \in D_k$ such that $s, t \in W_a^k$ and satisfy $s < t$ there. This relation is irreflexive. By the previous observation, if $s \to_k t$ holds, then so does $s \to_{k+1} t$. By ω-categoricity, there are only finitely many types of elements of W^2, hence for only finitely many k do there exist s and t such that $s \to_{k+1} t$ holds, but we do not have $s \to_k t$. Therefore there is $n_* < \omega$ such that for any $s, t \in W$, if $s \to_k t$ holds for some $k < \omega$, then it holds for $k = n_*$. We will write $\to$ for $\to_{n_*}$. Define also $D' = D_{n_*}$ and for $a \in D'$, $W'_a = W_a^{n_*}$.

Let

$$R_+(s) = \{t \in W : s \to t\},$$
$$R_-(s) = \{t \in W : t \to s\}$$

and

$$R(s) = R_+(s) \cup R_-(s) \cup \{s\}.$$

Note that $s \in R(t)$ if and only if $t \in R(s)$. Define also

$$R^2(s) = \{t \in W : (\exists u \in W) u \in R(s) \land t \in R(u)\}$$

and note that $R^2(s) \subseteq R(s)$ as witnessed by taking $u = s$.

Say that $s \in W$ is of *circular type* if there exists $a, b \in D'$ such that $s \in W'_a$ and such that W'_a and W'_b are in circular position. If $s \in W$ is not of circular type, say that it is of *linear type*.

Lemma 6.14. *If $s \in W$ is of circular type as witnessed by W'_a and W'_b, then $R^2(s) = W'_a \cup W'_b$. If $t \in R^2(s)$, then t is of circular type and $R^2(t) = R^2(s)$.*

Proof. We first prove that $R^2(s) = W'_a \cup W'_b$. If $t \in W'_a$, $t < s$ in W'_a, then $t \to s$ holds by definition, hence $t \in R(s)$, and similarly if $s < t$ or $t = s$. Hence $W'_a \subseteq R(s) \subseteq R^2(s)$. Now take $u \in W'_a \cap W'_b$, then $W'_b \subseteq R(u)$ and $u \in R(s)$, so $W'_b \subseteq R^2(s)$. Thus $W'_a \cup W'_b \subseteq R^2(s)$.

We then show $R(s) \subseteq W'_a \cup W'_b$. Take $t \in R(s)$ and take $c \in D'$ be such that s and t lie in W'_c (this exists by definition whether $s \to t$ or $t \to s$ holds). By Lemma 6.9, W'_c is a convex subset of the circular order $W'_a \cup W'_b$. It follows that $t \in W'_a \cup W'_b$ as required.

Now if $t \in R(s)$, then one of the pairs (W'_a, W'_b) or (W'_b, W'_a) witnesses that t is of circular type and by the previous paragraph $R(t) \subseteq W'_a \cup W'_b$. It follows that $R^2(s) \subseteq W'_a \cup W'_b$, hence $R^2(s) = W'_a \cup W'_b = R^2(t)$. Finally, if $t \in R^2(s)$, there is $u \in R(s)$ such that $t \in R(u)$. But then u is of circular type, then so is t and $R^2(s) = R^2(u) = R^2(t)$. $\square$

Lemma 6.15. *Assume that s is of linear type. Then the relation $\to$ defines a strict linear order on $R(s)$. For any $t \in R(s)$, t is of linear type and $R(t) = R^2(t) = R(s) = R^2(s)$.*

Proof. Let $s \to t \to u$. We wish to show $s \to u$. Let $a \in D'$ be such that $s < t$ holds in W'_a and let $b \in D'$ such that $t < u$ holds in W'_b. Then $t \in W'_a \cap W'_b$. As s is not of circular type, W'_a and W'_b are not in circular position. Hence either one is included in the other as a convex subset, or one is a simple continuation of the other. Let us consider all cases.

(a) $W'_a \subseteq W'_b$.

We then have $s < t$ in W'_b, hence $s < u$ holds in W'_b and $s \to u$.

(b) $W'_b \subseteq W'_a$.

In this case, we have $t < u$ in W'_a hence $s < u$ in W'_a and again $s \to u$ holds.

(c) W'_b is a simple continuation of W'_a.

Under this assumption, $W^{k+1}_{(a,b)}$ is well defined and $s < t < u$ holds there. Hence $s \to_{k+1} u$ and by hypothesis, we also have $s \to u$.

(d) W'_a is a simple continuation of W'_b.

It must be that $u \in W'_a$ and $s < t < u$ holds in W'_a so again $s \to u$ is true.

Let now $t \in R(s)$ and assume that $t \to u \to v$. Since $t \in R(s)$ and s is not of circular type, also t is not of circular type. Hence the previous result applies to t and we deduce $t \to v$. Therefore $\to$ is transitive on $R(s)$. Since it is irreflexive by construction, $\to$ defines a strict linear order in $R(s)$. It remains to show that $R(t) = R(s)$. Without loss of generality, we have $s \to t$. Take $u \in R(t)$. If $t \to u$, then $s \to u$ and $u \in R(s)$. Assume $u \to t$. Let $b \in D'$ be such that $u < t$ holds in W'_b. Then W'_a and W'_b are in good position (by Lemma 6.13) and contain t. Assume that the initial segment $x \leq t$ of W'_b is a convex subset of W'_a. Then $u \in W'_a$, so $u \in R(s)$. Otherwise, the initial segment $x \leq t$ of W'_a is included in W'_b, which gives us $s \in W'_b$ and again $u \in R(s)$. We have shown $R(t) \subseteq R(s)$. A similar argument gives us $R(s) \subseteq R(t)$ so $R(s) = R(t)$. Since t was an arbitrary element of $R(s)$, this implies that $R^2(s) = R(s)$ and then also $R^2(t) = R(t)$. □

We can now finish the proof of Theorem 6.5.

Let $E = R^2$. By lemmas 6.14 and 6.15, E is an equivalence relation on W. It is also 0-definable. Furthermore, in each E-class, either all elements are of linear type, or all elements are of circular type. Let E_0 be the quotient W/E, seen as a 0-definable subset of M^{eq}. Fix some $e \in E_0$ and let $W[e] \subseteq W$ be the E-class corresponding to e (we will also say *coded by* e). We will say that e, or $W[e]$, is of linear/circular type if all elements in $W[e]$ are such.

Assume first that e is of circular type and let $s \in W[e]$. By Lemma 6.14, we can write $W[e] = W'_a \cup W'_b$, where $a, b \in D'$ are such that $s \in W'_a$ and (W'_a, W'_b) is in circular position. The union $W'_a \cup W'_b$ admits a canonical circular order which induces each of the linear orders on W'_a and W'_b. If now $c, d \in D'$ are such that W'_c and W'_d are in circular position and $W[e] = W'_c \cup W'_d$, then the circular order on $W[e]$ induced by this decomposition is the same as the previous one. This follows from Lemma 6.9 which implies that both W'_c and W'_d are convex with respect to the circular order on $W'_a \cup W'_b$ induced by W'_a and W'_b; and conversely

W'_a and W'_b are convex with respect to the circular order given by W'_c and W'_d. Hence those circular orders coincide. It follows that the circular order on $W[e]$ given by any such decomposition is definable over e.

Assume now that e is of linear type. Then by Lemma 6.15 $\to$ defines a strict linear order on $W[e]$ which, on each W'_a included in $W[e]$, coincides with its canonical order.

To summarize: we have an equivalence relation E on W; every class e of E is equipped with either an e-definable linear order, or an e-definable circular order. Each W'_a, $a \in D'$ is a convex subset of one of those orders. This is also the case for each W_a, $a \in D$. Thus an element V_a, $a \in D$ of the original family is intertwined with a convex subset of one of those classes. This shows the first and last points of Theorem 6.5. It remains to show the other two, namely that the classes are minimal and any two are independent or in order-reversing bijection.

If $e \in W/E$ is circular, write $W[e] = W'_a \cup W'_b$ as above. Then $W[e]$ equipped with the circular order defined above has topological rank 1 since it is the union of two convex subsets W'_a and W'_b which each have topological rank 1. Assume that there is a cut $c \in \overline{W[e]}$ that is e-definable. By construction, the circular order $W[e]$ is covered by elements of the family $(W_a)_{a \in D}$. Hence there is $a \in D$ such that W_a is a convex subset of $W[e]$ and c falls inside W_a, so can be naturally identified with an element of $\overline{W_a}$. However $e \in W/E$ is the unique class containing W_a, so $e \in \mathrm{dcl}^{eq}(a)$ and $c \in \mathrm{dcl}^{eq}(a)$. This contradicts the fact that W_a is minimal over a.

The argument is similar for $e \in W/E$ linear. If $s < t \in W[e]$, then $s \to t$, hence the interval $s < x < t$ is included in W'_a for some $a \in D'$. Assume that there is a parameter-definable convex equivalence relation on $W[e]$ with infinitely many infinite classes. By ω-categoricity, there is $n < \omega$ such that every interval of $W[e]$ intersects either infinitely many infinite classes, or at most n many. Therefore, there is an interval of $W[e]$ which contains infinitely many classes. But then some W'_a, $a \in D'$ would contain infinitely many infinite classes which is impossible as it is weakly minimal. The argument that $W[e]$ is minimal is now the same as in the circular case: an e-definable cut of $W[e]$ would induce an a-definable cut of W_a for some $a \in D$ with $W_a \subseteq W[e]$. This proves the second point in Theorem 6.5.

Finally, if $e, e' \in W/E$, $e \neq e'$, then $W[e]$ and $W[e']$ are disjoint, hence by $\boxplus_3$ there can be no intertwining between a convex subset of $W[e]$ and a convex subset of $W[e']$.

Claim: Assume that there is an intertwining between a convex subset C of $W[e]$ and the reverse of a convex subset C' of $W[e']$. Then there is a definable order-reversing bijection between $W[e]$ and $W[e']$.

Proof: For $a \in D'$, let $\tilde{a} \in D'$ be such that $W'_{\tilde{a}}$ is in definable order-reversing bijection with W'_a, as given by Lemma 6.12.

Restricting C if necessary, C is a convex subset of some W'_a, $a \in D'$. Assume first that e is of circular type. Then $W[e] = W'_a \cup W'_b$ for some $a, b \in D'$ with W'_a, W'_b is circular position. Restricting C further, we may assume that it is included in either W'_a or W'_b, say it is the former. It follows that C' is in order-preserving bijection with a convex subset of $W'_{\tilde{a}}$, but then $C' \subseteq W'_{\tilde{a}}$. Therefore $W'_{\tilde{a}} \subseteq W[e']$. Now $W'_{\tilde{a}}$ and $W'_{\tilde{b}}$ are also in circular position so $W[e'] = W'_{\tilde{a}} \cup W'_{\tilde{b}}$ and there is a definable order-reversing bijection between $W[e]$ and $W[e']$.

The case where e is linear is treated in a similar way. As shown above, two paragraphs before the claim, every interval of $W[e]$ is included in some W'_a, $a \in D'$. If $C \subseteq W'_a$, then $C' \subseteq W'_{\tilde{a}}$. It follows that $W'_{\tilde{a}} \subseteq W[e']$. Now if I is an arbitrary interval of $W[e]$, let J be an interval containing both I and C and say that J is included in W'_b. Then C' is a convex subset of $W'_{\tilde{b}}$ and as above $W'_{\tilde{b}}$ is a convex subset of $W[e']$. It follows in particular that I admits a (necessarily unique) order-reversing bijection with a convex subset of $W[e']$ (compose the order-reversing bijection into $W'_{\tilde{b}}$ with the embedding of this order into $W[e']$). By gluing together those bijection for larger and larger I (which are compatible by uniqueness), we obtain a definable order-reversing bijection between $W[e]$ and $W[e']$.

It follows form the claim and the paragraph before it that any two E-classes are either independent or in order-reversing bijection. This therefore settles the third point of Theorem 6.5 and finishes the proof.

6.3 Analysis of a rank 1 structure

In this section we assume:

($\star$) M is an ω-categorical, rank 1, primitive, unstable NIP structure.

Under those assumptions, we will construct an interpretable set W equipped with a finite-to-one surjective map $W \to M$. The set W will be a disjoint union of finitely many linear and circular orders.

To begin, note that since M is primitive, it is transitive, that is has a unique 1-type. It follows that no element is algebraic over $\emptyset$ and $\mathrm{rk}(a) = 1$ for every $a \in M$. The relation $x \in \mathrm{acl}(y)$ is an equivalence relation on elements of M. This follows from the fact the rk defines a pregeometry on rank 1 structures as explained in Section 2.4, but can also be seen directly: transitivity holds in any structure and symmetry comes from the equivalence:

$$x \in \mathrm{acl}(y) \iff \mathrm{rk}(xy) = 1 \iff \mathrm{rk}(xy) < 2.$$

As M is assumed to be primitive, $\mathrm{acl}(a) = \{a\}$ for every singleton $a \in M$.

Assume that $\mathrm{opD}(M) \geq n$, then Fact 2.18 provides us with a finite tuple of parameters d (named A there) and a d-definable subset X_d transitive over d along with d-definable linear quasi-orders $\leq_{d,1}, \ldots, \leq_{d,n}$ on X_d. By definitions of quasi-orders ther are d-definable equivalence relations $E_{d,1}, \ldots, E_{d,n}$ on X_d such that each $\leq_{d,i}$ induces a linear order (still denoted $\leq_{d,i}$) on the quotient $V_{d,i} := X_d / E_{d,i}$. Note that each $E_{d,i}$ has infinitely many classes by the universality property stated at the end of Fact 2.18. Since M has rank 1 and X_d is transitive over d, all $E_{d,i}$-classes are finite. The quotients $(V_{d,i}, \leq_{d,i})$ also have rank 1 and are transitive, hence minimal over d.

Let $E_0(x, y)$ be the equivalence relation on X_d defined by $\mathrm{acl}(dx) = \mathrm{acl}(dy)$. Let $a \in X_d$ and for $i \leq n$, let a_i be the projection of a on $V_{d,i}$. Since $V_{d,i}$ is minimal over d, we have $\mathrm{acl}^{eq}(a_i d) \cap V_{d,i} = \{a_i\}$. Hence $\mathrm{acl}^{eq}(ad) \cap V_{d,i} = \{a_i\}$. Now all elements E_0-equivalent to a are in $\mathrm{acl}(ad)$. Therefore they must all project to a_i in $V_{d,i}$. Therefore E_0 is finer than $E_{d,i}$. Conversely, if a and b are $E_{d,i}$-equivalent, then $a \in \mathrm{acl}(db)$ and $b \in \mathrm{acl}(da)$, hence $\mathrm{acl}(da) = \mathrm{acl}(db)$ and a, b are E_0-equivalent. We thus see that $E_{d,i} = E_0$ for all i.

Define $V_d := X_d / E_0$ and let $\pi_d \colon X_d \to V_d$ be the canonical projection. Then V_d is equipped with n minimal d-definable linear orders. No two of those orders are equal, or reverse of each other (as ensured by Fact 2.18). We apply Corollary 3.28 to V_d equipped with its induced structure and deduce that there exists some finite tuple of parameters d' and a d'-definable subset $V'_{d'} \subseteq V_d$ such that $V'_{d'}$ is infinite and transitive over d' and any two orders $\leq_i$ and $\leq_j$, $i \neq j$ are independent when restricted to $V'_{d'}$. Replacing d by d' and X_d by $\pi_d^{-1}(V'_{d'})$, we can assume that the n orders are pairwise independent on V_d.

To place ourselves in the context of the previous section, pick arbitrarily n distinct elements $c_1, \ldots, c_n$ in $\mathrm{dcl}^{eq}(\emptyset)$. Let

$$D = \{(d', c_i) : \mathrm{tp}(d') = \mathrm{tp}(d), i \leq n\}.$$

We will write an element of D as (d', i) instead of (d', c_i). For $(d', i) \in D$, we have a (d', i)-interpretable set $V_{d',i}$ defined from the formula defining $V_{d,i}$ by replacing the parameter d by d' and that set is equipped with a minimal order $\leq_{d',i}$ obtained in the same way. Hence we have a uniformly definable family $(V_a)_{a \in D}$ of minimal linear orders. By Theorem 6.5, we can glue those orders together and obtain an interpretable set W equipped with an equivalence relation E satisfying the conclusion of that theorem. We will also make use of the equivalence relation $\sim$ defined before Lemma 6.6, along with the notation $[a, t]_\sim$ for the $\sim$-class of (a, t), for $a \in D$ and $t \in V_a$.

Claim 1: Let $e \in W/E$. Take any $a \in D$, say $a = (d, i)$, and $t \in X_d$ so that $[a, \pi_d(t)]_\sim$ belongs to $W[e]$. Then $[a, \pi_d(t)]_\sim$ is algebraic over (e, t).

Proof: Working over e, let $p(x, y) = \mathrm{tp}(a, t/e)$. Let f be the e-definable function on p which maps a pair $(a', t') \models p$ to $[a', \pi_{d'}(t)]_\sim$ (where $a' = (d', i')$). Proposition 6.1 gives $f(a, t) \in \mathrm{acl}^{eq}(e, t) \cup \mathrm{acl}^{eq}(e, a)$. Now we have $e \in \mathrm{acl}^{eq}(a)$ since e is the class of $V_{d,i}$ and is therefore definable over d. As X_d is transitive over d, hence over a, t is not algebraic over a.

Assume that $f(a, t) \in \mathrm{acl}^{eq}(e, a)$. Note that t is algebraic over $(a, \pi_d(t))$ since π_d has finite fibers. The canonical map from $V_{d,i} = \pi_d(X_d)$ to W is injective and maps $\pi_d(t)$ to $[a, \pi_d(t)]_\sim$. Hence t is algebraic over $(a, [a, \pi_d(t)]_\sim)$. Our assumption then implies that $t \in \mathrm{acl}(e, a)$. Next, we have $e \in \mathrm{acl}^{eq}(a)$ since e is the class to which $V_{d,i}$ maps in W and is therefore definable over d. So $t \in \mathrm{acl}(a)$, but this is absurd as X_d is transitive over a and infinite. So we must have $f(a, t) \in \mathrm{acl}^{eq}(e, t)$.

Claim 2: Let $a = (d, i)$ in D and $t \in X_d$, then t is algebraic over $[a, \pi_d(t)]_\sim$.

Proof: Let $u = [a, \pi_d(t)]_\sim$ and let $e \in W/E$ be the E-class of u. By the previous claim, $u \in \mathrm{acl}^{eq}(e, t)$. Also $u \notin \mathrm{acl}^{eq}(e)$ as $W[e]$ is minimal over e, hence also $t \notin \mathrm{acl}^{eq}(e)$. It follows that $\mathrm{rk}(u, e, t) = \mathrm{rk}(e, t) = \mathrm{rk}(e) + 1$. And $\mathrm{rk}(u, e) > \mathrm{rk}(e)$ so $\mathrm{rk}(u, e) = \mathrm{rk}(e) + 1 = \mathrm{rk}(u, e, t)$. Therefore by Proposition 2.4(7), $t \in \mathrm{acl}^{eq}(u, e)$ which equals $\mathrm{acl}^{eq}(u)$ since e is algebraic over u (indeed e is definable over u as it is the E-class of u).

Claim 3: There are finitely many E-classes.

Proof: Let $a = (d, i)$ in D and $t \in X_d$. Write $u = [a, \pi_d(t)]_\sim$. By Claim 1, u is algebraic over (e, t). Let $M(e) \subseteq M$ be the set of realizations of $\mathrm{tp}(t/e)$. Since t is not algebraic over e (as u is not), $M(e)$ is infinite.

Assume that there are infinitely many E-classes. As M has rank 1, there is an infinite subset $A \subseteq W/E$ such that the intersection $\bigcap_{e \in A} M(e)$ is infinite. Let $A_0 = \{e_1, \ldots, e_n\}$ be a finite subset of A such that $W[e_i]$ and $W[e_j]$ are independent for $i \neq j$. Let $M(A_0) = \bigcap_{e \in A_0} M(e)$. Assume first that all the orders $W[e]$, $e \in A_0$ are linear. In this case, $u \in \mathrm{dcl}(e, t)$. Let f_e be the e-definable function defined on $M(e)$ and sending t to u and let

$$X = \{(f_{e_1}(t), \ldots, f_{e_n}(t)) : t \in M(A_0)\}.$$

Then X is an infinite A_0-definable subset of the product of the independent orders $\prod_{i \neq n} W[e_i]$. Furthermore, since t is algebraic over $f_{e_i}(t)$ by Claim 2, each projection of X to $W[e_i]$ is finite-to-one. Let $t \in M(A_0)$ be non-algebraic over A_0. Then also each $f_{e_i}(t) \in W[e_i]$ is non-algebraic over A_0. For each i, let $W_i \subseteq W[e_i]$ be an A_0-definable convex subset containing $f_{e_i}(t)$ and minimal over A_0. Let $X' = X \cap \prod_{i \neq n} W_i$. Then X' is A_0-definable and non-empty. By Proposition 3.23, X' is dense in a boolean combination of closed definable subsets of each W_i. Since W_i is minimal over A_0, X' is dense in $\prod_{i \neq n} W_i$. It follows that for any $I \subseteq \{1, \ldots, n\}$, we can find $t, t' \in M(A_0)$ such that for $i < n$:

$$f_{e_i}(t) \leq_i f_{e_i}(t') \iff i \in I,$$

where $\leq_i$ is the order on $W[e_i]$.

Let $\phi(xx'; z)$, where x, x' range over M and z ranges over W/E and which expresses the fact that $f_z(x) \leq f_z(x')$ in $W[z]$. Then we see that we have at least 2^n ϕ-types over A_0. As n can be taken arbitrarily large, this shows that ϕ has the independence property and hence contradicts the NIP assumption.

The argument in the case where the $W[e]$'s are circular is similar. The main difference is that u need not be definable over (e, t) so we no longer have the function f_e. Instead define f_e as the function sending some t to the finite set of conjugates of u over e. If this set has one element, then the proof goes through, using the circular order at the end instead of the linear order. If $f_e(t)$ has more than one element, then we can for instance replace the formula ϕ by the formula $\phi'(xx'; z)$ expressing that the elements of $f_e(x)$ lie in a convex subset of $W[e]$ that does not contain any element in $f_e(x')$.

Claim 4: There is a 0-definable map $\pi \colon W \to M$ with finite fibers which maps each E-class surjectively on M.

Proof: It follows from claims 1 and 2, that if $t \in X_d$ and $a = (d, i)$, then $([a, \pi_d(t)]_\sim, e)$ is inter-algebraic with t. Since $e \in \mathrm{acl}^{eq}(\emptyset)$ by Claim 3, we have that $[a, \pi_d(t)]_\sim$ and t are inter-algebraic. For $t \in M$, we have seen that $\mathrm{acl}(t) = \{t\}$. We deduce that $(a, \pi_d(t)) \sim (b, \pi_d(u))$ implies $t = u$. Thus we have a 0-definable map $\pi\colon W \to M$ sending each $(a, \pi_d(t))$ to t. As M is primitive, each E-class maps surjectively onto M. Furthermore any $x \in W$ is algebraic over $\pi(x)$ by Claim 1, hence the map π has finite fibers.

In the proof of the claim, we saw that each map π_d is injective. Therefore the equivalence relation E_0 defined at the beginning of this subsection is trivial on each X_d and $V_d = X_d$. The orders V_a, $a \in D$ therefore have as universe definable subsets of M itself.

Given a point $a \in M$, define $W(a) = \pi^{-1}(a)$. Note that we also have $W(a) = \mathrm{acl}^{eq}(a) \cap W$ as $\mathrm{acl}^{eq}(a) \cap M = M$. For any $V \subseteq W$, define also $V(a) = \pi^{-1}(a) \cap V = \mathrm{acl}^{eq}(a) \cap V$.

Recall that the E-classes come in pairs that are in order-reversing bijection. Let $F \subseteq W/E$ be a set containing exactly one class from each such pair, so that $|F| = \frac{1}{2}|W/E|$ and the classes coded by elements of F are pairwise independent. For $V = W[e]$ an E-class, the cardinality of $V(t)$ does not depend on the choice of $t \in M$ by primitivity of M (consider the 0-definable equivalence relation on M for which $t, t' \in M$ are equivalent if $|V(t)| = |V(t')|$ for each E-class V). Let $n(e)$ be the cardinality of $V(t)$ for $t \in M$ and $V = W[e]$.

<u>Claim 5</u>: With notations as above, for each $e \in F$ choose $n(e)$ many disjoint open intervals of $W[e]$. Then there is $t \in M$ such that each interval selected contains a point of $\pi^{-1}(t)$.

Proof: Enumerate $F = \{e_0, \dots, e_{m-1}\}$ and for $i < m$, let $V_i = W[e_i]$ and $k_i = n(e_i)$. Then the V_i's are pairwise independent and minimal over$F$.

If $m = 1$, then the result follows at once from either Proposition 3.12 or Proposition 4.5, taking p to be the type of a tuple enumerating $V_0(t)$ for some $t \in M$. The general case follows from Theorem 4.11: Let $D \subseteq V_0^{k_0} \times \cdots \times V_{m-1}^{k_{m-1}}$ be the set of tuples that are an enumeration of $\pi^{-1}(t) \cap (V_0 \cup \ldots \cup V_{m-1})$ for some t. Then D is F-definable. Let $D_0 \subseteq D$ be transitive over F, then by Theorem 4.11 its closure is a product of its projection on each $V_i^{k_i}$. As in the case $m = 1$, we can find in each such projection a tuple which has one point in each of the selected intervals of V_i. The result follows.

<u>Claim 6</u>: The op-dimension of M is at least equal to $\frac{1}{2}|\pi^{-1}(t)|$ for some/any $t \in M$.

Proof: Fix a set F as above and let $W_F = \bigcup_{e \in F} W[e]$. Let $t \in M$ and let $m = |\pi^{-1}(t) \cap F| = \frac{1}{2}|\pi^{-1}(t)|$. Let $\bar{a}$ enumerate $\pi^{-1}(t) \cap F$. Then as $\bar{a} \subseteq \mathrm{acl}^{eq}(t)$, Fact 2.17 gives $\mathrm{opD}(\bar{a}) \leq \mathrm{opD}(\bar{a}t) = \mathrm{opD}(t)$. Now W_F is a union of pairwise independent orders. By Corollary 4.15, $\mathrm{opD}(\bar{a}) \geq m$. It follows that M has op-dimension at least m.

<u>Claim 7</u>: For $t \in M$, the fiber $\pi^{-1}(t)$ has size at least $2n$, where n is the same n as at the beginning of this subsection, namely the number of independent orders found on the the set X_d.

Proof: The set X_d is transitive over d and by construction is equipped with n many independent d-definable orders $\leq_1, \ldots, \leq_n$. For each $i \leq n$, the order $(X_d, \leq_i)$ admits a unique definable increasing embedding into one of the E-classes. The same is true of the reverse orders $(X_d, \geq_i)$. As those orders are independent, the convex hull of the images of those $2n$ embeddings are disjoint. It follows that a point $t \in X_d$ is sent by those embeddings to $2n$ different points, each of which lies in $\pi^{-1}(t)$.

Lemma 6.16. *Algebraic closure is trivial on M in the sense that* $\mathrm{acl}(A) = A$ *for each* $A \subseteq M$.

Proof. We already know by the first paragraphs of Section 6.3 that $\mathrm{acl}(a) = a$ for any single element $a \in M$. Let V be any E-class of W. Then V is definable over $\mathrm{acl}^{eq}(\emptyset)$. By Proposition 6.1, for any $A = \{a_1, \ldots, a_n\} \subseteq M$, we have $\mathrm{acl}^{eq}(A) \cap V = \bigcup_{i \leq n} \mathrm{acl}^{eq}(a_i) \cap V$. Any point in M is inter-algebraic with at least one point in V. It follows that $\mathrm{acl}^{eq}(A) \cap M = \bigcup_{i \leq n} \mathrm{acl}^{eq}(a_i) \cap M$, that is $\mathrm{acl}(A) = \bigcup_{i \leq n} \mathrm{acl}(a_i) = A$, since $\mathrm{acl}(a) = a$ for every $a \in M$. ☐

Corollary 6.17. *The only stable reduct of M is the trivial reduct to pure equality.*

Proof. By Proposition 2.11, any stable reduct of M is strongly minimal. By the previous lemma, algebraic closure is trivial on such a reduct (since the algebraic closure of a set cannot increase in a reduct). The only strongly minimal set with trivial algebraic closure is pure equality: in a strongly minimal set any two n-tuples of algebraically independent points have the same type, hence if algebraic closure is trivial, any two n-tuples of distinct points have the same type. ☐

Lemma 6.18. *There are three points $a, b, c \in M$ such that:*

- *there is a set $W_{or} \subseteq W$, definable over abc, which is a union of E-classes and contains exactly one class in each pair of classes in order-reversing bijection;*
- *for each $e \in W/E$,* $\mathrm{dcl}^{eq}(abce)$ *intersects $W[e]$ in at least 3 points;*
- *there is a formula $\theta(x, y; u)$ with parameters in abc such that for each $e \in W/E$, $\theta(x, y; e)$ defines a linear order on $W[e]$.*

Proof. Choose three points $a, b, c \in M$ so that for every class V, we have either $V(a) < V(b) < V(c)$ or $V(c) < V(b) < V(a)$ (meaning that those inequalities holds for any choice of one element in each tuple). This is possible by Theorem 4.11. If V and V' are two classes with an order-reversing definable bijection, then for exactly one of V or V' do we have $V(a) < V(b) < V(c)$. Take $W_{\mathrm{or}} \subseteq W$ to be the union of classes V for which $V(a) < V(b) < V(c)$.

If V is a linear class of code e, then $\mathrm{dcl}^{eq}(abce) \cap V = \mathrm{acl}^{eq}(abce) \cap V$, so $\mathrm{dcl}^{eq}(abce) \cap V$ has at least 3 elements. Let now V be a circular class in W of code $e \in W/E$. If we follow the circular order from any point in $V(a)$ to any point in $V(c)$, we encounter the points in $V(b)$ in the same order regardless of which points of $V(a)$ and $V(c)$ we choose. Therefore all the points of $V(b)$ are in $\mathrm{dcl}^{eq}(abce)$. The same is true for $V(a)$ and $V(c)$ by circularly permuting the roles of a, b, c. ☐

Proposition 6.19. *The structure M has finite op-dimension, bounded by the number of 4-types of elements of M.*

Proof. Assume that $\mathrm{opD}(M) \geq n$. Then applying the construction in this subsection starting with that value of n, we obtain a set W equivalence relation E and projection π such that $\pi^{-1}(t)$ has size at least $2n$ for each $t \in M$ (by Claim 7). It is therefore enough to show that the number of 4-types of elements of M is at least $\frac{1}{2}|\pi^{-1}(t)|$.

Let $a, b, c \in M$, $W_{\mathrm{or}} \subseteq W$ and $\theta(x, y; u)$ be as in the previous lemma. Recall the notation $n(e)$ which denotes the number of points in $\pi^{-1}(t) \cap W[e]$. Let $F = W_{\mathrm{or}}/E$. Let Σ be the set of functions $\sigma \colon F \to \omega$ satisfying $\sigma(e) \leq n(e)$ for each $e \in F$. For each $\sigma \in \Sigma$, we can find by Claim 5 a point $d_\sigma \in M$ such that $\pi^{-1}(d_\sigma)$ has exactly $\sigma(e)$ points in $W[e]$ that are larger than all the points in $\pi^{-1}(b)$ according to the linear order $\theta(x, y; e)$. (By construction, no point in $\pi^{-1}(b)$ is an extreme point according to that order.) The elements of F are not necessarily definable over abc, hence it could be that d_σ and $d_{\sigma'}$ have the same type over abc for $\sigma \neq \sigma'$. However, if those two points do have the same type, there must be a permutation ι of F such that $\sigma' = \sigma \circ \iota$, since this is something we can express with a first order formula over abc. Say that σ and σ' are conjugate if $\sigma' = \sigma \circ \iota$ for some permutation ι.

Claim: There are at least $\sum_{e \in F} n(e)$ conjugation classes of elements of Σ.

Proof: We prove this by induction on $\sum_{e \in F} n(e)$. First assume that $n(e) = 1$ for each $e \in F$, then there are $|F| + 1$ many conjugation classes: a conjugation class is given by how many 0s are in the image and this could be any number between 0 and $|F|$. Now assume for some $e_0 \in F$ we have $n(e_0) > 1$. By induction, there are at least $(\sum_{e \in F} n(e)) - 1$ conjugation classes of elements σ for which $\sigma(e_0) < n(e_0)$. Let $\sigma_0 \in \Sigma$ so that the maximal number of values of $e \in F$ are sent to $n(e_0)$. Then σ_0 cannot be conjugated to a σ for which $\sigma(e_0) < n(e_0)$, hence we have at least one extra class.

We conclude that there are at least $\sum_{e \in F} n(e)$ many 1-types over abc. As $\sum_{e \in F} n(e) = \frac{1}{2}|\pi^{-1}(t)|$, there are at least $\frac{1}{2}|\pi^{-1}(t)|$ many 4-types of elements of M as required. ☐

6.4 The skeletal structure

We know by Proposition 6.19 that the op-dimension of M is finite. Let $n = \mathrm{opD}(M)$. We can then apply the construction of the previous subsection starting with n many independent orders on the set X_d and hence obtain a set W, equivalence relation E and projection $\pi : W \to M$ such that $\pi^{-1}(t)$ has exactly $2n$ elements (by Claim 7 it has at least $2n$ elements, and at most $2n$ by Claim 6). To summarize the situation:

- We have a 0-interpretable set W equipped with a 0-definable equivalence relation E and a 0-definable surjective map $\pi : W \to M$.

- There are finitely many E-classes.
- For each $e \in W/E$, the class $W[e] \subseteq W$ coded by e is equipped with either an e-definable linear order or an e-definable circular order and as such is minimal over e.
- For any $e \in W/E$, there is a unique $\tilde{e}$ such that there is an e-definable order-reversing bijection between $W[e]$ and $W[\tilde{e}]$.
- For any $e \in W/E$ and any $e' \in W/E \setminus \{e, \tilde{e}\}$, the classes $W[e]$ and $W[e']$ are independent.
- For any $e \in W/E$ there is $0 < n(e) < \omega$ such that for any $t \in M$, the set $\pi^{-1}(t) \cap W[e]$ has size $n(e)$.
- For any $t \in M$, $|\pi^{-1}(t)| = 2\,\mathrm{opD}(M)$.

We think of W as a structure in its own right, when equipped with its full induced structure inherited from M (so that the notion of a 0-definable set is the same whether we think of W as a structure or as a definable subset of M^{eq}). The original structure M is a quotient of W and we can shift our focus from M to W: if we classify the possibilities for W (up to bi-definability), we will also have classified the possibilities for M.

A few words about the finite quotient W/E. This set is 0-definable in M^{eq} and hence the automorphism group $\mathrm{Aut}(M)$ acts on it. We are already aware of some relations on it that have to be preserved by this action (equivalently, that are 0-definable):

– the unary relation naming the classes that are linear (the complement being the circular classes);

– for each $k < n$, the unary relation naming the classes which contain exactly k points from $\pi^{-1}(a)$ for some/any $a \in M$;

– the equivalence relation on W/E with classes of size 2 composed of E-classes in order-reversing bijection.

It could be that this is the only structure on that set. At the other extreme, it could also be that W/E is rigid: $\mathrm{Aut}(M)$ acts trivially on it, equivalently $W/E \subseteq \mathrm{dcl}^{eq}(\emptyset)$. Any action between those two extremes is also possible. In the definition of the skeletal structure below, we simply include all the structure on W/E without analyzing it further.

Consider the reduct of W to the language consisting of:

- the equivalence relation E;
- for every $d \leq |W/E|$ and every 0-definable subset of $(W/E)^d$, a predicate naming the pullback of that set to W^d;
- a binary relation $\leq (x, y)$ which holds of a tuple (x, y) if and only if x, y are in the same linear E-class and x is less than y in that class;

- a ternary relation $C(x, y, z)$ which holds of a tuple (x, y, z) if and only if x, y, z are in the same circular E-class and are circularly ordered in this order;
- a relation $F(x, y)$ which holds if x is in some E-class e, y is in the E-class $\tilde{e}$ in order-reversing bijection with e and that bijection sends x to y;
- an equivalence relation E_π whose classes are the fibers of π.

We will call this structure the *skeletal structure* on W and denote the reduct to it by W_{Sk}. Note that the actual language of W_{Sk} is not precisely defined because of the second bullet point above and can depend on M. By an isomorphism between two skeletal structures W_{Sk} and W'_{Sk} we mean an isomorphism up to possibly renaming the predicates obtained from that second bullet point.

Let M and M' be two structures satisfying $(\star)$. Construct W and W' as above for each of them, along with equivalence relations E and E'. For $e \in W/E$ we use again the notation $n(e)$ to denote the number of elements in $\pi^{-1}(t) \cap W[e]$ for some/any $t \in M$, and define similarly $n(e')$ for $e' \in W'/E'$.

Lemma 6.20. *With notations as above, assume that we have a bijection $f : W/E \to W'/E'$ which sends circular classes to circular classes, sends linear classes to linear classes, preserving pairs in order-reversing bijection, such that $n(e) = n(f(e))$ for all $e \in W/E$ and such that 0-definable subsets of $(W'/E')^d$ are precisely the images of 0-definable subsets of $(W/E)^d$. Then there is an isomorphism g from W_{Sk} to W'_{Sk} whose image in the quotient by E is f.*

Proof. This is a straightforward back-and-forth argument using Claim 5: Assume that we have a partial isomorphism $g_0 : W_0 \subseteq W \to W'$, where W_0 is finite and g_0 respects f in the sense that it sends an element of a class e to an element of $f(e)$. Pick an element $a \in W$ and let a_* denote an enumeration of the E_π-class of a. We want to extend f so that its domain contains a. For that, it is enough to find some b_* enumerating an E_π-class in W' and satisfying certain inequalities between the coordinates and elements of $g_0(W_0)$. Let $d = |a_*|$. We can select d many open intervals $I_1, \dots, I_d$ of the various E-classes in such a way that any b_* enumerating an E_π class and having its i-th coordinate in I_i will satisfy the required inequalities. By Claim 5, we can always find a b_* satisfying those constraints. □

Proposition 6.21. *For a given number n, there are, up to isomorphism, finitely many possible skeletal structures W_{Sk} associated to structures M with at most n 4-types.*

Proof. Given a number n of 4-types, there are finitely many possibilities for the op-dimension of M by Proposition 6.19, hence also finitely many possibilities for the size of a fiber of π by Claim 6. The quotient W/E has size at most that of a fiber of π, hence its size is bounded in terms of n (in fact it is bounded by $2n$). Having fixed the size of W/E, there are finitely many possibilities for the choice of which classes are linear or circular, and for each class V, how many elements each fiber of π has in V. Given this data, there are finitely many possibilities for the remaining structure on W/E. By the previous lemma, this completely determines the skeletal structure. □

6.5 The additional local structure

In this subsection, we complete the description of W by showing that the structure not accounted for by the skeletal structure comes from finite local equivalence relations. In particular, it will follow that if all classes are linear, then there is no additional structure.

Let $F \subseteq W/E$ be a set containing exactly one point in every pair of classes in definable order-reversing bijection. Then any two different classes of F are independent (for instance by the fourth and fifth bullets at the start of Section 6.4). From now on, we work over F. Take some $m \in M$ and let m_* enumerate the intersection of $\pi^{-1}(m)$ with the classes in F.

Throughout this subsection, we work over $\mathrm{acl}^{eq}(\emptyset)$, which in particular contains W/E and hence the set F. Let W_* be the set in M^{eq} defined by $\mathrm{tp}(m_* / \mathrm{acl}^{eq}(\emptyset))$. We are now in the context of Section 5 and we use the terminology from there.

Lemma 6.22. *There is a finest finite local equivalence relation on W_*.*

Proof. Let $\mathcal{E}$ be a local equivalence relation and let $C_{\bar{t}}$ be a big cell of W_* defined as in Section 5. Then the relation $\mathcal{E}(C_{\bar{t}})$ is definable form $\bar{t}$ ($\mathcal{E}(C_{\bar{t}})$ is invariant under automorphisms fixing $\bar{t}$, since no other parameters were used in its definition, and hence definable over $\bar{t}$ by ω-categoricity). By ω-categoricity, there is a finest $\bar{t}$-definable equivalence relation on $C_{\bar{t}}$ with finitely many classes. Since W_* is partitioned in finitely many big cells and any local equivalence relation is determined by its restrictions to each of those, there are only finitely many local equivalence relations on W_*. Their intersection is the finest one. $\square$

Proposition 6.23. *Let $\mathcal{E}$ be the finest finite local equivalence relation on W_*, whose existence is guaranteed by the previous lemma. Then the number of classes of $\mathcal{E}$ is bounded by the number of 4-types of elements of M.*

Proof. First note that the structure M is still primitive over $\mathrm{acl}^{eq}(\emptyset)$: If say $E(x, y; a)$ is an equivalence relation relation with finitely many classes defined over $a \in \mathrm{acl}^{eq}(\emptyset)$, then the intersection $\bigcap E(x, y; a')$, where a' ranges over the finitely many conjugates of a under $\mathrm{Aut}(M)$ is a 0-definable equivalence relation with finitely many classes.

Take $x \in M$ and let $X \subseteq W_*$ be the union of the $\mathcal{E}$-classes that one can reach by starting with a point in $\mathrm{acl}^{eq}(x) \cap W_*$ and following a path of small cells. This set X is definable over $\mathrm{acl}^{eq}(\emptyset)x$ and given another $x' \in M$, the corresponding X' is either equal to X or disjoint from it. By primitivity of M, they have to be equal and we have $X = W_*$.

Let $a, b, c \in M$ be given by Lemma 6.18. Then the set F we used to define W_* is definable over abc (it is W_{or}/E in Lemma 6.18). Furthermore, each class V has three points $\alpha_V, \beta_V, \gamma_V$ definable over abc and hence admits a linear order definable over abc (for linear classes it is clear and for circular ones, use any one of the points as the minimal element for instance). It follows that $\mathrm{acl}^{eq}(Fabc) \cap W_* = \mathrm{dcl}^{eq}(abc) \cap W_*$ since algebraic and definable closures coincide on linear orders.

Define big cells $C_{\bar{t}}$ as in Section 5 using $\alpha_V, \beta_V, \gamma_V$. Let e be any $\mathcal{E}(C_{\bar{t}})$-class. Then any class in $\mathcal{E}(C_{\bar{s}})$ that one can reach from e following a sequence of transition maps $f_{\bar{t}',\bar{s}'}$ is definable from e (along with abc). By the previous two paragraphs, every class is reachable by such a sequence of transition maps starting with some point in $\mathrm{dcl}^{eq}(abc)$. Hence each class in each big cell $C_{\bar{t}}$ is definable over abc.

Let n be the number of $\mathcal{E}$-classes in some/any big cell. Two points in W_* that lie in the same big cell but in different $\mathcal{E}$-classes have different types over abc, since each $\mathcal{E}$-class is definable over abc. It follows that n is bounded by the number of types of elements of W_* over abc. □

Let $\phi(\bar{x}; y) = \phi(x_1, \ldots, x_k, y)$ be a formula over $\mathrm{acl}^{eq}(\emptyset)$, where y, as well as each x_i ranges over W_*. Fix $\bar{a} \in W_*^k$ and $b \in W_* \setminus \mathrm{acl}^{eq}(\bar{a})$. Let $U \subseteq W_*^k$ be a product of small cells containing $\bar{a}$ and $V \subseteq W_*$ a small cell containing b. Assume that U and V are small enough so that V is strongly disjoint from any small cell appearing in the product defining U.

Claim: The formula $\phi_{UV}(\bar{x}; y) \equiv \phi(\bar{x}; y) \wedge \bar{x} \in U \wedge y \in V$ is stable.

Proof: Assume not, then we can find sequences $(\bar{a}_i)_{i<\omega}$ in U and $(b_i)_{i<\omega}$ in V such that $\phi(\bar{a}_i; b_j)$ holds if and only if $i > j$. For every $j, n < \omega$, the set of realizations of $\mathrm{tp}(b_j/\bar{a}_{<n})$ is dense in an $\bar{a}_{<n}$-sector by Corollary 4.14. Since that sector has a point in V and U and V are strongly disjoint, it must contain V entirely. Hence the set of realizations of $\mathrm{tp}(b_j/\bar{a}_{<n})$ is dense in V. It follows that the set of realizations of the full type $\mathrm{tp}(b_j/\bar{a}_{<\omega})$ is dense in V.

For each coordinate i of W_*, let the formulas $(\zeta_i(y; \bar{d}_{i,k}) : k < \omega)$ define an increasing sequence of intervals on the i-th coordinate of $y \in W_*$ (where the tuples $\bar{d}_{i,k}$ are parameters from W_*). Recall that we let n_* denote the number of coordinates of W_*. Then the family

$$(\zeta_i(y; \bar{d}_{i,k}) : k < \omega, i < n_*)$$

forms an ird-pattern of size n_* inside V. Add to it the line $(\phi(y; \bar{a}_i) : i < \omega)$. This gives an ird-pattern of size $n_* + 1$: given $\eta \colon n_* + 1 \to \omega$ take b_η to be a realization of $\mathrm{tp}(b_{\eta(n_*)}/\bar{a}_{<\omega})$ such that for all $i < n_*$ and $k < \omega$, we have

$$\models \zeta_i(b_\eta; \bar{d}_{i,k}) \iff \eta(i) < k.$$

This is possible by density of $\mathrm{tp}(b_j/\bar{a}_{<\omega})$. We conclude that $\mathrm{opD}(W_*) \geq n_* + 1$. Now any element of W_* is inter-algebraic with an element of M. Thus by Fact 2.17, $\mathrm{opD}(W_*) = \mathrm{opD}(M) = n_*$: contradiction.

Claim: The formula $\phi(x_1, \ldots, x_k, y)$ is local.

Proof: Let $\bar{c}_U$ (resp. $\bar{c}_V$) be the tuple of end-points of the intervals in each E-class defining U (resp. V) and set $\bar{c} = \bar{c}_U \bar{c}_V$.

Let E_{UV} be the equivalence relation on V defined over $\bar{c}$ by:

$$b \; E_{UV} \; b' \iff (\forall \bar{a}' \in U)(\phi(\bar{a}', b) \leftrightarrow \phi(\bar{a}', b')).$$

Note that for the tuples that we consider ϕ is the same thing as ϕ_{UV}. We claim that E_{UV} has finitely many classes. To see this first note that if $b \in V$ and $\bar{a} \in U$, then we have $b \downarrow_{\bar{c}} \bar{c}\bar{a}$. This follows form Lemma 2.8: by construction of U and V, b cannot be algebraic over $\bar{a}\bar{c}$, hence $\mathrm{rk}(b/\bar{a}\bar{c}) \geq 1 = \mathrm{rk}(b/\bar{c})$.

Now, for $b \in V$, there are finitely many possibilities for $\mathrm{tp}(b/\bar{c})$. Fix such a type $p = \mathrm{tp}(b/\bar{c})$. By Fact 2.9 and the previous paragraph, the set

$$\{\mathrm{tp}_{\phi_{UV}}(b/U) : b \models p\}$$

is finite. Hence, there are finitely many possibilities for $\mathrm{tp}_{\phi_{UV}}(b/U)$, $b \in V$.

We now show that E_{UV} actually only depends on V and not on U. This will follow from similar argument as in Section 5. To simplify notation, we write e.g. $U \equiv_V U'$ to mean $\bar{c}_U \equiv_{\bar{c}_V} \bar{c}_{U'}$, where U' is defines from $c_{\bar{U}'}$ in the same way that U is defined from $c_{\bar{U}}$. If $U \equiv_V U'$ and $U' \subseteq U$, then E_{UV} and $E_{U'V}$ coincide, since they must have the same number of classes. Next, if $U \equiv_V U'$, are such that $U \cap U' \neq \emptyset$, then there is $U'' \subseteq U \cap U'$ such that $U'' \equiv_V U$ and we conclude that E_{UV} and $E_{U'V}$ coincide. Finally, any $U' \equiv_V U$ can be linked to U by a finite chain $U' = U_0, \dots, U_m = U$, with $U_i \equiv_V U$, $U_i \cap U_{i+1} \neq \emptyset$.

It follows that the relation E_{UV} is definable over $\bar{c}_V$ and depends only on V and $\mathrm{tp}(U/V)$. We can therefore write it as $E_{UV} = E_{\bar{c}_V}$. If $V' \subseteq V$, then $E_{\bar{c}_V}$ and $E_{\bar{c}_{V'}}$ coincide on V', hence $E_{\bar{c}_V}$ is a local equivalence relation (by definition of local equivalence relations). This relation depends only on $\mathrm{tp}(\bar{a}, b/\operatorname{acl}^{eq}(\emptyset))$.

Now, do the same starting with any type of tuple $(\bar{a}, b)$ and any permutation of the variables of ϕ. Let $\mathcal{E}_\phi$ be the intersection of all the local equivalence relations obtained. Then $\mathcal{E}_\phi$ is a finite local equivalence relation definable over $\operatorname{acl}^{eq}(\emptyset)$. Take now strongly disjoint small cells $C_1, \dots, C_{k+1}$ and two tuples $(a_1, \dots, a_{k+1}), (a'_1, \dots, a'_{k+1}) \in C_1 \times \cdots C_{k+1}$ such that $(a_i, a'_i) \in \mathcal{E}_\phi(C_i)$ for all $i \leq k+1$. Then by construction of $\mathcal{E}_\phi$, we can replace the a_i's by the a'_i's one by one in the formula $\phi(a_1, \dots, a_{k+1})$ without changing its truth value. Therefore by definition ϕ is a local formula.

6.6 Proof of the main theorems

We keep the same notation $M, W, \dots$ as in the previous section. Fix a finite set $A \subseteq M$ so that all elements of W/E are definable over A and each E-class has at least three points definable over A. Let $F \subseteq W/E$ be a set containing exactly one point from each pair of classes in order reversing bijection and let $W_F \subseteq W$ be the union of the classes in F. Then, over A, W_F is definable and W is just two copies of W_F (up to a definable bijection), so we can forget about W and focus on W_F instead.

Having named A, the induced structure on W_F is extremely simple: Each E-class splits into finitely many A-definable points and A-definable minimal convex subsets. Any such convex subset C is equipped with a definable linear order induced by the linear or circular order of the E-class to which it belongs. For each such C, let $\mathcal{E}(C)$ be the finest A-definable equivalence relation on

C. By minimality, it has finitely many classes, each of which is dense. Furthermore, by the argument in the proof of Proposition 6.23, each one of these classes is definable over A. Finally, if $R(x_1, \dots, x_m)$ is a local definable set on W_F, then if $(a_1, \dots, a_m), (b_1, \dots, b_m)$ are such that for each $i \leq m$, a_i and b_i lie in the same minimal convex set C_i and are $\mathcal{E}(C_i)$-equivalent, then we have

$$R(a_1, \dots, a_n) \iff R(b_1, \dots, b_m).$$

Consider the language L^0_A on W_F consisting of:

- a constant to name each A-definable element of W_F;
- for each minimal A-definable convex subset C of an E-class of W_F and each $\mathcal{E}(C)$-class e, a unary predicate $P_e(x)$ naming that class;
- for each such C, a binary predicate $\leq_C$ naming the linear order on C.

Since we know that apart from the E-classes and the orders, any additional structure on W_F is given by local formulas, we see that W_F admits elimination of quantifiers in L^0_A.

It remains to transfer that structure to M itself. For any $a \in M$, $\mathrm{acl}^{eq}(a) \cap W_F$ has size n equal to the op-dimension of M. Furthermore, every element of it lies in $\mathrm{dcl}^{eq}(Aa)$ (indeed, every such element is the k-th element of $\mathrm{acl}^{eq}(a)$ in some A-definable convex set C, for some k and C, and this can be expressed over Aa). For each $a \in A$, fix an enumeration $\mathrm{acl}^{eq}(a) \cap W_F = \{a_1, \dots, a_n\}$. We can further fix such an enumeration so that if $a \equiv_A b$, then $(a_1, \dots, a_n) \equiv_A (b_1, \dots, b_n)$.

Consider now the two-sorted structure with sorts M and W_F and the language L^1_A consisting of:

- the language L^0_A on the sort W_F;
- the projection $\pi\colon W \to M$;
- for each $i \leq m$, a function symbol f_i from M to W_F interpreted as $f_i(a) = a_i$, where a_i is as above.

Proposition 6.24. *The two-sorted structure (M, W) admits elimination of quantifiers in L^1_A. Furthermore the induced structure on M is inter-definable with the original structure on M with elements of A named.*

Proof. First notice that the skeletal structure on W as defined in Section 6.4 is quantifier-free definable from the L^1_A-structure: the equivalence relation E is in our language; every E-class is a union of sets named by predicates P_C, hence any pre-image of a subset of $(W/E)^d$ is L^1_A-quantifier-free definable; the orders on E-classes can be defined from the orders $\leq_C$; the order-reversing bijections are in the language and finally the equivalence relation E_π is quantifier-free definable from π. By Section 6.5, the additional structure on W is given by local formulas. Let $R(x_1, \dots, x_n)$ be such a relation. Let $C_1, \dots, C_n$ be E-classes.

Then by Proposition 5.18 and the fact that linear orders are simply connected, $R \cap C_1 \times \cdots \times C_n$ is a union of products of the form $X_1 \times \cdots X_n$, where $X_i \subseteq C_i$ is a $\mathcal{E}(C_i)$-class. All those classes are quantifier-free definable in L^1_A, hence so is R. This shows that the original structure on W after naming A is a reduct of the L^1_A-structure. Since the L^1_A-structure is itself a reduct of that structure, they are inter-definable. Hence the same is true for the structure on M since it is interpretable as the quotient of W by E_π.

Now quantifier-elimination is shown by a straightforward back-and-forth argument using Claim 5 and the density of each $\mathcal{E}(C)$-class. □

We can now easily obtain a language on the one-sorted structure M in which we have quantifier elimination: for each unary relation $R(x)$ in L^1_A and each $i \leq m$, let $R_i(x)$ be a unary relation over M interpreted in M as

$$M \models R_i(x) \iff (M, W) \models R(f_i(x)).$$

Introduce similarly a binary relation $R_{i,j}(x, y)$ for each binary relation $R(x, y) \in L^1_A$ and each $i, j \leq m$. This gives a binary language in which M admits elimination of quantifiers.

Lemma 6.25. *Let M be an ω-categorical structure. Assume that for some integer r, for any set $A \subseteq M$ of size r, the expansion of M naming every $\mathrm{acl}^{eq}(A)$-definable set is homogeneous in a finite relational language of arity at most m. Then M is finitely homogenizable.*

Proof. We need to show that for some integer k, any n-type $p(x_1, \ldots, x_n)$ is implied by the conjunction of its restrictions to sets of k variables. Fix an r-type q and $\bar{a} \models q$. Let $L_{\bar{a}} = \{\phi_1(\bar{x}_1), \ldots, \phi_l(\bar{x}_l)\}$ be a set of formulas with parameters in $\mathrm{acl}^{eq}(\bar{a})$ such that M has quantifier elimination in a language with a predicate for each of those formulas. Assume that $L_{\bar{a}}$ is closed under $\mathrm{Aut}(\mathrm{acl}^{eq}(\bar{a}))_{\bar{a}}$ (automorphisms fixing $\bar{a}$ pointwise) and that the maximal arity of those formulas is m. For any finite set $C \subset M$, define an equivalence relation $E^{\bar{a}}_C$ on $L_{\bar{a}}$ by saying that two formulas $\phi(\bar{x})$ and $\phi'(\bar{x})$ are $E^{\bar{a}}_C$-equivalent if they are conjugated over $\bar{a}$ and for any tuple $\bar{c}$ of elements of C, we have

$$M \models \phi(\bar{c}) \leftrightarrow \phi'(\bar{c}).$$

If a pair (ϕ, ϕ') is not in $E^{\bar{a}}_C$, then there is a subset $C_0 \subseteq C$ of size at most m such that (ϕ, ϕ') is not in $E^{\bar{a}}_{C_0}$. It follows that for any C, there is $C_* \subseteq C$ of size at most $l^2 m$ such that $E^{\bar{a}}_C = E^{\bar{a}}_{C_*}$. Since the numbers l and m depend only on $q = \mathrm{tp}(\bar{a})$, we can define $N(q) = l^2 m$.

Let $p = \mathrm{tp}(a_1, \ldots, a_n)$ be any type in finitely many variables and we want to show that for some k not depending on p, p is implied by its restrictions to subsets of k variables. We can assume that all the a_i's are distinct. Set $\bar{a} = (a_1, \ldots, a_r)$ and $q = \mathrm{tp}(\bar{a})$. Let $C = \{a_1, \ldots, a_n\}$ and take $C_* \subseteq \{a_1, \ldots, a_n\}$ of size at most $N(q)$ so that $E^{\bar{a}}_{C_*} = E^{\bar{a}}_C$. By construction of $E^{\bar{a}}_C$, for any subtuple $\bar{d}$ of $(a_1, \ldots, a_n)$, the type $\mathrm{tp}(\bar{d}/\bar{a}C_*)$ implies the quantifier-free $L_{\bar{a}}$-type of $\bar{d}$.

Since $L_{\bar{a}}$ is composed of relations of arity at most m the quantifier-free $L_{\bar{a}}$-type of $(a_1, \ldots, a_n)$ is implied by the conjunction of the quantifier-free $L_{\bar{a}}$-types of $(a_{i_1}, \ldots, a_{i_m})$ for $i_1, \ldots, i_m \leq n$. This quantifier-free $L_{\bar{a}}$-type is itself implied by $\mathrm{tp}(a_{i_1}, \ldots, a_{i_m}/\bar{a}C_*)$. Therefore the full type of the tuple $(a_1, \ldots, a_n)$ is implied by the conjunction of $\mathrm{tp}(a_{i_1}, \ldots, a_{i_m}, \bar{a}, \bar{c}_*)$ for $i_1, \ldots, i_m \leq n$, where $\bar{c}_*$ enumerates C_*. Thus $k := m + r + \max_q N(q)$ has the required property. □

Question 6.26. *In the previous lemma, can we replace "for any set $A \subseteq M$" by "for some set $A \subseteq M$"?*

Lemma 6.27. *Let M be an ω-categorical structure and fix some $r < \omega$. Then there are only finitely many reducts of M which are homogeneous in a relational language of arity at most r.*

Proof. A reduct M' of M that is homogeneous in a relational language of arity at most r is entirely determined by its r-types, or equivalently the orbits of $\mathrm{Aut}(M')$ acting on M^r. Any such orbit is a finite union of orbits of $\mathrm{Aut}(M)$ acting on M^r. Hence there are only finitely many possibilities. □

We can now prove our main theorem.

Theorem 6.28. *Let M satisfy $(\star)$ and assume that M has n 4-types. Then M is distal and is inter-definable with a structure M_0 in a finite relational language which is homogeneous and finitely axiomatizable. After naming a finite set of points, M admits elimination of quantifiers in a finite binary language. Furthermore, for a given n, there are finitely many possibilities for M_0.*

Proof. We have already seen that after naming a finite set A as above M admits quantifier elimination in a finite binary language. It is easy to see that the structure W in L^1_A satisfies the definition of distality given after definition 2.19 with $k = 4m$ (the quantifier-free type of an element a over a finite set $B = \mathrm{dcl}(B)$ given by the restriction of that type to $B_0 \subseteq B$ composed of the points in B closest to each point in $\mathrm{dcl}(a)$). Distality and non-distality are preserved under naming constants and are preserved by bi-interpretations, hence M is distal.

All W/E-classes are definable over $\mathrm{acl}^{eq}(\emptyset)$ and for any set $A \subseteq M$ of size 3, there are at least 3 $\mathrm{acl}^{eq}(A)$-definable elements in each E-class. We can then build the languages $L^0_{\mathrm{acl}^{eq}(A)}$ and $L^1_{\mathrm{acl}^{eq}(A)}$ in the same way that we built L^0_A and L^1_A, replacing everywhere A by $\mathrm{acl}^{eq}(A)$. We obtain quantifier elimination in L^1_A and hence the expansion of M obtained by naming all $\mathrm{acl}^{eq}(A)$-definable sets is finitely homogeneous. By Lemma 6.25, M itself is finitely homogenizable.

Assume that M is equipped with such a finite relational language L for which it is homogeneous. We have seen that after naming some appropriate finite set of points A, M becomes homogeneous in a binary language for which it is finitely axiomatizable. It follows that M is finitely axiomatizable in the language $L(A)$ equal to L augmented by a finite set of constants to name the elements of A. Then by quantifying on A, we see that M is finitely axiomatizable in L.

It remains to prove that there are only finitely many such M having n 4-types. By Proposition 6.21, there are finitely many possibilities for the skeletal structure. Then by Proposition 6.23, there are finitely many possibilities for the number of $\mathcal{E}$-classes. By Proposition 5.19, there are finitely many possibilities for $\mathcal{E}$ itself (given the skeletal structure). Hence there are finitely many possibilities for the L_A-structure described above. Since M is a reduct of that structure, we conclude by Lemma 6.27. □

6.7 Reducts

Using the classification, one can relatively easily describe the reducts of a given structure satisfying $(\star)$, at least when no local relations are present. (We have avoided giving an explicit classification of the possible local relations. This should be doable, but might be a bit tricky especially if non-trivial automorphisms of W/E are also present.)

Let M be such a structure and let W be the finite cover associated to it. Let M' be a reduct of M. If M' is stable, then by Corollary 6.17 it is pure equality. If M' is unstable, then we can construct a finite cover W' of it as above (with the properties listed at the beginning of Section 6.4). Then W' is interpretable in M.

In what follows, we work in M, over $\mathrm{acl}^{eq}(\emptyset)$. Let $2m = |W'/E'|$. Let $F \subseteq W/E$ be as usual a set containing exactly one class from each pair in order-reversing bijection and let $W_F \subseteq W$ be the union of the E-classes in F. Define similarly F' and $W'_{F'}$. Now $\mathrm{acl}^{eq}(a)$ contains n elements in W_F and say m elements in $W'_{F'}$. We have $\mathrm{opD}(a) = n$. Hence by Corollary 4.15 each E'-class V' in $W'_{F'}$ is intertwined with an E-class V in W_F and furthermore that intertwining must send points of $\mathrm{acl}^{eq}(a) \cap V'$ to points in $\mathrm{acl}^{eq}(a) \cap V$. It follows that V' is in increasing bijection with a (necessarily dense) subset of V. In particular, if E-classes are transitive over $\mathrm{acl}^{eq}(\emptyset)$, then the E'-classes are naturally a subset of the E-classes. We note however that the structure on W' induced from M' could be proper reduct of the one induced from M. For instance, classes that are linear seen in M could become circular in M'.

For a given W, one can then by inspection determine all the possibilities for W'. Instead of attempting to write a general statement, we will examine two special cases: the case where $M = (M; \leq_1, \ldots, \leq_n)$ is equipped with n independent linear orders and the case where W has just two circular orders in order-reversing bijection, each extending to a unique strong type over $\emptyset$.

Assume that $M = (M; \leq_1, \ldots, \leq_n)$ is the Fraïssé limit of sets equipped with n linear orders and define W and E as usual. Then W is composed of $2n$ linear orders pairwise in order-reversing bijection and otherwise independent, and the fibers of the projection $\pi \colon W \to M$ pick out exactly one element per linear order. By what we have explained, the E'-classes are a subset of the E-classes. The induced structure on W' from M' is a reduct of the structure induced from

M: some classes that are linear in the M-induced structure might be circular in the M'-induced structure. Also the automorphism group $\mathrm{Aut}(M')$ might induce more automorphisms of W'/E' as $\mathrm{Aut}(M)$ does (indeed W'/E' being a subset of W/E is fixed pointwise by $\mathrm{Aut}(M)$).

To summarize, one can associate to each reduct of M a triple (V_l, V_c, G) where V_l, V_c are two disjoint subsets of $\{1, \ldots, n\}$ of cardinalities m_l and m_c respectively, and G is a subgroup of the wreath product $\mathbb{Z}_2 \wr (\mathfrak{S}_{m_l} \times \mathfrak{S}_{m_c})$. The subsets V_l, V_c indicate respectively which of the n pairs of orders in W are kept as linear orders in W' and which are kept, but become circular in W'. The subgroup G is the group of automorphism on the quotient W'/E'. The reducts of M are completely classified by such triples and every triple corresponds to a reduct (the triple where $m_l = m_c = 0$ corresponding to the trivial reduct to pure equality).

For instance for $n = 2$, we have $3^2 = 9$ choices for the pair (V_l, V_r). If either of the two sets has cardinality 2, then we get 10 possibilities for G (the group $\mathbb{Z}_2 \wr \mathfrak{S}_2$ is isomorphic to the dihedral group D_8 and has 10 subgroups). If the two sets have cardinality 1, we get 5 possibilities for G corresponding to subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$, if one set has cardinality 1 and the other 0, we have two possibilities for G and finally, if both sets are empty, we have one possibility for G. Summing it all up, we obtain 10*2+5*2+2*4+1=39 reducts. We thus recover the result of Linman and Pinsker [LP15].

Let us now turn to the second example. Assume that W has two E-classes, which are circular, in order-reversing bijection, conjugated by an automorphism, and the fibers of the projection π contain exactly n points per class. The associated M can be obtained by taking the Fraïssé limit of separations relations with an equivalence relation F having classes of cardinality n and quotienting by F.

Let M' be an unstable reduct of M and W' its associated finite cover, which we again think of as interpreted in M. Let V be any one of the two E-classes of W. Every E'-class is in definable bijection with V. Since the map $\pi' \colon W' \to M'$ is also interpretable in M, fibers of π' have to contain at least n points from each E'-class (otherwise there would be in W an $\mathrm{acl}^{eq}(\emptyset)$-definable equivalence relation on V with classes of size $< n$, which is not the case). Hence as above, since algebraic closure cannot be larger in M' as it is in M, W' has two E'-classes in order reversing bijection and π' is n-to-one on each of them. But then we see that W' is isomorphic to W and there can be no additional automorphisms on the set of classes. So M' is equal to M.

This shows that M has no proper non-trivial reduct. This gives a new example of an infinite family of ω-categorical structures with no proper reduct, or equivalently of maximal closed (oligomorphic) permutation groups. (See e.g., [BM16] or [KS16] for more about maximal closed permutation groups.)

7 Binary structures and multi-orders

We say that a structure M is *binary* if it eliminates quantifiers in a finite binary relational language.

Lemma 7.1. *Let M be a binary structure. Then M has finite rank.*

Proof. Assume not and fix some integer N large enough. Then as $\mathrm{rk}(M) > N$, we can build:

- an increasing sequence of finite tuples $(c(n) : n < N)$;
- for each $n < N$, a $c(n)$-definable set D_n, transitive over $c(n)$;
- an infinite $c(n)$-definable set of parameters E_n, transitive over $c(n)$;
- a $c(n)$-uniformly definable family $(X_t : t \in E_n)$ of infinite subsets of D_n which is $k(n)$-inconsistent for some $k(n) < \omega$, such that if $n < N - 1$, then for some $t \in E_n$, $D_{n+1} \subseteq X_t$.

Why is this possible? If we drop the transitivity assumptions on D_n and E_n, then this is precisely the definition of rank, where we take D_n to be equal to some X_t built at stage $n - 1$ (and $D_0 = M$). We can enforce the transitivity hypotheses by first replacing D_n by a transitive $c(n)$-definable subset of it: since there are only finitely many such subsets, it must be true for one of them, say D'_n that the family $(X_t \cap D'_n : t \in E_n)$ has infinitely many infinite sets. Also, there are finitely many $c(n)$-definable transitive subsets of E_n. Again, there must be one of them, say E'_n for which $(X_t \cap D'_n : t \in E_n)$ has infinitely many infinite sets. Then replace E_n by E'_n and $(X_t : t \in E_n)$ by $(X_t \cap D'_n : t \in E'_n)$.

Claim: For each n, there are $x, y \in D_n$ such that for no $t \in E_n$ do we have both $x \in X_t$ and $y \in X_t$.

Proof: For any $x \in D_n$ there is a finite tuple $(t_1, \ldots, t_k)$ of elements of E_n such that x is in each X_{t_i} and in no other X_t. Since E_n is transitive over $c(n)$, no element of E_n is algebraic over $c(n)$ and we can find a tuple $(t'_1, \ldots, t'_k) \equiv (t_1, \ldots, t_k)$ with $t'_i \neq t_j$ for all $i, j \leq k$ (this follows from Neumann's separation lemma, see [EH93, Lemma 1.4(ii)]). Now take y so that $(y, t'_1, \ldots, t'_k) \equiv (x, t_1, \ldots, t_k)$.

For each n, let $\phi_n(x; y)$ be the formula expressing that for some $t \in E_n$, we have $x, y \in X_t$. This formula is definable over $c(n)$. We now need to get rid of the parameters in $\phi(x; y)$ and this where we use of the assumption that M is binary. By that assumption, we can write the formula $\phi_n(x; y)$ as a boolean combination of atomic formulas, each of which involves only two elements from the tuple $c(n)$ of parameters and the variables x, y. Any atomic formula involving two parameters is uniformly true or false, hence can be removed. Since all elements of D_n have the same type over $c(n)$, any atomic formula involving $c(n)$ and one of the variables x or y also has a constant truth value on D_n. Hence we can remove such terms from $\phi_n(x; y)$. In this way, we obtain a formula $\psi_n(x, y)$ which is equivalent to $\phi_n(x; y)$ when evaluated on elements from D_n and has no parameters.

For every n, there are $a, b \in D_n$ with $\neg\psi_n(a; b)$. However we must have $\psi_m(a; b)$ for all $m < n$. Hence the formulas $\psi_n(x; y)$ define distinct relations. Taking N large enough, this is a contradiction: there are up to logical equivalence only finitely many quantifier-free formulas one can construct from a finite relational language. □

Question 7.2. *Let M be a primitive binary structure. Must M have rank 1?*

We say that $(M, \leq)$ is *topologically primitive*, where $\leq$ is a linear order, if it does not admit a 0-definable convex non-trivial equivalence relation.

Lemma 7.3. *Let $(M, \leq, \ldots)$ be a ranked ω-categorical structure, where $\leq$ is a linear order on M. Assume that $(M, \leq)$ is topologically primitive. Then $(M, \leq)$ has topological rank 1.*

Proof. Assume that over parameters $\bar{a}$, there is some definable convex equivalence relation $E_{\bar{a}}$ with infinitely many classes. By ω-categoricity, the order induced by $\leq$ on the quotient $M/E_{\bar{a}}$ is not discrete. Thus there are $c < d$ in M such that there are infinitely many $E_{\bar{a}}$-classes between c and d. The relation $R(x, y)$ saying that for every $\bar{b} \equiv \bar{a}$, there are finitely many $E_{\bar{b}}$ classes between x and y is a 0-definable equivalence relation with convex classes. As M is topologically primitive, R is trivial: its classes are singletons. It follows that for every open interval I, we can find some $\bar{b} \equiv \bar{a}$ such that $E_{\bar{b}}$ has infinitely many classes in I. This easily implies that M has unbounded rank. □

Theorem 7.4. *Let $(M, \leq_1, \ldots, \leq_n)$ be a homogeneous multi-order such that no two orders $\leq_i$ and $\leq_j$ are equal or opposite of each other. Assume that each $(M, \leq_i)$ is topologically primitive, then M is the Fraïssé limit of finite sets equipped with n orders.*

Proof. The assumptions along with the previous lemmas imply that each order $(M, \leq_i)$ has topological rank 1 and is a complete type over $\emptyset$. Proposition 3.27 describes the possibilities. The only homogeneous structures in the list are the ones with no intertwining (other than equalities between orders), since the intertwining relations R_{ij} are not quantifier-free definable from the orders. □

The classification of imprimitive homogeneous multi-orders is carried out in [BS20], making further use of techniques from this paper.

More generally, a primitive set equipped with n orders definable in a binary structure satisfies the hypotheses of Proposition 3.27. This might help in classifying other classes of ordered homogeneous structures.

8 The general NIP case

We hope to be eventually able to classify all finitely homogeneous NIP structures, and possibly even all ω-categorical structures having polynomially many types over finite sets.

Conjecture 8.1. *Let M be finitely homogeneous and NIP, then:*

1. *The automorphism group* $\mathrm{Aut}(M)$ *acts oligomorphically on the space of types* $S_1(M)$.
2. *The structure M is interpretable in a distal, finitely homogeneous structure.*
3. *There is M' bi-interpretable with M whose theory is quasi-finitely axiomatizable.*
4. *If M is not distal, then its theory is not finitely axiomatizable.*

Points (2) and (3) each imply that there are only countably many such structures (for point (2), this follows from Theorem 8.3 below). If M is stable finitely homogeneous, then it is ω-stable and the conjecture is known to be true: (1) by [CHL85, Theorem 6.2], (2) by [Lac87], (3) by [Hru89] and (4) by [CHL85, Corollary 7.4].

Note that we cannot expect an analogue of Theorem 6.28: For $k < \omega$, let M_k be the Fraïssé limit of finite trees with $\leq k$ branching at each node. Then for $k \geq 4$, the structures M_k all have the same 4-types.

The previous conjecture was stated for the finitely homogeneous case, but we could have stated it also for ω-categorical structures with polynomially many types over finite sets, or finite dp-rank, which is *a priori* weaker. (For a definition of dp-rank, see e.g. [Sim15, Chapter 4].) However, even the stable case is then unkown.

Question 8.2. *Let M be ω-categorical, stable of finite dp-rank. Is M ω-stable?*

One intuition we have on NIP structures is that they are somehow combinations of stable and distal ones. At the very least, we expect that reasonable statements that hold true for stable and distal structures are true for all NIP structures. If M is finitely homogeneous and stable, then we know that it is quasi-finitely axiomatizable. Somewhat surprisingly, the distal case can be proved directly rather easily: see Theorem 8.3 below. We consider this as strong evidence towards this part of the conjecture. It is possible that the other parts could also be proved directly for distal structures, without having any kind of classification, but we have not managed to do so.

Theorem 8.3. *Let M be homogeneous in a finite relational language L and distal. Then the theory of M is finitely axiomatizable.*

Proof. Let r be the maximal arity of a relation in L. By distality, there is k such that for any finite set $A \subseteq M$ and element $a \in M$, there is $A_0 \subseteq A$ of size $\leq k$ with $\mathrm{tp}(a/A_0) \vdash \mathrm{tp}(a/A)$. Let $n_0 = kr + k + r + 1$. Consider the theory T_* composed of:

1. all formulas of the form $(\forall \bar{x})\phi(\bar{x})$, with $|\bar{x}| \leq n_0$ and ϕ quantifier-free that are true in M;
2. all formulas of the form $(\forall \bar{x})(\theta(\bar{x}) \to (\exists y)\phi(\bar{x}, y))$ with $|\bar{x}| \leq k$, $|y| = 1$ and θ, ϕ quantifier-free that are true in M.

Up to logical equivalence, T_* contains finitely many formulas. Since M is a model of T_*, that theory is consistent. Let N be any countable model of it and we will show that N is isomorphic to M.

Claim 0: Let

$$\Upsilon \equiv (\forall x, \bar{y}, \bar{z})(\theta(x, \bar{y}) \wedge \psi(\bar{y}, \bar{z}) \rightarrow \phi(x, \bar{z})),$$

with $|x| = 1$, $|\bar{y}| \leq k$ and where each of θ, ψ, ϕ is quantifier-free and describes a complete type. Then if M satisfies Υ, so does N.

Proof: Since the arity of L is bounded by r, $\phi(x, \bar{z})$ is a conjunction of formulas of the form $\phi'(x, \bar{z}')$, where $\bar{z}' \subseteq \bar{z}$ is a subtuple of size $\leq r$. For each such formula, we have

$$M \models (\forall x, \bar{y}, \bar{z}')(\theta(x, \bar{y}) \wedge \psi'(\bar{y}, \bar{z}') \rightarrow \phi'(x, \bar{z}'))$$

where $\psi'(\bar{y}, \bar{z}')$ is a complete quantifier-free formula implied by $\psi(\bar{y}, \bar{z})$ with variables $(\bar{y}, \bar{z}')$. This formula is in T_*, since it is universal and has less than n_0 variables, so N also satisfies it.

Claim 1: N satisfies the universal theory of M: for any finite set $B \subseteq N$, there is $B' \subseteq M$ which is isomorphic to it.

Proof: We prove the result by induction on the cardinality of B. For $|B| \leq n_0$, this follows from the construction of T_*. Assume that we know the result for some $n \geq n_0$ and are given a finite subset $B \subseteq N$ of size n and an additional point $d \in N$. We want to find an isomorphic copy of $B \cup \{d\}$ in M. Pick any r distinct elements $b_0, \ldots, b_{r-1}$ in B. For $i < r$, set $B_i = B \setminus \{b_i\}$. The set $B_i \cup \{d\}$ has an isomorphic copy in M. It follows by distality of M that there is $B_i' \subseteq B_i$ of size $\leq k$ such that

$$(\triangle) \qquad M \models \mathrm{tp}(d, B_i') \wedge \mathrm{tp}(B_i', B_i) \rightarrow \mathrm{tp}(d, B_i).$$

By Claim 0, N also satisfies that formula. Let $B_r = \bigcup_{i<r} B_i'$. By the case $n = kr + 1 < n_0$, the set $B_r \cup \{d\}$ is isomorphic to some $A_r \cup \{c\}$ in M. By homogeneity of M and induction, we can find $A \supseteq A_r$ such that $\mathrm{tp}(A_r, A) = \mathrm{tp}(B_r, B)$. For $i < r$, define A_i is the image of B_i under this isomorphism. By $(\triangle)$, which holds both in M and in N, we have $\mathrm{tp}(d, B_i) = \mathrm{tp}(c, A_i)$ for each A. Since the arity of the language is at most r and any r elements from Bd either lie in B or in some $B_i d$, we conclude that Bd and Ac are isomorphic. This finishes the induction.

We now show by back-and-forth that N is isomorphic to M. Assume we have a partial isomorphism f from a finite subset $A \subseteq M$ to N. Let $c \in M$. By distality, there is $A_0 \subseteq A$ of size $\leq k$ such that $\mathrm{tp}(c/A_0) \vdash \mathrm{tp}(c/A)$. Let B_0 be the image of A_0 in B. By assumption on T_*, there is $d \in N$ such that $\mathrm{tp}(d, B_0) = \mathrm{tp}(c, A_0)$. By Claim 0, we have $\mathrm{tp}(d, B) = \mathrm{tp}(c, A)$, hence we can extend the partial isomorphism f by setting $f(c) = d$. The back direction follows at once from Claim 1 and homogeneity of M. $\square$

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