DENSITY OF COMPRESSIBLE TYPES AND SOME CONSEQUENCES

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Abstract. We study compressible types in the context of (local and global) NIP. By extending a result in machine learning theory (the existence of a bound on the recursive teaching dimension), we prove density of compressible types. Using this, we obtain explicit uniform honest definitions for NIP formulas (answering a question of Eshel and the second author), and build compressible models in countable NIP theories.

1. Introduction

By the Sauer-Shelah lemma, if a formula $\phi(x;y)$ is NIP, then the number of ϕ-types over a finite set A is bounded by a polynomial in the cardinality of A. For a stable formula, this is a consequence of definability of types: one only needs to specify the parameters involved in the definition. In dense linear orders, the reason for this phenomenon is different: for any finite set A and element b, the $\leq$-type of b over A is implied by its restriction to some subset A_0 of size 2: the information of the full type can be *compressed* down to this subset of bounded size. A $\leq$-type over an infinite set cannot in general be compressed down to a finite set, however finite parts of it can be uniformly compressed; following [Sim20], we call such a type *compressible* (Definition 2.11). We expect NIP formulas to exhibit a combination of those two behaviours.

For NIP theories one manifestation of this philosophy is the result from [Sim20] that an arbitrary type has a generically stable part up to which it is compressible. Distal structures are (NIP) structures in which every type is compressible and hence this decomposition is trivial. For stable theories, compressible types turn out (Lemma 4.8) to be precisely types which are *l-isolated*, that is, isolated formula by formula. These play a role in Shelah's classification theory; one key property is that in a countable stable theory, an l-atomic model exists over any set [She90, IV.2.18(4),3.1(5),3.2(1)]. In this paper, we think of compressibility as an isolation notion and investigate its properties by analogy with the stable case. In order to obtain similar model-construction results, we need two basic properties: density of compressible types and transitivity of compressibility.

Density of compressible types over a set A means that every formula over A extends to a complete compressible type over A. We prove this for countable NIP theories (Corollary 3.21) by first considering the local setting of a single NIP formula ϕ, and showing that any finite partial ϕ-type extends to a complete compressible ϕ-type (Corollary 3.9). This is a combinatorial argument based on the proof by Chen, Cheng, and Tang [CCT16] of a bound on the "recursive teaching dimension" of a finite set system in terms of its VC-dimension. The existence of such a bound was used in [EK20] to prove uniform definability of types over finite sets (UDTFS) for an NIP formula in an arbitrary theory. We generalise this result (answering [EK20,

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Question 28]) by showing uniformity of honest definitions for NIP formulas, which was previously known only assuming NIP for the whole theory [CS15, Theorem 11]. For this, we first show that an arbitrary ϕ-type p is a *rounded average* of finitely many compressible types (Theorem 5.17). The rounded average of the compression schemes of these types gives an honest definition for p.

In fact, it turns out that full consistency of p is not required here: for large enough k we get uniform honest definitions for k-consistent families of instances of ϕ, which we dub *k-hypes* (Corollary 5.30). Using this, we also obtain in Theorem 5.36 uniform definability of ϕ-types which are *pseudofinite* in the sense that their positive and negative parts are pseudofinite (Definition 5.34).

In order to prove transitivity, namely that $\mathrm{tp}(AB/C)$ is compressible when $\mathrm{tp}(A/BC)$ and $\mathrm{tp}(B/C)$ are, we return to the global setting of an NIP theory and use the type-decomposition theorem from [Sim20]. We show in Proposition 6.23 that compressibility can be *rescoped* to an arbitrary subset of the domain: if $\mathrm{tp}(a/B)$ is compressible and $C \subseteq B$, then $\mathrm{tp}(a/C)$ is compressible in the language with constants for elements of B. We deduce transitivity in Proposition 6.25.

Finally, we conclude that for countable NIP theories (or even countable theories naming any set of constants) one can construct models which are compressible over arbitrary sets (Propositions 6.29 and 6.30). We give several applications:

- Given a definable unary set X whose induced structure is stable, and any model N of the theory of the induced structure, there is a model M of T such that $X(M) = N$ and moreover, if $N' \succ N$ then there is $M' \succ M$ such that $X(M') = N'$. This is Corollary 6.33.
- If the theory is not stable, we can extend models without realising any non-algebraic generically stable type (Corollary 6.39 and Remark 6.40).
- We analyse compressiblity in ACVF, showing that then any model M containing A whose residue field is algebraic over A is compressible over A (Example 6.41).

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2. Preliminaries

2.1. **Languages, formulas and types.**

Our notation is standard. We use $\mathcal{L}$ to denote a first order language and $\phi(x, y)$ to denote a formula ϕ with a partition of (perhaps a superset of) its free variables. When x is a (possibly infinite) tuple of variables and A is a set contained in some structure (perhaps in a collection of sorts), we write A^x to denote the tuples of the sort of x (and of length $|x|$) of elements from A; alternatively, one may think of A^x as the set of assignments of the variables x to A. When M is a structure and $A \subseteq M^x$, $b \in M^y$, we define $\phi(A, b) = \{a \in A \mid M \vDash \phi(a, b)\}$.

T will denote a complete theory in $\mathcal{L}$ (we do not really need T to be complete, but it is more convenient), and $\mathcal{U} \vDash T$ will be a monster model (a sufficiently large saturated model[1]). The word *small* means "of cardinality $< |\mathcal{U}|$". As usual, we will assume that all models, tuples and sets of parameters are small and are contained in (perhaps a collection of sorts from) $\mathcal{U}$ unless stated otherwise. Some results, such as Theorem 5.17, hold for any set, by considering a bigger monster model and applying the result there.

When $B \subseteq \mathcal{U}$, $\mathcal{L}(B)$ is the language $\mathcal{L}$ augmented with constants for elements from B, and $\mathcal{U}_B$ is the natural expansion of $\mathcal{U}$ to $\mathcal{L}(B)$. A *partial type* in variables x (perhaps infinite, perhaps from different sorts) over $B \subseteq \mathcal{U}$ is a set of $\mathcal{L}(B)$-formulas in x consistent with $\mathrm{Th}(\mathcal{U}_B)$ (i.e., formulas over B). For a partial type π over B and $C \subseteq B$, we use the notation $\pi|_C$ for the *restriction* of π to C, namely all formulas $\phi(x) \in \mathcal{L}(C)$ implied by π (i.e., $\pi \vdash \phi(x)$). Similarly, if x' is a sub-tuple of x, the restriction of π to x' is the partial type consisting of all formulas in x' implied by π.

A (complete) type over B is a maximal partial type over B. We denote the space of types over B in x by $S^x(B)$. It is a compact Hausdorff topological space in the logic topology (a basic open set has the form $\{p \in S^x(B) \mid \phi(x) \in p\}$). For $a \in \mathcal{U}^x$, write $\mathrm{tp}(a/B)$ for the type of a over B. $S(B)$ is the union of all types over B.

For an $\mathcal{L}$-formula $\phi(x, y)$, an *instance* of ϕ over $B \subseteq \mathcal{U}$ is a formula $\phi(x, b)$ where $b \in B^y$, and a (complete) ϕ*-type* over B is a maximal partial type consisting of instances and negations of instances of ϕ over B. We write $S_\phi(B)$ for the space of ϕ-types over B in x (in this notation we keep in mind the partition (x, y), and x is the first tuple there). As above, it is a compact Hausdorff topological space in the logic topology. For $a \in \mathcal{U}^x$, we write $\mathrm{tp}_\phi(a/B) \in S_\phi(B)$ for its ϕ-type over B. We also use the notation $\phi^1 = \phi$ and $\phi^0 = \neg\phi$. When $p(x) \in S(B)$ is a type, we write $p \restriction \phi \in S_\phi(B)$ for the complete ϕ-type over B implied by p. If Δ is a set of partitioned formulas $\phi(x, y)$, we define $S_\Delta(B)$ and the restriction $p \restriction \Delta \in S_\Delta(B)$ similarly.

We will also consider the case where $B \subseteq \mathcal{U}^y$ and (abusing notation) define $S_\phi(B)$ similarly — this should never cause a confusion.

If $\pi(x)$ is a small partial type (over some small set contained in $\mathcal{U}$), we write $S^\pi_\phi(B)$ for the closed subspace of $S^x_\phi(B)$ consisting of the ϕ-types which are consistent with π.

Generally we do not limit our discussion to finite tuples of variables (but in the context of ϕ-types for a formula ϕ this does not matter).

We write $A \subseteq_{\mathrm{fin}} B$ to mean that A is a finite subset of B.

2.2. **Global and invariant types.** For $A \subseteq \mathcal{U}$, an A**-invariant** type is a *global* type, i.e., a type over $\mathcal{U}$, which is invariant under the action of $\mathrm{Aut}(\mathcal{U}/A)$, the group of automorphisms fixing A pointwise.

For a sequence $(x_i)_{i\in I}$ and $j \in I$, we write $x_{<j}$ for $(x_i)_{i<j}$, and similarly for $x_{\leq j}$.

Definition 2.1. If $q(x)$ and $r(y)$ are A-invariant global types, then the type $(q \otimes r)(x, y)$ is defined to be $\mathrm{tp}(a, b/\mathcal{U})$ (in a bigger monster model) for any $b \vDash r$ and $a \vDash q|_{\mathcal{U}b}$ (here we understand q to mean its unique A-invariant extension to a bigger model). (This can also be defined without stepping outside of the monster model, see [Sim15a, Chapter 2].)

We define $q^{(n)}(x_{<n})$ for $n < \omega$ by induction: $q^{(1)}(x_0) = q(x_0)$,

$$q^{(n+1)}(x_{\leq n}) = q(x_n) \otimes q^{(n)}(x_{<n}),$$

and $q^{(\omega)}(x_{<\omega}) = \bigcup_{n<\omega} q^{(n)}$.

[1]There are set theoretic issues in assuming that such a model exists, but these are overcome by standard techniques from set theory that ensure the generalised continuum hypothesis from some point on while fixing a fragment of the universe. The reader can just accept this or alternatively assume that $\mathcal{U}$ is merely κ-saturated and κ-strongly homogeneous for large enough κ.

For any linear order $(X,<)$, we can define $q^{(X)}(x_i \mid i \in X)$ similarly, as the union of $q^{(X_0)}(x_i \mid i \in X_0)$ for every finite $X_0 \subseteq X$.

Fact 2.2. *[Sim15a, Chapter 2] Given a global A-invariant type q and a linear order $(X,<)$, $q^{(X)}$ is an A-invariant global type. In addition, it is the type of an indiscernible sequence over $\mathcal{U}$.*

For any small set $B \supseteq A$, $q^{(X)}|_B$ is given by $\operatorname{tp}((a_i \mid i \in X)/B)$ where $a_i \vDash q|_{Ba_{<i}}$. This is a **Morley sequence of** q **over** B *(indexed by X).*

2.3. VC-dimension and NIP.

Definition 2.3 (VC-dimension)**.** Let X be a set and $\mathcal{F} \subseteq \mathcal{P}(X)$. The pair $(X,\mathcal{F})$ is called a **set system**. We say that $A \subseteq X$ is **shattered** by $\mathcal{F}$ if for every $S \subseteq A$ there is $F \in \mathcal{F}$ such that $F \cap A = S$. A family $\mathcal{F}$ is said to be a **VC-class** on X if there is some $n < \omega$ such that no subset of X of size n is shattered by $\mathcal{F}$. In this case the **VC-dimension of** $\mathcal{F}$, denoted by $\operatorname{vc}(\mathcal{F})$, is the smallest integer n such that no subset of X of size $n+1$ is shattered by $\mathcal{F}$.

If no such n exists, we write $\operatorname{vc}(\mathcal{F}) = \infty$.

Definition 2.4. Suppose T is an $\mathcal{L}$-theory and $\phi(x,y)$ is a formula. Say $\phi(x,y)$ is **NIP** if for some/every $M \vDash T$, the family $\{\phi(M^x,a) \mid a \in M^y\}$ is a VC-class.

The theory T is **NIP** if all formulas are NIP.

Definition 2.5. Suppose T is an $\mathcal{L}$-theory and $\phi(x,y)$ is an NIP formula. Let $\operatorname{vc}(\phi)$ be the VC-dimension of $\{\phi(M^x,a) \mid a \in M^y\}$, where M is any (some) model of T. Note that this definition depends on the partition of variables.

Let ϕ^{opp} be the partitioned formula $\phi(y,x)$ (it is the same formula with the partition reversed). Let $\operatorname{vc}^*(\phi) = \operatorname{vc}(\phi^{\text{opp}})$ be the **dual** VC-dimension of ϕ.

Fact 2.6. *[Sim15a, Lemma 6.3] Suppose $\mathcal{F}$ is a VC-class on X. Let $\mathcal{F}^* = \{\{s \in \mathcal{F} \mid x \in s\} \mid x \in X\} \subseteq \mathcal{P}(\mathcal{F})$ be the* **dual** *of $\mathcal{F}$. Then $\mathcal{F}$ is a VC-class iff $\mathcal{F}^*$ is, and moreover* $\operatorname{vc}^*(\mathcal{F}) := \operatorname{vc}(\mathcal{F}^*) < 2^{\operatorname{vc}(\mathcal{F})+1}$.

Remark 2.7. By Fact 2.6, ϕ is NIP iff ϕ^{opp} is NIP, and $\operatorname{vc}(\phi^{\text{opp}}) = \operatorname{vc}^*(\phi) < 2^{\operatorname{vc}(\phi)+1}$.

By [Sim15a, Lemma 2.9], a Boolean combination of NIP formulas is NIP. In particular, if $\phi_i(x,y_i)$ is NIP for $i < k$ then so is $\bigwedge_{i<k} \phi_i(x,y_i)$. We end this subsection by giving an explicit bound on its VC-dimension; see also [DKL84, Theorem 9.2.6]. (This will be used only in Section 5.3.).

Definition 2.8. Let $\binom{s}{\le k} = \sum_{i\le k} \binom{s}{i}$, and let $B_{\text{vc}}(n,k) = \max\{s \in \mathbb{N} \mid \binom{s}{\le k}^n \ge 2^s\}$.

Remark 2.9. For all $1 \le n,k \in \mathbb{N}$, $B_{\text{vc}}(n,k) \ge n,k$.

Indeed, $\binom{n}{\le k} \ge 2$, so $\binom{n}{\le k}^n \ge 2^n$. Hence $B_{\text{vc}}(n,k) \ge n$. Similarly, to show that $B_{\text{vc}}(n,k) \ge k$, note that $\binom{k}{\le k}^n = 2^{kn} \ge 2^k$.

Lemma 2.10. *Let $k \in \mathbb{N}$, and let $\phi_1(x;y_0),\dots,\phi_n(x;y_{n-1})$ be partitioned formulas with $\operatorname{vc}(\phi_i) \le k$. Let $\theta(x;y_0,\dots,y_{n-1}) = \bigwedge_{i<n} \phi_i(x,y_i)$. Then $\operatorname{vc}(\theta) \le B_{\text{vc}}(n,k)$.*

Proof. Let $s > B_{\text{vc}}(n,k)$, and let $A_0 \subseteq \mathcal{U}^x$ with $|A_0| = s$. For each i, by Sauer-Shelah [Sim15a, Lemma 6.4], at most $\binom{s}{\le k}$ subsets of A_0 are defined by instances of ϕ_i; hence at most $\binom{s}{\le k}^n$ are defined by instances of θ. It follows from the definition of $B_{\text{vc}}(n,k)$ that θ does not shatter A_0. □

2.4. Compressible types.

Here we will review the basic properties of compressible types.

Definition 2.11. A type $p(x) \in S(A)$ is **compressible** if for any formula $\phi(x, y)$ there is a formula $\psi(x, z)$ such that for every finite set $A_0 \subseteq A$, there is some $c \in A^z$ such that

- $\psi(x, c) \in p$ and
- $\psi(x, c) \vdash (p \upharpoonright \phi)|_{A_0}$ (i.e., it implies the set $\{\phi(x, a) \mid \phi(x, a) \in p, \ a \in A_0^y\} \cup \{\neg\phi(x, a) \mid \phi(x, a) \notin p, \ a \in A_0^y\}$).

Suppose $A \subseteq \mathcal{U}$. Given $a \in \mathcal{U}^x$ any tuple, we let (A, a) be the structure with universe A and the induced structure coming from a-definable sets. In other words, for every formula $\phi(x, y)$, there is a relation $R_\phi(y)$ interpreted by $R_\phi(c)$ iff $\mathcal{U} \models \phi(a, c)$ for any $c \in A^y$. Note that if $M \equiv (A, a)$ then $M \cong (A', a)$ for some $A' \subseteq \mathcal{U}$, and moreover if $M \succ (A, a)$ then there is such an $A' \subseteq \mathcal{U}$ such that M and (A', a) are isomorphic over A. Thus, whenever we have such a structure, we will always assume it has the form (A', a) for some $A' \subseteq \mathcal{U}$.

This construction preserves useful information on the type $\operatorname{tp}(a/A)$. For example, recall that a type $p(x) \in S(A)$ is *definable* if for every formula $\phi(x, y)$, the set $\{a \in A^y \mid \phi(x, a) \in p\}$ is definable over A. It is easy to see that if $\operatorname{tp}(a/A)$ is definable and $(A', a') \equiv (A, a)$ then $\operatorname{tp}(a'/A')$ is also definable (with the same definition scheme). Moreover, we have:

Fact 2.12. *[Sim20, Lemma 3.2] If* $\operatorname{tp}(a/A)$ *is compressible and* $(A', a') \equiv (A, a)$*, then so is* $\operatorname{tp}(a'/A')$*.*

Compactness gives the following equivalent definition of compressibility:

Fact 2.13. *The type* $p = \operatorname{tp}(a/A)$ *is compressible iff for any (some)* $|A|^+$*-saturated elementary extension* $(A', a) \succ (A, a)$ *and any formula* $\phi(x, y)$*, there is some formula* $\psi(x, z)$ *and* $d \in (A')^z$ *such that* $\psi(a, d)$ *holds and* $\psi(x, d) \vdash (p \upharpoonright \phi)|_A$*.*

In fact, this was the original definition of compressibility in [Sim20, Definition 3.1].

We give another useful characterisation of compressible types. Recall that two types $p(x)$, $q(y)$ over A are *weakly orthogonal* if $p \cup q$ implies a complete type in x, y over A.

A type q is *finitely satisfiable* in some set A if every formula from q is realised in A. We write $S^x_{A\text{-fs}}(B) \subseteq S^x(B)$ for the subspace consisting of those types in x which are finitely satisfiable in A. Recall that such types can be extended to global types in $S^x_{A\text{-fs}}(\mathcal{U})$ (using ultrafilters). Note that $S^x_{A\text{-fs}}(B)$ is a closed (and hence compact) subspace of $S^x(B)$. As usual, omitting the x means taking all types (allowing infinite (small) tuples).

Fact 2.14. *[Sim20, Lemma 3.3] (T arbitrary) The following are equivalent for* a, A *and an* $|A|^+$*-saturated extension* $(A', a) \succ (A, a)$*:*

(1) $\operatorname{tp}(a/A)$ *is compressible.*
(2) For all $q(y) \in S_{A\text{-}fs}(A')$*,* $\operatorname{tp}(a/A')$ *(as a type in* x*) and* $q(y)$ *are weakly orthogonal.*
(3) For all $q(y) \in S_{A\text{-}fs}(A')$*,* $\operatorname{tp}(a/A')$ *(as a type in* x*) and* $q(y)$ *imply a complete type in* xy *over* $\emptyset$*.*

Corollary 2.15. *(T arbitrary) A type* $p(x) = \operatorname{tp}(a/A)$ *is compressible if and only if there is some (possibly infinite)* $d \subseteq \mathcal{U}$ *of length* $|d| \leq |T|$ *such that*

- $\operatorname{tp}(d/Aa)$ *is finitely satisfiable in* A*, and*
- *for every* $q \in S_{A\text{-}fs}(\mathcal{U})$*,*

$$q|_{Ad} \vdash q|_{Aa}.$$

Proof. Suppose $\mathrm{tp}(a/A)$ is compressible. Let $\phi(x, y, w)$ be an $\mathcal{L}$-formula. Then there is $\zeta(x, z)$ such that $\{\bigvee_{\epsilon<2} \forall x(\zeta(x, z) \to \phi(x, c', c)^\epsilon) \mid c' \in A^y, c \in A^w\} \cup \{\zeta(a, z)\}$ is finitely satisfiable in A, so let e_ϕ realise a completion in $S^z_{A\text{-fs}}(A)$. If $q \in S^y_{A\text{-fs}}(\mathcal{U})$ and $c \in A^w$, then $q|_{Ae_\phi}(y) \vdash \bigvee_{\epsilon<2} \forall x(\zeta(x, e_\phi) \to \phi(x, y, c)^\epsilon)$ by finite satisfiability, and $\vDash \zeta(a, e_\phi)$, so $q|_{Ae_\phi}(y) \vdash \phi(a, y, c)^\epsilon$ for some $\epsilon < 2$. So $d := (e_\phi)_{\phi(x,y,w)\in\mathcal{L}}$ is as required.

The other direction follows from Fact 2.14(3⇒1), since by saturation we can assume $d \subseteq A'$. □

3. Density of (local) compressibility

Here we will prove that (local) compressible types are dense. In Section 3.1 we prove an abstract version of this dealing with set systems of finite VC-dimension (generalising [CCT16, Lemma 4] to infinite sets). Then in Section 3.2 we deduce that locally compressible types are dense for NIP formulas, and in Section 3.3 we deduce that compressible types are dense in countable NIP theories.

3.1. Compressibility for set systems of finite VC-dimension.

Let A be a (possibly infinite) set. As usual, 2^A is the (Hausdorff compact) space of functions $A \to 2 = \{0, 1\}$ equipped with the product topology. Any $\mathcal{C} \subseteq 2^A$ naturally induces a set system on A (those sets whose characteristic functions are in $\mathcal{C}$) and as such has a VC-dimension $\mathrm{vc}(\mathcal{C})$. For $\mathcal{C} \subseteq 2^A$ and $B \subseteq A$, let $\mathcal{C}|_B := \{c|_B \mid c \in \mathcal{C}\}$, the set of restrictions to B.

Let $\mathcal{C} \subseteq 2^A$.

Definition 3.1.
- For $B \subseteq A$ and $c' \in \mathcal{C}|_B$, define the relativisation $\mathcal{C}_{c'} := \{c \in \mathcal{C} \mid c|_B = c'\} = \{c \in \mathcal{C} \mid c \supseteq c'\}$.
- For $c \in \mathcal{C}$ and $B, C \subseteq A$, write $c|_B \vdash_{\mathcal{C}} c|_C$ to mean that $c'|_C = c|_C$ for any $c' \in \mathcal{C}$ with $c'|_B = c|_B$.
- For $k < \omega$, say $c \in \mathcal{C}$ is **k-compressible in $\mathcal{C}$** if for any finite $A_0 \subseteq A$ there exists $A_1 \subseteq A$ with $|A_1| \leq k$ such that $c|_{A_1} \vdash_{\mathcal{C}} c|_{A_0}$.
- Say c is **compressible** in $\mathcal{C}$ if it is k-compressible for some $k < \omega$.

Remark 3.2. This terminology is originally inspired by, but does not precisely agree with, the terminology around compression schemes in the statistical learning literature.

Remark 3.3. Suppose $(P, \leq)$ is a directed partial order, and $c : P \to r < \omega$ is some colouring. Then there is some subset $X \subseteq P$ which is monochromatic ($X \subseteq c^{-1}(i)$ for some $i < r$) and cofinal (for all $p \in P$ there is some $q \in X$ such that $q \geq p$).

Indeed, if not, then for every $i < r$ there is some $p_i \in P$ such that $c(q) \neq i$ for all $q \geq p_i$. Let $p \geq p_i$ for all $i < r$. Then $c(p) \neq i$ for all $i < r$, contradiction.

The proof of the following theorem is an adaptation to the case of infinite A of the proof of [CCT16, Lemma 4], which proves it for finite A with the same bound on k.

Theorem 3.4. *For any $d < \omega$, let $k_{\mathrm{comp}}(d) := 2^{d+1}(d-2)+d+4$. For any set A, if $\mathcal{C} \subseteq 2^A$ is closed, non-empty, and has VC-dimension $\leq d$, then there exists $c \in \mathcal{C}$ which is $k_{\mathrm{comp}}(d)$-compressible in $\mathcal{C}$.*

Proof. The proof is by induction on d.

If $\mathrm{vc}(\mathcal{C}) = 0$, then $\mathcal{C}$ is a singleton $\{c\}$, and c is clearly 0-compressible in $\mathcal{C}$.

Suppose that $\mathrm{vc}(\mathcal{C}) = d + 1 > 0$.

Claim 3.5. *Let $k_0 := 2^{d+1}d + 1$. There is $D \subseteq A$ and $c' \in \mathcal{C}|_D$ which is k_0-compressible in $\mathcal{C}|_D$ such that $\mathrm{vc}(\mathcal{C}_{c'}) \leq d$.*

Proof. We may assume $|A| \geq k_0$, as otherwise the result is immediate (take $D = A$ and any $c' \in \mathcal{C}$).

Let $\mathcal{S} := \{c' \in \mathcal{C}|_D \mid D \subseteq A,\ c' \text{ is } k_0\text{-compressible in } \mathcal{C}|_D\}$. Equip $\mathcal{S}$ with the partial order of inclusion, i.e., $\mathcal{C}|_{D_1} \ni c'_1 \leq c'_2 \in \mathcal{C}|_{D_2}$ iff $D_1 \subseteq D_2$ and $c'_2|_{D_2} = c'_1$.

Then $\mathcal{S}$ is closed under unions of chains. Indeed, if $(c'_i)_{i\in I}$ is a chain with $c'_i \in \mathcal{C}|_{D_i}$, then the sets of extensions to $\mathcal{C}$, $\{c' \in \mathcal{C} : c'|_{D_i} = c'_i\}$, form a chain $\mathcal{F}$ of closed non-empty subsets of $\mathcal{C}$; but $\mathcal{C}$ is closed in 2^A hence compact, so $\mathcal{F}$ has non-empty intersection. Hence $\bigcup_{i\in I} c'_i \in \mathcal{C}|_{\bigcup_{i\in I} D_i}$. Meanwhile, $\bigcup_{i\in I} c'_i$ is k_0-compressible since each c'_i is.

So by Zorn's lemma, $\mathcal{S}$ has a maximal element $c' \in \mathcal{C}_D$.

We conclude by showing that $\mathrm{vc}(\mathcal{C}_{c'}) \leq d$.

Otherwise, $2^B \subseteq \mathcal{C}_{c'}$ for some $B \subseteq A$ with $|B| = d + 1$. Note that $B \cap D = \emptyset$. We claim that there is $e \in 2^B$ such that $c' \cup e \in \mathcal{S}$, contradicting maximality of c'.

Indeed, let $A_0 \subseteq_{\mathrm{fin}} D \cup B$ and let $D_0 := A_0 \cap D$. Then there is $D_1 \subseteq D$ with $|D_1| \leq k_0$ such that $c'|_{D_1} \vdash_{\mathcal{C}_D} c'|_{D_0}$. In fact we may take D_1 with $|D_1| = k_0$, since $|D| \geq k_0$ by maximality and the assumption that $|A| \geq k_0$. Since $\mathrm{vc}(\mathcal{C}) \leq d+1 = |B|$ and $2^B \subseteq \mathcal{C}_{c'}$, for each $a \in D_1$ there is $e_a \in 2^B$ such that $e_a \vdash_{\mathcal{C}} c'|_{\{a\}}$. By the choice of $k_0 = 2^{d+1}d+1$ and the pigeonhole principle, there exist $e \in 2^B$ and $E \subseteq D_1$ such that $|E| = d+1$ and $e \vdash_{\mathcal{C}} c'|_E$. Let $A_1 := (D_1 \cup B) \setminus E$, so $|A_1| = |D_1| = k_0$. Then

$$(c' \cup e)|_{A_1} \vdash_{\mathcal{C}} (c' \cup e)|_{A_0}. \tag{1}$$

In this way we obtain a 2^B-colouring of the partial order of finite subsets of $D \cup B$, where each finite $A_0 \subseteq D \cup B$ is coloured with an $e \in 2^B$ such that (1) holds for some $A_1 \subseteq D \cup B$ with $|A_1| = k_0$. By Remark 3.3 there is a cofinal monochromatic subset, yielding $e \in 2^B$ which is as required. □

Now by the induction hypothesis there is $c \in \mathcal{C}_{c'}$ which is $k_{\mathrm{comp}}(d)$-compressible in $\mathcal{C}_{c'}$. We conclude by showing that c is $(k_{\mathrm{comp}}(d) + k_0)$-compressible in $\mathcal{C}$; this gives the stated bound, since $(2^{d+1}(d-2) + d + 4) + (2^{d+1}d + 1) = 2^{(d+1)+1}((d + 1) - 2) + (d + 1) + 4$.

So suppose $A_0 \subseteq_{\mathrm{fin}} A$ and let $A_1 \subseteq A$ be such that $|A_1| \leq k_{\mathrm{comp}}(d)$ and $c|_{A_1} \vdash_{\mathcal{C}_{c'}} c|_{A_0}$. By compactness of $\mathcal{C}$ it follows that there is a finite subset $D_0 \subseteq D$ such that $c|_{A_1} \vdash_{\mathcal{C}_{c'|_{D_0}}} c|_{A_0}$.

Let D_1 be such that $|D_1| \leq k_0$ and $c'|_{D_1} \vdash_{\mathcal{C}|_D} c'|_{D_0}$ (which exists as $c' \in \mathcal{S}$). Then $c|_{A_1 \cup D_1} \vdash_{\mathcal{C}} c|_{A_0}$, as required. □

Remark 3.6. For finite A, the exponential dependency of $k_{\mathrm{comp}}(d)$ on d obtained in [CCT16] was improved to a quadratic dependency in [HWLW17]. Conjecturally it is even linear (see the introduction to [HWLW17]). The proof of this quadratic bound does not adapt so readily to the infinite case, and it would be interesting to find the best bound, and in particular to see whether Theorem 3.4 holds with a quadratic bound.

3.2. **Density of compressible local types.** In the following definition, we use the notation $p \vdash_\pi q$ for $p \cup \pi \vdash q$, where p, π are small partial types and q a finite partial type (this is compatible with the notation in Definition 3.1 when p, q are complete ϕ-types, and $\mathcal{C}$ is the set of ϕ-types consistent with π). As usual, we work in a complete $\mathcal{L}$-theory T.

Definition 3.7. Fix a formula $\phi(x, y)$, a parameter set $A \subseteq \mathcal{U}^y$ and a small partial type $\pi(x)$. Recall the notation $S^\pi_\phi(A)$ from Section 2.1.

- $p \in S^\pi_\phi(A)$ is **k-compressible modulo π** if it is compressible in $S^\pi_\phi(A)$ considered as a (closed) subspace of 2^A as in Definition 3.1: for any finite $A_0 \subseteq A$ there is $A_1 \subseteq A$ with $|A_1| \leq k$ such that $p|_{A_1}(x) \vdash_\pi p|_{A_0}(x)$.

- $p \in S^\pi_\phi(A)$ is **$\star$-compressible modulo** π if it is k-compressible modulo π for some $k < \omega$.
- $p \in S_\phi(A)$ is **k-** resp. **$\star$-compressible** if it is k- resp. $\star$-compressible modulo $x = x$.
- $S^\pi_{\phi\downarrow k}(A) \subseteq S^\pi_\phi(A)$ (respectively $S^\pi_{\phi\downarrow\star}(A)$) is the space of k-compressible (respectively $\star$-compressible) ϕ-types modulo π. When $A \subseteq \mathcal{U}$, $S^\pi_{\phi\downarrow k}(A) = S^\pi_{\phi\downarrow k}(A^y)$.

Remark 3.8. In terms of Borel complexity, if A is countable then $S^\pi_{\phi\downarrow\star}(A)$ is a $\mathbf{\Sigma^0_3}$-subset of $S^\pi_\phi(A)$: it is a countable union (going over all k) of countable intersections (going over all finite subsets of A) of countable unions (going over all subsets of A of size $\leq k$) of clopen sets (the implication).

Corollary 3.9. *Let $\phi(x, y)$ be a formula, $d \in \mathbb{N}$, $A \subseteq \mathcal{U}^y$ and $\pi(x)$ a small partial type. Suppose that $\phi(x, y)$ is NIP and that $\mathrm{vc}^*(\phi) \leq d$. Let $k = k_{\mathrm{comp}}(d)$ be as in Theorem 3.4.*

(i) $S^\pi_{\phi\downarrow k}(A) \neq \emptyset$.
(ii) *If $A' \subseteq A$, then any $p' \in S^\pi_{\phi\downarrow l}(A')$ extends to some $p \in S^\pi_{\phi\downarrow(l+k)}(A)$.*
(iii) *$S^\pi_{\phi\downarrow\star}(A)$ is dense in $S^\pi_\phi(A)$.*

Proof. (i) This is immediate from Theorem 3.4 by identifying $\mathcal{C}$ with $S^\pi_\phi(A)$.

(ii) By (i), there is some $p \in S^{\pi\cup p'}_{\phi\downarrow k}(A)$. Then if $A_0 \subseteq_{\mathrm{fin}} A$, there is $A_1 \subseteq A$ with $|A_1| \leq k$ such that $p' \cup p|_{A_1} \vdash_\pi p|_{A_0}$. By compactness, there is a finite $A'_0 \subseteq A'$ such that $p'|_{A'_0} \cup p|_{A_1} \vdash_\pi p|_{A_0}$, and then by l-compressibility modulo π of p' we have $p'|_{A'_1} \vdash_\pi p'|_{A'_0}$ for some $A'_1 \subseteq A'$ with $|A'_1| \leq l$. Then $p|_{A'_1\cup A_1} = p'|_{A'_1} \cup p|_{A_1} \vdash_\pi p|_{A_0}$. So p is $(l + k)$-compressible modulo π.

(iii) A basic open subset of $S^\pi_\phi(A)$ is of the form $S^{\pi\cup p'}_\phi(A)$ where $p' \in S^\pi_\phi(A')$ and $A' \subseteq_{\mathrm{fin}} A$. Clearly $p' \in S^\pi_{\phi\downarrow|A'|}(A')$, so by (ii) there is $p \in S^\pi_{\phi\downarrow(|A'|+k)}(A)$ extending p'.

□

Corollary 3.10. *The following are equivalent for a formula $\phi(x, y)$ and a partial type $\pi(x)$.*

(1) For some $\psi(x)$ such that $\pi \vdash \psi$, $\psi(x) \wedge \phi(x, y)$ is NIP.
(2) There exists $k < \omega$ such that for any set $A \subseteq \mathcal{U}^y$, $S^\pi_{\phi\downarrow k}(A) \neq \emptyset$.
(3) For any set $A \subseteq \mathcal{U}^y$, there exists $k < \omega$ such that $S^\pi_{\phi\downarrow k}(A) \neq \emptyset$.

Proof. (1) implies (2) is Corollary 3.9(i) (any type in $S^\pi_{\psi\wedge\phi\downarrow k}(A)$ naturally induces one in $S^\pi_{\phi\downarrow k}(A)$) and (2) implies (3) is clear.

$\neg$(1) implies $\neg$(3). By $\neg$(1), $\mathcal{U}^y$ is infinite. By compactness there is $A := \{a_i \mid i < \omega\} \subseteq \mathcal{U}^y$ such that for any $A' \subseteq A$, there is some $b_{A'} \vDash \pi$ such that $\phi(b_{A'}, a)$ holds iff $a \in A'$ for any $a \in A$. Suppose $p \in S^\pi_{\phi\downarrow k}(A)$ for some $k < \omega$. Then for some $A_0 \subseteq A$ of size $\leq k$, $p|_{A_0} \vdash_\pi p|_{a_{\leq k}}$. But setting $A' := \{a \in A_0 : \phi(x, a) \in p\} \cup \{a \in A \setminus A_0 : \neg\phi(x, a) \in p\}$, we have $b_{A'} \vDash p|_{A_0}$; but $b_{A'} \nvDash p|_a$ for any $a \in A \setminus A_0$, and $a_{\leq k} \not\subseteq A_0$ since $|A_0| \leq k$. □

This gives a new characterisation of NIP types.

Definition 3.11. We say that a partial type $\pi(x)$ has **IP** if there is a formula $\phi(x, y) \in \mathcal{L}$ which has IP as witnessed by realisations of π, i.e., if $\mathrm{vc}(\{\phi(\pi(\mathcal{U}), a) : a \in \mathcal{U}^y\}) = \infty$. A formula or a partial type is **NIP** if it does not have IP.

By compactness we have that:

Remark 3.12. A partial type $\pi(x)$ is NIP iff for every formula $\phi(x, y)$ there is a formula $\psi(x)$ implied by π such that $\psi(x) \wedge \phi(x, y)$ is NIP (as a formula over $\mathcal{U}$).

By Corollary 3.10 and Remark 3.12 we have:

Corollary 3.13. *A partial type $\pi(x)$ is NIP iff for every formula $\phi(x,y)$ and $A \subseteq \mathcal{U}^y$ there is $k < \omega$ such that $S^{\pi}_{\phi\downarrow k}(A) \neq \emptyset$.*

3.3. **Density of compressible types in countable NIP theories.** Now we turn from local types to types.

Definition 3.14. Let $\pi(x) \subseteq \pi'(x)$ be partial types over a parameter set $A \subseteq \mathcal{U}$ (perhaps contained in a collection of sorts).

- A formula $\zeta(x,z)$ **compresses** π **within** π' **with respect to** A if for any finite $A_0 \subseteq A$ there exists $a \in A^z$ such that
$$\pi'(x) \vdash \zeta(x,a) \vdash \pi|_{A_0}(x).$$
If π' is clear from the context we omit it.
- π is **compressible within** π' **with respect to** A if some ζ compresses π within π' with respect to A.
- π is **t-compressible**[2] with respect to A if π is compressible within π with respect to A.

Let $p(x) \in S(A)$.

(1) p is **compressible** if for each formula $\phi(x,y)$, the restriction $p \restriction \phi \in S_\phi(A)$ of p to a ϕ-type is compressible within p with respect to A.
(2) p is **strongly compressible** if for each formula ϕ there exists a finite set of formulas $\Delta \ni \phi$ such that $p \restriction \Delta$ is t-compressible with respect to A.

Remark 3.15. Note that the definition above of a compressible type is the same as Definition 2.11.

Remark 3.16. The reason we say "with respect to A" in the definition is because a partial type over A is also a partial type over any set containing A. In the future we will usually omit this since A will be clear from the context.

Remark 3.17. As we said in Section 2.1, we do not restrict ourselves to finitary types. Note that $p \in S^x(A)$ is compressible iff all of its restrictions to finite tuples of variables are compressible.

Remark 3.18. The relations between these definitions and the definitions for ϕ-types in Definition 3.7 are slightly subtle. In particular, for $A \subseteq \mathcal{U}^y$ and a ϕ-type $p \in S_\phi(A)$, the condition that p is $\star$-compressible (i.e., k-compressible for some k) is strictly stronger than the condition that p is t-compressible. For example, in $\mathrm{Th}(\mathbb{N}; <)$, the non-realised $(x=y)$-type in $S_{x=y}(\mathbb{N})$ is t-compressed by $x > z$, but is not k-compressible for any $k < \omega$.

Remark 3.19. Note that for a model M and $p \in S(M)$, p is (strongly) compressible iff its unique extension p^{eq} to M^{eq} is (strongly) compressible (by translating formulas in $\mathcal{L}^{\mathrm{eq}}$ to formulas in $\mathcal{L}$, see [Pil96, Lemma 1.1.4]).

Lemma 3.20. *Let $\pi(x)$ be a t-compressible partial type over a set $A \subseteq \mathcal{U}^y$, and let $\phi(x,y)$ be an NIP formula. Then there exists $p_\phi \in S_\phi(A)$ such that $\pi \cup p_\phi$ is consistent and t-compressible.*

Moreover, there is a formula $\xi(x,w)$ which is a Boolean combination (depending only on $\mathrm{vc}^*(\phi)$) of instances of ϕ and equality such that if $\zeta(x,w')$ t-compresses π (i.e., compresses π within itself) then $\zeta(x,w') \wedge \xi(x,w)$ t-compresses $\pi \cup p_\phi$.

[2]The letter 't' stands for totally, thoroughly, or typewise.

Proof. We may assume $|A| > 1$, as otherwise the result is clear.

By Corollary 3.9(i), there is $p_\phi \in S^{\pi}_{\phi\downarrow k}(A)$ for some $k < \omega$ depending only on $\mathrm{vc}^*(\phi)$. By a coding of finitely many formulas as one as in the proofs of e.g. [She90, Theorem II.2.12(1)] and [Gui12, Lemma 2.5], we obtain a formula $\xi(x,w)$ such that for any finite $A_0 \subseteq A$, there is $a \in A^w$ such that $p_\phi(x) \vdash \xi(x,a)$ and $\pi \cup \{\xi(x,a)\} \vdash p_\phi|_{A_0}$. Explicitly, we may take $\xi(x,w)$ with $w = (w^i_j)_{i<3,j<k}$ to be $\bigwedge_{j<k}(\phi(x,w^0_j) \leftrightarrow w^1_j = w^2_j)$. Then, for any finite A_0, there is some $(\epsilon_j)_{j<k} \in 2^k$ and $(c_j)_{j<k} \in A^k$ such that $\bigwedge_{j<k} \phi(x,c_j)^{\epsilon_j} \vdash p|_{A_0}$. Let $d_0 \neq d_1 \in A$. For $j < k$, let $a^0_j = c_j$, $a^1_j = d_0$, and let $a^2_j = d_0$ if $\epsilon_i = 1$ and otherwise let $a^2_j = d_1$. Finally, let $a = (a^i_j)_{i<3,j<k}$.

Now assume that $\zeta(x,w')$ is as in the lemma and fix some finite set $A_0 \subseteq A$. Let $a \in A^w$ be as above. By compactness there is a finite $A'_0 \subseteq A$ such that $A_0 \subseteq A'_0$ and $\pi|_{A'_0} \cup \{\xi(x,a)\} \vdash p_\phi|_{A_0}$, and so (by the assumption on ζ) there is $a' \in A^{w'}$ such that $(\pi \cup p_\phi)(x) \vdash (\zeta(x,a') \wedge \xi(x,a)) \vdash (\pi \cup p_\phi)|_{A_0}$. □

Corollary 3.21. *(T countable NIP) Suppose $A \subseteq \mathcal{U}$ is a set of parameters and x is a countable tuple of variables. Then, compressible types are dense in $S^x(A)$:*

If $\theta(x)$ is a consistent formula over A, then there exists a compressible type $p(x) \in S(A)$ with $p(x) \vdash \theta(x)$.

More generally, if $\pi(x)$ is a t-compressible partial type over A, then there exists a strongly compressible $p \in S(A)$ with $\pi \subseteq p$.

Proof. Clearly it is enough to prove the "more generally" part, so assume π is t-compressible and ζ compresses π within π.

Enumerate the formulas $\phi(x,y)$ as $(\phi_i(x,y_i))_{i<\omega}$ (where the y_i's are finite), with $\phi_0 = \zeta$. For $i < \omega$, let $\Delta_i = \{\phi_j \mid j < i\} \cup \{x = y\}$. Let $\pi_0 = \pi$. Recursively applying Lemma 3.20, let $p_{\phi_i} \in S_{\phi_i}(A)$ be such that $\pi_{i+1} := \pi_i \cup p_{\phi_i}$ is t-compressible, and moreover is compressed by a Boolean combination of formulas from Δ_{i+1}. Then each $\pi_i \restriction \Delta_i$ is t-compressible, and so $p := \bigcup_{i<\omega} \pi_i$ is strongly compressible. □

For an example showing the necessity of the countability assumption, see Remark 4.11 below.

Remark 3.22. It follows from Corollary 3.10 that Lemma 3.20 characterises ϕ being NIP (letting π be the empty type). However, Corollary 3.21 does not characterise NIP for countable theories. An easy example is $\mathrm{Th}(\mathbb{N},+,\cdot)$, and in fact any theory with IP in which $\mathrm{dcl}(A)$ is a model for any set A (given a consistent formula $\theta(x)$ over a set A, let $c \vDash \theta$ be in $\mathrm{dcl}(A)$, then $\mathrm{tp}(c/A)$ is compressible and even isolated).

Question 3.23. We could consider an apparently weaker notion of compressibility of a type: say $p \in S(A)$ is **weakly compressible** if for any formula $\phi(x,y)$ there is some formula $\zeta(x,z)$ such that for any finite $A_0 \subseteq A$ there is some $d \in \mathcal{U}^z$ such that $p \vdash \zeta(x,d)$ and $\zeta(x,d) \vdash (p \restriction \phi)|_{A_0}$. Note that if the base A is a model, then weak compressibility is equivalent to compressibility, but for general sets it is less clear. In Example 6.26 below we will see that if T is the theory of atomless Boolean algebras, this can fail. Is it true that if T is NIP then p is weakly compressible iff p is compressible?

4. Compressibility and stability

Here we discuss compressibility in the context of stability, in both the local and global senses, and point out that compressibility is equivalent to l-isolation (see Definition 4.6) in these contexts. The main results are:

- For stable formulas, k-compressibility is equivalent to k-isolation (Lemma 4.3).

- For stable types, compressibility is equivalent to l-isolation (Lemma 4.8), and in particular when T is stable these two notions are the same.
- For generically stable types, compressibility is equivalent to l-isolation (Proposition 4.14).

4.1. **Stable formulas.** Recall that a formula $\phi(x, y)$ is **stable** if it does not have the order property: there are no $(a_i, b_i)_{i<\omega}$ such that $\phi(a_i, b_j)$ holds iff $i < j$, and ϕ has **the strict order property** (**SOP**) if there is a sequence $(b_i)_{i<\omega}$ such that $(\phi(\mathcal{U}^x, b_i))_{i<\omega}$ forms a strictly decreasing sequence of definable sets (with respect to containment). A theory T is **stable** if no formula has the order property. Clearly if ϕ is stable, it is NIP.

Definition 4.1. Suppose $k < \omega$ and $p \in S_\phi(A)$ for some $A \subseteq \mathcal{U}^y$. Then p is k**-isolated** if for some $A_0 \subseteq A$ such that $|A_0| \leq k$, $p|_{A_0} \vdash p$. p is **isolated** if it is k-isolated for some k (this coincides with the usual topological definition).

Remark 4.2. Note that if $p \in S_\phi(A)$ is k-isolated then it is k-compressible. Also, if p is k-compressible and isolated, then p is k-isolated.

The following says in particular that under stability, k-compressibility and k-isolation are the same.

Lemma 4.3. *Let $\phi(x, y)$ be NIP. Then the following are equivalent:*

(i) ϕ is stable.

(ii) For any $B \subseteq \mathcal{U}^y$ and $p \in S_\phi(B)$ and $k \in \mathbb{N}$, if p is k-compressible then p is isolated (and hence k-isolated by Remark 4.2).

(iii) For all $k < \omega$ and $\overline{\epsilon} \in \{0, 1\}^k$,

$$\theta_{\overline{\epsilon}}(x, \overline{y}) := \bigwedge_{j<k} \phi(x, y_j)^{\epsilon_j}$$

does not have the strict order property.

Proof. (i) implies (iii) as a Boolean combination of stable formulas is stable (see e.g., [Pil96, Lemma 2.1]) and the strict order property implies the order property.

$\neg$(i) implies $\neg$(iii) by the proof of [Sim15a, Theorem 2.67] and the subsequent remark.

$\neg$(ii) implies $\neg$(iii): let $p \in S_\phi(B)$ be k-compressible but not k-isolated.

Inductively we find $\overline{b}^i \in B^k$ and $\overline{\epsilon}^i \in \{0, 1\}^k$ for $i < \omega$ such that we have $p(x) \vdash \theta_{\overline{\epsilon}^i}(x, \overline{b}^i)$ for all i, and $\theta_{\overline{\epsilon}^i}(x, \overline{b}^i) \vdash \theta_{\overline{\epsilon}^j}(x, \overline{b}^j)$ iff $i \geq j$. (Given i, since $\theta_{\overline{\epsilon}^i}(x, \overline{b}^i) \not\vdash p$, there is $b \in B$ such that $\theta_{\overline{\epsilon}^i}(x, \overline{b}^i) \not\vdash p|_b$. Let $\theta_{\overline{\epsilon}^{i+1}}(x, \overline{b}^{i+1}) \vdash p|_{\overline{b}^i b}$ be from p.) But then some $\overline{\epsilon}$ occurs infinitely often, and then $\theta_{\overline{\epsilon}}(x, \overline{y})$ has SOP.

$\neg$(iii) implies $\neg$(ii): suppose $\overline{b}^i \in \mathcal{U}^{\overline{y}}$ for $i < \omega$ and $\theta_{\overline{\epsilon}}(x, \overline{b}^i) \vdash \theta_{\overline{\epsilon}}(x, \overline{b}^j)$ iff $i \geq j$. Let $B = \{b^i_j \mid i < \omega,\ j < k\} \subseteq \mathcal{U}^y$, and note that $\{\theta_{\overline{\epsilon}}(x, \overline{b}^j) \mid j < \omega\}$ implies a complete type $p \in S_\phi(B)$. Then p is k-compressible but not isolated. □

4.2. **Stable types and theories.**

Definition 4.4. A partial type $\pi(x)$ over A is **stable** if every extension $p \in S(B)$, over every $B \supseteq A$, is definable.

It is well-known that T is stable if and only if every type is definable (see e.g., [TZ12, Corollary 8.3.2]), so T is stable if and only if every partial type is stable. For more on stable types (including equivalent definitions), see [ACP14, EK21]. We will use the following equivalence:

Fact 4.5. *[EK21, Remark 2.6] The following are equivalent for a partial type $\pi(x)$:*

(1) π is stable.

(2) For every formula $\phi(x, y)$ there is a formula $\psi(x)$ implied by π such that $\phi(x, y) \wedge \psi(x)$ is stable (as a formula over $\mathcal{U}$).

Under stability, the analogue of compressibility of a type is l-isolation.

Definition 4.6. A type $p(x) \in S(A)$ is **l-isolated** if for each formula $\phi(x, y)$ there is $\zeta \in p$ with $\zeta \vdash p \restriction \phi$.

Clearly, an l-isolated type is compressible. By considering the formula $x \neq y$, we easily obtain:

Remark 4.7. Any l-isolated type over a model is realised.

The following is analogous to (but not actually comparable with) Lemma 4.3.

Lemma 4.8. *(i) Suppose $p \in S(A)$ is compressible but not l-isolated. Then:*
(a) There are tuples a_i, b_i in A which witness the order property for some $\mathcal{L}$-formula.
(b) p is not stable.
In particular, if T is stable then any compressible type is l-isolated.
(ii) (T countable NIP) T is stable iff any compressible type is l-isolated, iff there is some ω-saturated model M such that every strongly compressible type over M is l-isolated.

Proof. For both (i.a) and (i.b), suppose $\phi(x, y)$ witnesses that p is not l-isolated and $\zeta(x, z)$ compresses $p \restriction \phi$.

(i.a) Let $\theta(y, z) = \bigvee_{\epsilon<2} \forall x(\zeta(x, z) \to \phi(x, y)^\epsilon)$. We recursively construct $a_i \in A^y$ and $b_i \in A^z$ for $i < \omega$, such that $\vDash \theta(a_i, b_j) \Leftrightarrow i < j$: if $a_{<i}$ and $b_{<i}$ are already defined, let b_i be such that $p \ni \zeta(x, b_i) \vdash (p \restriction \phi)|_{a_{<i}}(x)$, and let a_i be such that $\zeta(x, b_j) \nvdash (p \restriction \phi)|_{a_i}(x)$ for all $j \leq i$, which exists since $\bigwedge_{j \leq i} \zeta(x, b_j) \nvdash (p \restriction \phi)(x)$.

(i.b) For $\psi(x) \in p$, we show that $\theta(x, z) := \zeta(x, z) \wedge \psi(x)$ has the order property by recursively constructing $(a_i, b_i, c_i)_{i<\omega}$ such that $\vDash \theta(a_i, b_j)$ iff $i \geq j$ and $\bigwedge_{j<i} \theta(x, b_j) \in p$ and $\vDash \phi(a_i, c_i) \Leftrightarrow \phi(x, c_i) \notin p$. This is enough by Fact 4.5. Suppose we found $(a_j, b_j, c_j)_{j<i}$. Let b_i be such that $p \ni \zeta(x, b_i) \vdash (p \restriction \phi)|_{c_{<i}}(x)$. Since $\bigwedge_{j \leq i} \theta(x, b_j)$ does not isolate $p \restriction \phi$, there are some a_i, c_i such that $\vDash \bigwedge_{j \leq i} \theta(a_i, b_j)$, $\phi(x, c_i)^\epsilon \in p$ for some $\epsilon < 2$, and $\neg\phi(a_i, c_i)^\epsilon$ holds.

(ii) The implications from left to right follow by (i) and trivially, respectively. For the other direction, assume that T is not stable. By [Sim15a, Theorem 2.67], T has the SOP. So say $<$ is an $\emptyset$-definable (strict) preorder on an $\emptyset$-definable set D with infinite chains.

Let M be an ω-saturated model. So M contains an infinite chain C which we may assume is maximal. Since C is infinite, we can write $C = C_1 + C_2$ where either C_1 has no last element or C_2 has no first element (one of them may be empty).

Let $\pi(x)$ be the unique type in $<$ over C corresponding to the cut (C_1, C_2) (i.e., $\pi(x)$ is determined by $\{c_1 < x \mid c_1 \in C_1\} \cup \{x < c_2 \mid c_2 \in C_2\}$). Now, π is t-compressible (e.g., if both C_1, C_2 are nonempty, then $z < x < w$ compresses π within π and if C_1 is empty then $x < z$ compresses π within π). Hence by Corollary 3.21, π has a strongly compressible completion $q \in S(M)$. By maximality of C, π is not realised in M, so neither is q. So by Remark 4.7, q is not l-isolated.

(Note that we could have worked with a partial order instead of a preorder by passing to eq and using Remark 3.19.) ☐

Remark 4.9. From the proof of Lemma 4.8(ii) it follows that if $M \vDash T$ contains an infinite chain in an M-definable preorder D then there is $c \in D$ such that $\mathrm{tp}(c/M)$ is compressible but not l-isolated.

Remark 4.10. From Corollary 3.21, it follows that l-isolated types are dense in stable theories, but this was well-known and follows easily by 2-rank considerations, see [She90, Lemma IV.2.18(4)].

Remark 4.11. The following example demonstrates the necessity of the countability assumption on T in Corollary 3.21 even for stable theories.

Let κ be a cardinal, and consider κ colourings on a set X, with each colouring using the same colours, such that no point gets the same colour according to two different colourings, but apart from this restriction all possibilities are realised. We can formalise this in the language with a sort X, a sort C for the colours, and for each $i \in \kappa$ a function $f_i : X \to C$ giving the colour of an element according to the i-th colouring. The theory is axiomatised by saying there are infinitely many colours and, for each finite set $\{i_1, \ldots, i_n\} \subseteq \kappa$ enumerated without repetitions and each $m \geq 0$, an axiom

$$\forall c_1, \ldots, c_n \in C \; \forall x_1, \ldots, x_m \in X \; (\exists x \in X \; (\bigwedge_{j=1}^{n} f_{i_j}(x) = c_j \wedge \bigwedge_{j=1}^{m} x \neq x_j) \leftrightarrow \bigwedge_{j \neq k} c_j \neq c_k)$$

This axiomatises a complete consistent theory T with quantifier elimination in the given language. Indeed, restricting to any finite sublanguage $\mathcal{L}_0$ containing X, C and finitely many function symbols f_i, $T \restriction \mathcal{L}_0$ is the Fraïssé limit of the class of finite structures (X_0, C_0) where for every $f_i, f_j \in \mathcal{L}_0$ and all $x \in X_0$, if $f_i(x) = f_j(x)$ then $i = j$. It follows that T is stable; indeed, $|S^1(A)| \leq \lambda^\kappa$ for $|A| \leq \lambda$, so T is 2^κ-stable.

Now suppose $\kappa > \aleph_0$ and let $C_0 \subseteq C$ with $|C_0| = \aleph_0$. We claim that the formula $x \in X$ has no compressible (equivalently, by Lemma 4.8(i), l-isolated) completion $p \in S^x(C_0)$. Indeed, it is easy to see that p would have to include for each i a formula $f_i(x) = c_i$ for some $c_i \in C_0$, but then $c_i \neq c_j$ for $i \neq j$, contradicting $\kappa > |C_0|$.

Other (hints for) examples are given in [She90, Exercise IV.2.13] where Shelah also gives a *superstable*[3] counterexample, which we will describe briefly. Let $\mathcal{L} = \{E_\nu, P_s \mid \nu \in \omega^\omega, s \in \omega^{<\omega}\}$ where the P_s's are unary predicates and the E_ν's are binary relation symbols. Let M be the $\mathcal{L}$-structure whose universe is $\omega^\omega \times \omega$ where $P_s^M = \{(\eta, n) \in \omega^\omega \times \omega \mid s \lhd \eta\}$ and E_ν^M is an equivalence relation where $(\eta_1, n_1), (\eta_2, n_2)$ are equivalent iff ($\eta_1 = \eta_2$ or for some $n < \omega$, $\eta_1 \restriction n = \eta_2 \restriction n = \nu \restriction n$ and $\eta_1(n) = \eta_2(n) \neq \nu(n)$). Essentially, each class is infinite and two branches in the tree ω^ω are ν-equivalent if they divert from ν at the same point and in the same direction (starting the same cone), and ν is ν-equivalent only to itself. Let $T = \mathrm{Th}(M)$. It is not too hard to see that T has quantifier elimination. We leave it as exercise to check that for any set A, $S^1(A) \leq |A| + 2^{\aleph_0}$, and thus T is superstable.

Finally, working in $\mathcal{U}^{\mathrm{eq}}$ and letting $A = \{(\eta, n)/E_\nu \mid \eta \neq \nu, n < \omega\}$, there is no l-isolated type $p \in S^1(A)$ (in the home sort). Indeed, if $p(x)$ is such a type, then for every $\nu \in \omega^\omega$ there is some $\eta \neq \nu$ such that $x/E_\nu = (\eta, 0)/E_\nu$ is in p. It follows that for some $\nu \in \omega^\omega$, p must contain $\{P_s(x) \mid s \lhd \nu\}$. But then $p \vdash x/E_\nu \neq (\eta, 0)/E_\nu$ for all $\eta \neq \nu$, contradiction.

4.3. **Generically stable types.** Generically stable types are invariant types that exhibit stability-like behavior "generically" i.e., when considering their Morley sequences. This notion was first studied in the NIP context by Shelah [She04] (under the name "stable types") and then by Hrushovski and Pillay [HP11] and independently Usvyatsov [Usv09]. See also [Sim15a, Section 2.2.2]. It was defined in

[3]Recall that T is **superstable** if it is stable in all cardinals $\geq 2^{|T|}$.

general in [PT11] by Pillay and Tanović. See also [CG20] for more on generic stability outside of the NIP context.

Definition 4.12. We say that a global type p is **generically stable over** A if it is A-invariant and for every ordinal α, every $\phi(x)$ with parameters in $\mathcal{U}$ and every Morley sequence $(a_i)_{i<\alpha}$ of p over A, the set $\{i < \alpha \mid \mathcal{U} \vDash \phi(a_i)\}$ is finite or cofinite.

Fact 4.13. *Suppose p is generically stable over A. Then:*

(1) *[PT11, Proposition 2.1] p is A-definable and finitely satisfiable in every model containing A.*
(2) *[PT11, Proposition 2.1] If I is a Morley sequence of p over A then I is an indiscernible set (i.e., totally indiscernible).*
(3) *[CG20, Proposition 3.2] $p = \lim(a_i)_{i<\omega}$ for any Morley sequence $(a_i)_{i<\omega}$ of p over A, i.e., p is the limit type of any of its Morley sequences over A: $\theta(x) \in p$ iff $\theta(a_i)$ holds for all but finitely many $i < \omega$. (In [CG20, Proposition 3.2] it is stated over models, but it is also true over sets and follows directly from the definition; we leave this to the reader.)*

Moreover, by [Sim15a, Theorem 2.29] if T is NIP then each one of these conclusions is equivalent to generic stability for an A-invariant type.

Proposition 4.14. *Let $p \in S(\mathcal{U})$ be generically stable over $A \subseteq \mathcal{U}$, and suppose $p|_A$ is compressible. Then $p|_A$ is l-isolated.*

For the proof we will need the following observation. Recall the notations from Section 2.4.

Remark 4.15. Suppose $\mathrm{tp}(a/A)$ is l-isolated, and $(A, a) \equiv (A', a')$. Then $\mathrm{tp}(a'/A')$ is l-isolated. Indeed, if $\zeta(x, d) \in p$ isolates $p \restriction \phi$, then

$$(A, a) \vDash \exists d \in A^z (\zeta(a, d) \wedge \forall y (\bigwedge_{\epsilon<2} (\phi(a, y)^\epsilon \to \forall x (\zeta(x, d) \to \phi(x, y)^\epsilon)))).$$

Thus, the same is true in (A', a'), which suffices.

Proof of Proposition 4.14. Suppose $p \in S(\mathcal{U})$ is generically stable over A and $p|_A$ is compressible. Let $a \vDash p|_A$, and let M be a model containing Aa. Let $(M', A', a) \succ (M, A, a)$ be an $|M|^+$-saturated extension (in a language with a predicate P for A and constant symbols a), and let $(M'', A'', a) \succ (M', A', a)$ be an $|M'|^+$-saturated extension (with $M'' \subseteq \mathcal{U}$). Since p is definable over A by Fact 4.13(1), it follows that $p|_{A''} = \mathrm{tp}(a/A'')$.

Let $\phi(x, y)$ be any formula. Note that $(A'', a) \succ (A', a) \succ (A, a)$ and that (A'', a) is $|A'|^+$-saturated, so by Facts 2.12 and 2.13, there is some $d \in (A'')^z$ and some formula $\zeta(x, z)$ such that $\zeta(a, d)$ holds (so $\zeta(x, d) \in p$) and $\zeta(x, d) \vdash (p \restriction \phi)|_{A'}$. By Fact 4.13(3) and compactness there is some $N < \omega$ such that for every Morley sequence $(a_i)_{i<N}$ of p over A, $\zeta(a_i, d)$ holds for some $i < N$, and hence $a_i \vDash (p \restriction \phi)|_{A'}$.

Let $(a_i)_{i<n}$ be a Morley sequence of p over A of maximal length such that $a_i \nvDash (p \restriction \phi)|_{A'}$ for all $i < n$. For $i < n$, let $c_i \in (A')^y$ and $\epsilon_i < 2$ be such that $\phi(a_i, c_i)^{\epsilon_i}$ holds but $\neg\phi(x, c_i)^{\epsilon_i} \in p$. Then the following set of formulas over M' is inconsistent:

$$p^{(n+1)}(x_0, \ldots, x_n)|_A \cup \{\phi(x_i, c_i)^{\epsilon_i} \mid i < n\} \cup \neg\theta(x_n)$$

where $\theta(x) = \forall y \in P(\phi(a, y) \leftrightarrow \phi(x, y))$. By compactness (and saturation of M'), there is some formula $\psi(x_0, \ldots, x_n) \in p^{(n+1)}|_A$ such that $\psi \cup \{\phi(x_i, c_i)^{\epsilon_i} \mid i < n\} \vdash \theta(x_n)$.

Let $\chi(x) = \exists x_0 \dots x_{n-1}(\bigwedge_{i<n} \phi(x_i, c_i)^{\epsilon_i} \wedge \psi(x_0, \dots, x_{n-1}, x))$. Since M' is a model, $\chi(x) \vdash (p \restriction \phi)|_{A'}$. Also, since $\psi(a_0, \dots, a_{n-1}, x) \in p$, it follows that $\chi(x) \in p|_{A'}$.

Since ϕ was arbitrary, this means that $p|_{A'}$ is l-isolated, and hence by Remark 4.15, we are done. □

It is convenient to use the following definition.

Definition 4.16. Suppose $M \vDash T$. A type $p \in S(M)$ is **generically stable** if it has a global M-invariant extension which is generically stable over M.

Remark 4.17. If $p \in S(M)$ is generically stable then it has a unique M-invariant extension by [PT11, Proposition 2.1(iii)].

Note that if $p \in S(M)$ then it has a global M-invariant extension (e.g., a coheir). Thus, together with [CG20, Proposition 3.4], we get the following fact.

Fact 4.18. *If M is a model and $p \in S(M)$ is stable then p is generically stable.*

It follows that when the base is a model, Lemma 4.8(i.b) is implied by Proposition 4.14.

5. Rounded averages of compressible types and applications

Let Maj be the majority rule Boolean operator, i.e., for truth values $P_0, \dots, P_{n-1}$, let

$$\mathrm{Maj}_{i<n} P_i = \bigvee_{\substack{I_0 \subseteq n \\ |I_0| > n/2}} \bigwedge_{i \in I_0} P_i.$$

We just write Maj_i if n is clear.

More generally, for $\alpha \in (0,1)$, let Maj^α be the "greater than an α-fraction" Boolean operator, i.e.,

$$\mathrm{Maj}^\alpha_{i<n} P_i = \bigvee_{\substack{I_0 \subseteq n \\ |I_0| > \alpha n}} \bigwedge_{i \in I_0} P_i.$$

Definition 5.1. Suppose $\phi(x,y)$ is a formula, $B \subseteq \mathcal{U}^y$ and $p_0(x), \dots, p_{n-1}(x) \in S_\phi(B)$.

The **rounded average** of $p_0(x), \dots, p_{n-1}(x) \in S_\phi(B)$ is the following (possibly inconsistent) collection of formulas

$$\mathrm{rAvg}(p_i \mid i < n) := \{\phi(x,b)^\epsilon \mid b \in B, \epsilon < 2, \mathrm{Maj}_{i<n}(\phi(x,b)^\epsilon \in p_i(x))\}.$$

More generally, for $\alpha \in [\frac{1}{2}, 1)$, the **$\alpha$-rounded average** is the set

$$\mathrm{rAvg}_\alpha(p_i \mid i < n) := \{\phi(x,b)^\epsilon \mid b \in B, \epsilon < 2, \mathrm{Maj}^\alpha_{i<n}(\phi(x,b)^\epsilon \in p_i(x))\}.$$

The main result of this section is:

Theorem 5.2. *Let $\phi(x,y)$ be an NIP formula and suppose $\alpha \in [1/2, 1)$. Then there exist n and k depending only on $\mathrm{vc}(\phi)$ and α such that for $A \subseteq \mathcal{U}^y$, any $p \in S_{\phi^{\mathrm{opp}}}(A)$ is the α-rounded average of n types in $S_{\phi^{\mathrm{opp}} \downarrow k}(A)$.*

(We give a more precise and general statement in Theorem 5.17, allowing a partial type $\pi(x)$.)

We give some applications:

- Uniformity of honest definitions for NIP formulas, see Definition 5.22. This is Corollary 5.23.
- Uniform definability of pseudofinite types, see Theorem 5.36.

5.1. **Superdensity.** In this section we isolate a sufficient condition for proving Theorem 5.2, which uses the (p,q)-theorem (see Fact 5.4). We then apply it to retrieve UDTFS in Corollary 5.14, as a prelude to the proof of the uniformity of honest definitions in Corollary 5.23.

Definition 5.3. Suppose $q \leq p < \omega$. A set system $(X,\mathcal{F})$ has the (p,q)**-property** if for any $S \subseteq \mathcal{F}$ such that $|S| \geq p$, there exists $S_0 \subseteq S$ of size $|S_0| \geq q$ such that $\bigcap S_0 \neq \emptyset$.

Fact 5.4. *[Mat04] (The (p,q)-theorem) There exists a function $N_{\mathrm{pq}} : \mathbb{N}^2 \to \mathbb{N}$ such that for any $q \leq p < \omega$, if $(X,\mathcal{F})$ is a finite set system with the (p,q)-property such that every $s \in \mathcal{F}$ is nonempty and $\mathrm{vc}^*(\mathcal{F}) < q$, then there is $X_0 \subseteq X$ of size $|X_0| = N_{\mathrm{pq}}(p,q)$ such that $X_0 \cap s \neq \emptyset$ for all $s \in \mathcal{F}$.*

We isolate from the proof of [Sim15a, Corollary 6.11] the following immediate generalisation of the (p,q)-theorem to infinite set systems.

Lemma 5.5. *Let $\phi(x,y)$ be NIP. Let $p \geq q > \mathrm{vc}^*(\phi)$ be integers, and let $N = N_{\mathrm{pq}}(p,q)$. Let $A \subseteq \mathcal{U}^x$ and $B \subseteq \mathcal{U}^y$. Suppose that $\phi(A,b) \neq \emptyset$ for every $b \in B$, and that for every $B_0 \subseteq B$ with $|B_0| = p$ there exists $B_1 \subseteq B_0$ with $|B_1| = q$ such that for some $a \in A$ we have $\vDash \bigwedge_{b\in B_1} \phi(a,b)$.*

Then $\{\bigvee_{i<N} \phi(x_i,b) \mid b \in B\}$ is finitely satisfiable in A.

Proof. By the definition of finite satisfiability, it suffices to see this in the case that B is finite; but this case is a direct consequence of the (p,q)-theorem (Fact 5.4). $\square$

Suppose $\phi(x,y)$ is a formula, $A \subseteq \mathcal{U}^x$ and $N \in \mathbb{N}$. For variables $\bar{x} = (x_i \mid i < N)$ of the same sort of x, we denote by $S^{\bar{x}}_{\phi,A\text{-fs}}(\mathcal{U}^y)$ the space of Δ-types in $\bar{x}$ over $\mathcal{U}^y$ which are finitely satisfiable in A, where $\Delta = \{\phi(x_i,y) \mid i < N\}$. If $p(y) \in S_{\phi^{\mathrm{opp}}}(A)$ then for any $q \in S^{\bar{x}}_{\phi,A\text{-fs}}(\mathcal{U}^y)$, the product $q(\bar{x}) \otimes p(y)$ is the partial type $q(\bar{x}) \cup p(y) \cup \{\phi(x_i,y)^{\epsilon_i} \mid i < N\}$ where $\epsilon_i < 2$ is the truth value of $\phi(a_i,b)$ for some (any) $b \vDash p$ and $(a_i \mid i < N) \vDash q|_b$. Note that this is well-defined.

For $b \in \mathcal{U}^y$, $N \in \mathbb{N}$ and $S \subseteq S_{\phi^{\mathrm{opp}}}(A)$, we consider the following condition:

$$(\dagger)_{b,N,S} \begin{cases} \text{For every } q(\bar{x}) \in S^{\bar{x}}_{\phi,A\text{-fs}}(\mathcal{U}^y) \text{ where } \bar{x} = (x_i \mid i < N)\text{, there is } p \in S \\ \text{such that} \\ \qquad q(\bar{x}) \otimes p(y) \vdash \bigwedge_{i<N} (\phi(x_i,y) \leftrightarrow \phi(x_i,b)). \end{cases}$$

Definition 5.6. Suppose $\phi(x,y)$ is a formula and $A \subseteq \mathcal{U}^x$. A set $S \subseteq S_{\phi^{\mathrm{opp}}}(A)$ is **superdense** in $S_{\phi^{\mathrm{opp}}}(A)$ if $(\dagger)_{b,N,S}$ holds for every $b \in \mathcal{U}^y$ and $N \in \mathbb{N}$.

Remark 5.7. By considering realised types in A, it follows that any superdense set $S \subseteq S_{\phi^{\mathrm{opp}}}(A)$ is also dense.

Lemma 5.8. *Let $\alpha \in [1/2,1)$ and let $d \in \mathbb{N}$. Let $n \in \mathbb{N}$ be such that $(1-\alpha)n > 2^{d+1}-1$, and let $N = N_{\mathrm{pq}}(n,2^{d+1})$. Fix some formula $\phi(x,y)$ such that $\mathrm{vc}(\phi) \leq d$. Suppose $A \subseteq \mathcal{U}^x$, $b \in \mathcal{U}^y$ and $S \subseteq S_{\phi^{\mathrm{opp}}}(A)$ satisfy $(\dagger)_{b,N,S}$.*

Then $r := \mathrm{tp}_{\phi^{\mathrm{opp}}}(b/A)$ is the α-rounded average of n elements of S.

Proof. Since $\alpha \geq 1/2$, $r = \mathrm{rAvg}_\alpha(p_i \mid i < n)$ iff for all $a \in A$, $|\{i < n \mid \phi(a,y) \in r \Leftrightarrow \phi(a,y) \in p_i\}| > \alpha n$. Let $\phi'(x,y) = \phi(x,y) \leftrightarrow \phi(x,b)$ (as a formula over $\mathcal{U}$). In this notation, we must prove that for some $p_0,\ldots,p_{n-1} \in S$, for all $a \in A$, $|\{i < n \mid p_i(y) \vdash \phi'(a,y)\}| > \alpha n$.

Note that $\mathrm{vc}(\phi') = \mathrm{vc}(\phi) \leq d$; indeed, ϕ' and ϕ shatter the same subsets of $\mathcal{U}^x$.

Now assume the conclusion fails. In particular, $r \notin S$ (else it is the rounded average of n copies of itself). Let $B \subseteq \mathcal{U}^y$ be a set of realisations of the types in S.

By assumption, for every $B_0 \subseteq B$ with $|B_0| = n$, there exists $a \in A$ and $B_1 \subseteq B_0$ with $|B_1| \geq (1-\alpha)n > 2^{d+1} - 1 \geq \mathrm{vc}^*(\phi')$ such that $\neg\phi'(a, c)$ holds for all $c \in B_1$ (the last inequality follows from Remark 2.7).

Note that for $b' \in B$ we have $\mathrm{tp}_{\phi^{\mathrm{opp}}}(b'/A) \neq \mathrm{tp}_{\phi^{\mathrm{opp}}}(b/A)$ since $\mathrm{tp}_{\phi^{\mathrm{opp}}}(b/A) \notin S$, so $\neg\phi'(A, b') \neq \emptyset$.

So Lemma 5.5 applies to $\neg\phi'$ (with $p = n$, $q = 2^{d+1}$; note that $\mathrm{vc}^*(\phi') = \mathrm{vc}^*(\neg\phi')$) and hence $\pi(\bar{x}) := \{\bigvee_{i<N} \neg\phi'(x_i, c) \mid c \in B\}$ is finitely satisfiable in A, where $\bar{x} = (x_i \mid i < N)$.

Extend π to $q(\bar{x}) \in S^{\bar{x}}_{\phi, A\text{-fs}}(\mathcal{U}^y)$ (formally, first extend π to a global type finitely satisfiable in A, and then restrict to a global $\{\phi(x_i, y) \mid i < N\}$-type q. Then note that q implies π). Then for all $p \in S$, we have $q(\bar{x}) \otimes p(y) \vdash \bigvee_{i<N} \neg\phi'(x_i, y)$. However, $(\dagger)_{b,N,S}$ implies that for some $p \in S$, $q(\bar{x}) \otimes p(y) \vdash \bigwedge_{i<N} \phi'(x_i, y)$, contradiction. □

Remark 5.9. When $\alpha = 1/2$, $n := 2^{\mathrm{vc}(\phi)+2} - 1$ and $N := N_{\mathrm{pq}}(n, 2^{\mathrm{vc}(\phi)+1})$ work in Lemma 5.8.

Remark 5.10. From the proof of Lemma 5.8, we get something slightly stronger (under the same assumptions): either $r \in S$, or r is an α-rounded average of *distinct* types in S.

Remark 5.11. Lemma 5.8 admits a partial converse, for an arbitrary formula $\phi(x, y)$ and any n and any N: letting $\bar{x} = (x_i \mid i < N)$, if $\mathrm{tp}_\phi(b/A)$ is the rounded average of n elements from $S \subseteq S^y_{\phi^{\mathrm{opp}}}(A)$, then for every $q(\bar{x}) \in S^{\bar{x}}_{\phi, A\text{-fs}}(\mathcal{U}^y)$ there is $p(y) \in S$ such that

$$q(\bar{x}) \otimes p(y) \vdash \mathrm{Maj}_{i<N}(\phi(x_i, y) \leftrightarrow \phi(x_i, b)).$$

Indeed, suppose $\mathrm{tp}_{\phi^{\mathrm{opp}}}(b/A) = \mathrm{rAvg}(\mathrm{tp}(c_j/A) \mid i < n)$ where $\mathrm{tp}(c_j/A) \in S$ for $j < n$ and $q(\bar{x}) \in S^{\bar{x}}_{\phi, A\text{-fs}}(\mathcal{U}^y)$. If the conclusion fails, we get that for each $j < n$

$$q \vdash \neg\,\mathrm{Maj}_{i<N}(\phi(x_i, c_j) \leftrightarrow \phi(x_i, b)).$$

By finite satisfiability, there is $(a_0, \ldots, a_{N-1}) \in A^N$ satisfying this for every $j < n$. Hence $|\{(i, j) \in N \times n \mid \phi(a_i, c_j) \leftrightarrow \phi(a_i, b)\}| \leq \frac{1}{2}nN$. Then by the pigeonhole principle, for some $i < N$ we have

$$\vDash \neg\,\mathrm{Maj}_{j<n}(\phi(a_i, c_j) \leftrightarrow \phi(a_i, b)),$$

contradicting $\mathrm{tp}_{\phi^{\mathrm{opp}}}(b/A)$ being the rounded average of the $\mathrm{tp}(c_i/A)$.

In Section 5.3 we will prove that $S_{\phi^{\mathrm{opp}}\downarrow\star}(A)$ is superdense in $S_{\phi^{\mathrm{opp}}}(A)$ and even in a uniform way, as in the proof of Corollary 3.9(iii), which will imply Theorem 5.2 by Lemma 5.8. In the finite case we can already conclude the following, basically because superdensity is the same as density when A is finite.

Corollary 5.12. *Fix $\alpha \in [1/2, 1)$ and some $d \in \mathbb{N}$. Then there are $k, n \in \mathbb{N}$ (depending only on d, α) such that if $\phi(x, y)$ is a formula with $\mathrm{vc}(\phi) \leq d$, then for any finite $A \subseteq \mathcal{U}^x$, every $r(y) \in S_{\phi^{\mathrm{opp}}}(A)$ is the α-rounded average of n types in $S_{\phi^{\mathrm{opp}}\downarrow k}(A)$.*

Proof. Let n, N be as in Lemma 5.8, and let $k = k_{\mathrm{comp}}(d) + N$. Fix some $b \in \mathcal{U}^y$. By Lemma 5.8, to show the conclusion for $\mathrm{tp}_{\phi^{\mathrm{opp}}}(b/A)$, it is enough to show $(\dagger)_{b,N,S}$ with $S = S_{\phi^{\mathrm{opp}}\downarrow k}(A)$. But every $q(\bar{x}) \in S^{\bar{x}}_{\phi, A\text{-fs}}(\mathcal{U}^y)$ is realised in A (since A is finite), so it boils down to showing that for every $\bar{a} \in A^N$, there is some $p(y) \in S_{\phi^{\mathrm{opp}}\downarrow k}(A)$ such that $p(y) \vdash \bigwedge_{i<N}(\phi(a_i, y) \leftrightarrow \phi(a_i, b))$. This follows from the choice of k and Corollary 3.9(ii) applied to ϕ^{opp}. □

As a corollary we retrieve UDTFS. First recall the definition.

Definition 5.13 (UDTFS)**.** We say that $\phi(x,y)$ has **uniform definability of types over finite sets** (**UDTFS**) if there exists a formula $\psi(x,z)$ such that for every finite set $A \subseteq \mathcal{U}^x$ with $|A| \geq 2$ the following holds: for every $b \in \mathcal{U}^y$ there exist $c \in A^z$ such that $\psi(A,c) = \phi(A,b)$.

Every formula with UDTFS is easily NIP (see e.g., the proof of Theorem 14 in [EK20]). The proof of UDTFS for NIP formulas in [EK20] roughly goes by showing Corollary 5.12 with $\alpha = 1/2$, and deducing UDTFS from that (this is not stated explicitly in this language, see the proof of Theorem 14, (1) implies (2), (3) there). We omit the details here since we will prove uniformity of honest definitions in Corollary 5.23 below. In Theorem 5.36 we will extend UDTFS to pseudofinite types (see Definition 5.34).

Corollary 5.14. *The formula $\phi(x,y)$ is NIP iff it has UDTFS.*

We do point out that the proof here is, at least conceptually, simpler than the proof in [EK20]: both proofs use the finite version of Corollary 3.21, but here the only other ingredient is the (p,q)-theorem, while there both the VC-theorem and von Neumann's minimax theorem are used.

5.2. **A variant of Ramsey's theorem for finite subsets.** Here we will prove a variant of Ramsey's theorem for finite subsets of a cardinal. This result generalises Remark 3.3 for $(\mathcal{P}^{\text{fin}}(\kappa), \subseteq)$ in the same way that Ramsey's theorem generalises the pigeonhole principle. It will be used in the proof of Theorem 5.2.

For a partial order $(X, \leq)$ and $n \in \mathbb{N}$, let $X^n_< = \{(x_0, \ldots, x_{n-1}) \in X^n \mid x_0 < \ldots < x_{n-1}\}$ be the set of ordered chains of size n (in short, n**-chains**).

For κ a cardinal, let $\mathcal{P}^{\text{fin}}(\kappa)$ be the set of finite subsets of κ, partially ordered by inclusion.

Say $f : \mathcal{P}^{\text{fin}}(\kappa) \to \mathcal{P}^{\text{fin}}(\kappa)$ is **strictly increasing** if $s \subsetneq t \Rightarrow f(s) \subsetneq f(t)$ for all $s, t \in \mathcal{P}^{\text{fin}}(\kappa)$, and say f is **cofinal** if for all $s \in \mathcal{P}^{\text{fin}}(\kappa)$ there is $t \in \mathcal{P}^{\text{fin}}(\kappa)$ such that $f(t) \supseteq s$. Note that the image of an n-chain under a strictly increasing map is an n-chain.

Proposition 5.15. *Let κ be an infinite cardinal, $0 < n \in \mathbb{N}$ and let $c : \mathcal{P}^{\text{fin}}(\kappa)^n_< \to r < \omega$ be a finite colouring of the n-chains. Then there is a strictly increasing cofinal map $f : \mathcal{P}^{\text{fin}}(\kappa) \to \mathcal{P}^{\text{fin}}(\kappa)$ such that the image of all n-chains $f(\mathcal{P}^{\text{fin}}(\kappa)^n_<)$ is monochromatic, i.e., $|(c \circ f)(\mathcal{P}^{\text{fin}}(\kappa)^n_<)| = 1$.*

Proof. Denote by M the structure $(\mathcal{P}^{\text{fin}}(\kappa), \subseteq, (P_k)_{k<r})$ where $P_k = c^{-1}(k) \subseteq M^n_<$ for $k < r$. Let $N \succ M$ be an $|M|^+$-saturated extension. Let $\pi = \{x \supsetneq s \mid s \in M\}$. As π is finitely satisfiable in M, there is $q \in S_{M\text{-fs}}(N)$ extending π. Let $(a_{n-1}, \ldots, a_0) \models q^{(n)}|_M$ and let $\bar{a} = (a_0, \ldots, a_{n-1})$. Note that $\bar{a} \in N^n_<$ (because for all $a \models q|_M$ and all $b \in M$, $b \subsetneq a$), and hence for some $k < r$, $\bar{a} \in P^N_k$. We claim that this colour k works.

For $m < \omega$, let $S_m = \{s \in \mathcal{P}^{\text{fin}}(\kappa) \mid |s| = m\}$. By recursion on $m < \omega$ we define $f \restriction S_m$ such that for any $s \in S_m$:

(i) for any $1 \leq i \leq n$ and i-chain $s_0 \subsetneq \ldots \subsetneq s_{i-1} = s$, $(f(s_0), \ldots, f(s_{i-1}), a_i, \ldots, a_{n-1}) \in P^N_k$.

(ii) $f(s) \supseteq s$ and $f(s) \supsetneq f(t)$ for all $t \subsetneq s$.

The construction is possible because q is finitely satisfiable in M and since there only finitely many conditions to fulfill for each $s \in \mathcal{P}^{\text{fin}}(\kappa)$ (and because of the choice of $\bar{a}$): given $s \in S_m$ and a chain as in (i), by induction we have $q(x) \vdash P_k(f(s_0), \ldots, f(s_{i-2}), x, a_i, \ldots, a_{n-1})$; since there are only finitely many such chains to consider and also $q(x) \vdash x \supsetneq f(t) \cup s$ for any $t \subsetneq s$, we can find $f(s)$ by finite satisfiability.

Now the $i = n$ case of (i) implies that the image under f of any n-chain has colour k. Meanwhile (ii) implies that f is cofinal, and that f is strictly increasing. □

5.3. **The proofs of superdensity and of Theorem 5.2.** In this section we will prove Theorem 5.2, by proving superdensity of compressible types in a uniform way.

Proposition 5.16 (Superdensity of $\star$-compressible types). *Define* $k_{\mathrm{sd}} : \mathbb{N}^2 \to \mathbb{N}$ *by* $k_{\mathrm{sd}}(1, d) := k_{\mathrm{comp}}(d) + 2d + 2$*, and* $k_{\mathrm{sd}}(n, d) := n \cdot k_{\mathrm{sd}}(1, 2^{B_{\mathrm{vc}}(n, 2^{d+1})+1})$ *for* $n \neq 1$ *(see Definition 2.8).*

Let $\phi(x, y)$ *be a formula,* $d \in \mathbb{N}$*, and assume that* $\mathrm{vc}(\phi) \leq d$*. Let* $A \subseteq \mathcal{U}^x$*. Let* $\pi(y)$ *be a (small) partial type. Let* $b \in \pi(\mathcal{U})$*. For* $0 < n \in \mathbb{N}$*, let* $S = S^{\pi}_{\phi^{\mathrm{opp}} \downarrow k_{\mathrm{sd}}(n,d)}(A)$*.*

Then $(\dagger)_{b,n,S}$ *holds.*

Proof. Let $\bar{x} = (x_i \mid i < n)$ and let $q(\bar{x}) \in S^{\bar{x}}_{\phi, A\text{-fs}}(\mathcal{U}^y)$.

We first reduce to the case

$$(*) \qquad n = 1 \text{ and } q \vdash \phi(x, b).$$

For $i < n$, let $\epsilon_i < 2$ be such that $q \vdash \phi(x_i, b)^{\epsilon_i}$. Let $\phi'(\overline{x}, y) = \bigwedge_{i<n} \phi(x_i, y)^{\epsilon_i}$, and consider the global ϕ'-type $q'(\overline{x}) \in S^{\overline{x}}_{\phi'}(\mathcal{U}^y)$ implied by q, which is finitely satisfiable in A.

Now $\mathrm{vc}^*(\neg\phi) = \mathrm{vc}^*(\phi)$, and so by Lemma 2.10, and Remark 2.7, we have $\mathrm{vc}(\phi') < 2^{\mathrm{vc}^*(\phi')+1} \leq 2^{B_{\mathrm{vc}}(n, \mathrm{vc}^*(\phi))+1} < 2^{B_{\mathrm{vc}}(n, 2^{\mathrm{vc}(\phi)+1})+1} \leq 2^{B_{\mathrm{vc}}(n, 2^{d+1})+1} =: d'$. (Note that B_{vc} is increasing in the second variable.) Note that $q' \vdash \phi'(\overline{x}, b)$.

Let $k = k_{\mathrm{sd}}(1, d')$. Assuming the proposition in the (*) case, we obtain $p' \in S^{\pi}_{(\phi')^{\mathrm{opp}} \downarrow k}(A^n)$ such that $q'(\overline{x}) \otimes p'(y) \vdash \phi'(\overline{x}, y)$.

Claim. *p' implies a complete type* $p \in S^{\pi}_{\phi^{\mathrm{opp}} \downarrow k_{\mathrm{sd}}(n,d)}(A)$ *(which implies p').*

Proof. We first show that if $a \in A$ then $p'(y) \vdash \phi(a, y)^{\epsilon}$ for some ϵ.

Since $q'(\overline{x}) \otimes p'(y) \vdash \phi'(\overline{x}, y)$ and since q' is finitely satisfiable in A, there is some $(a_0, \ldots, a_{n-1}) \in A^n$ with $p'(y) \vdash \phi'(a_0, \ldots, a_{n-1}, y)$ (take some $c \vDash p'$, then since $\phi'(\overline{x}, c) \in q'(\overline{x})$, there is some $\bar{a} := (a_0, \ldots, a_{n-1}) \in A^n$ such that $\phi'(\bar{a}, c)$ holds, but as p' is complete, the same is true for any such c). Now, if $p'(y) \vdash \phi'(a, a_1, \ldots, a_{n-1}, y)$ then $p'(y) \vdash \phi(a, y)^{\epsilon_0}$. If $p'(y) \vdash \neg\phi'(a, a_1, \ldots, a_{n-1}, y)$ then by the choice of $(a_0, \ldots, a_{n-1})$, necessarily $p'(y) \vdash \phi(a, y)^{1-\epsilon_0}$.

So p' implies a complete type $p \in S^{\pi}_{\phi^{\mathrm{opp}}}(A)$. Let $A_0 \subseteq A$ be a finite subset. Then for some finite $A'_0 \subseteq A^n$, $p'|_{A'_0} \vdash p|_{A_0}$. By k-compressibility of p' modulo π, there is some $A'_1 \subseteq A^n$ of size k such that $p'|_{A'_1} \vdash_{\pi} p'|_{A'_0}$. Let $A_1 = \{a \in A \mid a \text{ appears in some } \bar{a} \in A'_1\}$. Then $|A_1| \leq nk$ and $p|_{A_1} \vdash p'|_{A'_1} \vdash_{\pi} p|_{A_0}$, so p is nk-compressible modulo π. Since $nk = k_{\mathrm{sd}}(n, d)$, we are done. □

Now, $q(\overline{x}) \otimes p(y) \vdash q'(\overline{x}) \otimes p'(y)$ and $q'(\overline{x}) \otimes p'(y) \vdash \phi'(\overline{x}, y) = \bigwedge_i \phi(x_i, y)^{\epsilon_i}$ so p is as required in $(\dagger)_{b,n,S}$.

It remains to prove the proposition assuming (*), so assume that $n = 1$ and $q(x) \vdash \phi(x, b)$.

If A is finite then q is realised in A, and we conclude (as in Corollary 5.12) by Corollary 3.9(ii) applied to ϕ^{opp} (note that $k_{\mathrm{comp}}(d) + 2d + 2 \geq k_{\mathrm{comp}}(d) + 1$ which would be enough in this case). So suppose $\kappa := |A| \geq \aleph_0$.

Let $\mathcal{F}^{\mathrm{fin}}(\kappa)$ be the filter on $\mathcal{P}^{\mathrm{fin}}(\kappa)$ generated by $\{X_s \mid s \in \mathcal{P}^{\mathrm{fin}}(\kappa)\}$ where $X_s = \{t \in \mathcal{P}^{\mathrm{fin}}(\kappa) \mid t \supseteq s\}$.

By [Sim15b, Lemma 2.9], q is the limit of a sequence of types realised in A:

$$q = \lim_{s \to \mathcal{F}^{\mathrm{fin}}(\kappa)} (\mathrm{tp}_{\phi}(a_s/\mathcal{U}^y))$$

where $a_s \in A$. (Note that in [Sim15b, Lemma 2.9], the type is over a model, but the same statement, with the same proof, works also over a set.)

This means that for any $c \in \mathcal{U}^y$, $\phi(x,c) \in q$ iff $\{s \mid \phi(a_s,c)\} \in \mathcal{F}^{\mathrm{fin}}(\kappa)$ iff for some $s \in \mathcal{P}^{\mathrm{fin}}(\kappa)$, $\phi(a_t,c)$ holds for all $t \supseteq s$. (Since q is complete, in fact the left-to-right implication suffices: $q = \lim_{s\to\mathcal{F}^{\mathrm{fin}}(\kappa)}(\mathrm{tp}_\phi(a_s/\mathcal{U}^y))$ iff we have that $\{s \mid \phi(a_s,c)\} \in \mathcal{F}^{\mathrm{fin}}(\kappa)$ whenever $\phi(x,c) \in q$.)

Since $q \vdash \phi(x,b)$, we may assume that $\vDash \phi(a_s,b)$ for all s (indeed, if s_0 is such that $\phi(a_s,b)$ for all $s \supseteq s_0$, then we can ensure this by replacing a_s with $a_{s\cup s_0}$).

Let $m = 2\,\mathrm{vc}(\phi)+3$. Let c^π_m be the 2^{2^m}-colouring of m-chains in $\mathcal{P}^{\mathrm{fin}}(\kappa)$ indicating which Boolean combinations of the corresponding m instances of ϕ are consistent with π, i.e., $c^\pi_m(s_0,\ldots,s_{m-1}) = \{(\epsilon_i)_{i<m} \in 2^m \mid \bigwedge_{i<m}\phi(a_{s_i},y)^{\epsilon_i} \nvdash_\pi \bot\}$. By Proposition 5.15, we may assume that c^π_m is constant; indeed, if f is as in Proposition 5.15, we may replace a_s with $a_{f(s)}$, and then q will still be the limit since f is cofinal. We will refer to the property that c^π_m is constant as **c^π_m-homogeneity**.

Identify $i \in \mathbb{N}$ with $\{0,\ldots,i-1\} \in \mathcal{P}^{\mathrm{fin}}(\kappa)$. Take a ϕ^{opp}-type $p_0(y) \in S^\pi_{\phi^{\mathrm{opp}}}((a_i)_{i<m})$ which first strictly alternates maximally and then is constantly true; i.e., $p_0(y) \vdash \phi(a_i,y) \leftrightarrow \neg\phi(a_{i+1},y)$ for $i<l$ and $p_0(y) \vdash \phi(a_i,y)$ for $i \in [l,m)$, and $l<m$ is maximal such that such a type exists. Note that $\mathrm{tp}(b/(a_i)_{i<m})$ is of this form with $l=0$, so some such p_0 exists (here we use the fact that $b \vDash \pi$). By c^π_m-homogeneity (in fact $c^\pi_{\mathrm{vc}(\phi)+1}$-homogeneity is enough) and the usual argument for bounding alternation number (see [Sim15a, Lemma 2.7]), we have $l \leq 2\,\mathrm{vc}(\phi) < m-2$.

Define

$$p_1(y) = p_0(y)|_{\{a_0,\ldots,a_l\}} \cup \{\phi(a_s,y) \mid l \subseteq s \in \mathcal{P}^{\mathrm{fin}}(\kappa)\},$$

and let $A' = \{a_0,\ldots,a_l\} \cup \{a_s \mid l \subseteq s \in \mathcal{P}^{\mathrm{fin}}(\kappa)\}$ be the domain of p_1.

Claim. *$p_1 \in S^\pi_{\phi^{\mathrm{opp}}\downarrow m-1}(A')$ and in particular is consistent with π.*

Proof. Suppose $A'_0 \subseteq_{\mathrm{fin}} A'$. Let $s_1 \in \mathcal{P}^{\mathrm{fin}}(\kappa)$ strictly contain all sets of the form $l \cup s \in \mathcal{P}^{\mathrm{fin}}(\kappa)$ such that $a_s \in A'_0$. Let $s_1 \subsetneq \ldots \subsetneq s_{m-l-2}$ be an $m-l-2$-chain starting with s_1. Let

$$p'_0(y) = p_1(y)|_{\{a_i \mid i\in[0,l]\cup\{s_1,\ldots,s_{m-l-2}\}\}}.$$

Then by c^π_m-homogeneity, p'_0 is consistent with π since p_0 is.

Suppose $l \subsetneq s_0 \subsetneq s_1$. We claim that $p'_0 \vdash_\pi \phi(a_{s_0},y)$.

Otherwise, by c^π_m-homogeneity,

$$p_1(y)|_{\{a_i \mid i\in[0,l]\cup[l+2,m)\}} \cup \{\neg\phi(a_{l+1},y)\}$$

is consistent with π. Since $l+2<m$, this contradicts the maximality of l.

In particular, $p'_0 \vdash_\pi p_1(y)|_{A'_0}$ and the latter is consistent with π. □

Now by Corollary 3.9(ii) applied to ϕ^{opp}, p_1 extends to $p \in S^\pi_{\phi\downarrow(m-1)+k_{\mathrm{comp}}(d)}(A)$. Since $m-1+k_{\mathrm{comp}}(d) \leq k_{\mathrm{sd}}(1,d)$, to conclude it is enough to show that $q(x)\otimes p(y) \vdash \phi(x,y)$. But this holds since $q = \lim_{s\to\mathcal{F}^{\mathrm{fin}}(\kappa)}(\mathrm{tp}_\phi(a_s/\mathcal{U}^y))$ and $p(y) \vdash \phi(a_s,y)$ for all $s \supseteq l$. □

We can now deduce Theorem 5.2.

Theorem 5.17. *Let $d \in \mathbb{N}$ and $\alpha \in (1/2,1]$. Then there exist n and k depending only on d,α such that the following holds. If $\phi(x,y)$ is a formula such that $\mathrm{vc}(\phi) \leq d$, then for any $A \subseteq \mathcal{U}^x$ and a (small) partial type $\pi(y)$, any $p \in S^\pi_{\phi^{\mathrm{opp}}}(A)$ is the α-rounded average of n types in $S^\pi_{\phi^{\mathrm{opp}}\downarrow k}(A)$.*

Namely, we may take $n := \min\{m \in \mathbb{N} \mid (1-\alpha)m > 2^{d+1}-1\}$ and $k := k_{\mathrm{sd}}(N_{\mathrm{pq}}(n,2^{d+1}),d)$.

Proof. By Lemma 5.8 it is enough to show $(\dagger)_{b,N,S}$ where $N = N_{\mathrm{pq}}(n, 2^{d+1})$ and $S = S^{\pi}_{\phi^{\mathrm{opp}}\downarrow k_{\mathrm{sd}}(N,d)}(A)$, which follows by Proposition 5.16. □

Remark 5.18. By Remark 5.10, in the context of Theorem 5.17, if p is not k-compressible then it is an α-rounded average of n *distinct* types in $S^{\pi}_{\phi^{\mathrm{opp}}\downarrow k}(A)$.

We give some immediate corollaries.

Corollary 5.19. *If $\phi(x, y)$ is NIP and $A \subseteq \mathcal{U}^y$ then $|S_{\phi^{\mathrm{opp}}\downarrow\star}(A)|+\aleph_0 = |S_{\phi^{\mathrm{opp}}}(A)|+\aleph_0$.*

Moreover, $|S_{\phi^{\mathrm{opp}}\downarrow k}(A)| + \aleph_0 = |S_{\phi^{\mathrm{opp}}}(A)| + \aleph_0$ for k from Theorem 5.17 and $S_{\phi^{\mathrm{opp}}\downarrow k}(A)$ is finite iff $S_{\phi^{\mathrm{opp}}}(A)$ is.

We can also improve Lemma 4.3 to add another equivalence:

Corollary 5.20. *The following are equivalent for an NIP formula $\phi(x, y)$:*

(i) ϕ is stable.
(ii) For any model $M \vDash T$ and any $k \in \mathbb{N}$, any $p \in S_{\phi\downarrow k}(M^y)$ is isolated.

Proof. (i) implies (ii) follows from Lemma 4.3.

$\neg$(i) implies $\neg$(ii): since ϕ is not stable, there is some (infinite) model M such that $|S_\phi(M)| > |M|$ (see e.g., [TZ12, Theorem 8.2.3]). By Corollary 5.19 (applied to ϕ^{opp}) $|S_{\phi\downarrow k}(M^y)| > |M|$ for some $k \in \mathbb{N}$. We conclude, since there are at most $|M|$ isolated ϕ-types over M. □

Remark 5.21. Suppose $\phi(x, y)$ is stable. Then we can replace k_{sd} in Proposition 5.16 by a linear (as opposed to exponential, see Remark 2.9) bound in terms of n with a simpler proof.

Let $\bar{x} = (x_i \mid i < n)$, $q(\bar{x}) \in S^{\bar{x}}_{\phi,A\text{-fs}}(\mathcal{U}^y)$, $\pi(y)$ and $b \in \pi(\mathcal{U})$ be as there. For $i < n$, let $q_i(x_i) = q \restriction \{\phi(x_i, y)\}$. As q_i is finitely satisfiable in A, by [TZ12, Exercise 8.3.6], q_i is definable by a Boolean combination of instances of ϕ^{opp} over A (the exercise assumes that T is stable but this is not necessary). The size of this Boolean combination depends only on ϕ (really only on the size of a maximal witness for the order property). Hence there is l depending only on ϕ, and $a_{i,j} \in A$, $\epsilon_{i,j} < 2$ for $j < l$, such that for some formula $\theta_i(y)$ of the form $\bigwedge_{j<l} \phi(a_{i,j}, y)^{\epsilon_{i,j}}$, $\theta_i(b)$ holds and if $b' \vDash \theta_i(y)$ then $\phi(x_i, b) \in q$ iff $\phi(x_i, b') \in q$. Let $\theta(y) = \bigwedge_{i<n} \theta_i$. Note that $\theta(b)$ holds, so that θ is consistent with π.

By Corollary 3.9(ii) applied to ϕ^{opp} (here we could use also the stable counterpart, using the 2-rank), there is some $p(y) \in S^{\pi}_{\phi^{\mathrm{opp}}\downarrow ln+k_{\mathrm{comp}}(\mathrm{vc}(\phi))}(A)$ such that $p(y) \vdash \theta(y)$. Thus,

$$q(\bar{x}) \otimes p(y) \vdash \bigwedge_{i<n} (\phi(x_i, y) \leftrightarrow \phi(x_i, b)).$$

Since q was arbitrary, we get $(\dagger)_{b,n,S}$ for $S := S^{\pi}_{\phi^{\mathrm{opp}}\downarrow ln+k_{\mathrm{comp}}(\mathrm{vc}(\phi))}(A)$.

5.4. **Local uniform honest definitions.**

In this section we will prove uniformity of honest definitions for NIP formulas.

Definition 5.22. [Sim15a, Definition 3.16 and Remark 3.14] Suppose $\phi(x, y)$ is a formula, $A \subseteq M^x$ is some set and $b \in \mathcal{U}^y$. Say that a formula $\psi(x, z)$ over $\emptyset$ (with z a tuple of variables each of the same sort as x) is an **honest definition** of $\mathrm{tp}_{\phi^{\mathrm{opp}}}(b/A)$ if for every finite $A_0 \subseteq A$ there is some $c \in A^z$ such

$$\phi(A_0, b) \subseteq \psi(A, c) \subseteq \phi(A, b).$$

In other words, for all $a \in A$, if $\psi(a, c)$ holds then so does $\phi(a, b)$ and for all $a \in A_0$ the other direction holds: if $\phi(a, b)$ holds then $\psi(a, c)$ holds.

It is proved in [Sim15a, Theorem 6.16], [CS15, Theorem 11] that if T is NIP then for every $\phi(x,y)$ there is a formula $\psi(y,z)$ that serves as an honest definition for any type in $S_\phi(A)$ provided that $|A| \geq 2$ (by [CS15, Remark 16] only some NIP is required of ϕ and formulas expressing consistency of Boolean combinations of $\mathrm{vc}(\phi)+1$ instances of ϕ). In this section we improve this by proving this result assuming only that ϕ is NIP.

Corollary 5.23. *Let $\phi(x,y)$ be NIP. Then there exists $\psi(x,z)$ such that if $A \subseteq \mathcal{U}^x$ with $|A| > 1$ and $b \in \mathcal{U}^y$, then $\psi(x,z)$ is an honest definition of $\mathrm{tp}_{\phi^{\mathrm{opp}}}(b/A)$.*
Namely,

$$\psi(x,(\overline{z},\overline{z}',\overline{z}'')) := \mathrm{Maj}_{i<n} \forall y \left(\bigwedge_{j<k} (\phi(z_{i,j},y) \leftrightarrow (z'_{i,j} = z''_{i,j})) \to \phi(x,y) \right),$$

where n and k are as in Theorem 5.17 with $d = \mathrm{vc}(\phi)$ and $\alpha = 1/2$.

Proof. By Theorem 5.17, $\mathrm{tp}_{\phi^{\mathrm{opp}}}(b/A)$ is the rounded average of k-compressible types $p_0,\ldots,p_{n-1} \in S_{\phi^{\mathrm{opp}}}(A)$.

Now we proceed as in the proof of Lemma 3.20: if $A_0 \subseteq_{\mathrm{fin}} A$, there are $D_i = (d_{i,j})_{j<k} \subseteq A$ for $1 \leq i \leq n$ such that $p_i|_{D_i} \vdash p_i|_{A_0}$. Let $c_0, c_1 \in A$ be distinct, let $d'_{i,j} = c_0$, and let $d''_{i,j} = c_0$ if $p_i(y) \vdash \phi(d_{i,j},y)$ and $d''_{i,j} = c_1$ otherwise, so $p_i|_{D_i}(y)$ is equivalent to $\bigwedge_{j<k}(\phi(d_{i,j},y) \leftrightarrow (d'_{i,j} = d''_{i,j}))$. Then $d := ((d_{i,j})_{ij}, (d'_{i,j})_{ij}, (d''_{i,j})_{ij})$ is as required:

For each $a \in A$, $\psi(a,d)$ holds iff $|\{i < n \mid p_i(y)|_{D_i} \vdash \phi(a,y)\}| > \frac{1}{2}n$. Thus, if $a \in A_0$ and $\phi(a,b)$ holds, then $|\{i < n \mid \phi(a,y) \in p_i\}| > \frac{1}{2}n$, and for each such $i < n$, $p_i|_{D_i} \vdash \phi(a,y)$ (by choice of D_i), so $\psi(a,d)$ holds. On the other hand, if $\psi(a,d)$ holds, then clearly $|\{i < n \mid \phi(a,y) \in p_i\}| > \frac{1}{2}n$, hence $\phi(a,b)$ holds. □

Remark 5.24. In fact, by a Löwenheim-Skolem argument, to prove Corollary 5.23 we require Theorem 5.17 only in the case that A is countable. The proof of this case of Theorem 5.17 is slightly simpler, in that we can use ω in place of $\mathcal{P}^{\mathrm{fin}}(\kappa)$ (using [Sim15b, Lemma 2.8] instead of [Sim15b, Lemma 2.9]), and the usual Ramsey theorem in place of Proposition 5.15.

Remark 5.25. If $\phi(x,y)$ is stable, one can use Theorem 5.17 similarly to get a new way to see definability of ϕ-types over arbitrary sets (since k-isolated types are definable).

5.5. **Hypes.**

Definition 5.26. Suppose $\phi(x,y)$ is a formula and $A \subseteq \mathcal{U}^y$. For $k \in \mathbb{N}$, a ***k*-hype**[4] **in ϕ over A**[5] is a collection Γ of instances of ϕ and $\neg\phi$ over A such that:

(1) It is k-consistent: if $S \subseteq \Gamma$ is of size $\leq k$, then S is consistent.
(2) For any $a \in A$, either $\phi(x,a) \in \Gamma$ or $\neg\phi(x,a) \in \Gamma$, but not both.

Suppose $\pi(x)$ is a (small) partial type. We say that Γ is k-consistent with π if in (1) we ask that $S \cup \pi$ is consistent.

Let $S^x_{\phi,k}(A)$ be the set of k-hypes in ϕ over A, and $S^\pi_{\phi,k}(A)$ the set of k-hypes which are k-consistent with π.

As with types, if Γ is a k-hype, we use the notation $\Gamma|_{A'}$ for the restriction of Γ to $A' \subseteq A$ with the obvious meaning.

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[4] The term "k-hype" stands for k-hypotype.

[5] As in Section 2.1, in this notation we have the partition in mind and x is the first tuple in the partition, so we do not specify x.

Suppose $\phi(x, y)$ is a formula, $\pi(y)$ a small partial type and $A \subseteq \mathcal{U}^x$. Let M be a small model containing A and the domain of π. We will consider the following auxiliary 2-sorted structure

$$\hat{A} = (A, S^{\pi}_{\phi^{\mathrm{opp}},k}(A), R)$$

where A and $S^{\pi}_{\phi^{\mathrm{opp}},k}(A)$ are in distinct sorts P, Q respectively and $R(x, z)$ is interpreted as $R(a, \Gamma) \Leftrightarrow \phi(a, y) \in \Gamma$ (so $R \subseteq P \times Q$). Let $\psi(x, z) = R(x, z)$.

Lemma 5.27. *Suppose $N \succ \hat{A}$ and $e \in Q^N$. Then:*

1. *If ϕ is NIP and $\mathrm{vc}(\phi) < k$, then ψ is NIP and $\mathrm{vc}(\psi) \le \mathrm{vc}(\phi)$.*
2. *Let $\Gamma_e = \{\phi(a, y)^{N \vDash R(a,e)} \mid a \in A\}$. Then $\Gamma_e \in S^{\pi}_{\phi^{\mathrm{opp}},k}(A)$.*
3. *If $\mathrm{tp}_{\psi^{\mathrm{opp}}}(e/A) \in S_{\psi^{\mathrm{opp}}\downarrow k}(A)$, then $\Gamma_e \in S^{\pi}_{\phi^{\mathrm{opp}}\downarrow k}(A)$, i.e., Γ_e is consistent with π and moreover k-compressible modulo π.*

Proof. (1) Suppose $C \subseteq_{\mathrm{fin}} A$ is of size $|C| = \mathrm{vc}(\phi) + 1$, and suppose that for any subset $C' \subseteq C$ there is some $\Gamma_{C'} \in S^{\pi}_{\phi^{\mathrm{opp}},k}(A)$ such that for all $c \in C$, $R(c, \Gamma_{C'})$ holds iff $c \in C'$. As $k \ge |C|$, for any $C' \subseteq C$, $\{\phi(c, y)^{R(c,\Gamma_{C'})} \mid c \in C\}$ is consistent; let $e_{C'}$ realise it. Then we get that $M \vDash \phi(c, e_{C'})$ iff $R(c, \Gamma_{C'})$ iff $c \in C'$. Thus C is shattered by $\phi(x, y)$ and has size $\mathrm{vc}(\phi) + 1$, contradiction

(2) Clearly (2) in Definition 5.26 holds. For (1), suppose $S = \Gamma_e|_{A_0}$ has size k. Then, as N is an elementary extension of $\hat{A}$, there is some $\Gamma \in S^{\pi}_{\phi^{\mathrm{opp}},k}(A)$ such that for all $a \in A_0$, $N \vDash R(a, e)$ iff $\hat{A} \vDash R(a, \Gamma)$. In particular, $S \subseteq \Gamma$. Since Γ is k-consistent, S is consistent.

For (3), suppose $A_0 \subseteq_{\mathrm{fin}} A$. Let $A_1 \subseteq A$ be of size k such that $\mathrm{tp}_{\psi^{\mathrm{opp}}}(e/A)|_{A_1} \vdash \mathrm{tp}_{\psi^{\mathrm{opp}}}(e/A)$. Since A_0 is arbitrary, it is enough to show (a) that there exists $d \vDash \Gamma_e|_{A_1} \cup \pi$ (in $\mathcal{U}$) and (b) that any such d satisfies $\Gamma_e|_{A_0}$. (a) follows from (2). For (b), let $q = \mathrm{tp}_{\phi^{\mathrm{opp}}}(d/A)$, so in particular, $q \in S^{\pi}_{\phi^{\mathrm{opp}},k}(A)$. Note that in $\hat{A}$, $q \vdash \mathrm{tp}_{\psi^{\mathrm{opp}}}(e/A)|_{A_1}$. Hence $q \vdash \mathrm{tp}_{\psi^{\mathrm{opp}}}(e/A)|_{A_0}$. Hence we have that $\phi(a, y)^{R(a,e)} \in q$ for any $a \in A_0$, or in other words, $d \vDash \Gamma_e|_{A_0} \cup \pi$. □

Remark 5.28. Suppose $\phi(x, y)$ is NIP and $k > \mathrm{vc}(\phi)$. Then by the (p, q)-theorem (Fact 5.4) and compactness, for any k-hype Γ, there are $N := N_{\mathrm{pq}}(k, k)$ types $p_0, \ldots, p_{N-1}$ such that $\Gamma \subseteq \bigcup_{i<N} p_i$.

The following extends Remark 5.28: not only are hypes covered by types, they are the rounded average of (compressible) types. It also generalises Theorem 5.17 to hypes.

Theorem 5.29. *Let $d \in \mathbb{N}$ and $\alpha \in (1/2, 1]$. Then there exist n and k depending only on d, α such that the following holds. If $\phi(x, y)$ is a formula such that $\mathrm{vc}(\phi) \le d$, then for any $A \subseteq \mathcal{U}^x$ and a (small) partial type $\pi(y)$, any k-hype $\Gamma \in S^{\pi}_{\phi^{\mathrm{opp}},k}(A)$ is the α-rounded average of n types in $S^{\pi}_{\phi^{\mathrm{opp}}\downarrow k}(A)$.*

In fact, k and n can be taken to be the numbers from Theorem 5.17.

Proof. Let k and n be as in Theorem 5.17, and note that by Remark 2.9, we have that $k > d$. Let $\phi(x, y)$ be such that $\mathrm{vc}(\phi) \le d < k$. Let Γ be a k-hype in ϕ^{opp} over A.

Consider the structure $\hat{A}$ as above, and let $\psi(x, z) = R(x, z)$. By Lemma 5.27(1), as $\mathrm{vc}(\phi) < k$, $\psi(x, z)$ is NIP and $\mathrm{vc}(\psi) \le \mathrm{vc}(\phi)$. By Theorem 5.17, any ψ^{opp}-type is the α-rounded average of n types in $S_{\psi^{\mathrm{opp}}\downarrow k}(A)$. In particular, this is true for $\mathrm{tp}_{\psi^{\mathrm{opp}}}(\Gamma/A)$; let $r_0, \ldots, r_{n-1}$ witness this. Let $e_i \vDash r_i$ for $i < n$ (in an elementary extension $N \succ \hat{A}$), and let $\Gamma_i = \Gamma_{e_i}$ be the corresponding k-hypes as in Lemma 5.27(2). By Lemma 5.27(3), we get that Γ_i is a type in $S^{\pi}_{\phi^{\mathrm{opp}}\downarrow k}(A)$.

Finally, since $\mathrm{tp}_{\psi^{\mathrm{opp}}}(\Gamma/A)$ is the α-rounded average of $r_0, \ldots r_{n-1}$, it follows by definition that Γ is the α-rounded average of $\Gamma_0, \ldots, \Gamma_{n-1}$. □

We deduce the existence of honest definitions for k-hypes.

Corollary 5.30. *Let $\phi(x,y)$ be NIP and let k be as in Theorem 5.29 for $d = \mathrm{vc}(\phi)$ and $\alpha = 1/2$. Then there exists $\psi(x,z)$ such that if $A \subseteq \mathcal{U}^x$ with $|A| > 1$ and $\Gamma \in S_{\phi^{\mathrm{opp}},k}(A)$ is a k-hype, then $\psi(x,z)$ is an honest definition of Γ in the sense that if $A_0 \subseteq_{\mathrm{fin}} A$, then there is some $d \in A^z$ such that:*

(1) If $a \in A_0$ and $\phi(a,y) \in \Gamma$ then $\psi(a,d)$ holds.
(2) For all $a \in A$, if $\psi(a,d)$ holds, then $\phi(a,y) \in \Gamma$.

Namely,

$$\psi(x,(\overline{z},\overline{z}',\overline{z}'')) := \mathrm{Maj}_{i<n} \forall y \left(\bigwedge_{j<k} (\phi(z_{i,j},y) \leftrightarrow (z'_{i,j} = z''_{i,j})) \to \phi(x,y) \right),$$

where n and k are as in Theorem 5.29.

Proof. The proof is the same as the one of Corollary 5.23, using Theorem 5.29. □

We relate hypes to the Shelah expansion which we now recall.

Definition 5.31. For a structure M, the Shelah expansion M^{Sh} of M is given by: for any formula $\phi(x,y)$ and any $b \in \mathcal{U}^y$, add a new relation $R_{\phi(x,b)}(x)$ interpreted as $\phi(M,b)$.

Fact 5.32. *[She09] If T is NIP then for any $M \vDash T$, M^{Sh} is NIP.*

Corollary 5.33. *Suppose T is NIP, and let $M \vDash T$. For each formula $\phi(x,y)$, let k_ϕ be as in Theorem 5.29 for $d = \mathrm{vc}(\phi)$ and $\alpha = 1/2$.*

Consider the expansion M^{hSh} of M given by naming for each partitioned $\mathcal{L}$-formula ϕ and each k_ϕ-hype $\Gamma \in S_{\phi^{\mathrm{opp}},k_\phi}(M^x)$ the set $R_\Gamma := \{a \in M^x \mid \phi(a,y) \in \Gamma\}$. Then M^{hSh} is interdefinable with M^{Sh} and in particular is NIP.

Proof. Since every ϕ^{opp}-type is in particular a k_ϕ-hype, every $R_{\phi(x,b)}(M)$ is definable in $M^{h\mathrm{Sh}}$.

For the other direction, fix some formula $\phi(x,y)$ and some k_ϕ-hype $\Gamma \in S_{\phi^{\mathrm{opp}},k}(M)$. By Theorem 5.29 (applied with $d = \mathrm{vc}(\phi)$ and $\alpha = 1/2$), there are $r_0,\ldots,r_{n-1} \in S_{\phi^{\mathrm{opp}}}(M)$ such that Γ is the rounded average of $r_0,\ldots,r_{n-1}$. Let $a_0 \vDash r_0,\ldots,a_{n-1} \vDash r_{n-1}$. Then $R_\Gamma(M) = \mathrm{Maj}_{i<n}(R_{\phi(x,a_i)}(M))$, and thus is definable in M^{Sh}. □

5.6. **UDTFS for pseudofinite types.** Here we extend UDTFS to *pseudofinite* types: every pseudofinite ϕ-type (Definition 5.34) is definable, and uniformly so.

Definition 5.34. Let $\mathcal{L}' = \mathcal{L} \cup \{P,Q\}$ where P,Q are predicates for subsets of $\mathcal{U}^x$.

Suppose $\phi(x,y)$ is an $\mathcal{L}$-formula, $M \vDash T$, $D \subseteq M^x$, and $p \in S_{\phi^{\mathrm{opp}}}(D)$. For $\epsilon < 2$, let $D^\epsilon = \{a \in D \mid \phi(a,y)^\epsilon \in p\}$. Then p is **pseudofinite** if for every $\mathcal{L}'$-sentence φ, if $(M,D^0,D^1) \vDash \varphi$ then there is an $\mathcal{L}'$-structure N such that $N \vDash \varphi$ and P^N, Q^N are finite.

Remark 5.35. In the notation of Definition 5.34, a type $p \in S_{\phi^{\mathrm{opp}}}(D)$ is pseudofinite iff there is a model $(N,E^0,E^1) \equiv (M,D^0,D^1)$ which is an ultraproduct $\prod_{i\in I} N_i/U$ such that $(E^\epsilon)^{N_i}$ is finite for each $i \in I$ and $\epsilon < 2$ (essentially the same proof as in [Vää03, Lemma 1] works).

Theorem 5.36. *Suppose $\phi(x,y)$ is NIP. Then there is a formula $\psi(x,z)$ such that whenever $M \vDash T$, $D \subseteq M^x$ is of size > 1 and $p \in S_{\phi^{\mathrm{opp}}}(D)$ is pseudofinite, p is definable by an instance of ψ over D^z.*

Moreover, if T has Skolem functions, then we can choose $\psi(x,z)$ to be NIP.

Proof. For the first part, let k and $\psi(x,z)$ be as in Corollary 5.30. Suppose that $p(y)$ is not definable by an instance of ψ over D^z. Working in the expansion (M, D^0, D^1) as in Definition 5.34, we get that in some $\mathcal{L}'$-structure (N, E^0, E^1), letting $E = E^0 \cup E^1$, the following hold:

- E is finite of size > 1.
- The formula $\phi(x,y)$ is NIP in N and its VC-dimension equals $\mathrm{vc}(\phi)$.
- The set of formulas $\Gamma = \{\phi(a,y) \mid a \in E^1\} \cup \{\neg\phi(a,y) \mid a \in E^0\}$ is a k-hype.
- E^1 is not definable in N by any instance of ψ over E^z.

However, by the choice of ψ and as E is finite, there is some $d \in E^z$ such that $\psi(E,d) = E^1$, contradiction.

For the second part, assuming that T has Skolem functions, we let

$$\psi(x, (\overline{z}, \overline{z}', \overline{z}'')) := \mathrm{Maj}_{i<n} \left(\phi(x, f(z_i, \overline{z}'_i, \overline{z}''_i))\right),$$

where n is as in Theorem 5.29 and f is a $\emptyset$-definable function such that T thinks that if $\exists y \bigwedge_{j<k}(\phi(z_{i,j},y) \leftrightarrow (z'_{i,j} = z''_{i,j}))$ then $f(\overline{z}_i, \overline{z}'_i, \overline{z}''_i) \vDash \bigwedge_{j<k}(\phi(z_{i,j},y) \leftrightarrow (z'_{i,j} = z''_{i,j}))$ (whose existence we assumed).

Note that $\psi(x,z)$ is NIP, since $\phi(x, f(z))$ is NIP; see also [EK20, Proof of Proposition 26]. To see that it works, assume not. Then using the same argument as above, we get an $\mathcal{L}'$-structure N with the same properties as above. Now, review the proof of Corollary 5.23. When the domain D of the k-compressible types p_i (for $i < n$) is finite, then p_i is in fact isolated by a conjunction of k instances of ϕ^{opp} or its negation, thus, putting the isolating parameters for z_i and coding the negations using z'_i, z''_i and two elements from D^N, we are done. □

Remark 5.37. Note that if p is realised in M, then Theorem 5.36 follows directly from UDTFS: in that case, in the proof one can replace the demand about Γ being a hype by it being a type.

Remark 5.38. Clearly Theorem 5.36 implies UDTFS (Corollary 5.14), and hence its conclusion implies that $\phi(x,y)$ is NIP (see e.g. the proof of Theorem 14 in [EK20]).

6. Compressibility as an isolation notion

In this section we study properties of compressibility seen as an isolation notion (mostly) under NIP, and in particular as a way to construct models analogous to constructible models in totally transcendental theories. Towards that we prove a transitivity result for compressibility in Proposition 6.25, which uses the type decomposition theorem from [Sim20].

As an application, we will show that if T is unstable and $M \vDash T$ is ω-saturated, then there are arbitrarily large elementary extensions N of M such that every generically stable type over M (see Definition 4.16) realised in N is realised in M (this is Corollary 6.39 and Remark 6.40).

6.1. **Monotonicity.**

Lemma 6.1. *Suppose b, c are finite tuples. If* $\mathrm{tp}(cb/A)$ *is compressible then so are* $\mathrm{tp}(c/Ab)$ *and* $\mathrm{tp}(b/A)$.

Proof. We start by showing that $\mathrm{tp}(c/Ab)$ is compressible. Suppose z is a tuple of variables such that $b \in \mathcal{U}^z$. Given a formula $\phi(x,y)$, let Φ be the set of all formulas of the form $\psi(x,z,y)$ we get from substituting variables from y by variables from yz in ϕ. Fix some formula $\psi(x,z,y) \in \Phi$. By assumption, there is some formula $\zeta_\psi(xz,w)$ that compresses $\mathrm{tp}_\psi(cb/A)$ (with the partition $\psi(xz,y)$). This means that for any $A_0 \subseteq_{\mathrm{fin}} A$ there is some $a_{\psi,A_0} \in A^w$ such that $\mathrm{tp}(cb/A) \vdash \zeta_\psi(xz, a_{\psi,A_0}) \vdash \mathrm{tp}_\psi(cb/A_0)$.

Then we have that $\mathrm{tp}(c/Ab) \vdash \bigwedge_{\psi\in\Phi} \zeta_\psi(x, ba_{\psi,A_0}) \vdash \mathrm{tp}_\phi(c/A_0 b)$, and this shows that $\mathrm{tp}(c/Ab)$ is compressible.

Now we show that $\mathrm{tp}(b/A)$ is compressible. Fix some formula $\phi(z, y)$. Suppose $\zeta(xz, w)$ compresses $\mathrm{tp}_\phi(cb/A)$. Fix some $A_0 \subseteq_{\mathrm{fin}} A$ and suppose $a \in A^w$ is such that $\mathrm{tp}(cb/A) \vdash \zeta(xy, a) \vdash \mathrm{tp}_\phi(cb/A_0)$. Note that if $a_0 \in A_0^y$ is such that $\vDash \forall x, z(\zeta(x, z, a) \to \phi(z, a_0)^\epsilon)$ for $\epsilon < 2$, then $\vDash \forall z(\exists x\zeta(x, z, a) \to \phi(z, a_0)^\epsilon)$, so $\mathrm{tp}(b/A) \vdash \exists x\zeta(x, z, a) \vdash \mathrm{tp}_\phi(b/A_0)$. □

Remark 6.2. We cannot hope for Lemma 6.1 to hold when b is infinite: every type over $\emptyset$ is compressible trivially, so if $\mathrm{tp}(a/B)$ is not compressible and b enumerates B, then $\mathrm{tp}(ab)$ is compressible while $\mathrm{tp}(a/b)$ is not.

However, the converse to Lemma 6.1 holds for infinite tuples as well (see Remark 6.27 below). For finite tuples this can be seen by a direct argument of this kind, but for infinite tuples we will need stronger tools which we will develop in the next section under NIP.

Definition 6.3. Suppose B, A are sets. We say B **is compressible over** A if $\mathrm{tp}(\bar{b}/A)$ is compressible where $\bar{b}$ is some (any) tuple enumerating B.

Remark 6.4. The set B is compressible over B (even isolated).

Remark 6.5. By Remark 3.17 B is compressible over A iff for every finite tuple b from B, $\mathrm{tp}(b/A)$ is compressible over A.

Lemma 6.6. *Given a set B and a tuple a, the set Ba is compressible over B if and only if* $\mathrm{tp}(a/B)$ *is compressible.*

Proof. Left to right follows from Lemma 6.1, so suppose that $\mathrm{tp}(a/B)$ is compressible. Let b be a tuple enumerating B. We must show that $\mathrm{tp}(ab/B)$ is compressible. By Remark 3.17, it is enough to prove that $\mathrm{tp}(ab'/B)$ is compressible where b' is a finite sub-tuple of b. Let y be a tuple of variables in the sort of b'. Let $\phi(x, y, z)$ be a formula and let $\psi(x, s)$ compress $\mathrm{tp}_{\phi(x,yz)}(a/B)$. Then $\psi(x, s) \wedge y = t$ compresses $\mathrm{tp}_{\phi(xy,z)}(ab'/B)$. □

6.2. **Type decomposition and rescoping compressibility.** Here we use the results from [Sim20] to prove that compressibility can be *rescoped* to an arbitrary subset of the domain (see Propositions 6.18 and 6.23).

For the remainder of Section 6 **we assume that T is NIP** unless otherwise specified.

We first recall the definition of a generically stable partial type. As opposed to previous sections, here a partial type does not have to be small, i.e., it is over $\mathcal{U}$. We call such partial types **global partial types**. As for global types, a global partial type π is A-invariant if it is invariant under automorphisms of $\mathcal{U}$ fixing A.

Remark 6.7. Suppose $\pi(x)$ is a global partial type. Then for any small set A, if $a \vDash \pi|_A$, then $\pi(x) \cup \mathrm{tp}(a/A)$ is consistent, and hence for any B there is some $a' \equiv_A a$ such that $a' \vDash \pi|_B$.

Definition 6.8. We say that a global partial type π is **ind-definable** over A if for every $\phi(x, y)$, the set $\{b \in \mathcal{U}^y \mid \phi(x, b) \in \pi\}$ is ind-definable over A, i.e., it is a union of A-definable sets.

Remark 6.9. [Sim20, Discussion after Definition 2.1] Note that $\pi(x)$ is ind-definable iff $\{\phi(x, c) \mid \pi \vdash \phi(x, c)\}$ is ind-definable.

Fact 6.10. *[Sim20, Lemma 2.2] Let $\pi(x)$ be an A-invariant global partial type. Then π is ind-definable over A if and only if the set $X = \{(a, \bar{b}) \mid \bar{b} \in \mathcal{U}^\omega, a \vDash \pi|_{A\bar{b}}\}$ is type-definable over A.*

Definition 6.11. Let $\pi(x)$ be a global partial type. We say that π is **generically stable over** A if π is ind-definable over A and the following holds:

(GS) if $(a_k \mid k<\omega)$ is such that $a_k \vDash \pi|_{Aa_{<k}}$ and $\pi \vdash \phi(x,b)$, then for all but finitely many values of k we have $\mathcal{U} \vDash \phi(a_k, b)$.

Remark 6.12. Note that a global type $p(x) \in S^x(\mathcal{U})$ is generically stable over A as in Definition 4.12 iff it is generically stable over A as a partial type (note that it is A-definable by Fact 4.13).

Remark 6.13. Much like in Remark 3.17, a global partial type $\pi(x)$ is generically stable iff its restriction to any finite sub-tuple x' of x is generically stable. (Note that we do not assume that π is ind-definable.)

Why? Clearly if the restrictions are all generically stable then π is, so we show the converse. Assume that π is generically stable and fix some finite $x' \subseteq x$. First note that $\pi \restriction x'$ is ind-definable, so we show (GS). Assume that $(a'_k \mid k<\omega)$ is such that $a'_k \vDash (\pi \restriction x')|_{Aa'_{<k}}$ and $\pi \vdash \phi(x', b)$. By induction on $n<\omega$ we construct sequences $(a_i \mid i<n)$ such that $a_i \restriction x' = a'_i$ and $a_i \vDash \pi|_{Aa_{<i}}$ for all $i<n$. Suppose we found such a sequence $(a_i \mid i<n)$. Since $a'_n \vDash (\pi \restriction x')|_{Aa'_{<k}}$, as in Remark 6.7 there is some $a''_n \vDash \pi|_{Aa_{<n}}$ such that $(a''_n \restriction x') \equiv_{Aa'_{<n}} a'_n$. Let σ be an automorphism fixing $Aa'_{<n}$ such that $\sigma(a''_n \restriction x') = a'_n$. Then $(\sigma(a_i) \mid i<n)\sigma(a''_n)$ is a sequence of length $n+1$ which is as required. By compactness and Fact 6.10, there is some sequence $(a_i \mid i<\omega)$ such that $a_i \restriction x' = a'_i$ and $a_i \vDash \pi|_{Aa_{<i}}$ for all $i<\omega$. By (GS) for π, for all but finitely many values of k we have $\vDash \phi(a'_k, b)$, as required.

Definition 6.14. We say that a global partial type $\pi(x)$ is **finitely satisfiable in** $A \subseteq \mathcal{U}$ if any formula implied by π has a realisation in A.

Fact 6.15. *Let $\pi(x)$ be a global partial type generically stable over A. Then:*

(FS) π is finitely satisfiable in every model containing A.

(NF) Let $\phi(x,b)$ be such that $\pi \vdash \phi(x,b)$ and take $a \vDash \pi|_A$ such that $\vDash \neg\phi(a,b)$. Then both $\operatorname{tp}(b/Aa)$ *and* $\operatorname{tp}(a/Ab)$ *fork over A.*

We now state [Sim20, Theorem 4.1] in the form we will use it below. Our formulation follows from the proof (rather than the statement) of [Sim20, Theorem 4.1], in particular from [Sim20, Proposition 4.7].

Fact 6.16. *[Sim20, Proposition 4.7] Given a type* $\operatorname{tp}(a/A)$ *and $q \in S_{A\text{-}fs}(\mathcal{U})$, there exists a global partial type $\pi(x)$ generically stable over A with $a \vDash \pi|_A$ such that if $(X,<)$ is an infinite linear order, $I \vDash q^{(X)}|_{Aa}$, $b \vDash q|_{AI}$ and $a \vDash \pi|_{AIb}$, then $b \vDash q|_{Aa}$.*

Remark 6.17. (i) In [Sim20, Proposition 4.7] the generically stable type constructed depends on q. Call it π_q. However, in the paragraph after [Sim20, Proposition 4.7], it is remarked that taking π to be the union of all the π_q works for all q.

(ii) It is not assumed in [Sim20] that the the tuple a above is finite.

(iii) Throughout the proof of [Sim20, Theorem 4.1], the sequences are assumed to be densely ordered without endpoints. In particular, that is the case for I above. However, the result is true for any infinite I. Indeed, suppose I, a, A, b are as in Fact 6.16. By Ramsey and compactness (and as I is infinite) there is an Aab-indiscernible sequence $I' = (a_i)_{i\in\mathbb{Q}}$ realising the EM-type of I over Aab. Since I is Aa-indiscernible, it follows that $I' \vDash q^{(\mathbb{Q})}|_{Aa}$, and since q is A-invariant, $b \vDash q|_{AI'}$. Also, by Fact 6.10, $a \vDash \pi|_{AI'b}$. So I' satisfies all the requirements of Fact 6.16 and is densely ordered with no endpoints, so $b \vdash q|_{Aa}$ as required.

(iv) In the context of Fact 6.16, it follows (by applying an automorphism; note that both q and π are A-invariant) that if $a' \vDash \pi|_{AIb} \cup \operatorname{tp}(a/AI)$, then $b \vDash q|_{Aa'}$.

Proposition 6.18. *Let $(X,<)$ be any (small) infinite linearly ordered set. A type* $\mathrm{tp}(a/A)$ *is compressible if and only if for every $q \in S_{A\text{-}fs}(\mathcal{U})$, if $I \models q^{(X)}|_{Aa}$, then*

$$q|_{AI} \vdash q|_{Aa}.$$

Moreover, if X has no first element then $q|_{AI} \vdash q|_{AIa}$.

Proof. That this condition implies compressibility follows from Fact 2.14(3⇒1), since I can be taken from A' (where (A', a) is saturated enough).

For the converse, assume compressibility and let q, I be as in the proposition. Let d be given by Corollary 2.15 so that $q|_{Ad} \vdash q|_{Aa}$ and $\mathrm{tp}(d/Aa)$ is finitely satisfiable in A. By perhaps changing d (by applying an automorphism fixing Aa), we may assume that $I \models q^{(X)}|_{Aad}$.

Let $b \models q|_{AI}$. We want to show that $b \models q|_{Aa}$.

Applying Fact 6.16 (and Remark 6.17(ii),(iv)) to $\mathrm{tp}(d/A)$, we obtain a generically stable over A global partial type π with $d \models \pi|_A$ such that $b \models q|_{Ad'}$ for any $d' \models \pi|_{AIb} \cup \mathrm{tp}(d/AI)$.

Now $\mathrm{tp}(d/Aa)$ is finitely satisfiable in, and hence does not fork over, A. Similarly, $\mathrm{tp}(I/Aad)$ is finitely satisfiable in A and so does not fork over Aa. By applying (NF) twice, it follows that $d \models \pi|_{AIa}$.

Hence $\pi' := \pi \cup \mathrm{tp}(d/AIa)$ is consistent by Remark 6.7. Let $\kappa := |\mathcal{L}(AIb)|^+$, and let $(d_i)_{i<\kappa}$ be a Morley sequence in π' over AIa, i.e., $d_i \models \pi'|_{AIad_{<i}}$. By (GS) and the choice of κ, for some $i < \kappa$ we have $d_i \models \pi|_{AIb}$. Then $d_i \models \pi|_{AIb} \cup \mathrm{tp}(d/AI)$, so $b \models q|_{Ad_i}$. But $d_i \equiv_{Aa} d$, so $q|_{Ad_i} \vdash q|_{Aa}$. So $b \models q|_{Aa}$, as required.

We conclude the "moreover" part. Suppose that X has no first element and that $I \models q^{(X)}|_{Aa}$, $b \models q|_{AI}$. By compactness we can find some I' of order type $\omega \times (X+1)$ (where $X+1$ is adding one more element in the end of X and the product is ordered lexicographically), such that $I' + I$ is indiscernible over Aa and over Ab. Thus, partitioning I' into $(X+1)$-sequences, we have that $I' \models (q^{(X+1)})^{(\omega)}|_{Aa}$, and $(I+b) \models q^{(X+1)}|_{AI'}$. Applying the first part to $q^{(X+1)}$, we have that $(I+b) \models q^{(X+1)}|_{Aa}$. It follows that $b \models q|_{AIa}$ as required. □

The following corollary will not be used in this paper.

Corollary 6.19. *A type $p = \mathrm{tp}(a/A) \in S(A)$ is compressible iff for any $q \in S_{A\text{-}fs}(\mathcal{U})$ and any Morley sequence $I := I_1 + (b) + I_2$ of q over A where I_1 has no first element, if $I_1 + I_2$ is a Morley sequence of q over Aa then so is I.*

Proof. Right to left is clear by Proposition 6.18, so suppose that p is compressible, and we are given I as above. Let $(b_i \mid 1 \leq i < n)$ be some finite subsequence from I_2 and let $b_0 = b$. Then by applying the "moreover" part of Proposition 6.18 inductively, $b_i \models q|_{AI_1 b_{<i} a}$. Since this is true for any $n < \omega$, I is Aa-indiscernible. □

Remark 6.20. One might call the condition in Corollary 6.19 **generic co-distality**: it is co-distality in a generic sense. For a definition of distal and co-distal types and a short discussion, see [EK21, Definition 4.21 and Remark 4.22].

Definition 6.21. Suppose $B, A \subseteq \mathcal{U}$ are (small) sets. Say that a type $p \in S(A)$ **is compressible up to** B if p is compressible in $\mathcal{U}_B$ (in the language $\mathcal{L}(B)$): in Definition 3.14, all the formulas are over B.

Remark 6.22. Suppose that $A \subseteq B$, $p \in S(A)$ and p is compressible up to B. Then p is compressible up to $C := B \setminus A$: given a formula $\phi(x,y)$ over C, there is an $\mathcal{L}$-formula $\psi(x,z,w,t)$ and $a \in A^w, c \in C^t$ such that $\psi(x,z,ac)$ compresses $p \upharpoonright \phi$. But then $\psi(x,z,w,c)$ compresses $p \upharpoonright \phi$.

Proposition 6.23. *If a type* $\mathrm{tp}(a/B)$ *is compressible and $A \subseteq B$, then* $\mathrm{tp}(a/A)$ *is compressible up to B.*

Proof. We use Proposition 6.18, so we are given $q \in S_{A\text{-fs}}(\mathcal{U}_B)$, and we need to show that if $I \vDash q^{(\omega)}|_{Aa}$ then $q|_{AI} \vdash q|_{Aa}$, all working in $\mathcal{U}_B$. So suppose that in $\mathcal{U}_B$, $b \vDash q|_{AI}$ and we need to show that $b \vDash q|_{Aa}$.

Taking the reduct to $\mathcal{L}$ (and identifying $q \upharpoonright \mathcal{L}$ with q), $q \in S_{A\text{-fs}}(\mathcal{U})$, $I \vDash q^{(\omega)}|_{Ba}$, $b \vDash q|_{BI}$ and we need to show that $b \vDash q|_{Ba}$. As $\mathrm{tp}(a/B)$ is compressible, Proposition 6.18 implies exactly that, and we are done. □

We can now generalise Corollary 3.21 to uncountable theories induced by adding constants to countable NIP theories (except that in the final clause we not obtain strength of the compressibility).

Corollary 6.24. *Suppose T is countable and let $B \subseteq \mathcal{U}$.*

Suppose $A \subseteq \mathcal{U}$ is a set of parameters and x is a countable tuple of variables. Then, compressible types are dense in $S^x(A)$ in $\mathrm{Th}(\mathcal{U}_B)$:

Working in $\mathcal{U}_B$, if $\theta(x)$ is a consistent ($\mathcal{L}_B$-)formula over A, then there exists a compressible type $p(x) \in S(A)$ with $p(x) \vdash \theta(x)$.

More generally, if, working in $\mathcal{U}_B$, π is a t-compressible partial type over A, then there exists a compressible $p \in S(A)$ with $\pi \subseteq p$.

Proof. Clearly it is enough to prove the "more generally" part.

Note first that π is t-compressible in $\mathcal{U}$ with respect to AB: if $\zeta(x, z, w)$ is such that $\zeta(x, z, b)$ compresses π within π in $\mathcal{U}_B$ (with respect to A) and $b \in B^w$, then $\zeta(x, z, w)$ compresses π within π with respect to AB. Indeed, given any $A_0B_0 \subseteq_{\text{fin}} AB$ there is some $d \in A^z$ such that (working in $\mathcal{U}$)

$$\pi \vdash \phi(x, d, b) \vdash \pi|_{A_0B} \vdash \pi|_{A_0B_0}.$$

By Corollary 3.21, there is a compressible type $p \in S(AB)$ containing π. By Proposition 6.23, p is compressible up to AB, which implies (by Remark 6.22) that p is compressible up to B, meaning that in $\mathcal{U}_B$, $p|_A \in S(A)$ (which equals p) is compressible. □

6.3. **Transitivity.** We continue to assume that T is NIP.

Proposition 6.25. *Suppose $A \subseteq B \subseteq C$, C is compressible over B, and B is compressible over A. Then C is compressible over A (recall Definition 6.3).*

Proof. By Remark 3.17, it is enough to show that $\mathrm{tp}(c/A)$ is compressible for any finite tuple c from C.

By Proposition 6.23, $\mathrm{tp}(c/A)$ is compressible up to B.

So given $\phi(x, y) \in \mathcal{L}$ (where c is of the sort of x), we get $\zeta(w, x, z) \in \mathcal{L}$, $b \in B^w$ such that for $A_0 \subseteq_{\text{fin}} A$ there is $a \in A^z$ such that

$$\mathrm{tp}(c/B) \vdash \zeta(b, x, a) \vdash \mathrm{tp}_{\phi(x,y)}(c/A_0).$$

Since $\mathrm{tp}(b/A)$ is compressible, for each $\epsilon < 2$ there are $\xi_\epsilon(w, z_\epsilon)$ and $a_\epsilon \in A^{z_\epsilon}$ such that

$$\mathrm{tp}(b/A) \vdash \xi_\epsilon(w, a_\epsilon) \vdash \mathrm{tp}_{\forall x(\zeta(w,x,z) \to \phi(x,y)^\epsilon)}(b/A_0 a).$$

Then

$$\mathrm{tp}(c/A) \vdash (\exists w(\zeta(w, x, a) \wedge \xi_0(w, a_0) \wedge \xi_1(w, a_1))) \vdash \mathrm{tp}_{\phi(x,y)}(c/A_0).$$

Hence $\mathrm{tp}(c/A)$ is compressible. □

Example 6.26. Proposition 6.25 is false without NIP. For example, let T be the theory of the countable atomless Boolean algebra. Let A be a countable set of pairwise disjoint elements $\{a_i \mid i < \omega\}$. For $i < \omega$, let $b_i = \bigcup a_{<i}$, and let $B = A \cup \{b_i \mid i < \omega\}$. Let $c \neq 1 \in \mathcal{U}$ contain all the elements from A. Let $C = Bc$. Then C is compressible over B, B is compressible over A, but C is not compressible over A.

Indeed, to show the first statement, it is enough to see that $p := \mathrm{tp}(c/B)$ is compressible by Lemma 6.6. This is true since $\zeta(x,z) := (z < x) \wedge (x \neq 1)$ compresses it: for every finite $B_0 \subseteq B$, let $i < \omega$ be such that $\bigcup B_0 \leq b_i$. Then $p \vdash (b_i < x) \wedge (x \neq 1) \vdash p|_{B_0}$ by quantifier elimination.

Since every tuple from B is in the definable closure of A, B is compressible over A (every finite tuple is isolated).

Finally, C is not compressible over A, since compressibility is monotonic (Lemma 6.1) and c is not compressible over A. Why? Suppose $\psi(x,z)$ compresses $\mathrm{tp}_{x \geq y}(c/A)$. Let $A_0 = a_{<|z|+2} \subseteq A$, and assume $d \in A^z$ is such that $\psi(c,d)$ holds and $\psi(x,d) \vdash b_{|z|+2} \leq x$. Let $i < \omega$ be such that $a_i \notin d$. Then $(\bigcup d < x) \wedge (x \neq 1) \vdash \psi(x,d)$ (since this implies $\mathrm{tp}(c/d)$), so $(\bigcup d \cup a_i) \vDash \psi(x,d)$ but $\neg(b_{|z|+2} \leq (\bigcup d \cup a_i))$ holds, contradiction.

Note that this example shows that in T, weak compressibility is different from compressibility (see Question 3.23 for the definition). Namely, $\mathrm{tp}(c/A)$ is weakly compressible (as witnessed by $\zeta(x,z)$) but not compressible.

Remark 6.27. The following rephrasing of Proposition 6.25 is worth mentioning explicitly: given (perhaps infinite) tuples c, b and a set A, if $\mathrm{tp}(c/Ab)$ and $\mathrm{tp}(b/A)$ are compressible, then $\mathrm{tp}(cb/A)$ is compressible. In this phrasing, this is a converse to Lemma 6.1 (see Remark 6.2).

This follows from Proposition 6.25 and Lemma 6.6. Note that Proposition 6.23 where $B \setminus A$ is finite can be seen with a direct argument (not using NIP), so for finite tuples c, b, the above can be easily proven and does not require NIP.

6.4. **Compressible models and applications.** In this section, T' is a countable NIP theory with monster model $\mathcal{U}$ in the language $\mathcal{L}'$, $F \subseteq \mathcal{U}$ is some small subset, and $T = \mathrm{Th}(\mathcal{U}_F)$ in the language $\mathcal{L} := \mathcal{L}'(F)$. In other words, T is a complete theory we get by naming constants in a countable NIP theory (whose monster model is still denoted by $\mathcal{U}$, abusing notation).

Definition 6.28. Say B is **compressibly constructible over** A if B can be enumerated as $B = (b_i)_{i<\alpha}$ for some ordinal α, such that $\mathrm{tp}(b_i/Ab_{<i})$ is compressible for all $i < \alpha$.

As with other isolation notions, the existence of compressibly constructible models follows straightforwardly from density. In fact this is an instance of the abstract result [She90, Theorem IV.3.1(5)], but we give the proof.

Proposition 6.29. *For any set A, there exists a model $M \supseteq A$ which is compressibly constructible over A and of cardinality $\leq |A| + |T|$.*

Moreover, if B is compressibly constructible over A then there is some model $M \supseteq B$ which is compressibly constructible over A and of cardinality $\leq |B| + |T|$.

Proof. Since A is compressibly constructible over A, it is enough to prove the "moreover" part.

Let $\lambda = |B|+|T|$. Construct an increasing chain $(B_i)_{i<\omega}$ as follows. Let $B_0 = B$, and given B_i, let $(\theta^i_j)_{j<\lambda}$ enumerate all consistent formulas over B_i. Construct a sequence $(b^i_j)_{j<\lambda}$ inductively by letting $b^i_j \vDash \theta^i_j$ be such that $\mathrm{tp}(b^i_j/B_i b^i_{<j})$ is compressible, using Corollary 6.24 (with $B = F$). Let $B_{i+1} = B_i \cup \{b^i_j \mid j < \lambda\}$. Finally, $M := \bigcup_{i<\omega} B_i$ is as required, by Tarski-Vaught. □

Thanks to Proposition 6.25, we also have the following instance of [She90, Theorem IV.3.2(1)].

Proposition 6.30. *If B is compressibly constructible over A, then B is compressible over A.*

Proof. Suppose $B = \{b_i \mid i < \alpha\}$ is an enumeration witnessing that B is compressibly constructible. Prove by induction on $\beta < \alpha$ that $B_{<\beta} = \{b_i \mid i < \beta\}$ is compressible over A. For the successor steps use Proposition 6.25 and Lemma 6.6, and for the limit steps use Remark 6.5. □

So compressible models exist over arbitrary sets. We give some applications.

6.4.1. *Realising models of a stable part.* Suppose A is some small set. Recall that **the induced structure on** A is the structure A_{ind} whose universe is A with the language consisting of a relation $R_{\phi(x)}$ for each $\mathcal{L}$-formula $\phi(x)$, where $R^{A_{\mathrm{ind}}}_{\phi(x)} = \{a \in A^x \mid \mathcal{U} \vDash \phi(a)\}$.

Lemma 6.31. *Suppose A is such that $\mathrm{Th}(A_{\mathrm{ind}})$ is stable (or just has stable quantifier-free formulas). Then there exists an l-atomic model M over A: for every finite tuple b from M, $\mathrm{tp}(b/A)$ is l-isolated.*

Proof. By Lemma 4.8(i.a), a compressible model over A is l-atomic over A, so this follows directly from Propositions 6.29 and 6.30. □

The next corollary deduces that the reduct map to a stable sort is surjective, and moreover elementary embeddings in the reduct theory can be lifted. Thanks to Anand Pillay and Martin Hils for suggesting that we consider this problem.

Remark 6.32. Suppose X is a $\emptyset$-definable set, and let $T_X := \mathrm{Th}(X(\mathcal{U})_{\mathrm{ind}})$. If T_X is stable then X is (uniformly) stably embedded by [Sim15a, Proposition 3.19]. It follows that stability of T_X is equivalent to assuming that X is stable as a partial type in the sense of Definition 4.4.

Corollary 6.33. *Suppose X is a $\emptyset$-definable subset of a sort of $\mathcal{U}$, and let $T_X := \mathrm{Th}(X(\mathcal{U})_{\mathrm{ind}})$. Suppose T_X is stable. Let $N \vDash T_X$. Then there exists $M \vDash T$ such that $N = X(M)_{\mathrm{ind}}$.*

Moreover, if $N_1 \prec N_2 \vDash T_X$, and $M_1 \vDash T$ is such that $X(M_1) = N_1$, then there is $M_2 \succ M_1$ such that $X(M_2) = N_2$.

Proof. Since $X(\mathcal{U})$ is a saturated model of T_X, we may assume that $N \prec X(\mathcal{U})$ and hence $N = A_{\mathrm{ind}}$ where $A \subseteq X(\mathcal{U})$ is the universe of N. By Lemma 6.31, there exists $M \prec \mathcal{U}$ which is l-atomic over A. In particular, $X(M)_{\mathrm{ind}}$ is l-atomic over A, and so $X(M) = A$ by Remark 4.7.

Now for the "moreover" part. Work in the language $\mathcal{L}(M_1)$ (adding constants for M_1). Then N_1 can be enriched to a model $N_1' \prec X(\mathcal{U}_{M_1})_{\mathrm{ind}}$ of $T_X' := \mathrm{Th}(X(\mathcal{U}_{M_1})_{\mathrm{ind}})$. By Remark 6.32, $X(\mathcal{U})$ is stably embedded, and since $M_1 \prec \mathcal{U}$ this expansion of N_1 is conservative: all new relations are already definable with parameters from N_1. Using these definitions, we can enrich N_2 to N_2' in such a way that $N_1' \prec N_2'$, and in particular, $N_2' \vDash T_X'$.

Now, T_X' is still stable, so we may apply the first part (recall that we allow naming any number of constants in T) and get some $M_2' \vDash \mathrm{Th}(\mathcal{U}_{M_1})$ such that $X(M_2') = N_2'$. Letting M_2 be the reduct to $\mathcal{L}$ (forgetting the constants for M_1), we get that $X(M_2) = N_2$ and $M_1 \prec M_2$ as required (this is a bit subtle, since what we really get is that there is an elementary embedding from M_1 to M_2, but then we can change M_2, fixing N_2, so that this embedding becomes the identity). □

Remark 6.34. The assumption that X is a subset of a sort of $\mathcal{U}$ is needed to make sense of the statement (otherwise, if X is e.g., a set of pairs and N_1 is not, then it is impossible that $X(M_1) = N_1$). This can be solved by instead asking for an isomorphism between $X(M_i)$ and N_i for $i = 1, 2$ such that the diagram commutes. This holds, since if X is contained in some product of sorts, then X could be replaced by its image in one sort of $\mathcal{U}^{\mathrm{eq}}$ under the canonical bijection.

Example 6.35. Let $M = (\mathrm{Aut}(\mathbb{Q},<),\mathbb{Q},<)$ in the language $\mathcal{L}$ which includes a predicate Q for $\mathbb{Q}$, a predicate G for the group of automorphisms $\mathrm{Aut}(\mathbb{Q},<)$ with the group operation $\cdot$ and the group action $e:(G\times Q)\to Q$. Let $T=\mathrm{Th}(M)$. Then, the induced structure on Q is just the order, because given two increasing tuples $a:=a_0,\dots,a_{n-1}$, $b:=b_0,\dots,b_{n-1}$, there is an automorphism σ of M mapping a to b (there is some automorphism $g\in\mathrm{Aut}(\mathbb{Q},<)$ mapping a to b; let σ be g on Q and conjugation by g on G). Hence, the theory $\mathrm{Th}(Q_{\mathrm{ind}})$ is a conservative extension of $\mathrm{DLO}=\mathrm{Th}(\mathbb{Q},<)$: all the relations are definable from the order.

However, T says that any two intervals have the same cardinality (because there is a bijection between them), and there are models of DLO in which this is not the case, and hence Corollary 6.33 does not hold in this case. This is not a surprise, since Q is not stably embedded (as the graph of any non-trivial automorphism is not definable using just the order, since dcl is trivial there), and moreover the theory is not NIP (for any finite tuple, there is an automorphism fixing any sub-tuple and moving the rest).

Example 6.35 raises the following question.

Question 6.36. Suppose that T is NIP and that $X\subseteq\mathcal{U}$ is $\emptyset$-definable and stably embedded. Is it true that every model M of $\mathrm{Th}(X(\mathcal{U})_{\mathrm{ind}})$ of the form $X(M)$ for some $M\vDash T$?

This question is related to Gaifman's categoricity conjecture, see Hodges [Hod11, Gai74].

6.4.2. *Extensions preserving a stable part.*

Corollary 6.37. *Suppose A is a set such that if M is a model containing A, then M contains an infinite chain in some M-definable preorder.*

Then there are arbitrarily large models which are compressible over A.

Proof. There exists a compressible model over A by Propositions 6.29 and 6.30. It is enough to show that any such model M can be strictly extended to a compressible model over A. By assumption M contains an infinite chain in some definable preorder. By Remark 4.9, there is some c such that $\mathrm{tp}(c/M)$ is compressible but not l-isolated, so $c\notin M$. Apply (the "moreover" part of) Proposition 6.29 to get some compressibly constructible model N over M containing Mc. Then N is compressible over M by Proposition 6.30. By transitivity (Proposition 6.25), N is compressible over A. □

Example 6.38. The following example shows that Corollary 6.37 requires the assumption that every model containing A contains an infinite chain; mere instability does not suffice. Let $\mathcal{L}=\{<\}$ where $<$ is a binary relation symbol. Let T say that $<$ is a partial order, that comparability is an equivalence relation having exactly one class of size n for each $0<n<\omega$, and that each class has maximal and minimal elements and is discretely ordered (every non-maximal element has a successor and every non-minimal element has a predecessor). Let M be the $\mathcal{L}$-structure with universe $\{(i,j)\mid i\in\mathbb{N}, j\leq i\}$ and with $<^M=\{(i,j_1),(i,j_2)\in M^2\mid i\in\mathbb{N}, j_1<j_2\}$. Then $M\vDash T$, and in fact every model of T is a disjoint union of (a copy of) M and infinite chains, each discretely ordered with a minimal and maximal element. Hence any two saturated models of T are isomorphic and thus T is complete. Moreover, T has quantifier elimination after adding the predecessor and successor functions by a back-and-forth argument. It follows that T is NIP (by e.g. showing that $x<y$ is NIP as there is a polynomial bound on every $(x<y)$-type over a finite subset of M).

Suppose $c \in \mathcal{U}$. Then $\mathrm{tp}(c/M)$ is generically stable (see Definition 4.16). Indeed, suppose $c \notin M$ and let p be a global coheir extending $\mathrm{tp}(c/M)$. Then any Morley sequence $I := (a_i)_{i<\omega}$ of p over M must be such that a_i, a_j are incomparable for $i \neq j$. Thus I is an indiscernible set by quantifier elimination, and hence p is generically stable by NIP and the "moreover" part of Fact 4.13.

Thus, if $\mathrm{tp}(c/M)$ is compressible then it is l-isolated by Proposition 4.14 and hence realised by Remark 4.7. Hence M is the only model compressible over M and the conclusion of Corollary 6.37 does not hold.

Corollary 6.39. *Let $M \vDash T$, and suppose that $<$ is an M-definable preorder on an M-definable set D and suppose that there is an infinite $<$-chain in $D(M)$. Then:*

(i) There exists $N \succ M$ with $D(N) \supsetneq D(M)$ such that if $a \in N$ and $\mathrm{tp}(a/M)$ is generically stable, then $a \in M$. (See Definition 4.16.)

(ii) For any cardinal λ there is $N \succ \mathcal{M}$ with $|D(N)| \geq \lambda$ such that if $a \in N$ and $\mathrm{tp}(a/M)$ is generically stable, then $a \in M$.

In particular, there are arbitrarily large elementary extensions N of M such that if S is M-definable and stable (see Definition 4.4), then $S(M) = S(N)$.

Proof. (ii) follows from (i) by taking the union of a suitably long elementary chain. We prove (i).

By Remark 4.9 there is some $c \in D(\mathcal{U})$ such that $\mathrm{tp}(c/M)$ is compressible but not l-isolated. In particular $c \notin M$. Apply (the "moreover" part of) Proposition 6.29 to get some compressibly constructible model $N \succ M$ over M containing Mc. By Proposition 4.14, if $a \in N$ and $\mathrm{tp}(a/M)$ is generically stable, then it is l-isolated and hence by Remark 4.7, it is realised.

The "in particular" part follows since for any $a \in S(\mathcal{U})$, $\mathrm{tp}(a/M)$ is stable and hence generically stable by Fact 4.18. □

Remark 6.40. Note that the condition in Corollary 6.39 holds whenever T is unstable and M is ω-saturated, since in this case T has the SOP ([Sim15a, Theorem 2.67]) and hence any ω-saturated model contains an infinite chain in some definable preorder.

6.4.3. *Compressible types in ACVF.*

Example 6.41. If $\mathcal{U} \vDash \mathrm{ACVF}$ and $A \subseteq \mathcal{U}^{\mathrm{eq}}$, then any model M containing A whose residue field is algebraic over A is compressible over A; i.e., if $M \prec \mathcal{U}$, $A \subseteq M^{\mathrm{eq}}$, and $k(M) = \mathrm{acl}^{\mathrm{eq}}(A) \cap k$, where $k = k(\mathcal{U})$ denotes the residue field, then M is compressible over A.

To see this, consider first a 1-type $\mathrm{tp}(a/B)$ over an $\mathrm{acl}^{\mathrm{eq}}$-closed set $B = \mathrm{acl}^{\mathrm{eq}}(B) \subseteq \mathcal{U}^{\mathrm{eq}}$, where $a \in \mathcal{U}$. We show that $\mathrm{tp}(a/B)$ is compressible iff $\mathrm{dcl}^{\mathrm{eq}}(Ba) \cap k = B \cap k$. By unique Swiss cheese decompositions [Hol95, Theorem 3.26], $\mathrm{tp}(a/B)$ is determined by knowing which of the balls defined over B contain a, i.e., $\mathrm{tp}(a/B)$ is implied by $\mathrm{tp}_{\in}(a/B)$ where $x \in y$ is the element relation between the valued field and the sort of balls (closed and open). One sees directly that this $\in$-type is compressible unless $\mathrm{tp}(a/B)$ is the generic type of a closed ball α over B which contains infinitely many open balls over B of radius $\gamma := \mathrm{radius}(\alpha)$. In that case, let $\beta_1 \neq \beta_2$ be distinct open subballs of α over B of radius γ (the existence of these two balls is our only use of our assumption that there are infinitely many such), and consider the map $\beta \mapsto \mathrm{res}(\frac{\beta-\beta_1}{\beta_2-\beta_1})$ for β an open subball of α of radius γ. This is a well-defined injective map to k defined over B, so genericity of $\mathrm{tp}(a/B)$ (and the fact that $B = \mathrm{acl}^{\mathrm{eq}}(B)$) implies that the image under this map of the open ball around a of radius γ is not in $B \cap k$, so we have $\mathrm{dcl}^{\mathrm{eq}}(Ba) \cap k \supsetneq B \cap k$. Conversely, if $\mathrm{tp}(a/B)$ is compressible, then since the residue field is a pure stably embedded algebraically closed field, we must have $\mathrm{dcl}^{\mathrm{eq}}(Ba) \cap k = B \cap k$.

The compressibility of M over A if $k(M) = \mathrm{acl}^{\mathrm{eq}}(A) \cap k$ now follows by first taking a compressible construction sequence which alternates taking $\mathrm{acl}^{\mathrm{eq}}$-closure and adding a single new element of M, and then applying Proposition 6.30.

If $A \cap k$ is infinite then conversely $k(M) = \mathrm{acl}^{\mathrm{eq}}(A) \cap k$ is necessary for compressibility of M over A, but this fails in general; for example, if A is finite, then any model containing A is of course compressible over A.

Remark 6.42. The argument in Example 6.41 is based on ideas from [BM21], and combining it with [BM21, Remark 3.12] actually yields an alternative proof of [BM21, Theorem 5.6]. Indeed, suppose K is a valued subfield of $\mathcal{U}$ with $\mathrm{res}(K)$ finite. Taking an elementary extension we may assume that $(\mathcal{U}, K)$ is sufficiently saturated. Let $A \subseteq K$. Then $k(K^{\mathrm{alg}}) = \mathrm{acl}^{\mathrm{eq}}(A) \cap k$ is the algebraic closure of the prime field, so by Example 6.41, K^{alg}, and in particular K, is compressible over A. By [BM21, Remark 3.12], it follows that K is distal in $\mathcal{U}$ in the sense defined in that paper. The same argument applies to K_r defined in [BM21, Theorem 5.6]. However, this proof does not yield bounds on the exponents of the resulting distal cell decompositions, whereas such bounds are obtained in [BM21, Remark 6.2] as a consequence of the more elementary methods of that paper.

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