ON 2D INCOMPRESSIBLE BOUSSINESQ SYSTEMS: GLOBAL STABILIZATION UNDER DYNAMIC BOUNDARY CONDITIONS

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ABSTRACT. This paper studies the stability of large data solutions to the 2D fully/partially dissipative Boussinesq systems on the unit square subject to various types of boundary conditions, including dynamic Couette flow. It is shown that under suitable conditions on the boundary data, solutions starting in H^3 exist globally in time and the differences between the solutions and their corresponding boundary data converge to zero in certain topology, as time goes to infinity. There is no smallness restrictions on initial data.

CONTENTS

1. INTRODUCTION

1.1. **Background information.** This paper investigates the stability and large-time behavior of smooth solutions to 2D Boussinesq systems subject to various boundary conditions such as the dynamic Couette flow. We simultaneously examine five Boussinesq systems with full or partial dissipation in order to understand how dissipation and the boundary conditions affect the stability and large-time behavior.

The Boussinesq systems considered here model buoyancy-driven fluids such as atmospheric and oceanographic flows and Rayleigh-Benard convection (see, e.g., [6, 11, 24, 27, 32]). Besides their wide applicability, the Boussinesq systems are also mathematically important. The 2D Boussinesq equations serve as a lower dimensional model of the 3D hydrodynamics equations. In fact, the 2D Boussinesq equations retain some key features of the 3D Euler and Navier-Stokes equations such as the vortex stretching mechanism and the inviscid 2D Boussinesq equations can be identified as the Euler equations for the 3D axisymmetric swirling flows [25]. More recent work of T. Elgindi and of T. Hou and his collaborators on the potential

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finite-time singularities of the 3D axisymmetric Euler and the 2D inviscid Boussinesq equations further affirm the connection between them (see [8, 9, 15, 16, 22, 23] and references therein).

The Boussinesq equations admit physically significant steady-state solutions such as the hydrostatic equilibrium and the shear flow. The stability problem on perturbations near these steady-states has recently attracted considerable interests. Doering, Wu, Zhao and Zheng [12] investigated the stability of the hydrostatic equilibrium to the 2D Boussinesq system with only kinematic dissipation and rigorously proved the global asymptotic stability. A followup work by Tao, Wu, Zhao and Zheng [31] resolves several important issues left open in [12]. In particular, [31] provides a precise description of the final buoyancy distribution in case of general initial conditions and the explicit decay rate of the velocity field. The paper of Castro, Córdoba and Lear successfully established the stability and large-time behavior on the 2D Boussinesq equations with velocity damping instead of dissipation [7]. The stability of the hydrostatic equilibrium is also established very recently for several Boussinesq systems with various partial dissipation or damping [1, 3, 17, 20, 21, 28, 33]. These papers discovered a remarkable stabilizing phenomenon implying that the buoyancy forcing can actually stabilize the fluids. Significant progress has also been made on the stability of special shear flows to the Boussinesq equations with partial dissipation [2, 4, 10, 13, 14, 19, 26, 30, 34, 36, 37].

1.2. **Aims and guiding examples.** This paper studies the stability of the Boussinesq systems with very natural boundary setups. The spatial domain Ω is taken to be the following box in $\mathbb{R}^2$,

$$\Omega = \left\{ \mathbf{x} = (x, y) \in \mathbb{R}^2 \,|\, 0 < x < 1,\ 0 < y < 1 \right\}.$$

When the top and bottom parts are moving at different speeds, the relative motion of the surfaces imposes a shear stress on the viscous fluid and induces a shear flow. It is then very natural to impose the dynamic Couette flow condition on the boundary. However, when the velocity equation is inviscid, the natural boundary condition is the no-penetration condition. These are the two types of boundary conditions we impose for the velocity. The temperature on the boundary is assumed to be either the stable hydrostatic temperature profile or the periodic boundary condition in the x-direction.

We consider five Boussinesq systems. The first one investigated here is the 2D Boussinesq system with full dissipation

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla P = \nu\Delta\mathbf{u} + \theta\mathbf{e}_2, & \mathbf{x}\in\Omega,\ t>0, \\ \nabla\cdot\mathbf{u} = 0, & \mathbf{x}\in\Omega,\ t>0, \\ \partial_t\theta + \mathbf{u}\cdot\nabla\theta = \kappa\Delta\theta, & \mathbf{x}\in\Omega,\ t>0; \\ (\mathbf{u},\theta)(\mathbf{x},0) = (\mathbf{u}_0,\theta_0)(\mathbf{x}), & \mathbf{x}\in\Omega; \\ \mathbf{u}(\mathbf{x},t) = (\alpha(t)y, 0)^{\mathrm{T}}, \quad \theta = \beta(t)y, & \mathbf{x}\in\partial\Omega,\ t>0, \end{cases} \tag{1.1}$$

where $\mathbf{u} = (u_1, u_2)^{\mathrm{T}}$ represents the fluid velocity, P the pressure, θ the temperature, and $\mathbf{e}_2 = (0,1)^{\mathrm{T}}$. The boundary condition for $\mathbf{u}$ represents the dynamic Couette flow while θ on the boundary $\partial\Omega$ obeys a dynamic hydrostatic balance profile. The aim here is to understand the stability and large-time behavior of general smooth solutions to (1.1).

When $\alpha(t) = \alpha_1$ and $\beta(t) = \beta_0$ are constants, $(u^{(0)}, \theta^{(0)})$ with $u^{(0)} = (\alpha(t)y, 0)^{\mathrm{T}}$ and $\theta^{(0)} = \beta(t)y$ is a steady-state solution of (1.1) with $P^{(0)} = \frac{\beta(t)}{2}y^2$. When $\beta_0 > 0$, $\theta^{(0)} = \beta(t)y$ is a stable temperature configuration. Here we take $\alpha(t)$ and $\beta(t)$ to be time dependent with

$$\alpha(t) \to \alpha_1 \quad \text{and} \quad 0 < \beta_1 < \beta(t) < \beta_2$$

for real numbers α_1, β_1 and β_2. Physically the time dependent boundary data can be regarded as a startup describing the approach to the Couette solution, which is not reached instantaneously.

To guide our investigations, we examine the solutions to two simple models. The first is the 1D heat equation

$$\begin{cases} \partial_t v = \nu \partial_y^2 v, & 0 < y < 1,\ t > 0; \\ v(y,0) = 0, & 0 < y < 1; \\ v(0,t) = 0, & v(1,t) = U,\ t > 0. \end{cases} \tag{1.2}$$

The solution of (1.2) is explicit,

$$v(y,t) = Uy - \frac{2U}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} e^{-n^2\pi^2\nu t} \sin(n\pi(1-y)).$$

Clearly, $v(y,t) \to Uy$ as $t \to \infty$. This simple example already reveals some of the transitory behavior such as oscillations before the flow converges to the Couette solution. In addition, v converges to the Couette flow Uy at the rate $e^{-\pi^2\nu t}$.

The Couette flow is one of the simplest shear flows. Shear flows are flows that point in one direction but depend on the variable in the other direction. The Couette flow is given by $\mathbf{U}(x,y) = (y,0)$.

The second example is given by the 2D Navier-Stokes equations defined in Ω,

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \nu \Delta \mathbf{u}, & \mathbf{x} \in \Omega,\ t > 0, \\ \nabla \cdot \mathbf{u} = 0, & \mathbf{x} \in \Omega,\ t > 0; \\ \mathbf{u}(\mathbf{x},t) = (c\,y, 0)^{\mathrm{T}}, & \mathbf{x} \in \partial\Omega,\ t > 0. \end{cases}$$

The perturbation

$$\widetilde{u} = u - (c\,y, 0)^{\mathrm{T}}$$

satisfies

$$\begin{cases} \partial_t \widetilde{\mathbf{u}} + \widetilde{\mathbf{u}} \cdot \nabla \widetilde{\mathbf{u}} + (c\,y,0)^{\mathrm{T}} \cdot \nabla \widetilde{\mathbf{u}} + (c\,\widetilde{u}_2, 0)^{\mathrm{T}} + \nabla P = \nu \Delta \widetilde{\mathbf{u}}, & \mathbf{x} \in \Omega,\ t > 0, \\ \nabla \cdot \widetilde{\mathbf{u}} = 0; & \mathbf{x} \in \Omega,\ t > 0; \\ \widetilde{\mathbf{u}}(\mathbf{x},t) = (0,0), & \mathbf{x} \in \partial\Omega,\ t > 0. \end{cases}$$

A basic L^2-level energy estimate reads

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\widetilde{\mathbf{u}}(t)\|_{L^2}^2 + 2\nu \|\nabla \widetilde{\mathbf{u}}(t)\|_{L^2}^2 = -2c \int_\Omega \widetilde{u}_1\, \widetilde{u}_2 \,\mathrm{d}\mathbf{x},$$

where $\widetilde{u}_1$ and $\widetilde{u}_2$ are the components of $\widetilde{\mathbf{u}}$. An application of the Poincaré inequality

$$\|\widetilde{\mathbf{u}}\|_{L^2(\Omega)} \le \frac{1}{\pi} \|\nabla \widetilde{\mathbf{u}}\|_{L^2(\Omega)}$$

leads to

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\widetilde{\mathbf{u}}(t)\|_{L^2}^2 + \left(2\nu - \frac{c}{\pi^2}\right) \|\widetilde{\mathbf{u}}\|_{L^2(\Omega)}^2 \le 0.$$

Therefore, the condition

$$|c| < 2\pi^2\nu, \tag{1.3}$$

guarantees the exponential decay of the perturbation

$$\|\widetilde{\mathbf{u}}(t)\|_{L^2}^2 \le \|\widetilde{\mathbf{u}}_0\|_{L^2}^2\, e^{\left(\frac{c - 2\pi^2\nu}{\pi^2}\right)t}.$$

The condition (1.3) on the size of the shear is needed in order to control the linear term $(c\,\widetilde{u}_2, 0)^{\mathrm{T}}$ in the equation of the perturbation. So a natural question is whether the smallness of c can be relaxed such that the Couette flow is still globally stable. This is one of the motivations of this work.

We remark that there are beautiful stability results near the Couette flow that make use of the enhanced dissipation generated by the Couette flow [2, 4, 10, 26, 34, 36, 37]. Due to the different circumstances here, we are not able to make use of the enhanced dissipation. Our main goal here is the global existence and regularity results. We make no smallness assumptions. In addition, in order to use the enhanced dissipation, one direction of the domain must be periodic. We have provided special examples of the solutions to several cases. These examples show that the results provided here are optimal.

1.3. **Summary of main results and ideas in the proofs.** We summarize the main results on the five Boussinesq systems examined in this paper and briefly explain how we prove these results. For the sake of simplicity, we sometimes use B.C. for Boundary Condition.

1.3.1. *Fully Dissipative System under Dynamic Couette Flow and Dirichlet B.C..* We now return to the stability problem on the Boussinesq system (1.1). Setting

$$\widehat{\mathbf{u}} \equiv (\alpha(t)y, 0)^{\mathrm{T}}, \qquad \widehat{\theta} \equiv \beta(t)y,$$

and

$$\mathbf{u} = \widehat{\mathbf{u}} + \widetilde{\mathbf{u}}, \qquad \theta = \widehat{\theta} + \widetilde{\theta},$$

we can convert (1.1) into a system in terms of $(\widetilde{\mathbf{u}}, \widetilde{\theta})$,

$$\begin{cases} \partial_t \widetilde{\mathbf{u}} + [(\widehat{\mathbf{u}} + \widetilde{\mathbf{u}}) \cdot \nabla](\widehat{\mathbf{u}} + \widetilde{\mathbf{u}}) + \nabla P = \nu \Delta \widetilde{\mathbf{u}} + (\widehat{\theta} + \widetilde{\theta})\mathbf{e}_2 - \partial_t \widehat{\mathbf{u}}, \\ \nabla \cdot \widetilde{\mathbf{u}} = 0, \\ \partial_t \widetilde{\theta} + (\widehat{\mathbf{u}} + \widetilde{\mathbf{u}}) \cdot \nabla(\widehat{\theta} + \widetilde{\theta}) = \kappa \Delta \widetilde{\theta} - \partial_t \widehat{\theta}. \end{cases} \tag{1.4}$$

We note that the Poincaré inequality applies to $\widetilde{\mathbf{u}}$. The Boussinesq system (1.4) obeyed by the perturbation $(\widetilde{\mathbf{u}}, \widetilde{\theta})$ appears to be much more complex than the two aforementioned examples, the behavior of $(\widetilde{\mathbf{u}}, \widetilde{\theta})$ does share some of the essential characteristics. In fact, the convergence of $\mathbf{u}$ to the Couette flow $(\alpha(t)y, 0)^T$ and θ to the hydrostatic profile $\beta(t)y$ can be established in the Sobolev setting H^3. More precisely, the following global existence and large-time behavior result holds.

Theorem 1.1. *Consider the initial-boundary value problem* (1.1). *Suppose that the initial data satisfy* $(\mathbf{u}_0, \theta_0) \in H^3$, $\nabla \cdot \mathbf{u}_0 = 0$, *and are compatible with the boundary conditions. Assume that* $\alpha, \beta : [0, \infty) \to \mathbb{R}$ *are smooth functions such that*

$$(\alpha(t) - \alpha_1) \in L^1(0, \infty), \quad \alpha'(t) \in L^1(0, \infty) \cap L^2(0, \infty), \quad \alpha''(t) \in L^2(0, \infty), \tag{1.5}$$

$$\beta_1 \le \beta(t) \le \beta_2, \quad \beta'(t) \in L^1(0, \infty) \cap L^2(0, \infty), \quad \beta''(t) \in L^2(0, \infty), \tag{1.6}$$

where α_1 *and* $0 < \beta_1 \le \beta_2$ *are constants. Suppose further that* $|\alpha_1| < 2\pi^2\nu$. *Then there exists a unique solution to* (1.1) *such that* $(\mathbf{u}, \theta) \in [L^\infty(0, \infty; H^3) \cap L^2(0, \infty; H^4)]^3$. *Moreover, the solution has the large-time behavior as* $t \to \infty$:

$$\|\mathbf{u}(t) - (\alpha(t)y, 0)^{\mathrm{T}}\|_{H^3} \to 0, \quad \|\theta(t) - \beta(t)y\|_{H^3} \to 0, \quad \|(\nabla P - \theta \mathbf{e}_2)(t)\|_{H^1} \to 0.$$

Remark 1.1. *We note that* (1.5) *implies* $\alpha(t) \to \alpha_1$ *as* $t \to \infty$. *The assumption* $|\alpha_1| < 2\pi^2\nu$ *indicates that the eventual Couette flow is dominated by the viscosity. This is consistent with the conventional study of Couette flow with constant coefficient. However, one could design* $\alpha(t) - \alpha_1$ *in whatever way one desires,* *such that the magnitude of the perturbation can be arbitrarily large up to any finite time, beyond which the perturbation decays to zero sufficiently fast. Hence, the uniform smallness of the magnitude of linear Couette flow, as was mentioned before, can be relaxed, such that the flow is still globally asymptotically stable. Moreover,* (1.6) *implies* $\beta(t)$ *stays positive for all time and converges to a positive constant as* $t \to \infty$. *Hence, problem* (1.1) *corresponds to the physical situation where natural convection arises and the flow eventually stratifies in the vertical direction with a linear profile in the temperature field. The general functions* $\alpha(t)$ *and* $\beta(t)$ *allow us to deal with Rayleigh-Bénard convection with time-dependent boundary conditions* [29].

The proof of this theorem is divided into eight steps. These steps establish upper bounds for $(\widetilde{\mathbf{u}}, \widetilde{\theta})$ in successively more and more regular functional spaces. The first step estimates a suitable combination of $\|\widetilde{\mathbf{u}}\|_{L^2}^2$ and $\|\widetilde{\theta}\|_{L^2}^2$ in order to eliminate the buoyancy forcing term. The second step controls $\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2$ and the integral of $\|\partial_t\nabla\widetilde{\mathbf{u}}\|_{L^2}^2$ by taking the L^2-inner product of the equation of $\widetilde{\mathbf{u}}$ with $\partial_t\widetilde{\mathbf{u}}$. This is followed by the estimates of $\|\nabla\widetilde{\theta}\|_{L^2}^2$ and $\|\partial_t\widetilde{\theta}\|_{L^2}^2$. The next few steps bound the norms of $(\widetilde{\mathbf{u}}, \widetilde{\theta})$ and its time derivative in higher Sobolev spaces. The convergence results in Theorem 1.1 are then shown as the consequences of the upper bounds.

1.3.2. *Fully Dissipative System under Dynamic Couette Flow and Neumann B.C.*. The second Boussinesq system investigated here assumes the form

$$\begin{cases} \partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla P = \nu\Delta\mathbf{u} + \theta\mathbf{e}_2, & \mathbf{x}\in\Omega,\ t>0,\\ \nabla\cdot\mathbf{u} = 0, & \mathbf{x}\in\Omega,\ t>0,\\ \partial_t\theta + \mathbf{u}\cdot\nabla\theta = \kappa\Delta\theta, & \mathbf{x}\in\Omega,\ t>0;\\ (\mathbf{u},\theta)(\mathbf{x},0) = (\mathbf{u}_0,\theta_0)(\mathbf{x}), & \mathbf{x}\in\Omega;\\ \mathbf{u}(\mathbf{x},t) = (\alpha(t)y, 0)^{\mathrm{T}}, \quad \nabla\theta\cdot\mathbf{n} = \beta(t)\mathbf{e}_2\cdot\mathbf{n}, & \mathbf{x}\in\partial\Omega,\ t>0. \end{cases} \tag{1.7}$$

Note that $\mathbf{e}_2 = \nabla y$, and hence the boundary condition for θ is equivalent to

$$\nabla(\theta - \beta(t)y)\cdot\mathbf{n}|_{\partial\Omega} = 0.$$

The Neumann boundary condition on θ reflects the physical setting that the vertical rate of change in the temperature at the top and the bottom of the domain is given by β.

The corresponding system governing the perturbation $(\widetilde{\mathbf{u}}, \widetilde{\theta})$ is given by

$$\begin{cases} \partial_t\widetilde{\mathbf{u}} + [(\widehat{\mathbf{u}} + \widetilde{\mathbf{u}})\cdot\nabla]\widetilde{\mathbf{u}} + (\alpha(t)\widetilde{u}_2, 0)^{\mathrm{T}} + \nabla\widetilde{P} = \nu\Delta\widetilde{\mathbf{u}} + \widetilde{\theta}\mathbf{e}_2 - \partial_t\widehat{\mathbf{u}},\\ \nabla\cdot\widetilde{\mathbf{u}} = 0,\\ \partial_t\widetilde{\theta} + (\widehat{\mathbf{u}} + \widetilde{\mathbf{u}})\cdot\nabla\widetilde{\theta} + \beta(t)\widetilde{u}_2 = \kappa\Delta\widetilde{\theta} - \partial_t\widehat{\theta}, \end{cases} \tag{1.8}$$

where $\widetilde{P} = P - \frac{\beta(t)}{2}y^2$, and $\widetilde{\mathbf{u}}$ and $\widetilde{\theta}$ satisfy the zero Dirichlet and Neumann boundary conditions, respectively. We are able to establish the global well-posedness and the large-time asymptotics, as stated in the following theorem.

Theorem 1.2. *Consider the initial-boundary value problem* (1.7). *Suppose that the initial data satisfy* $(\mathbf{u}_0, \theta_0) \in H^3$, $\nabla\cdot\mathbf{u}_0 = 0$, *and are compatible with the boundary conditions. Assume that* $\alpha, \beta : [0,\infty) \to$

$\mathbb{R}$ *are smooth functions such that*

$$\alpha(t) \in L^1(0,\infty), \quad \alpha'(t) \in L^1(0,\infty) \cap L^2(0,\infty), \quad \alpha''(t) \in L^2(0,\infty),$$
$$\beta_1 \le \beta(t) \le \beta_2, \quad \beta'(t) \in L^1(0,\infty) \cap L^2(0,\infty), \quad \beta''(t) \in L^2(0,\infty),$$

where $0 < \beta_1 \le \beta_2 < \infty$ *are constants. Then there exists a unique solution to* (1.7) *such that* $(\mathbf{u},\theta) \in [L^\infty(0,T;H^3) \cap L^2(0,T;H^4)]^3$ *for any finite* $T > 0$*. Moreover, the solution has the large-time behavior as* $t \to \infty$*:*

$$\|\mathbf{u}(t) - (\alpha(t)y,0)^{\mathrm{T}}\|_{H^2} \to 0, \quad \|\nabla\theta(t) - \beta(t)\mathbf{e}_2\|_{H^2} \to 0, \quad \|(\nabla P - \theta\mathbf{e}_2)(t)\|_{L^2} \to 0.$$

Mathematically the handling of the Boussinesq problem with θ satisfying the Neumann boundary condition is different from that for the Dirichlet condition in some technical aspects. In particular, since the temperature is not specified on the boundary of Ω, it is unknown whether the nonlinear term originated from the equation of $\widetilde{\theta}$,

$$-\int_\Omega \left[(\widehat{\mathbf{u}} + \widetilde{\mathbf{u}}) \cdot \nabla\widetilde{\theta}\right] \widetilde{\theta}\,\mathrm{d}\mathbf{x} = -\int_\Omega \left(\widehat{\mathbf{u}} \cdot \nabla\widetilde{\theta}\right) \widetilde{\theta}\,\mathrm{d}\mathbf{x} = \frac{\alpha(t)}{2}\int_0^1 \left[|\widetilde{\theta}(1,y,t)|^2 - |\widetilde{\theta}(0,y,t)|^2\right] y\,\mathrm{d}y,$$

is zero or not. For the same reason, since the total thermal flux on the boundary depends on time and the temperature on the two vertical sides of Ω, i.e.,

$$\int_{\partial\Omega} (\mathbf{u}\,\theta - \kappa\nabla\theta) \cdot \mathbf{n}\,\mathrm{d}s = \alpha(t)\int_0^1 [\theta(1,y,t) - \theta(0,y,t)] y\,\mathrm{d}y,$$

it is not clear whether the total thermal energy is conserved or not. Unlike the case of Dirichlet boundary condition, this prevents us from applying the Poincaré inequality to $\widetilde{\theta}$. These technical features leave a mark on the large-time behavior of the solution, which can be seen from the differences between the assumptions and conclusions of Theorem 1.1 and Theorem 1.2. The full proof of Theorem 1.2 is divided into six steps, which establish upper bounds for $(\widetilde{\mathbf{u}},\widetilde{\theta})$ in successively more and more regular functional spaces. The full proof is delicate and long, and is provided in Section 3.

1.3.3. *Semi-dissipative System under Dynamic Couette Flow and Periodic B.C.*. The third Boussinesq system examined here has no thermal diffusion. We impose the periodic boundary condition in the horizontal direction. More precisely, the Boussinesq system is given by

$$\begin{cases} \partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla P = \nu\Delta\mathbf{u} + \theta\mathbf{e}_2, & \mathbf{x} \in \Omega,\ t > 0, \\ \nabla\cdot\mathbf{u} = 0, & \mathbf{x} \in \Omega,\ t > 0, \\ \partial_t\theta + \mathbf{u}\cdot\nabla\theta = 0, & \mathbf{x} \in \Omega,\ t > 0; \\ (\mathbf{u},\theta)(\mathbf{x},0) = (\mathbf{u}_0,\theta_0)(\mathbf{x}), & \mathbf{x} \in \Omega; \\ \mathbf{u}(\mathbf{x},t) = (\alpha(t)y,0)^{\mathrm{T}}, \quad \theta(0,y,t) = \theta(1,y,t), & \mathbf{x} \in \partial\Omega,\ t > 0. \end{cases} \tag{1.9}$$

Here, for the sake of simplicity, we have written the boundary condition for $\mathbf{u}$ as

$$\mathbf{u}(\mathbf{x},t) = (\alpha(t)y,0)^{\mathrm{T}}.$$

It really means that $u_1(x,0,t) = 0$ and $u_1(x,1,t) = \alpha(t)$, and periodic in x.

(1.9) models viscous buoyancy driven fluids when the thermal diffusion can be ignored. This partial dissipation case is not only physically important but mathematically significant. It is one of the very first few cases that are studied for global regularity in the whole space [5, 18]. However, the well-posedness and the asymptotic behavior of this Boussinesq system under the dynamic Couette boundary conditions have never been studied and are not well understood. We are motivated to fill in this gap.

In the special case when $\alpha(t) \equiv 1$, the initial vertical velocity is identically equal to zero, and the initial horizontal velocity and temperature depend only on y, namely

$$\mathbf{u}_0(\mathbf{x}) = (u_{01}(y), 0), \quad \theta_0(\mathbf{x}) = \theta_0(y),$$

then the pressure gradient balances the buoyancy, and the horizontal velocity is deduced to satisfy the heat equation. The vertical velocity vanishes for all time. Then (1.9) admits the following special solution

$$\begin{aligned}
&u_1(x, y, t) = y - 2\sum_{n=1}^{\infty} a_n e^{-n^2\pi^2\nu t} \sin(n\pi(1-y)),\\
&u_2(x, y, t) \equiv 0,\\
&\theta(x, y, t) = \theta_0(y),\\
&P(x, y, t) = \int_0^y \theta_0(y')dy',
\end{aligned}$$

where

$$a_n = \frac{1}{\pi n} - \int_0^1 u_{01}(y) \sin(n\pi(1-y))\, dy.$$

It is very easy to check that

$$\begin{aligned}
&\|\mathbf{u}(t) - (\alpha(t)y, 0)^{\mathrm{T}}\|_{H^1}^2 = \sum_{n=1}^{\infty} \left(1 + n^2\pi^2\right) a_n^2\, e^{-2n^2\pi^2\nu t},\\
&\|\partial_t\mathbf{u}(t) - (\alpha'(t)y, 0)^{\mathrm{T}}\|_{L^2}^2 = \sum_{n=1}^{\infty} n^4\pi^4\nu^2\, a_n^2\, e^{-2n^2\pi^2\nu t},\\
&\nabla P - \theta\mathbf{e}_2 \equiv 0.
\end{aligned}$$

Trivially, as $t \to \infty$,

$$\|\mathbf{u}(t) - (\alpha(t)y, 0)^{\mathrm{T}}\|_{H^1} \to 0, \quad \|\partial_t\mathbf{u}(t) - (\alpha'(t)y, 0)^{\mathrm{T}}\|_{L^2} \to 0, \quad \|(\nabla P - \theta\mathbf{e}_2)(t)\|_{H^{-1}} \to 0.$$

Our main theorem for (1.9) states that this type of large-time behavior actually holds for much more general flows. This is remarkable.

Even though there is no thermal diffusion, we still manage to show the global existence and uniqueness in the Sobolev space H^3, but the convergence of $\mathbf{u}$ to the Couette flow is only at the H^1-level in space and in time as well. The pressure gradient converges to the buoyancy in a weak sense (in H^{-1}-norm). These are comparable to the phenomena associated with the case of zero-flow boundary condition studied in [12], but the underlying analysis is considerably more involved due to the presence of the dynamic Couette flow. More precisely, we prove the following result.

Theorem 1.3. *Consider the initial-boundary value problem* (1.9)*. Suppose that the initial data satisfy* $(\mathbf{u}_0, \theta_0) \in H^3$*,* $\nabla \cdot \mathbf{u}_0 = 0$*, and are compatible with the boundary conditions. Assume that* $\alpha : [0, \infty) \to \mathbb{R}$ *is a smooth function such that*

$$\alpha(t) \in L^1(0, \infty), \quad \alpha'(t) \in L^1(0, \infty) \cap L^2(0, \infty), \quad \alpha''(t) \in L^2(0, \infty).$$

Then there exists a unique solution to (1.9) *such that* $\mathbf{u} \in [L^\infty(0,T;H^3) \cap L^2(0,T;H^4)]^2$ *and* $\theta \in L^\infty(0,T;H^3)$ *for any finite* $T > 0$*. Moreover, the solution obeys the large-time behavior as* $t \to \infty$*:*

$$\|\mathbf{u}(t) - (\alpha(t)y, 0)^{\mathrm{T}}\|_{H^1} \to 0, \quad \|\partial_t \mathbf{u}(t) - (\alpha'(t)y, 0)^{\mathrm{T}}\|_{L^2} \to 0, \quad \|(\nabla P - \theta \mathbf{e}_2)(t)\|_{H^{-1}} \to 0.$$

Remark 1.2. *Using the idea of the proof of Theorem 1.1, we can show that under the condition* (1.5)*, the solution to* (1.9) *satisfies* $\|\mathbf{u}(t) - (\alpha(t)y, 0)^{\mathrm{T}}\|_{L^2} \to 0$ *as* $t \to \infty$*. However, the decay of the spatial and temporal derivatives of the velocity field is elusive, which in turn affects the convergence of the pressure gradient. This leaves an interesting question for future investigation.*

When there is no thermal diffusion, the estimates on the derivatives of the solution are not straightforward. In order to estimate the gradient of the velocity field, we make use of the dissipation to couple the estimate of $\nu\|\nabla\widetilde{u}\|_{L^2}$ with several other quantities such as $\|\partial_t \widetilde{u}\|_{L^2}$. More technical details can be found in the proof of Theorem 1.3.

1.3.4. *Semi-dissipative System under No-penetration and Dirichlet B.C.*. The fourth Boussinesq system investigated here is inviscid in the velocity equation. Naturally $\mathbf{u}$ satisfies the no-penetration boundary condition. The temperature satisfies a dynamic Dirichlet boundary condition. More precisely, we consider the following Boussinesq system

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla P = \theta \mathbf{e}_2, & \mathbf{x} \in \Omega,\ t > 0, \\ \nabla \cdot \mathbf{u} = 0, & \mathbf{x} \in \Omega,\ t > 0, \\ \partial_t \theta + \mathbf{u}\cdot\nabla\theta = \kappa\Delta\theta, & \mathbf{x} \in \Omega,\ t > 0; \\ (\mathbf{u}, \theta)(\mathbf{x}, 0) = (\mathbf{u}_0, \theta_0)(\mathbf{x}), & \mathbf{x} \in \Omega; \\ \mathbf{u}\cdot\mathbf{n} = 0, \quad \theta = \beta(t)y, & \mathbf{x} \in \partial\Omega,\ t > 0. \end{cases} \tag{1.10}$$

(1.10) models inviscid buoyancy driven fluids when the thermal diffusion plays an important role. The global existence and regularity problem in the whole space case is well-understood [5, 18]. But the bounded domain case deserves more investigations. Since the velocity satisfies the buoyancy-forced Euler equation, the no-penetration boundary condition $\mathbf{u}\cdot\mathbf{n} = 0$ is a physically natural setting. The boundary condition on θ corresponds to a natural linear profile configuration of Rayleigh-Benard convection [11].

We present a special solution of (1.10) to illustrate the potential behaviors of solutions. In the case when $\beta \equiv 1$, the initial horizontal velocity and temperature depend only on y, and the initial vertical velocity is identically equal to zero, namely

$$\mathbf{u}_0(\mathbf{x}) = (u_{01}(y), 0), \quad \theta_0(\mathbf{x}) = \theta_0(y),$$

It is not difficult to verify that (1.10) admits the following special solution

$$\begin{aligned} &u_1(x,y,t) = u_{01}(y), \\ &u_2(x,y,t) \equiv 0, \\ &\theta(x,y,t) = y - 2\sum_{n=1}^{\infty} b_n e^{-n^2\pi^2\kappa t}\sin(n\pi(1-y)), \\ &P(x,y,t) = \int_0^y \theta(y',t)dy', \end{aligned}$$

where

$$b_n = \frac{1}{\pi n} - \int_0^1 \theta_0(y)\, \sin(n\pi(1-y))\, dy.$$

It is very easy to check that

$$\|\theta(t) - (\beta(t)y, 0)^{\mathrm{T}}\|_{L^2}^2 = \sum_{n=1}^{\infty} b_n^2\, e^{-2n^2\pi^2\kappa t}.$$

Trivially, as $t \to \infty$,

$$\|\theta(t) - (\beta(t)y, 0)^{\mathrm{T}}\|_{L^2} \to 0.$$

Since the velocity equation is inviscid, there is no regularization and the velocity could grow in time. No precise behavior can be predicted. The temperature converges asymptotically to the linear profile. The behavior revealed by this example actually holds for much more general flows governed by (1.10), as Theorem 1.4 asserts.

As we show in Section 5, the system still possesses a unique global solution in the Sobolev space H^3. However, the large-time behavior of the velocity is unknown. The temperature is shown here to converge asymptotically to $\beta(t)y$. More precisely, we have the following theorem.

Theorem 1.4. *Consider the initial-boundary value problem* (1.10)*. Suppose that the initial data satisfy* $(\mathbf{u}_0, \theta_0) \in H^3$, $\nabla \cdot \mathbf{u}_0 = 0$*, and are compatible with the boundary conditions. Assume that* $\beta : [0, \infty) \to \mathbb{R}$ *is a smooth function such that*

$$\beta_1 \leq \beta(t) \leq \beta_2, \quad \beta'(t) \in L^1(0, \infty),$$

where $0 < \beta_1 \leq \beta_2 < \infty$ *are constants. Then there exists a unique solution to* (1.10) *such that* $\mathbf{u} \in [L^\infty(0, T; H^3)]^2$ *and* $\theta \in L^\infty((0, T); H^3) \cap L^2((0, T); H^4)$*. Moreover, the solution obeys the large-time behavior as* $t \to \infty$*:*

$$\|\theta(t) - \beta(t)y\|_{L^2} \to 0.$$

1.3.5. *Semi-dissipative System under No-penetration and Neumann B.C..* The last Boussinesq system considered here is also inviscid and with thermal diffusion. The difference is that the temperature satisfies a Neumann boundary condition. That is, we consider

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla P = \theta \mathbf{e}_2, & \mathbf{x} \in \Omega,\ t > 0, \\ \nabla \cdot \mathbf{u} = 0, & \mathbf{x} \in \Omega,\ t > 0, \\ \partial_t \theta + \mathbf{u} \cdot \nabla \theta = \kappa \Delta \theta, & \mathbf{x} \in \Omega,\ t > 0; \\ (\mathbf{u}, \theta)(\mathbf{x}, 0) = (\mathbf{u}_0, \theta_0)(\mathbf{x}), & \mathbf{x} \in \Omega; \\ \mathbf{u} \cdot \mathbf{n} = 0, \quad \nabla \theta \cdot \mathbf{n} = \mathbf{e}_2 \cdot \mathbf{n}, & \mathbf{x} \in \partial\Omega,\ t > 0. \end{cases} \tag{1.11}$$

Again since the velocity satisfies the buoyancy-forced Euler equation, the no-penetration boundary condition $\mathbf{u} \cdot \mathbf{n} = 0$ is a physically natural setting. The boundary condition on θ is equivalent to Neumann boundary condition

$$\mathbf{n} \cdot \nabla(\theta - y) = 0.$$

This type of boundary conditions enforce the linear profile but allow horizontal oscillations. Intuitively, θ should converge to the profile $y + \bar{\theta}$, where $\bar{\theta}$ denotes the spatial average of $\theta_0 - y$. We provide an example.

An explicit non-trivial example of this type of solutions are given by

$$
\begin{aligned}
&u_1(x,y,t) = 0,\\
&u_2(x,y,t) = -\frac{1}{a}e^{-at}\cos\pi x,\\
&\theta(x,y,t) = y + e^{-at}\cos\pi x,\\
&P(x,y,t) = \frac{1}{2}y^2,
\end{aligned}
$$

where a is a constant satisfying $a^2 - \kappa\pi^2 a + 1 = 0$. It is not hard to check that they indeed satisfy (1.11). Clearly, $\theta_0 = y + \cos\pi x$ and $\bar{\theta} = 0$. Thus, as $t \to \infty$,

$$\|\theta(t) - y - \bar{\theta}\|_{L^2} \leq e^{-at} \to 0.$$

Our theorem concludes that general flows governed by (1.11) also converges to this type of asymptotic profile. We are able to prove the following result.

Theorem 1.5. *Consider the initial-boundary value problem* (1.11). *Suppose that the initial data satisfy* $(\mathbf{u}_0, \theta_0) \in H^3$, $\nabla \cdot \mathbf{u}_0 = 0$, *and are compatible with the boundary conditions. Then there exists a unique solution to* (1.11) *such that* $\mathbf{u} \in [L^\infty(0,T;H^3)]^2$ *and* $\theta \in L^\infty((0,T);H^3) \cap L^2((0,T);H^4)$. *Moreover, the solution obeys the large-time behavior as* $t \to \infty$*:*

$$\|\theta(t) - y - \bar{\theta}\|_{L^2} \to 0,$$

where $\bar{\theta}$ *denotes the spatial average of* $\theta_0 - y$.

Remark 1.3. *In Theorems 1.4 and 1.5 the convergence of* θ *occurs only in* L^2. *Whether the convergence can be established in more regular functional spaces is unknown at this point. The proof constructed in* [35] *shows that under constant Dirichlet or homogeneous Neumann boundary condition, decaying of higher order frequencies of* θ *depends on the uniform-in-time estimate of* $\mathbf{u}$ *in* H^1, *which can be obtained with the help of the exponential decay of* θ *in* L^2. *However, under the thermal boundary conditions in* (1.10) *and* (1.11), *we are unable to prove the exponential decay of* θ *due to the lack of Poincaré inequality. The problem is beyond the scope of this paper and we leave it for the future.*

The rest of this paper is naturally divided into five sections with each one of them devoted to proving one of theorems stated above.

2. Fully Dissipative System under Dynamic Couette Flow and Dirichlet B.C.

This section proves Theorem 1.1 on the fully dissipative system under dynamic Couette flow and Dirichlet boundary condition.

Proof of Theorem 1.1. We define

$$\widehat{\mathbf{u}} \equiv (\alpha(t)y, 0)^{\mathrm{T}}, \qquad \widehat{\theta} \equiv \beta(t)y,$$

and let

$$\mathbf{u} = \widehat{\mathbf{u}} + \widetilde{\mathbf{u}}, \qquad \theta = \widehat{\theta} + \widetilde{\theta}.$$

By rewriting the original system of equations, (1.1), in terms of the ansatz and perturbed variables, we have

$$\begin{cases}\partial_t\widetilde{\mathbf{u}}+[(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla](\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})+\nabla P=\nu\Delta\widetilde{\mathbf{u}}+(\widehat{\theta}+\widetilde{\theta})\mathbf{e}_2-\partial_t\widehat{\mathbf{u}},\\ \nabla\cdot\widetilde{\mathbf{u}}=0,\\ \partial_t\widetilde{\theta}+(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla(\widehat{\theta}+\widetilde{\theta})=\kappa\Delta\widetilde{\theta}-\partial_t\widehat{\theta}.\end{cases}\tag{2.1}$$

By direct calculations, we can show that

$$(\widehat{\mathbf{u}}\cdot\nabla)\widehat{\mathbf{u}}=\mathbf{0},\quad(\widetilde{\mathbf{u}}\cdot\nabla)\widehat{\mathbf{u}}=(\alpha(t)\widetilde{u}_2,0)^{\mathrm{T}},\quad\widehat{\mathbf{u}}\cdot\nabla\widehat{\theta}=0,\quad\widetilde{\mathbf{u}}\cdot\nabla\widehat{\theta}=\beta(t)\widetilde{u}_2.$$

We use (2.1) to get

$$\begin{cases}\partial_t\widetilde{\mathbf{u}}+[(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla]\widetilde{\mathbf{u}}+(\alpha(t)\widetilde{u}_2,0)^{\mathrm{T}}+\nabla\widetilde{P}=\nu\Delta\widetilde{\mathbf{u}}+\widetilde{\theta}\mathbf{e}_2-\partial_t\widehat{\mathbf{u}},\\ \nabla\cdot\widetilde{\mathbf{u}}=0,\\ \partial_t\widetilde{\theta}+(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla\widetilde{\theta}+\beta(t)\widetilde{u}_2=\kappa\Delta\widetilde{\theta}-\partial_t\widehat{\theta},\end{cases}\tag{2.2}$$

where $\widetilde{P}=P-\frac{\beta(t)}{2}y^2$. Note that both the $\widetilde{\mathbf{u}}$ and $\widetilde{\theta}$ satisfy the zero Dirichlet boundary condition, and $\widehat{\mathbf{u}}$ is divergence free.

Step 1. Taking L^2 inner product of the first equation in (2.2) with $\widetilde{\mathbf{u}}$ and the third one with $\widetilde{\theta}$, we can show that, after applying the boundary conditions,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{\mathbf{u}}\|_{L^2}^2+\nu\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2=-\alpha(t)\int_\Omega\widetilde{u}_1\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}+\int_\Omega\widetilde{\theta}\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}-\alpha'(t)\int_\Omega\widetilde{u}_1\,y\,\mathrm{d}\mathbf{x},\tag{2.3}$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{\theta}\|_{L^2}^2+\kappa\|\nabla\widetilde{\theta}\|_{L^2}^2=-\beta(t)\int_\Omega\widetilde{\theta}\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}-\beta'(t)\int_\Omega\widetilde{\theta}\,y\,\mathrm{d}\mathbf{x}.\tag{2.4}$$

Multiplying (2.3) by $\beta(t)$, we can show that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\beta(t)\|\widetilde{\mathbf{u}}\|_{L^2}^2\right]+\nu\,\beta(t)\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2\\&=-\beta(t)\alpha(t)\int_\Omega\widetilde{u}_1\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}+\beta(t)\int_\Omega\widetilde{\theta}\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}-\beta(t)\alpha'(t)\int_\Omega\widetilde{u}_1\,y\,\mathrm{d}\mathbf{x}+\frac{\beta'(t)}{2}\|\widetilde{\mathbf{u}}\|_{L^2}^2.\end{aligned}\tag{2.5}$$

Taking the sum of (2.4) and (2.5), we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\beta(t)\|\widetilde{\mathbf{u}}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2\right]+\nu\,\beta(t)\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+\kappa\|\nabla\widetilde{\theta}\|_{L^2}^2\\&=-\beta(t)\alpha(t)\int_\Omega\widetilde{u}_1\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}-\beta'(t)\int_\Omega\widetilde{\theta}\,y\,\mathrm{d}\mathbf{x}-\beta(t)\alpha'(t)\int_\Omega\widetilde{u}_1\,y\,\mathrm{d}\mathbf{x}+\frac{\beta'(t)}{2}\|\widetilde{\mathbf{u}}\|_{L^2}^2.\end{aligned}\tag{2.6}$$

The first term on the right-hand side of (2.6) can be shown to satisfy

$$\begin{aligned}\left|\beta(t)\alpha(t)\int_\Omega\widetilde{u}_1\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}\right|&\le\beta(t)|\widetilde{\alpha}(t)|\int_\Omega|\widetilde{u}_1|\,|\widetilde{u}_2|\,\mathrm{d}\mathbf{x}+\beta(t)|\alpha_1|\int_\Omega|\widetilde{u}_1|\,|\widetilde{u}_2|\,\mathrm{d}\mathbf{x}\\&\le\frac{\beta(t)}{2}|\widetilde{\alpha}(t)|\,\|\widetilde{\mathbf{u}}\|_{L^2}^2+\beta(t)\frac{|\alpha_1|}{2}\|\widetilde{\mathbf{u}}\|_{L^2}^2\\&\le\frac{\beta(t)}{2}|\widetilde{\alpha}(t)|\,\|\widetilde{\mathbf{u}}\|_{L^2}^2+\beta(t)\frac{|\alpha_1|}{2\pi^2}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2,\end{aligned}\tag{2.7}$$

where $\widetilde{\alpha}(t) = \alpha(t) - \alpha_1$, and the Poincaré inequality was applied (note that the diameter of Ω is 1, hence the optimal constant in the Poincaré inequality is $\frac{1}{\pi}$). The second and third terms on the right-hand side of (2.6) are estimated by using the Hölder inequality as

$$\left|\beta'(t)\int_\Omega \widetilde{\theta}\, y\,\mathrm{d}\mathbf{x}\right| + \left|\beta(t)\alpha'(t)\int_\Omega \widetilde{u}_1\, y\,\mathrm{d}\mathbf{x}\right| \le |\beta'(t)|\,\|\widetilde{\theta}\|_{L^2}\|y\|_{L^2} + \beta(t)\,|\alpha'(t)|\,\|\widetilde{\mathbf{u}}\|_{L^2}\|y\|_{L^2}. \tag{2.8}$$

Substituting (2.7) and (2.8) into (2.6), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\beta(t)\|\widetilde{\mathbf{u}}\|_{L^2}^2 + \|\widetilde{\theta}\|_{L^2}^2\Big] + \beta(t)\Big(\nu - \frac{|\alpha_1|}{2\pi^2}\Big)\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2 + \kappa\|\nabla\widetilde{\theta}\|_{L^2}^2\\ &\le \frac{1}{2}\Big(|\widetilde{\alpha}(t)| + \frac{|\beta'(t)|}{\beta(t)}\Big)\beta(t)\|\widetilde{\mathbf{u}}\|_{L^2}^2 + \frac{\sqrt{3}}{3}\,|\beta'(t)|\,\|\widetilde{\theta}\|_{L^2} + \frac{\sqrt{3}}{3}\,\beta(t)\,|\alpha'(t)|\,\|\widetilde{\mathbf{u}}\|_{L^2}.\end{aligned} \tag{2.9}$$

Using the Cauchy inequality, we can further show that

$$\begin{aligned}&\frac{\sqrt{3}}{3}\,|\beta'(t)|\,\|\widetilde{\theta}\|_{L^2} + \frac{\sqrt{3}}{3}\,\beta(t)\,|\alpha'(t)|\,\|\widetilde{\mathbf{u}}\|_{L^2}\\ &\le \frac{1}{2}\frac{|\beta'(t)|}{\beta(t)}\,\|\widetilde{\theta}\|_{L^2}^2 + \frac{\beta(t)}{2}\,|\alpha'(t)|\,\|\widetilde{\mathbf{u}}\|_{L^2}^2 + \frac{\beta(t)}{6}\big(|\beta'(t)| + |\alpha'(t)|\big).\end{aligned} \tag{2.10}$$

Using (2.10), (2.9) becomes

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\beta(t)\|\widetilde{\mathbf{u}}\|_{L^2}^2 + \|\widetilde{\theta}\|_{L^2}^2\Big] + \beta(t)\Big(\nu - \frac{|\alpha_1|}{2\pi^2}\Big)\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2 + \kappa\|\nabla\widetilde{\theta}\|_{L^2}^2\\ &\le \frac{1}{2}\Big(|\widetilde{\alpha}(t)| + |\alpha'(t)| + \frac{|\beta'(t)|}{\beta(t)}\Big)\Big[\beta(t)\|\widetilde{\mathbf{u}}\|_{L^2}^2 + \|\widetilde{\theta}\|_{L^2}^2\Big] + \frac{\beta(t)}{6}\big(|\beta'(t)| + |\alpha'(t)|\big)\\ &\le \frac{1}{2}\Big(|\widetilde{\alpha}(t)| + |\alpha'(t)| + \frac{|\beta'(t)|}{\beta_1}\Big)\Big[\beta(t)\|\widetilde{\mathbf{u}}\|_{L^2}^2 + \|\widetilde{\theta}\|_{L^2}^2\Big] + \frac{\beta_2}{6}\big(|\beta'(t)| + |\alpha'(t)|\big),\end{aligned} \tag{2.11}$$

where we used the assumption $0 < \beta_1 \le \beta(t) \le \beta_2$. Applying Grönwall's inequality to (2.11) and using the assumptions in Theorem 1.1, we can show that

$$\|\widetilde{\mathbf{u}}(t)\|_{L^2}^2 + \|\widetilde{\theta}(t)\|_{L^2}^2 + \int_0^t \Big(\|\nabla\widetilde{\mathbf{u}}(\tau)\|_{L^2}^2 + \|\nabla\widetilde{\theta}(\tau)\|_{L^2}^2\Big)\,\mathrm{d}\tau \le C, \quad \forall\, t > 0, \tag{2.12}$$

where the constant on the right-hand side is independent of t. It may depend on various norms of $\alpha(t)$ and $\beta(t)$.

Step 2. Taking L^2 inner product of the first equation in (2.2) with $\partial_t\widetilde{\mathbf{u}}$, we deduce that

$$\begin{aligned}\frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2 + \|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2 &= \int_\Omega \widetilde{\theta}\mathbf{e}_2\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x} - \int_\Omega \partial_t\widehat{\mathbf{u}}\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x} - \int_\Omega (\alpha(t)\widetilde{u}_2, 0)^{\mathrm{T}}\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}\\ &\quad - \int_\Omega [(\widehat{\mathbf{u}}\cdot\nabla)\widetilde{\mathbf{u}}]\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x} - \int_\Omega [(\widetilde{\mathbf{u}}\cdot\nabla)\widetilde{\mathbf{u}}]\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}\\ &\le \|\widetilde{\theta}\|_{L^2}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2} + \|\partial_t\widehat{\mathbf{u}}\|_{L^2}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2} + \|\alpha(t)\widetilde{u}_2\|_{L^2}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}\\ &\quad + \|\widehat{\mathbf{u}}\|_{L^\infty}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2} + \|\widetilde{\mathbf{u}}\|_{L^4}\|\nabla\widetilde{\mathbf{u}}\|_{L^4}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}\\ &\le \frac{1}{4}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2 + 4\big(\|\widetilde{\theta}\|_{L^2}^2 + \|\partial_t\widehat{\mathbf{u}}\|_{L^2}^2 + \|\alpha(t)\widetilde{u}_2\|_{L^2}^2 + \|\widehat{\mathbf{u}}\|_{L^\infty}^2\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2\big)\\ &\quad + \|\widetilde{\mathbf{u}}\|_{L^4}\|\nabla\widetilde{\mathbf{u}}\|_{L^4}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2},\end{aligned} \tag{2.13}$$

where the Cauchy inequality is applied. For the last term (triple product) on the right hand side of (2.13), using Ladyzhenskaya's and Poincaré's inequalities, we can show that

$$\|\widetilde{\mathbf{u}}\|_{L^4}\|\nabla\widetilde{\mathbf{u}}\|_{L^4}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2} \le C\|\widetilde{\mathbf{u}}\|_{L^2}^{\frac12}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}\|\Delta\widetilde{\mathbf{u}}\|_{L^2}^{\frac12}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2} \tag{2.14}$$

for some constant $C>0$ depending only on Ω. According to classic estimates for the Stokes equation, we deduce from the first equation in (2.2) that

$$\begin{aligned}
&\|\Delta\widetilde{\mathbf{u}}\|_{L^2}\\
&\le C\big(\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}+\|\widehat{\mathbf{u}}\|_{L^\infty}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}+\|\widetilde{\mathbf{u}}\|_{L^4}\|\nabla\widetilde{\mathbf{u}}\|_{L^4}+\|\alpha(t)\widetilde{u}_2\|_{L^2}+\|\widetilde{\theta}\|_{L^2}+\|\partial_t\widehat{\mathbf{u}}\|_{L^2}\big)\\
&\le C\big(\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}+\|\widehat{\mathbf{u}}\|_{L^\infty}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}+\|\widetilde{\mathbf{u}}\|_{L^2}^{\frac12}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}\|\Delta\widetilde{\mathbf{u}}\|_{L^2}^{\frac12}+\|\alpha(t)\widetilde{u}_2\|_{L^2}+\|\widetilde{\theta}\|_{L^2}+\|\partial_t\widehat{\mathbf{u}}\|_{L^2}\big)\\
&\le \frac12\|\Delta\widetilde{\mathbf{u}}\|_{L^2}+C\big(\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}+\|\widehat{\mathbf{u}}\|_{L^\infty}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}+\|\widetilde{\mathbf{u}}\|_{L^2}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+\|\alpha(t)\widetilde{u}_2\|_{L^2}+\|\widetilde{\theta}\|_{L^2}+\|\partial_t\widehat{\mathbf{u}}\|_{L^2}\big),
\end{aligned}$$

where (2.14) and the Cauchy inequality are applied. Hence, we have

$$\|\Delta\widetilde{\mathbf{u}}\|_{L^2}\le C\big(\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}+\|\widehat{\mathbf{u}}\|_{L^\infty}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}+\|\widetilde{\mathbf{u}}\|_{L^2}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+\|\alpha(t)\widetilde{u}_2\|_{L^2}+\|\widetilde{\theta}\|_{L^2}+\|\partial_t\widehat{\mathbf{u}}\|_{L^2}\big). \tag{2.15}$$

Substituting (2.15) into (2.14), we obtain

$$\begin{aligned}
\|\widetilde{\mathbf{u}}\|_{L^4}\|\nabla\widetilde{\mathbf{u}}\|_{L^4}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2} &\le C\|\widetilde{\mathbf{u}}\|_{L^2}^{\frac12}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}\big(\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^{\frac12}+\|\widehat{\mathbf{u}}\|_{L^\infty}^{\frac12}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^{\frac12}+\|\widetilde{\mathbf{u}}\|_{L^2}^{\frac12}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}\\
&\qquad+\|\alpha(t)\widetilde{u}_2\|_{L^2}^{\frac12}+\|\widetilde{\theta}\|_{L^2}^{\frac12}+\|\partial_t\widehat{\mathbf{u}}\|_{L^2}^{\frac12}\big)\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}\\
&\le C\|\widetilde{\mathbf{u}}\|_{L^2}^{\frac12}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^{\frac32}+C\|\widetilde{\mathbf{u}}\|_{L^2}^{\frac12}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}\big(\|\widehat{\mathbf{u}}\|_{L^\infty}^{\frac12}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^{\frac12}\\
&\qquad+\|\widetilde{\mathbf{u}}\|_{L^2}^{\frac12}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}+\|\alpha(t)\widetilde{u}_2\|_{L^2}^{\frac12}+\|\widetilde{\theta}\|_{L^2}^{\frac12}+\|\partial_t\widehat{\mathbf{u}}\|_{L^2}^{\frac12}\big)\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}.
\end{aligned} \tag{2.16}$$

Using Young's inequality, we can show that

$$C\|\widetilde{\mathbf{u}}\|_{L^2}^{\frac12}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^{\frac32}\le\frac18\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2+C\|\widetilde{\mathbf{u}}\|^2\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^4,$$

and

$$\begin{aligned}
&C\|\widetilde{\mathbf{u}}\|_{L^2}^{\frac12}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}\big(\|\widehat{\mathbf{u}}\|_{L^\infty}^{\frac12}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^{\frac12}+\|\widetilde{\mathbf{u}}\|_{L^2}^{\frac12}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}+\|\alpha(t)\widetilde{u}_2\|_{L^2}^{\frac12}+\|\widetilde{\theta}\|_{L^2}^{\frac12}+\|\partial_t\widehat{\mathbf{u}}\|_{L^2}^{\frac12}\big)\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}\\
&\le\frac18\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2+C\|\widetilde{\mathbf{u}}\|_{L^2}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2\big(\|\widehat{\mathbf{u}}\|_{L^\infty}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}+\|\widetilde{\mathbf{u}}\|_{L^2}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+\|\alpha(t)\widetilde{u}_2\|_{L^2}+\|\widetilde{\theta}\|_{L^2}+\|\partial_t\widehat{\mathbf{u}}\|_{L^2}\big)\\
&\le\frac18\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2+C\big(\|\widehat{\mathbf{u}}\|_{L^\infty}^2\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+\|\widetilde{\mathbf{u}}\|_{L^2}^2\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^4+\|\alpha(t)\widetilde{u}_2\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2+\|\partial_t\widehat{\mathbf{u}}\|_{L^2}^2\big).
\end{aligned}$$

Substituting the above estimates into (2.16), we have

$$\begin{aligned}
&\|\widetilde{\mathbf{u}}\|_{L^4}\|\nabla\widetilde{\mathbf{u}}\|_{L^4}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}\\
&\le\frac14\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2+C\big(\|\widehat{\mathbf{u}}\|_{L^\infty}^2\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+\|\widetilde{\mathbf{u}}\|_{L^2}^2\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^4+\|\alpha(t)\widetilde{u}_2\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2+\|\partial_t\widehat{\mathbf{u}}\|_{L^2}^2\big).
\end{aligned} \tag{2.17}$$

Substituting (2.17) into (2.13), we have

$$\begin{aligned}
&\frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+\frac12\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2\\
&\le C\big(\|\widehat{\mathbf{u}}\|_{L^\infty}^2\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+\|\widetilde{\mathbf{u}}\|_{L^2}^2\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^4+\|\alpha(t)\widetilde{u}_2\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2+\|\partial_t\widehat{\mathbf{u}}\|_{L^2}^2\big).
\end{aligned} \tag{2.18}$$

Note that since $\lim_{t\to\infty}\alpha(t)=\alpha_1$ and $\alpha'(t)\in L^1(0,\infty)$, it holds that

$$\begin{aligned}|\alpha(t)| &= \Big|\alpha_1-\int_t^\infty \alpha'(\tau)\mathrm{d}\tau\Big|\\ &\le |\alpha_1|+\int_0^\infty |\alpha'(\tau)|\mathrm{d}\tau\\ &\le C, \qquad\qquad \forall\, 0\le t<\infty,\end{aligned}\tag{2.19}$$

where the constant C is independent of t. Using (2.19), we can show that

$$\begin{aligned}\|\widehat{\mathbf{u}}\|_{L^\infty}^2\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2 &= \|\alpha(t)y\|_{L^\infty}^2\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2\\ &\le |\alpha(t)|^2\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2\\ &\le C\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2,\end{aligned}\tag{2.20}$$

and

$$\begin{aligned}\|\alpha(t)\widetilde{u}_2\|_{L^2}^2 &\le |\alpha(t)|^2\|\widetilde{\mathbf{u}}\|_{L^2}^2\\ &\le C\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2,\end{aligned}\tag{2.21}$$

where we applied the Poincaré inequality. Using (2.12) and the Poincaré inequality, we can show that

$$\|\widetilde{\mathbf{u}}\|_{L^2}^2\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^4+\|\widetilde{\theta}\|_{L^2}^2\le C\big(\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^4+\|\nabla\widetilde{\theta}\|_{L^2}^2\big).\tag{2.22}$$

Substituting (2.20), (2.21), (2.22), and $\|\partial_t\widehat{\mathbf{u}}\|_{L^2}^2=\frac{1}{3}|\alpha'(t)|^2$ into (2.18), we obtain

$$\frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+\frac12\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2\le C\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2\Big(\frac{\nu}{2}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2\Big)+C\big(\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+\|\nabla\widetilde{\theta}\|_{L^2}^2+|\alpha'(t)|^2\big).\tag{2.23}$$

Applying Grönwall's inequality to (2.23), using (2.12) and the assumption $\alpha'(t)\in L^2(0,\infty)$, we infer that

$$\|\nabla\widetilde{\mathbf{u}}(t)\|_{L^2}^2\le C,\quad \forall\, t>0.\tag{2.24}$$

Substituting (2.24) into (2.23), then integrating the result with respect to time, we have

$$\int_0^t\|\partial_t\widetilde{\mathbf{u}}(\tau)\|_{L^2}^2\mathrm{d}\tau\le C,\quad\forall\, t>0.\tag{2.25}$$

Step 3. Taking L^2 inner product of the third equation in (2.2) with $-\Delta\widetilde{\theta}$, we infer that

$$\begin{aligned}\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\widetilde{\theta}\|_{L^2}^2+\kappa\|\Delta\widetilde{\theta}\|_{L^2}^2 &= \int_\Omega(\partial_t\widehat{\theta})(\Delta\widetilde{\theta})\mathrm{d}\mathbf{x}+\int_\Omega\beta(t)\widetilde{u}_2\,\Delta\widetilde{\theta}\,\mathrm{d}\mathbf{x}+\int_\Omega[(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla\widetilde{\theta}]\Delta\widetilde{\theta}\,\mathrm{d}\mathbf{x}\\ &\le\frac{\kappa}{4}\|\Delta\widetilde{\theta}\|_{L^2}^2+\frac{1}{\kappa}\big(\|\partial_t\widehat{\theta}\|_{L^2}^2+\|\beta(t)\widetilde{u}_2\|_{L^2}^2\big)+\|(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla\widetilde{\theta}\|_{L^2}\|\Delta\widetilde{\theta}\|_{L^2}.\end{aligned}\tag{2.26}$$

For the last term on the right-hand side of (2.26), using the Sobolev embedding: $H^1\hookrightarrow L^4$, Ladyzhenskaya's, Poincaré's and Young's inequalities, we can show that

$$\begin{aligned}\|(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla\widetilde{\theta}\|_{L^2}\|\Delta\widetilde{\theta}\|_{L^2} &\le\|\widehat{\mathbf{u}}\|_{L^\infty}\|\nabla\widetilde{\theta}\|_{L^2}\|\Delta\widetilde{\theta}\|_{L^2}+\|\widetilde{\mathbf{u}}\|_{L^4}\|\nabla\widetilde{\theta}\|_{L^4}\|\Delta\widetilde{\theta}\|_{L^2}\\ &\le\|\widehat{\mathbf{u}}\|_{L^\infty}\|\nabla\widetilde{\theta}\|_{L^2}\|\Delta\widetilde{\theta}\|_{L^2}+\|\nabla\widetilde{\mathbf{u}}\|_{L^2}\|\nabla\widetilde{\theta}\|_{L^2}^{\frac12}\|\Delta\widetilde{\theta}\|_{L^2}^{\frac32}\\ &\le\frac{\kappa}{4}\|\Delta\widetilde{\theta}\|_{L^2}^2+C\big(\|\widehat{\mathbf{u}}\|_{L^\infty}^2\|\nabla\widetilde{\theta}\|_{L^2}^2+\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^4\|\nabla\widetilde{\theta}\|_{L^2}^2\big).\end{aligned}\tag{2.27}$$

Substituting (2.27) into (2.26), we deduce that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\widetilde{\theta}\|_{L^2}^2+\frac{\kappa}{2}\|\Delta\widetilde{\theta}\|_{L^2}^2\\
&\le C\big(\|\widehat{\mathbf{u}}\|_{L^\infty}^2\|\nabla\widetilde{\theta}\|_{L^2}^2+\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^4\|\nabla\widetilde{\theta}\|_{L^2}^2+\|\partial_t\widehat{\theta}\|_{L^2}^2+\|\beta(t)\widetilde{u}_2\|_{L^2}^2\big)\\
&\le C\big(|\alpha(t)|^2\|\nabla\widetilde{\theta}\|_{L^2}^2+\|\nabla\widetilde{\theta}\|_{L^2}^2+|\beta'(t)|^2+|\beta(t)|^2\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2\big)\\
&\le C\big(\|\nabla\widetilde{\theta}\|_{L^2}^2+|\beta'(t)|^2+\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2\big),
\end{aligned}
\tag{2.28}
$$

where (2.19), (2.24), the boundedness of $\beta(t)$, and the Poincaré inequality are applied. Integrating (2.28) with respect to time and using (2.12) and the integrability of $\beta'(t)$, we have

$$
\|\nabla\widetilde{\theta}(t)\|_{L^2}^2+\int_0^t\|\Delta\widetilde{\theta}(\tau)\|_{L^2}^2\mathrm{d}\tau\le C,\quad\forall\, t>0.
\tag{2.29}
$$

Using the third equation in (2.2), we can show that

$$
\begin{aligned}
\|\partial_t\widetilde{\theta}\|_{L^2}^2&\le 5\big(\|\widehat{\mathbf{u}}\cdot\nabla\widetilde{\theta}\|_{L^2}^2+\|\widetilde{\mathbf{u}}\cdot\nabla\widetilde{\theta}\|_{L^2}^2+\|\beta(t)\widetilde{u}_2\|_{L^2}^2+\|\kappa\Delta\widetilde{\theta}\|_{L^2}^2+\|\partial_t\widehat{\theta}\|_{L^2}^2\big)\\
&\le C\big(\|\widehat{\mathbf{u}}\|_{L^\infty}^2\|\nabla\widetilde{\theta}\|_{L^2}^2+\|\widetilde{\mathbf{u}}\|_{L^4}^2\|\nabla\widetilde{\theta}\|_{L^4}^2+|\beta(t)|^2\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+\|\Delta\widetilde{\theta}\|_{L^2}^2+|\beta'(t)|^2\big)\\
&\le C\big(|\alpha(t)|^2\|\nabla\widetilde{\theta}\|_{L^2}^2+\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2\|\Delta\widetilde{\theta}\|_{L^2}^2+\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+\|\Delta\widetilde{\theta}\|_{L^2}^2+|\beta'(t)|^2\big)\\
&\le C\big(\|\nabla\widetilde{\theta}\|_{L^2}^2+\|\Delta\widetilde{\theta}\|_{L^2}^2+\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+|\beta'(t)|^2\big).
\end{aligned}
\tag{2.30}
$$

Integrating (2.30) with respect to time and using (2.12), (2.29), and the integrability of $\beta'(t)$, we infer that

$$
\int_0^t\|\partial_t\widetilde{\theta}\|_{L^2}^2\mathrm{d}\tau\le C,\quad\forall\, t>0,
\tag{2.31}
$$

where the constant on the right-hand side is independent of t.

Step 4. Taking ∂_t of the first equation in (2.2), we have

$$
\begin{aligned}
&\partial_{tt}\widetilde{\mathbf{u}}+[\partial_t(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla]\widetilde{\mathbf{u}}+[(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla]\partial_t\widetilde{\mathbf{u}}+(\alpha'(t)\widetilde{u}_2,0)^{\mathrm{T}}+(\alpha(t)\partial_t\widetilde{u}_2,0)^{\mathrm{T}}+\nabla\partial_t\widetilde{P}\\
&=\nu\Delta\partial_t\widetilde{\mathbf{u}}+\partial_t\widetilde{\theta}\mathbf{e}_2-\partial_{tt}\widehat{\mathbf{u}}.
\end{aligned}
\tag{2.32}
$$

Taking L^2 inner product of (2.32) with $\partial_t\widetilde{\mathbf{u}}$, we infer that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2+\nu\|\nabla\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2\\
&=\int_\Omega\partial_t\widetilde{\theta}(\mathbf{e}_2\cdot\partial_t\widetilde{\mathbf{u}})\mathrm{d}\mathbf{x}-\int_\Omega\partial_{tt}\widehat{\mathbf{u}}\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}-\int_\Omega\big([\partial_t(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla]\widetilde{\mathbf{u}}\big)\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}\\
&\quad-\int_\Omega(\alpha'(t)\widetilde{u}_2,0)^{\mathrm{T}}\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}-\int_\Omega(\alpha(t)\partial_t\widetilde{u}_2,0)^{\mathrm{T}}\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}.
\end{aligned}
\tag{2.33}
$$

Since $\nabla\cdot(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})=0$ and $\widetilde{\mathbf{u}}|_{\partial\Omega}=\mathbf{0}$, the third term on the right-hand side of (2.33) can be written as

$$
-\int_\Omega\big([\partial_t(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla]\widetilde{\mathbf{u}}\big)\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}=\int_\Omega\widetilde{\mathbf{u}}\cdot\big([\partial_t(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla]\partial_t\widetilde{\mathbf{u}}\big)\mathrm{d}\mathbf{x}.
\tag{2.34}
$$

Then (2.33) implies

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2+\nu\|\nabla\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2\\
&=\underbrace{\int_\Omega \partial_t\widetilde{\theta}(\mathbf{e}_2\cdot\partial_t\widetilde{\mathbf{u}})\mathrm{d}\mathbf{x}}_{\equiv I_1}-\underbrace{\int_\Omega \partial_{tt}\widehat{\mathbf{u}}\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}}_{\equiv I_2}+\underbrace{\int_\Omega \widetilde{\mathbf{u}}\cdot\big([\partial_t(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla]\partial_t\widetilde{\mathbf{u}}\big)\mathrm{d}\mathbf{x}}_{\equiv I_3}\\
&\quad-\underbrace{\int_\Omega (\alpha'(t)\widetilde{u}_2,0)^{\mathrm{T}}\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}}_{\equiv I_4}-\underbrace{\int_\Omega (\alpha(t)\partial_t\widetilde{u}_2,0)^{\mathrm{T}}\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}}_{\equiv I_5}.
\end{aligned}
\tag{2.35}
$$

Using Cauchy inequality, we can show that

$$
\begin{aligned}
&|I_1|+|I_2|+|I_4|+|I_5|\\
&\le \frac{1}{2}\big(\|\partial_t\widetilde{\theta}\|_{L^2}^2+\|\partial_{tt}\widehat{\mathbf{u}}\|_{L^2}^2+\|\alpha'(t)\widetilde{u}_2\|_{L^2}^2+\|\alpha(t)\partial_t\widetilde{u}_2\|_{L^2}^2\big)+2\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2\\
&\le \frac{1}{2}\big(\|\partial_t\widetilde{\theta}\|_{L^2}^2+|\alpha''(t)|^2+|\alpha'(t)|^2\|\widetilde{\mathbf{u}}\|_{L^2}^2+|\alpha(t)|^2\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2\big)+2\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2\\
&\le C\big(\|\partial_t\widetilde{\theta}\|_{L^2}^2+|\alpha''(t)|^2+|\alpha'(t)|^2+\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2\big),
\end{aligned}
\tag{2.36}
$$

where we used (2.12) and (2.19). Using elementary inequalities, (2.12) and (2.24), we can show that

$$
\begin{aligned}
|I_3|&\le \|\widetilde{\mathbf{u}}\|_{L^2}\|\partial_t\widehat{\mathbf{u}}\|_{L^\infty}\|\nabla\partial_t\widetilde{\mathbf{u}}\|_{L^2}+\|\widetilde{\mathbf{u}}\|_{L^4}\|\partial_t\widetilde{\mathbf{u}}\|_{L^4}\|\nabla\partial_t\widetilde{\mathbf{u}}\|_{L^2}\\
&\le C|\alpha'(t)|\,\|\nabla\partial_t\widetilde{\mathbf{u}}\|_{L^2}+C\|\nabla\widetilde{\mathbf{u}}\|_{L^2}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_t\widetilde{\mathbf{u}}\|_{L^2}^{\frac{3}{2}}\\
&\le C\big(|\alpha'(t)|^2+\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2\big)+\frac{\nu}{2}\|\nabla\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2.
\end{aligned}
\tag{2.37}
$$

Substituting (2.36) and (2.37) into (2.35), we obtain

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2+\frac{\nu}{2}\|\nabla\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2\le C\big(\|\partial_t\widetilde{\theta}\|_{L^2}^2+|\alpha''(t)|^2+|\alpha'(t)|^2+\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2\big).
\tag{2.38}
$$

Integrating (2.38) with respect to t and using (2.31), (2.25) and the assumptions for $\alpha(t)$, we infer that

$$
\|\partial_t\widetilde{\mathbf{u}}(t)\|_{L^2}^2+\int_0^t\|\nabla\partial_t\widetilde{\mathbf{u}}(\tau)\|_{L^2}^2\mathrm{d}\tau\le C,\quad \forall\, t>0.
\tag{2.39}
$$

As a consequence of (2.15) and previous estimates, we have

$$
\begin{aligned}
\|\widetilde{\mathbf{u}}\|_{H^2}&\le C\big(\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}+\|\widehat{\mathbf{u}}\|_{L^\infty}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}+\|\widetilde{\mathbf{u}}\|_{L^2}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+\|\alpha(t)\widetilde{u}_2\|_{L^2}+\|\widetilde{\theta}\|_{L^2}+\|\partial_t\widehat{\mathbf{u}}\|_{L^2}\big)\\
&\le C\big(\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}+|\alpha(t)|\,\|\nabla\widetilde{\mathbf{u}}\|_{L^2}+\|\widetilde{\mathbf{u}}\|_{L^2}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+|\alpha(t)|\,\|\widetilde{\mathbf{u}}\|_{L^2}+\|\widetilde{\theta}\|_{L^2}+|\alpha'(t)|\big)\\
&\le C\big(1+|\alpha'(t)|\big),\quad \forall\, t>0,
\end{aligned}
\tag{2.40}
$$

where we used (2.39), (2.19), (2.24), and (2.12). Since $\alpha'(t)\in L^2(0,\infty)$ and $\alpha''(t)\in L^2(0,\infty)$, we can show that

$$
\begin{aligned}
\int_0^\infty\Big|\frac{\mathrm{d}}{\mathrm{d}t}[\alpha'(t)]^2\Big|\mathrm{d}t&=2\int_0^\infty|\alpha'(t)|\,|\alpha''(t)|\mathrm{d}t\\
&\le 2\|\alpha'\|_{L^2(0,\infty)}\|\alpha''\|_{L^2(0,\infty)}\\
&\le C,
\end{aligned}
$$

which, together with $\alpha'(t) \in L^2(0,\infty)$, implies that $[\alpha'(t)]^2 \in W^{1,1}(0,\infty)$. Hence, $\lim_{t\to\infty}[\alpha'(t)]^2 = 0$. As a result, we have

$$\begin{aligned}[\alpha'(t)]^2 &= -2\int_t^\infty \alpha'(\tau)\alpha''(\tau)\mathrm{d}\tau \\ &\le 2\|\alpha'\|_{L^2(0,\infty)}\|\alpha''\|_{L^2(0,\infty)} \\ &\le C,\end{aligned} \tag{2.41}$$

for some constant which is independent of t. Using (2.41), we use (2.40) to get

$$\|\widetilde{\mathbf{u}}\|_{H^2} \le C, \quad \forall\, t>0. \tag{2.42}$$

Moreover, applying estimates of the Stokes equation, the Poincaré inequality and (2.19), we can show that

$$\begin{aligned}\|\widetilde{\mathbf{u}}(t)\|_{H^3}^2 &\le C\big(\|\partial_t\widetilde{\mathbf{u}}\|_{H^1}^2 + \|[(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla]\widetilde{\mathbf{u}}\|_{H^1}^2 + \|\alpha(\tau)\widetilde{\mathbf{u}}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2 + \|\partial_t\widehat{\mathbf{u}}\|_{H^1}^2\big) \\ &\le C\big(\|\nabla\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2 + \|(\widehat{\mathbf{u}},\widetilde{\mathbf{u}})\|_{H^2}^2\|\nabla\widetilde{\mathbf{u}}\|_{H^1}^2 + \|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2 + \|\nabla\widetilde{\theta}\|_{L^2}^2 + |\alpha'(t)|^2\big).\end{aligned} \tag{2.43}$$

Applying (2.19), (2.42), (2.40), and Poincaré inequality, we can show that

$$\begin{aligned}&\|(\widehat{\mathbf{u}},\widetilde{\mathbf{u}})\|_{H^2}^2\|\nabla\widetilde{\mathbf{u}}\|_{H^1}^2 \\ &\le C\|\nabla\widetilde{\mathbf{u}}\|_{H^1}^2 \\ &\le C\big(\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2 + |\alpha(t)|^2\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2 + \|\widetilde{\mathbf{u}}\|_{L^2}^2\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^4 + |\alpha(t)|^2\|\widetilde{\mathbf{u}}\|_{L^2}^2 + \|\widetilde{\theta}\|_{L^2}^2 + |\alpha'(t)|^2\big) \\ &\le C\big(\|\nabla\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2 + \|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2 + \|\nabla\widetilde{\theta}\|_{L^2}^2 + |\alpha'(t)|^2\big).\end{aligned} \tag{2.44}$$

Using (2.44), we use (2.43) to get

$$\|\widetilde{\mathbf{u}}(t)\|_{H^3}^2 \le C\big(\|\nabla\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2 + \|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2 + \|\nabla\widetilde{\theta}\|_{L^2}^2 + |\alpha'(t)|^2\big). \tag{2.45}$$

Integrating (2.45) with respect to t, we obtain

$$\int_0^t \|\widetilde{\mathbf{u}}(\tau)\|_{H^3}^2\mathrm{d}\tau \le C, \qquad \forall\, t>0, \tag{2.46}$$

where we used (2.39), (2.12), and the integrability of $\alpha'(t)$, and the constant is independent of t.

Step 5. Taking L^2 inner product of (2.32) with $\partial_{tt}\widetilde{\mathbf{u}}$, we calculate

$$\begin{aligned}&\frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2 + \|\partial_{tt}\widetilde{\mathbf{u}}\|_{L^2}^2 \\ &= \underbrace{\int_\Omega \partial_t\widetilde{\theta}\mathbf{e}_2\cdot\partial_{tt}\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}}_{\equiv J_1} - \underbrace{\int_\Omega \partial_{tt}\widehat{\mathbf{u}}\cdot\partial_{tt}\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}}_{\equiv J_2} - \underbrace{\int_\Omega \big([\partial_t(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla]\widetilde{\mathbf{u}}\big)\cdot\partial_{tt}\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}}_{\equiv J_3} \\ &\quad - \underbrace{\int_\Omega \big([(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla]\partial_t\widetilde{\mathbf{u}}\big)\cdot\partial_{tt}\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}}_{\equiv J_4} - \underbrace{\int_\Omega (\alpha'(t)\widetilde{u}_2,0)^{\mathrm{T}}\cdot\partial_{tt}\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}}_{\equiv J_5} - \underbrace{\int_\Omega (\alpha(t)\partial_t\widetilde{u}_2,0)^{\mathrm{T}}\cdot\partial_{tt}\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}}_{\equiv J_6}.\end{aligned} \tag{2.47}$$

Using Cauchy inequality, we can show that

$$|J_1| + |J_2| + |J_5| + |J_6| \le C\big(\|\partial_t\widetilde{\theta}\|_{L^2}^2 + |\alpha''(t)|^2 + |\alpha'(t)|^2 + \|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2\big) + \frac{1}{4}\|\partial_{tt}\widetilde{\mathbf{u}}\|_{L^2}^2. \tag{2.48}$$

Using Hölder's inequality, we can show that

$$\begin{aligned}|J_3|+|J_4| &\le \|\partial_t\widehat{\mathbf{u}}\|_{L^\infty}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}\|\partial_{tt}\widetilde{\mathbf{u}}\|_{L^2}+\|\partial_t\widetilde{\mathbf{u}}\|_{L^4}\|\nabla\widetilde{\mathbf{u}}\|_{L^4}\|\partial_{tt}\widetilde{\mathbf{u}}\|_{L^2}\\ &\quad+\big(\|\widehat{\mathbf{u}}\|_{L^\infty}+\|\widetilde{\mathbf{u}}\|_{L^\infty}\big)\|\nabla\partial_t\widetilde{\mathbf{u}}\|_{L^2}\|\partial_{tt}\widetilde{\mathbf{u}}\|_{L^2}\\ &\le C\big(|\alpha'(t)|^2+\|\nabla\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2\big)+\frac{1}{4}\|\partial_{tt}\widetilde{\mathbf{u}}\|_{L^2}^2,\end{aligned}\tag{2.49}$$

where we used the Sobolev embedding $H^2\hookrightarrow L^\infty$ and (2.42). Substituting (2.48) and (2.49) into (2.47), we obtain

$$\frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2+\frac{1}{2}\|\partial_{tt}\widetilde{\mathbf{u}}\|_{L^2}^2\le C\big(\|\partial_t\widetilde{\theta}\|_{L^2}^2+|\alpha''(t)|^2+|\alpha'(t)|^2+\|\partial_t\widetilde{\mathbf{u}}\|_{H^1}^2\big).\tag{2.50}$$

Integrating (2.50) with respect to t, we infer that

$$\|\nabla\partial_t\widetilde{\mathbf{u}}(t)\|_{L^2}^2+\int_0^t\|\partial_{tt}\widetilde{\mathbf{u}}(\tau)\|_{L^2}^2\mathrm{d}\tau\le C,\quad\forall\, t>0.\tag{2.51}$$

As a consequence of (2.45), (2.51), (2.24), (2.29), and (2.41), we infer that

$$\|\widetilde{\mathbf{u}}(t)\|_{H^3}\le C,\quad\forall\, t>0.\tag{2.52}$$

Furthermore, using the uniform integrability of $\|\partial_{tt}\widetilde{\mathbf{u}}\|_{L^2}^2$ and an iteration argument, we can show that

$$\int_0^t\|\widetilde{\mathbf{u}}(\tau)\|_{H^4}^2\mathrm{d}\tau\le C,\quad\forall\, t>0.\tag{2.53}$$

This completes the proof for the spatial regularity of the velocity field.

Step 6. Taking ∂_t of the third equation in (2.2), we have

$$\partial_{tt}\widetilde{\theta}+\partial_t(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla\widetilde{\theta}+(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla\partial_t\widetilde{\theta}+\beta'(t)\widetilde{u}_2+\beta(t)\partial_t\widetilde{u}_2=\kappa\Delta\partial_t\widetilde{\theta}-\partial_{tt}\widehat{\theta}.\tag{2.54}$$

Taking L^2 inner product of (2.54) with $\partial_t\widetilde{\theta}$, we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t\widetilde{\theta}\|_{L^2}^2+\kappa\|\nabla\partial_t\widetilde{\theta}\|_{L^2}^2\\ &=-\int_\Omega\partial_{tt}\widehat{\theta}\,\partial_t\widetilde{\theta}\,\mathrm{d}\mathbf{x}-\int_\Omega\Big[\partial_t(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla\widetilde{\theta}\Big]\partial_t\widetilde{\theta}\,\mathrm{d}\mathbf{x}-\int_\Omega\beta'(t)\widetilde{u}_2\,\partial_t\widetilde{\theta}\,\mathrm{d}\mathbf{x}-\int_\Omega\beta(t)\partial_t\widetilde{u}_2\,\partial_t\widetilde{\theta}\,\mathrm{d}\mathbf{x}\\ &=-\int_\Omega\partial_{tt}\widehat{\theta}\,\partial_t\widetilde{\theta}\,\mathrm{d}\mathbf{x}+\int_\Omega\widetilde{\theta}\,\partial_t(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla\partial_t\widetilde{\theta}\,\mathrm{d}\mathbf{x}-\int_\Omega\beta'(t)\widetilde{u}_2\,\partial_t\widetilde{\theta}\,\mathrm{d}\mathbf{x}-\int_\Omega\beta(t)\partial_t\widetilde{u}_2\,\partial_t\widetilde{\theta}\,\mathrm{d}\mathbf{x}\\ &\le C\big(|\beta''(t)|^2+|\beta'(t)|^2+\|\partial_t\widetilde{\mathbf{u}}\|_{H^1}^2+\|\partial_t\widetilde{\theta}\|_{L^2}^2+|\alpha'(t)|^2\big)+\frac{\kappa}{2}\|\nabla\partial_t\widetilde{\theta}\|_{L^2}^2,\end{aligned}\tag{2.55}$$

which implies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t\widetilde{\theta}\|_{L^2}^2+\frac{\kappa}{2}\|\nabla\partial_t\widetilde{\theta}\|_{L^2}^2\le C\big(|\beta''(t)|^2+|\beta'(t)|^2+\|\partial_t\widetilde{\mathbf{u}}\|_{H^1}^2+\|\partial_t\widetilde{\theta}\|_{L^2}^2+|\alpha'(t)|^2\big).\tag{2.56}$$

Integrating (2.56) with respect to t, and using the previous estimates, we can show that

$$\|\partial_t\widetilde{\theta}(t)\|_{L^2}^2+\int_0^t\|\nabla\partial_t\widetilde{\theta}(\tau)\|_{L^2}^2\mathrm{d}\tau\le C,\quad\forall\, t>0.\tag{2.57}$$

According to the third equation in (2.2), standard elliptic estimates and the Poincaré inequality,

$$\begin{aligned}\|\widetilde{\theta}\|_{H^2}&\lesssim\|\partial_t\widetilde{\theta}\|_{L^2}+\|(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla\widetilde{\theta}\|_{L^2}+\|\beta(t)\widetilde{\mathbf{u}}\|_{L^2}+\|\partial_t\widehat{\theta}\|_{L^2}\\ &\lesssim\|\partial_t\widetilde{\theta}\|_{L^2}+\|(\widehat{\mathbf{u}},\widetilde{\mathbf{u}})\|_{L^\infty}\|\nabla\widetilde{\theta}\|_{L^2}+\|\widetilde{\mathbf{u}}\|_{L^2}+|\beta'(t)|\\ &\lesssim\|\partial_t\widetilde{\theta}\|_{L^2}+\|\nabla\widetilde{\theta}\|_{L^2}+\|\widetilde{\mathbf{u}}\|_{L^2}+|\beta'(t)|,\end{aligned}\tag{2.58}$$

and

$$
\begin{aligned}
\|\widetilde{\theta}\|_{H^3} \lesssim \|\Delta\widetilde{\theta}\|_{H^1} &\lesssim \|\partial_t\widetilde{\theta}\|_{H^1} + \|(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla\widetilde{\theta}\|_{H^1} + \|\beta(t)\widetilde{\mathbf{u}}\|_{H^1} + \|\partial_t\widehat{\theta}\|_{H^1} \\
&\lesssim \|\nabla\partial_t\widetilde{\theta}\|_{L^2} + \|(\widehat{\mathbf{u}},\widetilde{\mathbf{u}})\|_{H^2}\|\nabla\widetilde{\theta}\|_{H^1} + \|\nabla\widetilde{\mathbf{u}}\|_{L^2} + |\beta'(t)| \\
&\lesssim \|\nabla\partial_t\widetilde{\theta}\|_{L^2} + \|\Delta\widetilde{\theta}\|_{L^2} + \|\nabla\widetilde{\mathbf{u}}\|_{L^2} + |\beta'(t)| \\
&\lesssim \|\nabla\partial_t\widetilde{\theta}\|_{L^2} + \|\nabla\widetilde{\theta}\|_{L^2} + \|\nabla\widetilde{\mathbf{u}}\|_{L^2} + |\beta'(t)|.
\end{aligned} \tag{2.59}
$$

As a result of (2.57) and previous estimates, we have

$$
\|\widetilde{\theta}(t)\|_{H^2}^2 + \int_0^t \|\widetilde{\theta}(\tau)\|_{H^3}^2 \mathrm{d}\tau \le C, \quad \forall\, t>0. \tag{2.60}
$$

Step 7. Taking L^2 inner product of (2.54) with $-\Delta\partial_t\widetilde{\theta}$, we can show that

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\partial_t\widetilde{\theta}\|_{L^2}^2 + \frac{\kappa}{2}\|\Delta\partial_t\widetilde{\theta}\|_{L^2}^2 \le C\big(|\beta''(t)|^2 + |\beta'(t)|^2 + |\alpha'(t)|^2 + \|\partial_t\widetilde{\mathbf{u}}\|_{H^1}^2 + \|\nabla\partial_t\widetilde{\theta}\|_{L^2}^2\big). \tag{2.61}
$$

Integrating (2.61) with respect to t and using previous estimates, we can show that

$$
\|\nabla\partial_t\widetilde{\theta}(t)\|_{L^2}^2 + \int_0^t \|\Delta\partial_t\widetilde{\theta}(\tau)\|_{L^2}^2 \mathrm{d}\tau \le C, \quad \forall\, t>0, \tag{2.62}
$$

which, together with standard elliptic estimates, implies that

$$
\|\widetilde{\theta}(t)\|_{H^3}^2 + \int_0^t \|\widetilde{\theta}(\tau)\|_{H^4}^2 \mathrm{d}\tau \le C, \quad \forall\, t>0. \tag{2.63}
$$

This completes the proof for the regularity of the temperature field.

Step 8. It remains to prove the time decaying of the perturbation. First, according to (2.45),

$$
\|\widetilde{\mathbf{u}}\|_{H^3} \lesssim \|\nabla\partial_t\widetilde{\mathbf{u}}\|_{L^2} + \|\nabla\widetilde{\mathbf{u}}\|_{L^2} + \|\nabla\widetilde{\theta}\|_{L^2} + |\alpha'(t)|. \tag{2.64}
$$

Hence, it suffices to prove the temporal decaying of the quantities on the right-hand side of (2.64). Note that according to (2.50),

$$
\left|\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2\right| \le C\big(\|\partial_{tt}\widetilde{\mathbf{u}}\|_{L^2}^2 + \|\partial_t\widetilde{\theta}\|_{L^2}^2 + \|\partial_t\widetilde{\mathbf{u}}\|_{H^1}^2 + |\alpha''(t)|^2 + |\alpha'(t)|^2\big). \tag{2.65}
$$

Integrating (2.65) with respect to t, we infer that

$$
\int_0^t \left|\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\partial_t\widetilde{\mathbf{u}}(\tau)\|_{L^2}^2\right| \mathrm{d}\tau \le C, \quad \forall\, t>0, \tag{2.66}
$$

which, together with (2.39), implies $\|\nabla\partial_t\widetilde{\mathbf{u}}(t)\|_{L^2}^2 \in W^{1,1}(0,\infty)$. Hence, $\|\nabla\partial_t\widetilde{\mathbf{u}}(t)\|_{L^2}^2 \to 0$, as $t\to\infty$. In a similar fashion, we can show that $\|\nabla\widetilde{\mathbf{u}}(t)\|_{L^2}^2 \to 0$ and $\|\nabla\widetilde{\theta}(t)\|_{L^2}^2 \to 0$, as $t\to\infty$. Since $\alpha'(t)\to 0$ as $t\to\infty$ (see the arguments preceding (2.41)), we conclude that $\|\widetilde{\mathbf{u}}(t)\|_{H^3}\to 0$, as $t\to\infty$. In terms of the original variables, we see that

$$
\|\mathbf{u}(t) - (\alpha(t)y, 0)^{\mathrm{T}}\|_{H^3} \to 0, \quad \text{as} \quad t\to\infty.
$$

In addition, the decaying of $\|\widetilde{\theta}(t)\|_{H^3}$ follows from (2.59) and similar arguments as above. In terms of the original variables, we see that

$$
\|\theta(t) - \beta(t)y\|_{H^3} \to 0, \quad \text{as} \quad t\to\infty.
$$

Furthermore, as a result of the first equation in (2.2) and the decaying of the velocity field, we have

$$
\begin{aligned}
&\|\nabla \widetilde{P} - \widetilde{\theta}\mathbf{e}_2\|_{H^1} \\
\lesssim\ & \|\partial_t \widetilde{\mathbf{u}}\|_{H^1} + \|\widehat{\mathbf{u}}\|_{W^{1,\infty}}\|\nabla\widetilde{\mathbf{u}}\|_{H^1} + \|\widetilde{\mathbf{u}}\|_{W^{1,\infty}}\|\nabla\widetilde{\mathbf{u}}\|_{H^1} + \|\widetilde{\mathbf{u}}\|_{H^1} + \|\Delta\widetilde{\mathbf{u}}\|_{H^1} + |\beta'(t)| \\
\lesssim\ & \|\nabla\partial_t\widetilde{\mathbf{u}}\|_{L^2} + \|\widetilde{\mathbf{u}}\|_{H^3} + |\beta'(t)| \to 0, \quad \text{as} \quad t\to\infty.
\end{aligned}
$$

In terms of the original variables, we have

$$
\|(\nabla P - \theta\mathbf{e}_2)(t)\|_{H^1} \to 0, \quad \text{as} \quad t\to\infty.
$$

This completes the proof of Theorem 1.1. □

3. Fully Dissipative System under Dynamic Couette Flow and Neumann B.C.

This section is devoted to proving Theorem 1.2.

Proof of Theorem 1.2. First of all, we note that $\mathbf{e}_2 = \nabla y$, and hence the boundary condition for θ is equivalent to

$$
\nabla(\theta - \beta(t)y)\cdot\mathbf{n}|_{\partial\Omega} = 0. \tag{3.1}
$$

Again, we denote

$$
\widehat{\mathbf{u}} \equiv (\alpha(t)y, 0)^{\mathrm{T}}, \qquad \widehat{\theta} \equiv \beta(t)y,
$$

and let

$$
\mathbf{u} = \widehat{\mathbf{u}} + \widetilde{\mathbf{u}}, \qquad \theta = \widehat{\theta} + \widetilde{\theta}.
$$

The perturbed system in this case is identical to (2.2) (except boundary conditions), which reads

$$
\begin{cases}
\partial_t\widetilde{\mathbf{u}} + [(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla]\widetilde{\mathbf{u}} + (\alpha(t)\widetilde{u}_2, 0)^{\mathrm{T}} + \nabla\widetilde{P} = \nu\Delta\widetilde{\mathbf{u}} + \widetilde{\theta}\mathbf{e}_2 - \partial_t\widehat{\mathbf{u}},\\
\nabla\cdot\widetilde{\mathbf{u}} = 0,\\
\partial_t\widetilde{\theta} + (\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla\widetilde{\theta} + \beta(t)\widetilde{u}_2 = \kappa\Delta\widetilde{\theta} - \partial_t\widehat{\theta},
\end{cases} \tag{3.2}
$$

where $\widetilde{P} = P - \frac{\beta(t)}{2}y^2$, and $\widetilde{\mathbf{u}}$ and $\widetilde{\theta}$ satisfy the zero Dirichlet and Neumann boundary conditions, respectively.

Step 1. Calculating L^2 inner products, we can show that

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{\mathbf{u}}\|_{L^2}^2 + \nu\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2 = -\alpha(t)\int_\Omega \widetilde{u}_1\,\widetilde{u}_2\,\mathrm{d}\mathbf{x} + \int_\Omega \widetilde{\theta}\,\widetilde{u}_2\,\mathrm{d}\mathbf{x} - \alpha'(t)\int_\Omega \widetilde{u}_1\,y\,\mathrm{d}\mathbf{x}, \tag{3.3}
$$

and

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{\theta}\|_{L^2}^2 + \kappa\|\nabla\widetilde{\theta}\|_{L^2}^2 = -\beta(t)\int_\Omega \widetilde{\theta}\,\widetilde{u}_2\,\mathrm{d}\mathbf{x} - \beta'(t)\int_\Omega \widetilde{\theta}\,y\,\mathrm{d}\mathbf{x} - \int_\Omega\Big[(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla\widetilde{\theta}\Big]\,\widetilde{\theta}\,\mathrm{d}\mathbf{x}. \tag{3.4}
$$

For the last term on the right hand side of (3.4), due to the divergence free condition and zero boundary condition for $\widetilde{\mathbf{u}}$, we have

$$
-\int_\Omega\Big[(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla\widetilde{\theta}\Big]\,\widetilde{\theta}\,\mathrm{d}\mathbf{x} = -\int_\Omega\Big(\widehat{\mathbf{u}}\cdot\nabla\widetilde{\theta}\Big)\,\widetilde{\theta}\,\mathrm{d}\mathbf{x},
$$

where the integral on the right hand side does not vanish, due to the boundary condition of $\widehat{\mathbf{u}}$, which is one of the major differences between the problems (1.1) and (1.7). Then (3.4) becomes

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{\theta}\|_{L^2}^2 + \kappa\|\nabla\widetilde{\theta}\|_{L^2}^2 = -\beta(t)\int_\Omega \widetilde{\theta}\,\widetilde{u}_2\,\mathrm{d}\mathbf{x} - \beta'(t)\int_\Omega \widetilde{\theta}\,y\,\mathrm{d}\mathbf{x} - \int_\Omega\Big(\widehat{\mathbf{u}}\cdot\nabla\widetilde{\theta}\Big)\,\widetilde{\theta}\,\mathrm{d}\mathbf{x}. \tag{3.5}
$$

Multiplying (3.3) by $\beta(t)$, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\beta(t)\|\widetilde{\mathbf{u}}\|_{L^2}^2\right]+\nu\,\beta(t)\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2\\ =&\;-\beta(t)\alpha(t)\int_\Omega \widetilde{u}_1\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}+\beta(t)\int_\Omega\widetilde{\theta}\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}-\beta(t)\alpha'(t)\int_\Omega\widetilde{u}_1\,y\,\mathrm{d}\mathbf{x}+\frac{\beta'(t)}{2}\|\widetilde{\mathbf{u}}\|_{L^2}^2.\end{aligned}\tag{3.6}$$

Taking the sum of (3.5) and (3.6), we can show that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\beta(t)\|\widetilde{\mathbf{u}}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2\right]+\nu\,\beta(t)\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+\kappa\|\nabla\widetilde{\theta}\|_{L^2}^2\\ =&\;-\beta(t)\alpha(t)\int_\Omega \widetilde{u}_1\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}-\beta'(t)\int_\Omega\widetilde{\theta}\,y\,\mathrm{d}\mathbf{x}-\beta(t)\alpha'(t)\int_\Omega\widetilde{u}_1\,y\,\mathrm{d}\mathbf{x}\\ &+\frac{\beta'(t)}{2}\|\widetilde{\mathbf{u}}\|_{L^2}^2-\int_\Omega\Big(\widehat{\mathbf{u}}\cdot\nabla\widetilde{\theta}\Big)\,\widetilde{\theta}\,\mathrm{d}\mathbf{x}\\ \le&\;-\beta(t)\alpha(t)\int_\Omega \widetilde{u}_1\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}-\beta'(t)\int_\Omega\widetilde{\theta}\,y\,\mathrm{d}\mathbf{x}-\beta(t)\alpha'(t)\int_\Omega\widetilde{u}_1\,y\,\mathrm{d}\mathbf{x}\\ &+\frac{\beta'(t)}{2}\|\widetilde{\mathbf{u}}\|_{L^2}^2+\frac{\kappa}{2}\|\nabla\widetilde{\theta}\|_{L^2}^2+\frac{1}{2\kappa}\|\widehat{\mathbf{u}}\|_{L^\infty}^2\|\widetilde{\theta}\|_{L^2}^2,\end{aligned}\tag{3.7}$$

which implies

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\beta(t)\|\widetilde{\mathbf{u}}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2\right]+\nu\,\beta(t)\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+\frac{\kappa}{2}\|\nabla\widetilde{\theta}\|_{L^2}^2\\ \le&\;-\beta(t)\alpha(t)\int_\Omega \widetilde{u}_1\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}-\beta'(t)\int_\Omega\widetilde{\theta}\,y\,\mathrm{d}\mathbf{x}-\beta(t)\alpha'(t)\int_\Omega\widetilde{u}_1\,y\,\mathrm{d}\mathbf{x}\\ &+\frac{\beta'(t)}{2}\|\widetilde{\mathbf{u}}\|_{L^2}^2+\frac{1}{2\kappa}\|\widehat{\mathbf{u}}\|_{L^\infty}^2\|\widetilde{\theta}\|_{L^2}^2.\end{aligned}\tag{3.8}$$

Similar to (2.11), we can show that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\beta(t)\|\widetilde{\mathbf{u}}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2\right]+\nu\,\beta(t)\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+\kappa\|\nabla\widetilde{\theta}\|_{L^2}^2\\ \le&\; C\big(|\alpha(t)|+|\alpha'(t)|+|\beta'(t)|\big)\left[\beta(t)\|\widetilde{\mathbf{u}}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2\right]+C\big(|\beta'(t)|+|\alpha'(t)|\big).\end{aligned}\tag{3.9}$$

Applying Grönwall's inequality to (3.9), we can show that

$$\|\widetilde{\mathbf{u}}(t)\|_{L^2}^2+\|\widetilde{\theta}(t)\|_{L^2}^2+\int_0^t\big(\|\nabla\widetilde{\mathbf{u}}(\tau)\|_{L^2}^2+\|\nabla\widetilde{\theta}(\tau)\|_{L^2}^2\big)\mathrm{d}\tau\le C,\quad\forall\,t>0.\tag{3.10}$$

Step 2. Taking L^2 inner product of the first equation in (3.2) with $\partial_t\widetilde{\mathbf{u}}$, we have

$$\begin{aligned}\frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2=&\int_\Omega\widetilde{\theta}\mathbf{e}_2\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}-\int_\Omega\partial_t\widehat{\mathbf{u}}\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}-\int_\Omega(\alpha(t)\widetilde{u}_2,0)^{\mathrm{T}}\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}\\ &-\int_\Omega[(\widehat{\mathbf{u}}\cdot\nabla)\widetilde{\mathbf{u}}]\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}-\int_\Omega[(\widetilde{\mathbf{u}}\cdot\nabla)\widetilde{\mathbf{u}}]\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}.\end{aligned}\tag{3.11}$$

Note that due to the lack of Poincaré inequality for $\widetilde{\theta}$, the first term on the right-hand side of (3.11) can not be estimated by using the Cauchy inequality as in (2.13)–(2.23). Hence, we need to derive an alternative estimate for such a term. We proceed as the following:

$$
\begin{aligned}
\int_\Omega \widetilde{\theta}\mathbf{e}_2\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega \widetilde{\theta}\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}\Big) - \int_\Omega (\partial_t\widetilde{\theta})\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega \widetilde{\theta}\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}\Big) + \int_\Omega \big[(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla\widetilde{\theta} + \beta(t)\widetilde{u}_2 - \kappa\Delta\widetilde{\theta} + \partial_t\widehat{\theta}\big]\widetilde{u}_2\,\mathrm{d}\mathbf{x}\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega \widetilde{\theta}\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}\Big) + \int_\Omega \big[-\widetilde{\theta}(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}}) + \kappa\nabla\widetilde{\theta}\big]\cdot\nabla\widetilde{u}_2\,\mathrm{d}\mathbf{x} + \int_\Omega \big[\beta(t)\widetilde{u}_2 + \partial_t\widehat{\theta}\big]\widetilde{u}_2\,\mathrm{d}\mathbf{x},
\end{aligned}
\tag{3.12}
$$

where we used the third equation in (3.2), the divergence-free condition and zero boundary condition for $\widetilde{u}_2$. Substituting (3.12) into (3.11), we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\nu}{2}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2 + \int_\Omega \widetilde{\theta}\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}\Big) + \|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2\\
&= \underbrace{\int_\Omega \big[-\widetilde{\theta}(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}}) + \kappa\nabla\widetilde{\theta}\big]\cdot\nabla\widetilde{u}_2\,\mathrm{d}\mathbf{x}}_{\equiv K_1} + \underbrace{\int_\Omega \big[\beta(t)\widetilde{u}_2 + \partial_t\widehat{\theta}\big]\widetilde{u}_2\,\mathrm{d}\mathbf{x}}_{\equiv K_2} \underbrace{-\int_\Omega \partial_t\widehat{\mathbf{u}}\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}}_{\equiv K_3}\\
&\quad \underbrace{-\int_\Omega (\alpha(t)\widetilde{u}_2, 0)^{\mathrm{T}}\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}}_{\equiv K_4} \underbrace{-\int_\Omega [(\widehat{\mathbf{u}}\cdot\nabla)\widetilde{\mathbf{u}}]\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}}_{\equiv K_5} \underbrace{-\int_\Omega [(\widetilde{\mathbf{u}}\cdot\nabla)\widetilde{\mathbf{u}}]\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}}_{\equiv K_6}.
\end{aligned}
\tag{3.13}
$$

By a direct calculation, we can show that

$$
\begin{aligned}
K_1 &= \int_\Omega \big[-\widetilde{\theta}(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}}) + \kappa\nabla\widetilde{\theta}\big]\cdot\nabla\widetilde{u}_2\,\mathrm{d}\mathbf{x}\\
&= \int_\Omega \big(\widehat{\mathbf{u}}\cdot\nabla\widetilde{\theta}\big)\,\widetilde{u}_2\,\mathrm{d}\mathbf{x} + \int_\Omega \big(-\widetilde{\theta}\,\widetilde{\mathbf{u}} + \kappa\nabla\widetilde{\theta}\big)\cdot\nabla\widetilde{u}_2\,\mathrm{d}\mathbf{x},
\end{aligned}
\tag{3.14}
$$

from which we can show that

$$
\begin{aligned}
|K_1| &\le \|\widehat{\mathbf{u}}\|_{L^\infty}\|\nabla\widetilde{\theta}\|_{L^2}\|\widetilde{u}_2\|_{L^2} + \|\widetilde{\theta}\|_{L^4}\|\widetilde{\mathbf{u}}\|_{L^4}\|\nabla\widetilde{u}_2\|_{L^2} + \kappa\|\nabla\widetilde{\theta}\|_{L^2}\|\nabla\widetilde{u}_2\|_{L^2}\\
&\le C|\alpha(t)|\,\|\nabla\widetilde{\theta}\|_{L^2}\|\nabla\widetilde{u}_2\|_{L^2} + C\big(\|\widetilde{\theta}\|_{L^2}^{\frac12}\|\nabla\widetilde{\theta}\|_{L^2}^{\frac12} + \|\widetilde{\theta}\|_{L^2}\big)\|\widetilde{\mathbf{u}}\|_{L^2}^{\frac12}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^{\frac12}\|\nabla\widetilde{u}_2\|_{L^2}\\
&\quad + \kappa\|\nabla\widetilde{\theta}\|_{L^2}\|\nabla\widetilde{u}_2\|_{L^2}\\
&\le C\|\nabla\widetilde{\theta}\|_{L^2}\|\nabla\widetilde{\mathbf{u}}\|_{L^2} + C\|\widetilde{\theta}\|_{L^2}^{\frac12}\|\nabla\widetilde{\theta}\|_{L^2}^{\frac12}\|\widetilde{\mathbf{u}}\|_{L^2}^{\frac12}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^{\frac32} + C\|\widetilde{\theta}\|_{L^2}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2\\
&\quad + \kappa\|\nabla\widetilde{\theta}\|_{L^2}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}\\
&\le C\big(\|\nabla\widetilde{\theta}\|_{L^2}^2 + \|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2\big),
\end{aligned}
\tag{3.15}
$$

where the assumptions for $\alpha(t)$, Poincaré inequality, Ladyzhenskaya inequality, (3.10), and Young's inequality are applied. For K_2, we can show that

$$
\begin{aligned}
|K_2| &\le |\beta(t)|\,\|\widetilde{\mathbf{u}}\|_{L^2}^2 + \|\partial_t\widehat{\theta}\|_{L^2}\|\widetilde{\mathbf{u}}\|_{L^2}\\
&\le C\big(\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2 + |\beta'(t)|\,\|\nabla\widetilde{\mathbf{u}}\|_{L^2}\big)\\
&\le C\big(\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2 + |\beta'(t)|^2\big).
\end{aligned}
\tag{3.16}
$$

For K_3, K_4 and K_5, we can show that

$$\begin{aligned}|K_3|+|K_4|+|K_5| &\le \big(\|\partial_t\widehat{\mathbf{u}}\|_{L^2}+|\alpha(t)|\,\|\widetilde{\mathbf{u}}\|_{L^2}+\|\widehat{\mathbf{u}}\|_{L^\infty}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}\big)\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}\\ &\le C\big(|\alpha'(t)|+\|\nabla\widetilde{\mathbf{u}}\|_{L^2}+\|\nabla\widetilde{\mathbf{u}}\|_{L^2}\big)\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}\\ &\le \frac{1}{4}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2+C\big(|\alpha'(t)|^2+\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2\big).\end{aligned}\tag{3.17}$$

Similar to (2.16), and using (3.10) and the Cauchy inequality, we can show that

$$\begin{aligned}|K_6| &\le \|\widetilde{\mathbf{u}}\|_{L^4}\|\nabla\widetilde{\mathbf{u}}\|_{L^4}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}\\ &\le C\|\nabla\widetilde{\mathbf{u}}\|_{L^2}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^{\frac{3}{2}}+C\|\nabla\widetilde{\mathbf{u}}\|_{L^2}\big(\|\nabla\widetilde{\mathbf{u}}\|_{L^2}+|\alpha'(t)|^{\frac{1}{2}}+1\big)\|\partial_t\widetilde{\mathbf{u}}\|_{L^2},\end{aligned}\tag{3.18}$$

where the terms on the right-hand side can be estimated as

$$C\|\nabla\widetilde{\mathbf{u}}\|_{L^2}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^{\frac{3}{2}}\le\frac{1}{8}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2+C\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^4,$$

and

$$\begin{aligned}&C\|\nabla\widetilde{\mathbf{u}}\|_{L^2}\big(\|\nabla\widetilde{\mathbf{u}}\|_{L^2}+|\alpha'(t)|^{\frac{1}{2}}+1\big)\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}\\ &\le\frac{1}{8}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2+C\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2\big(\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+|\alpha'(t)|+1\big)\\ &\le\frac{1}{8}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2+C\big(\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^4+\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+|\alpha'(t)|^2\big).\end{aligned}$$

Substituting the above estimates into (3.18), we have

$$|K_6|\le\frac{1}{4}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2+C\big(\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^4+\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+|\alpha'(t)|^2\big).\tag{3.19}$$

Substituting (3.15), (3.16), (3.17) and (3.19) into (3.13), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\nu}{2}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+\int_\Omega\widetilde{\theta}\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}\Big)+\frac{1}{2}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2\\ &\le C\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^4+C\big(\|\nabla\widetilde{\mathbf{u}}\|^2+\|\nabla\widetilde{\theta}\|^2+|\alpha'(t)|^2+|\beta'(t)|^2\big).\end{aligned}\tag{3.20}$$

Note that the quantity inside the temporal derivative in (3.20) is not necessarily positive, which prevents us from applying Grönwall's inequality. To regain the accessibility of Grönwall's inequality, we resort to (3.9), which, after applying (3.10), reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\beta(t)\|\widetilde{\mathbf{u}}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2\big]+2\nu\,\beta(t)\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+2\kappa\|\nabla\widetilde{\theta}\|_{L^2}^2\le C\big(|\alpha(t)|+|\alpha'(t)|+|\beta'(t)|\big).\tag{3.21}$$

Note that

$$\beta(t)\|\widetilde{\mathbf{u}}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2\ge\min\{\beta_1,1\}\big(\|\widetilde{\mathbf{u}}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2\big)\equiv\widetilde{C}\big(\|\widetilde{\mathbf{u}}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2\big).\tag{3.22}$$

Multiplying (3.20) by $\widetilde{C}$, then adding the result to (3.21), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big[\beta(t)\|\widetilde{\mathbf{u}}\|_{L^2}^2+\|\widetilde{\theta}\|^2+\widetilde{C}\int_\Omega\widetilde{\theta}\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}+\frac{\widetilde{C}\nu}{2}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2\Big]\\ &\quad+2\nu\,\beta(t)\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+2\kappa\|\nabla\widetilde{\theta}\|_{L^2}^2+\frac{\widetilde{C}}{2}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2\\ &\le C\big(\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^4+\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+\|\nabla\widetilde{\theta}\|_{L^2}^2+|\alpha(t)|+|\alpha'(t)|+|\beta'(t)|\big).\end{aligned}\tag{3.23}$$

Note that according to (3.22), it holds that

$$\beta(t)\|\widetilde{\mathbf{u}}\|_{L^2}^2+\|\widetilde{\theta}\|^2+\widetilde{C}\int_\Omega \widetilde{\theta}\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}\geq \widetilde{C}\big(\|\widetilde{\mathbf{u}}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2+\int_\Omega \widetilde{\theta}\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}\big)\geq \frac{\widetilde{C}}{2}\big(\|\widetilde{\mathbf{u}}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2\big).$$

Hence, (3.23) becomes

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big[\beta(t)\|\widetilde{\mathbf{u}}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2+\widetilde{C}\int_\Omega \widetilde{\theta}\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}+\frac{\widetilde{C}\nu}{2}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2\Big]\\&\qquad+2\nu\,\beta(t)\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+2\kappa\|\nabla\widetilde{\theta}\|_{L^2}^2+\frac{\widetilde{C}}{2}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2\\&\leq C\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2\Big[\beta(t)\|\widetilde{\mathbf{u}}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2+\widetilde{C}\int_\Omega \widetilde{\theta}\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}+\frac{\widetilde{C}\nu}{2}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2\Big]\\&\qquad+C\big(\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+\|\nabla\widetilde{\theta}\|_{L^2}^2+|\alpha(t)|+|\alpha'(t)|+|\beta'(t)|\big).\end{aligned}\tag{3.24}$$

Applying Grönwall's inequality to (3.23) and using (3.10) and the assumptions for $\alpha(t)$ and $\beta(t)$, we can show that

$$\begin{aligned}&\Big[\beta\|\widetilde{\mathbf{u}}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2+\widetilde{C}\int_\Omega \widetilde{\theta}\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}+\frac{\widetilde{C}\nu}{2}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2\Big](t)\\&\quad+\int_0^t\Big(2\nu\,\beta\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+2\kappa\|\nabla\widetilde{\theta}\|_{L^2}^2+\frac{\widetilde{C}}{2}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2\Big)(\tau)\mathrm{d}\tau\leq C,\quad\forall\,t>0.\end{aligned}\tag{3.25}$$

In particular, we obtain from (3.25) that

$$\|\nabla\widetilde{\mathbf{u}}(t)\|_{L^2}^2+\int_0^t\|\partial_t\widetilde{\mathbf{u}}(\tau)\|_{L^2}^2\mathrm{d}\tau\leq C,\quad\forall\,t>0.\tag{3.26}$$

Step 3. Similar to (2.26), we can show that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\widetilde{\theta}\|_{L^2}^2+\kappa\|\Delta\widetilde{\theta}\|_{L^2}^2\\&=\int_\Omega(\partial_t\widehat{\theta})(\Delta\widetilde{\theta})\mathrm{d}\mathbf{x}+\int_\Omega\beta(t)\widetilde{u}_2\,\Delta\widetilde{\theta}\,\mathrm{d}\mathbf{x}+\int_\Omega[(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla\widetilde{\theta}]\Delta\widetilde{\theta}\,\mathrm{d}\mathbf{x}\\&\leq\frac{\kappa}{4}\|\Delta\widetilde{\theta}\|_{L^2}^2+C\big(|\beta'(t)|^2+\|\widetilde{\mathbf{u}}\|_{L^2}^2\big)+\|(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla\widetilde{\theta}\|_{L^2}\|\Delta\widetilde{\theta}\|_{L^2},\end{aligned}\tag{3.27}$$

where the last term on the right-hand side can be estimated as

$$\begin{aligned}&\|(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla\widetilde{\theta}\|_{L^2}\|\Delta\widetilde{\theta}\|_{L^2}\\&\leq\|\widehat{\mathbf{u}}\|_{L^\infty}\|\nabla\widetilde{\theta}\|_{L^2}\|\Delta\widetilde{\theta}\|_{L^2}+\|\widetilde{\mathbf{u}}\|_{L^4}\|\nabla\widetilde{\theta}\|_{L^4}\|\Delta\widetilde{\theta}\|_{L^2}\\&\leq C\|\nabla\widetilde{\theta}\|_{L^2}\|\Delta\widetilde{\theta}\|_{L^2}+C\|\nabla\widetilde{\mathbf{u}}\|_{L^2}\Big(\|\nabla\widetilde{\theta}\|_{L^2}^{\frac12}\|D^2\widetilde{\theta}\|_{L^2}^{\frac12}+\|\nabla\widetilde{\theta}\|_{L^2}\Big)\|\Delta\widetilde{\theta}\|_{L^2}.\end{aligned}\tag{3.28}$$

Note that since $\nabla\widetilde{\theta}\cdot\mathbf{n}|_{\partial\Omega}=0$ and $\nabla\times(\nabla\widetilde{\theta})=\mathbf{0}$, then $\nabla\widetilde{\theta}$ satisfies the Caldéron-Zygmund inequality:

$$\|\nabla\widetilde{\theta}\|_{H^s}\leq C\|\Delta\widetilde{\theta}\|_{H^{s-1}},\quad\forall\,s\geq1.\tag{3.29}$$

Hence, (3.28) becomes

$$\|(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla\widetilde{\theta}\|_{L^2}\|\Delta\widetilde{\theta}\|_{L^2}\leq\frac{\kappa}{4}\|\Delta\widetilde{\theta}\|_{L^2}^2+C\big(\|\nabla\widetilde{\theta}\|_{L^2}^2+\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^4\|\nabla\widetilde{\theta}\|_{L^2}^2\big).\tag{3.30}$$

Substituting (3.30) into (3.27), we can show that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\widetilde{\theta}\|_{L^2}^2+\frac{\kappa}{2}\|\Delta\widetilde{\theta}\|_{L^2}^2 &\le C\big(|\beta'(t)|^2+\|\widetilde{\mathbf{u}}\|_{L^2}^2+\|\nabla\widetilde{\theta}\|_{L^2}^2+\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^4\|\nabla\widetilde{\theta}\|_{L^2}^2\big)\\ &\le C\big(|\beta'(t)|^2+\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+\|\nabla\widetilde{\theta}\|_{L^2}^2\big),\end{aligned}\tag{3.31}$$

where we used the Poincaré inequality and (3.26). Similar to (2.29), (2.30) and (2.31), we can show that

$$\|\nabla\widetilde{\theta}(t)\|_{L^2}^2+\int_0^t\big(\|\nabla\widetilde{\theta}(\tau)\|_{H^1}^2+\|\partial_t\widetilde{\theta}(\tau)\|_{L^2}^2\big)\mathrm{d}\tau\le C,\quad\forall\, t>0.\tag{3.32}$$

Step 4. Repeating the arguments in **Step 4** of Section 2, we can show that

$$\|\partial_t\widetilde{\mathbf{u}}(t)\|_{L^2}^2+\|\widetilde{\mathbf{u}}(t)\|_{H^2}+\int_0^t\|\nabla\partial_t\widetilde{\mathbf{u}}(\tau)\|_{L^2}^2\mathrm{d}\tau\le C,\quad\forall\, t>0.\tag{3.33}$$

Note that due to the lack of the Poincaré inequality for $\widetilde{\theta}$, the uniform temporal integrability of $\|D^2\widetilde{\mathbf{u}}\|_{L^2}^2$ can not be accessed as in Section 2, which is the major difference between the problems (1.1) and (1.7), and affects the subsequent asymptotic analysis in this section. As a result, we have

$$\int_0^t\|\widetilde{\mathbf{u}}(\tau)\|_{H^3}^2\mathrm{d}\tau\le C(1+t),\quad\forall\, t>0.\tag{3.34}$$

In a similar fashion, by repeating the arguments in **Step 5** of Section 2, we can show that

$$\|\nabla\partial_t\widetilde{\mathbf{u}}(t)\|_{L^2}^2+\|\widetilde{\mathbf{u}}(t)\|_{H^3}^2+\int_0^t\|\partial_{tt}\widetilde{\mathbf{u}}(\tau)\|_{L^2}^2\mathrm{d}\tau\le C,\quad\forall\, t>0,\tag{3.35}$$

and

$$\int_0^t\|\widetilde{\mathbf{u}}(\tau)\|_{H^4}^2\mathrm{d}\tau\le C(1+t),\quad\forall\, t>0.\tag{3.36}$$

Step 5. For $\widetilde{\theta}$, by repeating the arguments in **Step 6** of Section 2, we can show that

$$\|\partial_t\widetilde{\theta}(t)\|_{L^2}^2+\int_0^t\|\nabla\partial_t\widetilde{\theta}(\tau)\|_{L^2}^2\mathrm{d}\tau\le C,\quad\forall\, t>0.\tag{3.37}$$

Similar to (2.58), by using (3.29), we can show that

$$\|\widetilde{\theta}\|_{H^2}\lesssim\|\widetilde{\theta}\|_{L^2}+\|\nabla\widetilde{\theta}\|_{H^1}\lesssim\|\widetilde{\theta}\|_{L^2}+\|\partial_t\widetilde{\theta}\|_{L^2}+\|\nabla\widetilde{\theta}\|_{L^2}+\|\widetilde{\mathbf{u}}\|_{L^2}+|\beta'(t)|,\tag{3.38}$$

$$\|\nabla\widetilde{\theta}\|_{H^2}\lesssim\|\Delta\widetilde{\theta}\|_{H^1}\lesssim\|\partial_t\widetilde{\theta}\|_{L^2}+\|\nabla\partial_t\widetilde{\theta}\|_{L^2}+\|\nabla\widetilde{\theta}\|_{L^2}+\|\nabla\widetilde{\mathbf{u}}\|_{L^2}+|\beta'(t)|,\tag{3.39}$$

and

$$\|\widetilde{\theta}\|_{H^3}\lesssim\|\widetilde{\theta}\|_{L^2}+\|\Delta\widetilde{\theta}\|_{H^1}\lesssim\|\widetilde{\theta}\|_{L^2}+\|\partial_t\widetilde{\theta}\|_{L^2}+\|\nabla\partial_t\widetilde{\theta}\|_{L^2}+\|\nabla\widetilde{\theta}\|_{L^2}+\|\nabla\widetilde{\mathbf{u}}\|_{L^2}+|\beta'(t)|.\tag{3.40}$$

As a result of the previous estimates, we have

$$\|\widetilde{\theta}(t)\|_{H^2}^2+\int_0^t\|\nabla\widetilde{\theta}(\tau)\|_{H^2}^2\,\mathrm{d}\tau\le C,\quad\forall\, t>0,\tag{3.41}$$

and

$$\int_0^t\|\widetilde{\theta}(\tau)\|_{H^3}^2\,\mathrm{d}\tau\le C(t+1),\quad\forall\, t>0,\tag{3.42}$$

where the second estimate is due to the lack of the Poincaré inequality.

Moreover, by repeating the arguments in **Step 7** of Section 2, we can show that

$$\|\nabla\partial_t\widetilde{\theta}(t)\|_{L^2}^2+\int_0^t\|\Delta\partial_t\widetilde{\theta}(\tau)\|_{L^2}^2\mathrm{d}\tau\le C,\quad\forall\, t>0,\tag{3.43}$$

which implies

$$\|\widetilde{\theta}(t)\|_{H^3}^2 + \int_0^t \|\nabla\widetilde{\theta}(\tau)\|_{H^3}^2 \,\mathrm{d}\tau \le C, \quad \forall\, t > 0, \tag{3.44}$$

and

$$\int_0^t \|\widetilde{\theta}(\tau)\|_{H^4}^2 \,\mathrm{d}\tau \le C(t+1), \quad \forall\, t > 0. \tag{3.45}$$

Step 6. To derive the temporal decaying of the perturbation, we note that since

$$\begin{aligned}\int_0^t \Big|\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\widetilde{\mathbf{u}}(\tau)\|^2\Big|\mathrm{d}\tau &\lesssim \int_0^t \|\nabla\widetilde{\mathbf{u}}(\tau)\|_{L^2}\|\nabla\partial_t\widetilde{\mathbf{u}}(\tau)\|_{L^2}\mathrm{d}\tau\\ &\lesssim \int_0^t \|\nabla\widetilde{\mathbf{u}}(\tau)\|_{L^2}^2\mathrm{d}\tau + \int_0^t \|\nabla\partial_t\widetilde{\mathbf{u}}(\tau)\|_{L^2}^2\mathrm{d}\tau,\end{aligned}$$

as a consequence of (3.10) and (3.33), it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\widetilde{\mathbf{u}}(t)\|_{L^2}^2 \in L^1(0,\infty),$$

which, together with (3.10), implies

$$\|\nabla\widetilde{\mathbf{u}}(t)\|_{L^2}^2 \in W^{1,1}(0,\infty).$$

Hence, $\|\nabla\widetilde{\mathbf{u}}(t)\|_{L^2}^2 \to 0$, as $t\to\infty$, which, together with Poincaré inequality, implies that $\|\widetilde{\mathbf{u}}(t)\|_{H^1}^2 \to 0$, as $t\to\infty$. Since $\|\widetilde{\mathbf{u}}(t)\|_{H^3}$ is uniformly bounded with respect to time (cf. (3.33)), according to the Gagliardo-Nirenberg interpolation inequality:

$$\|D^2\widetilde{\mathbf{u}}\|_{L^2} \le C\big(\|D^3\widetilde{\mathbf{u}}\|_{L^2}^{\frac{2}{3}}\|\widetilde{\mathbf{u}}\|_{L^2}^{\frac{1}{3}} + \|\widetilde{\mathbf{u}}\|_{L^2}\big),$$

and the decaying of $\|\widetilde{\mathbf{u}}(t)\|_{H^1}$, we infer that $\|\widetilde{\mathbf{u}}(t)\|_{H^2}\to 0$, as $t\to\infty$. In terms of the original variables, we see that

$$\|\mathbf{u}(t) - (\alpha(t)y,0)^{\mathrm{T}}\|_{H^2} \to 0, \quad \text{as} \quad t\to\infty.$$

Moreover, by using similar arguments as in deriving (2.50), we can show that

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2\Big| \le C\big(\|\partial_{tt}\widetilde{\mathbf{u}}\|_{L^2}^2 + \|\partial_t\widetilde{\theta}\|_{L^2}^2 + |\alpha''(t)|^2 + \|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2 + \|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2 + \|\nabla\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2\big). \tag{3.46}$$

It then follows from (3.35), (3.32), (3.10), (3.26) and (3.33) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2 \in L^1(0,\infty),$$

which, together with (3.33), implies

$$\|\nabla\partial_t\widetilde{\mathbf{u}}(t)\|_{L^2}^2 \in W^{1,1}(0,\infty).$$

Hence, $\|\nabla\partial_t\widetilde{\mathbf{u}}(t)\|_{L^2}^2 \to 0$, as $t\to\infty$, which, together with Poincaré inequality, implies that $\|\partial_t\widetilde{\mathbf{u}}(t)\|_{H^1}^2 \to 0$, as $t\to\infty$. In terms of the original variables, we have

$$\|\partial_t\mathbf{u}(t) - (\alpha'(t)y,0)^{\mathrm{T}}\|_{H^1} \to 0, \quad \text{as} \quad t\to\infty.$$

On the other hand, according to (3.29) and the third equation in (3.2), we have

$$\|\nabla\widetilde{\theta}\|_{H^2} \lesssim \|\Delta\widetilde{\theta}\|_{H^1} \lesssim \|\partial_t\widetilde{\theta}\|_{H^1} + \|\widehat{\mathbf{u}}\cdot\nabla\widetilde{\theta}\|_{H^1} + \|\widetilde{\mathbf{u}}\cdot\nabla\widetilde{\theta}\|_{H^1} + \|\beta(t)\widetilde{\mathbf{u}}\|_{H^1} + \|\partial_t\widehat{\theta}\|_{H^1}.$$

Note that, because of the uniform boundedness of $\|\widetilde{\mathbf{u}}(t)\|_{H^3}$ and Sobolev embedding,

$$\begin{aligned}
\|\widehat{\mathbf{u}}\cdot\nabla\widetilde{\theta}\|_{H^1} &\lesssim \|\widehat{\mathbf{u}}\|_{L^\infty}\|\nabla\widetilde{\theta}\|_{H^1} + \|\nabla\widehat{\mathbf{u}}\|_{L^\infty}\|\nabla\widetilde{\theta}\|_{L^2} \\
&\lesssim \|\Delta\widetilde{\theta}\|_{L^2} \\
&\lesssim \|\partial_t\widetilde{\theta}\|_{L^2} + \|\widehat{\mathbf{u}}\cdot\nabla\widetilde{\theta}\|_{L^2} + \|\widetilde{\mathbf{u}}\cdot\nabla\widetilde{\theta}\|_{L^2} + \|\beta(t)\widetilde{\mathbf{u}}\|_{L^2} + \|\partial_t\widehat{\theta}\|_{L^2} \\
&\lesssim \|\partial_t\widetilde{\theta}\|_{L^2} + \|\widehat{\mathbf{u}}\|_{L^\infty}\|\nabla\widetilde{\theta}\|_{L^2} + \|\widetilde{\mathbf{u}}\|_{L^\infty}\|\nabla\widetilde{\theta}\|_{L^2} + \|\widetilde{\mathbf{u}}\|_{L^2} + |\beta'(t)| \\
&\lesssim \|\partial_t\widetilde{\theta}\|_{L^2} + \|\nabla\widetilde{\theta}\|_{L^2} + \|\widetilde{\mathbf{u}}\|_{L^2} + |\beta'(t)|,
\end{aligned}$$

and similarly,

$$\begin{aligned}
\|\widetilde{\mathbf{u}}\cdot\nabla\widetilde{\theta}\|_{H^1} &\lesssim \|\widetilde{\mathbf{u}}\|_{L^\infty}\|\nabla\widetilde{\theta}\|_{H^1} + \|\nabla\widetilde{\mathbf{u}}\|_{L^\infty}\|\nabla\widetilde{\theta}\|_{L^2} \\
&\lesssim \|\partial_t\widetilde{\theta}\|_{L^2} + \|\nabla\widetilde{\theta}\|_{L^2} + \|\widetilde{\mathbf{u}}\|_{L^2} + |\beta'(t)|.
\end{aligned}$$

Hence,

$$\|\nabla\widetilde{\theta}\|_{H^2} \lesssim \|\partial_t\widetilde{\theta}\|_{H^1} + \|\nabla\widetilde{\theta}\|_{L^2} + \|\widetilde{\mathbf{u}}\|_{H^1} + |\beta'(t)|.$$

Using (3.31) and (3.10), we can show that $\|\nabla\widetilde{\theta}(t)\|_{L^2}^2 \in W^{1,1}(0,\infty)$, which implies $\|\nabla\widetilde{\theta}(t)\|_{L^2} \to 0$, as $t\to\infty$. In addition, the decaying of $\|\partial_t\widetilde{\theta}(t)\|_{H^1}$ follows from (3.32), (3.37), (3.43), and similar arguments as in deriving (2.56) and (2.61). Since $\|\widetilde{\mathbf{u}}(t)\|_{H^2}$ and $|\beta'(t)|$ converge to zero as $t\to\infty$, we conclude that $\|\nabla\widetilde{\theta}(t)\|_{H^2}\to\infty$, as $t\to\infty$. In terms of the original variables, we see that

$$\|\nabla\theta(t) - \beta(t)\mathbf{e}_2\|_{H^2} \to 0, \quad \text{as} \quad t\to\infty.$$

Furthermore, as a result of the first equation in (3.2) and the decaying of the velocity field, we have

$$\begin{aligned}
\|\nabla\widetilde{P} - \widetilde{\theta}\mathbf{e}_2\|_{L^2} &\lesssim \|\partial_t\widetilde{\mathbf{u}}\|_{L^2} + \|\widehat{\mathbf{u}}\|_{L^\infty}\|\nabla\widetilde{\mathbf{u}}\|_{L^2} + \|\widetilde{\mathbf{u}}\|_{L^\infty}\|\nabla\widetilde{\mathbf{u}}\|_{L^2} + \|\alpha(t)\widetilde{\mathbf{u}}\|_{L^2} + \|\Delta\widetilde{\mathbf{u}}\|_{L^2} + \|\beta'(t)y\|_{L^2} \\
&\lesssim \|\partial_t\widetilde{\mathbf{u}}\|_{L^2} + \|\widetilde{\mathbf{u}}\|_{H^2} + |\beta'(t)| \to 0, \quad \text{as} \quad t\to\infty.
\end{aligned}$$

In terms of the original variables, we see that

$$\|(\nabla P - \theta\mathbf{e}_2)(t)\|_{L^2} \to 0, \quad \text{as} \quad t\to\infty.$$

This completes the proof of Theorem 1.2. □

4. SEMI-DISSIPATIVE SYSTEM UNDER DYNAMIC COUETTE FLOW AND PERIODIC B.C.

This section proves Theorem 1.3.

Proof of Theorem 1.3. The *a priori* estimates leading to the global well-posedness of classical solutions to (1.9) can be established by using similar arguments as those in [12], we focus on deriving the decay estimate of the solution. For this purpose, we let

$$\widehat{\mathbf{u}} \equiv (\alpha(t)y, 0)^{\mathrm{T}}, \qquad \widehat{\theta} \equiv y,$$

and

$$\mathbf{u} = \widehat{\mathbf{u}} + \widetilde{\mathbf{u}}, \qquad \theta = \widehat{\theta} + \widetilde{\theta}.$$

Then the perturbed system reads

$$\begin{cases}
\partial_t\widetilde{\mathbf{u}} + [(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla]\widetilde{\mathbf{u}} + (\alpha(t)\widetilde{u}_2, 0)^{\mathrm{T}} + \nabla\widetilde{P} = \nu\Delta\widetilde{\mathbf{u}} + \widetilde{\theta}\mathbf{e}_2 - \partial_t\widehat{\mathbf{u}}, \\
\nabla\cdot\widetilde{\mathbf{u}} = 0, \\
\partial_t\widetilde{\theta} + (\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla\widetilde{\theta} = -\widetilde{u}_2,
\end{cases} \tag{4.1}$$

where $\widetilde{P} = P - \frac{1}{2}y^2$. The perturbed velocity field $\widetilde{\mathbf{u}}$ satisfies the zero Dirichlet boundary condition, and $\widetilde{\theta}$ satisfies the periodic boundary conditions on the two sides of the 2D square.

Step 1. Taking L^2 inner products of the first equation in (4.1) with $\widetilde{\mathbf{u}}$, and the second one with $\widetilde{\theta}$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{\mathbf{u}}\|_{L^2}^2 + \nu\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2 = -\alpha(t)\int_\Omega \widetilde{u}_1\,\widetilde{u}_2\,\mathrm{d}\mathbf{x} + \int_\Omega \widetilde{\theta}\,\widetilde{u}_2\,\mathrm{d}\mathbf{x} - \alpha'(t)\int_\Omega \widetilde{u}_1\,y\,\mathrm{d}\mathbf{x}, \tag{4.2}$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{\theta}\|_{L^2}^2 = -\int_\Omega \widetilde{\theta}\,\widetilde{u}_2\,\mathrm{d}\mathbf{x} - \int_\Omega \big[(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla\widetilde{\theta}\big]\widetilde{\theta}\,\mathrm{d}\mathbf{x}. \tag{4.3}$$

For the second term on the right-hand side of (4.3), note that because of the incompressibility and boundary conditions,

$$\begin{aligned}\int_\Omega \big[(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla\widetilde{\theta}\big]\widetilde{\theta}\,\mathrm{d}\mathbf{x} &= \frac{1}{2}\int_{\partial\Omega} |\widetilde{\theta}|^2\,(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\mathbf{n}\,\mathrm{d}s\\ &= \frac{1}{2}\int_{\partial\Omega} |\widetilde{\theta}|^2\,\widehat{\mathbf{u}}\cdot\mathbf{n}\,\mathrm{d}s,\end{aligned}$$

where, according to the geometry of Ω and the periodic boundary conditions satisfied by $\widetilde{\theta}$,

$$\begin{aligned}\frac{1}{2}\int_{\partial\Omega} |\widetilde{\theta}|^2\,\widehat{\mathbf{u}}\cdot\mathbf{n}\,\mathrm{d}s &= \frac{\alpha(t)}{2}\int_0^1 \big[|\widetilde{\theta}(1,y,t)|^2 - |\widetilde{\theta}(0,y,t)|^2\big]y\,\mathrm{d}y\\ &= 0.\end{aligned}$$

Hence, (4.3) becomes

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{\theta}\|_{L^2}^2 = -\int_\Omega \widetilde{\theta}\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}. \tag{4.4}$$

Taking the sum of (4.2) and (4.4), we can show that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\widetilde{\mathbf{u}}\|_{L^2}^2 + \|\widetilde{\theta}\|_{L^2}^2\big) + \nu\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2\\ &= -\alpha(t)\int_\Omega \widetilde{u}_1\,\widetilde{u}_2\,\mathrm{d}\mathbf{x} - \alpha'(t)\int_\Omega \widetilde{u}_1\,y\,\mathrm{d}\mathbf{x}\\ &\le |\alpha(t)|\,\|\widetilde{\mathbf{u}}\|_{L^2}^2 + |\alpha'(t)|\,\|\widetilde{\mathbf{u}}\|_{L^2}^2 + \frac{|\alpha'(t)|}{4}\,\|y\|_{L^2}^2\\ &\le \big(|\alpha(t)| + |\alpha'(t)|\big)\big(\|\widetilde{\mathbf{u}}\|_{L^2}^2 + \|\widetilde{\theta}\|_{L^2}^2\big) + \frac{|\alpha'(t)|}{12}.\end{aligned} \tag{4.5}$$

Aapplying Grönwall's inequality and using the assumptions for $\alpha(t)$, we can show that

$$\|\widetilde{\mathbf{u}}(t)\|_{L^2}^2 + \|\widetilde{\theta}(t)\|_{L^2}^2 + \int_\Omega \|\widetilde{\mathbf{u}}(\tau)\|_{H^1}^2\,\mathrm{d}\tau \le C, \quad \forall\, t>0, \tag{4.6}$$

where Poincaré inequality is applied for $\widetilde{\mathbf{u}}$.

Step 2. Taking L^2 inner product of the first equation in (4.1) with $\partial_t\widetilde{\mathbf{u}}$, we have

$$\begin{aligned}&\frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2 + \|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2\\ &= \int_\Omega \widetilde{\theta}\mathbf{e}_2\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x} - \int_\Omega \partial_t\widehat{\mathbf{u}}\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x} - \int_\Omega (\alpha(t)\widetilde{u}_2,0)^{\mathrm{T}}\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}\\ &\quad - \int_\Omega [(\widehat{\mathbf{u}}\cdot\nabla)\widetilde{\mathbf{u}}]\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x} - \int_\Omega [(\widetilde{\mathbf{u}}\cdot\nabla)\widetilde{\mathbf{u}}]\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}.\end{aligned} \tag{4.7}$$

For the first term on the right-hand side of (4.7), we can show that

$$\begin{aligned}\int_\Omega \widetilde{\theta}\mathbf{e}_2 \cdot \partial_t \widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega \widetilde{\theta}\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}\Big) - \int_\Omega (\partial_t\widetilde{\theta})\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega \widetilde{\theta}\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}\Big) + \int_\Omega \big[(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla\widetilde{\theta} + \widetilde{u}_2\big]\widetilde{u}_2\,\mathrm{d}\mathbf{x}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega \widetilde{\theta}\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}\Big) - \int_\Omega \widetilde{\theta}(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla\widetilde{u}_2\,\mathrm{d}\mathbf{x} + \|\widetilde{u}_2\|_{L^2}^2,\end{aligned} \tag{4.8}$$

by using which (4.7) implies

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\nu}{2}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2 - \int_\Omega \widetilde{\theta}\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}\Big) + \|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2\\ =& -\int_\Omega \widetilde{\theta}(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla\widetilde{u}_2\,\mathrm{d}\mathbf{x} + \|\widetilde{u}_2\|_{L^2}^2 - \int_\Omega \partial_t\widehat{\mathbf{u}}\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x} - \int_\Omega (\alpha(t)\widetilde{u}_2, 0)^{\mathrm{T}}\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}\\ &-\int_\Omega [(\widehat{\mathbf{u}}\cdot\nabla)\widetilde{\mathbf{u}}]\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x} - \int_\Omega [(\widetilde{\mathbf{u}}\cdot\nabla)\widetilde{\mathbf{u}}]\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}.\end{aligned} \tag{4.9}$$

To estimate the first term on the right-hand side of (4.9), we first note that by taking L^2 inner product of the third equation in (1.9) with $p\,|\theta|^{p-2}\,\theta$ for any $p \geq 2$, one can show that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_{L^p}^p &= -\int_\Omega \mathbf{u}\cdot\nabla\left(|\theta|^p\right)\mathrm{d}\mathbf{x}\\ &= -\int_{\partial\Omega} |\theta|^p\,\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}s\\ &= -\alpha(t)\int_0^1 \left(|\theta(1,y,t)|^p - |\theta(0,y,t)|^p\right)y\,\mathrm{d}y\\ &= 0,\end{aligned}$$

where the boundary conditions for $\mathbf{u}$ and θ are applied. This show that

$$\|\theta(t)\|_{L^p} = \|\theta_0\|_{L^p}, \quad \forall\, p \geq 2, \quad \forall\, t > 0, \tag{4.10}$$

and hence

$$\|\theta(t)\|_{L^\infty} \leq \|\theta_0\|_{L^\infty}, \quad \forall\, t > 0. \tag{4.11}$$

Since $\widetilde{\theta} = \theta - \widehat{\theta} = \theta - y$, we infer from (4.11) that

$$\|\widetilde{\theta}(t)\|_{L^\infty} \leq \|\theta_0\|_{L^\infty} + 1, \quad \forall\, t > 0. \tag{4.12}$$

Using (4.12), we estimate the first term on the right-hand side of (4.9) as

$$\begin{aligned}\Big| -\int_\Omega \widetilde{\theta}(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla\widetilde{u}_2\,\mathrm{d}\mathbf{x}\Big| &\leq \|\widetilde{\theta}\|_{L^\infty}\big(\|\widehat{\mathbf{u}}\|_{L^2} + \|\widetilde{\mathbf{u}}\|_{L^2}\big)\|\nabla\widetilde{\mathbf{u}}\|_{L^2}\\ &\leq C\big(|\alpha(t)| + \|\nabla\widetilde{\mathbf{u}}\|_{L^2}\big)\|\nabla\widetilde{\mathbf{u}}\|_{L^2}\\ &\leq C\big(|\alpha(t)|^2 + \|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2\big),\end{aligned} \tag{4.13}$$

where Poincaré and Cauchy inequalities are applied. Note that the 3rd–6th terms on the right-hand side of (4.9) are identical to K_3–K_6 on the right-hand side of (3.13), respectively. Hence, substituting (4.13), (3.17), and (3.19) into (4.9), we can show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\nu}{2}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2 - \int_\Omega \widetilde{\theta}\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}\Big) + \frac{1}{2}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2 \leq C\big(|\alpha(t)| + |\alpha'(t)| + \|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2 + \|\nabla\widetilde{\mathbf{u}}\|_{L^2}^4\big), \tag{4.14}$$

where we used the boundedness of $|\alpha(t)|$ and $|\alpha'(t)|$. To implement Grönwall's inequality, we observe that by applying (4.6) to (4.5), it holds that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\widetilde{\mathbf{u}}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2\big)+\nu\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2\le C\big(|\alpha(t)|+|\alpha'(t)|\big). \tag{4.15}$$

Taking the sum of (4.14) and (4.15), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\nu}{2}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+\frac{1}{2}\|\widetilde{\theta}\|_{L^2}^2-\int_\Omega\widetilde{\theta}\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}+\frac{1}{2}\|\widetilde{\mathbf{u}}\|_{L^2}^2\Big)+\frac{1}{2}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2+\nu\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2\\&\le C\big(|\alpha(t)|+|\alpha'(t)|+\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^4\big).\end{aligned} \tag{4.16}$$

Note that

$$G(t)\equiv\frac{\nu}{2}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+\frac{1}{2}\|\widetilde{\theta}\|_{L^2}^2-\int_\Omega\widetilde{\theta}\,\widetilde{u}_2\,\mathrm{d}\mathbf{x}+\frac{1}{2}\|\widetilde{\mathbf{u}}\|_{L^2}^2\ge\frac{\nu}{2}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2.$$

Hence, (4.16) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}G(t)+\frac{1}{2}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2+\nu\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2\le C\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2\,G(t)+C\big(|\alpha(t)|+|\alpha'(t)|+\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2\big). \tag{4.17}$$

Applying Grönwall's inequality and using (4.6), we can show that

$$\|\nabla\widetilde{\mathbf{u}}(t)\|_{L^2}^2+\int_0^t\|\partial_t\widetilde{\mathbf{u}}(\tau)\|_{L^2}^2\mathrm{d}\tau\le C,\quad\forall\,t>0. \tag{4.18}$$

Step 3. Taking ∂_t of the first equation in (4.1), then taking L^2 inner product of the resulting equation with $\partial_t\widetilde{\mathbf{u}}$, we have can show that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2+\nu\|\nabla\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2\\&=\int_\Omega\partial_t\widetilde{\theta}(\mathbf{e}_2\cdot\partial_t\widetilde{\mathbf{u}})\mathrm{d}\mathbf{x}-\int_\Omega\partial_{tt}\widehat{\mathbf{u}}\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}+\int_\Omega\widetilde{\mathbf{u}}\cdot\big([\partial_t(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla]\partial_t\widetilde{\mathbf{u}}\big)\mathrm{d}\mathbf{x}\\&\quad-\int_\Omega(\alpha'(t)\widetilde{u}_2,0)^{\mathrm{T}}\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}-\int_\Omega(\alpha(t)\partial_t\widetilde{u}_2,0)^{\mathrm{T}}\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}.\end{aligned} \tag{4.19}$$

For the first term on the right-hand side of (4.19), using the third equation in (1.9), we have

$$\int_\Omega\partial_t\widetilde{\theta}(\mathbf{e}_2\cdot\partial_t\widetilde{\mathbf{u}})\mathrm{d}\mathbf{x}=\int_\Omega\partial_t\theta(\mathbf{e}_2\cdot\partial_t\widetilde{\mathbf{u}})\mathrm{d}\mathbf{x}=-\int_\Omega\nabla\cdot(\mathbf{u}\,\theta)(\mathbf{e}_2\cdot\partial_t\widetilde{\mathbf{u}})\,\mathrm{d}\mathbf{x}.$$

Since $\partial_t\widetilde{\mathbf{u}}$ satisfies the zero boundary condition, it holds that

$$-\int_\Omega\nabla\cdot(\mathbf{u}\,\theta)(\mathbf{e}_2\cdot\partial_t\widetilde{\mathbf{u}})\,\mathrm{d}\mathbf{x}=\int_\Omega\theta\,\mathbf{u}\cdot\nabla(\mathbf{e}_2\cdot\partial_t\widetilde{\mathbf{u}})\,\mathrm{d}\mathbf{x}=\int_\Omega\theta\,\mathbf{u}\cdot\nabla\partial_t\widetilde{u}_2\,\mathrm{d}\mathbf{x}. \tag{4.20}$$

Substituting (4.20) into (4.19), we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2+\nu\|\nabla\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2\\&=\int_\Omega\theta\,\mathbf{u}\cdot\nabla\partial_t\widetilde{u}_2\,\mathrm{d}\mathbf{x}-\int_\Omega\partial_{tt}\widehat{\mathbf{u}}\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}+\int_\Omega\widetilde{\mathbf{u}}\cdot\big([\partial_t(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\cdot\nabla]\partial_t\widetilde{\mathbf{u}}\big)\mathrm{d}\mathbf{x}\\&\quad-\int_\Omega(\alpha'(t)\widetilde{u}_2,0)^{\mathrm{T}}\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}-\int_\Omega(\alpha(t)\partial_t\widetilde{u}_2,0)^{\mathrm{T}}\cdot\partial_t\widetilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}.\end{aligned} \tag{4.21}$$

The first term on the right-hand side of (4.21) can be estimated by using (4.11) as

$$\begin{aligned}\Big|\int_\Omega \theta\,\mathbf{u}\cdot\nabla\partial_t\widetilde{u}_2\,\mathrm{d}\mathbf{x}\Big| &\le \|\theta_0\|_{L^\infty}\big(\|\widehat{\mathbf{u}}\|_{L^2}+\|\widetilde{\mathbf{u}}\|_{L^2}\big)\|\nabla\partial_t\widetilde{\mathbf{u}}\|_{L^2}\\ &\le \frac{\nu}{4}\|\nabla\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2+C\big(|\alpha(t)|+\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2\big),\end{aligned}\tag{4.22}$$

where Poincaré inequality is applied. The remaining terms on the right-hand side of (4.21) can be estimated by using (4.18) and similar arguments as in Section 2. Hence, (4.21) implies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2+\frac{\nu}{2}\|\nabla\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2\le C\big(|\alpha(t)|+|\alpha'(t)|^2+|\alpha''(t)|^2+\|\partial_t\widetilde{\mathbf{u}}\|_{L^2}^2\big).\tag{4.23}$$

Integrating (4.23) with respect to t and using (4.18), we infer that

$$\|\partial_t\widetilde{\mathbf{u}}(t)\|_{L^2}^2+\int_0^t\|\nabla\partial_t\widetilde{\mathbf{u}}(\tau)\|_{L^2}^2\mathrm{d}\tau\le C,\quad \forall\, t>0.\tag{4.24}$$

As a consequence of (2.15) and the previous energy estimates, we see that $\|\Delta\widetilde{\mathbf{u}}(t)\|_{L^2}$, and hence $\|\widetilde{\mathbf{u}}(t)\|_{H^2}$ is uniformly bounded in time.

Step 4. Applying the arguments in **Step 6** of Section 3 and using (4.6) and (4.24), we can show that $\|\nabla\widetilde{\mathbf{u}}(t)\|_{L^2}^2\in W^{1,1}(0,\infty)$, which implies $\|\nabla\widetilde{\mathbf{u}}(t)\|_{L^2}^2\to 0$, as $t\to\infty$. As a consequence of Poincaré inequality, we have $\|\widetilde{\mathbf{u}}(t)\|_{H^1}^2\to 0$, as $t\to\infty$. Moreover, the decaying of $\|\partial_t\widetilde{\mathbf{u}}(t)\|_{L^2}^2$ follows from (4.18), (4.23) and (4.24). In terms of the original variables, we see that

$$\|\mathbf{u}(t)-(\alpha(t)y,0)^{\mathrm{T}}\|_{H^1}+\|\partial_t\mathbf{u}-(\alpha'(t)y,0)^{\mathrm{T}}\|_{L^2}\to 0,\quad \text{as}\quad t\to\infty.$$

Lastly, it follows from the first equation in (4.1) and decaying of the velocity field that

$$\begin{aligned}&\|(\nabla\widetilde{P}-\widetilde{\theta}\mathbf{e}_2)(t)\|_{H^{-1}}\\ \lesssim\ &\|\partial_t\widetilde{\mathbf{u}}(t)\|_{L^2}+\|(\widehat{\mathbf{u}}+\widetilde{\mathbf{u}})\otimes\widetilde{\mathbf{u}}(t)\|_{L^2}+\|\alpha(t)\widetilde{\mathbf{u}}(t)\|_{L^2}+\|\nabla\widetilde{\mathbf{u}}(t)\|_{L^2}+\|\partial_t\widehat{\mathbf{u}}(t)\|_{L^2}\\ \lesssim\ &\|\partial_t\widetilde{\mathbf{u}}(t)\|_{L^2}+\big(\|\widehat{\mathbf{u}}(t)\|_{L^\infty}+\|\widetilde{\mathbf{u}}(t)\|_{L^\infty}\big)\|\widetilde{\mathbf{u}}(t)\|_{L^2}+\|\widetilde{\mathbf{u}}(t)\|_{L^2}+\|\nabla\widetilde{\mathbf{u}}(t)\|_{L^2}+|\alpha'(t)|\\ \lesssim\ &\|\partial_t\widetilde{\mathbf{u}}(t)\|_{L^2}+\|\nabla\widetilde{\mathbf{u}}(t)\|_{L^2}+|\alpha'(t)|\to 0,\quad \text{as}\quad t\to\infty.\end{aligned}$$

In terms of the original variables, we have

$$\|(\nabla P-\theta\mathbf{e}_2)(t)\|_{H^{-1}}\to 0,\quad \text{as}\quad t\to\infty.$$

This completes the proof of Theorem 1.3. □

Remark 4.1. *It is worth mentioning that, instead of assuming $\alpha(t)\in L^1(0,\infty)$, one could impose the condition: $\alpha(t)-\alpha_1\in L^1(0,\infty)$, where α_1 is a constant satisfying $|\alpha_1|<2\pi^2\nu$ (see Theorem 1.1). In this case, by using similar arguments as those in Section 2, one can show that $\|\widetilde{\mathbf{u}}(t)-(\alpha(t)y,0)^{\mathrm{T}}\|_{L^2}\to 0$, as $t\to\infty$. However, it is not clear whether the decaying of the first order derivatives of the perturbation can be established. This is caused by the cubic nonlinearity in* (4.8)*, whose estimate is given by* (4.13)*. Due to the lack of the uniform temporal integrability of $\widetilde{\theta}$, we cannot split $\alpha(t)$ into $\alpha(t)-\alpha_1$ and α_1, then proceed in the same way as in the H^1-level energy estimate in Section 2.*

5. Semi-dissipative System under No-penetration and Dirichlet B.C.

We prove Theorem 1.4 in this section.

Proof of Theorem 1.4. By following similar arguments as those in [35], we can establish the global well-posedness of classical solutions to (1.10). We shall focus on deriving the decay estimate of the solution. Rewriting the third equation in (1.10) in terms of the perturbation: $\widetilde{\theta} = \theta - \beta(t)y$, we have

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\widetilde{P} = \widetilde{\theta}\mathbf{e}_2, \\ \nabla\cdot\mathbf{u} = 0, \\ \partial_t\widetilde{\theta} + \mathbf{u}\cdot\nabla\widetilde{\theta} + \beta(t)u_2 = \kappa\Delta\widetilde{\theta} - \beta'(t)y. \end{cases} \tag{5.1}$$

Calculating L^2 inner products, we can show that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|_{L^2}^2 = \int_\Omega \widetilde{\theta}\mathbf{e}_2\cdot\mathbf{u}\,\mathrm{d}\mathbf{x} = \int_\Omega \widetilde{\theta}\,u_2\,\mathrm{d}\mathbf{x}, \tag{5.2}$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{\theta}\|_{L^2}^2 + \kappa\|\nabla\widetilde{\theta}\|_{L^2}^2 = -\beta(t)\int_\Omega \widetilde{\theta}\,u_2\,\mathrm{d}\mathbf{x} - \beta'(t)\int_\Omega y\,\widetilde{\theta}\,\mathrm{d}\mathbf{x}. \tag{5.3}$$

Multiplying (5.2) with $\beta(t)$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big[\beta(t)\|\mathbf{u}\|_{L^2}^2\big] = \beta(t)\int_\Omega \widetilde{\theta}\,u_2\,\mathrm{d}\mathbf{x} + \frac{\beta'(t)}{2}\|\mathbf{u}\|_{L^2}^2. \tag{5.4}$$

Taking the sum of (5.3) and (5.4), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\beta(t)\|\mathbf{u}\|_{L^2}^2 + \|\widetilde{\theta}\|_{L^2}^2\big] + 2\kappa\|\nabla\widetilde{\theta}\|_{L^2}^2 = \beta'(t)\|\mathbf{u}\|_{L^2}^2 - 2\beta'(t)\int_\Omega y\,\widetilde{\theta}\,\mathrm{d}\mathbf{x}, \tag{5.5}$$

where two terms on the right-hand side can be estimated as

$$\begin{aligned} \Big|\beta'(t)\|\mathbf{u}\|_{L^2}^2 - 2\beta'(t)\int_\Omega y\,\widetilde{\theta}\,\mathrm{d}\mathbf{x}\Big| &\le \frac{|\beta'(t)|}{\beta_1}\,\beta(t)\|\mathbf{u}\|_{L^2}^2 + |\beta'(t)|\big(\|y\|_{L^2}^2 + \|\widetilde{\theta}\|_{L^2}^2\big) \\ &\le C|\beta'(t)|\big(\beta(t)\|\mathbf{u}\|_{L^2}^2 + \|\widetilde{\theta}\|_{L^2}^2\big) + \frac{|\beta'(t)|}{3}. \end{aligned}$$

Substituting the above estimate into (5.5), then applying Grönwall's inequality and using the assumptions for $\beta(t)$, we infer that

$$\|\mathbf{u}(t)\|_{L^2}^2 + \|\widetilde{\theta}(t)\|_{L^2}^2 + \int_0^t \|\widetilde{\theta}(\tau)\|_{H^1}^2\mathrm{d}\tau \le C, \quad \forall\, t>0. \tag{5.6}$$

To derive the decaying of the perturbation, we let

$$Y(t) \equiv \big(\|\widetilde{\theta}(t)\|_{L^2}^2\big)^2.$$

It follows from (5.6) that

$$\int_0^t Y(\tau)\mathrm{d}\tau \le C, \quad \forall\, t>0. \tag{5.7}$$

On the other hand, according to (5.3), we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}Y(t) &= 2\|\widetilde{\theta}(t)\|_{L^2}^2\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{\theta}(t)\|_{L^2}^2 \\ &= -\|\widetilde{\theta}(t)\|_{L^2}^2\Big(\kappa\|\nabla\widetilde{\theta}\|_{L^2}^2 + \beta(t)\int_\Omega \widetilde{\theta}\,u_2\,\mathrm{d}\mathbf{x} + \beta'(t)\int_\Omega y\,\widetilde{\theta}\,\mathrm{d}\mathbf{x}\Big), \end{aligned}$$

which implies

$$\begin{aligned}\Big|\frac{\mathrm{d}}{\mathrm{d}t}Y(t)\Big| &\le \|\widetilde{\theta}(t)\|_{L^2}^2\big(\kappa\|\nabla\widetilde{\theta}\|_{L^2}^2+\beta(t)\|\widetilde{\theta}\|_{L^2}\|\widetilde{\mathbf{u}}\|_{L^2}+|\beta'(t)|\,\|y\|_{L^2}\|\widetilde{\theta}\|_{L^2}\big)\\ &\le C\big(\|\nabla\widetilde{\theta}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2+|\beta'(t)|\big)\\ &\le C\big(\|\nabla\widetilde{\theta}\|_{L^2}^2+|\beta'(t)|\big),\end{aligned}$$

where (5.6) and Poincaré inequality are applied. Integrating the above inequality with respect to t and using (5.6), we have

$$\int_0^t \Big|\frac{\mathrm{d}}{\mathrm{d}t}Y(\tau)\Big|\mathrm{d}\tau \le C, \quad \forall\, t>0,$$

which, together with (5.7), implies

$$Y(t)\in W^{1,1}(0,\infty).$$

Therefore, it holds that

$$Y(t)\to 0, \quad \text{as} \quad t\to\infty.$$

From the definition of $Y(t)$ we conclude that

$$\|\widetilde{\theta}(t)\|_{L^2}\to 0, \quad \text{as} \quad t\to\infty.$$

In terms of the original variables, we have

$$\|\theta(t)-\beta(t)y\|_{L^2}\to 0, \quad \text{as} \quad t\to\infty.$$

This completes the proof of Theorem 1.4. □

6. Semi-dissipative System under No-penetration and Neumann B.C.

This section proves Theorem 1.5.

Proof of Theorem 1.5. Again, the global well-posedness of classical solutions to (1.11) follows from similar arguments in [35]. To derive the decay estimate, we set

$$\widetilde{\theta}\equiv\theta-y.$$

Then it follows from the third equation in (1.11) that

$$\partial_t\widetilde{\theta}=-\nabla\cdot(\mathbf{u}\,\theta-\kappa\nabla\widetilde{\theta}).$$

Since $\mathbf{u}\cdot\mathbf{n}=0$ and $\nabla\widetilde{\theta}\cdot\mathbf{n}=0$ on $\partial\Omega$, integrating the above equation over Ω, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\widetilde{\theta}(\mathbf{x},t)\,\mathrm{d}\mathbf{x}=0,$$

which implies

$$\int_\Omega\widetilde{\theta}(\mathbf{x},t)\,\mathrm{d}\mathbf{x}=\int_\Omega\widetilde{\theta}_0(\mathbf{x})\,\mathrm{d}\mathbf{x}=\int_\Omega[\theta_0(\mathbf{x})-y]\,\mathrm{d}\mathbf{x}\equiv\bar{\theta}.$$

Let

$$\breve{\theta}\equiv\widetilde{\theta}-\bar{\theta}.$$

Then we have (note that $|\Omega|=1$)

$$\int_\Omega\breve{\theta}(\mathbf{x},t)\,\mathrm{d}\mathbf{x}=0.$$

Moreover, since $\bar{\theta}$ is a constant, it holds that $\nabla\breve{\theta}\cdot\mathbf{n}=0$ on $\partial\Omega$, and

$$\partial_t\breve{\theta}+\mathbf{u}\cdot\nabla\breve{\theta}=\kappa\Delta\breve{\theta}-u_2. \tag{6.1}$$

In addition, using the definition of $\breve{\theta}$, we rewrite the first equation in (1.11) as

$$\partial_t\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla\breve{P}=\breve{\theta}\mathbf{e}_2, \tag{6.2}$$

where $\breve{P}=P-\frac{y^2}{2}-\bar{\theta}\,y$. Taking L^2 inner products of (6.1) with $\breve{\theta}$ and (6.2) with $\mathbf{u}$, then adding the results, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\mathbf{u}\|_{L^2}^2+\|\breve{\theta}\|_{L^2}^2\big)+\kappa\|\nabla\breve{\theta}\|_{L^2}^2=0. \tag{6.3}$$

Since $\breve{\theta}$ is mean free, integrating (6.3) with respect to time and utilizing Poincaré inequality, we can show that

$$\|\mathbf{u}(t)\|_{L^2}^2+\|\breve{\theta}(t)\|_{L^2}^2+\int_0^t\|\breve{\theta}(\tau)\|_{H^1}^2\mathrm{d}\tau\le C,\quad\forall\,t>0. \tag{6.4}$$

Let

$$Z(t)\equiv\big(\|\breve{\theta}(t)\|_{L^2}^2\big)^2.$$

Then, following the arguments in Section 5 and using (6.4), we can show that

$$Z(t)\in W^{1,1}(0,\infty),$$

which implies $\|\breve{\theta}(t)\|_{L^2}\to 0$, as $t\to\infty$. In terms of the original variables, we see that

$$\|\theta(t)-y-\bar{\theta}\|_{L^2}\to 0,\quad\text{as}\quad t\to\infty.$$

This completes the proof of Theorem 1.5. □

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