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Matrix-valued Allen–Cahn equation and the Keller–Rubinstein–Sternberg problem

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Abstract In this paper, we consider the sharp interface limit of a matrix-valued Allen–Cahn equation, which takes the form:

$$\partial_t \mathbf{A} = \Delta \mathbf{A} - \varepsilon^{-2}(\mathbf{A}\mathbf{A}^{\mathrm{T}}\mathbf{A} - \mathbf{A}) \quad \text{with} \quad \mathbf{A} : \Omega \subset \mathbb{R}^m \to \mathbb{R}^{n\times n}.$$

We show that the sharp interface system is a two-phases flow system: the interface evolves according to the motion by mean curvature; in the two bulk phase regions, the solution obeys the heat flow of harmonic maps with values in $O^+(n)$ and $O^-(n)$ (represent the sets of $n \times n$ orthogonal matrices with determinant $+1$ and -1 respectively); on the interface, the phase matrices on two sides satisfy a novel mixed boundary condition. The above result provides

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a solution to the Keller–Rubinstein–Sternberg's problem in the $O(n)$ setting. Our proof relies on two key ingredients. First, in order to construct the approximate solutions by matched asymptotic expansions, as the standard approach does not seem to work, we introduce the notion of quasi-minimal connecting orbits. They satisfy the usual leading order equations up to some small higher order terms. In addition, the linearized systems around these quasi-minimal orbits needs to be solvable up to some good remainders. These flexibilities are needed for the possible "degenerations" and higher dimensional kernels for the linearized operators on matrix-valued functions due to intriguing boundary conditions at the sharp interface. The second key point is to establish a spectral uniform lower bound estimate for the linearized operator around approximate solutions. To this end, we introduce additional decompositions to reduce the problem into the coercive estimates of several linearized operators for scalar functions and some singular product estimates which are accomplished by exploring special cancellation structures between eigenfunctions of these linearized operators.

1 Introduction

1.1 Background and related results

The phase transition problem has drawn great interest in both analysis and applications. The simplest model for the phase transition is the scalar Allen–Cahn equation, which was introduced by Allen–Cahn [3] to model the motion of antiphase boundaries in crystalline solids. Let $u : \Omega \subseteq \mathbb{R}^m \to \mathbb{R}$, and $F(u)$ is a potential function with two wells (e.g. $F(u) = (u^2-1)^2/4$). The equation reads as follows

$$\partial_t u = \Delta u - \frac{1}{\varepsilon^2} F'(u). \tag{1.1}$$

As $\varepsilon \to 0$, the domain Ω will be separated into two regions $\Omega_\pm$, where $u \to \pm 1$ respectively. Moreover, the interface between these two regions evolves according to the mean curvature flow. This sharp interface limit has been rigorously justified in both static and dynamic cases by numerous authors via different methods, see [30,31] for the static case, and [8,11,13,15,21,38] for dynamical problems.

In [35,36], Rubinstein–Sternberg–Keller introduced a vector-valued system for fast reaction and slow diffusion:

$$\partial_t u = \Delta u - \frac{1}{\varepsilon^2} \partial_u F(u), \tag{1.2}$$

where $u : \Omega \subseteq \mathbb{R}^m \to \mathbb{R}^N$ is a phase-indicator function, and the nonnegative, smooth potential function $F : \mathbb{R}^N \to \mathbb{R}$ vanishes exactly on two disjoint

connected sub-manifolds in $\mathbb{R}^N$. By a formal asymptotic expansion, they found that, when $\varepsilon \to 0$, the interface moves by its mean curvature, while away from the interface u tends to the heat flow of harmonic maps into the sub-manifolds (potential wells).

There is a lack of rigorous analysis for the Keller–Rubinstein–Sternberg problem (1.2) in general. A few studies are for some special situations and the problem has remained largely open including the $O(n)$-model. In [26], Lin–Pan–Wang analyzed the asymptotic behaviour as $\varepsilon \to 0$ for the energy minimizing static solutions to a class of the Keller–Rubinstein–Sternberg problem. One found a non-standard mixed Dirichlet–Neumann boundary condition for the phase field along the interface. A regularity theory for minimizing maps of the limit problem was addressed in [27].

For asymptotics of the gradient flow (1.2), Bronsard–Stoth [9] studied the radially symmetric case for a special $\mathbb{R}^2$-valued problem, and a Neumann-jump boundary condition for the limit system was derived. In [16], without the radially symmetric assumption, Fei–Wang–Zhang–Zhang rigorously justified the asymptotic limit for a physical $\mathbb{R}^{3\times 3}$-valued model, which describes the isotropic-nematic phase transition for liquid crystals (the radially symmetric case is studied in [29]). For the latter problem, we refer to [22,28] for more recent progress. There are several interesting studies for related problems, see for examples, [7,19,20,23,33], most of which consider the case when the minimizing set of F consists of finite isolated points.

In this paper, through a careful and systematic analysis of the matrix-valued Allen–Cahn equations, we obtain these intriguing boundary conditions for the phase field along the sharp interface in evolution. In fact, a much clearer picture emerges. The boundary conditions for the phase field (and also for the limiting problem) are actually dictated by the geometric properties of the potential energy wells and the so-called minimal connecting orbits. The existence and geometric properties of these minimal-connecting-orbits (often infinitely many if only one of the ends is fixed) will also show up in deriving matched asymptotic expansions and the construction of higher order approximate solutions with drastic variations of analytic difficulties. It is something that does not appear in the scalar case or several vector-valued cases, which has been studied by various authors before. It also illustrates why there are different types of boundary conditions along the interface in earlier works [16,22,26–28], for examples. Let us describe briefly our method below. Some related discussions are in Sect. 3 of the paper.

Following [35,36] and [26], we consider the energy functional for $\Omega \subset \mathbb{R}^m$:

$$E_\varepsilon(u) = \int_\Omega \left(|\nabla u|^2 + \frac{1}{\varepsilon^2} F(u) \right) \mathrm{d}x,$$

where the potential function $F(u)$ is smooth for $u \in \mathbb{R}^N$ and satisfies the properties that

$$c_1 d^2(u, \Sigma) \leqslant F(u) \leqslant C_2 d^2(u, \Sigma)$$

for $u \in \Sigma_{\delta_0}$ (the δ_0-neighborhood of Σ), and that $F(u) \geqslant c_3$ whenever $d(u, \Sigma) \geqslant \delta_0$. Here $\Sigma = \Sigma_+ \cap \Sigma_-$ is the union of two disjoint, smooth, compact and connected submanifolds in $\mathbb{R}^N$ on which F vanishes. For any two points p_+, p_- in $\mathbb{R}^N$, one can define the distance with weight $\sqrt{F/2}$:

$$d_F(p^+, p^-) = \inf \Big\{ \int_{\mathbb{R}} \Big(|\xi'(t)|^2 + F(\xi) \Big) \mathrm{d}t : \xi \in H^1(\mathbb{R}, \mathbb{R}^N), \\ \xi(\pm\infty) = p_\pm \Big\},$$

and let

$$c_0^F = \inf_{q^+ \in \Sigma_+, q^- \in \Sigma_-} d_F(q^+, q^-).$$

Under some very nature assumptions, one has the following expansion for the energy $E_\varepsilon(u)$ for the so-called well-prepared data u:

$$E_\varepsilon(u) = \frac{c_0^F}{\varepsilon} \mathcal{H}^{m-1}(\Gamma_u) + D(u) + O_\varepsilon(1).$$

Here Γ_u is a sharp interface between the two phases Σ_+ and Σ_-, and $D(u)$ is the Dirichlet energy of the map from $\Omega_\pm(u)$ (the sub-regions of Ω which are separated by Γ_u) into $\Sigma_\pm$.

If one considers energy minimizers (as in [26]), then it is intuitively clear that Γ_u will be an area minimizing surface, and $u : \Omega_\pm(u) \to \Sigma_\pm$ will be energy minimizing maps. There will also be natural boundary conditions for u on the two sides of the sharp interface.

The difficult point for the gradient flow for this type of energy functional is that the sharp interface motion of $\{\Gamma(t)\}$ and the heat flow of harmonic map in the bulk are in the same time scale (unlike the dynamics of Ginzburg–Landau vortices [24]). These two motions may be coupled in $O(\varepsilon)$ order terms (hence it does not show up in the first order formal asymptotic expansions [35,36]); and due to the interfacial energy being of order $O(\varepsilon^{-1})$, it may lead to undesired $O(1)$ changes in motion of the interface. This coupling occurs through the intriguing boundary conditions for the maps at the sharp interface. As phases evolve in the bulk, boundary values of $u_\pm$ at the sharp interface would change accordingly (and even if they are of the size $O(\varepsilon)$, they have to be counted as explained

above). The latter may alter the weights in calculating of the (weighted) surface area of the sharp interface, that is the d_F distance between points $u_\pm(x,t)$ from two sides of $\Gamma(t)$. Of course, if the weighted distance between any pair of points $p_\pm \in \Sigma_\pm$ is the same absolute minimum, then there will be no such effect in coupling and the weight for the sharp interface area will not change in the evolution. This is the case we shall say that $F(u)$ is *fully minimally paired* (see discussions in Sect. 3). The case that u is a scalar function, and the cases that have been considered in many previous works for the vector-valued functions (including [16,22,28]) are all so that $F(u)$ being *fully minimally paired*. From this point of view, [26] studied a class of problem which is only partially minimally paired, but in a very specific way. The $O(n)$-model we study in this paper has a great generality for situations of partially minimally paired $F(u)$, and it is kind of a generic situation. We refer to the discussions in Sect. 3.

In this paper we shall concentrate on the study of the evolution of well-prepared initial data for the matrix-valued Allen–Cahn equations. The construction of approximate solutions in this paper is based on a new modified matched asymptotic expansion method. It turns out to be one of the most difficult points of this paper. The new method provides a great flexibility and it improves substantially the earlier ones [1,2,13,16] in the study of the sharp interface limit problems. On the other hand, we will use the setups in Alikakos–Bates–Chen [2] as it fits better for a more geometric description of our problem.

It is worth pointing out that though by now there are several elegant and powerful approaches to sharp interface limit problems, to our best knowledge along with many attempts, it remains unclear whether these arguments could work in a general vector-valued setting (e.g., the Keller–Rubinstein–Sternberg problem. Roughly speaking, the geometric measure theoretic approach via varifold flows relies on the key monotonicity formula. The latter uses the so-called "discrepancy function" in [21] and many improvements up to date working only for the case that phase fields stay in a one-dimensional submanifold in a neighborhood of the sharp interface. The recent approach using the idea of relative entropy or modulated energy [18,22] (see also [25]) has been applied to study sharp interface limit problems, but it is not clear also whether it works for general partially minimally paired situations. Other geometric measure theoretic approaches using energy concentrations such as that by Ambrosio and Soner [4] will need both the so-called concentration energy density lower bound and the energy equal partition (quantization). One meets the similar challenges, whether any of these approaches can be modified to work for the general sharp phase transition problems remain to be a fascinating problem.

1.2 Presentation of the problem and main results

We consider a matrix-valued Allen–Cahn equation with a small parameter ε in the bounded domain $\Omega = I_1 \times I_2 \cdots \times I_m \subset \mathbb{R}^m$ with periodic boundary conditions, which was introduced in [34]:

$$\partial_t \mathbf{A}^\varepsilon = \Delta \mathbf{A}^\varepsilon - \varepsilon^{-2} f(\mathbf{A}^\varepsilon), \tag{1.3}$$

where

$$f(\mathbf{A}^\varepsilon) = \mathbf{A}^\varepsilon (\mathbf{A}^\varepsilon)^{\mathrm{T}} \mathbf{A}^\varepsilon - \mathbf{A}^\varepsilon \tag{1.4}$$

with $(\mathbf{A}^\varepsilon)^{\mathrm{T}}$ denoting the transpose of $\mathbf{A}^\varepsilon \in \mathbb{R}^{n\times n} \triangleq \mathbb{M}_n$. The system (1.3), which was first introduced by [34], could be viewed as the gradient flow for the energy functional

$$\mathcal{E}(\mathbf{A}, \nabla \mathbf{A}) = \int_\Omega \Big(\frac{1}{2} \|\nabla \mathbf{A}\|^2 + \varepsilon^{-2} F(\mathbf{A}) \Big) \mathrm{d}x, \quad F(\mathbf{A}) = \frac{1}{4} \|\mathbf{A}^{\mathrm{T}} \mathbf{A} - \mathbf{I}\|^2, \tag{1.5}$$

where $\mathbf{I}$ is the identity matrix. In this paper, we are interested in the asymptotical behavior of solutions to the system (1.3) when $\varepsilon \to 0$.

Note that $F(\mathbf{A})$ attains its minimum at the orthogonal group $O(n) = O^+(n) \cup O^-(n)$, where $O^\pm(n)$ denotes the set of orthogonal matrices with determinant ± 1. Formally, one has that, in the limit of $\varepsilon \to 0$, the domain Ω can be divided into two disjoint parts Ω_t^+ and Ω_t^- with the property that

$$\mathbf{A}^\varepsilon(x,t) \to \mathbf{A}_\pm(x,t) \in O^\pm(n), \quad \text{for } x \in \Omega_t^\pm.$$

Then we need to determine the evolution of the interface $\Gamma_t = \partial\Omega_t^+ \cap \partial\Omega_t^-$, the evolution of $\mathbf{A}_\pm$ in $\Omega_t^\pm$, and the boundary conditions of $\mathbf{A}_\pm$ on Γ. Indeed, the limit sharp interface model of the system (1.3) takes the form:

$$\partial_t \mathbf{A}_\pm = \Delta \mathbf{A}_\pm - \sum_{i=1}^m \partial_i \mathbf{A}_\pm \mathbf{A}_\pm^{\mathrm{T}} \partial_i \mathbf{A}_\pm \quad (\mathbf{A}_\pm \in O^\pm(n)), \qquad \text{in } \Omega_t^\pm, \tag{1.6a}$$

$$V = \kappa, \qquad \text{on } \Gamma_t, \tag{1.6b}$$

$$(\mathbf{A}_+, \mathbf{A}_-) \text{ is a minimal pair}, \qquad \text{on } \Gamma_t, \tag{1.6c}$$

$$\frac{\partial \mathbf{A}_+}{\partial \nu} = \frac{\partial \mathbf{A}_-}{\partial \nu}, \qquad \text{on } \Gamma_t. \tag{1.6d}$$

Here V, κ, and ν are the normal velocity, the mean curvature and the unit outward normal vector of Γ_t respectively. The condition (1.6c) is equivalent to

that for each $x \in \Gamma_t$, there exists $\mathbf{n}(x,t) \in S^{n-1}$ such that $\mathbf{A}_+ = \mathbf{A}_-(\mathbf{I}-2\mathbf{nn})$; see Definition 3.4 and Lemma 3.7.

The heat flow of harmonic maps (1.6a) and the mean curvature flow (1.6b) for the interface are rather natural, which have been formally argued in [35,36]. The relation (1.6c) has been obtained in [26] for minimizing equilibrium solutions. The boundary condition (1.6d) is new and special for this matrix-valued Allen–Cahn equation on $O(n)$ due to the underlying geometric structure and properties of minimal connecting orbits (see also [26] for different boundary conditions in another geometric setting). We shall provide a derivation in Appendix A.1.

In the case of $n = 2$, the minimal pair condition holds for any $(\mathbf{A}_-, \mathbf{A}_+) \in O^-(2) \times O^+(2)$. So, the condition (1.6c) is redundant. The Neumann-jump condition in this case (1.6d) is reduced to the usual Neumann boundary condition for functions on both sides:

$$\partial_\nu \mathbf{A}_- = \partial_\nu \mathbf{A}_+ = 0.$$

Indeed, in this case, we can write

$$\mathbf{A}_+ = \begin{pmatrix} \cos\alpha_+ & \sin\alpha_+ \\ -\sin\alpha_+ & \cos\alpha_+ \end{pmatrix}, \quad \mathbf{A}_- = \begin{pmatrix} \cos\alpha_- & \sin\alpha_- \\ \sin\alpha_- & -\cos\alpha_- \end{pmatrix}$$

($\alpha_\pm$ are the rotation angles), then (1.6d) becomes

$$\partial_\nu \alpha_+ = \partial_\nu \alpha_- = 0. \tag{1.7}$$

This boundary condition (1.7) for $n = 2$ has been observed in [39], in which the asymptotic dynamics of (1.3) under different time scales are studied.

The main goal of this paper is to provide a rigorous justification of the limit from the regularized system (1.3) to the sharp interface model (1.6), which is also called the sharp interface limit. The proofs will follow the approach in de Mottoni-Schatzman [13] and Alikakos-Bates-Chen [2]: we first construct an approximate solution $\mathbf{A}^K$ solving (1.3) up to sufficiently high order small terms; and then we prove a spectral lower bound for the linearized operator around $\mathbf{A}^K$; and finally we estimate the difference $\mathbf{A}^\varepsilon - \mathbf{A}^K$.

Our main results are stated as follows. The first main result is concerned with the existence of approximate solutions, whose proof turns out to be the most difficult for this article.

Theorem 1.1 *Assume that* $(\Gamma_t, \mathbf{A}_+, \mathbf{A}_-)$ *is a smooth solution on* $[0, T]$ *to the sharp interface system (1.6). Then for any* $K \geqslant 1$, *there exists* $\mathbf{A}^K$ *such that*

$$\partial_t \mathbf{A}^K = \Delta \mathbf{A}^K - \varepsilon^{-2} f(\mathbf{A}^K) + \mathfrak{R}^{K-1}, \tag{1.8}$$

where $\mathfrak{R}^{K-1} \sim O(\varepsilon^{K-1})$, *and for any* (x,t) *with* $x \in \Omega_t^{\pm}$, $\|\mathbf{A}^K(x,t) - \mathbf{A}_{\pm}(x,t)\| \to 0$ *as* $\varepsilon \to 0$.

Remark 1.2 The existence of smooth solutions to (1.6) is not a trivial issue in general due to the complicated boundary conditions and we will address it and some related issues in a forthcoming paper. But for this $O(n)$ model, the situation is very similar to that of problem studied in [27], and the well-posedness has been established by the second author and Wang in [27].

The second main result is the spectral lower bound for the linearized operator around the approximate solution $\mathbf{A}^K$. Compared with the proof in the scalar case, the proof is much more involved, and several new ideas are introduced to obtain the desired conclusions.

Theorem 1.3 *Assume that* $\mathbf{A}^K (K \geqslant 1)$ *is an approximate solution constructed as in Theorem 1.1. Then there exists* $C_0 > 0$ *such that, for any* $\mathbf{A} \in H^1(\Omega)$ *and* $t \in [0,T]$ *it holds*

$$\int_\Omega \Big(\frac{1}{2}\|\nabla \mathbf{A}\|^2 + \frac{1}{\varepsilon^2}\mathcal{H}_{\mathbf{A}^K}\mathbf{A} : \mathbf{A}\Big)\mathrm{d}x \geqslant -C_0 \int_\Omega \|\mathbf{A}\|^2 \mathrm{d}x, \tag{1.9}$$

where

$$\mathcal{H}_{\mathbf{B}}\mathbf{A} = \mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{A} + \mathbf{A}\mathbf{B}^{\mathrm{T}}\mathbf{B} + \mathbf{B}\mathbf{A}^{\mathrm{T}}\mathbf{B} - \mathbf{A} \tag{1.10}$$

is the linearized operator of f *defined in (1.4).*

With the help of Theorem 1.1 and Theorem 1.3, the following nonlinear stability result follows easily via an energy method.

Theorem 1.4 *Let* $L = 3([\frac{m}{2}] + 2)$ *and* $K = L + 1$. *Assume that* $\mathbf{A}^K$ *is an approximate solution constructed as in Theorem 1.1, and* $\mathbf{A}^\varepsilon$ *is a solution to (1.3) with initial data* $\mathbf{A}^\varepsilon(\cdot,0)$ *satisfying*

$$\bar{\mathcal{E}}(\mathbf{A}^\varepsilon(\cdot,0) - \mathbf{A}^K(\cdot,0)) \leqslant C_0\varepsilon^{2L},$$

where

$$\bar{\mathcal{E}}(\mathbf{A}) = \sum_{i=1}^{[\frac{m}{2}]+1} \varepsilon^{6i} \int_\Omega \|\nabla^i \mathbf{A}\|^2 \mathrm{d}x.$$

Then there exists constants $\varepsilon_0, C_1 > 0$ *such that for* $\forall \varepsilon \leqslant \varepsilon_0$ *it holds*

$$\bar{\mathcal{E}}(\mathbf{A}^\varepsilon(\cdot,t) - \mathbf{A}^K(\cdot,t)) \leqslant C_1\varepsilon^{2L}, \quad \text{for } \forall t \in [0,T]. \tag{1.11}$$

1.3 Main difficulties, key ideas and outline of the proof

The proofs are based on two key ingredients: construction of approximate solutions and the spectral lower bound estimate. Compared with the scalar case (e.g. [2,13]) or the case that the potential function $F(u)$ is fully minimally paired (e.g. [16]), there exist several serious difficulties, for which we need to introduce various new arguments to overcome them. Let us give a sketch here.

Assume $(\Gamma_t, \mathbf{A}_+, \mathbf{A}_-)$ is a smooth solution of (1.6) on $[0, T]$. To proceed, let $d(x, \Gamma_t)$ be the signed distance from x to Γ_t, and $\nu = \nabla d|_{\Gamma_t}$ be the unit outer normal of Ω_t^-. We denote

$$\Gamma(\delta) = \{(x,t) \in \Omega \times [0,T] : |d(x,\Gamma_t)| < \delta\},$$
$$Q_\pm = \{(x,t) \in \Omega \times [0,T] : d(x,\Gamma_t) \gtrless 0\}.$$

We also write $\Gamma = \{(x,t) \in \Omega \times [0,T] : x \in \Gamma_t\}$ for simplicity.

1.3.1 Construction of approximate solutions

As in [2], we construct approximate solutions via different expansions in the two regions: outer region $Q_\pm \backslash \Gamma(\delta/2)$ and inner region $\Gamma(\delta)$.

In $Q_\pm \backslash \Gamma(\delta/2)$, we assume that the solution of (1.3) has the form:

$$\mathbf{A}^\varepsilon(x,t) = \mathbf{A}_O^\varepsilon(x,t) := \sum_{i=0}^{+\infty} \varepsilon^i \mathbf{A}_\pm^{(i)}(x,t), \tag{1.12}$$

where $\mathbf{A}_\pm^{(i)}(x,t)$ are smooth functions defined in $Q_\pm$. Substituting the above expansion into (1.3) and collecting the terms with same power of ε, we can obtain the equations for $\mathbf{A}_\pm^{(i)}(x,t)(i \geqslant 0)$, where the leading order equation for $\mathbf{A}_\pm^{(0)}(x,t)$ is actually the equation (1.6a). The boundary/jump conditions for $\mathbf{A}_\pm^{(i)}(x,t)$ on Γ_t will be determined to ensure the solvability of expansions in inner regions (particularly on Γ_t). Moreover, we will use the value of $\mathbf{A}_\pm^{(i)}(x,t)$ not only in the domain $Q_\pm$, but also in $Q_\pm \cup \Gamma(\delta)$. Hence, we extend $\mathbf{A}_\pm^{(i)}(x,t)$ from $Q_\pm$ to $Q_\pm \cup \Gamma(\delta)$ smoothly.

In the region near the interface, we try to find functions for $(x,t) \in \Gamma(\delta)$

$$\mathbf{A}_I^\varepsilon(z,x,t) = \mathbf{A}_I^{(0)}(z,x,t) + \varepsilon \mathbf{A}_I^{(1)}(z,x,t) + \varepsilon^2 \mathbf{A}_I^{(k)}(z,x,t) + \cdots, \tag{1.13}$$
$$d^\varepsilon(x,t) = d_0(x,t) + \varepsilon d_1(x,t) + \varepsilon^2 d_2(x,t) + \cdots, \tag{1.14}$$

such that $d^\varepsilon(x,t)$ is a signed distance function with respect to a surface Γ^ε and $\widetilde{\mathbf{A}}_I^\varepsilon(x,t) = \mathbf{A}_I^\varepsilon\big(\varepsilon^{-1} d^\varepsilon(x,t), x, t\big)$ solves (1.3) in $\Gamma(\delta)$. Here $\mathbf{A}_I^{(k)}(z,x,t)$ and $d_k(x,t)$ $(k \geqslant 0)$ are functions independent of ε and $d_0(x,t) = d(x,t)$.

To match the two expansions, we require that $\mathbf{A}_I^{(k)}(z,x,t)$ tends to $\mathbf{A}_\pm^{(k)}(x,t)$ exponentially as $z \to \pm\infty$. Then the approximate solution in the whole domain Ω can be constructed by gluing $\mathbf{A}_O^\varepsilon$ and $\widetilde{\mathbf{A}}_I^\varepsilon$ in the overlapped region $\Gamma(\delta) \setminus \Gamma(\delta/2)$.

Once we substitute the inner expansion (1.13) into (1.3), the leading order ($O(\varepsilon^{-2})$) system is an ODE system in z, which reads as

$$-\partial_z^2 \mathbf{A}_I^{(0)} + f(\mathbf{A}_I^{(0)}) = 0, \quad \mathbf{A}_I^{(0)}(\pm\infty, x, t) = \mathbf{A}_\pm^{(0)}(x,t) = \mathbf{A}_\pm(x,t), \tag{1.15}$$

for given $(x,t) \in \Gamma(\delta)$. However, as the discussion in Sect. 3.2, the existence of solution highly depends on the boundary data $\mathbf{A}_\pm^{(0)}$. In particular, it has no solution unless $\mathbf{A}_+^{\mathrm{T}}\mathbf{A}_-$ is symmetric. More importantly, only when $(\mathbf{A}_+, \mathbf{A}_-)$ is a minimal pair, (1.15) can have a stable solution which takes the form

$$\mathbf{\Theta}(\mathbf{A}_+, \mathbf{A}_-; z) := s(z)\mathbf{A}_+ + (1 - s(z))\mathbf{A}_-, \qquad s(z) = 1 - (1 + e^{\sqrt{2}z})^{-1}. \tag{1.16}$$

Such a $\mathbf{\Theta}(\mathbf{A}_+, \mathbf{A}_-; z)$ with $(\mathbf{A}_+, \mathbf{A}_-)$ being a minimal pair is called a minimal connecting orbit (see Definition 3.1 for the precise definition).

The boundary condition (1.6c) gives us that $(\mathbf{A}_+(x,t), \mathbf{A}_-(x,t))$ forms a minimal pair for $(x,t) \in \Gamma$. As we have to perform expansion in $\Gamma(\delta)$, a natural question is:

Whether $\mathbf{A}_\pm(x,t)$ *can be smoothly extended to* $\Gamma(\delta)$ *such that* $(\mathbf{A}_+(x,t), \mathbf{A}_-(x,t))$ *remains to be a minimal pair for every* $(x,t) \in \Gamma(\delta)$?
Unfortunately, due to the *partially minimally paired* (cf. Definition 3.5) nature of this problem, for general solutions of the system (1.6), such an extension does not exist unless $n = 2$. [It partially explains difficulties for generalizing the so-called relative entropy method to the Keller–Rubinstein–Sternberg problem]. Thus, for $(x,t) \in \Gamma(\delta) \setminus \Gamma$, one can not expect that $(\mathbf{A}_+(x,t), \mathbf{A}_-(x,t))$ is a minimal pair, and consequently one can not find a solution to (1.15) for $(x,t) \in \Gamma(\delta) \setminus \Gamma$.

More troublesomely, the equations of $\mathbf{A}_I^{(k)}(z,x,t)$ $(k \geqslant 1)$, whose main part is the linearization of (1.15), take the form

$$\mathcal{L}_{\mathbf{A}_I^{(0)}} \mathbf{A}_I^{(k)} := -\partial_z^2 \mathbf{A}_I^{(k)} + \mathcal{H}_{\mathbf{A}_I^{(0)}} \mathbf{A}_I^{(k)} = \mathbf{F}_k. \tag{1.17}$$

This system may have no solutions, unless $\mathbf{A}_I^{(0)} = \mathbf{\Theta}(\mathbf{A}_+, \mathbf{A}_-; z)$ with $(\mathbf{A}_+, \mathbf{A}_-)$ being a minimal pair. Therefore, for $(x,t) \in \Gamma(\delta) \setminus \Gamma$, the construction of $\mathbf{A}_I^{(k)}(z,x,t)$ $(k \geqslant 1)$ can not proceed either.

The solvability of (1.15) and (1.17) have thus become a major obstacle for the construction of approximate solutions. As a result, the traditional matched asymptotic expansion method does not work for our problem. We remark that, in several sharp interface problems such as the scalar Allen–Cahn problem [13] or the isotropic-nematic interface problem [16], such difficulties do not exist, because any pair of points from the two potential wells of F is a minimal pair (that is, $F(u)$ is fully minimally paired, see Definition 3.5). In such cases any smooth extension will work, and the phase field equations on the two sides of the sharp interfaces are decoupled.

Now let us explain our main ideas.

To solve (1.15), a key idea is to construct a profile $\mathbf{A}_0(z, x, t)$ which fulfills the boundary conditions and "almost" satisfies (1.15). Here "almost" means the remainder

$$\mathbf{R}_{\mathbf{A}_0}(z, x, t) = -\partial_z^2 \mathbf{A}_0 + f(\mathbf{A}_0)$$

is small and can be absorbed into the equations of the next orders by using the relation of $z = d^\varepsilon/\varepsilon$. Precisely, we require that it has the form

$$\begin{aligned} &\mathbf{R}_{\mathbf{A}_0}(z, x, t) = d_0(x, t)\mathbf{G}(z, x, t) \\ &\qquad = \varepsilon(z - d_1 - \varepsilon d_2 - \varepsilon^2 d_3 - \cdots)\mathbf{G}(z, x, t), \\ &\text{with } \mathbf{G}(z, x, t) \text{ smooth, and } \mathbf{G}(z, x, t) = O(e^{-\alpha_0|z|}) \\ &\text{as } z \to \pm\infty \text{ for some } \alpha_0 > 0. \end{aligned} \tag{1.18}$$

Then it can be viewed as source terms for the systems of the next orders.

The requirement ensuring (1.17) to be solvable is much more restrictive. Applying a similar idea as above, one may expect to find a minimal connecting orbit $\widetilde{\mathbf{\Theta}}(\widetilde{\mathbf{A}}_+, \widetilde{\mathbf{A}}_-; z)$ such that the difference

$$\mathcal{L}_{\mathbf{A}_I^{(0)}} \mathbf{A}_I^{(k)} - \mathcal{L}_{\widetilde{\mathbf{\Theta}}} \mathbf{A}_I^{(k)}$$

can be absorbed into the expansions of the next orders. However, once $\big(\mathbf{A}_I^{(0)}(-\infty), \mathbf{A}_I^{(0)}(+\infty)\big)$ is not a minimal pair, it is hard, and in general impossible, to find such a $\widetilde{\mathbf{\Theta}}$ so that the difference decays exponentially to zero in z. Our idea is to introduce a profile for $(x, t) \in \Gamma(\delta)$:

$$\begin{aligned} &\mathbf{\Theta}(z, x, t) = \Phi(z, x, t)\mathbf{P}_0(z, x, t) \\ &\text{with} \quad \mathbf{P}_0(z, x, t) = \mathbf{I} - 2s(z)\mathbf{n}(x, t)\mathbf{n}(x, t), \end{aligned}$$

where $\mathbf{n}(x,t)$ is smoothly extended from Γ to $\Gamma(\delta)$ with $\partial_\nu \mathbf{n} = 0$ on Γ and

$$\begin{aligned}
&\Phi(-\infty, x, t) = \mathbf{A}_-(x,t),\\
&\Phi(+\infty, x, t) = \mathbf{A}_+(x,t)(\mathbf{I} - 2\mathbf{n}(x,t)\mathbf{n}(x,t)),\\
&\Phi(z,x,t) \in O^-(n), \quad \Phi(z,x,t) \to \Phi(\pm\infty, x, t)\\
&\text{exponentially as } z \to \pm\infty.
\end{aligned}$$

By choosing $\Phi(z,x,t)$ suitably, we can obtain $\partial_z^2\Phi, \partial_z\Phi = O(d_0^2 e^{-\alpha_0|z|})$. Thus

$$\mathbf{R}_{\mathbf{\Theta}}(z,x,t) = \partial_z^2\mathbf{\Theta} - f(\mathbf{\Theta}) = \partial_z^2\Phi\mathbf{P}_0 + 2\partial_z\Phi\partial_z\mathbf{P}_0$$

has the form in (1.18). A more crucial observation is that, if we write

$$\mathbf{A}_I^{(k)}(z,x,t) = \Phi(z,x,t)\mathbf{P}_k(z,x,t),$$

then we have

$$\mathcal{L}_{\mathbf{\Theta}}\mathbf{A}_I^{(k)} = \Phi\mathcal{L}_{\mathbf{P}_0}\mathbf{P}_k - \partial_z^2\Phi\mathbf{P}_k - 2\partial_z\Phi\partial_z\mathbf{P}_k.$$

Since $\mathbf{P}_0$ is a trivial minimal connecting orbit, $\mathcal{L}_{\mathbf{P}_0}$ can be explicitly inverted by diagonalizing it into several differential operators $\mathcal{L}_i (1 \leqslant i \leqslant 5)$ (cf. (4.5)) acting on scalar parameter functions. Thus, $\mathcal{L}_{\mathbf{\Theta}}$ is solvable up to some $O(d_0^2 e^{-\alpha_0|z|})$ terms, which can also be absorbed into the next higher order systems. These key properties enable us to construct solutions to the expansions of each order. Such profiles $\mathbf{\Theta}$ are called *quasi-minimal connecting orbits*, which play a crucial role in the whole construction of inner expansions. We refer to Sect. 3 for details on motivation and construction of quasi-minimal connecting orbits, and to Sect. 4 for diagonalizing of $\mathcal{L}_{\mathbf{P}_0}$.

The above procedure increases dramatically the complexity of the inner expansions, and thus the process for solving the system in the expansion has to be carefully examined. This will be accomplished in Sects. 5–6 with a sketched illustration in Fig. 1.

1.3.2 Spectral lower bound estimate

Another key ingredient in our analysis is to prove a spectral lower bound estimate for the linearized operator $-\Delta + \mathcal{H}_{\mathbf{A}^K}$, which is clearly more difficult than the scalar case. This is carried out in Sect. 7.

First, we restrict the inequality into the region near the interface (see (7.1)), as it holds on regions away from the interface by direct estimation. Then we introduce two transformations, one for coordinates and the other for matrix

fields, to reduce the problem into a matrix-valued inequality on a 1-D interval. See (7.6) in Sect. 7.1.

By using a decomposition based on the diagonalization of $\mathcal{L}_{\mathbf{P}_0}$ along with some further simplifications, a matrix-valued inequality can be reduced to two scalar bilinear estimates for singular crossing terms and correction terms respectively; see (7.15) and (7.17) in Sect. 7.2.

As these terms are both singular, they can not be estimated directly. To overcome this issue, we have to introduce several decompositions based on the eigenfunctions of the scalar linearized operators $\{\mathcal{L}_i (1 \leqslant i \leqslant 5)\}$. Then the singularities can be removed by careful analysis on the weights and by employing some delicate cancellation structures between these eigenfunctions.

Furthermore, the above decompositions also lead us to develop some weighted coercive estimates for the linearized operators. These coercive estimates give new and elementary proofs for the spectral estimates of the operators $\{\mathcal{L}_i\}$, which do not rely on the maximum/comparison principle or Harnack inequalities, and that might have their own interest.

Finally, we would like to remark that, although the analysis in this paper is carried out in the $O(n)$-setting, the ideas, on both construction of approximate solutions and spectral analysis, are rather flexible to be applied to other partially minimally paired Keller–Rubinstein–Sternberg problems.

1.4 Notations

- For any two matrices $\mathbf{A}$ and $\mathbf{B}$, we denote $\mathbf{A} : \mathbf{B} = \mathbf{A}_{ij}\mathbf{B}_{ij}$, $\|\mathbf{A}\|^2 = \mathbf{A} : \mathbf{A}$ and $\mathbf{A} \perp \mathbf{B}$ means $\mathbf{A} : \mathbf{B} = 0$.
- For any two vectors $\mathbf{m}$ and $\mathbf{n}$, we use $\mathbf{mn}$ to denote $\mathbf{m}\otimes\mathbf{n}$ when no ambiguity is possible.
- We use $\mathbf{nA}$ to denote the vector $(\mathbf{n}_j\mathbf{A}_{ji})_{1\leqslant i\leqslant n}$, and $\mathbf{An}$ to denote the vector $(\mathbf{A}_{ij}\mathbf{n}_j)_{1\leqslant i\leqslant n}$. Then $\mathbf{Amn}$ is understood as $(\mathbf{A}_{ik}\mathbf{m}_k\mathbf{n}_j)_{1\leqslant i,j\leqslant n}$ and similarly $\mathbf{mnA} = (\mathbf{m}_i\mathbf{n}_k\mathbf{A}_{kj})_{1\leqslant i,j\leqslant n}$.
- $\mathbb{M}_n$: the space of $n \times n$ matrices.
- $\mathbb{S}_n$, $\mathbb{A}_n$: the spaces of symmetric and asymmetric $n \times n$ matrices.
- $O(n)$, $O^{\pm}(n)$: $n \times n$ orthogonal group, the set of $n \times n$ orthogonal matrices with determinant ± 1.
- $O(e^{-\alpha_0|z|})$ denotes the terms which can be bounded by $C|z|^k e^{-\alpha_0|z|}$ for some $k \geqslant 0$ as $z \to \infty$.

The following simple fact (Jacobi identity) will be constantly used:

$$\text{for } \mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{M}_n, \quad (\mathbf{AB}) : \mathbf{C} = \mathbf{A} : (\mathbf{CB}^{\mathrm{T}}) = \mathbf{B} : (\mathbf{A}^{\mathrm{T}}\mathbf{C}). \tag{1.19}$$

2 Outer expansion

2.1 Formal outer expansion

We perform *outer expansion* in $Q_\pm$ rather than $Q_\pm \setminus \Gamma(\delta/2)$ by using the Hilbert expansion method as in [40,41]. Assume that

$$\mathbf{A}^\varepsilon(x,t) = \sum_{i=0}^{+\infty} \varepsilon^i \mathbf{A}_\pm^{(i)}(x,t), \qquad \text{for } (x,t) \in Q_\pm. \tag{2.1}$$

Substituting it into (1.3), one can find the leading order $O(\frac{1}{\varepsilon^2})$ equation reads as

$$\mathbf{A}_\pm^{(0)}(\mathbf{A}_\pm^{(0)})^{\mathrm{T}}\mathbf{A}_\pm^{(0)} = \mathbf{A}_\pm^{(0)},$$

which is satisfied by taking

$$\mathbf{A}_\pm^{(0)} = \mathbf{A}_\pm \in O^\pm(n). \tag{2.2}$$

Now we assume that

$$\mathbf{A}^\varepsilon(x,t) = \mathbf{A}_\pm \mathbf{U}^\varepsilon(x,t) = \mathbf{A}_\pm \sum_{i=0}^{+\infty} \varepsilon^i \mathbf{U}_\pm^{(i)}(x,t), \qquad \text{for } (x,t) \in Q_\pm.$$

Here $\mathbf{U}_\pm^{(0)} = \mathbf{I}$. A direct calculation yields that

$$\begin{aligned} \mathbf{A}_\pm^{\mathrm{T}} f(\mathbf{A}^\varepsilon) &= f(\mathbf{U}^\varepsilon) \\ &= \varepsilon\big(\mathbf{U}_\pm^{(1)} + (\mathbf{U}_\pm^{(1)})^{\mathrm{T}}\big) + \sum_{k\geqslant 1} \varepsilon^{k+1}\Big(\big(\mathbf{U}_\pm^{(k+1)} + (\mathbf{U}_\pm^{(k+1)})^{\mathrm{T}}\big) \\ &\quad + \mathbf{B}_\pm^{(k)} + \mathbf{C}_\pm^{(k-1)}\Big), \end{aligned} \tag{2.3}$$

where

$$\mathbf{B}_\pm^{(k)} = \sum_{\{i,j,l\}=\{0,1,k\}} \mathbf{U}_\pm^{(i)}\big(\mathbf{U}_\pm^{(j)}\big)^{\mathrm{T}}\mathbf{U}_\pm^{(l)}, \qquad \mathbf{C}_\pm^{(k-1)} = \sum_{\substack{i+j+l=k+1,\\ 0\leqslant i,j,l\leqslant k-1}} \mathbf{U}_\pm^{(i)}\big(\mathbf{U}_\pm^{(j)}\big)^{\mathrm{T}}\mathbf{U}_\pm^{(l)}. \tag{2.4}$$

Here and in what follows we use the convention that $\mathbf{U}_\pm^{(i)} = 0$ for $i < 0$. Note that

$$\mathbf{C}_{\pm}^{(0)} = 0, \quad \mathbf{C}_{\pm}^{(1)} = \mathbf{U}_{\pm}^{(1)}\big(\mathbf{U}_{\pm}^{(1)}\big)^{\mathrm{T}}\mathbf{U}_{\pm}^{(1)}, \tag{2.5}$$

and $\mathbf{C}_{\pm}^{(k-1)}$ only involves $\mathbf{U}_{\pm}^{(0)}$, $\mathbf{U}_{\pm}^{(1)}$, ..., $\mathbf{U}_{\pm}^{(k-1)}$.

On the other hand, we define the linear operator

$$\begin{aligned}\mathcal{J}_{\pm}\Psi &= \mathbf{A}_{\pm}^{\mathrm{T}}\Big(\partial_t(\mathbf{A}_{\pm}\Psi) - \Delta(\mathbf{A}_{\pm}\Psi)\Big)\\ &= \partial_t\Psi - \Delta\Psi + \mathbf{A}_{\pm}^{\mathrm{T}}(\partial_t\mathbf{A}_{\pm} - \Delta\mathbf{A}_{\pm})\Psi - 2\mathbf{A}_{\pm}^{\mathrm{T}}\nabla\mathbf{A}_{\pm}\nabla\Psi.\end{aligned} \tag{2.6}$$

Then we have

$$\mathbf{A}_{\pm}^{\mathrm{T}}(\partial_t\mathbf{A}^{\varepsilon} - \Delta\mathbf{A}^{\varepsilon}) = \mathcal{J}_{\pm}\mathbf{U}^{\varepsilon}. \tag{2.7}$$

Substituting (2.3) and (2.7) into (1.3), and then equating the $O(\varepsilon^k)(k \geq -1)$ system yields that

$$O(\varepsilon^{-1}): \qquad \mathbf{U}_{\pm}^{(1)} + \big(\mathbf{U}_{\pm}^{(1)}\big)^{\mathrm{T}} = 0, \tag{2.8}$$

$$O(\varepsilon^k)(k \geqslant 0): \quad \mathbf{U}_{\pm}^{(k+2)} + \big(\mathbf{U}_{\pm}^{(k+2)}\big)^{\mathrm{T}} = -\mathcal{J}_{\pm}\mathbf{U}_{\pm}^{(k)} - \mathbf{B}_{\pm}^{(k+1)} - \mathbf{C}_{\pm}^{(k)}. \tag{2.9}$$

In the sequel, we will use the decomposition

$$\mathbf{M}_{\pm}^{(k)} = \frac{1}{2}\big(\mathbf{U}_{\pm}^{(k)} + (\mathbf{U}_{\pm}^{(k)})^{\mathrm{T}}\big) \in \mathbb{S}_n, \quad \mathbf{V}_{\pm}^{(k)} = \frac{1}{2}\big(\mathbf{U}_{\pm}^{(k)} - (\mathbf{U}_{\pm}^{(k)})^{\mathrm{T}}\big) \in \mathbb{A}_n,$$

and solve $\mathbf{M}_{\pm}^{(k)}$, $\mathbf{V}_{\pm}^{(k)}$ separately.

2.2 The leading order equation

It yields from (2.8) that

$$\mathbf{M}_{\pm}^{(1)} = 0. \tag{2.10}$$

The equation (2.9) for $k = 0$ gives us that

$$2\mathbf{M}_{\pm}^{(2)} = -\mathcal{J}_{\pm}\mathbf{U}_{\pm}^{(0)} - \mathbf{B}_{\pm}^{(1)},$$

which leads to

$$\mathcal{J}_{\pm}\mathbf{U}_{\pm}^{(0)} + \mathbf{B}_{\pm}^{(1)} \in \mathbb{S}_n.$$

Since $\mathbf{M}_{\pm}^{(1)} = 0$, we have that $\mathbf{U}_{\pm}^{(1)} = \mathbf{V}_{\pm}^{(1)}$ is antisymmetric, and thus from (2.4):

$$\mathbf{B}_{\pm}^{(1)} = \mathbf{V}_{\pm}^{(1)}(\mathbf{V}_{\pm}^{(1)})^{\mathrm{T}} = -(\mathbf{V}_{\pm}^{(1)})^2 \in \mathbb{S}_n, \tag{2.11}$$

which implies

$$\mathcal{J}_{\pm}\mathbf{U}_{\pm}^{(0)} \in \mathbb{S}_n. \tag{2.12}$$

As $\mathbf{U}_{\pm}^{(0)} = \mathbf{I}$, we deduce from (2.7) that

$$\mathbf{A}_{\pm}^{\mathrm{T}}(\partial_t \mathbf{A}_{\pm} - \Delta \mathbf{A}_{\pm}) \in \mathbb{S}_n,$$

which is actually equivalent to **the heat flow of harmonic maps to** $O^{\pm}(n)$ given in (1.6a).

2.3 The next order equations

For general k, (2.9) can be equivalently written as

$$\mathcal{J}_{\pm}\mathbf{U}_{\pm}^{(k)} + \mathbf{B}_{\pm}^{(k+1)} + \mathbf{C}_{\pm}^{(k)} \in \mathbb{S}_n, \tag{2.13}$$

$$\mathbf{M}_{\pm}^{(k+2)} = -\frac{1}{2}\Big(\mathcal{J}_{\pm}\mathbf{U}_{\pm}^{(k)} + \mathbf{B}_{\pm}^{(k+1)} + \mathbf{C}_{\pm}^{(k)}\Big). \tag{2.14}$$

Equation (2.14) implies that $\mathbf{M}_{\pm}^{(i)}$ is uniquely determined from $\mathbf{A}_{\pm}^{(j)} (0 \leqslant j \leqslant i-1)$ for $i \geqslant 2$.

From the definition (2.4) of $\mathbf{B}_{\pm}^{(k)}$ and (2.10), we have for $k \geqslant 2$,

$$\mathbf{B}_{\pm}^{(k)} = \mathbf{V}_{\pm}^{(1)}(\mathbf{U}_{\pm}^{(k)})^{\mathrm{T}} + (\mathbf{U}_{\pm}^{(k)})^{\mathrm{T}}\mathbf{V}_{\pm}^{(1)} \triangleq \mathcal{B}_{\pm}\mathbf{U}_{\pm}^{(k)}. \tag{2.15}$$

One can directly verify that

$$\mathcal{B}_{\pm}\mathbf{V} \in \mathbb{S}_n, \quad \text{for } \mathbf{V} \in \mathbb{A}_n. \tag{2.16}$$

Therefore, it follows from (2.13) and (2.16) that

$$\mathcal{J}_{\pm}\mathbf{U}_{\pm}^{(k)} + \mathcal{B}_{\pm}\mathbf{M}_{\pm}^{(k+1)} + \mathbf{C}_{\pm}^{(k)} \in \mathbb{S}_n,$$

which further gives

$$\mathcal{J}_{\pm}\mathbf{U}_{\pm}^{(k)} - \frac{1}{2}\mathcal{B}_{\pm}\Big(\mathcal{J}_{\pm}\mathbf{U}_{\pm}^{(k-1)} + \mathbf{B}_{\pm}^{(k)} + \mathbf{C}_{\pm}^{(k-1)}\Big) + \mathbf{C}_{\pm}^{(k)} \in \mathbb{S}_n. \tag{2.17}$$

For $k \geqslant 2$, once $\{\mathbf{U}_\pm^{(i)}\}_{0\leqslant i\leqslant k-1}$ is determined, the above equation indeed gives a heat flow type evolution equation for $\mathbf{V}_\pm^{(k)}$:

$$\begin{aligned}&\mathcal{J}_\pm \mathbf{V}_\pm^{(k)} - \frac{1}{2}\mathcal{B}_\pm^2 \mathbf{V}_\pm^{(k)} + \mathbf{C}_\pm^{(k)} + \mathcal{J}_\pm \mathbf{M}_\pm^{(k)} \\ &\quad - \frac{1}{2}\mathcal{B}_\pm\Big(\mathcal{J}_\pm \mathbf{U}_\pm^{(k-1)} + \mathcal{B}_\pm \mathbf{M}_\pm^{(k)} + \mathbf{C}_\pm^{(k-1)}\Big) \in \mathbb{S}_n,\end{aligned} \tag{2.18}$$

since $\mathbf{M}_\pm^{(k)}$ is already given by (2.14).

In addition, when $k \geqslant 3$, the equation (2.18) is linear for $\mathbf{V}_\pm^{(k)}$, since the coefficients in the operator $\mathcal{B}_\pm$ (see (2.15)) and $\mathbf{C}_\pm^{(k)}$ depend only on $\mathbf{U}_\pm^{(1)}$ and $\mathbf{U}_\pm^{(2)}$.

For $k = 1$ or 2, (2.17) or (2.18) seems to be nonlinear at a first glance. However, by a careful checking, we could find that it is also a linear equation for $k = 1$ or 2.

Indeed, for $k = 1$, from (2.11), (2.15) and (2.5) one has that

$$\mathcal{B}_\pm \mathbf{B}_\pm^{(1)} = \mathbf{V}_\pm^{(1)}(\mathbf{B}_\pm^{(1)})^{\mathrm{T}} + (\mathbf{B}_\pm^{(1)})^{\mathrm{T}}\mathbf{V}_\pm^{(1)} = -2(\mathbf{V}_\pm^{(1)})^3 = 2\mathbf{C}_\pm^{(1)}.$$

Thus, the equation (2.17) for $k = 1$ is reduced to

$$\mathcal{J}_\pm \mathbf{V}_\pm^{(1)} - \frac{1}{2}\mathcal{B}_\pm \mathcal{J}_\pm \mathbf{U}_\pm^{(0)} \in \mathbb{S}_n, \tag{2.19}$$

which is apparently a linear equation for $\mathbf{V}_\pm^{(1)}$.

For $k = 2$, the only nonlinear (in $\mathbf{V}_\pm^{(2)}$) terms are contained in $\mathbf{C}_\pm^{(2)}$, which can be written as

$$\begin{aligned}&\mathbf{U}_\pm^{(0)}\big(\mathbf{U}_\pm^{(2)}\big)^{\mathrm{T}}\mathbf{U}_\pm^{(2)} + \mathbf{U}_\pm^{(2)}\big(\mathbf{U}_\pm^{(0)}\big)^{\mathrm{T}}\mathbf{U}_\pm^{(2)} + \mathbf{U}_\pm^{(2)}\big(\mathbf{U}_\pm^{(2)}\big)^{\mathrm{T}}\mathbf{U}_\pm^{(0)} \\ &\quad = \big(\mathbf{V}_\pm^{(2)}\big)^{\mathrm{T}}\mathbf{V}_\pm^{(2)} + \mathbf{V}_\pm^{(2)}\big(\mathbf{V}_\pm^{(2)}\big)^{\mathrm{T}} + (\mathbf{V}_\pm^{(2)})^2 + \text{ linear terms} \\ &\quad = \text{ symmetric terms } + \text{ linear terms }.\end{aligned}$$

Therefore, by eliminating symmetric terms, (2.18) for $k = 2$ is indeed a linear equation of $\mathbf{V}_\pm^{(2)}$.

Note that for each $\mathbf{A}_\pm^{(k)}$ or $\mathbf{U}_\pm^{(k)}$, the symmetric part $\mathbf{M}_\pm^{(k)}$ is solved explicitly from (2.14). Thus, we do not need boundary/jump conditions for $\mathbf{M}_\pm^{(k)}$ on Γ_t. While, the antisymmetric part $\mathbf{V}_\pm^{(k)}$ is solved from a linear heat-flow type equation, thus their boundary/jump conditions on Γ_t are needed. These conditions will be determined in the inner expansion to ensure that outer/inner expansions match each other in the overlap region.

Once $\mathbf{A}_\pm|_{Q_\pm}$, $\mathbf{M}_\pm^{(k)}|_{Q_\pm}$, $\mathbf{V}_\pm^{(k)}|_{Q_\pm}$ are determined, we extend them to $\Gamma(\delta)$, such that

$$\mathbf{A}_\pm : \Gamma(\delta) \cup Q_\pm \mapsto O^\pm(n), \quad \mathbf{M}_\pm^{(k)} : \Gamma(\delta) \cup Q_\pm \mapsto \mathbb{S}_n,$$
$$\mathbf{V}_\pm^{(k)} : \Gamma(\delta) \cup Q_\pm \mapsto \mathbb{A}_n$$

are all smooth functions. Then $\mathbf{A}_\pm^{(k)}(x,t) = \mathbf{A}_\pm(\mathbf{M}_\pm^{(k)} + \mathbf{V}_\pm^{(k)})$ are also smooth in $\Gamma(\delta) \cup Q_\pm$.

3 Minimal pair and quasi-minimal connecting orbits

3.1 Motivation

The aim of the inner expansion is to find a good approximation, up to any order of ε, to the exact solution in the region $\Gamma(\delta)$ near the interface. As explained in Sect. 1.3, the main strategy used here is that, we try to find functions $\mathbf{A}_I^{(k)}(z,x,t)$ and $d_k(x,t)$ $(k \geqslant 0)$ such that

$$\widetilde{\mathbf{A}}_I^\varepsilon(x,t) = \mathbf{A}_I^\varepsilon\big(\varepsilon^{-1}d^\varepsilon(x,t), x, t\big)$$

solves the original equation (1.3) in $\Gamma(\delta)$, with $\mathbf{A}_I^\varepsilon\big(z,x,t)$ defined in (1.13) and $d^\varepsilon(x,t)$ given by (1.14) which is a signed distance function with respect to a surface Γ_t^ε. In addition, we require that, for $(x,t) \in \Gamma(\delta)$, $i, j, k \geqslant 0$ and some $\alpha_0 > 0$,

$$|\partial_t^i \partial_x^j \partial_z^l (\mathbf{A}_I^{(k)}(z,x,t) - \mathbf{A}_\pm^{(k)}(x,t))| = O(e^{-\alpha_0|z|}) \ \text{ as } z \to \pm\infty. \tag{3.1}$$

Since we would like to approximate the sharp interface system (1.6), $d_0(x,t)$ is naturally taken as the signed distance function to Γ_t. Thus, $\nabla d_0 \cdot \nabla$ on Γ_t is the normal derivative ∂_ν. In addition, if ε is sufficiently small, Γ_t^ε should be a good approximation of Γ_t. As d^ε is a signed distance function, one has $|\nabla d^\varepsilon|^2 = 1$, which gives

$$|\nabla d_0|^2 = 1, \quad \nabla d_0 \cdot \nabla d_1 = 0, \tag{3.2}$$
$$2\nabla d_0 \cdot \nabla d_k = -\sum_{1 \leqslant j \leqslant k-1} \nabla d_j \cdot \nabla d_{k-j}, \quad k \geq 2. \tag{3.3}$$

Substituting the expansions (1.13)–(1.14) into the following equation

$$\partial_t \mathbf{A}^\varepsilon = \Delta \mathbf{A}^\varepsilon - \varepsilon^{-2} f(\mathbf{A}^\varepsilon),$$

we find that, to eliminate the leading $O(\frac{1}{\varepsilon^2})$ order terms, $\mathbf{A} = \mathbf{A}_I^{(0)}$ should satisfy

$$\partial_z^2 \mathbf{A} = f(\mathbf{A}), \qquad \mathbf{A}(\pm\infty) = \mathbf{A}_\pm(x,t). \tag{3.4}$$

In addition, the $O(\varepsilon^{k-2})$ $(k \geqslant 1)$ system gives that $\mathbf{A}_I^{(k)}$ satisfies an equation with the form

$$\mathcal{L}_{\mathbf{A}} \mathbf{A}_I^{(k)} = \mathbf{F}, \qquad \mathbf{A}_I^{(k)}(\pm\infty) = \mathbf{A}_\pm^{(k)}(x,t), \tag{3.5}$$

where $\mathbf{F}$ contains terms determined before, and $\mathcal{L}_{\mathbf{A}}$ is the linearized operator of (3.4) around $\mathbf{A}$ defined by

$$\mathcal{L}_{\mathbf{A}} \Psi := -\partial_z^2 \Psi + \mathbf{A}\mathbf{A}^{\mathrm{T}}\Psi + \mathbf{A}\Psi^{\mathrm{T}}\mathbf{A} + \Psi\mathbf{A}^{\mathrm{T}}\mathbf{A} - \Psi. \tag{3.6}$$

Therefore, the existence of solutions to the systems (3.4) and (3.5) are at the heart of the inner expansion. As we will show in Sects. 3.2 and 4, when $(\mathbf{A}_+, \mathbf{A}_-)$ is a minimal pair, or equivalently, $\mathbf{A}_+ = \mathbf{A}_-(\mathbf{I} - 2\mathbf{nn})$ for some $\mathbf{n} \in S^{n-1}$, one can find directly a solution $\mathbf{A}$ to (3.4) with (3.5) being solvable.

The boundary condition (1.6c) gives us that $(\mathbf{A}_+(x,t), \mathbf{A}_-(x,t))$ forms a minimal pair for $(x,t) \in \Gamma$. However, after a smooth extension, $(\mathbf{A}_+(x,t), \mathbf{A}_-(x,t))$ may not be a minimal pair in general for $(x,t) \in \Gamma(\delta)$. This is the main obstacle to the construction of approximated solutions in the inner expansion. To overcome this difficulty, we construct a solution $\boldsymbol{\Theta}$ which satisfies (3.4) up to some "good" remainders, which decay exponentially fast in z-variable and vanish on Γ. More importantly, the corresponding $\mathcal{L}_{\boldsymbol{\Theta}}$ is also solvable up to some "good" remainders. Such a solution is called *quasi-minimal connecting orbit* (see Sect. 3.3).

3.2 Minimal pair and minimal connecting orbits

We start from a general nonnegative smooth potential function $F : \mathbb{R}^N \to \mathbb{R}_{\geqslant 0}$ which vanishes exactly on two disjoint, compact, connected, smooth Riemannian submanifolds $\Sigma^\pm \subset \mathbb{R}^N$ without boundaries. A simple choice of such potential function $F(u)$ is giving by the square of the distance from u to $\Sigma^+ \cup \Sigma^-$, for u near $\Sigma^+ \cup \Sigma^-$, otherwise it could be a positive constant, see for example [26]. Giving two points $p_\pm \in \Sigma^\pm$, the solution of the following ODE

$$\partial_z^2 u = \partial_u F, \qquad u(\pm\infty) = p_\pm, \tag{3.7}$$

describes the way of phase transition from the state p_- to another state p_+. The existence of solutions to (3.7) is so called the *heteroclinic connection*

problem, which has been studied extensively for the case of $\Sigma_\pm = \{p_\pm\}$; see [20,32,38,42] and the references therein for examples.

Definition 3.1 A solution of (3.7) is called a *connecting orbit*, and $p_\pm$ are called its *ends*.

In particular, we are interested in minimal connecting orbits, which is defined as follows.

Definition 3.2 A solution of (3.7) is called a *minimal connecting orbit* [26], if it minimizes the energy

$$\mathfrak{e}(u) = \int_{\mathbb{R}} \frac{1}{2}|u'|^2 + F(u)\mathrm{d}z \tag{3.8}$$

among $H^1_{p_\pm}(\mathbb{R}) := \{u \in H^1_{loc}(\mathbb{R}) : \lim_{z\to\pm\infty} u = p_\pm\}$.

Remark 3.3 Let the trajectory of u be defined as $\mathrm{Traj}(u) = \{u(z)| -\infty < z < +\infty\}$. By using an argument as in [32,42], one can show that $u(z)$ is a minimal connecting orbit, only if:

(1) The closure of $\mathrm{Traj}(u)$ is a minimal geodesic curve connecting (p_-, p_+) with the weight $\sqrt{2F}$ in $\mathbb{R}^N$;
(2) $\mathrm{Traj}(u)$ contains no other points in $\Sigma_- \cup \Sigma_+$.

Indeed, for $\gamma = \mathrm{Traj}(w)$ with $w \in C(I) \cap H^1_{loc}(I)$ (I is a interval in $\mathbb{R}$), we define

$$e_F(\gamma) := \int_I \sqrt{2F(w(z))}|w'(z)|\mathrm{d}z,$$

which is independent of the parametrization of γ. One can deduce that for $u \in C^1(\Omega)$ with $u(\pm\infty) \in \Sigma_\pm$, $\mathfrak{e}(u) \geqslant e_F(\mathrm{Traj}(u))$, and the equality holds only if $\frac{1}{2}|\partial_z u|^2 = F(u)$. This is a simple example of the so-called self-dual solutions. If u solves (3.7), then $\partial_z(\frac{1}{2}|\partial_z u|^2 - F(u)) = 0$ and hence $\frac{1}{2}|\partial_z u|^2 = F(u)$. Thus $e_F(\mathrm{Traj}(u)) = \mathfrak{e}(u)$. As u minimizes $\mathfrak{e}$ with given ends (p_-, p_+), one has that $\mathrm{Traj}(u)$ minimizes e_F with given ends (p_-, p_+). Otherwise one can find another w with same ends $p_\pm$ such that $e_F(\mathrm{Traj}(w)) < e_F(\mathrm{Traj}(u))$. By suitably regularizing and reparametrizing w, we could assume $w \in H^1_{p_\pm}(\mathbb{R})$ and obtain that $\mathfrak{e}(w) < \mathfrak{e}(u)$ which contradicts with the minimality of u.

In addition, if u is a solution to (3.7) and $\mathrm{Traj}(u)$ contains some other point in $\Sigma_\pm$, we can assume $u(z_0) = p_0 \in \Sigma_+ \cup \Sigma_-$. Then $(u(z), u'(z))(z \geqslant z_0)$ is solution to the ODE system

$$w_1' = w_2, \quad w_2' = \partial_u F(w_1), \quad \text{for } z \geqslant z_0$$

with initial data $w_1(z_0) = p_0$, $w_2(z_0) = 0$. However, this system obviously admits a trivial solution $w_1(z) \equiv p_0$, $w_2(z) \equiv 0$, which contradicts with the uniqueness of solution (note that $\partial_u F$ is Lipschitz).

Conversely, for a minimizing geodesic (with weight $\sqrt{2F}$) connecting $(p_-, p_+) \in \Sigma_- \times \Sigma_+$ which does not visit some other point in $\Sigma_- \cup \Sigma_+$, one can suitably reparametrize it to obtain a minimal connecting orbit, which solves (3.7). We refer to the proof of [42, Theorem 3.1] or [32, Theorem 3] for details.

With some mild assumptions on the potential function F, the existences of minimal connecting orbits can be proved by variational methods as in [26, 32,42]. However, it is well-known that, for a general given pair $(p_-, p_+) \in \Sigma_- \times \Sigma_+$, a minimal connecting orbit (and even a connecting orbit) with ends $p_\pm$ may not exist.

Definition 3.4 A pair $(p_-, p_+) \in \Sigma_- \times \Sigma_+$ connected by a minimal connecting orbit is called a *minimal pair*.

Definition 3.5 If any pair $(p_-, p_+) \in \Sigma_- \times \Sigma_+$ is a minimal pair, we say that F is *fully minimally paired*. Otherwise, i.e., if there exists $(p_-, p_+) \in \Sigma_- \times \Sigma_+$ which is not a minimal pair, we say that F is *partially minimally paired*.

Most of classic models studied previously are fully minimally paired. For example, for the scalar Allen–Cahn energy $F = \frac{1}{4}(1-u^2)^2 : \mathbb{R} \to \mathbb{R}_{\geqslant 0}$, $\Sigma_\pm = \{\pm 1\}$. Obviously, F is fully minimally paired. For the isotropic-nematic phase transition problem in liquid crystals [16,22,28], the energy $F : \mathbb{Q} \to \mathbb{R}$ ($\mathbb{Q}$ denotes the space of 3×3 symmetric trace free matrices) takes the form:

$$F(\mathbf{Q}) = \frac{a}{2}|\mathbf{Q}|^2 - \frac{b}{3}\mathrm{tr}\mathbf{Q}^3 + \frac{c}{4}|\mathbf{Q}|^4, \quad a, b, c > 0,\ b^2 = 27ac.$$

One has

$$\Sigma_- = \{0\}, \quad \Sigma_+ = \left\{ s_+\left(\mathbf{nn} - \frac{1}{3}\mathbf{I}\right) : \mathbf{n} \in S^2, s_+ = \frac{b + \sqrt{b^2 - 24ac}}{4c} \right\}.$$

As $(0, \mathbf{Q}_*)$ is a minimal pair for $\forall\ \mathbf{Q}_* \in \Sigma_+$, F is fully minimally paired.

Another geometric example is that $\Sigma_\pm$ are the linked spheres S^k, S^l in S^{k+l+1}:

$$F(u) = \Big((|u_1|^2 - 1)^2 + |u_2|^2\Big)\Big(|u_1|^2 + (|u_2|^2 - 1)^2\Big), \qquad u = (u_1, u_2) \in \mathbb{R}^{k+1} \times \mathbb{R}^{l+1}.$$

Clearly, this problem is fully minimally paired. It seems hopeful to generalize the relative entropy arguments [18] for the fully minimally paired case to

obtain a short and elegant proof of the main result, see for example [22]. On the other hand, the class of $F(u)$ considered in [26] is at the exactly other extreme. In the latter case, there are two compact, smooth sub-manifolds $M_\pm$ of $\Sigma_\pm$ respectively, such that there is a smooth diffeomorphism between points p_-, p_+ in $M_\pm$ so that the corresponding points p_-, p_+ form a minimal pair.

For the problem (1.3) considered in this paper, $F = \frac{1}{4}\|\mathbf{A}\mathbf{A}^{\mathrm{T}} - \mathbf{I}\|^2$, and the equation (3.7) becomes

$$\partial_z^2 \mathbf{A} = \mathbf{A}\mathbf{A}^{\mathrm{T}}\mathbf{A} - \mathbf{A}, \qquad \mathbf{A}(\pm\infty) = \mathbf{A}_\pm \in O^\pm(n), \tag{3.9}$$

which is the Euler-Lagrange equation to the one dimensional energy functional:

$$\int_{\mathbb{R}} \frac{1}{2}\|\partial_z \mathbf{A}\|^2 + \frac{1}{4}\|\mathbf{A}\mathbf{A}^{\mathrm{T}} - \mathbf{I}\|^2 \mathrm{d}z. \tag{3.10}$$

In this case, we have an explicit characterization of minimal pairs, which implies that this problem is partially minimally paired unless $n = 2$.

Lemma 3.6 *For $(\mathbf{A}_+, \mathbf{A}_-) \in O^+(n) \times O^-(n)$, the following statements are equivalent:*

(i) $\|\mathbf{A}_+ - \mathbf{A}_-\| = \min_{(\mathbf{A},\mathbf{B})\in O^+(n)\times O^-(n)} \|\mathbf{A} - \mathbf{B}\|$;
(ii) $\|\mathbf{A}_+ - \mathbf{A}_-\| = 2$;
(iii) $\mathbf{A}_- = \mathbf{A}_+(\mathbf{I} - 2\mathbf{n} \otimes \mathbf{n})$ *for some* $\mathbf{n} \in S^{n-1}$;

Proof First, we show that, if $\mathbf{B} \in O^-(n)$, then $\mathrm{tr}\,\mathbf{B} \leqslant n-2$, and equality holds only when $\mathbf{B} = \mathbf{I} - 2\mathbf{n}\otimes\mathbf{n}$ for some $\mathbf{n} \in S^{n-1}$. For this, we assume $\mathbf{B} = \mathbf{R}\tilde{\mathbf{B}}\mathbf{R}^{\mathrm{T}}$ where $\mathbf{R} \in O(n)$ and $\tilde{\mathbf{B}} = \mathrm{diag}\,\{\lambda_1, \lambda_2, \cdots, \lambda_k, J_1, \cdots, J_l\}(k + 2j = n)$ is quasi-diagonal with $\lambda_i \in \{\pm 1\}(1 \leqslant i \leqslant k)$, $J_j = \begin{pmatrix} \cos\theta_j & \sin\theta_j \\ -\sin\theta_j & \cos\theta_j \end{pmatrix} (1 \leqslant j \leqslant l)$. As $\det\tilde{\mathbf{B}} = \det\mathbf{B} = -1$, at least one of λ_i equals to -1. Thus, $\mathrm{tr}\,\mathbf{B} \leqslant n-2$. Equality holds only if $\tilde{\mathbf{B}} = \mathrm{diag}\,\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ and only one of λ_i takes value -1, which implies $\mathbf{B} = \mathbf{I} - 2\mathbf{n}\otimes\mathbf{n}$ for some $\mathbf{n} \in S^{n-1}$.

From the above claim, as $\mathbf{A}_+^{\mathrm{T}}\mathbf{A}_- \in O^-(n)$, we have $\|\mathbf{A}_- - \mathbf{A}_+\|^2 = \mathrm{tr}\,((\mathbf{A}_- - \mathbf{A}_+)^{\mathrm{T}}(\mathbf{A}_- - \mathbf{A}_+)) = \mathrm{tr}\,(2\mathbf{I} - \mathbf{A}_+^{\mathrm{T}}\mathbf{A}_- - \mathbf{A}_-^{\mathrm{T}}\mathbf{A}_+) = 2n - 2\mathrm{tr}\,\mathbf{A}_+^{\mathrm{T}}\mathbf{A}_- \geqslant 4$. In addition, equality holds if and only if $\mathbf{A}_+^{\mathrm{T}}\mathbf{A}_- = \mathbf{I} - 2\mathbf{n}\mathbf{n}$ with some $\mathbf{n} \in S^{n-1}$, which concludes the lemma. □

Lemma 3.7 *A pair $(\mathbf{A}_-, \mathbf{A}_+) \in O^-(n) \times O^+(n)$ is a* minimal pair, *if and only if $\|\mathbf{A}_+ - \mathbf{A}_-\| = 2$.*

Proof See Appendix A.2. □

Remark 3.8 Lemmas 3.6 and 3.7 imply that $(\mathbf{A}_-, \mathbf{A}_+) \in O^-(n) \times O^+(n)$ is a minimal pair if and only if $\mathbf{A}_+ - \mathbf{A}_- = 2(\mathbf{A}_+\mathbf{n}) \otimes \mathbf{n}$ for some $\mathbf{n} \in S^{n-1}$. This implies that $\mathbf{A}_+$ and $\mathbf{A}_-$ are rank-one connected which is a crucial condition in solid mechanics (e.g., the seminal work [6] of Ball-James). A good reason for this to be the case may be because they all involve phase transitions between two minimums of potential energies. For our problem, it implies formally that (at the blow-up) tangent planes of the sharp interface are described by rank-one connections. One of the difficulties in the Keller–Rubinstein–Sternberg's problem is that there are infinitely many such rank-one connections (if only one of the two endpoints is given) which can be parametrized by the sphere S^{n-1}.

Let

$$s(z) = 1 - (1 + e^{\sqrt{2}z})^{-1}, \tag{3.11}$$

which solves

$$s'' = 2s(1-s)(1-2s) \quad \text{for } z \in \mathbb{R}; \quad s(+\infty) = 1,\ s(-\infty) = 0. \tag{3.12}$$

Apparently, for $k \geqslant 0$, $\partial_z^k(s(z) - s(\pm\infty)) = O(e^{-\sqrt{2}|z|})$ as $z \to \pm\infty$. In the sequel, we will choose $\alpha_0 \in (0, \sqrt{2}]$. Note that all solutions of (3.11) are given by $\{s_\tau(z) = s(z+\tau) : \tau \in \mathbb{R}\}$.

Lemma 3.9 *All minimal connecting orbits are given by*

$$\mathbf{\Theta}_\tau(\mathbf{A}_+, \mathbf{A}_-; z) := s_\tau(z)\mathbf{A}_+ + (1 - s_\tau(z))\mathbf{A}_-, \tag{3.13}$$

with $(\mathbf{A}_+, \mathbf{A}_-)$ *being a minimal pair and* $s(z)$ *defined in (3.11).*

Proof We defer the proof to Appendix A.2. □

Remark 3.10 We remark that, for any pair $(\mathbf{A}_-, \mathbf{A}_+) \in O^-(n) \times O^+(n)$ with $\mathbf{A}_+^{\mathrm{T}}\mathbf{A}_-$ symmetric, $\mathbf{\Theta}_\tau(\mathbf{A}_+, \mathbf{A}_-; z)$ defined in (3.13) are solutions to (3.9), i.e., connecting orbits. However, only if $\mathbf{A}_+^{\mathrm{T}}\mathbf{A}_- = \mathbf{I} - 2\mathbf{nn}$ for some $\mathbf{n} \in S^{n-1}$, $\mathbf{\Theta}_\tau(\mathbf{A}_+, \mathbf{A}_-; z)$ are minimal connecting orbits. In this example, the dimension of $O(n)$ is $n(n-1)/2$, its symmetric group is S^{n-1}. Thus for every point p_+ in $O^+(n)$ there is an embedded S^{n-1} in $O^-(n)$ which is minimum (and equal) distance to p_+. The similar statement is also true in the other way for points in $O^-(n)$. This is a rather interesting (and also typical at least locally) partially minimally paired situation, and leads to many mixed type boundary conditions.

3.3 Quasi-minimal connecting orbit

As discussed at the beginning of this section, in general, for $(x,t) \in \Gamma(\delta) \setminus \Gamma$, one can not expect that $(\mathbf{A}_-(x,t), \mathbf{A}_+(x,t))$ is a minimal pair. Thus, the solution to (3.4) may not exist. To this end, we construct a profile $\boldsymbol{\Theta}$ which approximately satisfies (3.4) for $(x,t) \in \Gamma(\delta)$.

We assume that there exists a smooth vector field $\mathbf{n}(x,t) : \Gamma \to S^{n-1}$ such that $\mathbf{A}_- = \mathbf{A}_+(\mathbf{I} - 2\mathbf{nn})$ on Γ (in general this assumption may be not true and this issue will be discussed in Remark 6.1). Then we extend $\mathbf{n}(x,t)$ to be a smooth S^{n-1}-valued function in $\Gamma(\delta)$ with $\partial_\nu \mathbf{n} = 0$ on Γ. Define smooth orthogonal matrices

$$\Phi_-(x,t) = \mathbf{A}_-(x,t), \quad \Phi_+(x,t) = \mathbf{A}_+(x,t)(\mathbf{I} - 2\mathbf{n}(x,t)\mathbf{n}(x,t)),$$
$$\text{for } (x,t) \in \Gamma(\delta).$$

It holds that $\Phi_+(x,t) = \Phi_-(x,t)$ on the interface Γ. Moreover, as $\partial_\nu \mathbf{n}(x,t) = 0$ on Γ, the boundary condition (1.6d) ensures that

$$\partial_\nu \Phi_+(x,t) = \partial_\nu \Phi_-(x,t) \quad \text{for } (x,t) \in \Gamma,$$

which implies that

$$\|\Phi_+(x,t) - \Phi_-(x,t)\| \leqslant C d_0^2(x,t) \quad \text{for } (x,t) \in \Gamma(\delta).$$

This quadratical vanishing property near the interface is very important.

Let $\overline{\Phi}(x,t;\tau)(\tau \in [0,1]$ be a geodesic on $O^-(n)$ connecting $\Phi_-(x,t)$ and $\Phi_+(x,t)$:

$$\overline{\Phi}(x,t;0) = \Phi_-(x,t), \quad \overline{\Phi}(x,t;1) = \Phi_+(x,t),$$
$$\|\partial_\tau \overline{\Phi}(x,t;\tau)\| = \text{const. for } \tau \in (0,1).$$

Then we reparameterize the geodesic as

$$\Phi(z,x,t) = \overline{\Phi}(x,t;\bar{\eta}(z)), \tag{3.14}$$

where $\bar{\eta}(z)$ is a monotonic increasing function which tends to 0 (or 1) exponentially fast as $z \to -\infty$ (or $+\infty$). In particular, we can choose $\bar{\eta}(z) = s(z)$. Apparently, one has that for $k \geqslant 0$

$$\|\partial_z^k(\Phi(z,x,t) - \Phi_\pm(x,t))\| = O(e^{-\alpha_0|z|} d_0^2(x,t)), \text{ as } z \to \pm\infty, \ d_0 \to 0. \tag{3.15}$$

We define

$$\mathbf{\Theta}(z,x,t)=\Phi(z,x,t)\mathbf{P}_0(z,x,t)\quad\text{with}\quad\mathbf{P}_0=(\mathbf{I}-2s(z)\mathbf{n}(x,t)\mathbf{n}(x,t)). \tag{3.16}$$

Thus, one has

$$f(\mathbf{\Theta})=\mathbf{\Theta}\mathbf{\Theta}^{\mathrm{T}}\mathbf{\Theta}-\mathbf{\Theta}=4s(s-1)(1-2s)\Phi(z,x,t)\mathbf{nn},$$

which gives

$$\partial_z^2\mathbf{\Theta}-f(\mathbf{\Theta})=\partial_z^2\Phi(z,x,t)\mathbf{P}_0+2\partial_z\Phi(z,x,t)\partial_z\mathbf{P}_0.$$

Moreover, the linearized operator around $\mathbf{\Theta}$ can be written as

$$\mathcal{L}_{\mathbf{\Theta}}\mathbf{A}=\Phi(z,x,t)\mathcal{L}_{\mathbf{P}_0}\mathbf{P}+\partial_z^2\Phi(z,x,t)\mathbf{P}+2\partial_z\Phi(z,x,t)\partial_z\mathbf{P}$$

for $\mathbf{A}=\Phi\mathbf{P}$.

Due to the construction of Φ, $\partial_z^2\Phi(z,x,t)$, $\partial_z\Phi(z,x,t)$ are of order $O(e^{-\alpha_0|z|}d_0^2(x,t))$. Therefore, $\mathbf{\Theta}$ satisfies (3.4) up to some terms which decay exponentially fast in z-variable and quadratically vanish on Γ. More importantly, as $\Phi\in O^-(n)$ is invertible and $\mathcal{L}_{\mathbf{P}_0}$ is solvable (see Sect. 4 below), $\mathcal{L}_{\mathbf{\Theta}}$ is also solvable up to some small remainders. These crucial properties enable us to modify the original equation and solve the expanding systems exactly.

The profile $\mathbf{\Theta}$ defined in (3.16) is called a *quasi-minimal connecting orbit*, and we will use it as the leading order approximation in the inner region.

4 Diagonalization of the linearized operator

To solve the linearized ODE system

$$\mathcal{L}_{\mathbf{P}_0}\mathbf{P}:=-\partial_z^2\mathbf{P}+\mathbf{P}_0\mathbf{P}_0^{\mathrm{T}}\mathbf{P}+\mathbf{P}_0\mathbf{P}^{\mathrm{T}}\mathbf{P}_0+\mathbf{P}\mathbf{P}_0^{\mathrm{T}}\mathbf{P}_0-\mathbf{P}=\mathbf{F}.$$

with $\mathbf{P}_0(z,x,t)=\mathbf{I}-2s(z)\mathbf{n}(x,t)\mathbf{n}(x,t)$ and $\mathbf{n}(x,t)\in S^{n-1}$, we need to make a diagonalization to $\mathcal{L}_{\mathbf{P}_0}$. Here and in what follows, we simply write $\mathcal{L}_{\mathbf{P}_0}$ as $\mathcal{L}$ when no ambiguity is caused.

4.1 An orthogonal decomposition of $\mathbb{M}_n$

For $\mathbf{n} = \mathbf{n}(x,t)$ fixed, we introduce

$$\begin{aligned}
&\mathbb{V}_1 = \big\{\lambda\mathbf{nn}\big|\ \lambda\in\mathbb{R}\big\},\ \ \mathbb{V}_2 = \big\{(\mathbf{nl}+\mathbf{ln})\big|\ \mathbf{l}\cdot\mathbf{n}=0, \mathbf{l}\in\mathbb{R}^n\big\},\\
&\mathbb{V}_3 = \big\{(\mathbf{nl}-\mathbf{ln})\big|\ \mathbf{l}\cdot\mathbf{n}=0, \mathbf{l}\in\mathbb{R}^n\big\},\\
&\mathbb{V}_4 = \mathrm{span}\,\big\{(\mathbf{lm}-\mathbf{ml})\big|\ \mathbf{l}\cdot\mathbf{m}=\mathbf{l}\cdot\mathbf{n}=\mathbf{m}\cdot\mathbf{n}=0, \mathbf{l},\mathbf{m}\in\mathbb{R}^n\big\},\\
&\mathbb{V}_5 = \mathrm{span}\,\big\{\mathbf{ll}, (\mathbf{lm}+\mathbf{ml})\big|\ \mathbf{l}\cdot\mathbf{m}=\mathbf{l}\cdot\mathbf{n}=\mathbf{m}\cdot\mathbf{n}=0, \mathbf{l},\mathbf{m}\in\mathbb{R}^n\big\},
\end{aligned}\tag{4.1}$$

all of which are dependent on $\mathbf{n}(x,t)$. Clearly, $\mathbb{M}_n = \oplus_{i=1}^5 \mathbb{V}_i$, $\mathbb{A}_n = \mathbb{V}_3\oplus\mathbb{V}_4$, $\mathbb{S}_n = \mathbb{V}_1\oplus\mathbb{V}_2\oplus\mathbb{V}_5$ and $\dim(\mathbb{V}_1,\ \mathbb{V}_2,\ \mathbb{V}_3,\ \mathbb{V}_4,\ \mathbb{V}_5) = \big(1,\ n-1,\ n-1,\ \frac{1}{2}(n-1)(n-2),\ \frac{1}{2}n(n-1)\big)$. Moreover, if $(\mathbf{A}_-, \mathbf{A}_+)$ is a minimal pair with $\mathbf{A}_- = \mathbf{A}_+(\mathbf{I}-2\mathbf{nn})$, then

$$\mathbf{A}_+\mathbb{V}_i = \mathbf{A}_-\mathbb{V}_i (i=1,4,5),\quad \mathbf{A}_+\mathbb{V}_2 = \mathbf{A}_-\mathbb{V}_3,\quad \mathbf{A}_+\mathbb{V}_3 = \mathbf{A}_-\mathbb{V}_2.$$

Let $\mathcal{P}_i : \mathbb{M}_n \to \mathbb{V}_i$ $(1\leqslant i\leqslant 5)$ be the projection operators. Then one has

$$\begin{aligned}
\mathcal{P}_1\mathbf{A} &= \mathbf{nnAnn} = \mathbf{nn}(\mathbf{A}:\mathbf{nn}),\\
\mathcal{P}_2\mathbf{A} &= \frac{1}{2}\big[\mathbf{nn}(\mathbf{A}+\mathbf{A}^{\mathrm{T}})(\mathbf{I}-\mathbf{nn}) + (\mathbf{I}-\mathbf{nn})(\mathbf{A}+\mathbf{A}^{\mathrm{T}})\mathbf{nn}\big],\\
\mathcal{P}_3\mathbf{A} &= \frac{1}{2}\big[\mathbf{nn}(\mathbf{A}-\mathbf{A}^{\mathrm{T}})(\mathbf{I}-\mathbf{nn}) + (\mathbf{I}-\mathbf{nn})(\mathbf{A}-\mathbf{A}^{\mathrm{T}})\mathbf{nn}\big],\\
\mathcal{P}_4\mathbf{A} &= \frac{1}{2}(\mathbf{I}-\mathbf{nn})\big(\mathbf{A}-\mathbf{A}^{\mathrm{T}}\big)(\mathbf{I}-\mathbf{nn}),\\
\mathcal{P}_5\mathbf{A} &= \frac{1}{2}(\mathbf{I}-\mathbf{nn})\big(\mathbf{A}+\mathbf{A}^{\mathrm{T}}\big)(\mathbf{I}-\mathbf{nn}).
\end{aligned}\tag{4.2}$$

One can directly check from (4.2) that

$$\begin{aligned}
(\mathbf{I}-2\mathbf{nn})\mathcal{P}_2\mathbf{A} &= -\mathcal{P}_3\big((\mathbf{I}-2\mathbf{nn})\mathbf{A}\big),\\
(\mathbf{I}-2\mathbf{nn})\mathcal{P}_3\mathbf{A} &= -\mathcal{P}_2\big((\mathbf{I}-2\mathbf{nn})\mathbf{A}\big),\\
\mathcal{P}_4((\mathbf{I}-2\mathbf{nn})\mathbf{A}) &= (\mathbf{I}-2\mathbf{nn})\mathcal{P}_4\mathbf{A} = \mathcal{P}_4\mathbf{A}.
\end{aligned}\tag{4.3}$$

These projection operators play important roles throughout this paper. We remark that for our later use, $\mathbf{n}$ and the corresponding decompositions (4.1) may depend on (x,t).

4.2 Diagonalization

Now we solve the ODE system

$$\mathcal{L}\mathbf{P}(z,x,t)=\mathbf{F}(z,x,t). \tag{4.4}$$

A crucial observation is that the system (4.4) can be diagonalized into several scalar ODEs via the above orthogonal decomposition of $\mathbb{M}_n$.

We denote

$$\mathbf{U}_i(z)=\mathcal{P}_i\mathbf{P}(z),\quad \mathbf{V}_i(z)=\mathcal{P}_i\mathbf{F}(z)\quad \text{for } 1\leqslant i\leqslant 5.$$

From the fact

$$\mathbf{P}_0\mathbf{P}_0^{\mathrm{T}}=\mathbf{P}_0^{\mathrm{T}}\mathbf{P}_0=\mathbf{I}-4s(1-s)\mathbf{nn}$$

and (4.2), we deduce that

$$\begin{aligned}\mathbf{P}_0\mathbf{P}_0^{\mathrm{T}}\mathbf{P}+\mathbf{P}\mathbf{P}_0^{\mathrm{T}}\mathbf{P}_0&=2\mathbf{P}-4s(1-s)(\mathbf{nnP}+\mathbf{Pnn})\\&=2\mathbf{P}-8s(1-s)\mathbf{nn}(\mathbf{P}:\mathbf{nn})-4s(1-s)\\&\quad\times((\mathbf{I}-\mathbf{nn})\mathbf{Pnn}+\mathbf{nnP}(\mathbf{I}-\mathbf{nn}))\\&=2\mathbf{P}-8s(1-s)\mathbf{U}_1-4s(1-s)(\mathbf{U}_2+\mathbf{U}_3),\\\mathbf{P}_0\mathbf{P}^{\mathrm{T}}\mathbf{P}_0&=(\mathbf{I}-2s\mathbf{nn})\mathbf{P}^{\mathrm{T}}(\mathbf{I}-2s\mathbf{nn})\\&=\mathbf{P}^{\mathrm{T}}+4s^2\mathbf{U}_1-4s\mathbf{U}_1-2s(\mathbf{U}_2-\mathbf{U}_3).\end{aligned}$$

Using the fact that $\mathbf{U}_i^{\mathrm{T}}=\mathbf{U}_i(i=1,2,5)$ and $\mathbf{U}_j^{\mathrm{T}}=-\mathbf{U}_j(j=3,4)$, we find

$$\mathcal{P}_i\mathcal{L}\mathbf{P}=\mathcal{L}_i\mathcal{P}_i\mathbf{P}\quad \text{for } 1\leqslant i\leqslant 5,$$

where

$$\begin{aligned}&\mathcal{L}_iu=-\partial_z^2u+\kappa_i(s)u,\\&\text{with } \kappa_1(s)=2(1-6s+6s^2),\quad \kappa_2(s)=2(1-s)(1-2s),\\&\qquad\kappa_3(s)=2s(2s-1),\quad \kappa_4(s)=0,\quad \kappa_5(s)=2.\end{aligned} \tag{4.5}$$

Thus the system (4.4) can be reduced to

$$\mathcal{L}_i\mathbf{U}_i=\mathbf{V}_i\quad \text{for } 1\leqslant i\leqslant 5. \tag{4.6}$$

Let

$$\theta_1(z) = s'(z) = \sqrt{2}s(1-s), \quad \theta_2(z) = s(z),$$
$$\theta_3(z) = 1 - s(z), \quad \theta_4(z) \equiv 1. \tag{4.7}$$

Then from (3.12), it is easy to see that

$$\kappa_i(s(z)) = \frac{\theta_i''(z)}{\theta_i(z)}, \quad 1 \leqslant i \leqslant 4.$$

Thus, we obtain for $1 \leqslant i \leqslant 4$ that:

$$\mathcal{L}_i u = -\frac{1}{\theta_i}\partial_z\Big(\theta_i^2\partial_z\Big(\frac{u}{\theta_i}\Big)\Big).$$

It is direct to check that for a bounded function $u \in C^2(\mathbb{R})$, $\mathcal{L}_i u = 0$ if and only if $u = \lambda\theta_i$. Thus, if we define

$$\text{Null}\,\mathcal{L} = \text{span}\big\{\theta_i(z)\mathbf{E}_i (1 \leqslant i \leqslant 4) : \text{for } \forall \text{ constant } \mathbf{E}_i \in \mathbb{V}_i\big\},$$

then for a bounded C^2 function Ψ, $\mathcal{L}\Psi = 0$ if and only if $\Psi \in \text{Null}\,\mathcal{L}$.

Let $\alpha_0 \in (0, \sqrt{2}]$. Define the spaces:

$$\mathcal{S}_J(\alpha_0, k) = \big\{f \in C^J(\mathbb{R}) : f^{\pm} = \lim_{z\to\pm\infty} f(z) \text{ exist},$$
$$\text{and for } \forall j \in [0, J],\ |\partial_z^j(f(z) - f^{\pm})| \lesssim |z|^k e^{-\alpha_0|z|}, \text{ as } z \to \pm\infty\big\}.$$
$$\mathcal{S}_{J,L,M}(\alpha_0, k) = \big\{f(\cdot, x, t) \in \mathcal{S}_J(\alpha_0, k) : \text{for } \forall\ (j, l, m) \in [0, J] \times[0, L]\times[0, M],$$
$$\big|\partial_z^j\partial_x^l\partial_t^m\big(f(z, x, t) - f^{\pm}(x, t)\big)\big| \lesssim |z|^k e^{-\alpha_0|z|}, \text{ as } z \to \pm\infty\big\}.$$

Lemma 4.1 *Assume* $\mathbf{F}(z, x, t) \in \mathcal{S}_{J,L,M}(\alpha_0, k)$ *with*

$$(B2) : \mathcal{P}_2\mathbf{F}^+(x, t) = 0; \quad (B3) : \mathcal{P}_3\mathbf{F}^-(x, t) = 0; \quad (B4) : \mathcal{P}_4\mathbf{F}^{\pm}(x, t) = 0,$$

and orthogonal conditions:

$$(\mathrm{O}i) : \int_{\mathbb{R}} \theta_i(z)\mathbf{F}(z) : \mathbf{E}_i \mathrm{d}z = 0, \ \forall \text{ constant } \mathbf{E}_i \in \mathbb{V}_i, \quad 1 \leqslant i \leqslant 4.$$

Then for (4.4), there exists a unique bounded solution $\mathbf{P}^*(z, x, t) \in \mathcal{S}_{J+2,L,M}(\alpha_0, k+1)$ *satisfying* $\partial_z\mathbf{P}^*(0, x, t) : (\mathbf{nn})(x, t) = 0$ *and*

$$(S2) : \mathcal{P}_2\mathbf{P}^{*+}(x, t) = 0, \quad (S3) : \mathcal{P}_3\mathbf{P}^{*-}(x, t) = 0, \quad (S4) : \mathcal{P}_4\mathbf{P}^{*\pm}(x, t) = 0.$$

In addition, all bounded solutions to (4.4) satisfying $\partial_z \mathbf{P}(0,x,t) : (\mathbf{nn})(x,t) = 0$ *are given by*

$$\mathbf{P}(z,x,t) = s(z)\mathbf{Q}_2(x,t) + (1-s(z))\mathbf{Q}_3(x,t) + \mathbf{Q}_4(x,t) + \mathbf{P}^*(z,x,t), \tag{4.8}$$

with $(\mathbf{Q}_2, \mathbf{Q}_3, \mathbf{Q}_4) \in \mathbb{V}_2 \times \mathbb{V}_3 \times \mathbb{V}_4$.

Remark 4.2 Conditions (B2)-(B4) ensure that the integrals in (O1)-(O4) are finite.

Remark 4.3 One can see that $\mathbf{P}^*(z,x,t)$ shares the same regularity in (x,t) with $\mathbf{F}(z,x,t)$.

Proof The results can be deduced from Lemma A.2. □

4.3 Cubic-null cancellation

For $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3 \in \mathbb{M}_n$, we define the trilinear form

$$\mathbf{T}_f(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3) = \sum_{\{i,j,k\}=\{1,2,3\}} \mathbf{A}_i \mathbf{A}_j^{\mathrm{T}} \mathbf{A}_k. \tag{4.9}$$

Then $\mathbf{T}_f(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3) : \mathbf{A}_4$ keeps the same if we exchange any $\mathbf{A}_i$ and $\mathbf{A}_j$ $(1 \leqslant i, j \leqslant 4)$. The following cancellation relation plays an important role in closing the expansion system of each order.

Lemma 4.4 *For* $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3 \in \mathrm{Null}\ \mathcal{L}$, *we have*

$$\int_{\mathbb{R}} \mathbf{T}_f(\mathbf{P}_0, \mathbf{Q}_1, \mathbf{Q}_2) : \mathbf{Q}_3 \mathrm{d}z = 0,$$

where the integral is understood as $\lim_{R\to+\infty} \int_{-R}^{R} (\cdot)\mathrm{d}z$ *if necessary.*

Proof For the convenience, the left hand side is denoted by $\mathfrak{I}(\mathbf{P}_0, \mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3)$. It suffices to prove the case of $\mathbf{Q}_2 = \mathbf{Q}_3 = \mathbf{B}$, since we have by (1.19) that

$$\begin{aligned} 2\mathfrak{I}(\mathbf{P}_0, \mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3) = {} & \mathfrak{I}(\mathbf{P}_0, \mathbf{Q}_1, \mathbf{Q}_2 + \mathbf{Q}_3, \mathbf{Q}_2 + \mathbf{Q}_3) \\ & - \mathfrak{I}(\mathbf{P}_0, \mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_2) - \mathfrak{I}(\mathbf{P}_0, \mathbf{Q}_1, \mathbf{Q}_3, \mathbf{Q}_3). \end{aligned}$$

Assuming $\mathbf{B}(z) = \sum_{i=1}^{4} \mathbf{B}_i(z)$ with $\mathbf{B}_i(z) \in \mathbb{V}_i$. When $\mathbf{Q}_1 = \lambda s' \mathbf{nn}$, we have

$$\begin{aligned}&\mathfrak{I}(\mathbf{P}_0,\mathbf{Q}_1,\mathbf{B},\mathbf{B})\\&\quad=2\lambda\int_{\mathbb{R}}s'\Big((1-2s)(|\mathbf{Bn}|^2+|\mathbf{nB}|^2)+|\mathbf{B}^2:\mathbf{nn}|^2-2s|\mathbf{B}:\mathbf{nn}|^2\Big)\mathrm{d}z.\end{aligned}$$

It is easy to check that $(\mathbf{B}_i\mathbf{B}_j):\mathbf{nn}=0$ for different $i,\ j$, and $\mathbf{B}_4\mathbf{n}=\mathbf{nB}_4=0$. Thus,

$$\begin{aligned}&\mathfrak{I}(\mathbf{P}_0,\mathbf{Q}_1,\mathbf{B},\mathbf{B})\\&=2\lambda\sum_{i=1}^{3}\int_{\mathbb{R}}s'\Big((1-2s)\big[|\mathbf{B}_i\mathbf{n}|^2+|\mathbf{nB}_i|^2\big]+|\mathbf{B}_i^2:\mathbf{nn}|^2-2s|\mathbf{B}_i:\mathbf{nn}|^2\Big)\mathrm{d}z\\&=\sum_{i=1}^{3}\mathfrak{I}(\mathbf{P}_0,\mathbf{Q}_1,\mathbf{B}_i,\mathbf{B}_i).\end{aligned}$$

For $\mathbf{B}_2=s(\mathbf{nl}+\mathbf{ln})$ with $\mathbf{l}\perp\mathbf{n}$, we have

$$\begin{aligned}\mathfrak{I}(\mathbf{P}_0,\mathbf{Q}_1,\mathbf{B}_2,\mathbf{B}_2)&=\lambda\int_{\mathbb{R}}\Big(4s'(1-2s)s^2+2s's^2\Big)|\mathbf{l}|^2\mathrm{d}z\\&=\lambda\int_{\mathbb{R}}(s^3(1-s))'|\mathbf{l}|^2\mathrm{d}z=0.\end{aligned}$$

Similarly, we can prove that $\mathfrak{I}(\mathbf{P}_0,\mathbf{Q}_1,\mathbf{B}_3,\mathbf{B}_3)=0$ for $\mathbf{B}_3=(1-s)(\mathbf{nl}-\mathbf{ln})$ with $\mathbf{l}\perp\mathbf{n}$. For $\mathbf{B}_1=\mu s'\mathbf{nn}$, we have

$$\mathfrak{I}(\mathbf{P}_0,\mathbf{Q}_1,\mathbf{B}_1,\mathbf{B}_1)=\lambda\mu^2\int_{\mathbb{R}}6(s')^3(1-2s)\mathrm{d}z=0.$$

Therefore, $\mathfrak{I}(\mathbf{P}_0,\mathbf{Q}_1,\mathbf{B},\mathbf{B})=0$ for $\mathbf{Q}_1=\lambda s'\mathbf{nn}$ and $\mathbf{B}\in\mathrm{Null}\,\mathcal{L}$.

For $\mathbf{Q}_1=s\mathbf{E}_2+(1-s)\mathbf{E}_3+\mathbf{E}_4$ with $\mathbf{E}_i\in\mathbb{V}_i$, by direct calculations (see Lemma A.3), we get

$$\begin{aligned}&\mathfrak{I}(\mathbf{P}_0,\mathbf{Q}_1,\mathbf{B},\mathbf{B})\\&=\int_{\mathbb{R}}\Big\{\,2s\mathbf{E}_2:\big[(2s-1)(\mathbf{B}_3\mathbf{B}_4+\mathbf{B}_4\mathbf{B}_3)+(3-4s)(\mathbf{B}_1\mathbf{B}_2+\mathbf{B}_2\mathbf{B}_1)\big]\\&\quad+2(1-s)\mathbf{E}_3:\big[(1-4s)(\mathbf{B}_1\mathbf{B}_3+\mathbf{B}_3\mathbf{B}_1)+(1-2s)(\mathbf{B}_2\mathbf{B}_4+\mathbf{B}_4\mathbf{B}_2)\big]\\&\quad+2(1-2s)\mathbf{E}_4:(\mathbf{B}_2\mathbf{B}_3+\mathbf{B}_3\mathbf{B}_2)\Big\}\mathrm{d}z.\end{aligned}$$

By taking $\mathbf{B}_i(z) = \lambda_i \theta_i(z)\tilde{\mathbf{E}}_i$ with $\tilde{\mathbf{E}}_i \in \mathbb{V}_i$, we can show that all the integrals vanish due to the following fact:

$$\begin{aligned}\int_{\mathbb{R}} s(2s-1)(1-s)\mathrm{d}z &= \int_{\mathbb{R}} 2s^2(3-4s)s'\mathrm{d}z \\ &= \int_{\mathbb{R}} 2(1-s)^2(1-4s)s'\mathrm{d}z = 0.\end{aligned}$$

The proof is completed. □

5 Inner expansion

5.1 Formal inner expansion

Formally, we write the inner expansion as

$$\begin{aligned}\mathbf{A}_I^\varepsilon(z,x,t) &= \Phi(z,x,t)\mathbf{P}^\varepsilon(z,x,t) \\ &= \Phi(z,x,t)\big(\mathbf{P}_0 + \varepsilon\mathbf{P}_1 + \varepsilon^2\mathbf{P}_2 + \cdots\big)(z,x,t),\end{aligned}$$

where $\mathbf{P}_0(z,x,t) = \mathbf{I} - 2s(z)\mathbf{n}(x,t)\mathbf{n}(x,t)$ and Φ is given by (3.14). We should keep in mind that $\mathbf{P}_k$ has to satisfy the matching conditions for $(x,t) \in \Gamma(\delta)$:

$$\Phi_\pm \mathbf{P}_k(\pm\infty,x,t) = \mathbf{A}_\pm^{(k)}, \quad \text{or equivalently} \quad \mathbf{P}_k(\pm\infty,x,t) = \mathbf{P}_0^\pm \mathbf{U}_\pm^{(k)}. \tag{5.1}$$

Let $\widetilde{\mathbf{P}}^\varepsilon(x,t) = \mathbf{P}^\varepsilon(d^\varepsilon/\varepsilon,x,t)$, $\widetilde{\Phi}(x,t) = \Phi(d^\varepsilon/\varepsilon,x,t)$ and $\widetilde{\mathbf{A}}_I^\varepsilon = \widetilde{\mathbf{P}}^\varepsilon\widetilde{\Phi}$. Then we have

$$\begin{aligned}\partial_t\widetilde{\mathbf{A}}_I^\varepsilon - \Delta\widetilde{\mathbf{A}}_I^\varepsilon + \varepsilon^{-2}f(\widetilde{\mathbf{A}}_I^\varepsilon) &= \widetilde{\Phi}(\partial_t\widetilde{\mathbf{P}}^\varepsilon - \Delta\widetilde{\mathbf{P}}^\varepsilon + \varepsilon^{-2}f(\widetilde{\mathbf{P}}^\varepsilon)) \\ &\quad + (\partial_t - \Delta)\widetilde{\Phi}\widetilde{\mathbf{P}}^\varepsilon - 2\nabla\widetilde{\Phi}\nabla\widetilde{\mathbf{P}}^\varepsilon.\end{aligned}$$

To ensure $\widetilde{\mathbf{A}}_I^\varepsilon$ solves (1.3) in $\Gamma(\delta)$, one has that

$$\begin{aligned}&\Big\{\varepsilon^{-2}\big[-\partial_z^2\mathbf{P}^\varepsilon + f(\mathbf{P}^\varepsilon) - \Phi^{\mathrm{T}}\partial_z^2\Phi\mathbf{P}^\varepsilon - 2\Phi^{\mathrm{T}}\partial_z\Phi\partial_z\mathbf{P}^\varepsilon\big] \\ &\quad+\varepsilon^{-1}\big[(\partial_t d^\varepsilon - \Delta d^\varepsilon)(\partial_z\mathbf{P}^\varepsilon + \Phi^{\mathrm{T}}\partial_z\Phi) - 2\Phi^{\mathrm{T}}\nabla d^\varepsilon\nabla\partial_z(\Phi\mathbf{P}^\varepsilon)\big] \\ &\quad+\Phi^{\mathrm{T}}(\partial_t - \Delta)(\Phi\mathbf{P}^\varepsilon)\Big\}\Big|_{z=d^\varepsilon/\varepsilon} = 0.\end{aligned} \tag{5.2}$$

As in [2], we will regard z as an independent variable and $(x,t) \in \Gamma(\delta)$ as parameters. Then we will solve a series of ODE systems with respect to z for

$\mathbf{P}_k(z,x,t)$ $(k=0,1,2,\cdots)$. Note that we can add any terms vanishing on $\{d^\varepsilon=\varepsilon z\}$ on the left hand side, which does not change the equation (5.2).

Let $\widetilde{\mathbf{G}}^\varepsilon(z,x,t)$ and $\mathbf{H}^{\pm,\varepsilon}(x,t)$ be matrix-valued functions to be determined later. We choose a fixed smooth and nonnegative function $\eta(z)$ satisfying: $\eta(z)=0$ if $z\leqslant -1$, $\eta(z)=1$ if $z\geqslant 1$, $\eta'(z)\geqslant 0$. Let

$$a_0=\int_{\mathbb{R}}(s')^2\mathrm{d}z,\qquad a_i=\int_{\mathbb{R}}\eta'\theta_i\mathrm{d}z\quad(1\leqslant i\leqslant 3). \tag{5.3}$$

Consider the following modified system

$$\begin{aligned}&\varepsilon^{-2}\Big\{-\partial_z^2\mathbf{P}^\varepsilon+f(\mathbf{P}^\varepsilon)-\Phi^{\mathrm{T}}\partial_z^2\Phi\mathbf{P}^\varepsilon-2\Phi^{\mathrm{T}}\partial_z\Phi\partial_z\mathbf{P}^\varepsilon+(d^\varepsilon-\varepsilon z)\widetilde{\mathbf{G}}^\varepsilon\Big\}\\&\quad+\varepsilon^{-1}\Big\{(\partial_t d^\varepsilon-\Delta d^\varepsilon)(\partial_z\mathbf{P}^\varepsilon+\Phi^{\mathrm{T}}\partial_z\Phi)-2\Phi^{\mathrm{T}}\big(\nabla d^\varepsilon\nabla\partial_z(\Phi\mathbf{P}^\varepsilon)\big)\Big\}\\&\quad+\Phi^{\mathrm{T}}(\partial_t-\Delta)(\Phi\mathbf{P}^\varepsilon)+\mathbf{H}^{+,\varepsilon}(x,t)\eta_M^+(z)+\mathbf{H}^{-,\varepsilon}(x,t)\eta_M^-(z)=0\end{aligned} \tag{5.4}$$

for any $(z,x,t)\in\mathbb{R}\times\Gamma(\delta)$. Following the idea in [2], we choose $\eta_M^\pm(z)=\eta(-M\pm z)$ with

$$M=\|d_1\|_{C^0(\Gamma_t)}+2.$$

Then one has $\eta(-M\pm d^\varepsilon(x,t)/\varepsilon)=0$ for $(x,t)\in\Gamma(\delta)\cap Q_\mp$. $\mathbf{H}^{\pm,\varepsilon}(x,t)$ will be chosen as in (5.11) with (5.19) and (5.20), which imply $\mathbf{H}^{\pm,\varepsilon}(x,t)=0$ for $(x,t)\in\Gamma(\delta)\cap Q_\pm$. Thus, one has

$$\mathbf{H}^{+,\varepsilon}(x,t)\eta_M^+(z)+\mathbf{H}^{-,\varepsilon}(x,t)\eta_M^-(z)\Big|_{z=d^\varepsilon(x,t)/\varepsilon}=0,\qquad\text{for any }(x,t)\in\Gamma(\delta).$$

As a result, all the modified terms vanish on $\{d^\varepsilon=\varepsilon z\}$, and will not change the system (5.2).

From the definition of Φ, we can deduce that $\partial_z(\Phi^{\mathrm{T}}\partial_\nu\Phi)=0$ on Γ, which gives

$$\Phi^{\mathrm{T}}\partial_\nu\Phi=\Phi_-^{\mathrm{T}}\partial_\nu\Phi_-\triangleq\overline{\mathbf{W}}\in\mathbb{V}_4\quad\text{on }\Gamma. \tag{5.5}$$

Indeed, as $\Phi_+-\Phi_-=O(d_0^2)$, we have $\partial_z\Phi=O(d_0^2)$ and $\partial_z(\Phi^{\mathrm{T}}\partial_\nu\Phi)=\Phi^{\mathrm{T}}\partial_z\partial_\nu\Phi=\partial_\nu(\Phi^{\mathrm{T}}\partial_z\Phi)=0$ on Γ. See Appendix A.1 for the proof of $\overline{\mathbf{W}}\in\mathbb{V}_4$.

Moreover, since it holds $\Phi^{\mathrm{T}}\partial_z^2\Phi=0$ and $\Phi^{\mathrm{T}}\partial_z\Phi=0$ on Γ, we can assume that

$$\Phi^{\mathrm{T}}\partial_z^2\Phi=d_0(x,t)\Phi_1(z,x,t),\quad 2\Phi^{\mathrm{T}}\partial_z\Phi=d_0(x,t)\Phi_2(z,x,t), \tag{5.6}$$

where on Γ the definitions are interpreted as

$$\Phi_1(z,x,t) = \lim_{d_0\to 0} d_0^{-1}\Phi^{\mathrm{T}}\partial_z^2\Phi = \partial_\nu(\Phi^{\mathrm{T}}\partial_z^2\Phi),$$
$$\Phi_2(z,x,t) = \lim_{d_0\to 0} 2d_0^{-1}\Phi^{\mathrm{T}}\partial_z\Phi = 2\partial_\nu(\Phi^{\mathrm{T}}\partial_z\Phi).$$

It is direct to check that

$$\Phi_1(z,x,t),\ \ \Phi_2(z,x,t) = O(e^{-\alpha_0|z|})d_0(x,t). \tag{5.7}$$

Before going to the next steps, let us recall from (3.15) and (5.7) that

$$\partial_z\Phi = O(d_0^2),\quad \Phi_1,\ \Phi_2,\ \nabla d_0\cdot\nabla\partial_z\Phi,\ \nabla d_0\cdot\nabla\mathbf{n} = O(d_0) \quad \text{for } (x,t)\in\Gamma(\delta), \tag{5.8}$$

which will be repeatedly used in the sequel.

Now we take

$$\widetilde{\mathbf{G}}^\varepsilon(z,x,t) = (\Phi_1\mathbf{P}^\varepsilon + \Phi_2\partial_z\mathbf{P}^\varepsilon) + \mathbf{G}^\varepsilon(z,x,t), \tag{5.9}$$
$$\text{with } \mathbf{G}(z,x,t) = \sum_{k\geqslant 1}\varepsilon^k\Big(\mathbf{G}_k(x,t)\eta'(z) + \mathbf{L}_k(x,t)\eta''(z)\Big), \tag{5.10}$$
$$\mathbf{H}^{\pm,\varepsilon}(x,t) = \sum_{k\geqslant 0}\varepsilon^k\mathbf{H}_k^\pm(x,t). \tag{5.11}$$

At each $(x,t)\in\Gamma(\delta)$, we will choose

$$\mathbf{G}_k(x,t)\in\mathbb{V}_1\oplus\mathbb{V}_2\oplus\mathbb{V}_3\oplus\mathbb{V}_4,\quad \mathbf{L}_k(x,t)\in\mathbb{V}_4,$$
$$\mathbf{H}_k^+\in\mathbb{V}_2\oplus\mathbb{V}_4,\quad \mathbf{H}_k^-\in\mathbb{V}_3\oplus\mathbb{V}_4,$$

which will be precisely defined later.

Remark 5.1 The role of $(\Phi_1\mathbf{P}^\varepsilon+\Phi_2\partial_z\mathbf{P}^\varepsilon)$ in $\widetilde{\mathbf{G}}^\varepsilon$ is to leave the small error terms into the next orders. Otherwise, the obtained ODE systems are too complicated to solve. The term $\mathbf{G}_k$ is used to ensure the orthogonal conditions (O1)-(O4) of $\mathbf{F}_k$ on $\Gamma(\delta)\setminus\Gamma$, while $\mathbf{H}_k^\pm$ is used to ensure the boundary conditions (B2)-(B4), and $\mathbf{L}_k$ is introduced to characterize the variation of "normal derivative" of $\mathcal{P}_4\mathbf{P}_k$.

Now we substitute the expansion into (5.4) and collect the terms of the same order.

- The $O(\varepsilon^{-2})$ system takes the form

$$-\partial_z^2\mathbf{P}_0 + \mathbf{P}_0\mathbf{P}_0^{\mathrm{T}}\mathbf{P}_0 - \mathbf{P}_0 = 0, \tag{5.12}$$

which is satisfied by

$$\mathbf{P}_0(z,x,t)=\mathbf{I}-2s(z)\mathbf{n}(x,t)\mathbf{n}(x,t). \tag{5.13}$$

- The $O(\varepsilon^{-1})$ system reads as

$$\begin{aligned}
&-\partial_z^2\mathbf{P}_1+\mathbf{P}_0\mathbf{P}_0^{\mathrm{T}}\mathbf{P}_1+\mathbf{P}_0\mathbf{P}_1^{\mathrm{T}}\mathbf{P}_0+\mathbf{P}_1\mathbf{P}_0^{\mathrm{T}}\mathbf{P}_0-\mathbf{P}_1\\
&\quad=-(\partial_t d_0-\Delta d_0)(\Phi^{\mathrm{T}}\partial_z\Phi\mathbf{P}_0+\partial_z\mathbf{P}_0)\\
&\qquad+2\Phi^{\mathrm{T}}\nabla d_0\cdot\nabla_x\partial_z(\Phi\mathbf{P}_0)\\
&\qquad-(\Phi_1\mathbf{P}_0+\Phi_2\partial_z\mathbf{P}_0)(d_1-z)+\mathbf{G}_1\eta' d_0+\mathbf{L}_1\eta'' d_0\\
&\quad\triangleq\mathbf{F}_1+\mathbf{G}_1\eta' d_0+\mathbf{L}_1\eta'' d_0.
\end{aligned}\tag{5.14}$$

- The $O(\varepsilon^{k-2})$ $(k\geqslant 2)$ system takes the form

$$\begin{aligned}
&-\partial_z^2\mathbf{P}_k+\mathbf{P}_0\mathbf{P}_0^{\mathrm{T}}\mathbf{P}_k+\mathbf{P}_0\mathbf{P}_k^{\mathrm{T}}\mathbf{P}_0+\mathbf{P}_k\mathbf{P}_0^{\mathrm{T}}\mathbf{P}_0-\mathbf{P}_k\\
&\quad=-(\partial_t d_0-\Delta d_0)(\Phi^{\mathrm{T}}\partial_z\Phi\mathbf{P}_{k-1}+\partial_z\mathbf{P}_{k-1})+2\Phi^{\mathrm{T}}\nabla d_0\cdot\nabla_x\partial_z(\Phi\mathbf{P}_{k-1})\\
&\qquad-(\partial_t d_{k-1}-\Delta d_{k-1})(\Phi^{\mathrm{T}}\partial_z\Phi\mathbf{P}_0+\partial_z\mathbf{P}_0)+2\Phi^{\mathrm{T}}\nabla d_{k-1}\cdot\nabla_x\partial_z(\Phi\mathbf{P}_0)\\
&\qquad-\sum_{\substack{i+j=k-1\\1\leqslant i,j\leqslant k-2}}(\partial_t d_i-\Delta d_i)(\Phi^{\mathrm{T}}\partial_z\Phi\mathbf{P}_j+\partial_z\mathbf{P}_j)\\
&\qquad+2\Phi^{\mathrm{T}}\sum_{\substack{i+j=k-1\\1\leqslant i,j\leqslant k-2}}\nabla d_i\cdot\nabla_x\partial_z(\Phi\mathbf{P}_j)\\
&\qquad-\Phi^{\mathrm{T}}(\partial_t-\Delta)(\Phi\mathbf{P}_{k-2})-\sum_{\substack{i+j+l=k\\0\leqslant i,j,l\leqslant k-1}}\mathbf{P}_i\mathbf{P}_j^{\mathrm{T}}\mathbf{P}_l+\mathbf{H}_{k-2}^{+}\eta_M^{+}+\mathbf{H}_{k-2}^{-}\eta_M^{-}\\
&\qquad-\sum_{\substack{i+j=k\\1\leqslant i,j\leqslant k-1}}(\Phi_1\mathbf{P}_i+\Phi_2\partial_z\mathbf{P}_i)(d_j-\delta_j^1 z)+(\Phi_1\mathbf{P}_0+\Phi_2\partial_z\mathbf{P}_0)d_k\\
&\qquad+(\mathbf{G}_1\eta'+\mathbf{L}_1\eta'')d_{k-1}+\sum_{\substack{i+j=k\\2\leqslant i,j\leqslant k-2}}(\mathbf{G}_i\eta'+\mathbf{L}_i\eta'')(d_j-\delta_j^1 z)\\
&\qquad+(\mathbf{G}_k\eta'+\mathbf{L}_k\eta'')d_0\\
&\quad\triangleq\ \mathbf{F}_k+(\mathbf{G}_k\eta'+\mathbf{L}_k\eta'')d_0.
\end{aligned}\tag{5.15}$$

5.2 Determination of $\mathbf{H}_k^{\pm}$ and $\mathbf{G}_k$

The equation (5.14) – (5.15) can be written as

$$\mathcal{L}(\mathbf{P}_k + \mathbf{L}_k d_0\eta) = \mathbf{F}_k + \mathbf{G}_k \eta' d_0. \tag{5.16}$$

Now we determine $\mathbf{H}_k^{\pm}$ and $\mathbf{G}_k$ in order to ensure that the system for the inner expansion is solvable. Let $\mathbf{P}_k^{\pm}(x,t) = \mathbf{P}_k(\pm\infty, x, t)$. Firstly, we choose

$$\mathbf{H}_{k-2}^{+} = -(\mathcal{P}_2 + \mathcal{P}_4)\Big(\Phi_+^{\mathrm{T}}(\partial_t - \Delta)(\Phi_+\mathbf{P}_{k-2}^{+}) + \sum_{\substack{i+j+l=k\\ 0\leqslant i,j,l\leqslant k-1}} \mathbf{P}_i^{+}(\mathbf{P}_j^{+})^{\mathrm{T}}\mathbf{P}_l^{+}\Big), \tag{5.17}$$

$$\mathbf{H}_{k-2}^{-} = -(\mathcal{P}_3 + \mathcal{P}_4)\Big(\Phi_-^{\mathrm{T}}(\partial_t - \Delta)(\Phi_-\mathbf{P}_{k-2}^{-}) + \sum_{\substack{i+j+l=k\\ 0\leqslant i,j,l\leqslant k-1}} \mathbf{P}_i^{-}(\mathbf{P}_j^{-})^{\mathrm{T}}\mathbf{P}_l^{-}\Big), \tag{5.18}$$

for all $k \geqslant 2$. Thus, we have

$$\mathcal{P}_2 \lim_{z\to+\infty} \mathbf{F}_k = \mathcal{P}_3 \lim_{z\to-\infty} \mathbf{F}_k = \mathcal{P}_4 \lim_{z\to\pm\infty} \mathbf{F}_k = 0,$$

which are necessary for solvability of $\mathbf{P}_k$; see (B2)-(B4) in Lemma 4.1.

From (5.1), one has $\Phi_+\mathbf{P}_k^{+} = \mathbf{A}_+^{(k)} = \mathbf{A}_+\mathbf{U}_+^{(k)}$ and $\Phi_+^{\mathrm{T}}\mathbf{A}_+ = \mathbf{I} - 2\mathbf{nn}$. Thus, we get

$$\begin{aligned}
\mathbf{H}_{k-2}^{+} =& -(\mathcal{P}_2 + \mathcal{P}_4)\Big\{(\mathbf{I} - 2\mathbf{nn})\Big(\mathbf{A}_+^{\mathrm{T}}(\partial_t - \Delta)(\mathbf{A}_+\mathbf{U}_+^{(k-2)}) \\
&+ \sum_{\substack{i+j+l=k\\ 0\leqslant i,j,l\leqslant k-1}} \mathbf{U}_+^{(i)}(\mathbf{U}_+^{(j)})^{\mathrm{T}}\mathbf{U}_+^{(l)}\Big)\Big\} \\
=& (\mathbf{I} - 2\mathbf{nn})(\mathcal{P}_3 + \mathcal{P}_4)\Big(\mathcal{J}_+\mathbf{U}_+^{(k-2)} + \mathbf{B}_+^{(k-1)} + \mathbf{C}_+^{(k-2)}\Big) \\
=& 0, \qquad \text{for } x \in Q_+ \cap \Gamma(\delta).
\end{aligned} \tag{5.19}$$

Similarly, one has

$$\mathbf{H}_{k-2}^{-} = 0, \qquad \text{for } x \in Q_- \cap \Gamma(\delta). \tag{5.20}$$

In particular, we have

$$\mathbf{H}_{k-2}^{\pm} = 0 \text{ on } \Gamma_t, \quad \text{for } k \geqslant 2. \tag{5.21}$$

From (5.14)–(5.15), we know that

$$\mathbf{F}_k = \widetilde{\mathbf{F}}_k(\Phi, \{d_i, \mathbf{P}_i, \mathbf{G}_i, \mathbf{L}_i : i \leqslant k-1\}) + (\Phi_1\mathbf{P}_0 + \Phi_2\partial_z\mathbf{P}_0)d_k, \quad (5.22)$$

where $\widetilde{\mathbf{F}}_k$ is an explicitly given function, and the last term vanishes on Γ.

Once we have determined $\mathbf{F}_k$ on $\Gamma(\delta)$ by (5.22) with $\mathbf{F}_k|_\Gamma$ satisfying (O1)-(O4), $\mathbf{G}_k$ can be uniquely defined as

$$\mathbf{G}_k = \begin{cases} -d_0^{-1}\int_{\mathbb{R}} \big(\sum_{i=1}^3 a_i^{-1}\theta_i\mathcal{P}_i + \mathcal{P}_4\big)\mathbf{F}_k \mathrm{d}z, & (x,t) \in \Gamma(\delta)\setminus\Gamma, \\ -\partial_\nu \int_{\mathbb{R}} \big(\sum_{i=1}^3 a_i^{-1}\theta_i\mathcal{P}_i + \mathcal{P}_4\big)\mathbf{F}_k \mathrm{d}z, & (x,t) \in \Gamma. \end{cases} \quad (5.23)$$

Here $a_i (i = 1, 2, 3)$ are constants defined in (5.3). One can directly check that $\mathbf{F}_k + \mathbf{G}_k\eta' d_0$ satisfies (B2)–(B4) and (O1)–(O4) in $\Gamma(\delta)$. Moreover, we have

Lemma 5.2 $(\mathbf{G}_k : \mathbf{nn})|_{\Gamma_t}$ *is independent of* d_k.

Proof From (5.22) and (5.23), it suffices to prove that

$$\lim_{d_0\to 0} d_0^{-1}(\Phi_1\mathbf{P}_0 + \Phi_2\partial_z\mathbf{P}_0) : \mathbf{nn} = 0. \quad (5.24)$$

We recall the definition (5.6) for Φ_1 and Φ_2. Obviously, $\Phi^{\mathrm{T}}\partial_z\Phi$ is antisymmetric in $\Gamma(\delta)$, and then so do Φ_2 and $\lim_{d_0\to 0} d_0^{-1}\Phi_2$. Therefore,

$$\Phi_2\partial_z\mathbf{P}_0 : \mathbf{nn} = -2s'\Phi_2 : \mathbf{nn} = 0 \qquad \text{in } \Gamma(\delta).$$

Moreover, we have

$$\Phi^{\mathrm{T}}\partial_z^2\Phi = \partial_z(\Phi^{\mathrm{T}}\partial_z\Phi) - \partial_z\Phi^{\mathrm{T}}\partial_z\Phi.$$

The first term is antisymmetric, and the second term is of order $O(d_0^4)$. This proves (5.24). □

We can also obtain the explicit expression for $\mathbf{G}_k : \mathbf{nn}$ on Γ from (5.23):

$$\mathbf{G}_k : \mathbf{nn} = -a_1^{-1}\partial_\nu \int_{\mathbb{R}} s'\widetilde{\mathbf{F}}_k : \mathbf{nn}\mathrm{d}z, \quad (5.25)$$

which only depends on Φ and $\{d_i, \mathbf{P}_i, \mathbf{G}_i, \mathbf{L}_i : i \leqslant k-1\}$. In particular, $(\mathbf{G}_1 : \mathbf{nn})|_{\Gamma_t}$ only relies on $\Phi, d_0, \mathbf{P}_0$. We will use these facts for the derivation of the equations of d_1 and $d_k (k \geqslant 2)$.

6 Solving the systems for outer/inner expansion

In this section, we first solve the systems for the outer/inner expansions, and then construct the approximate solutions by the gluing method.

Recall that $\mathbf{P}_k$ for $k \geqslant 1$ solves

$$\mathcal{L}(\mathbf{P}_k + \mathbf{L}_k d_0 \eta) = \mathbf{F}_k + \mathbf{G}_k \eta' d_0, \tag{6.1}$$

together with the boundary condition

$$\mathbf{P}_k(\pm\infty, x, t) = \mathbf{P}_0^{\pm}\mathbf{U}_{\pm}^{(k)}. \tag{6.2}$$

By Lemma 4.1, the solution $\mathbf{P}_k$ of (6.1) can be written as

$$\begin{aligned}\mathbf{P}_k = {} & s(z)\mathbf{P}_{k,2}(x,t) + (1-s(z))\mathbf{P}_{k,3}(x,t) + \mathbf{P}_{k,4}(x,t) \\ & - \mathbf{L}_k(x,t)d_0(x,t)\eta(z) + \mathbf{P}_k^*(z,x,t),\end{aligned} \tag{6.3}$$

where $\mathbf{P}_{k,i}(x,t) \in \mathbb{V}_i$ and $\mathbf{P}_k^*(z,x,t)$ is uniquely solved by

$$\begin{aligned}&\mathcal{L}\mathbf{P}_k^*(z,x,t) = \mathbf{F}_k + \mathbf{G}_k\eta' d_0, \\ &\mathcal{P}_2\mathbf{P}_k^*(+\infty,x,t) = \mathcal{P}_3\mathbf{P}_k^*(-\infty,x,t) = \mathcal{P}_4\mathbf{P}_k^*(-\infty,x,t) = 0.\end{aligned} \tag{6.4}$$

Due to the matching condition (6.2), one has for $(x,t) \in \Gamma(\delta)$ that

$$\begin{aligned}\mathcal{P}_4\mathbf{P}_k(-\infty,x,t) &= \mathbf{P}_{k,4}(x,t) \\ &= \mathcal{P}_4\Big(\mathbf{P}_k(+\infty,x,t) - \mathbf{P}_k^*(+\infty,x,t)\Big) + \mathbf{L}_k d_0,\end{aligned} \tag{6.5}$$

and

$$\begin{aligned}\mathbf{P}_{k,2}(x,t) &= \mathcal{P}_2\mathbf{P}_k(+\infty,x,t) = \mathcal{P}_2(\mathbf{I}-2\mathbf{nn})\mathbf{U}_+^{(k)} \\ &= -(\mathbf{I}-2\mathbf{nn})\mathcal{P}_3\mathbf{U}_+^{(k)} = -(\mathbf{I}-2\mathbf{nn})\mathcal{P}_3\mathbf{V}_+^{(k)},\end{aligned} \tag{6.6}$$

$$\mathbf{P}_{k,3}(x,t) = \mathcal{P}_3\mathbf{P}_k(-\infty,x,t) = \mathcal{P}_3\mathbf{U}_-^{(k)} = \mathcal{P}_3\mathbf{V}_-^{(k)}. \tag{6.7}$$

Here we have used the relation $\mathcal{P}_2\big((\mathbf{I}-2\mathbf{nn})\mathbf{A}\big) = -(\mathbf{I}-2\mathbf{nn})\mathcal{P}_3\mathbf{A}$ in (4.3). Moreover, by (6.2):

$$\begin{aligned}&\mathcal{P}_4\mathbf{P}_k(-\infty,x,t) = \mathcal{P}_4\mathbf{U}_-^{(k)} = \mathcal{P}_4\mathbf{V}_-^{(k)}, \\ &\mathcal{P}_4\mathbf{P}_k(+\infty,x,t) = \mathcal{P}_4\Big((\mathbf{I}-2\mathbf{nn})\mathbf{U}_+^{(k)}\Big) = \mathcal{P}_4\mathbf{U}_+^{(k)} = \mathcal{P}_4\mathbf{V}_+^{(k)}.\end{aligned}$$

We then infer from (6.5) that

$$\mathcal{P}_4\mathbf{V}_-^{(k)} = \mathcal{P}_4\Big(\mathbf{V}_+^{(k)}(x,t) - \mathbf{P}_k^*(+\infty,x,t)\Big), \quad \text{for } (x,t)\in\Gamma, \tag{6.8}$$

$$\partial_\nu\mathcal{P}_4\mathbf{V}_-^{(k)} = \partial_\nu\mathcal{P}_4\Big(\mathbf{V}_+^{(k)}(x,t) - \mathbf{P}_k^*(+\infty,x,t)\Big) + \mathbf{L}_k, \quad \text{for } (x,t)\in\Gamma. \tag{6.9}$$

This gives two boundary conditions on Γ for $\mathbf{V}_-^{(k)}$ and $\mathbf{V}_+^{(k)}$. The determination of $\mathbf{L}_k$ and other boundary conditions for $\mathbf{V}_-^{(k)}$ and $\mathbf{V}_+^{(k)}$ will be derived from conditions (O2)–(O4) for $\mathbf{F}_{k+1}$, which will be explained in Sect. 6.3.3.

6.1 Solving the equation of $\mathbf{P}_1$

Since $\partial_z\Phi$, $\partial_z\mathbf{P}_0$, Φ_1, Φ_2, η', $\eta''\in\mathcal{S}_{J,L,M}(\alpha_0,0)$, one has $\mathbf{F}_1\in\mathcal{S}_{J,L,M}(\alpha_0,1)$ and the boundary condition (B2)-(B4) for $\mathbf{F}_1$ are automatically satisfied.

Firstly, one has $\partial_z\Phi,\ \Phi_1,\ \Phi_2,\ \partial_\nu\partial_z\Phi,\ \partial_\nu\mathbf{n}=0$ on Γ. In addition, it holds that

$$2\Phi^{\mathrm{T}}\nabla d_0\cdot\nabla_x\partial_z(\Phi\mathbf{P}_0) = 2\Phi^{\mathrm{T}}\partial_\nu\Phi\partial_z\mathbf{P}_0 = -4s'\overline{\mathbf{W}}\mathbf{n}\mathbf{n} = 0,$$

as $\overline{\mathbf{W}}\in\mathbb{V}_4$. Thus, we get

$$\mathbf{F}_1 = -(\partial_t d_0 - \Delta d_0)\partial_z\mathbf{P}_0 \qquad \text{on } \Gamma. \tag{6.10}$$

Therefore, the orthogonal condition (O1) for $\mathbf{F}_1$ on Γ is equivalent to

$$a_0(\partial_t d_0 - \Delta d_0) = 0, \tag{6.11}$$

which means that Γ_t **evolves according to the mean curvature flow**. This in turn gives that $\mathbf{F}_1 = 0$ on Γ. Then the orthogonal conditions (O2)–(O4) for $\mathbf{F}_1$ on Γ are automatically satisfied.

Now we have $\mathbf{F}_1 = 0$ on Γ. So we can define $\mathbf{F}_1$ and $\mathbf{G}_1$ in $\Gamma(\delta)$ by (5.14) and (5.23) respectively. Note that d_1 has not been determined yet. However, a remarkable consequence of Lemma 5.2 is that, as we will see in (6.17), d_1 satisfies an equation, which only depends on the zeroth order terms $(\Phi, d_0, \mathbf{n})$. Therefore, from Lemma 4.1, we can write

$$\begin{aligned}\mathbf{P}_1 = {} & s(z)\mathbf{P}_{1,2}(x,t) + (1-s(z))\mathbf{P}_{1,3}(x,t)\\ & + \mathbf{P}_{1,4}(x,t) - \mathbf{L}_1(x,t)d_0(x,t)\eta(z) + \mathbf{P}_1^*(z,x,t),\end{aligned} \tag{6.12}$$

with $\mathbf{P}_1^*(z,x,t)$ uniquely determined by (6.4). Note that $\mathbf{P}_1^*(z,x,t)$ depends only on the zeroth order terms $(\Phi, d_0, \mathbf{n})$. Moreover,as $\mathbf{F}_1|_\Gamma = 0$, one has $\mathbf{P}_1^*|_\Gamma = 0$ and hence $\mathbf{P}_1|_\Gamma \in \text{Null}\,\mathcal{L}$.

6.2 Solving the equation of $\mathbf{P}_2$

Again, we yield from (5.8) and (5.21) that on Γ,

$$\begin{aligned}\mathbf{F}_2 = {} & 2\Phi^{\mathrm{T}}\nabla d_0\cdot\nabla_x\partial_z(\Phi\mathbf{P}_1) - (\partial_t d_1 - \Delta d_1)\partial_z\mathbf{P}_0 + 2\Phi^{\mathrm{T}}\nabla d_1\cdot\nabla_x\partial_z(\Phi\mathbf{P}_0)\\ & - \Phi^{\mathrm{T}}(\partial_t - \Delta_x)(\Phi\mathbf{P}_0) - (\mathbf{P}_0\mathbf{P}_1^{\mathrm{T}}\mathbf{P}_1 + \mathbf{P}_1\mathbf{P}_0^{\mathrm{T}}\mathbf{P}_1 + \mathbf{P}_1\mathbf{P}_1^{\mathrm{T}}\mathbf{P}_0)\\ & + (\mathbf{G}_1\eta' + \mathbf{L}_1\eta'')(d_1 - z).\end{aligned} \tag{6.13}$$

We can directly check that the right hand side exponentially tends to its values at $\pm\infty$.

6.2.1 The equation of d_1

The orthogonal condition (O1) on Γ gives us that

$$0 = \int_{\mathbb{R}} \mathbf{F}_2 : \partial_z\mathbf{P}_0 \mathrm{d}z.$$

The above equation gives an evolution equation for d_1 on Γ. Surprisingly, it is independent of the choice of $\mathbf{P}_1$. As we mentioned above, this in turn gives a closed solution (5.23) ($k=1$) for $\mathbf{G}_1$.

Firstly, due to (5.8), we have $\partial_z\Phi = 0$, $\partial_\nu\partial_z\Phi = 0$ on Γ, and thus

$$2\Phi^{\mathrm{T}}\nabla d_0\cdot\nabla_x\partial_z(\Phi\mathbf{P}_1) = 2\Phi^{\mathrm{T}}\partial_\nu\Phi\partial_z\mathbf{P}_1 + 2\partial_z(\partial_\nu\mathbf{P}_1).$$

Again, as $\Phi^{\mathrm{T}}\partial_\nu\Phi = \overline{\mathbf{W}} \in \mathbb{V}_4$, one has $\Phi^{\mathrm{T}}\partial_\nu\Phi\partial_z\mathbf{P}_1 : \partial_z\mathbf{P}_0 = 0$. From (6.12), one has $\partial_z\partial_\nu\mathbf{P}_1 : \mathbf{nn} = 0$ and thus $\partial_z(\partial_\nu\mathbf{P}_1) : \partial_z\mathbf{P}_0 = 0$. Therefore, we obtain

$$2\Phi^{\mathrm{T}}\nabla d_0\cdot\nabla_x\partial_z(\Phi\mathbf{P}_1) : \partial_z\mathbf{P}_0 = 0. \tag{6.14}$$

As $\mathbf{P}_1|_\Gamma \in \text{Null}\,\mathcal{L}$, thus by Lemma 4.4 we have on Γ that

$$\int_{\mathbb{R}} (\mathbf{P}_0\mathbf{P}_1^{\mathrm{T}}\mathbf{P}_1 + \mathbf{P}_1\mathbf{P}_0^{\mathrm{T}}\mathbf{P}_1 + \mathbf{P}_1\mathbf{P}_1^{\mathrm{T}}\mathbf{P}_0) : \partial_z\mathbf{P}_0 \mathrm{d}z = 0. \tag{6.15}$$

Since $\partial_z\Phi$ vanishes on Γ quadratically in d_0 and $\Phi^{\mathrm{T}}\nabla d_1\cdot\nabla_x\Phi$ is antisymmetric, we have

$$2\Phi^{\mathrm{T}}\nabla d_1\cdot\nabla_x\partial_z(\Phi\mathbf{P}_0)=2\Phi^{\mathrm{T}}\nabla d_1\cdot\nabla_x\Phi\partial_z\mathbf{P}_0+2\nabla d_1\cdot\nabla_x\partial_z\mathbf{P}_0$$
$$=2s'\Phi^{\mathrm{T}}\nabla d_1\cdot\nabla_x\Phi\mathbf{nn}+2s'\nabla d_1\cdot\nabla_x(\mathbf{nn}) \quad (6.16)$$

which is orthogonal to $\mathbf{nn}$. Therefore, combining the above conclusions we obtain that

$$4a_0(\partial_t d_1-\Delta d_1)-\int_{\mathbb{R}}(d_1-z)s'\mathbf{G}_1:\mathbf{nn}\mathrm{d}z=\mathfrak{F}(\Phi,d_0,\mathbf{n})\qquad\text{on }\Gamma, \quad (6.17)$$

which is linear due to (5.25) with $k=1$. Then we can extend d_1 from Γ to $\Gamma(\delta)$ uniquely by $\nabla d_0\cdot\nabla d_1=0$.

6.2.2 *Determining* $\mathbf{F}_1,\mathbf{G}_1,\mathbf{P}_1^*$ *in* $\Gamma(\delta)$

After d_1 is determined in $\Gamma(\delta)$, we define $\mathbf{F}_1,\mathbf{G}_1$ in $\Gamma(\delta)$ as in (5.14) and (5.23). Then $\mathbf{P}_1^*$ can be determined from (6.4).

6.2.3 *The equations for* $\mathbf{P}_{1,i}(2\leqslant i\leqslant 4),\mathbf{L}_1$ *on* Γ

The conditions (O2)–(O4) for $\mathbf{F}_2|_\Gamma$ give us that

$$\int_{\mathbb{R}}\Big(2\Phi^{\mathrm{T}}\partial_\nu\Phi\partial_z\mathbf{P}_1+2\partial_z(\partial_\nu\mathbf{P}_1)-\Phi^{\mathrm{T}}(\partial_t-\Delta_x)(\Phi\mathbf{P}_0)$$
$$+(\mathbf{G}_1\eta'+\mathbf{L}_1\eta'')(d_1-z)\Big):\mathbf{Q}\mathrm{d}z=0, \quad (6.18)$$

for $\mathbf{Q}(z,x,t)=s\mathbf{E}_2,(1-s)\mathbf{E}_3,\mathbf{E}_4$ with $\mathbf{E}_j\in\mathbb{V}_j(j=2,3,4)$. Here we have used the fact that $\mathbf{P}_1|_{\Gamma_t}\in$ Null $\mathcal{L}$ and the cubic cancellation relation Lemma 4.4.

By (5.5), $\Phi^{\mathrm{T}}\partial_\nu\Phi$ is independent of z on Γ. Thus, we have

$$\int_{\mathbb{R}}2\Phi^{\mathrm{T}}\partial_\nu\Phi\partial_z\mathbf{P}_1:\mathbf{E}_4\mathrm{d}z=2\Phi^{\mathrm{T}}\partial_\nu\Phi\int_{\mathbb{R}}\partial_z\mathbf{P}_1:\mathbf{E}_4\mathrm{d}z=0\quad\text{on }\Gamma. \quad (6.19)$$

Noting that $\partial_\nu\mathbf{n}=0$ on Γ, we have $\partial_\nu\mathbf{P}_{1,2}(x,t),\partial_\nu\mathbf{P}_{1,3}(x,t)\perp\mathbf{E}_4$. Thus

$$\int_{\mathbb{R}}\Big(2\partial_z(\partial_\nu\mathbf{P}_1)+\mathbf{L}_1\eta''(d_1-z)\Big):\mathbf{E}_4\mathrm{d}z$$
$$=\int_{\mathbb{R}}\Big(\mathbf{L}_1\eta''(d_1-z)-2\mathbf{L}_1\eta'\Big):\mathbf{E}_4\mathrm{d}z=-\mathbf{L}_1:\mathbf{E}_4. \quad (6.20)$$

Therefore, (6.18) with $\mathbf{Q}(z,x,t)=\mathbf{E}_4$ gives us that

$$\mathbf{L}_1=-\int_{\mathbb{R}}\mathcal{P}_4\Big(\Phi^{\mathrm{T}}(\partial_t-\Delta_x)(\Phi\mathbf{P}_0)-\mathbf{G}_1\eta'(d_1-z)\Big)\mathrm{d}z. \tag{6.21}$$

Recalling (2.12), we get $\lim_{z\to\pm\infty}\Phi^{\mathrm{T}}(\partial_t-\Delta_x)(\Phi\mathbf{P}_0)=\mathbf{A}_-^{\mathrm{T}}(\partial_t-\Delta_x)\mathbf{A}_\pm\perp\mathbb{V}_4$, which gives

$$\mathcal{P}_4\Big(\Phi^{\mathrm{T}}(\partial_t-\Delta_x)(\Phi\mathbf{P}_0)\Big)\to 0\ \text{ (exponentially), as } z\to\pm\infty. \tag{6.22}$$

Thus, the integral in right side of (6.21) is well defined.

Since

$$\partial_z\partial_\nu\mathbf{P}_1:s\mathbf{E}_2=s'\partial_\nu\mathbf{P}_{1,2}:s\mathbf{E}_2,\quad \Phi^{\mathrm{T}}\partial_\nu\Phi\mathbf{P}_{1,2}:\mathbf{E}_2=0,$$

we have

$$\begin{aligned}0=&\int_{\mathbb{R}}\Big(2\Phi^{\mathrm{T}}\partial_\nu\Phi\partial_z\mathbf{P}_1+2\partial_z(\partial_\nu\mathbf{P}_1)-\Phi^{\mathrm{T}}(\partial_t-\Delta_x)(\Phi\mathbf{P}_0)\\&+(\mathbf{G}_1\eta'+\mathbf{L}_1\eta'')(d_1-z)\Big):s\mathbf{E}_2\mathrm{d}z\\=&\int_{\mathbb{R}}\Big(-\overline{\mathbf{W}}\mathbf{P}_{1,3}(s^2)'+(s^2)'\partial_\nu\mathbf{P}_{1,2}-s\Phi^{\mathrm{T}}(\partial_t-\Delta_x)(\Phi\mathbf{P}_0)\\&+\mathbf{G}_1\eta'(d_1-z)s\Big):\mathbf{E}_2\mathrm{d}z,\end{aligned}$$

which gives

$$\begin{aligned}&\partial_\nu\mathbf{P}_{1,2}-\mathcal{P}_2(\overline{\mathbf{W}}\mathbf{P}_{1,3})\\&+\int_{\mathbb{R}}s\mathcal{P}_2\Big(-\Phi^{\mathrm{T}}(\partial_t-\Delta_x)(\Phi\mathbf{P}_0)+\mathbf{G}_1\eta'(d_1-z)\Big)\mathrm{d}z=0.\end{aligned} \tag{6.23}$$

Similarly, we have on Γ that

$$\begin{aligned}&\partial_\nu\mathbf{P}_{1,3}-\mathcal{P}_3(\overline{\mathbf{W}}\mathbf{P}_{1,2})\\&+\int_{\mathbb{R}}(1-s)\mathcal{P}_3\Big(-\Phi^{\mathrm{T}}(\partial_t-\Delta_x)(\Phi\mathbf{P}_0)+\mathbf{G}_1\eta'(d_1-z)\Big)\mathrm{d}z=0.\end{aligned} \tag{6.24}$$

Similar to (6.22), we have

$$\mathcal{P}_2\Big(\Phi^{\mathrm{T}}(\partial_t-\Delta_x)(\Phi\mathbf{P}_0)\Big)\to 0\ \text{ (exponentially), as } z\to\infty,$$
$$\mathcal{P}_3\Big(\Phi^{\mathrm{T}}(\partial_t-\Delta_x)(\Phi\mathbf{P}_0)\Big)\to 0\ \text{ (exponentially), as } z\to-\infty,$$

which imply that the integrals in right side of (6.23) and (6.24) are well defined.

6.2.4 *Solving* $\mathbf{V}_{\pm}^{(1)}$ *in* $Q_{\pm}$

Using $[\partial_\nu, \mathbf{I}-2\mathbf{nn}]=0$, the equations (6.6)–(6.9), (6.21), (6.23)–(6.24) together shows that for $(x,t)\in\Gamma$

$$\begin{aligned}
\mathcal{P}_4\mathbf{V}_-^{(1)} &= \mathcal{P}_4\mathbf{V}_+^{(1)},\\
\partial_\nu(\mathcal{P}_4\mathbf{V}_-^{(1)}) &= \partial_\nu\mathcal{P}_4\mathbf{V}_+^{(1)}+\mathbf{L}_1,\\
\partial_\nu(\mathcal{P}_3\mathbf{V}_+^{(1)})-\mathcal{P}_3\big(\overline{\mathbf{W}}\mathcal{P}_3\mathbf{V}_-^{(1)}\big) &= (\mathbf{I}-2\mathbf{nn})\int_{\mathbb{R}} s\mathcal{P}_2\Big(-\Phi^{\mathrm{T}}(\partial_t-\Delta_x)(\Phi\mathbf{P}_0)\\
&\qquad +\mathbf{G}_1\eta'(d_1-z)\Big)\mathrm{d}z,\\
\partial_\nu(\mathcal{P}_3\mathbf{V}_-^{(1)})+\mathcal{P}_3\big(\overline{\mathbf{W}}\mathcal{P}_3\mathbf{V}_+^{(1)}\big) &= \int_{\mathbb{R}}(1-s)\mathcal{P}_3\Big(\Phi^{\mathrm{T}}(\partial_t-\Delta_x)(\Phi\mathbf{P}_0)\\
&\qquad -\mathbf{G}_1\eta'(d_1-z)\Big)\mathrm{d}z, \qquad (6.25)
\end{aligned}$$

which provide boundary conditions for $\mathbf{V}_+^{(1)}$ and $\mathbf{V}_-^{(1)}$ on Γ. As $\mathbf{V}_\pm^{(1)}=(\mathcal{P}_3+\mathcal{P}_4)\mathbf{V}_\pm^{(1)}\in\mathbb{A}_n$, these conditions are complete. So $\mathbf{V}_+^{(1)}|_{Q_+}$ and $\mathbf{V}_-^{(1)}|_{Q_-}$ can be solved from a linear parabolic system.

6.2.5 *Determining* $\mathbf{P}_1$ *in* $\Gamma(\delta)$

We extend $\mathbf{V}_+^{(1)}$ and $\mathbf{V}_-^{(1)}$ to be smooth antisymmetric matrix-valued functions on $\Gamma(\delta)$, and let

$$\mathbf{P}_{1,2}(x,t)=-(\mathbf{I}-2\mathbf{nn})\mathcal{P}_3\mathbf{V}_+^{(1)}, \qquad (6.26)$$
$$\mathbf{P}_{1,3}(x,t)=\mathcal{P}_3\mathbf{V}_-^{(1)}, \qquad (6.27)$$
$$\mathbf{L}_1(x,t)=d_0^{-1}\mathcal{P}_4\big(\mathbf{V}_-^{(1)}-\mathbf{V}_+^{(1)}+\mathbf{P}_1^*(+\infty,x,t)\big), \qquad (6.28)$$
$$\mathbf{P}_{1,4}(x,t)=\mathcal{P}_4\mathbf{V}_-^{(1)}=\mathcal{P}_4\big(\mathbf{V}_+^{(1)}-\mathbf{P}_1^*(+\infty,x,t)\big)+\mathbf{L}_1d_0. \qquad (6.29)$$

In addition, we define $\mathbf{P}_1(z,x,t)$ as in (6.12). Then it satisfies the matching conditions (5.1).

6.2.6 *Determining* $\mathbf{F}_2, \mathbf{G}_2, \mathbf{P}_2^*$ *in* $\Gamma(\delta)$

$\mathbf{F}_2$ is determined by (5.22) with $k = 2$, and $\mathbf{G}_2$ is determined by (5.23) which is derived from the orthogonal condition (O1)-(O4) for $\mathbf{F}_2$. Thus, we can solve $\mathbf{P}_2^*(z, x, t)$ from (6.4) and write $\mathbf{P}_2(z, x, t)$ as in (6.3) with $k = 2$. We remark that $\mathbf{P}_2^*(z, x, t)$ depends only on $(\Phi, d_0, \mathbf{n}, \mathbf{P}_1, d_1)$.

6.3 Solving the equation of $\mathbf{P}_k$ ($k \geqslant 3$)

Repeating the above procedure, we can solve $(\mathbf{P}_k, d_k)$ step by step. Assume that in $\Gamma(\delta)$

$$\{d_i,\ \mathbf{P}_i,\ \mathbf{L}_i,\ \mathbf{G}_i | 0 \leqslant i \leqslant k-1\},$$

has been known. Then we solve $d_k, \mathbf{P}_k, \mathbf{L}_k, \mathbf{G}_k$ in $\Gamma(\delta)$.

Firstly, we obtain $\mathbf{F}_k|_\Gamma$ and $\mathbf{P}_k^*|_\Gamma$ from (5.22) and (6.4) respectively. One can also determine $(\mathbf{G}_k : \mathbf{nn})|_{\Gamma_t}$ from (5.25). Note that all these terms are independent of d_k.

On Γ, from (5.15), we can write

$$\begin{aligned}
\mathbf{F}_{k+1} =& -(\partial_t d_k - \Delta d_k)\partial_z \mathbf{P}_0 + 2\Phi^{\mathrm{T}}\nabla d_0 \cdot \nabla_x \partial_z(\Phi \mathbf{P}_k) + 2\Phi^{\mathrm{T}}\nabla d_k \cdot \nabla_x \partial_z(\Phi \mathbf{P}_0) \\
&+ \sum_{\substack{i+j=k \\ 1\leqslant i,j\leqslant k-1}} \Big(-(\partial_t d_i - \Delta d_i)\partial_z \mathbf{P}_j + 2\Phi^{\mathrm{T}}\nabla d_i \cdot \nabla_x \partial_z(\Phi \mathbf{P}_j) \Big) \\
&- \sum_{\substack{i+j+l=k+1 \\ 0\leqslant i,j,l\leqslant k}} \mathbf{P}_i \mathbf{P}_j^{\mathrm{T}} \mathbf{P}_l - \Phi^{\mathrm{T}}(\partial_t - \Delta)(\Phi \mathbf{P}_{k-1}) + (\mathbf{G}_1 \eta' + \mathbf{L}_1 \eta'') d_k \\
&+ (\mathbf{G}_k \eta' + \mathbf{L}_k \eta'')(d_1 - z) + \sum_{i=2}^{k-1} (\mathbf{G}_i \eta' + \mathbf{L}_i \eta'') d_{k+1-i}.
\end{aligned}$$

6.3.1 *The equation for* d_k

We use the condition (O1) for $\mathbf{F}_{k+1}|_\Gamma$. Let $\mathbf{P}_k^\top = \mathbf{P}_k - \mathbf{P}_k^*$. Similar to the derivation of (6.14), we can obtain

$$2\Phi^{\mathrm{T}}\nabla d_0 \cdot \nabla_x \partial_z(\Phi \mathbf{P}_k^\top) : \partial_z \mathbf{P}_0 = 0.$$

Moreover, similar to (6.15) and (6.16), we can obtain

$$\begin{aligned}
&\int_{\mathbb{R}} \mathbf{T}_f(\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_k^\top) : \partial_z \mathbf{P}_0 \mathrm{d}z = 0, \\
&2\Phi^{\mathrm{T}}\nabla d_k \cdot \nabla_x \partial_z(\Phi \mathbf{P}_0) : \mathbf{nn} \\
&\quad = 2s'\big(\Phi^{\mathrm{T}}\nabla d_k \cdot \nabla_x \Phi \mathbf{nn} + \nabla d_k \cdot \nabla_x(\mathbf{nn})\big) : \mathbf{nn} = 0.
\end{aligned} \tag{6.30}$$

Therefore, the equation for d_k on Γ reads as

$$4a_0(\partial_t d_k - \Delta d_k) + d_k \int_{\mathbb{R}} \mathbf{G}_1 : \mathbf{n}\mathbf{n}\eta' s' \mathrm{d}z = \int_{\mathbb{R}} \Big\{ 2\Phi^{\mathrm{T}} \nabla d_0 \cdot \nabla_x \partial_z (\Phi \mathbf{P}_k^*) - \sum_{\substack{i+j=k \\ 1 \leqslant i,j \leqslant k-1}} \Big((\partial_t d_i - \Delta d_i) \partial_z \mathbf{P}_j + 2\Phi^{\mathrm{T}} \nabla d_i \cdot \nabla_x \partial_z (\Phi \mathbf{P}_j) \Big) - \sum_{\substack{i+j+l=k+1 \\ 0 \leqslant i,j,l \leqslant k}} \mathbf{P}_i \mathbf{P}_j^{\mathrm{T}} \mathbf{P}_l - \Phi^{\mathrm{T}} (\partial_t - \Delta)(\Phi \mathbf{P}_{k-1}) + \mathbf{G}_k \eta'(d_1 - z) + \sum_{i=2}^{k-1} s' \mathbf{G}_i \eta' d_{k+1-i} \Big\} : \mathbf{n}\mathbf{n} s' \mathrm{d}z. \tag{6.31}$$

Note that, from (6.30), the right hand side is independent of $\mathbf{P}_k$, and from Lemma 5.2 or (5.25), it is also independent of d_k.

After $d_k|_\Gamma$ is determined, d_k can be extended to $\Gamma(\delta)$ by the ODE (cf. (3.3))

$$2\nabla d_0 \cdot \nabla d_k + \sum_{i=1}^{k-1} \nabla d_i \cdot \nabla d_{k-i} = 0. \tag{6.32}$$

6.3.2 *Determining* $\mathbf{F}_k, \mathbf{G}_k, \mathbf{P}_k^*$ *in* $\Gamma(\delta)$

$\mathbf{F}_k$, $\mathbf{G}_k$ and $\mathbf{P}_k^*$ in $\Gamma(\delta)$ can be determined by (5.22), (5.23) and (6.4) accordingly.

6.3.3 *The equation for* $\mathbf{P}_{k,i} (2 \leqslant i \leqslant 4)$, $\mathbf{L}_k$ *on* Γ

We use the condition (O2)–(O4) for $\mathbf{F}_{k+1}$ on Γ. For $\mathbf{Q}(z, x, t) = s\mathbf{E}_2$, $(1 - s)\mathbf{E}_3$, $\mathbf{E}_4$ with $\mathbf{E}_i \in \mathbb{V}_i$, we have

$$\int_{\mathbb{R}} \Big\{ 2\Phi^{\mathrm{T}} \partial_\nu \Phi \partial_z \mathbf{P}_k^\top + 2\partial_z (\partial_\nu \mathbf{P}_k^\top) + \mathbf{L}_k \eta''(d_1 - z) \Big\} : \mathbf{Q} \mathrm{d}z + \int_{\mathbb{R}} \Big\{ \sum_{\substack{i+j=k, \\ i \geqslant 1}} \Big(- (\partial_t d_i - \Delta d_i) \partial_z \mathbf{P}_j + 2\Phi^{\mathrm{T}} \nabla d_i \cdot \nabla_x \partial_z (\Phi \mathbf{P}_j) \Big) - \sum_{\substack{i+j+l=k+1 \\ 0 \leqslant i,j,l \leqslant k}} \mathbf{P}_i \mathbf{P}_j^{\mathrm{T}} \mathbf{P}_l + 2\Phi^{\mathrm{T}} \nabla d_0 \cdot \nabla_x \partial_z (\Phi \mathbf{P}_k^*) - \Phi^{\mathrm{T}} (\partial_t - \Delta)(\Phi \mathbf{P}_{k-1}) + (\mathbf{G}_1 \eta' + \mathbf{L}_1 \eta'') d_k + \mathbf{G}_k \eta'(d_1 - z) + \sum_{i=2}^{k-1} (\mathbf{G}_i \eta' + \mathbf{L}_i \eta'') d_{k+1-i} \Big\} : \mathbf{Q} \mathrm{d}z = 0.$$

We denote the last three lines by $\int_{\mathbb{R}} \mathbf{T}_k(z,x,t) : \mathbf{Q}(z,x,t)\mathrm{d}z$, in which all terms are known functions due to $\int_{\mathbb{R}} \mathbf{T}_f(\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_k^\top) : \mathbf{Q}\mathrm{d}z = 0$.

Taking $\mathbf{Q} = \mathbf{E}_4$ and using the same argument as that in (6.19)–(6.20), one has

$$\int_{\mathbb{R}} \left\{ 2\Phi^{\mathrm{T}} \partial_v \Phi \partial_z \mathbf{P}_k^\top + 2\partial_z(\partial_v \mathbf{P}_k^\top) + \mathbf{L}_k \eta''(d_1 - z) \right\} : \mathbf{E}_4 \mathrm{d}z = -\mathbf{L}_k : \mathbf{E}_4.$$

Thus, we get

$$\partial_v \mathcal{P}_4 \mathbf{P}_k^+ - \partial_v \mathcal{P}_4 \mathbf{P}_k^- = -\mathbf{L}_k = \mathcal{P}_4 \int_{\mathbb{R}} \mathbf{T}_k(z,x,t)\mathrm{d}z. \tag{6.33}$$

Similarly, we have

$$\partial_v \mathbf{P}_{k,2} - \mathcal{P}_2(\bar{\mathbf{W}} \mathbf{P}_{k,3}) + \mathcal{P}_2 \int_{\mathbb{R}} s(z)\mathbf{T}_k(z,x,t)\mathrm{d}z = 0, \tag{6.34}$$

$$\partial_v \mathbf{P}_{k,3} - \mathcal{P}_3(\bar{\mathbf{W}} \mathbf{P}_{k,2}) + \mathcal{P}_3 \int_{\mathbb{R}} (1 - s(z))\mathbf{T}_k(z,x,t)\mathrm{d}z = 0. \tag{6.35}$$

6.3.4 *Solving* $\mathbf{V}_+^{(k)}$ *and* $\mathbf{V}_-^{(k)}$ *in* $Q_\pm$

The equations (6.8)–(6.9), (6.33)–(6.35) together give that

$$\begin{aligned}
\mathcal{P}_4 \mathbf{V}_+^{(k)} - \mathcal{P}_4 \mathbf{V}_-^{(k)} &= \mathcal{P}_4 \mathbf{P}_k^*(+\infty, x, t), \\
\partial_v(\mathcal{P}_4 \mathbf{V}_+^{(k)}) - \partial_v(\mathcal{P}_4 \mathbf{V}_-^{(k)}) &= \mathcal{P}_4 \int_{\mathbb{R}} \mathbf{T}_k(z,x,t)\mathrm{d}z, \\
\partial_v(\mathcal{P}_3 \mathbf{V}_+^{(k)}) - \mathcal{P}_3\left(\bar{\mathbf{W}} \mathcal{P}_3 \mathbf{V}_-^{(k)}\right) &= (\mathbf{I} - 2\mathbf{nn}) \int_{\mathbb{R}} s\mathcal{P}_2 \mathbf{T}_k(z,x,t)\mathrm{d}z, \\
\partial_v(\mathcal{P}_3 \mathbf{V}_-^{(k)}) + \mathcal{P}_3\left(\bar{\mathbf{W}} \mathcal{P}_3 \mathbf{V}_+^{(k)}\right) &= -\int_{\mathbb{R}} (1-s)\mathcal{P}_3 \mathbf{T}_k(z,x,t)\mathrm{d}z,
\end{aligned} \tag{6.36}$$

which offer complete boundary conditions for $\mathbf{V}_+^{(k)}$ and $\mathbf{V}_-^{(k)}$ on Γ. Thus combining (2.18) we can solve $\mathbf{V}_+^{(k)}|_{Q_+}$ and $\mathbf{V}_-^{(k)}|_{Q_-}$.

6.3.5 *Determining* $\mathbf{P}_k$ *in* $\Gamma(\delta)$

Afterwards, as in Sect. 6.2.5, we extend $\mathbf{V}_+^{(k)}$ and $\mathbf{V}_-^{(k)}$ to be smooth antisymmetric matrix-valued functions in $\Gamma(\delta)$, and let

$$\mathbf{P}_{k,2}(x,t) = -(\mathbf{I} - 2\mathbf{nn})\mathcal{P}_3 \mathbf{V}_+^{(k)}, \tag{6.37}$$

$$\mathbf{P}_{k,3}(x,t) = \mathcal{P}_3 \mathbf{V}_-^{(k)}, \tag{6.38}$$

$$\mathbf{L}_k(x,t) = d_0^{-1}(\mathcal{P}_4 \mathbf{V}_-^{(k)} - \mathcal{P}_4 \mathbf{V}_+^{(k)}), \tag{6.39}$$

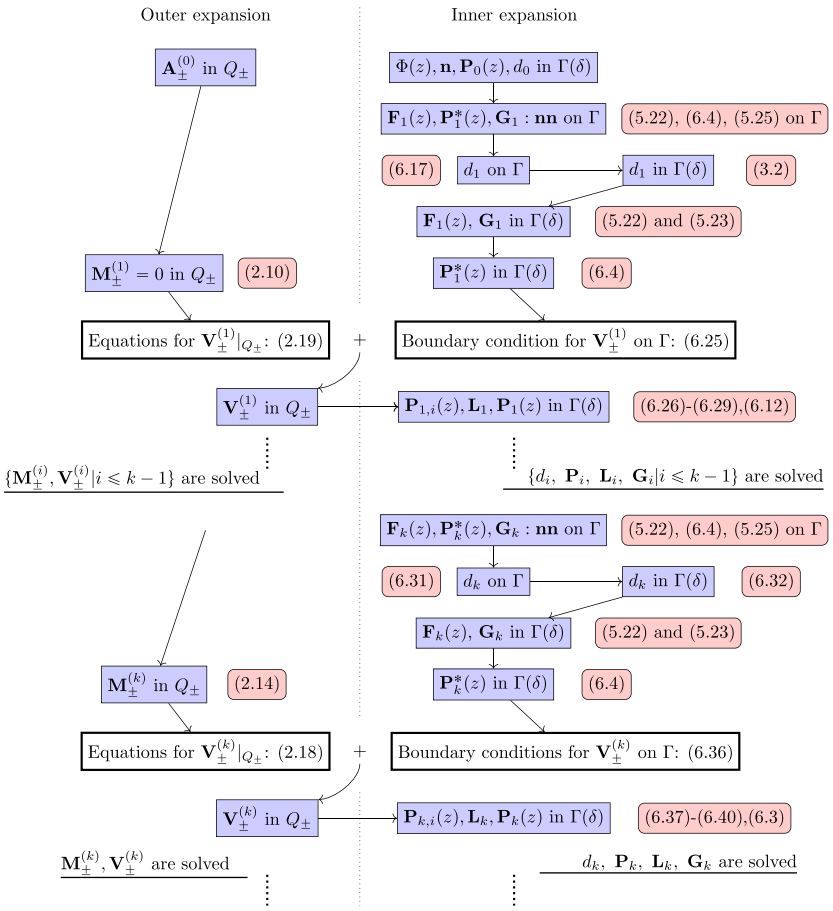

Fig. 1 The whole procedure to solve the outer and inner expansion systems

$$\mathbf{P}_{k,4}(x,t) = \mathcal{P}_4 \mathbf{V}_-^{(k)} = \mathcal{P}_4\Big(\mathbf{V}_+^{(k)} - \mathbf{P}_k^*(+\infty, x, t)\Big) + \mathbf{L}_k d_0. \tag{6.40}$$

Moreover, now we can define $\mathbf{P}_k(z, x, t)$ as in (6.12) and the matching conditions (5.1) are satisfied.

As a result, we have solved $d_k, \mathbf{P}_k, \mathbf{L}_k, \mathbf{G}_k$ in $\Gamma(\delta)$. Therefore, by repeating the above steps, the expansion system can be solved from an induction argument. Note that, in each step, we only need to solve a linear system whose well-posedness can be shown directly. The whole procedure is illustrated in Fig. 1.

Remark 6.1 The minimal paired condition (1.6c) and Lemma 3.7 give us that there exists a smooth map $\mathbf{N}(x,t) : \Gamma \to \{\mathbf{nn} : \mathbf{n} \in S^{n-1}\}$ such that $\mathbf{A}_-(x,t) = \mathbf{A}_+(x,t)(\mathbf{I} - 2\mathbf{N}(x,t))$ for all $(x,t) \in \Gamma$. For given $(x_0, t_0) \in \Gamma$ and any neighbourhood $U \subset \Gamma$ of (x_0, t_0), there exists a smooth vector field

$\mathbf{n}(x,t) : U \to S^{n-1}$ such that $\mathbf{N}(x,t) = (\mathbf{nn})(x,t)$. We have assumed that such a lifting map, which keeps the regularity, exists globally in Γ. This assumption is made just for simplicity and clarity of presentations, as our previous analysis does not rely on the particular choice of $\mathbf{n}$ or $-\mathbf{n}$. For example, the condition $\partial_\nu \mathbf{n} = 0$ can be replaced by $\partial_\nu \mathbf{N} = 0$, which will not cause any obstacles in our arguments, and the decomposition and projections in (4.1) and (4.2) indeed depend only on $\mathbf{N} = \mathbf{nn}$.

6.4 Gluing the two expansions and the proof of Theorem 1.1

Now we glue the outer expansion and inner expansion to obtain the approximate solutions in the whole region Ω. Let

$$\mathbf{A}_O^K(x,t) = \sum_{k=0}^{K} \varepsilon^k \left(\mathbf{A}_+^{(k)} \chi_{Q_+} + \mathbf{A}_-^{(k)} \chi_{Q_-}\right) \quad \text{for } (x,t) \in Q_\pm.$$

Then it holds that for $(x,t) \in Q_\pm$,

$$\begin{aligned}
&(\partial_t - \Delta)\mathbf{A}_O^K - \varepsilon^{-2} f(\mathbf{A}_O^K) \\
&= \sum_{k=0}^{K-2} \varepsilon^k \Big\{ (\partial_t - \Delta)\mathbf{A}_\pm^{(k)} - \sum_{i+j+l=k+2} \mathbf{A}_\pm^{(i)} (\mathbf{A}_\pm^{(j)})^{\mathrm{T}} \mathbf{A}_\pm^{(l)} \Big\} + O(\varepsilon^{K-1}) \\
&= O(\varepsilon^{K-1}).
\end{aligned}$$

For $(x,t) \in \Gamma(\delta)$, we define:

$$d^K(x,t) = \sum_{k=0}^{K} \varepsilon^k d_k(x,t),$$

$$\mathbf{A}_I^K(x,t) = \Phi(\varepsilon^{-1} d^K, x, t) \sum_{k=0}^{K} \varepsilon^k \mathbf{P}_k(\varepsilon^{-1} d^K, x, t),$$

$$\mathbf{G}^K = \sum_{k=1}^{K} \varepsilon^k \mathbf{G}_k(x,t), \quad \mathbf{L}^K = \sum_{k=1}^{K} \varepsilon^k \mathbf{L}_k(x,t), \quad \mathbf{H}^{\pm,K} = \sum_{k=0}^{K-2} \varepsilon^k \mathbf{H}_k^\pm(x,t),$$

with $d_k, \Phi, \mathbf{P}_k, \mathbf{G}_k, \mathbf{L}_k, \mathbf{H}_k^\pm$ defined in Sects. 5–6. Then

$$|\nabla d^K|^2 = 1 + \sum_{1 \leqslant i,j \leqslant K, i+j \geqslant K+1} \varepsilon^{i+j} \nabla d_i \nabla d_j = 1 + O(\varepsilon^{K+1}),$$

and for $(x,t)\in\Gamma(\delta)$, we have

$$\begin{aligned}
&(\partial_t-\Delta)\mathbf{A}_I^K-\varepsilon^{-2}f(\mathbf{A}_I^K)\\
&\quad=\Big\{\varepsilon^{-2}\Phi\big(-\partial_z^2\mathbf{P}^K+f(\mathbf{P}^K)-\Phi^{\mathrm{T}}\partial_z^2\Phi\mathbf{P}^K-2\Phi^{\mathrm{T}}\partial_z\Phi\partial_z\mathbf{P}^K\big)\\
&\qquad+\varepsilon^{-1}\Phi\big[(\partial_t d^K-\Delta d^K)(\partial_z\mathbf{P}^K+\Phi^{\mathrm{T}}\partial_z\Phi)-2\Phi^{\mathrm{T}}\nabla d^K\nabla\partial_z(\Phi\mathbf{P}^K)\big]\\
&\qquad+(\partial_t-\Delta)(\Phi\mathbf{P}^K)\Big\}\Big|_{z=\varepsilon^{-1}d^K}+O(\varepsilon^{K-1})\\
&\quad=\Big\{\varepsilon^{-2}\Phi\big[-\partial_z^2\mathbf{P}^K+f(\mathbf{P}^K)+(d^K-\varepsilon z-d_0)(\Phi_1\mathbf{P}^K+\Phi_2\partial_z\mathbf{P}^K)\\
&\qquad+(d^K-\varepsilon z)(\mathbf{G}^K\eta'+\mathbf{L}^K\eta'')\big]\\
&\qquad+\varepsilon^{-1}\Phi\big[(\partial_t d^K-\Delta d^K)(\partial_z\mathbf{P}^K+\Phi^{\mathrm{T}}\partial_z\Phi)-2\Phi^{\mathrm{T}}\nabla d^K\nabla\partial_z(\Phi\mathbf{P}^K)\big]\\
&\qquad+(\partial_t-\Delta)(\Phi\mathbf{P}^K)+\Phi\mathbf{H}^{+,K}\eta_M^+\Phi\mathbf{H}^{-,K}\eta_M^-\Big\}\Big|_{z=\varepsilon^{-1}d^K}+O(\varepsilon^{K-1})\\
&\quad=O(\varepsilon^{K-1}).
\end{aligned}$$

Here in the last equality, we used the expansion systems (5.12)–(5.15), which imply that all $O(\varepsilon^k)(k\leqslant K-2)$ terms are cancelled by each other. Moreover, for $(x,t)\in\Gamma(\delta)$, due to the matching condition (3.1), we have

$$|\partial_t^i\partial_x^j\partial_z^l(\mathbf{A}_I^K-\mathbf{A}_O^K)|\leqslant Ce^{-\alpha_0|d^K(x,t)|/\varepsilon}\leqslant Ce^{-\alpha_0|d_0(x,t)|/\varepsilon}. \tag{6.41}$$

Therefore, if we define

$$\mathbf{A}^K(x,t)=\big\{1-\tilde\chi\big(d_0(x,t)\delta^{-1}\big)\big\}\mathbf{A}_O^K(x,t)+\tilde\chi\big(d_0(x,t)\delta^{-1}\big)\mathbf{A}_I^K(x,t),$$

where $\tilde\chi$ is a smooth nonnegative function satisfying $\operatorname{supp}\tilde\chi\subset(-1,1)$ and $\tilde\chi(z)=1$ for $|z|\leqslant 1/2$, then it holds that

$$(\partial_t-\Delta)\mathbf{A}^K-\varepsilon^{-2}f(\mathbf{A}^K)=\mathfrak{R}^{K-1}\sim O(\varepsilon^{K-1}),$$

in the whole domain Ω. Moreover, we have $\partial_x^i\mathfrak{R}^{K-1}=O(\varepsilon^{K-1-i})$ for $i\in\mathbb{N}$. This finishes the proof of Theorem 1.1.

7 Spectral lower bound estimate for the linearized operator

This section is devoted to proving Theorem 1.3, i.e., inequality (1.9) for $\mathbf{A}\in H^1(\Omega)$. Obviously, it suffices to consider ε small enough. The proof is accomplished by five steps, which are done in Sects. 7.1–7.5 respectively:

Step 1: reduce to 1-D interval. By introducing two transformations, one for coordinates and the other for matrix fields, we reduce the problem into inequalities on a 1-D interval.

Step 2: decompose into scalar inequalities. We use a bases decomposition to reduce a matrix-valued problem to two scalar bilinear estimates for cross terms and correction terms respectively; see (7.15) and (7.17).

Step 3: coercive estimates and endpoints L^∞-control. We develop the coercive estimates for the scalar linearized operators $\mathcal{L}_i$(Lemmas 7.1 and 7.5), L^∞-control at endpoints (Lemmas 7.7 and 7.8).

Step 4: estimate the cross terms. The cross terms involving $\mathcal{L}_4$ or $\mathcal{L}_5$ can be controlled directly as we have strong coercive estimates for these two operators; see Proposition 7.9. However, the coercive estimates for $\mathcal{L}_1$-$\mathcal{L}_3$ are relatively weak, thus the same method could not be applied to control the corresponding cross terms, which becomes somewhat technical.

Motivated by [16], we observe that the weights in all cross terms involving $\mathcal{L}_1$ are indeed small by using the homogeneous Neumann boundary condition of $\mathbf{n}$. This is the key that it enables us to remove the singularity of cross terms involving $\mathcal{L}_1$; see Lemma 7.10 and Proposition 7.11.

The estimates for the cross terms involving $\mathcal{L}_2$ and $\mathcal{L}_3$ are much more involved, since neither do we have strong coercive estimates, nor the weights are small. We accomplish it by a product estimate (see Proposition 7.13), which is proved by applying a symmetric structure for the eigenfunctions of $\mathcal{L}_2$ and $\mathcal{L}_3$ (see Lemma 7.14).

Step 5: estimate the correction terms. We explicitly decompose the singular correction terms. Then the inequality is reduced to some new product estimates similar to Lemma 7.14. The proof of these product estimate also rely on the important cancellation structures between the first eigenfunctions of $\mathcal{L}_i (1 \leqslant i \leqslant 4)$; see Lemmas 7.15.

7.1 Reduction to inequalities on an interval

First of all, we choose ε small enough such that

$$\Gamma_t(\delta/8) \subset \Gamma_t^K(\delta/4) := \{x : |d^K(x,t)| < \delta/4\} \subset \Gamma_t(\delta/2), \quad \text{for } t \in [0,T].$$

For $x \in \Omega_t^{\pm}$, $\mathbf{A}^{(0)} \in \mathbb{O}_n$ and $\mathbf{A}^{(1)} \in \mathbf{A}_+\mathbb{A}_n$ or $\mathbf{A}_-\mathbb{A}_n$. For $\mathbf{A} \in H^1(\Omega)$, we perform the decomposition:

$$\mathbf{A} = \mathbf{K} + \mathbf{J} \qquad \text{with } \mathbf{K} \in \mathbf{A}_\pm\mathbb{A}_n,\ \mathbf{J} \in \mathbf{A}_\pm\mathbb{S}_n.$$

Then we have

$$\begin{aligned} \mathcal{H}_{\mathbf{A}^{(0)}}\mathbf{A} : \mathbf{A} &= \mathcal{H}_{\mathbf{A}^{(0)}}\mathbf{J} : \mathbf{J} = 2|\mathbf{J}|^2, \\ \mathbf{T}_f(\mathbf{A}^{(0)}, \mathbf{A}^{(1)}, \mathbf{A}) : \mathbf{A} &= \mathbf{T}_f(\mathbf{A}^{(0)}, \mathbf{A}^{(1)}, \mathbf{J}) : \mathbf{J} + 2\mathbf{T}_f(\mathbf{A}^{(0)}, \mathbf{A}^{(1)}, \mathbf{J}) : \mathbf{K}, \end{aligned}$$

where $\mathbf{T}_f$ is defined in (4.9). Thus, for $x \in \Omega_t^{\pm}$, one can find a constant C such that

$$\varepsilon^{-2}\mathcal{H}_{\mathbf{A}^{(0)}}\mathbf{A} : \mathbf{A} + \varepsilon^{-1}\mathbf{T}_f(\mathbf{A}^{(0)}, \mathbf{A}^{(1)}, \mathbf{A}) : \mathbf{A} \geqslant -C\|\mathbf{K}\|^2 \geqslant -C\|\mathbf{A}\|^2.$$

In addition, in $\Omega \setminus \Gamma_t^K(\delta/4)$, due to the matching condition (6.41), we have

$$\begin{aligned}\mathbf{A}^K - \mathbf{A}^{(0)} - \varepsilon\mathbf{A}^{(1)} &= \tilde{\chi}\big(d/\delta\big)(\mathbf{A}_I^K - \mathbf{A}_O^K) + O(\varepsilon^2)\\ &= O(e^{-\frac{\alpha_0\delta}{4\varepsilon}}) + O(\varepsilon^2) = O(\varepsilon^2).\end{aligned}$$

Thus, in $\Omega \setminus \Gamma_t^K(\delta/4)$, one has that

$$\begin{aligned}\varepsilon^{-2}\mathcal{H}_{\mathbf{A}^K}\mathbf{A} : \mathbf{A} &= \varepsilon^{-2}\mathcal{H}_{\mathbf{A}^{(0)}}\mathbf{A} : \mathbf{A} + \varepsilon^{-1}\mathbf{T}_f(\mathbf{A}^{(0)}, \mathbf{A}^{(1)}, \mathbf{A}) : \mathbf{A} + O(\|\mathbf{A}\|^2)\\ &\geqslant -C\|\mathbf{A}\|^2.\end{aligned}$$

For $x \in \Gamma_t^K(\delta/4)$, we let

$$\mathbf{A}_0 = \big(\Phi\mathbf{P}_0\big)(\varepsilon^{-1}d^K(x,t), x, t), \qquad \mathbf{A}_1 = \big(\Phi\mathbf{P}_1\big)(\varepsilon^{-1}d^K(x,t), x, t).$$

Then

$$\mathbf{A}^K = \mathbf{A}_I^K = \mathbf{A}_0 + \varepsilon\mathbf{A}_1 + O(\varepsilon^2).$$

Therefore, it suffices to prove that

$$\begin{aligned}&\int_{\Gamma_t^K(\frac{\delta}{4})} \Big\{\|\nabla\mathbf{A}\|^2 + \frac{1}{\varepsilon^2}\mathcal{H}_{\mathbf{A}_0}\mathbf{A} : \mathbf{A} + \frac{1}{\varepsilon}\mathbf{T}_f(\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}) : \mathbf{A}\Big\}\mathrm{d}x\\ &\geq -C\int_{\Gamma_t^K(\frac{\delta}{4})} \|\mathbf{A}\|^2\mathrm{d}x. \qquad (7.1)\end{aligned}$$

By a standard density argument, we can assume that $\mathbf{A} \in C^1(\Gamma_t^K(\frac{\delta}{4}))$.

For given $t \in [0, T]$ and $\sigma \in \Gamma_t^K$, $r \in [-\delta/4, \delta/4]$, we define $x(\sigma, r) \in \Omega$ by $x(\sigma, 0) = \sigma \in \Gamma_t^K$ and

$$\partial_r x(\sigma, r) = \frac{\nabla d^K}{|\nabla d^K|^2} \circ (x(\sigma, r), t).$$

Then $\frac{\mathrm{d}}{\mathrm{d}r}(d^K(x(\sigma, r)) - r) = 0$. As $d^K(x(\sigma, 0)) = 0$, we have

$$d^K(x(\sigma, r), t) = r. \qquad (7.2)$$

Thus for small δ, $(\sigma, r) \mapsto x(\sigma, r)$ is a bijective mapping from $\Gamma_t^K \times (-\delta/4, \delta/4)$ to $\Gamma_t^K(\delta/4)$. Let $J(\sigma, r) = \det(\frac{\partial x(\sigma,r)}{\partial(\sigma,r)})$ be the Jacobian of the mapping. Then

$$J|_{r=0} = 1,\ J(\sigma, r) = 1 + O(r),\ \mathrm{d}x = J\mathrm{d}\sigma\mathrm{d}r,\ \text{ and}$$
$$\partial_r f = \partial_r x \cdot \nabla f = \frac{\nabla d^K \cdot \nabla f}{|\nabla d^K|^2}.$$

Therefore, as $|\nabla d^K|^2 = 1 + O(\varepsilon^{K+1})$, we have

$$|\nabla f|^2 \geqslant \left|\frac{\nabla d^K}{|\nabla d^K|} \cdot \nabla f\right|^2 \geqslant (\partial_r f)^2 + O(\varepsilon^{K+1})|\nabla f|^2. \tag{7.3}$$

Let $I(\delta) = [-\delta/4, \delta/4]$. The inequality (7.1) is equivalent to

$$\int_{\Gamma_t^K}\int_{I(\delta)} \Big(\|\nabla \mathbf{A}\|^2 + \varepsilon^{-2}\big(\mathcal{H}_{\mathbf{A}_0}\mathbf{A} : \mathbf{A}\big) + \varepsilon^{-1}\big(\mathbf{T}_f(\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}) : \mathbf{A}\big)\Big) J(\sigma, r)\mathrm{d}r\mathrm{d}\sigma$$
$$\geq -C\int_{\Gamma_t^K}\int_{I(\delta)} \|\mathbf{A}\|^2 J(\sigma, r)\mathrm{d}r\mathrm{d}\sigma. \tag{7.4}$$

Using (7.3), it suffices to prove that for each $\sigma \in \Gamma_t^K$,

$$\int_{I(\delta)} \Big(\|\partial_r \mathbf{A}\|^2 + \varepsilon^{-2}\big(\mathcal{H}_{\mathbf{A}_0}\mathbf{A} : \mathbf{A}\big) + \varepsilon^{-1}\big(\mathbf{T}_f(\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}) : \mathbf{A}\big)\Big) J\mathrm{d}r$$
$$\geq -C\int_{I(\delta)} \|\mathbf{A}\|^2 J\mathrm{d}r. \tag{7.5}$$

Let $\mathbf{B}(x, t) = \Phi^{\mathrm{T}}\big(\varepsilon^{-1}d^K(x, t), x, t\big)\mathbf{A}(x, t)$ or $\mathbf{A}(x, t) = \Phi\big(\varepsilon^{-1}d^K(x, t), x, t\big)\mathbf{B}(x, t)$. Then

$$\|\partial_r \mathbf{A}\|^2 = \|\partial_r \mathbf{B}\|^2 + \|\partial_r \Phi\mathbf{B}\|^2 + 2\Phi\partial_r\mathbf{B} : \partial_r\Phi\mathbf{B},$$
$$\mathcal{H}_{\mathbf{A}_0}\mathbf{A} : \mathbf{A} = \mathcal{H}_{\mathbf{P}_0}\mathbf{B} : \mathbf{B}, \quad \mathbf{T}_f(\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}) : \mathbf{A} = \mathbf{T}_f(\mathbf{P}_0, \mathbf{P}_1, \mathbf{B}) : \mathbf{B}.$$

Therefore, (7.5) is reduced to

$$\int_{I(\delta)} \Big(\|\partial_r \mathbf{B}\|^2 + \frac{1}{\varepsilon^2}\mathcal{H}_{\mathbf{P}_0}\mathbf{B} : \mathbf{B} + \frac{1}{\varepsilon}\mathbf{T}_f(\mathbf{P}_0, \mathbf{P}_1, \mathbf{B}) : \mathbf{B} + 2(\Phi\partial_r\mathbf{B}) : (\partial_r\Phi\mathbf{B})\Big) J\mathrm{d}r$$
$$\geq -C\int_{I(\delta)} \|\mathbf{B}\|^2 J\mathrm{d}r, \tag{7.6}$$

which can be concluded from the following two inequalities:

$$\int_{I(\delta)} (\Phi\partial_r \mathbf{B}) : (\partial_r \Phi \mathbf{B}) J \mathrm{d}r \leqslant \frac{1}{4}\int_{I(\delta)} \left(\|\partial_r \mathbf{B}\|^2 + \frac{1}{\varepsilon^2}\mathcal{H}_{\mathbf{P}_0}\mathbf{B} : \mathbf{B} \right) J \mathrm{d}r + C\int_{I(\delta)} \|\mathbf{B}\|^2 J \mathrm{d}r, \tag{7.7}$$

$$\frac{1}{\varepsilon}\int_{I(\delta)} \mathbf{T}_f(\mathbf{P}_0, \mathbf{P}_1, \mathbf{B}) : \mathbf{B} J \mathrm{d}r \leqslant \frac{1}{4}\int_{I(\delta)} \left(\|\partial_r \mathbf{B}\|^2 + \frac{1}{\varepsilon^2}\mathcal{H}_{\mathbf{P}_0}\mathbf{B} : \mathbf{B} \right) J \mathrm{d}r + C\int_{I(\delta)} \|\mathbf{B}\|^2 J \mathrm{d}r. \tag{7.8}$$

In the sequel, without loss of generality, we will assume $\delta/4 = 1$ and let $I = [-1, 1]$ to simplify the notations.

7.2 Reduction to inequalities for scalar functions

Recall that $\mathbb{V}_i$ is a finite dimensional space which only depends on $\mathbf{n}(x, t)$. So for given $\sigma \in \Gamma_t^K$, we can choose $\{\mathbf{E}_\alpha : \alpha \in \Lambda_i\}$ to be a set of complete orthogonal bases of $\mathbb{V}_i$ which are smooth in r. Let $\Lambda = \cup_{i=1}^5 \Lambda_i$. Then we can write

$$\mathbf{B} = \sum_{\alpha\in\Lambda} p_\alpha \mathbf{E}_\alpha.$$

As $\mathbf{P}_0 = (\mathbf{I} - 2s_\varepsilon(r)\mathbf{nn})$ with $s_\varepsilon(\cdot) = s((\cdot)/\varepsilon)$, a direct calculation(see (4.5)) leads to

$$\mathcal{H}_{\mathbf{P}_0}\mathbf{B} : \mathbf{B} = \sum_{i=1}^5 \sum_{\alpha\in\Lambda_i} \kappa_i(s_\varepsilon(r)) p_\alpha^2, \tag{7.9}$$

where κ_i are defined in (4.5). Moreover,

$$\begin{aligned} \|\partial_r \mathbf{B}\|^2 &= \|\sum_{\alpha\in\Lambda} (\partial_r p_\alpha \mathbf{E}_\alpha + p_\alpha \partial_r \mathbf{E}_\alpha)\|^2 \geqslant \sum_\alpha |\partial_r p_\alpha|^2 + 2\sum_{\alpha\neq\beta} \partial_r p_\alpha p_\beta \mathbf{E}_\alpha : \partial_r \mathbf{E}_\beta \\ &= \sum_\alpha |\partial_r p_\alpha|^2 + \sum_{\alpha\neq\beta} (\partial_r p_\alpha p_\beta - p_\alpha \partial_r p_\beta)\mathbf{E}_\alpha : \partial_r \mathbf{E}_\beta. \end{aligned} \tag{7.10}$$

Then we obtain

$$\begin{aligned}
&\int_I \Big(\|\partial_r \mathbf{B}\|^2 + \varepsilon^{-2}\mathcal{H}_{\mathbf{P}_0}\mathbf{B} : \mathbf{B}\Big) J\,\mathrm{d}r + C\int_I \|\mathbf{B}\|^2 J\,\mathrm{d}r \\
&\geqslant \int_I \sum_{i=1}^{5}\sum_{\alpha\in\Lambda_i}\Big(|\partial_r p_\alpha|^2 + \frac{1}{\varepsilon^2}\kappa_i(s_\varepsilon(r))p_\alpha^2\Big) J\,\mathrm{d}r + C\int_I \sum_{\alpha} |p_\alpha|^2 J\,\mathrm{d}r \\
&\quad + \int_I \sum_{\alpha\neq\beta}(\partial_r p_\alpha p_\beta - p_\alpha \partial_r p_\beta)\mathbf{E}_\alpha : \partial_r \mathbf{E}_\beta J\,\mathrm{d}r. \qquad (7.11)
\end{aligned}$$

Since for $\alpha \in \Lambda_i (i = 1, 2, 5)$, $\beta \in \Lambda_j (j = 3, 4)$, one has $\mathbf{E}_\alpha \in \mathbb{S}_n$, $\partial_r \mathbf{E}_\beta \in \mathbb{A}_n$ which yields that $\mathbf{E}_\alpha : \partial_r \mathbf{E}_\beta = 0$. Thus, we only have to consider the case $\alpha, \beta \in \Lambda_1 \cup \Lambda_2 \cup \Lambda_5$ or $\alpha, \beta \in \Lambda_3 \cup \Lambda_4$.

Now we remove J via the endpoint estimates established in Lemmas 7.7 and 7.8. Let $q_\alpha = J^{\frac{1}{2}} p_\alpha$, and introduce the quadratic forms for $q \in H^1(I)$:

$$\mathcal{Q}_i(q) = \int_I \Big(|\partial_r q|^2 + \frac{1}{\varepsilon^2}\kappa_i(s_\varepsilon(r))q^2\Big)\mathrm{d}r, \qquad (7.12)$$

which are related to the scaled linearized operators

$$\mathcal{L}_{i,\varepsilon} = -\partial_r^2 + \frac{1}{\varepsilon^2}\kappa_i(s_\varepsilon(r)).$$

Then by using Lemmas 7.7 and 7.8 to control $|q_\alpha(\pm 1)|$, we obtain for sufficiently small ε that

$$\begin{aligned}
\int_I |\partial_r p_\alpha|^2 J\,\mathrm{d}r &= \int_I |\partial_r (J^{-\frac{1}{2}} q_\alpha)|^2 J\,\mathrm{d}r \\
&= \int_I |\partial_r q_\alpha|^2 + \Big[|\partial_r (J^{-\frac{1}{2}})|^2 J - \partial_r(\partial_r J^{-\frac{1}{2}} J^{\frac{1}{2}})\Big] q_\alpha^2 \mathrm{d}r \\
&\quad + \partial_r J^{-\frac{1}{2}} J^{\frac{1}{2}} q_\alpha^2\Big|_{-1}^{1} \\
&\geqslant \int_I |\partial_r q_\alpha|^2 \mathrm{d}r - \frac{1}{4}\mathcal{Q}_i(q_\alpha) - C\int_I q_\alpha^2 \mathrm{d}r.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&\int_I \sum_{i=1}^{5}\sum_{\alpha\in\Lambda_i}\Big(|\partial_r p_\alpha|^2 + \frac{1}{\varepsilon^2}\kappa_i(s_\varepsilon(r))p_\alpha^2\Big) J\,\mathrm{d}r + C\int_I \sum_{\alpha} |p_\alpha|^2 J\,\mathrm{d}r \\
&\geqslant \frac{3}{4}\sum_{i=1}^{5}\sum_{\alpha\in\Lambda_i}\mathcal{Q}_i(q_\alpha) + C\int_I \sum_{\alpha} |q_\alpha|^2 \mathrm{d}r. \qquad (7.13)
\end{aligned}$$

Let $\mathbf{W} = \Phi^{\mathrm{T}}\partial_r\Phi$ which is antisymmetric. Notice that

$$\begin{aligned}(\Phi\partial_r\mathbf{B}) : (\partial_r\Phi\mathbf{B}) = \partial_r\mathbf{B} : (\mathbf{W}\mathbf{B}) &= \sum_{\alpha,\beta} p_\beta(\partial_r p_\alpha \mathbf{E}_\alpha + p_\alpha\partial_r\mathbf{E}_\alpha) : (\mathbf{W}\mathbf{E}_\beta)\\ &= \frac{1}{2}\sum_{\alpha\neq\beta}(\partial_r p_\alpha p_\beta - p_\alpha\partial_r p_\beta)\mathbf{E}_\alpha : (\mathbf{W}\mathbf{E}_\beta)\\ &\quad + \sum_{\alpha,\beta} p_\alpha p_\beta\partial_r\mathbf{E}_\alpha : (\mathbf{W}\mathbf{E}_\beta), \qquad (7.14)\end{aligned}$$

as well as that

$$\int_I \sum_{\alpha,\beta} p_\alpha p_\beta\partial_r\mathbf{E}_\alpha : (\mathbf{W}\mathbf{E}_\beta)J\mathrm{d}r \leqslant C\int_I\sum_\alpha |p_\alpha|^2 J\mathrm{d}r = C\int_I\sum_\alpha |q_\alpha|^2\mathrm{d}r.$$

Then combining (7.11), (7.13), (7.14) and

$$\begin{aligned}(\partial_r p_\alpha p_\beta - p_\alpha\partial_r p_\beta)J &= \partial_r(q_\alpha J^{-\frac{1}{2}})q_\beta J^{\frac{1}{2}} - q_\alpha\partial_r(q_\beta J^{-\frac{1}{2}})J^{\frac{1}{2}}\\ &= \partial_r q_\alpha q_\beta - q_\alpha\partial_r q_\beta,\end{aligned}$$

the inequality (7.7) can be deduced from

$$\int_I\sum_{\alpha\neq\beta}(\partial_r q_\alpha q_\beta - q_\alpha\partial_r q_\beta)a(r)\mathrm{d}r \leqslant \frac{1}{4}\sum_{i=1}^5\sum_{\alpha\in\Lambda_i}\mathcal{Q}_i(q_\alpha) + C\int_I\sum_\alpha |q_\alpha|^2\mathrm{d}r, \qquad (7.15)$$

for $a(r) = \mathbf{E}_\alpha : \partial_r\mathbf{E}_\beta$ or $\mathbf{E}_\alpha : (\mathbf{W}\mathbf{E}_\beta)$.

To obtain (7.8), we introduce

$$\tilde{\mathbf{B}} = J^{1/2}\mathbf{B} = \sum_{\alpha\in\Lambda} q_\alpha\mathbf{E}_\alpha. \qquad (7.16)$$

Then (7.8) can be deduced from (7.15) and

$$\int_I \varepsilon^{-1}\mathbf{T}_f(\mathbf{P}_0,\mathbf{P}_1,\tilde{\mathbf{B}}) : \tilde{\mathbf{B}}\mathrm{d}r \leqslant \frac{1}{4}\sum_{i=1}^5\sum_{\alpha\in\Lambda_i}\mathcal{Q}_i(q_\alpha) + C\int_I\sum_\alpha |q_\alpha|^2\mathrm{d}r. \qquad (7.17)$$

The left hand side of (7.15) and (7.17) are called cross terms and correction terms of the next order, which will be proved in Sects. 7.4 and 7.5 respectively. Again, in the sequel, we will assume that ε is sufficiently small.

7.3 Estimates for quadratic forms $\mathcal{Q}_i$

Here and in what follows, we write $L^2 = L^2(I) = L^2([-1,1])$. Moreover, we will always assume the functions $q, q_i \in H^1(I)$.

7.3.1 Coercive estimates

As $\kappa_4 = 0, \kappa_5 = 2$, the following estimate is obvious:

$$\|\partial_r q\|_{L^2}^2 = \mathcal{Q}_4(q), \quad \|\partial_r q\|_{L^2}^2 + \varepsilon^{-2}\|q\|_{L^2}^2 \leqslant \mathcal{Q}_5(q). \tag{7.18}$$

For $1 \leqslant i \leqslant 3$, we let $\theta_{i,\varepsilon}(r) = \theta_i(r/\varepsilon)$. Recalling the definition (4.7), we have

$$(\theta_{1,\varepsilon}, \theta_{2,\varepsilon}, \theta_{3,\varepsilon})(r) = (\theta_\varepsilon, s_\varepsilon, 1 - s_\varepsilon)(r), \tag{7.19}$$

where $s_\varepsilon(r) = s(r/\varepsilon)$ and $\theta_\varepsilon(r) = s'(r/\varepsilon) = \sqrt{2}s_\varepsilon(1 - s_\varepsilon)$. Then for $\mathcal{Q}_i(q)(1 \leqslant i \leqslant 3)$, we have

Lemma 7.1 *Let $q_i = \theta_{i,\varepsilon}\bar{q}_i$ for $i = 1, 2, 3$. Then there exists $C_0 > 0$ such that*

$$\frac{1}{4}\int_I \theta_{i,\varepsilon}^2\big(\partial_r \bar{q}_i\big)^2 \mathrm{d}r \leqslant \mathcal{Q}_i(q_i) + \frac{C_0}{\varepsilon}e^{-\frac{2\sqrt{2}}{\varepsilon}}\int_I q_i^2 \mathrm{d}r. \tag{7.20}$$

The above lemma is a direct corollary of Lemmas 7.2 and 7.3. It gives another version of the first eigenvalue estimate for $\mathcal{L}_{i,\varepsilon}$, which has been proved in [12] (for $i = 1$) and [16]. The method presented here uses only elementary decompositions and does not rely on the maximum/comparison principle or the Harnack inequality. Note that no boundary condition is needed here and the lower bounds $-\frac{C_0}{\varepsilon}e^{-\frac{2\sqrt{2}}{\varepsilon}}$ for $i = 2, 3$ is optimal. We can also deduce an optimal lower bound $-\frac{C_0}{\varepsilon^2}e^{-\frac{2\sqrt{2}}{\varepsilon}}$ for the first eigenvalue of $\mathcal{L}_{1,\varepsilon}$ from Lemma 7.3.

Lemma 7.2 *Let $q_i = \theta_{i,\varepsilon}\bar{q}_i$ for $i = 2, 3$. Then for any $\nu_0 > 0$, there exists $C_0(\nu_0), C_1(\nu_0) > 0$ such that*

$$\mathcal{Q}_i(q_i) \geqslant \Big(\frac{1}{2} + e^{-\frac{\sqrt{2}}{\varepsilon}}C_1(\nu_0) - \nu_0\Big)\int_I \theta_{i,\varepsilon}^2\big(\partial_r \bar{q}_i\big)^2 \mathrm{d}r - \frac{1}{\varepsilon}e^{-\frac{2\sqrt{2}}{\varepsilon}}C_0(\nu_0)\int_I \theta_{i,\varepsilon}^2 \bar{q}_i^2 \mathrm{d}r. \tag{7.21}$$

Proof Using the fact that $\varepsilon^2\partial_r^2 s_\varepsilon = s_\varepsilon \kappa_2(s_\varepsilon)$, we arrive at

$$\begin{aligned}\mathcal{Q}_2(q_2) &= \int_I \Big(\big[\partial_r s_\varepsilon \bar{q}_2 + s_\varepsilon \partial_r \bar{q}_2\big]^2 + \varepsilon^{-2}\kappa_2(s_\varepsilon) s_\varepsilon^2 \bar{q}_2^2 \Big) \mathrm{d}r \\ &= s_\varepsilon \partial_r s_\varepsilon \bar{q}_2^2 \Big|_{-1}^{1} + \int_I s_\varepsilon^2 (\partial_r \bar{q}_2)^2 \mathrm{d}r \\ &\geqslant -(s_\varepsilon \partial_r s_\varepsilon \bar{q}_2^2)(-1) + \int_I s_\varepsilon^2 (\partial_r \bar{q}_2)^2 \mathrm{d}r.\end{aligned}$$

As $s_\varepsilon(r) = 1/(1 + e^{-\sqrt{2}r/\varepsilon})$, one can directly get

$$\begin{aligned}&(s_\varepsilon \partial_r s_\varepsilon)(-1) = \varepsilon^{-1}\sqrt{2} s_\varepsilon^2 (1 - s_\varepsilon)(-1) \leqslant \frac{\sqrt{2}}{\varepsilon} e^{-\frac{2\sqrt{2}}{\varepsilon}}, \\ &\int_{-1}^{0} s_\varepsilon^{-2} \mathrm{d}r = \int_{-1}^{0} (1 + e^{-\frac{\sqrt{2}}{\varepsilon}})^2 \mathrm{d}r = \frac{\varepsilon e^{\frac{2\sqrt{2}}{\varepsilon}}}{2\sqrt{2}} \Big(1 + O\big(e^{-\sqrt{2}/\varepsilon}\big) \Big).\end{aligned}$$

Moreover, from the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}\bar{q}_2(0)^2 &\leq \frac{\nu_1^2}{2} \int_0^1 \big(\partial_r \bar{q}_2\big)^2 \mathrm{d}r + C(\nu_1) \int_0^1 \bar{q}_2^2 \mathrm{d}r \\ &\leq 2\nu_1^2 \int_0^1 \big(s_\varepsilon \partial_r \bar{q}_2\big)^2 \mathrm{d}r + C(\nu_1) \int_0^1 s_\varepsilon^2 \bar{q}_2^2 \mathrm{d}r.\end{aligned}$$

Thus choosing $\nu_1 = \nu_0/2$ and ε sufficiently small, we obtain

$$\begin{aligned}&(s_\varepsilon \partial_r s_\varepsilon \bar{q}_2^2)(-1) \\ &\leqslant (s_\varepsilon \partial_r s_\varepsilon)(-1) \Big(|\bar{q}_2(0)| + \Big| \int_{-1}^{0} \partial_r \bar{q}_2 \mathrm{d}r \Big| \Big)^2 \\ &\leqslant (s_\varepsilon \partial_r s_\varepsilon)(-1) \Big((1 + \nu_1^{-1}) |\bar{q}_2(0)|^2 + (1 + \nu_1) \Big| \int_{-1}^{0} (s_\varepsilon \partial_r \bar{q}_2)^2 \mathrm{d}r \Big| \Big| \int_{-1}^{0} s_\varepsilon^{-2} \mathrm{d}r \Big| \Big) \\ &\leqslant \frac{\sqrt{2}}{\varepsilon} e^{-\frac{2\sqrt{2}}{\varepsilon}} \Big(3\nu_1 \int_0^1 s_\varepsilon^2 \big(\partial_r \bar{q}_2\big)^2 \mathrm{d}r + C(\nu_1) \int_0^1 s_\varepsilon^2 \bar{q}_2^2 \mathrm{d}r \Big) \\ &\quad + \Big(\frac{1 + \nu_1}{2} + O\big(e^{-\frac{\sqrt{2}}{\varepsilon}}\big) \Big) \int_{-1}^{0} (s_\varepsilon \partial_r \bar{q}_2)^2 \mathrm{d}r \\ &\leqslant \Big(\frac{1}{2} + \nu_0 + O\big(e^{-\frac{\sqrt{2}}{\varepsilon}}\big) \Big) \int_I s_\varepsilon^2 \big(\partial_r \bar{q}_2\big)^2 \mathrm{d}r + \frac{1}{\varepsilon} e^{-\frac{2\sqrt{2}}{\varepsilon}} C_1(\nu_0) \int_I s_\varepsilon^2 \bar{q}_2^2 \mathrm{d}r.\end{aligned}$$

Then (7.21) for $i = 2$ follows immediately. The case of $i = 3$ can be proved in a similar way. □

Lemma 7.3 *Let $q_1 = \theta_{1,\varepsilon}\bar{q}_1$. For any $\nu_0 > 0$, there exists $C_0(\nu_0)$, $C_1(\nu_0) > 0$ such that*

$$\begin{aligned}\mathcal{Q}_1(q_1) \geqslant & \Big(\frac{1}{2}+e^{-\frac{\sqrt{2}}{\varepsilon}}C_1(\nu_0)-\nu_0\Big)\int_I \theta_{1,\varepsilon}^2\big(\partial_r\bar{q}_1\big)^2\mathrm{d}r \\ & -\frac{1}{\varepsilon^2}e^{-\frac{2\sqrt{2}}{\varepsilon}}C_0(\nu_0)\int_I \theta_{1,\varepsilon}^2\bar{q}_1^2\mathrm{d}r. \end{aligned} \tag{7.22}$$

Remark 7.4 The constant $\frac{1}{2}$ on the right hand side of (7.21) and (7.22) is optimal.

Proof As $\theta_{1,\varepsilon} = \theta_\varepsilon = \sqrt{2}s_\varepsilon(1-s_\varepsilon)$, one can get

$$\begin{aligned}\mathcal{Q}_1(q_1) &= \int_I \Big(\big[\partial_r\theta_\varepsilon\bar{q}_1 + \theta_\varepsilon\partial_r\bar{q}_1\big]^2 + \varepsilon^{-2}\kappa_1(s_\varepsilon)\theta_\varepsilon^2\bar{q}_1^2\Big)\mathrm{d}r \\ &= (\theta_\varepsilon\partial_r\theta_\varepsilon\bar{q}_1^2)\Big|_{-1}^1 + \int_I \theta_\varepsilon^2(\partial_r\bar{q}_1)^2\mathrm{d}r. \end{aligned} \tag{7.23}$$

It suffices to estimate $(\theta_\varepsilon\partial_r\theta_\varepsilon\bar{q}_1^2)|_{-1}^1$. Direct calculations give that

$$|(\theta_\varepsilon\partial_r\theta_\varepsilon)(\pm1)| = \frac{2\sqrt{2}}{\varepsilon}|s_\varepsilon^2(1-s_\varepsilon)^2(1-2s_\varepsilon)|(\pm1) \leqslant \frac{2\sqrt{2}}{\varepsilon}e^{-\frac{2\sqrt{2}}{\varepsilon}},$$

$$\int_0^1 \theta_\varepsilon^{-2}\mathrm{d}r = \frac{1}{2}\int_0^1 e^{2\sqrt{2}r/\varepsilon}\mathrm{d}r\Big(1+O\big(e^{-\frac{\sqrt{2}}{\varepsilon}}\big)\Big) = \frac{\varepsilon e^{\frac{2\sqrt{2}}{\varepsilon}}}{4\sqrt{2}}\Big(1+O\big(e^{-\frac{\sqrt{2}}{\varepsilon}}\big)\Big).$$

Moreover, from the Gagliardo-Nirenberg inequality and a scaling argument, we have

$$\begin{aligned}|\bar{q}_1(0)|^2 \leq & \frac{\nu_1^2}{100}\varepsilon\int_{-\varepsilon}^{\varepsilon}\big(\partial_r\bar{q}_1\big)^2\mathrm{d}r + C(\nu_1)\frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\bar{q}_1^2\mathrm{d}r \\ \leq & 2\nu_1^2\varepsilon\int_{-\varepsilon}^{\varepsilon}\theta_\varepsilon^2\big(\partial_r\bar{q}_1\big)^2\mathrm{d}r + \varepsilon^{-1}C(\nu_1)\int_{-\varepsilon}^{\varepsilon}\theta_\varepsilon^2\bar{q}_1^2\mathrm{d}r. \end{aligned}$$

Thus choosing $\nu_1 = \nu_0/2$ and ε sufficiently small, we obtain

$$\begin{aligned}&|(\theta_\varepsilon\partial_r\theta_\varepsilon\bar{q}_1^2)(1)| \\ &\leqslant |(\theta_\varepsilon\partial_r\theta_\varepsilon)(1)|\Big(|\bar{q}_1(0)| + \Big|\int_0^1\partial_r\bar{q}_1\mathrm{d}r\Big|\Big)^2 \\ &\leqslant |(\theta_\varepsilon\partial_r\theta_\varepsilon)(1)|\Big((1+\nu_1^{-1})|\bar{q}_1(0)|^2+(1+\nu_1)\Big[\int_0^1(\theta_\varepsilon\partial_r\bar{q}_1)^2\mathrm{d}r\Big]\int_0^1\theta_\varepsilon^{-2}\mathrm{d}r\Big)\end{aligned}$$

$$\begin{aligned}
&\leqslant \frac{2\sqrt{2}}{\varepsilon} e^{-\frac{2\sqrt{2}}{\varepsilon}} \Big(3\nu_1 \varepsilon \int_{-\varepsilon}^{\varepsilon} \theta_\varepsilon^2 \big(\partial_r \bar{q}_1\big)^2 \mathrm{d}r + \varepsilon^{-1} C(\nu_1) \int_{-\varepsilon}^{\varepsilon} \theta_\varepsilon^2 \bar{q}_1^2 \mathrm{d}r \Big) \\
&\quad + \Big(\frac{1+\nu_1}{2} + O\big(e^{-\frac{\sqrt{2}}{\varepsilon}}\big) \Big) \int_0^1 (\theta_\varepsilon \partial_r \bar{q}_1)^2 \mathrm{d}r \\
&\leqslant \Big(\frac{1}{2} + \nu_0 + O\big(e^{-\frac{\sqrt{2}}{\varepsilon}}\big) \Big) \int_I \theta_\varepsilon^2 \big(\partial_r \bar{q}_1\big)^2 \mathrm{d}r + \frac{1}{\varepsilon^2} e^{-\frac{2\sqrt{2}}{\varepsilon}} C_1(\nu_0) \int_I \theta_\varepsilon^2 \bar{q}_1^2 \mathrm{d}r.
\end{aligned} \tag{7.24}$$

The same estimate holds for the value at -1. Thus, (7.22) follows. □

The following lemma gives a new proof for the second eigenvalue estimate of $\mathcal{L}_{1,\varepsilon}$.

Lemma 7.5 *Let $q_1 = \mu\theta_\varepsilon + \hat{q}_1$ with $\mu \in \mathbb{R}$ and $\int_I \theta_\varepsilon \hat{q}_1 \mathrm{d}r = 0$. Then there exists $c_0 > 0$ such that for ε small:*

$$\mathcal{Q}_1(q_1) + o(\varepsilon^2) \int_I q_1^2 \mathrm{d}r \geqslant \frac{c_0}{\varepsilon^2} \int_I \hat{q}_1^2 \mathrm{d}r. \tag{7.25}$$

Proof Let $\bar{q}_1 = q_1/\theta_\varepsilon - \mu = \hat{q}_1/\theta_\varepsilon$. Then it follows from (7.22) that

$$\mathcal{Q}_1(q_1) + o(\varepsilon^2) \int_I q_1^2 \mathrm{d}r \geqslant \frac{1}{4} \int_I \theta_\varepsilon^2 \big(\partial_r \bar{q}_1\big)^2 \mathrm{d}r.$$

Now we prove that for $\int_I \theta_\varepsilon^2 \bar{q}_1 \mathrm{d}r = 0$ and some $c_0 > 0$,

$$\int_I \theta_\varepsilon^2 \big(\partial_r \bar{q}_1\big)^2 \mathrm{d}r \geqslant \frac{c_0}{\varepsilon^2} \int_I \theta_\varepsilon^2 \bar{q}_1^2 \mathrm{d}r = \frac{c_0}{\varepsilon^2} \int_I \hat{q}_1^2 \mathrm{d}r.$$

Let

$$\begin{aligned}
&A_1 = \int_0^1 \theta_\varepsilon^2(\tau) \bar{q}_1^2(\tau) \mathrm{d}\tau, \quad B_1 = \int_0^1 \theta_\varepsilon^2 \big(\partial_r \bar{q}_1\big)^2 \mathrm{d}r, \quad D_1 = \int_0^1 \theta_\varepsilon^2(\tau) \bar{q}_1(\tau) \mathrm{d}\tau, \\
&A_2 = \int_{-1}^0 \theta_\varepsilon^2(\tau) \bar{q}_1^2(\tau) \mathrm{d}\tau, \quad B_2 = \int_{-1}^0 \theta_\varepsilon^2 \big(\partial_r \bar{q}_1\big)^2 \mathrm{d}r, \quad D_2 = \int_{-1}^0 \theta_\varepsilon^2(\tau) \bar{q}_1(\tau) \mathrm{d}\tau.
\end{aligned}$$

Assume that $A_1 + A_2 > 0$. We have

$$\begin{aligned}
\int_0^1 \theta_\varepsilon^2(\tau) \bar{q}_1^2(\tau) \mathrm{d}\tau &= \int_0^1 \theta_\varepsilon^2(\tau) \bar{q}_1(\tau) \Big(\bar{q}_1(0) + \int_0^\tau \partial_r \bar{q}_1 \mathrm{d}r \Big) \mathrm{d}\tau \\
&= \bar{q}_1(0) D_1 + \int_0^1 \theta_\varepsilon^2(\tau) \bar{q}_1(\tau) \Big(\int_0^\tau \partial_r \bar{q}_1 \mathrm{d}r \Big) \mathrm{d}\tau \leqslant \bar{q}_1(0) D_1 \\
&\quad + \Big(\int_0^1 \theta_\varepsilon^2(\tau) \bar{q}_1^2(\tau) \mathrm{d}\tau \Big)^{\frac{1}{2}} \Big(\int_0^1 \theta_\varepsilon^2(\tau) \Big(\int_0^\tau \partial_r \bar{q}_1 \mathrm{d}r \Big)^2 \mathrm{d}\tau \Big)^{\frac{1}{2}}.
\end{aligned}$$

On the other hand, we have

$$\int_0^1 \theta_\varepsilon^2(\tau)\Big(\int_0^\tau \partial_r \bar{q}_1 \mathrm{d}r\Big)^2 \mathrm{d}\tau \leqslant \int_0^1 \theta_\varepsilon^2(\tau)\Big(\int_0^\tau \theta_\varepsilon(\partial_r \bar{q}_1)^2 \mathrm{d}r\Big)\Big(\int_0^\tau \frac{1}{\theta_\varepsilon}\mathrm{d}r\Big)\mathrm{d}\tau$$
$$= \int_0^1 \theta_\varepsilon(\partial_r \bar{q}_1)^2 I_0(r)\mathrm{d}r$$

with

$$I_0(r) = \int_r^1 \theta_\varepsilon^2(\tau)\Big(\int_0^\tau \frac{1}{\theta_\varepsilon}(y)\mathrm{d}y\Big)\mathrm{d}\tau$$
$$= \varepsilon^2 \int_{\frac{r}{\varepsilon}}^{\frac{1}{\varepsilon}} \theta^2(z)\Big(\int_0^z \frac{1}{\theta}(w)\mathrm{d}w\Big)\mathrm{d}z \leqslant C\varepsilon^2\theta_\varepsilon(r).$$

Thus

$$A_1 \leqslant \bar{q}_1(0)D_1 + C\varepsilon A_1^{1/2}B_1^{1/2}.$$

Similarly, we have

$$A_2 \leqslant \bar{q}_1(0)D_2 + C\varepsilon A_2^{1/2}B_2^{1/2}.$$

As $D_1 + D_2 = 0$, we get

$$A_1 + A_2 \leqslant C\varepsilon A_1^{1/2}B_1^{1/2} + C\varepsilon A_2^{1/2}B_2^{1/2} \leqslant C\varepsilon(A_1 + A_2)^{1/2}(B_1 + B_2)^{1/2},$$

which concludes our lemma. □

As $|\partial_r \theta_\varepsilon| \leqslant \frac{C}{\varepsilon}\theta_\varepsilon$, and $|\partial_r \hat{q}_1| \leqslant |\partial_r \theta_\varepsilon \bar{q}_1| + |\theta_\varepsilon \partial_r \bar{q}_1|$, we have the following

Corollary 7.6 *Let $q_1 = \mu\theta_\varepsilon + \hat{q}_1$ with $\mu \in \mathbb{R}$ and $\int_I \theta_\varepsilon \hat{q}_1 \mathrm{d}r = 0$. Then there exists $c_0 > 0$ such that:*

$$\mathcal{Q}_1(q_1) + o(\varepsilon^2)\int_I q_1^2 \mathrm{d}r \geqslant c_0 \int_I (\partial_r \hat{q}_1)^2 \mathrm{d}r. \tag{7.26}$$

7.3.2 Endpoints L^∞ estimates

Again, we assume that ε is sufficiently small.

Lemma 7.7 (L^∞ control) *For $i = 2, 3, 4, 5$, and any $\nu_0 > 0$, there exists $C(\nu_0)$ such that*

$$\|q\|^2_{L^\infty([-1,1])} \leqslant \nu_0 \mathcal{Q}_i(q) + C(\nu_0)\int_I q^2 \mathrm{d}r.$$

Proof The claims for $i = 4, 5$ are obvious, since $\kappa_4 = 0, \kappa_5 \geqslant 0$. For $i = 2$, let $q = s_\varepsilon \bar{q}$. Then

$$\|\bar{q}\|^2_{L^\infty([0,1])} \leqslant \nu_0 \int_0^1 s_\varepsilon^2 (\partial_r \bar{q})^2 \mathrm{d}r + C(\nu_0) \int_0^1 s_\varepsilon^2 \bar{q}^2 \mathrm{d}r.$$

For $r \in [-1, 0]$, we have

$$\begin{aligned} |q(r)| = s_\varepsilon(r)|\bar{q}(r)| &\leqslant s_\varepsilon(r)(|\bar{q}(0)| + \int_0^r |\partial_r \bar{q}| \mathrm{d}r) \\ &\leqslant \frac{1}{2}|\bar{q}(0)| + s_\varepsilon(r)\Big(\int_0^r s_\varepsilon^2 |\partial_r \bar{q}|^2 \mathrm{d}r\Big)^{1/2}\Big(\int_0^r s_\varepsilon^{-2} \mathrm{d}r\Big)^{1/2} \\ &\leqslant \frac{1}{2}|\bar{q}(0)| + O(\sqrt{\varepsilon})\Big(\int_0^r s_\varepsilon^2 |\partial_r \bar{q}|^2 \mathrm{d}r\Big)^{1/2}. \end{aligned}$$

Thus using Lemma 7.1, we obtain the claim for $i = 2$. The proof for $i = 3$ is similar. □

The above lemma is not true for $i = 1$. However, we have the following estimate.

Lemma 7.8 (Endpoints control) *There exists $C > 0$ such that*

$$|q(\pm 1)|^2 \leqslant C\varepsilon \Big\{ \mathcal{Q}_1(q) + \int_I q^2 \mathrm{d}r \Big\}.$$

Proof Let $q = \theta_\varepsilon \bar{q}$. Note that $\theta_\varepsilon^2(\pm 1) \leqslant C\varepsilon |(\theta_\varepsilon \partial_r \theta_\varepsilon)(\pm 1)|$, then we can get the result from (7.24) and Lemma 7.3. □

7.4 Estimate for cross terms

Now the inequality (7.15) is a consequence of the following Propositions 7.9, 7.11, 7.12 and 7.13 by letting $a(r) = \mathbf{E}_\alpha : \partial_r \mathbf{E}_\beta$ or $\mathbf{E}_\alpha : \mathbf{W}\mathbf{E}_\beta$.

Proposition 7.9 *Assume i or $j \in \{4, 5\}$. Then for any $\nu_0 > 0$ there exists $C_0 = C_0(\nu_0, \|a\|_{W^{1,\infty}}) > 0$ such that*

$$\int_I \big(\partial_r q_\alpha q_\beta - q_\alpha \partial_r q_\beta\big) a(r) \mathrm{d}r \leq \nu_0 \big(\mathcal{Q}_i(q_\alpha) + \mathcal{Q}_j(q_\beta)\big) + C_0 \int_I \big(q_\alpha^2 + q_\beta^2\big) \mathrm{d}r.$$

Proof Assume $j = 4$ or 5. Then $\kappa_j \geqslant 0$ and $\mathcal{Q}_j(q_\beta) \geqslant \|\partial_r q_\beta\|_{L^2}^2$. Thus

$$\begin{aligned}
\int_I q_\alpha \partial_r q_\beta a(r)\mathrm{d}r &\leqslant \nu_0 \int_I (\partial_r q_\beta)^2 \mathrm{d}r + C(\nu_0, |a|_{L^\infty}) \int_I q_\alpha^2 \mathrm{d}r,\\
\int_I \partial_r q_\alpha q_\beta a(r)\mathrm{d}r &\leqslant -\int_I (q_\alpha \partial_r q_\beta a + q_\alpha q_\beta \partial_r a)\mathrm{d}r + (q_\alpha q_\beta a)\Big|_{-1}^{1}\\
&\leqslant \nu_0 \int_I (\partial_r q_\beta)^2 \mathrm{d}r + C(\nu_0, \|a\|_{W^{1,\infty}})\\
&\quad \times \int_I (q_\alpha^2 + q_\beta^2)\mathrm{d}r + (q_\alpha q_\beta a)\Big|_{-1}^{1}.
\end{aligned}$$

Then the result follows from Lemmas 7.7 and 7.8. □

Now we turn to crossing terms involving elements in $\mathbb{V}_1$. Assume $\mathbf{E}_\beta = \mathbf{E}_1 \in \mathbb{V}_1$. The following lemma shows that the variation of $\mathbf{E}_1$ and Φ along the normal direction ∇d^K is very small, which is key to bounding the crossing terms involving elements in $\mathbb{V}_1$.

Lemma 7.10 *There exists constant C_1 such that for $(x, t) \in \Gamma^K(\delta)$, one has that*

$$|\partial_r \mathbf{E}_1| \leqslant C_1(d^K(x,t) + \varepsilon), \qquad |\Phi^{\mathrm{T}} \partial_r \Phi \mathbf{E}_1| \leqslant C_1(d^K(x,t) + \varepsilon).$$

Proof Recalling $\partial_r = (1 + O(\varepsilon^{K+1}))\nabla d^K \cdot \nabla$, we can replace ∂_r with $\nabla d^K \cdot \nabla$. On the other hand, as $\|d^K - d_0\|_{C^1(\Gamma^K(\delta))} \leqslant C\varepsilon$, one has

$$\begin{aligned}
|(\nabla d^K \cdot \nabla)\mathbf{E}_1| &\leqslant |(\nabla d_0 \cdot \nabla)\mathbf{E}_1| + C_1\varepsilon \leqslant |(\nabla d_0 \cdot \nabla)\mathbf{E}_1|\Big|_{d_0=0} + C_1(d_0 + \varepsilon)\\
&= C_1(d_0 + \varepsilon).
\end{aligned}$$

In the last equality, we have used the fact that $\partial_\nu \mathbf{n}|_{d_0=0} = 0$. Similarly, we have

$$\begin{aligned}
|\Phi^{\mathrm{T}}(\nabla d^K \cdot \nabla)\Phi\mathbf{E}_1| &= |\Phi^{\mathrm{T}}(\nabla d_0 \cdot \nabla)\Phi\mathbf{E}_1| + C_1\varepsilon\\
&\leqslant |\Phi^{\mathrm{T}}(\nabla d_0 \cdot \nabla)\Phi\mathbf{E}_1|\Big|_{d_0=0} + C_1(d_0 + \varepsilon)\\
&= C_1(d_0 + \varepsilon),
\end{aligned}$$

as it holds on $\Gamma_0 = \{d_0(x,t) = 0\}$ that

$$(\Phi^{\mathrm{T}}(\nabla d_0 \cdot \nabla)\Phi\mathbf{E}_1)\Big|_{d_0=0} = (\mathbf{A}_-^{\mathrm{T}} \partial_\nu \mathbf{A}_- \mathbf{nn})\Big|_{d_0=0} = 0.$$

due to the boundary condition (1.6d). The proof is finished. □

Proposition 7.11 *Assume that i or $j = 1$ and $|a(r)| \leqslant C_a(r+\varepsilon)$. Then for any $\nu_0 > 0$ there exists $C_0 = C_0(\nu_0, C_a, \|a\|_{W^{1,\infty}}) > 0$ such that*

$$\int_I \big(\partial_r q_\alpha q_\beta - q_\alpha \partial_r q_\beta\big) a(r) \mathrm{d}r \leq \nu_0 \big(\mathcal{Q}_i(q_\alpha) + \mathcal{Q}_j(q_\beta)\big) + C_0 \int_I \big(q_\alpha^2 + q_\beta^2\big) \mathrm{d}r. \tag{7.27}$$

Proof Assume $j = 1$. Firstly, we have

$$\int_I \big(\partial_r q_\alpha q_\beta - q_\alpha \partial_r q_\beta\big) a(r) \mathrm{d}r \leqslant q_\alpha q_\beta a\Big|_{-1}^{1} - \int_I \big(2 q_\alpha \partial_r q_\beta a(r) + q_\alpha q_\beta \partial_r a(r)\big) \mathrm{d}r.$$

From Lemmas 7.7 and 7.8, it suffices to estimate $\int_I q_\alpha \partial_r q_\beta a(r) \mathrm{d}r$. Let

$$q_\beta = \mu_0 \theta_\varepsilon + \hat{q}_\beta, \text{ with } \mu_0 \in \mathbb{R} \text{ and } \int_I \theta_\varepsilon \hat{q}_\beta \mathrm{d}r = 0.$$

Then using the fact that $\partial_r \theta_\varepsilon = \frac{\sqrt{2}}{\varepsilon} \theta_\varepsilon (1 - 2 s_\varepsilon)$, we get

$$\begin{aligned} \int_I q_\alpha \partial_r q_\beta a(r) \mathrm{d}r &= \mu_0 \int_I q_\alpha \partial_r \theta_\varepsilon a(r) \mathrm{d}r + \int_I q_\alpha \partial_r \hat{q}_\beta a(r) \mathrm{d}r \\ &= \mu_0 \sqrt{2} \int_I q_\alpha \theta_\varepsilon (1 - 2 s_\varepsilon) \frac{a(r)}{\varepsilon} \mathrm{d}r + \int_I q_\alpha \partial_r \hat{q}_\beta a(r) \mathrm{d}r \\ &\leqslant C \|q_\alpha\|_{L^2} \Big(|\mu_0| \|\varepsilon^{-1} \theta_\varepsilon a\|_{L^2} + \|\partial_r \hat{q}_\beta\|_{L^2} \Big). \end{aligned}$$

As $a(r) \leqslant C_a(r + \varepsilon)$, we have

$$|\mu_0| \|\varepsilon^{-1} \theta_\varepsilon a\|_{L^2} \leqslant C_a |\mu_0| \big(\|\theta_\varepsilon\|_{L^2} + \|\theta_\varepsilon r/\varepsilon\|_{L^2} \big) \leqslant C |\mu_0| \|\theta_\varepsilon\|_{L^2} \leqslant C \|q_\beta\|_{L^2}.$$

Moreover, due to Corollary 7.6, we can control $\|\partial_r \hat{q}_\beta\|_{L^2}^2$ by the right hand side of (7.27). □

Proposition 7.12 *Let $i = 2$ or 3. For any $\nu_0 > 0$ there exists $C_0 = C_0(\nu_0, \|a\|_{L^\infty}) > 0$ such that*

$$\int_I \big(\partial_r q_\alpha q_\beta - q_\alpha \partial_r q_\beta\big) a(r) \mathrm{d}r \leq \nu_0 \big(\mathcal{Q}_i(q_\alpha) + \mathcal{Q}_i(q_\beta)\big) + C_0 \int_I \big(q_\alpha^2 + q_\beta^2\big) \mathrm{d}r.$$

Proof We assume $i = 2$, as the case $i = 3$ can be proved similarly. We use the decompositions:

$$q_\alpha(r) = \theta_{2,\varepsilon}(r) \bar{q}_\alpha(r) = s_\varepsilon(r) \bar{q}_\alpha(r), \quad q_\beta(r) = s_\varepsilon(r) \bar{q}_\beta(r).$$

Then $\partial_r q_\alpha q_\beta - q_\alpha \partial_r q_\beta = s_\varepsilon^2(\partial_r \bar{q}_\alpha \bar{q}_\beta - \bar{q}_\alpha \partial_r \bar{q}_\beta)$. Thus, using Cauchy-Schwartz inequality, we get

$$\int_I \big(\partial_r q_\alpha q_\beta - q_\alpha \partial_r q_\beta\big) a(r)\mathrm{d}r \le \int_I s_\varepsilon^2 \big[\nu_0\big((\partial_r \bar{q}_\alpha)^2 + (\partial_r \bar{q}_\beta)^2\big) + C_0(\bar{q}_\alpha^2 + \bar{q}_\beta^2)\big]\mathrm{d}r.$$

which yields the conclusion by Lemma 7.1. □

The estimate of crossing terms for $\mathbb{V}_2$ and $\mathbb{V}_3$ is more subtle.

Proposition 7.13 *For any $\nu_0 > 0$ there exists $C_0 = C_0(\nu_0, |a|_{L^\infty}) > 0$ such that*

$$\int_I \big(\partial_r q_2 q_3 - q_2 \partial_r q_3\big) a(r)\mathrm{d}r \leqslant \nu_0\big(\mathcal{Q}_2(q_2) + \mathcal{Q}_3(q_3)\big) + C_0 \int_I \big(q_2^2 + q_3^2\big)\mathrm{d}r. \tag{7.28}$$

Proof We use the decompositions:

$$\begin{aligned} q_2(r) &= \theta_{2,\varepsilon}(r)\bar{q}_2(r) = s_\varepsilon(r)\bar{q}_2(r), \\ q_3(r) &= \theta_{3,\varepsilon}(r)\bar{q}_3(r) = (1 - s_\varepsilon(r))\bar{q}_3(r). \end{aligned} \tag{7.29}$$

Using Lemma 7.1, it suffices to prove that the left side is bounded by

$$\sum_{i=2,3} \int_I \theta_{i,\varepsilon}^2 \Big(\nu_0 (\partial_r \bar{q}_i)^2 + C_0 \bar{q}_i^2\Big)\mathrm{d}r.$$

One has

$$\begin{aligned} \partial_r q_2 q_3 - q_2 \partial_r q_3 &= \big(\partial_r s_\varepsilon \bar{q}_2 + s_\varepsilon \partial_r \bar{q}_2\big)(1 - s_\varepsilon)\bar{q}_3 \\ &\quad - s_\varepsilon \bar{q}_2\big[-\partial_r s_\varepsilon \bar{q}_3 + (1 - s_\varepsilon)\partial_r \bar{q}_3\big] \\ &= \partial_r s_\varepsilon \bar{q}_2 \bar{q}_3 + \theta_{2,\varepsilon}\theta_{3,\varepsilon}\big(\partial_r \bar{q}_2 \bar{q}_3 - \bar{q}_2 \partial_r \bar{q}_3\big). \end{aligned}$$

Using Cauchy-Schwartz inequality, one gets

$$\Big|\int_I \theta_{2,\varepsilon}\theta_{3,\varepsilon}\big(\partial_r \bar{q}_2 \bar{q}_3 - \bar{q}_2 \partial_r \bar{q}_3\big) a(r)\mathrm{d}r\Big| \leqslant \frac{1}{2}\sum_{i=2,3} \int_I \theta_{i,\varepsilon}^2 \Big(\nu_0 (\partial_r \bar{q}_i)^2 + C_0 \bar{q}_i^2\Big)\mathrm{d}r.$$

Then together with the next Lemma 7.14, we immediately obtain Proposition 7.13. □

Lemma 7.14 *Let $a(r) \in L^\infty([-1,1])$. Then for any $\nu_0 > 0$ there exists $C_0 = C_0(\nu_0, |a|_{L^\infty}) > 0$ such that*

$$\Big|\int_I \partial_r s_\varepsilon \bar{q}_2 \bar{q}_3 a(r)\mathrm{d}r\Big| \le \sum_{i=2,3} \int_I \theta_{i,\varepsilon}^2 \Big(\nu_0 (\partial_r \bar{q}_i)^2 + C_0 \bar{q}_i^2\Big)\mathrm{d}r. \tag{7.30}$$

Proof Recall $\theta_{2,\varepsilon} = s_\varepsilon$ and $\theta_{3,\varepsilon} = 1 - s_\varepsilon$. Let

$$I_1 = \|s_\varepsilon \bar{q}_2\|_{L^2}^2 + \|(1 - s_\varepsilon)\bar{q}_3\|_{L^2}^2, \quad I_2 = \|s_\varepsilon(\partial_r \bar{q}_2)\|_{L^2}^2 + \|(1 - s_\varepsilon)(\partial_r \bar{q}_3)\|_{L^2}^2.$$

By the Gagliardo-Nirenberg inequality, one has that for any $\nu_1 > 0$, there exists $C(\nu_1) > 0$ such that

$$\|\bar{q}_2\|_{L^\infty([0,1])} \le \frac{\nu_1}{2}\|\partial_r \bar{q}_2\|_{L^2[0,1]} + C(\nu_1)\|\bar{q}_2\|_{L^2[0,1]} \le \nu_1 I_2^{\frac{1}{2}} + C(\nu_1) I_1^{\frac{1}{2}}.$$

Similarly, it holds the same estimate for $\|\bar{q}_3\|_{L^\infty([-1,0])}$. Then

$$\left| \int_{-1}^{0} \partial_r s_\varepsilon \bar{q}_2 \bar{q}_3 a(r) \mathrm{d}r \right| \le C\Big(\int_{-1}^{0} \partial_r s_\varepsilon |\bar{q}_2| \mathrm{d}r \Big)\big(\nu_1 I_2^{\frac{1}{2}} + C(\nu_1) I_1^{\frac{1}{2}}\big).$$

On the other hand, we have

$$\begin{aligned} \int_{-1}^{0} \partial_r s_\varepsilon |\bar{q}_2| \mathrm{d}r &\le \int_{-1}^{0} s_\varepsilon |\partial_r \bar{q}_2| \mathrm{d}r + |(s_\varepsilon \bar{q}_2)(0)| + |(s_\varepsilon \bar{q}_2)(-1)| \\ &\le I_2^{\frac{1}{2}} + \nu_1 I_2^{\frac{1}{2}} + C(\nu_1) I_1^{\frac{1}{2}} + |(s_\varepsilon \bar{q}_2)(-1)|, \end{aligned} \tag{7.31}$$

and

$$\begin{aligned} |(s_\varepsilon \bar{q}_2)(-1)| &\le s_\varepsilon(-1)\Big(|\bar{q}_2(0)| + \int_{-1}^{0} |\partial_r \bar{q}_2| \mathrm{d}r \Big) \\ &\le |\bar{q}_2(0)| + \int_{-1}^{0} s_\varepsilon |\partial_r \bar{q}_2| \mathrm{d}r \le \nu_1 I_2^{\frac{1}{2}} + C(\nu_1) I_1^{\frac{1}{2}} + I_2^{\frac{1}{2}}. \end{aligned} \tag{7.32}$$

With the help of (7.31)–(7.32), one has that for any $\nu_0 > 0$, there exists $C_0 > 0$ such that

$$\left| \int_{-1}^{0} \partial_r s_\varepsilon \bar{q}_2 \bar{q}_3 a(r) \mathrm{d}r \right| \le \nu_0 I_2 + C_0 I_1.$$

Similarly we can get the same estimate for the integral on $[0, 1]$. Then the proof is completed. □

7.5 Estimate for correction terms

Now we prove (7.17). Recall from (6.12) that

$$\begin{aligned}\mathbf{P}_1 = s_\varepsilon(r)\mathbf{P}_{1,2}(x,t) + (1-s_\varepsilon)\mathbf{P}_{1,3}(x,t) + \mathbf{P}_{1,4}(x,t)\\ - \mathbf{L}_1(x,t)d_0(x,t)\eta(r/\varepsilon) + \mathbf{P}_1^*(r/\varepsilon,x,t).\end{aligned}$$

Using the exponential decay in z of $\mathbf{P}_1^*(z,x,t)$, and the fact that $|\mathbf{P}_1^*(z,x,t)|_{d_0(x,t)=0} = 0$, we have

$$\left|\frac{1}{\varepsilon}\mathbf{P}_1^*\Big(\frac{r}{\varepsilon},x,t\Big)\right| \leqslant C\frac{|d_0|}{\varepsilon}e^{-\frac{\alpha_0|r|}{\varepsilon}} \leqslant C.$$

In the last inequality, we have used $d_0 = d^K - \sum_{1\leqslant k\leqslant K}\varepsilon^k d_k = r + O(\varepsilon)$ by (7.2). Thus, the terms containing $\mathbf{P}_1^*$ can be controlled by $\int |\mathbf{B}|^2 J\mathrm{d}r$. Note that $\mathbf{L}_1(x,t)d_0(x,t)\eta(\frac{r}{\varepsilon}) \in \mathbb{V}_4$ and $\partial_r\big(\mathbf{L}_1(x,t)d_0(x,t)\eta(\frac{r}{\varepsilon})\big)$ is bounded. Thus, without the loss of generality, we only need to consider

$$\mathbf{P}_1(r) = s_\varepsilon(r)\mathbf{Q}_2(r) + (1-s_\varepsilon)\mathbf{Q}_3(r) + \mathbf{Q}_4(r) \quad \text{with} \quad \mathbf{Q}_i \in \mathbb{V}_i (i=2,3,4).$$

Now we calculate $\mathbf{T}_f(\mathbf{P}_0,\mathbf{P}_1,\tilde{\mathbf{B}}) : \tilde{\mathbf{B}}$ in (7.17). First of all, any term in $\int \varepsilon^{-1}\mathbf{T}_f(\mathbf{P}_0,\mathbf{P}_1,\tilde{\mathbf{B}}) : \tilde{\mathbf{B}}dr$ containing $\mathcal{P}_5\tilde{\mathbf{B}}$ can be bounded by the right hand side of (7.8). So, we only need to consider the terms in $\mathbb{V}_i (1 \leqslant i \leqslant 4)$.

Consider $\tilde{\mathbf{B}} = \sum_{i=1}^4 \mathbf{B}_i$ with $\mathbf{B}_i \in \mathbb{V}_i (1 \leqslant i \leqslant 4)$. By Lemma A.3, we have

$$\begin{aligned}&\mathbf{T}_f(\mathbf{P}_0,\mathbf{P}_1,\tilde{\mathbf{B}}) : \tilde{\mathbf{B}}\\ &= 2s_\varepsilon(2s_\varepsilon-1)\mathbf{Q}_2 : (\mathbf{B}_3\mathbf{B}_4 + \mathbf{B}_4\mathbf{B}_3) + 2s_\varepsilon(3-4s_\varepsilon)\mathbf{Q}_2 : (\mathbf{B}_1\mathbf{B}_2 + \mathbf{B}_2\mathbf{B}_1)\\ &\quad + 2(1-s_\varepsilon)(1-4s_\varepsilon)\mathbf{Q}_3 : (\mathbf{B}_1\mathbf{B}_3 + \mathbf{B}_3\mathbf{B}_1)\\ &\quad + 2(1-s_\varepsilon)(1-2s_\varepsilon)\mathbf{Q}_3 : (\mathbf{B}_2\mathbf{B}_4 + \mathbf{B}_4\mathbf{B}_2)\\ &\quad + 2(1-2s_\varepsilon)\mathbf{Q}_4 : (\mathbf{B}_2\mathbf{B}_3 + \mathbf{B}_3\mathbf{B}_2)\\ &\triangleq L_{34} + L_{12} + L_{13} + L_{24} + L_{23}.\end{aligned}$$

Now using the decomposition (7.16), we can write:

$$\mathbf{B}_i(r) = \sum_{\alpha\in\Lambda_i} q_\alpha(r)\mathbf{E}_\alpha(r), \qquad i = 1,2,3,4.$$

We use $\mathbf{E}_i$ to denote an element in $\mathbb{V}_i (1 \leqslant i \leqslant 4)$. Then the integral $\frac{1}{\varepsilon}\int L_{34}$ can be written as a summation of terms with form

$$\frac{1}{\varepsilon}\int_I s_\varepsilon(2s_\varepsilon-1)q_3q_4\mathbf{Q}_2 : (\mathbf{E}_3\mathbf{E}_4 + \mathbf{E}_4\mathbf{E}_3)\mathrm{d}r. \tag{7.33}$$

Letting $q_i = \theta_{i,\varepsilon}\bar{q}_i$ with $\theta_{i,\varepsilon}$ defined in (7.19), and using (7.34) below and Lemma 7.2, we bound (7.33) by

$$\nu_0\big(\mathcal{Q}_3(q_3)+\mathcal{Q}_4(q_4)\big)+C(\nu_0)\int_I (q_3^2+q_4^2)\mathrm{d}r, \quad \text{for } \forall \nu_0>0.$$

Similarly, by using (7.35)–(7.37) below, Lemma 7.14 along with Lemma 7.1, we can bound

$$\frac{1}{\varepsilon}\int_I L_{24}, \quad \frac{1}{\varepsilon}\int_I L_{12}, \quad \frac{1}{\varepsilon}\int_I L_{13}, \quad \frac{1}{\varepsilon}\int_I L_{23},$$

by the right hand side of (7.8). Then (7.8) follows easily.

Lemma 7.15 *For any $\nu_0 > 0$ there exists $C_0 = C_0(\nu_0, \|a\|_{W^{1,\infty}}) > 0$ such that ($\theta_{4,\varepsilon} = 1$)*

$$\frac{1}{\varepsilon}\Big|\int_I s_\varepsilon(1-s_\varepsilon)(2s_\varepsilon-1)\bar{q}_3\bar{q}_4 a\mathrm{d}r\Big| \le \sum_{i=3,4}\int_I \theta_{i,\varepsilon}^2\big[\nu_0(\partial_r\bar{q}_i)^2+C_0\bar{q}_i^2\big]\mathrm{d}r, \tag{7.34}$$

$$\frac{1}{\varepsilon}\Big|\int_I (1-s_\varepsilon)(1-2s_\varepsilon)s_\varepsilon\bar{q}_2\bar{q}_4 a\mathrm{d}r\Big| \le \sum_{i=2,4}\int_I \theta_{i,\varepsilon}^2\big[\nu_0(\partial_r\bar{q}_i)^2+C_0\bar{q}_i^2\big]\mathrm{d}r, \tag{7.35}$$

$$\frac{1}{\varepsilon}\Big|\int_I s_\varepsilon^2(3-4s_\varepsilon)\theta_\varepsilon\bar{q}_1\bar{q}_2 a\mathrm{d}r\Big| \le \sum_{i=1,2}\int_I \theta_{i,\varepsilon}^2\big[\nu_0(\partial_r\bar{q}_i)^2+C_0\bar{q}_i^2\big]\mathrm{d}r, \tag{7.36}$$

$$\frac{1}{\varepsilon}\Big|\int_I (1-s_\varepsilon)^2(1-4s_\varepsilon)\theta_\varepsilon\bar{q}_1\bar{q}_3 a\mathrm{d}r\Big| \le \sum_{i=1,3}\int_I \theta_{i,\varepsilon}^2\big[\nu_0(\partial_r\bar{q}_i)^2+C_0\bar{q}_i^2\big]\mathrm{d}r. \tag{7.37}$$

Proof The left side of (7.34) can be written as

$$\begin{aligned}&\Big|\int_I \partial_r\big[s_\varepsilon(1-s_\varepsilon)\big]\bar{q}_3\bar{q}_4 a\mathrm{d}r\Big|\\ &\le\Big|\int_I s_\varepsilon(1-s_\varepsilon)\big[\partial_r\bar{q}_3\bar{q}_4 a+\bar{q}_3\partial_r\bar{q}_4 a+\bar{q}_3\bar{q}_4\partial_r a\big]\mathrm{d}r\Big|+\Big(s_\varepsilon(1-s_\varepsilon)\bar{q}_3\bar{q}_4 a\Big)\Big|_{-1}^{1}.\end{aligned}$$

Then (7.34) follows from Cauchy-Schwartz inequality and Lemma 7.7. (7.35) can be obtained in a similar way.

The left side of (7.36) can be estimated as

$$\begin{aligned}\left|\int_I \partial_r\big[s_\varepsilon^3(1-s_\varepsilon)\big]\bar{q}_1\bar{q}_2 a\mathrm{d}r\right| &= \left|\int_I \partial_r\big(s_\varepsilon^2\theta_\varepsilon\big)\bar{q}_1\bar{q}_2 a\mathrm{d}r\right| \\ &\leqslant \left|\int_I s_\varepsilon^2\theta_\varepsilon\big[\partial_r\bar{q}_1\bar{q}_2 a+\bar{q}_1\partial_r\bar{q}_2 a+\bar{q}_1\bar{q}_2\partial_r a\big]\mathrm{d}r\right| \\ &\quad+\Big(s_\varepsilon^2\theta_\varepsilon\bar{q}_1\bar{q}_2 a\Big)\Big|_{-1}^{1}.\end{aligned}$$

Then the claim follows from the Cauchy-Schwartz inequality and Lemmas 7.7 and 7.8. As $(1-s_\varepsilon)^2(1-4s_\varepsilon)\theta_\varepsilon/\varepsilon = \partial_r((1-s_\varepsilon)^3 s_\varepsilon)$, we can prove (7.37) similarly. □

Remark 7.16 The proof of Lemma 7.15 relies heavily on the fact that, all the weights can be written as derivatives of some good functions. These functions have factors which consist of production of corresponding eigenfunctions, thus enable us to use integrating by parts to remove the $O(\varepsilon^{-1})$ singularities. The mechanism behind such coincidence is the cubic null cancellation Lemma 4.4.

8 Uniform error estimates

Let $\mathbf{A}^\varepsilon$ be a solution to (1.3) and $\mathbf{A}^K$ be the approximate solution constructed in Sect. 6. Define

$$\Psi = \frac{1}{\varepsilon^L}(\mathbf{A}^\varepsilon - \mathbf{A}^K).$$

Then we have

$$\partial_t\Psi = \Delta\Psi - \varepsilon^{-2}\mathcal{H}_{\mathbf{A}^K}\Psi - \frac{\varepsilon^{L-2}}{2}\mathbf{T}_f(\mathbf{A}^K,\Psi,\Psi) + \varepsilon^{2L-2}\Psi\Psi^{\mathrm{T}}\Psi - \mathfrak{R}^\varepsilon, \tag{8.1}$$

where $\mathfrak{R}^\varepsilon$ is independent of Ψ which satisfies $\partial^i\mathfrak{R}^\varepsilon = O(\varepsilon^{K-L-i-1})$ for $i \geqslant 0$. We choose $L = 3([\frac{m}{2}]+1)+3$, and $K \geqslant L+1$.

Let

$$\bar{\mathcal{E}}(\Psi) = \sum_{i=0}^{[\frac{m}{2}]+1}\varepsilon^{6i}\int_\Omega \|\partial^i\Psi\|^2\mathrm{d}x.$$

Then one has

$$\varepsilon^{L-2}\|\Psi\|_{L^\infty} \leqslant \varepsilon^{3([\frac{m}{2}]+1)+1}\|\Psi\|_{L^\infty} \leqslant C\varepsilon\bar{\mathcal{E}}(\Psi)^{\frac{1}{2}}.$$

With the help of Theorem 1.3, standard energy estimates yield that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \|\Psi\|^2 \mathrm{d}x &\leqslant C(1+\varepsilon\bar{\mathcal{E}}(\Psi)^{\frac{1}{2}}+\varepsilon^2\bar{\mathcal{E}}(\Psi))\int_\Omega \|\Psi\|^2\mathrm{d}x + C\\ &\leqslant C\big(1+\bar{\mathcal{E}}(\Psi)+\varepsilon^2\bar{\mathcal{E}}(\Psi)^2\big).\end{aligned}$$

Applying $\partial^i (0\leqslant i\leqslant [\frac{m}{2}]+1)$ on the equation (8.1), we get

$$\begin{aligned}\partial_t\partial^i\Psi = \Delta\partial^i\Psi - \varepsilon^{-2}\mathcal{H}_{\mathbf{A}^K}\partial^i\Psi + \varepsilon^{-2}[\mathcal{H}_{\mathbf{A}^K},\partial^i]\Psi\\ -\frac{1}{2}\varepsilon^{L-2}\partial^i\mathbf{T}_f(\mathbf{A}^K,\Psi,\Psi)+\varepsilon^{2L-2}\partial^i(\Psi\Psi^{\mathrm{T}}\Psi)-\partial^i\mathfrak{R}^\varepsilon,\end{aligned}$$

where

$$\varepsilon^{-2}[\mathcal{H}_{\mathbf{A}^K},\partial^i]\Psi = \varepsilon^{-2}\sum_{j+l+k=i,k\leqslant i-1,j\leqslant l}\mathbf{T}_f(\partial^j\mathbf{A}^K,\partial^l\mathbf{A}^K,\partial^k\Psi),$$

whose L^2-norm can be bounded by

$$C\varepsilon^{-2-j-l}\|\partial^k\Psi\|_{L^2}\leqslant\varepsilon^{-3k-j-l-2}\bar{\mathcal{E}}^{1/2}(\Psi)\leqslant\varepsilon^{-3i}\bar{\mathcal{E}}^{1/2}(\Psi).$$

Moreover

$$\begin{aligned}\|\varepsilon^{L-2}\partial^i\mathbf{T}_f(\mathbf{A}^K,\Psi,\Psi)\|_{L^2} &= \|\varepsilon^{L-2}\sum_{j+k+l=i,k\leqslant l}\mathbf{T}_f(\partial^j\mathbf{A}^K,\partial^k\Psi,\partial^l\Psi)\|_{L^2}\\ &\leqslant C\varepsilon^{1-3i}\bar{\mathcal{E}}(\Psi),\\ \|\varepsilon^{2L-2}\partial^i(\Psi\Psi^{\mathrm{T}}\Psi)\|_{L^2} &\leqslant \varepsilon^{2L-2}\|\Psi\|^2_{H^{[\frac{m}{2}]+1}}\|\Psi\|_{H^i}\leqslant C\varepsilon^{2-3i}\bar{\mathcal{E}}^{3/2}(\Psi),\\ \|\partial^i\mathfrak{R}^\varepsilon_k\|_{L^2} &\leqslant \varepsilon^{K-L-i-1}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varepsilon^{6i}\|\partial^i\Psi\|^2\mathrm{d}x &\leqslant C(1+\bar{\mathcal{E}}(\Psi)^{\frac{1}{2}}+\varepsilon\bar{\mathcal{E}}(\Psi)+\varepsilon^2\bar{\mathcal{E}}^{3/2}(\Psi))\|\partial^i\Psi\|_{L^2}\\ &\leqslant C(1+\bar{\mathcal{E}}(\Psi)+\varepsilon^2\bar{\mathcal{E}}(\Psi)^2).\end{aligned}$$

Summing i from 0 to $[\frac{m}{2}]+1$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\mathcal{E}}(\Psi)\leqslant C(1+\bar{\mathcal{E}}(\Psi)+\varepsilon^2\bar{\mathcal{E}}(\Psi)^2).$$

Then Theorem 1.4 can be concluded by a direct continuation argument.

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Appendix A

A.1. Formal derivation of Neumann jump condition

Assume that

$$\begin{aligned} \mathbf{A}^\varepsilon &\to \mathbf{A}_\pm \text{ strongly in } L^2(\Omega_t^\pm);\\ \partial_t \mathbf{A}^\varepsilon \rightharpoonup \partial_t \mathbf{A}_\pm,\ \nabla \mathbf{A}^\varepsilon &\rightharpoonup \nabla \mathbf{A}_\pm \text{ weakly in } L^2(\Omega_t^\pm). \end{aligned}$$

The equation (1.3) yields that

$$(\mathbf{A}^\varepsilon)^{\mathrm{T}}(\partial_t \mathbf{A}^\varepsilon - \Delta \mathbf{A}^\varepsilon) - (\partial_t (\mathbf{A}^\varepsilon)^{\mathrm{T}} - \Delta (\mathbf{A}^\varepsilon)^{\mathrm{T}})\mathbf{A}^\varepsilon = 0,$$

which gives

$$(\mathbf{A}^\varepsilon)^{\mathrm{T}}\partial_t \mathbf{A}^\varepsilon - \partial_t (\mathbf{A}^\varepsilon)^{\mathrm{T}}\mathbf{A}^\varepsilon = \nabla \cdot ((\mathbf{A}^\varepsilon)^{\mathrm{T}}\nabla \mathbf{A}^\varepsilon - \nabla (\mathbf{A}^\varepsilon)^{\mathrm{T}}\mathbf{A}^\varepsilon).$$

Testing the above equation with a smooth matrix-valued function Ψ and taking the limit $\varepsilon \to 0$, one gets

$$\begin{aligned} &\int_{\Omega_t^+} \Big((\mathbf{A}_+^{\mathrm{T}}\partial_t \mathbf{A}_+ - \partial_t \mathbf{A}_+^{\mathrm{T}}\mathbf{A}_+)\Psi + (\mathbf{A}_+^{\mathrm{T}}\nabla \mathbf{A}_+ - \nabla \mathbf{A}_+^{\mathrm{T}}\mathbf{A}_+) \cdot \nabla \Psi \Big) \mathrm{d}x \\ &+ \int_{\Omega_t^-} \Big((\mathbf{A}_-^{\mathrm{T}}\partial_t \mathbf{A}_- - \partial_t \mathbf{A}_-^{\mathrm{T}}\mathbf{A}_-)\Psi + (\mathbf{A}_-^{\mathrm{T}}\nabla \mathbf{A}_- - \nabla \mathbf{A}_-^{\mathrm{T}}\mathbf{A}_-) \cdot \nabla \Psi \Big) \mathrm{d}x = 0. \end{aligned}$$

As $\mathbf{A}_\pm$ obeys the harmonic map heat flow to $O^\pm(n)$ on $\Omega_\pm$, one immediately gets the boundary condition

$$\mathbf{A}_+^{\mathrm{T}}\partial_\nu \mathbf{A}_+ - \partial_\nu \mathbf{A}_+^{\mathrm{T}}\mathbf{A}_+ = \mathbf{A}_-^{\mathrm{T}}\partial_\nu \mathbf{A}_- - \partial_\nu \mathbf{A}_-^{\mathrm{T}}\mathbf{A}_-.$$

Similarly, we have

$$\mathbf{A}_+\partial_\nu \mathbf{A}_+^{\mathrm{T}} - \partial_\nu \mathbf{A}_+\mathbf{A}_+^{\mathrm{T}} = \mathbf{A}_-\partial_\nu \mathbf{A}_-^{\mathrm{T}} - \partial_\nu \mathbf{A}_-\mathbf{A}_-^{\mathrm{T}}.$$

These two relations give (1.6d) under the minimal pair relation (1.6c).

Assume that $(\mathbf{A}_-, \mathbf{A}_+)$ is a minimal pair, i.e., $\mathbf{A}_+ = \mathbf{A}_-(\mathbf{I} - 2\mathbf{nn})$ for some $\mathbf{n} \in S^{n-1}$. We prove that the following three boundary conditions are equivalent:

$$
\begin{aligned}
(i): &\quad \begin{cases} \mathbf{A}_+^{\mathrm{T}}\partial_\nu \mathbf{A}_+ - \partial_\nu \mathbf{A}_+^{\mathrm{T}}\mathbf{A}_+ = \mathbf{A}_-^{\mathrm{T}}\partial_\nu \mathbf{A}_- - \partial_\nu \mathbf{A}_-^{\mathrm{T}}\mathbf{A}_-, \\ \mathbf{A}_+\partial_\nu \mathbf{A}_+^{\mathrm{T}} - \partial_\nu \mathbf{A}_+\mathbf{A}_+^{\mathrm{T}} = \mathbf{A}_-\partial_\nu \mathbf{A}_-^{\mathrm{T}} - \partial_\nu \mathbf{A}_-\mathbf{A}_-^{\mathrm{T}}; \end{cases} \\
(ii): &\quad \partial_\nu \mathbf{A}_+ = \partial_\nu \mathbf{A}_-; \\
(iii): &\quad \mathbf{A}_+^{\mathrm{T}}\partial_\nu \mathbf{A}_+ = \mathbf{A}_-^{\mathrm{T}}\partial_\nu \mathbf{A}_- = \mathbf{W}, \quad \text{for some } \mathbf{W} \in \mathbb{V}_4 \text{ (cf. (4.1))}.
\end{aligned}
$$

$(i) \Rightarrow (iii)$: As $\mathbf{A}_+^{\mathrm{T}}\partial_\nu \mathbf{A}_+$ and $\mathbf{A}_-^{\mathrm{T}}\partial_\nu \mathbf{A}_-$ are both antisymmetric, the first equation gives

$$\mathbf{A}_+^{\mathrm{T}}\partial_\nu \mathbf{A}_+ = \mathbf{A}_-^{\mathrm{T}}\partial_\nu \mathbf{A}_- = \mathbf{W} \in \mathbb{A}_n.$$

Similarly, we have $\partial_\nu \mathbf{A}_+\mathbf{A}_+^{\mathrm{T}} = \partial_\nu \mathbf{A}_-\mathbf{A}_-^{\mathrm{T}}$. Therefore, we get

$$\mathbf{A}_+\mathbf{W}\mathbf{A}_+^{\mathrm{T}} = \partial_\nu \mathbf{A}_+\mathbf{A}_+^{\mathrm{T}} = \partial_\nu \mathbf{A}_-\mathbf{A}_-^{\mathrm{T}} = \mathbf{A}_-\mathbf{W}\mathbf{A}_-^{\mathrm{T}},$$

which implies $(\mathbf{I} - 2\mathbf{nn})\mathbf{W}(\mathbf{I} - 2\mathbf{nn}) = \mathbf{W}$. Thus, $\mathbf{nW} = 0$, i.e. $\mathbf{W} \in \mathbb{V}_4$.

$(ii) \Rightarrow (iii)$: Assume that $\partial_\nu \mathbf{A}_\pm = \mathbf{A}_\pm \mathbf{W}_\pm$ with $\mathbf{W}_\pm \in \mathbb{A}_n$. Then we have $\mathbf{A}_+\mathbf{W}_+ = \mathbf{A}_-\mathbf{W}_-$, or equivalently $(\mathbf{I} - 2\mathbf{nn})\mathbf{W}_+ = \mathbf{W}_-$. As $\mathbf{W}_\pm$ are both antisymmetric, one has $\mathbf{n} \cdot \mathbf{W}_+ = 0$ and $\mathbf{W}_+ = \mathbf{W}_-$, which implies that $\mathbf{W}_+ = \mathbf{W}_- \in \mathbb{V}_4$.

By reversing the above derivations, we immediately obtain $(iii) \Rightarrow (i), (ii)$.

A.2. A sketch proof of Lemmas 3.7 and 3.9

We define for a matrix $\mathbf{A} \in \mathbb{M}_n$:

$$\rho(\mathbf{A}, O(n)) = \min_{\mathbf{B} \in O(n)} \|\mathbf{A} - \mathbf{B}\|.$$

Lemma A.1 *For $\mathbf{A} \in \mathbb{M}_n$ and $\rho = \rho(\mathbf{A}, O(n))$, one has: (i) if $\rho \leqslant 1$, then $F(\mathbf{A}) \geqslant \frac{1}{4}\rho^2(2-\rho)^2$, and equality holds if and only if $\mathbf{A} = \mathbf{B}(\mathbf{I} - \rho\mathbf{nn})$ for some $\mathbf{B} \in O(n)$ and $\mathbf{n} \in S^{n-1}$; (ii) If $\rho > 1$, then $F(\mathbf{A}) > \frac{1}{4}$.*

Proof We perform the singular value decomposition for $\mathbf{A}$ as $\mathbf{A} = \mathbf{U}\Lambda\mathbf{V}^T$ where $\mathbf{U}, \mathbf{V} \in O(n)$ and $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ is diagonal with $\lambda_i \geqslant 0$. Apparently,

$$F(\mathbf{A}) = F(\Lambda) = \frac{1}{4}\sum_{i=1}^{n}(\lambda_i^2 - 1)^2,$$

$$\rho^2(\mathbf{A}, O(n)) = \rho^2(\Lambda, O(n)) = \|\Lambda - \mathbf{I}\|^2 = \sum_{i=1}^{n}(\lambda_i - 1)^2.$$

If $\rho^2 = \sum_{i=1}^{n} |\lambda_i - 1|^2 \leqslant 1$, then for each i, $\lambda_i \geqslant 1 - \rho$, which implies

$$F(\mathbf{A}) = \frac{1}{4}\sum_{i=1}^{n}(\lambda_i^2 - 1)^2 \geqslant \frac{1}{4}\min_i(\lambda_i + 1)^2 \sum_{k=1}^{n} |\lambda_k - 1|^2 \geqslant \frac{1}{4}(2-\rho)^2\rho^2.$$

Equality holds only if $\lambda_i \equiv 1$ except one of them takes value $1 - \rho$. This implies $\mathbf{A} = \mathbf{B}(\mathbf{I} - \rho\mathbf{n}\mathbf{n})$ for some $\mathbf{B} \in O(n)$ and $\mathbf{n} \in S^{n-1}$. If $\rho^2 > 1$, then $F(\mathbf{A}) = \frac{1}{4}\sum_{i=1}^{n}(\lambda_i^2 - 1)^2 \geqslant \frac{1}{4}\sum_{i=1}^{n}(\lambda_i - 1)^2 = \frac{1}{4}\rho^2 > \frac{1}{4}$. □

For any curve $\gamma = \{\mathbf{B}(z) : a < z < b\}$ in $\mathbb{M}_n$, the quantity

$$e_F(\gamma) = \int_\gamma \sqrt{F(\mathbf{B}(z))/2}\|\mathbf{B}'(z)\|\mathrm{d}z$$

is independent of the parametrization of γ. If $\mathbf{A}(z)(z \in \mathbb{R})$ is a minimal connecting orbit, then

$$e_F(\mathrm{Traj}(\mathbf{A})) = \min_{\gamma(\pm 1)\in\Sigma_\pm} e_F(\gamma).$$

Define

$$\tilde{F}(\mathbf{A}) = \begin{cases} \dfrac{1}{4}\rho^2(2-\rho)^2, & \text{if } \rho := \rho(\mathbf{A}, O(n)) \leqslant 1; \\ \dfrac{1}{4}, & \text{if } \rho(\mathbf{A}, O(n)) \geqslant 1. \end{cases}$$

Then by Lemma A.1, one get $\tilde{F}(\mathbf{A}) \leqslant F(\mathbf{A})$ for $\mathbf{A} \in \mathbb{M}_n$. Note that $\tilde{F}$ can be viewed as a continuous function of $\rho(\mathbf{A}, O(n))$. This enables us to apply the arguments in [26, Theorem 2.1] to obtain that: $\mathbf{A}(z)$ is a minimal connecting orbit with respect to $\tilde{F}$ if and only if

$$\mathbf{A}(z) = \Theta_\tau(\mathbf{A}_+, \mathbf{A}_-; z) := s_\tau(z)\mathbf{A}_+ + (1 - s_\tau(z))\mathbf{A}_-, \\ \text{with } \mathbf{A}_\pm \in O^\pm(n) \text{ and } \|\mathbf{A}_+ - \mathbf{A}_-\| = 2.$$

Here $\tau \in \mathbb{R}$ is a constant and $s_\tau(z) = s(z + \tau)$ is a translation of $s(z)$.

Then for a minimal connecting orbit $\mathbf{A}(z)$ with respect to F, we have

$$e_F(\mathrm{Traj}(\mathbf{A})) \geqslant e_{\tilde{F}}(\mathrm{Traj}(\mathbf{A})) \geqslant \min_{\gamma(\pm 1)\in\Sigma_\pm} e_{\tilde{F}}(\gamma)$$

$$= e_{\tilde{F}}(\mathrm{Traj}(\Theta_\tau(\mathbf{A}_+, \mathbf{A}_-; z))) = e_F(\mathrm{Traj}(\Theta_\tau(\mathbf{A}_+, \mathbf{A}_-; z))).$$

Thus all the inequalities hold as equalities which implies $\mathrm{Traj}(\mathbf{A}) = \{t\mathbf{A}_+ + (1-t)\mathbf{A}_- : t \in (0,1)\}$ for some $(\mathbf{A}_+, \mathbf{A}_-) \in O^+(n) \times O^-(n)$ satisfying $\|\mathbf{A}_+ - \mathbf{A}_-\| = 2$. This gives Lemma 3.7. In addition, if we write $\mathbf{A}(z) = \tilde{s}(z)\mathbf{A}_+ + (1-\tilde{s}(z))\mathbf{A}_-$ and substitute it into equation (3.9), then $\tilde{s}$ has to be a solution of (3.11) and thus $\tilde{s} = s_\tau$ for some $\tau \in \mathbb{R}$, which yields Lemma 3.9.

A.3. Solvability of scalar ODEs

We collect the results on solving the ODEs (cf. (4.5)):

$$\mathcal{L}_i u_i(z,x,t) = f_i(z,x,t) \quad (i = 1, 2, \cdots, 5), \tag{A.1}$$

in $\mathbb{R}$ for $(x,t) \in \Gamma(\delta)$, which have been proved in [2] ($i = 1$) and [17]. We take $\alpha_0 \in (0, \sqrt{2}]$.

Lemma A.2 *Assume $f_i(\cdot, x, t) \in \mathcal{S}_{J,L,M}(\alpha_0, k)$ for $1 \leqslant i \leqslant 5$ with*

$$f_2^+(x,t) = 0, \quad f_3^-(x,t) = 0, \quad f_4^\pm(x,t) = 0,$$
$$\text{and} \quad \int_{\mathbb{R}} f_j(z,x,t)\theta_j(z)\mathrm{d}z = 0 (1 \leqslant j \leqslant 4).$$

Then (A.1) has a unique bounded solution $u_i^*(\cdot, x, t) \in \mathcal{S}_{J+2,L,M}(\alpha_0, k+1)$ which satisfies

$$u_1^*(0,x,t) = 0, \quad u_2^{*+}(x,t) = 0, \quad u_3^{*-}(x,t) = 0, \quad u_4^{*-}(x,t) = 0.$$

Precisely, the solution can be written as

$$u_1^*(z,x,t) = \theta_1(z) \int_0^z \theta_1^{-2}(\varsigma) \int_\varsigma^{+\infty} f_1(\tau,x,t)\theta_1(\tau)\mathrm{d}\tau\mathrm{d}\varsigma,$$
$$u_2^*(z,x,t) = -\theta_2(z) \int_z^{+\infty} \theta_2^{-2}(\varsigma) \int_\varsigma^{+\infty} f_2(\tau,x,t)\theta_2(\tau)\mathrm{d}\tau\mathrm{d}\varsigma,$$
$$u_3^*(z,x,t) = -\theta_3(z) \int_{-\infty}^z \theta_3^{-2}(\varsigma) \int_{-\infty}^\varsigma f_3(\tau,x,t)\theta_3(\tau)\mathrm{d}\tau\mathrm{d}\varsigma,$$
$$u_4^*(z,x,t) = \int_{-\infty}^z (\tau - z) f_4(\tau,x,t)\mathrm{d}\tau,$$
$$u_5^*(z,x,t) = \frac{1}{2\sqrt{2}} \int_0^{+\infty} e^{-\sqrt{2}\tau} \big[f_5(z+\tau,x,t) + f_5(z-\tau,x,t)\big]\mathrm{d}\tau,$$

and we have

$$u_1^{*\pm}(x,t)=\frac{1}{2}f_1^{\pm}(x,t),\quad u_2^{*-}(x,t)=\frac{1}{2}f_2^{-}(x,t),$$
$$u_3^{*+}(x,t)=\frac{1}{2}f_3^{+}(x,t),\quad u_5^{*\pm}(x,t)=\frac{1}{2}f_5^{\pm}(x,t).$$

Moreover, all bounded solutions of (A.1) are given by:

$$u_i(z,x,t)=u_i^*(z,x,t)+a_i(x,t)\theta_i(z),\quad \text{for } 1\leqslant i\leqslant 4,$$

and u_5^* is the only bounded solution to (A.1) for $i=5$.

A.4. A key formula of trilinear form $\mathbf{T}_f$

We present a lemma, which was used in the proof of Lemma 4.4 and the estimate for correction terms in Sect. 7.5. Recall from (4.9) that

$$\begin{aligned}\mathbf{T}_f(\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}_3)=&(\mathbf{A}_1\mathbf{A}_2^{\mathrm{T}}+\mathbf{A}_2\mathbf{A}_1^{\mathrm{T}})\mathbf{A}_3+(\mathbf{A}_3\mathbf{A}_1^{\mathrm{T}}+\mathbf{A}_1\mathbf{A}_3^{\mathrm{T}})\mathbf{A}_2\\&+(\mathbf{A}_2^{\mathrm{T}}\mathbf{A}_3+\mathbf{A}_2\mathbf{A}_3^{\mathrm{T}})\mathbf{A}_1.\end{aligned}$$

Lemma A.3 *Let $\mathbf{P}_1=\sum_{i=2}^4\theta_i\mathbf{E}_i$ and $\mathbf{B}=\sum_{i=1}^4\mathbf{B}_i$ with $\mathbf{E}_i,\mathbf{B}_i\in\mathbb{V}_i$. Then*

$$\begin{aligned}&\mathbf{T}_f(\mathbf{P}_0,\mathbf{P}_1,\mathbf{B}):\mathbf{B}\\&=2s(2s-1)\mathbf{E}_2:(\mathbf{B}_3\mathbf{B}_4+\mathbf{B}_4\mathbf{B}_3)+2s(3-4s)\mathbf{E}_2:(\mathbf{B}_1\mathbf{B}_2+\mathbf{B}_2\mathbf{B}_1)\\&\quad+2(1-s)(1-4s)\mathbf{E}_3:(\mathbf{B}_1\mathbf{B}_3+\mathbf{B}_3\mathbf{B}_1)\\&\quad+2(1-s)(1-2s)\mathbf{E}_3:(\mathbf{B}_2\mathbf{B}_4+\mathbf{B}_4\mathbf{B}_2)\\&\quad+2(1-2s)\mathbf{E}_4:(\mathbf{B}_2\mathbf{B}_3+\mathbf{B}_3\mathbf{B}_2).\end{aligned}\tag{A.2}$$

Proof Through a direct calculation, we have

$$\begin{aligned}&\mathbf{T}_f(\mathbf{P}_0,\mathbf{P}_1,\mathbf{B}):\mathbf{B}\\&=(\mathbf{P}_0\mathbf{P}_1^{\mathrm{T}}+\mathbf{P}_1\mathbf{P}_0^{\mathrm{T}}):(\mathbf{B}\mathbf{B}^{\mathrm{T}})+(\mathbf{P}_0^{\mathrm{T}}\mathbf{P}_1+\mathbf{P}_1^{\mathrm{T}}\mathbf{P}_0):(\mathbf{B}^{\mathrm{T}}\mathbf{B})+2(\mathbf{B}^{\mathrm{T}}\mathbf{P}_1):(\mathbf{P}_0\mathbf{B}).\end{aligned}\tag{A.3}$$

Moreover, one has

$$\begin{aligned}\mathbf{P}_0\mathbf{P}_1^{\mathrm{T}}+\mathbf{P}_1\mathbf{P}_0^{\mathrm{T}}&=(1-s)(\mathbf{P}_1^{\mathrm{T}}+\mathbf{P}_1)+s\Big((\mathbf{I}-2\mathbf{n}\mathbf{n})\mathbf{P}_1^{\mathrm{T}}+\mathbf{P}_1(\mathbf{I}-2\mathbf{n}\mathbf{n})\Big)\\&=2s(1-s)\big(\mathbf{E}_2-(\mathbf{I}-2\mathbf{n}\mathbf{n})\mathbf{E}_3\big).\end{aligned}$$

Here, we remark that $(\mathbf{I}-2\mathbf{nn})\mathbf{E}_3 \in \mathbb{V}_2$ is symmetric. Similarly, we have

$$\mathbf{P}_0^{\mathrm{T}}\mathbf{P}_1 + \mathbf{P}_1^{\mathrm{T}}\mathbf{P}_0 = 2s(1-s)\big(\mathbf{E}_2 + (\mathbf{I}-2\mathbf{nn})\mathbf{E}_3\big).$$

Therefore, if we let $\mathbf{B} = \mathbf{D} + \mathbf{W}$ with $\mathbf{D} \in \mathbb{S}_n$ and $\mathbf{W} \in \mathbb{A}_n$, the first two terms in (A.3) read as

$$4s(1-s)\Big(\mathbf{E}_2 : (\mathbf{D}^2 - \mathbf{W}^2) + (\mathbf{I}-2\mathbf{nn})\mathbf{E}_3 : (\mathbf{DW}-\mathbf{WD})\Big).$$

The last term in (A.3) equals to

$$\begin{aligned}
2\mathbf{P}_1 : (\mathbf{B}\mathbf{P}_0\mathbf{B}) &= 2\mathbf{P}_1 : (\mathbf{D}\mathbf{P}_0\mathbf{D} + \mathbf{D}\mathbf{P}_0\mathbf{W} + \mathbf{W}\mathbf{P}_0\mathbf{D} + \mathbf{W}\mathbf{P}_0\mathbf{W}) \\
&= 2s\mathbf{E}_2 : (\mathbf{D}\mathbf{P}_0\mathbf{D} + \mathbf{W}\mathbf{P}_0\mathbf{W}) \\
&\quad + 2((1-s)\mathbf{E}_3 + \mathbf{E}_4) : (\mathbf{D}\mathbf{P}_0\mathbf{W} + \mathbf{W}\mathbf{P}_0\mathbf{D}).
\end{aligned}$$

As $\mathbf{D} = \mathbf{B}_1 + \mathbf{B}_2$, $\mathbf{W} = \mathbf{B}_3 + \mathbf{B}_4$, we have

$$\begin{aligned}
\mathbf{D}^2 &= \mathbf{B}_1^2 + \mathbf{B}_1\mathbf{B}_2 + \mathbf{B}_2\mathbf{B}_1 + \mathbf{B}_2^2, \\
\mathbf{W}^2 &= \mathbf{B}_3 + \mathbf{B}_4\mathbf{B}_3 + \mathbf{B}_3\mathbf{B}_4 + \mathbf{B}_4^2, \\
\mathbf{DW} - \mathbf{WD} &= (\mathbf{B}_1\mathbf{B}_3 - \mathbf{B}_3\mathbf{B}_1) + (\mathbf{B}_2\mathbf{B}_3 - \mathbf{B}_3\mathbf{B}_2) \\
&\quad + \mathbf{B}_2\mathbf{B}_4 - \mathbf{B}_4\mathbf{B}_2, \\
(\mathbf{I}-2\mathbf{nn})(\mathbf{DW}-\mathbf{WD}) &= -(\mathbf{B}_1\mathbf{B}_3 + \mathbf{B}_3\mathbf{B}_1) + (\mathbf{I}-2\mathbf{nn})(\mathbf{B}_2\mathbf{B}_3 - \mathbf{B}_3\mathbf{B}_2) \\
&\quad - \mathbf{B}_2\mathbf{B}_4 - \mathbf{B}_4\mathbf{B}_2, \\
\mathbf{D}\mathbf{P}_0\mathbf{D} &= (\mathbf{B}_1 + \mathbf{B}_2)(\mathbf{I}-2s\mathbf{nn})(\mathbf{B}_1 + \mathbf{B}_2) \\
&= (1-2s)\mathbf{B}_1^2 + (1-2s)(\mathbf{B}_1\mathbf{B}_2 + \mathbf{B}_2\mathbf{B}_1) \\
&\quad + \mathbf{B}_2^2 - 2s\mathbf{B}_2\mathbf{nn}\mathbf{B}_2, \\
\mathbf{W}\mathbf{P}_0\mathbf{W} &= (\mathbf{B}_3 + \mathbf{B}_4)(\mathbf{I}-2s\mathbf{nn})(\mathbf{B}_3 + \mathbf{B}_4) \\
&= \mathbf{B}_3^2 - 2s\mathbf{B}_3\mathbf{nn}\mathbf{B}_3 + \mathbf{B}_4\mathbf{B}_3 + \mathbf{B}_3\mathbf{B}_4 + \mathbf{B}_4^2, \\
\mathbf{D}\mathbf{P}_0\mathbf{W} + \mathbf{W}\mathbf{P}_0\mathbf{D} &= (\mathbf{B}_1 + \mathbf{B}_2)(\mathbf{I}-2s\mathbf{nn})(\mathbf{B}_3 + \mathbf{B}_4) \\
&\quad + (\mathbf{B}_3 + \mathbf{B}_4)(\mathbf{I}-2s\mathbf{nn})(\mathbf{B}_1 + \mathbf{B}_2) \\
&= (1-2s)(\mathbf{B}_1\mathbf{B}_3 + \mathbf{B}_3\mathbf{B}_1 + \mathbf{B}_2\mathbf{B}_3 + \mathbf{B}_3\mathbf{B}_2) \\
&\quad + \mathbf{B}_2\mathbf{B}_4 + \mathbf{B}_4\mathbf{B}_2.
\end{aligned}$$

Then (A.2) follows directly as we have

$$\begin{aligned}
&\mathbf{E}_2 : \mathbf{B}_i^2 = 0, \ \text{for } i = 1,2,3,4; \quad \mathbf{E}_2 : (\mathbf{B}_i\mathbf{nn}\mathbf{B}_i) = 0, \ \text{for } i = 2,3; \\
&\mathbf{E}_3 : (\mathbf{B}_2\mathbf{B}_3 + \mathbf{B}_3\mathbf{B}_2) = 0, \quad \mathbf{E}_3 : \big[(\mathbf{I}-2\mathbf{nn})(\mathbf{B}_2\mathbf{B}_3 - \mathbf{B}_3\mathbf{B}_2)\big] = 0; \\
&\mathbf{E}_4 : \mathbf{B}_1\mathbf{B}_3 = \mathbf{E}_4 : \mathbf{B}_3\mathbf{B}_1 = \mathbf{E}_4 : \mathbf{B}_4\mathbf{B}_2 = \mathbf{E}_4 : \mathbf{B}_2\mathbf{B}_4 = 0.
\end{aligned}$$

□

A.5. Existence of solutions to the system for $\mathbf{V}_{\pm}^{(k)}$

We give a sketch procedure to solve the system (2.19) with (6.25) for $\mathbf{V}_{\pm}^{(1)}$ and the system (2.18) with (6.36) for $\mathbf{V}_{\pm}^{(k)}$. For simplicity, we drop the superscript k and write $\mathbf{V}_{\pm} = \mathbf{V}_{\pm}^{(k)}$ for $k \geqslant 1$. Then the equation (2.18) can be written as

$$\partial_t \mathbf{V}_{\pm} - \Delta \mathbf{V}_{\pm} + \mathbf{B}_{\pm,i}\partial_i \mathbf{V}_{\pm} + \mathbf{C}_{\pm}\mathbf{V}_{\pm} + \mathbf{J}_{\pm} = 0, \quad \text{in } Q_{\pm}, \tag{A.4}$$

where $\mathbf{B}_{\pm,i}, \mathbf{J}_{\pm}$ are known matrices and $\mathbf{C}_{\pm} : \mathbb{M}_n \to \mathbb{M}_n$ is a known linear map. Noticing that the equation (2.19) is linear for $\mathbf{V}_{\pm}^{(1)}$, it can also be rewritten as the form (A.4). As for $\mathbf{A} \in \mathbb{A}_n$, it holds that $\mathbf{A} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^{\mathrm{T}})$. Thus, one has

$$\mathcal{P}_3\mathbf{A} = \mathbf{nnA}(\mathbf{I} - \mathbf{nn}) + (\mathbf{I} - \mathbf{nn})\mathbf{Ann}, \quad \mathcal{P}_4\mathbf{A} = (\mathbf{I} - \mathbf{nn})\mathbf{A}(\mathbf{I} - \mathbf{nn}). \tag{A.5}$$

According to the fact that $\partial_\nu(\mathbf{nn}) = 0$ on Γ, ∂_ν commutes with $\mathcal{P}_3$ and $\mathcal{P}_4$. Thus, the interface jump conditions (6.25) or (6.36) can be written as

$$\begin{aligned}
&\mathcal{P}_4\mathbf{V}_+ - \mathcal{P}_4\mathbf{V}_- = \mathbf{K}_1,\\
&\mathcal{P}_4(\partial_\nu\mathbf{V}_+) - \mathcal{P}_4(\partial_\nu\mathbf{V}_-) = \mathbf{K}_2,\\
&\mathcal{P}_3(\partial_\nu\mathbf{V}_+) - \mathcal{P}_3\Big(\bar{\mathbf{W}}\mathcal{P}_3\mathbf{V}_-\Big) = \mathbf{K}_3,\\
&\mathcal{P}_3(\partial_\nu\mathbf{V}_-) + \mathcal{P}_3\Big(\bar{\mathbf{W}}\mathcal{P}_3\mathbf{V}_+\Big) = \mathbf{K}_4,
\end{aligned} \tag{A.6}$$

where $\mathbf{K}_1, \mathbf{K}_2 \in \mathbb{V}_4, \mathbf{K}_3, \mathbf{K}_4 \in \mathbb{V}_3$ are given smooth functions on $\Gamma = \cup_{t\geqslant 0}\Gamma_t$. The system (A.4)–(A.6) is a parabolic transmission-like system with an interface moving along the mean curvature flow. The transmission problem with a fixed interface has been studied by many works; see [10,37] for the elliptic case and [5,14] for parabolic problems for example. To treat the moving interface, we transform the problem into a parabolic transmission-like system with a fixed interface via introducing time-dependent maps.

First, we consider the simpler case $\mathbf{K}_1 = 0$. We can assume that there is a smooth (in both space and time variables) map $\Phi(\cdot, t) : \Gamma_0 \to \Gamma_t$ for $t \in [0, T]$. Then one can extend it to Ω, which is still denoted by $\Phi(\cdot, t)$, such that $\Phi_{\pm}(\cdot, t) = \Phi(\cdot, t)|_{\Omega_0^{\pm}} : \Omega_0^{\pm} \to \Omega_t^{\pm}$ are smooth non-degenerate diffeomorphism. Let $\mathbf{W}_{\pm}(y, t) = \mathbf{V}_{\pm}(\Phi_{\pm}(y, t), t)$, $\Psi_{\pm}(\cdot, t) : \Omega_t^{\pm} \to \Omega_0^{\pm}$ be the inverse of $\Phi_{\pm}$ and $A_{\pm,i}^{j} = \partial_i\Psi_{\pm,j} \circ \Phi_{\pm}$ for $1 \leqslant i, j \leqslant m$. We also use the notations $D_i = \frac{\partial}{\partial_{y_i}}$, and $\hat{N}_k^{\pm} = \hat{\nu}_j A_{\pm,i}^{j} A_{\pm,i}^{k}/|\hat{\nu}_j A_{\pm}^{j}|$ with $\hat{\nu}$ being the unit

outer normal of Ω_0^+ (the Einstein summation convention is used). By suitably choosing Φ, one may assume that

$$\hat{a}^{\pm} := \det(\nabla\Phi) = \det(A_{\pm}^{-1}) \in [\lambda_0, \lambda_0^{-1}] \quad \text{for some } \lambda_0 > 0,$$

and $(\hat{a}^{\pm} A^j_{\pm,i} A^k_{\pm,i})_{1\leqslant j,k\leqslant m}$ is uniformly elliptic in the sense of

$$\int_{\Omega_0^{\pm}} \hat{a}^{\pm} A^j_{\pm,i} A^k_{\pm,i} D_j \mathbf{W}_{\pm} : D_k \mathbf{W}_{\pm} \geqslant 2\lambda_1 \int_{\Omega_0^{\pm}} |D\mathbf{W}_{\pm}|^2 \quad \text{for some } \lambda_1 > 0. \tag{A.7}$$

Then the system can be rewritten into a new system of the following type in a fixed domain:

$$\begin{aligned} &\partial_t \mathbf{W}_{\pm} - A^k_{\pm,i} D_k (A^j_{\pm,i} D_j \mathbf{W}_{\pm}) + \hat{\mathbf{B}}_{\pm,i} D_i \mathbf{W}_{\pm} + \hat{\mathbf{C}}_{\pm} \mathbf{W}_{\pm} + \hat{\mathbf{J}}_{\pm} = 0, \\ &(y,t) \in \Omega_0^{\pm} \times [0,T], \end{aligned} \tag{A.8}$$

and the boundary conditions are reduced to

$$\begin{aligned} &\mathcal{P}_4 \mathbf{W}_+ - \mathcal{P}_4 \mathbf{W}_- = 0, \\ &\mathcal{P}_4 (D_{\hat{N}^+} \mathbf{W}_+) - \mathcal{P}_4 (D_{\hat{N}^-} \mathbf{W}_-) = \mathbf{K}_2 \circ \Phi, \\ &\mathcal{P}_3 (D_{\hat{N}^+} \mathbf{W}_+) - \mathcal{P}_3 \Big(\bar{\mathbf{W}} \circ \Phi \mathcal{P}_3 \mathbf{W}_- \Big) = \mathbf{K}_3 \circ \Phi, \\ &\mathcal{P}_3 (D_{\hat{N}^-} \mathbf{W}_-) + \mathcal{P}_3 \Big(\bar{\mathbf{W}} \circ \Phi \mathcal{P}_3 \mathbf{W}_+ \Big) = \mathbf{K}_4 \circ \Phi, \end{aligned} \tag{A.9}$$

for $y \in \Gamma_0$ and $t \geqslant 0$. Here $\mathcal{P}_3$, $\mathcal{P}_4$ are defined by (A.5) with $\mathbf{n}$ being replaced by $\mathbf{n} \circ \Phi$.

By using a similar method as in [5,14], we can obtain the existence and uniqueness of weak solutions to the above system. The regularity of solutions can be deduced from the following a priori energy estimate. Here we omit the details and left them to the interested readers.

Proposition A.4 *When $\mathbf{K}_1 = 0$, there exists $\lambda_0, C > 0$ such that for smooth solutions of (A.8)–(A.9), it holds*

$$\frac{\mathrm{d}}{\mathrm{d}t} E_k(t) + \lambda_0 F_k(t) \leqslant C(1 + E_k(t)),$$

with

$$E_k(t) = \frac{1}{2}\sum_{i=0}^{k}\int_{\Omega_0^\pm} \hat{a}^\pm(y,t)|\partial_t^i \mathbf{W}_\pm|^2 \mathrm{d}y,$$
$$F_k(t) = \frac{1}{2}\sum_{i=0}^{k}\int_{\Omega_0^\pm} \hat{a}^\pm(y,t)|D\partial_t^i \mathbf{W}_\pm|^2 \mathrm{d}y.$$

Here we use the notation $\int_{\Omega_0^\pm} f_\pm = \int_{\Omega_+} f_+ + \int_{\Omega_-} f_-$ *for simplicity.*

Proof We give the proof for $k = 0$. Direct calculations give us that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_0(t) &= \int_{\Omega_0^\pm}\Big(\hat{a}^\pm \mathbf{W}_\pm : \partial_t \mathbf{W}_\pm + \frac{1}{2}\partial_t \hat{a}^\pm |\mathbf{W}_\pm|^2\Big)\mathrm{d}y\\ &= -\int_{\Omega_0^\pm} \hat{a}^\pm A^j_{\pm,i} A^k_{\pm,i}\partial_i \mathbf{W}_\pm : \partial_k \mathbf{W}_\pm \mathrm{d}y\\ &\quad + \int_{\Omega_0^\pm} \hat{a}^\pm\Big(-\hat{\mathbf{B}}_{\pm,i} D_i \mathbf{W}_\pm + (\frac{1}{2}\partial_t \hat{a}^\pm - \hat{\mathbf{C}}_\pm)\mathbf{W}_\pm - \hat{\mathbf{J}}_\pm\Big) : \mathbf{W}_\pm \mathrm{d}y\\ &\quad + \int_{\Gamma_0}(D_{\hat{N}^+}\mathbf{W}_+ : \mathbf{W}_+ - D_{\hat{N}^-}\mathbf{W}_- : \mathbf{W}_-)\hat{a}^\pm |\hat{v}_j A^j_\pm| \mathrm{d}\sigma(y).\end{aligned}$$

Here we should note that $\hat{a}^\pm|\hat{v}_j A^j_\pm|\mathrm{d}\sigma(y) = \mathrm{d}\sigma(\Phi(y))$ for $y \in \Gamma_0$ and it is independent of the symbol $+$ or $-$. So one may denote $\hat{a}^\pm|\hat{v}_j A^j_\pm|$ by $\hat{a}|\hat{v}_j A^j|$ for $y \in \Gamma_0$. We have

$$\begin{aligned}&\int_{\Omega_0^\pm} \hat{a}^\pm\Big(-\hat{\mathbf{B}}_{\pm,i} D_i \mathbf{W}_\pm + (\frac{1}{2}\partial_t \hat{a}^\pm - \hat{\mathbf{C}}_\pm)\mathbf{W}_\pm - \hat{\mathbf{J}}_\pm\Big) : \mathbf{W}_\pm\\ &\quad \leqslant -c_0\int_{\Omega_0^\pm}|D\mathbf{W}_\pm|^2 + C\int_{\Omega_0^\pm}|\mathbf{W}_\pm|^2,\end{aligned}$$

for some $c_0, C > 0$. Using the fact that $D_{\hat{N}}\mathbf{W} : \mathbf{W} = \mathcal{P}_4 D_N \mathbf{W} : \mathcal{P}_4\mathbf{W} + \mathcal{P}_3 D_N \mathbf{W} : \mathcal{P}_3\mathbf{W}$, we get

$$\begin{aligned}&\int_{\Gamma_0}(D_{\hat{N}^+}\mathbf{W}_+ : \mathbf{W}_+ - D_{\hat{N}^-}\mathbf{W}_- : \mathbf{W}_-)\hat{a}|\hat{v}_j A^j|\mathrm{d}\sigma(y)\\ &\quad = \int_{\Gamma_0}(\mathcal{P}_4 D_{\hat{N}^+}\mathbf{W}_+ : \mathcal{P}_4\mathbf{W}_+ - \mathcal{P}_4 D_{\hat{N}^-}\mathbf{W}_- : \mathcal{P}_4\mathbf{W}_-)\hat{a}|\hat{v}_j A^j|\mathrm{d}\sigma(y)\\ &\qquad + \int_{\Gamma_0}(\mathcal{P}_3 D_{\hat{N}^+}\mathbf{W}_+ : \mathcal{P}_3\mathbf{W}_+ - \mathcal{P}_3 D_{\hat{N}^-}\mathbf{W}_- : \mathcal{P}_3\mathbf{W}_-)\hat{a}|\hat{v}_j A^j|\mathrm{d}\sigma(y)\end{aligned}$$

$$= \int_{\Gamma_0} (\mathbf{K}_2 \circ \Phi : \mathcal{P}_4 \mathbf{W}_+ + \mathbf{K}_3 \circ \Phi : \mathcal{P}_3 \mathbf{W}_+ - \mathbf{K}_4 \circ \Phi : \mathcal{P}_3 \mathbf{W}_-) \hat{a} |\hat{v}_j A^j| \mathrm{d}\sigma(y).$$

Applying the trace theorem and interpolation inequalities for $H^1(\Omega_0^\pm)$ functions, they can be bounded by

$$\delta \int_{\Omega_0^\pm} |D\mathbf{W}_\pm|^2 \mathrm{d}y + C_\delta \Big(1 + \int_{\Omega_0^\pm} |\mathbf{W}_\pm|^2 \mathrm{d}y\Big).$$

Therefore, by (A.7) and choosing δ sufficiently small, we have for some $c_1 > 0$ that

$$\frac{\mathrm{d}}{\mathrm{d}t} E_0(t) + c_1 \int_{\Omega_0^\pm} \hat{a}^\pm(y,t) |D\mathbf{W}_\pm|^2 \mathrm{d}y \leqslant C(1 + E_0(t)).$$

For general $k \geqslant 1$, the estimate can be obtained similarly by noticing that

$$\|\partial_t^i \mathbf{W}_\pm\|^2_{H^2(\Omega_\pm)} \leqslant C\Big(1 + \sum_{j=0}^{i+1} \int_{\Omega_0^\pm} |\partial_t^j \mathbf{W}_\pm|^2 \mathrm{d}y\Big), \quad \text{for } 0 \leqslant i \leqslant k-1,$$

and the boundary terms can be controlled by applying the trace theorem and interpolation inequalities as before. □

For general $\mathbf{K}_1$, we can extend $\mathbf{K}_1$ to be a smooth $\mathbb{A}_n$-valued function in Ω_-, which is denoted by $\tilde{\mathbf{K}}_1$. Then let

$$\mathbf{V}_*(x,t) = \zeta(d_0(x,t))(\mathbf{I} - \mathbf{nn})\tilde{\mathbf{K}}_1(\mathbf{I} - \mathbf{nn}),$$

where $\zeta \in C^\infty((-\infty, 0])$ satisfies $\zeta = 1$ in $[-\delta_0/2, 0]$ and $\zeta = 0$ for $z \leqslant -\delta_0$. Note that $\mathbf{n}$ is defined in $\Gamma(\delta_0)$ with $\partial_\nu \mathbf{n} = 0$. We obtain that $\mathbf{V}_*$ is well-defined and smooth with

$$\mathcal{P}_4 \mathbf{V}_* = \mathbf{K}_1, \quad \mathcal{P}_4 \partial_\nu \mathbf{V}_* = \partial_\nu \mathbf{K}_*, \quad \mathcal{P}_3 \mathbf{V}_* = 0, \quad \mathcal{P}_3 \partial_\nu \mathbf{V}_* = 0.$$

By considering the new unknowns $(\tilde{\mathbf{V}}_+, \tilde{\mathbf{V}}_-) = (\mathbf{V}_+, \mathbf{V}_- + \mathbf{V}_*)$, we can reduce the problem to the case of $\mathbf{K}_1 = 0$.

References

1. Abels, H., Liu, Y.: Sharp interface limit for a Stokes/Allen–Cahn system. Arch. Ration. Mech. Anal. **229**, 417–502 (2018)

2. Alikakos, N.D., Bates, P.W., Chen, X.: Convergence of the Cahn–Hilliard equation to the Hele–Shaw model. Arch. Ration. Mech. Anal. **128**, 165–205 (1994)
3. Allen, S., Cahn, J.: A microscopic theory for antiphase motion and its application to antiphase domain coarsening. Acta Metall. **27**, 1084–1095 (1979)
4. Ambrosio, L., Soner, M.: A measure-theoretic approach to higher codimension mean curvature flows. Dedicated to Ennio De Giorgi. Ann. Scuola Norm. Sup. Pisa Cl. Sci.(4) **25**, 27–49 (1997)
5. Amann, H.: Maximal regularity of parabolic transmission problems. J. Evol. Equ. **21**, 3375–3420 (2021)
6. Ball, J.M., James, R.D.: Fine phase mixtures as minimizers of energy. Arch. Ration. Mech. Anal. **100**, 13–52 (1987)
7. Bethuel, F.: Asymptotics for two-dimensional vectorial Allen–Cahn systems. arXiv:2003.10189
8. Bronsard, L., Kohn, R.V.: Motion by mean curvature limit of Ginzburg–Landau as the singular dynamics. J. Differ. Equ. **237**, 211–237 (1991)
9. Bronsard, L., Stoth, B.: The singular limit of a vector-valued reaction–diffusion process. Trans. Am. Math. Soc. **350**, 4931–4953 (1998)
10. Caffarelli, L., Soria-Carro, M., Stinga, P.: Regularity for $C^{1,\alpha}$ interface transmission problems. Arch. Ration. Mech. Anal. **240**, 265–294 (2021)
11. Chen, X.: Generation and propagation of interfaces for reaction-diffusion equations. J. Differ. Equ. **96**, 116–141 (1992)
12. Chen, X.: Spectrum for the Allen–Cahn, Cahn–Hilliard, and phase-field equations for generic interfaces. Comm. Partial Differ. Eqs. **19**, 1371–1395 (1994)
13. de Mottoni, P., Schatzman, M.: Geometrical evolution of developed interfaces. Trans. Am. Math. Soc. **347**, 1533–1589 (1995)
14. Dong, H., Xu, L.: Gradient estimates for divergence form parabolic systems from composite materials. Calc. Var. Partial Differ. Equ. 60, Paper No. 98 (2021)
15. Evans, L.C., Soner, H.M., Souganidis, P.E.: Phase transitions and generalized motion by mean curvature. Comm. Pure Appl. Math. **45**, 1097–1123 (1992)
16. Fei, M., Wang, W., Zhang, P., Zhang, Z.: Dynamics of the nematic-isotropic sharp interface for the liquid crystal. SIAM J. Appl. Math. **75**, 1700–1724 (2015)
17. Fei, M., Wang, W., Zhang, P., Zhang, Z.: On the isotropic-nematic phase transition for the liquid crystal. Peking Math. Jour. **1**, 141–219 (2018)
18. Fischer, J., Laux, T., Simon, T.: Convergence rates of the Allen–Cahn equation to mean curvature flow: a short proof based on relative entropies. SIAM J. Math. Anal. **52**, 6222–6233 (2020)
19. Fischer, J., Marveggio, A.: Quantitative convergence of the vectorial Allen–Cahn equation towards multiphase mean curvature flow. arXiv:2203.17143
20. Fonseca, I., Tartar, L.: The gradient theory of phase transitions for systems with two potential wells. Proc. R. Soc. Edinb. Sect. A **111**, 89–102 (1989)
21. Ilmanen, T.: Convergence of the Allen-Cahn equation to Brakke motion by mean curvature. J. Differ. Geom. **38**, 417–461 (1993)
22. Laux, T., Liu, Y.: Nematic-isotropic phase transition in liquid crystals: a variational derivation of effective geometric motions. Arch. Ration. Mech. Anal. **241**, 1785–1814 (2021)
23. Laux, T., Simon, T.M.: Convergence of the Allen–Cahn equation to multiphase mean curvature flow. Comm. Pure Appl. Math. **71**, 1597–1647 (2018)
24. Lin, F.-H.: Some dynamical properties of Ginzburg-Landau vortices. Comm. Pure Appl. Math. **49**, 323–359 (1996)
25. Lin, F.-H.: Complex Ginzburg–Landau equations and dynamics of vortices, filaments, and codimension-2 submanifolds. Comm. Pure Appl. Math. **51**, 385–441 (1998)
26. Lin, F.-H., Pan, X., Wang, C.: Phase transition for potentials of high-dimensional wells. Comm. Pure Appl. Math. **65**, 0833–0888 (2012)

27. Lin, F.-H., Wang, C.: Harmonic maps in connection of phase transitions with higher dimensional potential wells. Chin. Ann. Math. Ser. B **40**, 781–810 (2019)
28. Lin, F.-H., Wang, C.: Isotropic-nematic phase transition and liquid crystal droplets. Comm. Pure Appl. Math. https://doi.org/10.1002/cpa.22050
29. Majumdar, A., Milewski, P.A., Spicer, A.: Front propagation at the nematic-isotropic transition temperature. SIAM J. Appl. Math. **76**, 1296–1320 (2016)
30. Modica, L.: The gradient theory of phase transitions and the minimal interface criterion. Arch. Ration. Mech. Anal. **98**, 123–142 (1987)
31. Modica, L., Mortola, S.: Il limite nella Γ-convergenza di una famiglia di funzionali ellittici. Boll. Un. Mat. Ital. A (5) **14**, 526–529 (1977)
32. Monteil, A., Santambrogio, F.: Metric methods for heteroclinic connections. Math. Methods Appl. Sci. **41**, 1019–1024 (2018)
33. Moser, M.: Convergence of the scalar- and vector-valued Allen–Cahn Equation to mean curvature flow with 90°-contact angle in higher dimensions. arXiv:2105.07100
34. Osting, B., Wang, D.: A diffusion generated method for orthogonal matrix-valued fields. Math. Comput. **89**, 515–550 (2020)
35. Rubinstein, J., Sternberg, P., Keller, J.: Fast reaction, slow diffusion, and curve shortening. SIAM J. Appl. Math. **49**, 116–133 (1989)
36. Rubinstein, J., Sternberg, P., Keller, J.: Reaction–diffusion processes and evolution to harmonic maps. SIAM J. Appl. Math. **49**, 1722–1733 (1989)
37. Schechter, M.: A generalization of the problem of transmission. Annali della Scuola Normale Superiore di Pisa - Classe di Scienze, Serie **3**(14), 207–236 (1960)
38. Sternberg, P.: The effect of a singular perturbation on nonconvex variational problems. Arch. Ration. Mech. Anal. **101**, 209–260 (1988)
39. Wang, D., Osting, B., Wang, X.-P.: Interface dynamics for an Allen–Cahn-type equation governing a matrix-valued field. Multiscale Model. Simul. **17**, 1252–1273 (2019)
40. Wang, W., Zhang, P., Zhang, Z.: Rigorous derivation from Landau-de Gennes theory to Ericksen–Leslie theory. SIAM J. Math. Anal. **47**, 127–158 (2015)
41. Wang, W., Zhang, P., Zhang, Z.: The small Deborah number limit of the Doi-Onsager equation to the Ericksen-Leslie equation. Comm. Pure Appl. Math. **68**, 1326–1398 (2015)
42. Zuniga, A., Sternberg, P.: On the heteroclinic connection problem for multi-well gradient systems. J. Differ. Equ. **261**, 3987–4007 (2016)