DENSITY-CONSTRAINED CHEMOTAXIS AND HELE-SHAW FLOW

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ABSTRACT. We consider a model of congestion dynamics with chemotaxis, where the density of cells follows the chemical signal it generates, while observing an incompressibility constraint (incompressible parabolic-elliptic Patlak-Keller-Segel model). We show that when the chemical diffuses slowly and attracts the cells strongly, then the dynamics of the congested cells is well approximated by a surface-tension driven free boundary problem. More precisely, we rigorously establish the convergence of the solution to the characteristic function of a set whose evolution is determined by the classical Hele-Shaw free boundary problem with surface tension.

The problem is set in a bounded domain, which leads to an interesting analysis on the limiting boundary conditions. Namely, we prove that the assumption of Robin boundary conditions for the chemical potential leads to a contact angle condition for the free interface (in particular Neumann boundary conditions lead to an orthogonal contact angle condition, while Dirichlet boundary conditions lead to a tangential contact angle condition).

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1. INTRODUCTION

1.1. **A model for chemotaxis with density constraint.** The classical parabolic-elliptic Patlak-Keller-Segel model for chemotaxis reads:

$$\begin{cases} \partial_t \rho - \mu \Delta \rho + \chi \operatorname{div}(\rho \nabla \phi) = 0, \\ \eta \Delta \phi + \theta \rho - \sigma \phi = 0, \end{cases}$$

where ρ denotes the cell density and ϕ the concentration of some chemical. The nonnegative parameters μ and η are the cell and chemical diffusivity, χ is the cell sensitivity, and θ and σ describe the production and degradation of the chemical (see [14], [23], [10]).

In this model, the diffusion competes with the aggregating potential ϕ, leading to the well-known phenomena of concentration and finite time blow-up of the density (see e.g. [5], [12]). In order to investigate the behavior of the density ρ after saturation occurs we take into account the incompressibility of the cells by imposing a constraint $\rho \leq \rho_M$. We replace ρ with ρ/ρ_M and ϕ^ε with $\phi^\varepsilon/(\rho_M \theta)$ so that $\rho_M = \theta = 1$ and denote $\bar{\chi} = \chi \rho_M \theta$. We are then led to the equation (see [26, 15] for details):

$$(1.1) \quad \begin{cases} \partial_t \rho - \mu \Delta \rho + \bar{\chi} \operatorname{div}(\rho \nabla \phi^\varepsilon) - \Delta p = 0, & \rho \leq 1 \\ \sigma \phi^\varepsilon - \eta \Delta \phi^\varepsilon = \rho, \end{cases}$$

where the pressure p is a Lagrange multiplier for the contraint $\rho \leq 1$, and satisfies

$$p \geq 0, \qquad p(1 - \rho) = 0 \text{ a.e.}$$

Similar models have been used in particular in the study of congested crowd motion (see [26]). The conditions on p can also be expressed by writing $p \in P(\rho)$ with

$$(1.2) \qquad P(\rho) := \begin{cases} 0 & 0 \le \rho < 1; \\ [0, \infty) & \rho = 1, \end{cases}$$

which is sometimes referred to as the Hele-Shaw graph.

In a companion paper [15], we proved the existence and uniqueness of a weak solution for (1.1) and investigated its relation to some free boundary problems. In this paper, we will investigate the singular limit of strong attraction ($\chi \gg \mu$) and small chemical diffusion ($\eta \ll 1$) and prove that the model is asymptotically close to a Hele-Shaw free boundary problem with surface tension (1.8). This establishes the first rigorous link between a general Chemotaxis system and Hele-Shaw flow with surface tension, to the best of our knowledge.

We also aim to analyze the behavior of the solutions of (1.1) near a fixed boundary, by setting the problem in a bounded domain $\Omega \subset \mathbb{R}^d$. In particular we are interested in the effect on the dynamics of different absorption rates of the chemical at the boundary. For full generality, we will use Robin boundary conditions for ϕ^ε with a fixed parameter for absorption rate. For the density, we impose Neumann boundary conditions which ensure the conservation of cell density.

Above discussions, by setting $\eta = \varepsilon^2$ and $\bar{\chi} = \varepsilon^{-1}$ for small $\varepsilon > 0$, lead to the system (1.3)-(1.4):

$$(1.3) \qquad \begin{cases} \partial_t \rho - \mu \Delta \rho + \operatorname{div}(\varepsilon^{-1} \rho \nabla \phi^\varepsilon - \nabla p) = 0, & \text{in } \Omega \times (0, \infty), \qquad p \in P(\rho); \\ (-\mu \nabla \rho + \varepsilon^{-1} \rho \nabla \phi^\varepsilon - \nabla p) \cdot n = 0, & \text{on } \partial\Omega \times (0, \infty); \\ \rho(x, 0) = \rho_{in}(x) & \text{in } \Omega, \end{cases}$$

with ϕ^ε solving

$$(1.4) \qquad \begin{cases} \sigma \phi^\varepsilon - \varepsilon^2 \Delta \phi^\varepsilon = \rho & \text{in } \Omega; \\ \alpha \phi^\varepsilon + \beta \varepsilon \nabla \phi^\varepsilon \cdot n = 0 & \text{on } \partial\Omega, \end{cases}$$

where n denotes the outward normal on $\partial\Omega$. Note that the scaling of the continuity equation can also be obtained by rescaling the time variable so that we observe the evolution of ρ at a time scale $\bar{t} \sim \varepsilon^{-1}/\bar{\chi}$, under the assumption that $\mu = \mathcal{O}(\varepsilon\bar{\chi})$.

Throughout the paper, we assume that α, β and σ are constants satisfying

$$\sigma > 0, \quad \alpha \ge 0, \quad \beta \ge 0, \quad \alpha + \beta > 0.$$

The assumption $\sigma > 0$ is important. When $\sigma = 0$, the function $\bar{\phi}^\varepsilon = \varepsilon^2 \phi^\varepsilon$ is the usual Newtonian potential (up to the boundary condition on $\partial\Omega$), which does not localize in the limit $\varepsilon \to 0$: see the discussion below (1.8). By contrast, when $\sigma > 0$, we have $\phi^\varepsilon \sim \frac{1}{\sigma}\rho$ when $\varepsilon \ll 1$ and the effect of $\varepsilon^{-1}\nabla\phi^\varepsilon$ on the dynamic of the saturated regions is akin to that of surface tension.

1.2. **Relation to Hele-Shaw free boundary problems.** When $\mu = 0$, (1.3)-(1.4) is a weak formulation for the free boundary problem

$$(1.5) \qquad \begin{cases} \rho \in [0, 1), \quad p = 0, \quad \partial_t \rho + \varepsilon^{-1}\operatorname{div}(\rho\nabla\phi^\varepsilon) = 0 & \text{in } \Omega \setminus \Omega_s(t); \\ \rho = 1, \quad p > 0, \quad \Delta p = \varepsilon^{-1}\Delta\phi^\varepsilon & \text{in } \Omega_s(t), \end{cases}$$

where $\Omega_s(t) = \{\rho(t) = 1\}$ denotes the saturated density set and the free boundary $\Sigma(t) = \partial\Omega_s(t) \cap \Omega$ moves according to the velocity law

$$(1.6) \qquad (1 - \rho|_{\Omega_s^C})V = (-\nabla p + \varepsilon^{-1}\nabla\phi^\varepsilon) \cdot \nu|_{\Omega_s}.$$

Here V denotes the outward normal velocity of $\Sigma(t)$ and ν denotes the outward normal of $\Omega_s(t)$. In particular, when the density is a characteristic function $\rho(x,t) = \chi_{\Omega_s(t)}(x)$, we recognize the usual one phase Hele-Shaw problem without surface tension, which we can write with the modified pressure $q = p + \varepsilon^{-1}\rho\left(\frac{1}{2\sigma} - \phi^\varepsilon\right)$, as

$$(1.7) \qquad \begin{cases} \Delta q = 0 \text{ in } \Omega_s(t), \qquad q = \varepsilon^{-1}\left(\frac{1}{2\sigma} - \phi^\varepsilon\right) \text{ on } \Sigma(t); \\ V = -\nabla q \cdot \nu \quad \text{ on } \Sigma(t). \end{cases}$$

In other words, in this fully saturated regime, the chemotaxis system (1.3)-(1.4) can be seen as a free boundary problem describing the motion of the region occupied by the cell, driven by the chemical concentration ϕ^ε and the pressure variable p. Since we obtained (1.3)-(1.4) by imposing the constraint $\rho \leq 1$, but without requiring $\rho \in \{0,1\}$, it is not clear that we should actually have $\rho(x,t) = \chi_{\Omega_s(t)}(x)$ in general. In [15], we proved that if ρ is a characteristic function at $t = 0$, then this remains true at positive times for (1.3)-(1.4) when $\mu = 0$. On the other hand when $\mu > 0$ the density is never a characteristic function. Indeed in this case the saturated set interacts with the unsaturated part of the density by a Richards-type problem, as shown in [15].

Nevertheless, we will show in this paper that, in the limit $\varepsilon \to 0$, the effect of the attractive potential is strong enough to ensure the convergence of ρ to a characteristic function $\chi_{\Omega_s(t)}(x)$ for all $\mu \geq 0$. We will then show that the asymptotic dynamic of $\Omega_s(t)$ is described by the Hele-Shaw free boundary problem with surface tension

$$(1.8) \qquad \begin{cases} \Delta q = 0 \text{ in } \Omega_s(t), \qquad q = \frac{\kappa}{4\sigma^{3/2}} \text{ on } \Sigma(t); \\ V = -\nabla q \cdot \nu \quad \text{ on } \Sigma(t), \end{cases}$$

where κ denotes the mean curvature of the free boundary $\Sigma(t)$ (taken to be positive when $\Omega_s(t)$ is convex). Formally, we can get (1.8) from (1.7) by proving that the quantity $\varepsilon^{-1}\left(\frac{1}{2\sigma} - \phi^\varepsilon\right)$ is an approximation of the mean-curvature of Ω_s when $\varepsilon \ll 1$.

(1.8) is a classical model, originally describing the motion of the interface separating two immiscible fluids in a Hele-Shaw cell. While the problem has been derived in various frameworks (see for instance [22, 2, 8, 16]), our paper appears to be the first to establish a rigorous connection to chemotaxis, or more generally to a second-order (or, when $\mu = 0$, a first-order) equation with nonlocal drift.

Note that, as σ tends to zero, the weight on surface tension grows to infinity in (1.8). Thus heuristically we expect that the limit density support will re-adjust itself into a ball at time scale of order $\sigma^{3/2}$ (and instantly when $\sigma = 0$). This is consistent with the convergence to radial solutions of (1.1) when $\sigma = \mu = 0$: see [4] and [10] for further discussions.

1.3. **The presence of bounded domain.** Our result for $\mu = 0$ bears similarities with [13], where the emergence of surface tension and derivation of a Muskat problem is studied via a variational approximation. In that paper, the potential ϕ^ε solves $\phi_t^\varepsilon - \Delta\phi^\varepsilon = \rho$ (in $\mathbb{R}^d$) and instead of the Keller-Segel system, the authors considers a discrete-time approximations constructed via a JKO scheme. A similar variational analysis is performed in [17] for the L^2-based thresholding scheme.

Note that both [13] and [17] consider the setting of periodic torus or entire $\mathbb{R}^d$ for the interaction energy, in which case ϕ^ε can be written as a convolution with the heat kernel. Such a representation of ϕ^ε, as well as the symmetry of the heat kernel in space variables, played an important role in the analysis of the aforementioned papers, in particular when deriving the weak limit equation. The fact that our problem is set in a bounded domain presents an interesting challenge to this analysis. In particular this necessitates a more PDE-oriented proof of Proposition 5.2, replacing corresponding proofs in [17] and [13]. Our result appears to be the first that links a Keller-Segel system with a Hele-Shaw flow with surface tension, regardless of the choice of the domain. This connection was also suggested in the

very recent paper [6], where the incompressible limit of a generalized version of Keller-Segel system is investigated. The model is a variant of Cahn-Hilliard equation, which can be seen as a diffuse-interface approximation of our Hele-Shaw flow with surface tension.

Another novel feature of our analysis, also related to the bounded domain, is the characterization of the free boundary behavior near the fixed boundary $\partial\Omega$. Of particular interest, in the context of the singular limit $\varepsilon \to 0$, is how the Robin boundary conditions imposed on ϕ^ε play a role in the dynamics of $\Omega_s(t)$. We will show that (1.8) must be supplemented by the contact angle condition

$$(1.9) \qquad \cos(\theta) = \gamma := -\min\left(1, \frac{2\alpha}{\alpha + \sqrt{\sigma}\beta}\right) \quad \text{on } \Sigma(t) \cap \partial\Omega,$$

where θ is the angle formed by the free surface $\Sigma(t)$ and the fixed boundary $\partial\Omega$, measured from inside of the set $\Omega_s(t)$, and the fixed boundary $\partial\Omega$ at the triple junction $\Sigma(t) \cap \partial\Omega$ (see Figure 1). In particular, for Neuman condition $\alpha = 0$ (zero absorption of chemicals), the contact must be orthogonal, while for Dirichlet condition $\beta = 0$ (and whenever the absortion rate α is bigger or equal than $\sqrt{\sigma}\beta$), the contact must be tangential. Let us note that a similar contact angle condition was derived recently in [11] for the geometric mean-curvature flow.

1.4. **Notations and definitions.** Throughout the paper Ω is a smooth bounded domain in $\mathbb{R}^d$.

We will use the following definition of weak solutions of (1.3)-(1.4), as in [15]:

Definition 1.1. *The pair of functions (ρ, p) is a weak solution of (1.3)-(1.4) if $\rho \in L^1((0, \infty); L^\infty(\Omega)) \cap C^{1/2}([0, \infty); H^{-1}(\Omega))$, $p \in L^2((0, \infty); H^1(\Omega))$ with*

$$0 \le \rho \le 1, \quad p \ge 0, \quad (1 - \rho)p = 0 \quad a.e. \text{ in } \Omega \times (0, \infty)$$

and the following holds:

$$(1.10) \qquad \int_\Omega \rho_{in}(x)\zeta(x, 0)\, dx + \int_0^\infty \int_\Omega (\rho\, \partial_t\zeta + \rho v \cdot \nabla\zeta)\, dx\, dt = 0$$

for any function $\zeta \in C_c^\infty(\overline{\Omega} \times [0, \infty))$ and for some $v \in (L^2(\Omega \times (0, \infty), d\rho))^d$ satisfying

$$(1.11) \qquad \int_0^\infty \int_\Omega (\rho v \cdot \xi - \varepsilon^{-1}\rho\nabla\phi^\varepsilon \cdot \xi - \mu\rho\operatorname{div}\xi - p\operatorname{div}\xi)\, dx\, dt = 0$$

for any vector field $\xi \in C_c^\infty(\overline{\Omega} \times (0, \infty); \mathbb{R}^d)$ such that $\xi \cdot n = 0$ on $\partial\Omega$ and with ϕ^ε given by (1.4).

One can also define solutions on finite time interval in parallel way, with considering the time integral over $(0, T)$ and test functions vanishing outside of $[0, T)$.

Equality (1.10) is the usual weak formulation for the continuity equation $\partial_t\rho + \operatorname{div}(\rho v) = 0$ with Neumann boundary conditions and initial condition ρ_{in}. Equation (1.11) is equivalent to the equality $\rho v = \varepsilon^{-1}\rho\nabla\phi^\varepsilon - \nabla p$ in $L^2(\Omega \times (0, \infty))$. It is written in this way to make it easy to compare with Definition 1.3 below (see (1.15)).

In [15] we prove the existence and uniqueness of a weak solution in the sense of Definition 1.1, using the fact that it is a gradient flow with respect to the Wasserstein metric. Here the free energy is given by

$$\mu \int_\Omega \rho \log\rho\, dx - \frac{1}{2\varepsilon}\int_\Omega \rho\, \phi^\varepsilon\, dx \qquad \text{with the constraint } \rho \le 1,$$

where ϕ^ε given by (1.4). This energy structure of the equation will also play a key role in this paper. Because this energy does not behave well when $\varepsilon \ll 1$, we will work instead with the functional

$$\mathscr{F}_\varepsilon(\rho) = \begin{cases} \mu \int_\Omega \rho \log\rho\, dx + \frac{1}{2\sigma\varepsilon}\int_\Omega \rho\,(1 - \sigma\phi^\varepsilon)\, dx, & \text{if } 0 \le \rho(x) \le 1 \text{ a.e.;} \\ \infty & \text{otherwise .} \end{cases}$$

Since $\int_\Omega \rho \, dx$ is preserved by the equation we are only adding a constant to the energy, but this constant is important when $\varepsilon \ll 1$ (it was proved in [21] that $\mathscr{F}_\varepsilon(\rho)$ is bounded uniformly in ε when $\rho = \chi_E \in BV(\Omega; \{0,1\})$). The following result was proved in [15]:

Theorem 1.2 ([15]). *For any $\varepsilon > 0$ and any initial condition ρ_{in} satisfying*

$$0 \leq \rho_{in} \leq 1 \quad a.e. \ in \ \Omega,$$

there exists a unique $(\rho^\varepsilon, p^\varepsilon)$ weak solution of (1.3)-(1.4) in the sense of Definition 1.1. Furthermore, ρ^ε satisfies the energy inequality

$$(1.12) \qquad \mathscr{F}_\varepsilon(\rho^\varepsilon(t)) + \int_0^t \int_\Omega |v^\varepsilon|^2 \rho^\varepsilon \, dx \, dt \leq \mathscr{F}_\varepsilon(\rho_{in}) \quad \forall t > 0$$

with v^ε defined as in Definition 1.1.

The goal of this paper is to show that when $\varepsilon \ll 1$, the solution of (1.3)-(1.4) given by Theorem 1.2 converges to the solution of the following Hele-Shaw problem with surface tension:

$$(1.13) \qquad \begin{cases} \Delta q = 0 & \text{in } \Omega_s(t); \\ q = \frac{\kappa}{4\sigma^{3/2}} & \text{on } \Sigma(t) = \partial\Omega_s(t) \cap \Omega; \\ \nabla q \cdot n = 0 & \text{on } \partial\Omega \cap \Omega_s(t); \\ V = -\nabla q \cdot \nu & \text{on } \Sigma(t), \end{cases}$$

together with the contact angle condition (1.9). Recall that n and ν respectively denote the outward normal of Ω and $\Omega_s(t)$ at their boundary points.

The definition of a weak solution of (1.13)-(1.9) is parallel to the Definition 1.1.

Definition 1.3. *The pair of functions (ρ, q) is a weak solution of (1.13)-(1.9) if*

$$\rho \in L^\infty(0, T; BV(\Omega; \{0,1\})) \cap C^{1/2}(0, T; H^{-1}(\Omega)), \qquad q \in L^2(0, T; (C^s(\Omega))^*)$$

for some $s > 0$ and the followings hold:

$$(1.14) \qquad \int_\Omega \rho_{in}(x)\zeta(x, 0) \, dx + \int_0^T \int_\Omega \rho \, \partial_t \zeta + \rho v \cdot \nabla \zeta \, dx \, dt = 0$$

for any function $\zeta \in C_c^\infty(\overline\Omega \times [0, T))$ and for some $v \in (L^2(\Omega \times (0, T), d\rho))^d$ satisfying

$$(1.15) \qquad \int_0^T \int_\Omega \rho v \cdot \xi - q \operatorname{div} \xi(t) \, dx \, dt = -\frac{1}{4\sigma^{3/2}} \int_0^T \int_\Omega [\operatorname{div} \xi - \nu \otimes \nu : D\xi] \, |\nabla \rho| \, dt$$

$$+ \frac{\gamma}{4\sigma^{3/2}} \int_0^T \int_{\partial\Omega} [\operatorname{div} \xi - n \otimes n : D\xi] \, \rho \, d\mathcal{H}^{n-1}(x) \, dt$$

for any vector field $\xi \in C_c^\infty(\overline\Omega \times (0, T); \mathbb{R}^d)$ such that $\xi \cdot n = 0$ on $\partial\Omega$. [1]

This definition, similar to the one given in [13, 16, 17], warrants several comments.

(1) The condition $\rho \in L^\infty(0, T; BV(\Omega; \{0,1\}))$ implies that for a.e. $t > 0$ we have $\rho(t) = \chi_{\Omega_s(t)}$ for a set $\Omega_s(t) \subset \Omega$ with finite perimeter.

(2) In (1.15), $\nu = \frac{\nabla \rho}{|\nabla \rho|}$ stands for the L^∞ density of $\nabla \rho$ with respect to the total variation $|\nabla \rho|$ (which exists by Radon-Nikodym's differentiation theorem). Since $\rho(t) = \chi_{\Omega_s(t)} \in BV$, it is also the measure theoretic normal to the boundary $\Sigma(t) = \partial\Omega_s(t)$. In particular, the term $(\nu \otimes \nu : D\xi) \, |\nabla \rho|$ is of the form $f(x, \lambda/|\lambda|) d|\lambda|$ with f continuous and 1-homogeneous and $\lambda = \nabla \rho$. The integral in (1.15) thus makes sense (see for example [9]).

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[1] The integral $\int_\Omega q \operatorname{div} \xi \, dx$ above should be understood as the duality bracket $\langle q, \operatorname{div} \xi \rangle_{(C^s(\Omega))^*, C^s(\Omega)}$.
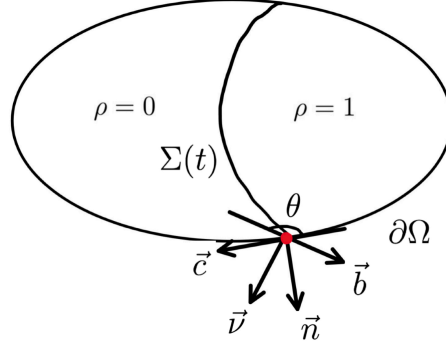
FIGURE 1.

(3) Note that q has very low regularity in this definition. Since $q \sim \kappa$ along $\Sigma(t)$, we cannot expect much more regularity on q without improving the regularity of the free boundary $\Sigma(t)$.

(4) As in Definition 1.1, (1.14) is simply the weak formulation for the continuity equation $\partial_t \rho +$ div $(\rho v) = 0$ with Neumann boundary conditions and initial data ρ_{in}. Since $\rho = \chi_{\Omega_s(t)}$, it encodes the velocity law $V = v \cdot \nu$, the incompressibility condition div $v = 0$ in $\Omega_s(t)$ and the Neumann condition $v \cdot \nu = 0$ on $\partial\Omega \cap \Omega_s(t)$.

(5) By taking test functions ξ supported in either $\{\rho = 0\}$ or $\{\rho = 1\}$, we see that Equation (1.15) implies $\nabla q = 0$ in $\{\rho(t) = 0\}$ and $v = -\nabla q$ in $\Omega_s(t) = \{\rho(t) = 1\}$. Subtracting a constant if needed, we can in particular assume that $q = 0$ in $\{\rho(t) = 0\}$. For general test functions ξ, and taking into account the right hand side of (1.15) we further get the surface tension condition $q = \frac{\kappa}{4\sigma^{3/2}}$ on $\Sigma(t)$ and the contact angle condition (1.9). This can be seen by using the classical formula (for a smooth interface Σ):

$$(1.16) \qquad \int_\Sigma \text{div}\, \xi - \nu \otimes \nu : D\xi = \int_\Sigma \kappa \xi \cdot \nu + \int_\Gamma b \cdot \xi,$$

where ν is the normal vector to Σ, κ denotes the mean curvature of Σ and b is the conormal vector along $\Gamma = \partial\Sigma$. Indeed, formally at least, the right hand side of (1.15) is (using the fact that $\xi \cdot n = 0$ on $\partial\Omega$):

$$-\frac{1}{4\sigma^{3/2}}\left[\int_\Sigma \kappa\xi \cdot \nu + \int_\Gamma \vec{b} \cdot \xi\right] + \frac{\gamma}{4\sigma^{3/2}}\left[\int_{\partial\Omega\cap\Omega_s} \kappa\xi \cdot n + \int_\Gamma \vec{c} \cdot \xi\right]$$

$$= \frac{1}{4\sigma^{3/2}}\left[-\int_\Sigma \kappa\xi \cdot \nu + \int_\Gamma \gamma\vec{c} \cdot \xi - \vec{b} \cdot \xi\right],$$

where $\vec{b}$ and $\vec{c}$ are unit conormal vectors along $\Gamma = \partial\Sigma \cap \partial\Omega$: $\vec{b}$ is tangent to Σ while $\vec{c}$ is tangent to $\partial\Omega$ (see Figure 1).

Integration by parts in (1.15) thus reveals that the jump of q across Σ must be equal to $\frac{1}{4\sigma^{3/2}}\kappa$. Since $q = 0$ in $\{\rho(t) = 0\}$, we get $q = \frac{1}{4\sigma^{3/2}}\kappa$ along $\Sigma = \partial E \cap \Omega$. Finally, the cancellation of the lower dimensional integral requires that the component of the vector $\vec{b} - \gamma\vec{c}$ that is tangential to $\partial\Omega$ must vanish. In particular, we must have $\left[\vec{b} - \gamma\vec{c}\right] \cdot \vec{c} = 0$ and so $\vec{c} \cdot \vec{b} = \gamma$, which gives the contact angle condition

$$\cos\theta = \gamma.$$

(6) A simple computation show that (1.11) is equivalent to

$$\int_0^T \int_\Omega \rho v \cdot \xi - q \operatorname{div} \xi \, dx \, dt = \int_0^T \int_\Omega \varepsilon^{-1} \left(\frac{1}{2\sigma} - \phi^\varepsilon \right) \xi \cdot \nabla \rho \, dt$$

with $q = p + \varepsilon^{-1} \rho \left(\frac{1}{2\sigma} - \phi^\varepsilon \right) + \mu \rho$. Passing to the limit in the right hand side of this equation to derive (1.15) will be the main result of the second part of the paper (see Proposition 5.2) and is at the heart of the relation between the Hele-Shaw model with active potential (1.6) and the Hele-Shaw model with surface tension (1.13). We will see in particular that the contact angle γ depends on the boundary condition for the potential ϕ^ε and is given by

$$\gamma = -\min \left(1, \frac{2\alpha}{\alpha + \sqrt{\sigma}\beta} \right).$$

1.5. **Energy.** The proof of the convergence will require some assumptions about the convergence of the energy. Before stating this assumption, we need to recall a few important facts about the singular part of the energy

$$(1.17) \qquad \mathscr{J}_\varepsilon(\rho) := \begin{cases} \dfrac{1}{2\sigma\varepsilon} \displaystyle\int_\Omega \rho \left(1 - \sigma\phi^\varepsilon \right) dx & \text{if } 0 \leq \rho(x) \leq 1 \text{ a.e. ;} \\ +\infty & \text{otherwise,} \end{cases}$$

where ϕ^ε solves (1.4). The properties of $\mathscr{J}_\varepsilon$ when $\varepsilon \ll 1$ were studied by two of the authors in [21]. The first important result is the following:

Proposition 1.4 ([21])**.** *Let Ω be a bounded open set with $C^{1,\alpha}$ boundary. Given a set $E \subset \Omega$ with finite perimeter $P(E, \Omega) < \infty$, we have*

$$(1.18) \qquad \lim_{\varepsilon \to 0} \mathscr{J}_\varepsilon(\chi_E) = \frac{1}{4\sigma^{3/2}} \left[\int_\Omega |\nabla \chi_E| + \int_{\partial\Omega} \frac{2\alpha}{\alpha + \sqrt{\sigma}\beta} \chi_E(x) \, d\mathcal{H}^{n-1}(x) \right].$$

We point out that while the result is proved only for $\sigma = 1$ in [21], but it can be easily extended to $\sigma \neq 1$ by scaling. More precisely with $\bar{\phi}^\varepsilon = \sigma\phi^\varepsilon$, $\bar{\varepsilon} = \varepsilon/\sqrt{\sigma}$ and $\bar{\beta} = \sqrt{\sigma}\beta$, equation (1.4) become

$$\begin{cases} \bar{\phi}^\varepsilon - \bar{\varepsilon}^2 \Delta \bar{\phi}^\varepsilon = \rho & \text{in } \Omega; \\ \alpha \bar{\phi}^\varepsilon + \bar{\beta}\bar{\varepsilon}\nabla\bar{\phi}^\varepsilon \cdot n = 0 & \text{on } \partial\Omega, \end{cases}$$

which is the equation studied in [21].

Above proposition identifies the limit of $\mathscr{J}_\varepsilon(\chi_E)$. However, this functional is not lower-semicontinuous when $\frac{2\alpha}{\alpha + \sqrt{\sigma}\beta} > 1$ and cannot be the Γ-limit of $\mathscr{J}_\varepsilon$. We can in fact prove:

Theorem 1.5. *Let Ω be a bounded open set with $C^{1,\alpha}$ boundary. The functional $\mathscr{J}_\varepsilon$ Γ-converges, when $\varepsilon \to 0$ to*

$$\mathscr{J}_0(\rho) := \begin{cases} \dfrac{1}{4\sigma^{3/2}} \left[\displaystyle\int_\Omega |\nabla\rho| + \int_{\partial\Omega} \min\left(1, \frac{2\alpha}{\alpha + \sqrt{\sigma}\beta} \right) \rho \, d\mathcal{H}^{n-1}(x) \right] & \text{if } \rho \in \mathrm{BV}(\Omega; \{0, 1\}); \\ \infty & \text{otherwise.} \end{cases}$$

This theorem is proved in [21] when $\mathscr{J}_\varepsilon$ is restricted to characteristic functions, so we show in Appendix C how the proof can be generalized to our more general framework. This extension requires a new formulation of the energy $\mathscr{J}_\varepsilon$, see (2.2).

In the absence of boundary term, the gradient flow associated to the energy $\mathscr{J}_0$ with respect to the Wassertein distance was proved to lead to the Hele-Shaw flow (1.13) in [2, 22]. The Γ-convergence result therefore suggests that the solution of the gradient flow associated to the energy $\mathscr{J}_\varepsilon$ (which corresponds

to equation (1.3)) converges when $\varepsilon \to 0$ to a solution of (1.13) (supplemented with the boundary condition (1.9)). This is the result that we want to make precise in the present paper.

1.6. **Main results.** We are now able to state the main result:

Theorem 1.6. *Given an initial data* $\rho_{in} = \chi_{E_{in}} \in BV(\Omega; \{0,1\})$, $\mu \geq 0$ *and a sequence* $\varepsilon_n \to 0$, *let* $(\rho^{\varepsilon_n}, p^{\varepsilon_n})$ *be the unique solution of* (1.3)-(1.4) *given by Theorem 1.2. Then for any given* $T > 0$, *along a subsequence, the density* $\rho^{\varepsilon_n}(x,t)$ *converges strongly in* $L^\infty((0,T); L^1(\Omega))$ *to*

$$\rho(x,t) \in L^\infty((0,\infty); \mathrm{BV}(\Omega; \{0,1\}))$$

and the modified pressure variable q^{ε_n} *(defined by* (5.1)*) converges to* q *weak-$*$ in* $L^2((0,T); (C^s(\Omega))^*)$ *for any* $s > 0$. *Furthermore,* ρ *satisfies the continuity equation* (1.14) *for some velocity function* $v(x,t)$ *as well as the energy dissipation property*

$$(1.19) \qquad\qquad \mathscr{J}_0(\rho(t)) + \int_0^t \int_\Omega |v|^2 \rho \, dx \, dt \leq \mathscr{J}_0(\rho_{in}).$$

Finally, if the following energy convergence assumption holds:

$$(1.20) \qquad\qquad \lim_{n\to\infty} \int_0^T \mathscr{J}_{\varepsilon_n}(\rho^{\varepsilon_n}(t)) \, dt = \int_0^T \mathscr{J}_0(\rho(t)) \, dt,$$

then the limit pair (ρ, q) *also satisfies the pressure equation* (1.15) *on* $(0,T)$. *Thus it follows that* (ρ, p) *is a weak solution of* (1.13)-(1.9) *in the sense of Definition 1.3, with initial condition* ρ_{in} *and contact angle*

$$\gamma = -\min\left(1, \frac{2\alpha}{\alpha + \sqrt{\sigma}\beta}\right).$$

The result also holds if we consider a sequence of initial data $\rho_{in}^\varepsilon = \chi_{E_{in}^\varepsilon}$ bounded in $BV(\Omega)$, converging strongly to $\rho_{in} = \chi_{E_{in}}$ in L^1 and satisfying $\lim \mathscr{J}_{\varepsilon_n}(\rho_{in}^\varepsilon) \to \mathscr{J}_0(\rho_{in})$. The existence of such a sequence, for any finite perimeter set E_{in}, is proved in [21] (Proposition 5.3) as part of the Γ convergence result.

We note that Theorem 1.6 is a conditional result, since it requires the energy convergence assumption (1.20). The analysis of [21] implies that we always have

$$\liminf_{n\to\infty} \int_0^T \mathscr{J}_{\varepsilon_n}(\rho^{\varepsilon_n}(t)) \, dt \geq \int_0^T \mathscr{J}_0(\rho(t)) \, dt,$$

so (1.20) ensures that there is no loss of boundary between phases in the limit. This assumption is rather natural and is similar to the one required for instance in [19, 17, 13].

Weaker notions of solutions using the theory of varifolds have been obtained in related frameworks without the assumption (1.20) (see for example [3]). This would be an interesting direction to explore for our problem as well.

1.7. **Outline of the paper.** We begin with deriving two alternative formulas for the energy $\mathscr{J}_\varepsilon$ in the next section, which play a crucial role in our analysis. Section 3 collects the main a priori estimates for the ε-solutions. The proof of Theorem 1.6 is then split between sections 4 and 5. In Section 4, we prove Proposition 4.1 which gives the first part of the theorem, namely the strong convergence in L^1 of the density toward a characteristic function which satisfies the continuity equation (1.14) and the energy inequality (1.19). Section 5 completes the proof of Theorem 1.6 by deriving equation (1.15) under condition (1.20). The main step is Proposition 5.2 which shows that the convergence of the energy (1.20) implies the convergence of the first variation. In the last section, we briefly recall the construction of the JKO scheme used in [15] to prove the existence of weak solutions to (1.3)-(1.4) (Theorem 1.2) and we state a convergence result similar to Theorem 1.6 for a discrete-time approximation: such a result is of independent interest for numerical applications.

2. Alternative formulas for $\mathscr{J}_\varepsilon$

A crucial tool in our analysis will be a couple of alternative formulas for the energy $\mathscr{J}_\varepsilon$. We recall that the total energy of the model, $\mathscr{F}_\varepsilon$, is given by

$$\mathscr{F}_\varepsilon(\rho) = \mu \int_\Omega \rho \log \rho \, dx + \mathscr{J}_\varepsilon(\rho),$$

where $\mathscr{J}_\varepsilon$ is defined by (1.17) and plays key role in the analysis when $\varepsilon \ll 1$. For ρ satisfying the constraint $0 \le \rho \le 1$, we have

$$\mathscr{J}_\varepsilon(\rho) = \frac{1}{2\sigma\varepsilon} \int_\Omega \rho \, (1 - \sigma\phi^\varepsilon) \, dx = \frac{1}{2\sigma\varepsilon} \int_\Omega \rho(x) - \sigma \int_\Omega \int_\Omega G_\varepsilon(x, y)\rho(y) \, dy \, dx,$$

where G_ε is the Green's function associated to equation (1.4) A similar energy functional is used in [17, 13] with G_ε is the heat kernel in $\mathbb{R}^d$. However, we will rely on some different formulations for $\mathcal{J}_\varepsilon$ which make use of the particular equation solved by the function ϕ^ε in our model: First, we write

$$\mathscr{J}_\varepsilon(\rho) = \frac{1}{2\sigma\varepsilon} \int_\Omega (\rho - \rho^2) + (\rho^2 - 2\sigma\phi^\varepsilon\rho + (\sigma\phi^\varepsilon)^2) - (\sigma\phi^\varepsilon)^2 + \sigma\phi^\varepsilon\rho \, dx$$

$$= \frac{1}{2\sigma\varepsilon} \int_\Omega \rho(1 - \rho) + (\rho - \sigma\phi^\varepsilon)^2 - \sigma\phi^\varepsilon(\sigma\phi^\varepsilon - \rho) \, dx.$$

Using equation (1.4) for ϕ^ε implies

$$\mathscr{J}_\varepsilon(\rho) = \frac{1}{2\sigma\varepsilon} \int_\Omega \rho(1 - \rho) + (\rho - \sigma\phi^\varepsilon)^2 - \sigma\phi^\varepsilon(\varepsilon^2 \Delta\phi^\varepsilon) \, dx$$

$$= \frac{1}{2\sigma\varepsilon} \int_\Omega \rho(1 - \rho) \, dx + \frac{1}{2\sigma\varepsilon} \int_\Omega (\rho - \sigma\phi^\varepsilon)^2 \, dx + \frac{\varepsilon}{2} \int_\Omega |\nabla\phi^\varepsilon|^2 \, dx + \frac{1}{2} \int_{\partial\Omega} \frac{\alpha}{\beta} |\phi^\varepsilon|^2 \, d\mathcal{H}^{n-1}(x)$$

when $\beta \ne 0$ and

$$\mathscr{J}_\varepsilon(\rho) = \frac{1}{2\sigma\varepsilon} \int_\Omega \rho(1 - \rho) \, dx + \frac{1}{2\sigma\varepsilon} \int_\Omega (\rho - \sigma\phi^\varepsilon)^2 \, dx + \frac{\varepsilon}{2} \int_\Omega |\nabla\phi^\varepsilon|^2 \, dx$$

when $\beta = 0$. Alternatively, we can write the more symmetric formula (for any α, $\beta \ge 0$ with $\alpha + \beta > 0$):

$$\mathscr{J}_\varepsilon(\rho) = \frac{1}{2\sigma\varepsilon} \int_\Omega \rho(1 - \rho) \, dx + \frac{1}{2\sigma\varepsilon} \int_\Omega (\rho - \sigma\phi^\varepsilon)^2 \, dx + \frac{\varepsilon}{2} \int_\Omega |\nabla\phi^\varepsilon|^2 \, dx$$

$$\text{(2.1)} \qquad + \frac{\varepsilon^2}{2} \int_{\partial\Omega} \frac{\beta}{\alpha + \beta} |\nabla\phi^\varepsilon \cdot n|^2 \, d\mathcal{H}^{n-1}(x) + \frac{1}{2} \int_{\partial\Omega} \frac{\alpha}{\alpha + \beta} |\phi^\varepsilon|^2 \, d\mathcal{H}^{n-1}(x).$$

This formula played a key role in the proof of Proposition 1.4 and Theorem 1.5 in [21]. Thanks to the constraint $0 \le \rho \le 1$, all the terms in this formula are non-negative (without the constraint, the first term will favor values of ρ larger than 1). Furthermore, in the regime $\varepsilon \ll 1$, the first term will be bounded only for characteristic functions. This observation will be crucial when proving that the limiting density is a characteristic function (even though ρ^ε may not be). We also note that the scaling of the following two terms is consistent with the scaling of the classical Modica-Mortola regularization of the perimeter functional.

Our analysis will also require a slight variation of this formula: we can write

$$\rho(1 - \rho) + (\rho - \sigma\phi^\varepsilon)^2 = \rho + \sigma^2(\phi^\varepsilon)^2 - 2\rho\sigma\phi^\varepsilon = (1 - \rho)(\sigma\phi^\varepsilon)^2 + \rho(1 - \sigma\phi^\varepsilon)^2$$

leading to the formula

$$\mathscr{J}_\varepsilon(\rho) = \frac{1}{2\sigma\varepsilon} \int_\Omega (1-\rho)(\sigma\phi^\varepsilon)^2 + \rho(1-\sigma\phi^\varepsilon)^2 \, dx + \frac{\varepsilon}{2} \int_\Omega |\nabla\phi^\varepsilon|^2 \, dx$$

$$(2.2) \qquad + \frac{\varepsilon^2}{2} \int_{\partial\Omega} \frac{\beta}{\alpha+\beta} |\nabla\phi^\varepsilon \cdot n|^2 \, d\mathcal{H}^{n-1}(x) + \frac{1}{2} \int_{\partial\Omega} \frac{\alpha}{\alpha+\beta} |\phi^\varepsilon|^2 \, d\mathcal{H}^{n-1}(x).$$

Here also we note that all the terms are non-negative when $0 \le \rho \le 1$. This formula will in particular be crucial in proving the strong convergence of ρ^ε and when deriving the pressure equation (1.15) (see Section 5).

3. A priori estimates

We now derive the a priori estimates that will be used to prove the convergence of ρ^ε. Here and below, we denote

$$E^\varepsilon := \rho^\varepsilon v^\varepsilon.$$

Lemma 3.1. *Let $\rho_{in} \in \mathrm{BV}(\Omega; \{0,1\})$ and $(\rho^\varepsilon, p^\varepsilon)$ be the unique solution of* (1.3)-(1.4) *given by Theorem 1.2. There exists a constant C depending only on $\int_\Omega |\nabla\rho_{in}|$ (and in particular independent of ε) such that for all $\varepsilon > 0$ we have:*
(i) $\mathscr{J}_\varepsilon(\rho^\varepsilon(t)) \le C$ *for all $t > 0$.*
(ii) $\int_0^\infty \int_\Omega |v^\varepsilon|^2 \rho^\varepsilon \, dx \, dt \le C$ *and* $\|E^\varepsilon\|_{L^2(\Omega \times (0,\infty))} \le C$.
(iii) $\|\rho^\varepsilon(t) - \rho^\varepsilon(s)\|_{H^{-1}(\Omega)} \le C\sqrt{t-s}$, *for any $0 \le s \le t$.*

Proof. We recall that

$$\mathscr{F}_\varepsilon(\rho) = \mu \int_\Omega \rho \log \rho \, dx + \mathscr{J}_\varepsilon(\rho),$$

where (since $0 \le \rho \le 1$) $-C \le \rho \log \rho \le 0$. The energy inequality (1.12) thus implies

$$\mathscr{F}_\varepsilon(\rho^\varepsilon(t)) \le \mathscr{F}_\varepsilon(\rho_{in}) \le \mathscr{J}_\varepsilon(\rho_{in}) \quad \forall t > 0.$$

Using Proposition 1.4, we see that when $\rho_{in} = \chi_{E_{in}} \in BV(\Omega; \{0,1\})$, we have $\mathscr{J}_\varepsilon(\rho_{in}) \le C$ for some constant C independent on ε. We deduce

$$(3.1) \qquad \mathscr{J}_\varepsilon(\rho^\varepsilon(t)) \le \mathscr{J}_\varepsilon(\rho_{in}) - \mu \int_\Omega \rho \log \rho \, dx \le C.$$

The energy inequality also gives

$$\int_0^\infty \int_\Omega |v^\varepsilon|^2 \rho^\varepsilon \, dx \, dt \le \mathscr{F}_\varepsilon(\rho_{in}) \le \mathscr{J}_\varepsilon(\rho_{in})$$

and since $\rho^\varepsilon \le 1$ (ii) follows immediately.

Finally, for a given test function $\psi \in H_0^1(\Omega)$, the continuity equation (1.10) implies

$$\int_\Omega \rho^\varepsilon(x,t)\psi(x) \, dx - \int_\Omega \rho^\varepsilon(x,s)\psi(x) \, dx = \int_s^t \int_\Omega \rho^\varepsilon v^\varepsilon \cdot \nabla\psi \, dx \, d\tau,$$

and so (since $\rho^\varepsilon \le 1$):

$$\left| \int_\Omega \left(\rho^\varepsilon(x,t) - \rho^\varepsilon(x,s) \right)\psi(x) \, dx \right| \le \left(\int_s^t \int_\Omega |v^\varepsilon|^2 d\rho^\varepsilon \right)^{1/2} \left(\int_s^t \int_\Omega \rho^\varepsilon |\nabla\psi|^2 \, dx \, d\tau \right)^{1/2}$$

$$\le \|\psi\|_{H^1(\Omega)} \left(\int_0^t \int_\Omega |v^\varepsilon|^2 d\rho^\varepsilon \right)^{1/2} (t-s)^{1/2},$$

and (iii) now follows from (ii). $\square$

We also need some estimates on ϕ^ε, solution of (1.4). The comparison principle applied to the Robin boundary value problem (1.4) (see e.g. Section 4 in [24]) immediately gives

$$(3.2) \qquad\qquad 0 \le \phi^\varepsilon(x) \le 1/\sigma \text{ in } \Omega$$

and multiplying (1.4) by ϕ^ε and integrating leads to the estimate (recall that $\int_\Omega \rho^\varepsilon \, dx = 1$)

$$(3.3) \qquad \sigma\|\phi^\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon^2 \|\nabla\phi^\varepsilon\|_{L^2(\Omega)}^2 \le \int_\Omega \rho^\varepsilon \phi^\varepsilon \, dx \le \frac{1}{\sigma}\int_\Omega \rho^\varepsilon \, dx = \frac{1}{\sigma}\int_\Omega \rho_{in} \, dx.$$

4. Strong Convergence of ρ^ε and continuity equation.

The main result of this section is the following proposition, which yields the first part of Theorem 1.6:

Proposition 4.1. *Let* $\rho_{in}(x) = \chi_{E_{in}} \in BV(\Omega; \{0,1\})$ *and* $\rho^\varepsilon(x,t)$ *the unique solution of* (1.3)-(1.4) *given by Theorem 1.2. Then the followings holds for any sequence* $\varepsilon_n \to 0$:
(i) *There exists a subsequence (still denoted* ε_n) *along which* $\rho^{\varepsilon_n}(t)$ *converges uniformly with respect to* t *in* $H^{-1}(\Omega)$ *to* $\rho(t)$ *and* E^{ε_n} *converges weakly in* $L^2(\Omega \times (0,\infty))$ *to* E.
(ii) *There exists* $v \in (L^2(\Omega \times (0,\infty), d\rho))^d$ *such that* $E = \rho v$ *and the continuity equation* (1.14) *holds.*
(iii) *Up to another subsequence,* $\rho^{\varepsilon_n}(t)$ *converges to* $\rho(t)$ *strongly in* $L^1(\Omega)$, *uniformly in* t. *Furthermore, for all* $t > 0$ *we have*

$$\rho(t) \in BV(\Omega; \{0,1\})$$

(that is $\rho(t)$ *is the characteristic function of a set of finite perimeter) and the energy inequality* (1.19) *holds.*

Proof. First note that Lemma 3.1 (iii) together with Arzelà-Ascoli's theorem yields the uniform convergence of $\rho^{\varepsilon_n}(t)$ in $H^{-1}(\Omega)$, and Lemma 3.1 (ii) gives the weak convergence of E^{ε_n}.

Next, we can pass to the limit in (1.10) to get

$$\int_\Omega \rho_{in}(x)\zeta(x,0)\,dx + \int_0^\infty \int_\Omega \rho\,\partial_t\zeta + E \cdot \nabla\zeta \, dx = 0$$

for any function $\zeta \in C_c^\infty([0,\infty) \times \overline{\Omega})$. Thus ρ solves the continuity equation

$$(4.1) \qquad \begin{cases} \partial_t \rho + \operatorname{div} E = 0; \\ \rho(x,0) = \rho_{in}(x). \end{cases}$$

To complete the proof of (ii) and derive (1.14), we just need to show that E can be written in the form ρv. We use an argument that can be found, for example, in [20]: For a scalar measure μ and a vectorial measure F, we define the function

$$\Theta : (\mu, F) \mapsto \begin{cases} \displaystyle\int_0^T \int_\Omega \frac{|F|^2}{\mu} & \text{if } F \ll \mu \text{ a.e. } t \in [0,T]; \\ +\infty & \text{otherwise.} \end{cases}$$

This function Θ is lower semi-continuous for the weak convergence of measure (see [1], Theorem 2.34). Together with the uniform bound $\Theta(\rho^{\varepsilon_n}, E^{\varepsilon_n}) = \int_0^T \int_\Omega \rho^{\varepsilon_n} |v^{\varepsilon_n}|^2 \le C$ (see Lemma 3.1 (ii)), it implies that E is absolutely continuous with respect to ρ and that there exists $v(t, \cdot) \in (L^2(d\rho(t)))^d$ such that $E = \rho v$. Inserting this in (4.1) yields (1.14).

The rest of the proof is devoted to (iii). The fact that we can get stronger convergence for the density is non-trivial: it is due to the fact that the energy $\mathscr{J}_\varepsilon$ controls the BV norm of ϕ^ε, which is close to ρ^ε when $\varepsilon \ll 1$. To see this, we introduce the function

$$(4.2) \qquad F(t) := \int_0^t 2\min(\tau, 1-\tau)\, d\tau = \begin{cases} t^2 & \text{for } 0 \le t \le 1/2; \\ 2t - t^2 - \frac{1}{2} & \text{for } 1/2 \le t \le 1. \end{cases}$$

We then have $F'(\sigma\phi^\varepsilon) = 2\min(\sigma\phi^\varepsilon, 1 - \sigma\phi^\varepsilon)$ and so

$$\frac{1}{\sigma^{3/2}}|\nabla F(\sigma\phi^\varepsilon)| \le 2\frac{1}{\sqrt{\sigma}}|\nabla\phi^\varepsilon|\min(\sigma\phi^\varepsilon, 1 - \sigma\phi^\varepsilon)$$

$$\le \frac{1}{\sigma\varepsilon}\min((\sigma\phi^\varepsilon)^2, (1 - \sigma\phi^\varepsilon)^2) + \varepsilon|\nabla\phi^\varepsilon|^2$$

$$\le \frac{1}{\sigma\varepsilon}\left[(1-\rho)(\sigma\phi^\varepsilon)^2 + \rho(1-\sigma\phi^\varepsilon)^2\right] + \varepsilon|\nabla\phi^\varepsilon|^2,$$

as long as $0 \le \rho \le 1$. This inequality, together with the formula (2.2) for $\mathscr{J}_\varepsilon$, implies

$$(4.3) \qquad \frac{1}{2\sigma^{3/2}}\int_\Omega |\nabla F(\sigma\phi^\varepsilon)|\, dx \le \mathscr{J}_\varepsilon(\rho^\varepsilon).$$

Note that in [21] a similar inequality was derived when ρ is a characteristic function. The computation above extends this important property of $\mathscr{J}_\varepsilon$ to the case where $0 \le \rho \le 1$ by using the formula (2.2).

Inequality (4.3) shows that the boundedness of the energy $\mathscr{J}_{\varepsilon_n}(\rho^{\varepsilon_n})$ implies some a priori estimates for the auxiliary function

$$\psi^n := 2F(\sigma\phi^{\varepsilon_n}).$$

More precisely, (4.3) and Lemma 3.1 (i) imply that

$$(4.4) \qquad \psi^n \text{ is bounded in } L^\infty((0,T); BV(\Omega)).$$

Next, we can write

$$(4.5) \qquad \psi^n = [2F(\sigma\phi^{\varepsilon_n}) - 2F(\rho^{\varepsilon_n})] + [2F(\rho^{\varepsilon_n}) - \rho^{\varepsilon_n}] + \rho^{\varepsilon_n}.$$

We are going to show that the first two terms in the right hand side go to zero (uniformly in t):

- Formula (2.1) and the energy bound (Lemma 3.1 (i)) imply

$$\|\rho^{\varepsilon_n}(t) - \sigma\phi^{\varepsilon_n}(t)\|_{L^2(\Omega)}^2 \le 2\sigma\varepsilon_n \mathscr{J}_\varepsilon(\rho^{\varepsilon_n}(t)) \le 2\sigma\varepsilon_n \mathscr{J}_\varepsilon(\rho_{in}) \le C\varepsilon_n.$$

  Since F is Lipschitz, we deduce

$$(4.6) \qquad \|2F(\rho^{\varepsilon_n}(t)) - 2F(\sigma\phi^{\varepsilon_n}(t))\|_{L^2(\Omega)}^2 \le C\|\rho^{\varepsilon_n}(t) - \sigma\phi^{\varepsilon_n}(t)\|_{L^2(\Omega)}^2 \le C\varepsilon_n \qquad \forall t > 0.$$

- When ρ is a characteristic function, we have $2F(\rho) = \rho$ and so the second term in (4.5) vanishes. When $\rho \in (0,1)$, we can use the fact that $|2F(\rho) - \rho| \le C\delta$ whenever $\rho < \delta$ or $\rho > 1 - \delta$ and use the energy to control the set where $\delta \le \rho \le 1 - \delta$. Indeed, formula (2.1) and the energy bound (Lemma 3.1 (i)) imply

$$(4.7) \qquad |\{\delta \le \rho^{\varepsilon_n} \le 1 - \delta\}| \le \frac{C}{\delta(1-\delta)}\int_\Omega \rho^{\varepsilon_n}(1 - \rho^{\varepsilon_n})\, dx \le \frac{C\sigma\varepsilon_n}{\delta(1-\delta)}\mathscr{J}_\varepsilon(\rho^{\varepsilon_n}(t)) \le C\frac{\varepsilon_n}{\delta(1-\delta)}.$$

  Choosing $\delta = \sqrt{\varepsilon_n}$ we conclude that

$$\int_\Omega |2F(\rho^{\varepsilon_n}) - \rho^{\varepsilon_n}|\, dx \le \int_{\{\sqrt{\varepsilon_n} \le \rho^{\varepsilon_n} \le 1 - \sqrt{\varepsilon_n}\}} |2F(\rho^{\varepsilon_n}) - \rho^{\varepsilon_n}|\, dx + C|\Omega|\sqrt{\varepsilon_n}$$

$$\le C|\{\sqrt{\varepsilon_n} \le \rho^{\varepsilon_n} \le 1 - \sqrt{\varepsilon_n}\}| + C|\Omega|\sqrt{\varepsilon_n}$$

$$\le C\sqrt{\varepsilon_n},$$

and so

$$(4.8) \qquad \|2F(\rho^{\varepsilon_n(t)}) - \rho^{\varepsilon_n}(t)\|_{L^2(\Omega)} \le C\varepsilon_n^{1/4} \quad \forall t > 0.$$

Since we already know that ρ^{ε_n} converges uniformly in t, with respect to the $H^{-1}(\Omega)$ norm, to ρ, we deduce from (4.5), (4.6) and (4.8) that

$$(4.9) \qquad \psi^n(t) \to \rho(t) \text{ in } H^{-1}(\Omega), \text{ uniformly in } t.$$

Using a Lions-Aubin compactness type result (see Lemma B.1), (4.4) and (4.9) yield

$$\psi^n \to \rho \qquad \text{strongly in } L^\infty((0,T); L^1(\Omega)).$$

In particular, the lower semicontinuity of the BV norm and (4.4) imply that $\rho \in L^\infty((0,T); BV(\Omega))$.

Finally, using (4.5) together with (4.6) and (4.8) we see that

$$\|\rho^{\varepsilon_n}(t) - \psi^n(t)\|_{L^2(\Omega)} \le C\varepsilon_n^{1/4},$$

so the strong convergence of ψ^n also implies that

$$\rho^{\varepsilon_n} \to \rho \qquad \text{strongly in } L^\infty((0,T); L^1(\Omega)).$$

It remains to show that ρ is a characteristic function. We note that given $t > 0$, we can extract a subsequence which converges a.e. in Ω and (4.7) implies that $|\{\delta \le \rho(t) \le 1 - \delta\}| = 0$ for all $\delta > 0$. We deduce that $\rho(x,t) \in \{0,1\}$ a.e. $x \in \Omega$ (for all $t > 0$).

Finally, we can pass to the limit in (1.12) to get the energy inequality (1.19): The lower-semicontinuity of Θ allows us to pass to the limit in the dissipation and Proposition 1.4 gives (since $\mathscr{F}_\varepsilon(\rho_{in}) = \mathscr{J}_\varepsilon(\rho_{in})$ when ρ_{in} is a characteristic function)

$$\lim_{\varepsilon \to 0} \mathscr{F}_\varepsilon(\rho_{in}) = \mathscr{J}_0(\rho_{in}).$$

The liminf property of Proposition C.1 and the strong convergence of ρ^ε then yield:

$$\liminf_{\varepsilon \to 0} \mathscr{F}_\varepsilon(\rho^\varepsilon) \ge \liminf_{\varepsilon \to 0} \mu \int_\Omega \rho^\varepsilon \log \rho^\varepsilon \, dx + \liminf_{\varepsilon \to 0} \mathscr{J}_\varepsilon(\rho^\varepsilon)$$

$$\ge \mu \int_\Omega \rho \log \rho \, dx + \mathscr{J}_0(\rho)$$

$$\ge \mathscr{J}_0(\rho).$$

This completes the proof of Proposition 4.1.

$\square$

5. Convergence of the first variation

5.1. **Convergence of the first variation and proof of Theorem 1.6.** To prove Theorem 1.6, the only remaining step now is to pass to the limit in equation (1.11) under the energy convergence assumption (1.20) to derive (1.15). We recall (1.11) here:

$$\int_0^\infty \int_\Omega E^\varepsilon \cdot \xi \, dx \, dt = \int_0^\infty \int_\Omega \varepsilon^{-1} \rho^\varepsilon \nabla \phi^\varepsilon \cdot \xi + \mu \rho^\varepsilon \mathrm{div}\, \xi + p^\varepsilon \mathrm{div}\, \xi \, dx \, dt.$$

Note that when $\varepsilon \ll 1$, neither the term $\varepsilon^{-1}\rho^\varepsilon \nabla \phi^\varepsilon$ nor the function p^ε (or its gradient ∇p^ε) are bounded. As explained in the introduction, it is the modified pressure, defined by $p^\varepsilon + \varepsilon^{-1}\left(\frac{1}{2\sigma} - \phi^\varepsilon\right)\rho^\varepsilon$ which is

expected to converge to the pressure in the Hele-Shaw model with surface tension. We can also include the term $\mu\rho^\varepsilon$ in this modified pressure and normalize it to have zero average. We thus set:

$$(5.1) \qquad q^\varepsilon = p^\varepsilon + \varepsilon^{-1}\left(\frac{1}{2\sigma} - \phi^\varepsilon\right)\rho^\varepsilon + \mu\rho^\varepsilon + m^\varepsilon$$

where $m^\varepsilon(t)$ (constant in x) is chosen so that

$$(5.2) \qquad \int_\Omega q^\varepsilon(x,t)\,dx = 0 \quad \forall t > 0.$$

After a straightforward computation, using the fact that ξ is tangential on $\partial\Omega$, we can rewrite (1.11) as

$$(5.3) \qquad \int_0^\infty \int_\Omega E^\varepsilon \cdot \xi\,dx\,dt = \int_0^\infty \int_\Omega -\rho^\varepsilon\mathrm{div}\left(\varepsilon^{-1}\left(\frac{1}{2\sigma} - \phi^\varepsilon\right)\xi\right) + q^\varepsilon\mathrm{div}\,\xi\,dx\,dt.$$

Passing to the limit in (5.3) is the objective of this section. The main challenge is with the middle term, which we can write (with a slight abuse of notation since $\nabla\rho^\varepsilon \notin L^1$):

$$(5.4) \qquad \int_0^\infty \int_\Omega \varepsilon^{-1}\left(\frac{1}{2\sigma} - \phi^\varepsilon\right)\xi\cdot\nabla\rho^\varepsilon\,dx\,dt,$$

which gives rise to the mean-curvature and the contact angle condition in the limit $\varepsilon \to 0$. This term is related to the first variation of the energy $\mathscr{J}_\varepsilon$ defined by (1.17). The key result of this section is Proposition 5.7 below, which states that the assumption on the convergence of the energy (1.20) implies the convergence of the first variation. Similar results have been proved for different energy functionals. In particular, a classical result of Reshetnyak [25] gives that if χ_{E_ε} converges to χ_E strongly in L^1, then the convergence of the perimeter

$$P(E_\varepsilon) := \int |\nabla\chi_{E_\varepsilon}| \to \int |\nabla\chi_E|,$$

implies the convergence of its first variation to

$$\int (\mathrm{div}\,\xi - \nu\otimes\nu : D\xi)|D\chi_E|.$$

A similar result was proved by Luckhaus and Modica [18] for the Ginzburg-Landau functional

$$E_\varepsilon^1(\rho) = \int_\Omega \varepsilon|\nabla\rho|^2 + \frac{1}{\varepsilon}(1-\rho^2)^2\,dx$$

and by Laux and Otto [17] with

$$E_\varepsilon^2(\rho) = \varepsilon^{-1}\int \rho(1 - G_\varepsilon\star\rho)\,dx$$

when G_ε is the Gaussian kernel of variance ε^2.

Crucially, both E_ε^1 and E_ε^2 are regularizations of the perimeter functional. In our case, we recall (see Theorem 1.5) that the energy functional $\mathscr{J}_\varepsilon$ Γ-converges to the perimeter functional as well, together with a boundary term. Furthermore, it is easy to check that (5.4) is the first variation of $\mathscr{J}_\varepsilon(\rho)$ for a perturbation defined by

$$(5.5) \qquad \begin{cases} \partial_s\rho_s + \nabla\rho_s\cdot\xi = 0 \\ \rho_s|_{s=0} = \rho. \end{cases}$$

Remark 5.1. *Note that* (5.5) *preserves the constraint* $\rho_s \leq 1$ *but not the condition* $\int \rho_s = 1$ *(unless* $\operatorname{div} \xi = 0$*). Alternatively, we could consider the first variation of* $\mathscr{J}_\varepsilon(\rho)$ *for a perturbation defined by*

$$
(5.6) \qquad \begin{cases} \partial_s \rho_s + \operatorname{div}(\rho_s \xi) = 0 \\ \rho_s|_{s=0} = \rho. \end{cases}
$$

which leads to the integral $\varepsilon^{-1} \int_\Omega \rho\, \xi \nabla(\frac{1}{2\sigma} - \phi^\varepsilon)\, dx$. *The two integrals are the same when* $\operatorname{div} \xi = 0$, *but this second integral does not, in general, converge when* $\varepsilon \to 0$. *This is due to the fact that* (5.6) *does not preserve characteristic functions and the energy* $\mathscr{J}_\varepsilon$ *blows up in that case for* $\varepsilon \ll 1$.

The key result of this section, which will allow us to complete the proof of Theorem 1.6 is the following:

Proposition 5.2. *Given a sequence of functions* $\rho^\varepsilon \in L^1(\Omega)$ *satisfying* $0 \leq \rho^\varepsilon \leq 1$ *and such that* $\rho^\varepsilon \to \rho$ *strongly in* $L^1(\Omega)$ *with* $\rho \in \mathrm{BV}(\Omega; \{0,1\})$ *and*

$$
(5.7) \qquad \lim_{\varepsilon \to 0} \mathscr{J}_\varepsilon(\rho^\varepsilon) = \mathscr{J}_0(\rho),
$$

we have, for all $\xi \in C^1(\Omega; \mathbb{R}^d)$ *satisfying* $\xi \cdot n = 0$ *on* $\partial\Omega$,

$$
(5.8) \qquad \left| \varepsilon^{-1} \int_\Omega \operatorname{div}\left(\left(\frac{1}{2\sigma} - \phi^\varepsilon \right) \xi \right) \rho^\varepsilon \right| \leq C \|D\xi\|_{L^\infty(\Omega)} \mathscr{J}_\varepsilon(\rho^\varepsilon)
$$

and

$$
\lim_{\varepsilon \to 0} \varepsilon^{-1} \int_\Omega \operatorname{div}\left(\left(\frac{1}{2\sigma} - \phi^\varepsilon \right) \xi \right) \rho^\varepsilon
$$
$$
(5.9)
$$
$$
= \frac{1}{4\sigma^{3/2}} \left[\int_\Omega [\operatorname{div}\xi - \nu \otimes \nu : D\xi]\, |\nabla\rho| + \min\left(1, \frac{2\alpha}{\alpha + \sqrt{\sigma}\beta}\right) \int_{\partial\Omega} [\operatorname{div}\xi - n \otimes n : D\xi]\, \rho\, d\mathcal{H}^{n-1}(x) \right]
$$

where $\nu = \frac{\nabla\rho}{|\nabla\rho|}$ *and* n *denotes the outward normal unit vector to the fixed boundary* $\partial\Omega$.

Note that we can replace (5.7) with the equivalent condition

$$
\lim_{\varepsilon \to 0} \mathscr{F}_\varepsilon(\rho^\varepsilon) = \mathscr{F}_0(\rho).
$$

Indeed a bound on $\mathscr{J}_\varepsilon(\rho^\varepsilon)$ or $\mathscr{F}_\varepsilon(\rho^\varepsilon)$ implies that $\rho \in \mathrm{BV}(\Omega; \{0,1\})$ (see the proof of Proposition 4.1) so that $\mathscr{F}_0(\rho) = \mathscr{J}_0(\rho)$ and $\int \rho^\varepsilon \log \rho^\varepsilon \to 0$.

Proceeding as in [17], we can check that this proposition imply the following time-dependent version:

Corollary 5.3. *Given a sequence of functions* $\rho^\varepsilon(x,t) \in L^1(\Omega \times (0,T))$ *satisfying* $0 \leq \rho^\varepsilon \leq 1$ *and such that* $\rho^\varepsilon \to \rho$ *in* $L^1(\Omega \times (0,T))$ *and*

$$
\lim_{\varepsilon \to 0} \int_0^T \mathscr{J}_\varepsilon(\rho^\varepsilon)\, dt = \int_0^T \mathscr{J}_0(\rho)\, dt,
$$

we have, for all $\xi \in C^1(\Omega \times (0,T); \mathbb{R}^d)$ *satisfying* $\xi \cdot n = 0$ *on* $\partial\Omega$,

$$
\lim_{\varepsilon \to 0} \varepsilon^{-1} \int_0^T \int_\Omega \operatorname{div}\left(\left(\frac{1}{2\sigma} - \phi^\varepsilon \right) \xi \right) \rho^\varepsilon\, dt
$$
$$
= \frac{1}{4\sigma^{3/2}} \left[\int_0^T \int_\Omega [\operatorname{div}\xi - \nu \otimes \nu : D\xi]\, |\nabla\rho|\, dt \right.
$$
$$
(5.10) \qquad \left. + \min\left(1, \frac{2\alpha}{\alpha + \sqrt{\sigma}\beta}\right) \int_0^T \int_{\partial\Omega} [\operatorname{div}\xi - n \otimes n : D\xi]\, \rho\, d\mathcal{H}^{n-1}(x)\, dt \right].
$$

With this corollary, we can now complete the proof of Theorem 1.6.

End of the proof of Theorem 1.6. We can now pass to the limit in (5.3) and thus complete the proof of Theorem 1.6. Recall that E^{ε_n} converges weakly in L^2 and the convergence of the first term in the right hand side follows from Corollary 5.3: we can use the Corollary here since Proposition 4.1 gives the strong convergence of ρ^{ε_n} in L^1 and we are assuming condition (1.20). We thus only need to explain why the pressure q^ε converges.

Note that (5.3) and (5.8) imply

$$(5.11) \qquad \left| \int_0^T \int_\Omega q^\varepsilon \operatorname{div} \xi \, dx \, dt \right| \leq \|E^\varepsilon\|_{L^2(\Omega \times (0,T))} \|\xi\|_{L^2(\Omega \times (0,T))} + C \mathscr{J}_\varepsilon(\rho^\varepsilon) \|D\xi\|_{L^1((0,T);L^\infty(\Omega))}.$$

To obtain a bound on q^ε, we proceed as in [13]: For $\varphi \in C^s(\Omega)$, let u solve

$$\begin{cases} \Delta u = \varphi - \fint_\Omega \varphi \, dx & \text{in } \Omega; \\ \nabla u \cdot n = 0 & \text{on } \partial\Omega, \end{cases}$$

and take $\xi = \nabla u$ as a test function in (5.11) to get (using the a priori estimates of Lemma 3.1 and classical Schauder estimates)

$$\left| \int_0^T \int_\Omega q^\varepsilon \Delta u \, dx \, dt \right| \leq C \|D^2 u\|_{L^2((0,T);L^\infty(\Omega))} \leq C \|u\|_{L^2((0,T);C^{2,s}(\Omega))} \leq C \|\varphi\|_{L^2((0,T);C^s(\Omega))}$$

for any $s > 0$. Using (5.2) we deduce

$$\left| \int_0^T \int_\Omega q^\varepsilon \varphi \, dx \, dt \right| \leq C \|\varphi\|_{L^2((0,T);C^s(\Omega))},$$

which implies that q^ε is uniformly bounded in $L^2((0,T); (C^s(\Omega))^*)$ and has a weak-* limit q. We can now pass to the limit in (5.3), for ξ smooth enough, and derive (1.15). $\qquad\square$

5.2. **Proof of Proposition 5.2.** As noted earlier, if we set $\bar{\phi}^\varepsilon = \sigma \phi^\varepsilon$, $\bar\varepsilon = \varepsilon/\sqrt{\sigma}$ and $\bar\beta = \sqrt{\sigma}\beta$, equation (1.4) becomes

$$\begin{cases} \bar{\phi}^\varepsilon - \bar\varepsilon^2 \Delta \bar{\phi}^\varepsilon = \rho & \text{in } \Omega; \\ \alpha \bar{\phi}^\varepsilon + \bar\beta \bar\varepsilon \nabla \bar{\phi}^\varepsilon \cdot n = 0 & \text{on } \partial\Omega. \end{cases}$$

It is therefore enough to prove the result when $\sigma = 1$ and use the fact that

$$\varepsilon^{-1} \int_\Omega \operatorname{div} \left(\left(\frac{1}{2\sigma} - \phi^\varepsilon \right) \xi \right) \rho^\varepsilon = \bar\varepsilon^{-1} \frac{1}{\sigma^{3/2}} \int_\Omega \operatorname{div} \left(\left(\frac{1}{2} - \bar{\phi}^\varepsilon \right) \xi \right) \rho^\varepsilon$$

to get the result when $\sigma > 0$.

The proof of Proposition 5.2 makes use of the following lemma. We postpone the proof of the lemma, which is elementary, to the end of this section.

Lemma 5.4. *Given $\rho(x)$ such that $0 \leq \rho \leq 1$ and ϕ^ε solution of (1.4) with $\sigma = 1$, we have the following formulas, for all $\xi \in C^1(\Omega; \mathbb{R}^d)$:*
(i) If $\alpha = 0$ or $\beta = 0$ (Neumann or Dirichlet boundary condition for ϕ^ε) then

$$\varepsilon^{-1} \int_\Omega \operatorname{div} ((1 - 2\phi^\varepsilon)\xi) \rho = \varepsilon^{-1} \int_\Omega \left[(1-\rho)(\phi^\varepsilon)^2 + \rho(1-\phi^\varepsilon)^2 \right] \operatorname{div} \xi \, dx + \varepsilon \int_\Omega |\nabla \phi^\varepsilon|^2 \operatorname{div} \xi \, dx$$

$$(5.12) \qquad\qquad\qquad - 2\varepsilon \int_\Omega \nabla \phi^\varepsilon \otimes \nabla \phi^\varepsilon : D\xi \, dx.$$

(ii) If $\beta \neq 0$, then

$$\varepsilon^{-1} \int_\Omega \operatorname{div}\left((1-2\phi^\varepsilon)\xi\right)\rho = \varepsilon^{-1} \int_\Omega \left[(1-\rho)(\phi^\varepsilon)^2 + \rho(1-\phi^\varepsilon)^2\right] \operatorname{div}\xi \, dx$$

$$+ \varepsilon \int_\Omega |\nabla\phi^\varepsilon|^2 \operatorname{div}\xi \, dx + \int_{\partial\Omega} \frac{\alpha}{\beta}|\phi^\varepsilon|^2 \operatorname{div}\xi \, d\mathcal{H}^{n-1}(x)$$

(5.13)
$$- 2\varepsilon \int_\Omega \nabla\phi^\varepsilon \otimes \nabla\phi^\varepsilon : D\xi \, dx - \int_{\partial\Omega} \frac{\alpha}{\beta}\phi^{\varepsilon\,2} n \otimes n : D\xi \, d\mathcal{H}^{n-1}(x).$$

Idea of the proof of Proposition 5.2: While the proof of this Proposition 5.2 may appear long and technical, the idea is quite simple. We recall that when $\alpha = 0$ (Neumann boundary condition), we have (see (2.2)):

$$\mathscr{I}_\varepsilon(\rho^\varepsilon) = \frac{\varepsilon^{-1}}{2} \int_\Omega (1-\rho^\varepsilon)(\phi^\varepsilon)^2 + \rho^\varepsilon(1-\phi^\varepsilon)^2 \ dx + \frac{\varepsilon}{2} \int_\Omega |\nabla\phi^\varepsilon|^2 \, dx$$

$$= \frac{1}{2} \int_\Omega u_\varepsilon^2 + v_\varepsilon^2 \, dx,$$

where we denoted $u_\varepsilon = \varepsilon^{-1/2}\left[(1-\rho^\varepsilon)(\phi^\varepsilon)^2 + \rho^\varepsilon(1-\phi^\varepsilon)^2\right]^{1/2}$ and $v_\varepsilon = \varepsilon^{1/2}|\nabla\phi^\varepsilon|$. The boundedness of the energy implies that u_ε and v_ε are bounded in L^2 and in order to pass to the limit in the first two terms in (5.12), we need to show the convergence of $\int[u_\varepsilon^2 + v_\varepsilon^2]\operatorname{div}\xi \, dx$.

We now use the argument introduced in [21] (to prove the Γ-convergence of the energy), which is itself an adaption of the classical Modica-Mortola approach: Using the fact that $u_\varepsilon \geq \min(\phi^\varepsilon, 1-\phi^\varepsilon)$ and introducing the function F such that $F'(\phi^\varepsilon) = 2\min(\phi^\varepsilon, 1-\phi^\varepsilon)$, (already used in Proposition 4.1, see (4.2)) we find:

(5.14)
$$|\nabla F(\phi^\varepsilon)| = 2\min(\phi^\varepsilon, 1-\phi^\varepsilon)|\nabla\phi^\varepsilon| \leq 2u_\varepsilon v_\varepsilon \leq u_\varepsilon^2 + v_\varepsilon^2.$$

Since the limit ρ of ρ^ε is a characteristic function and we know that $\phi^\varepsilon \to \rho$, we have $F(\phi^\varepsilon) \to F(\rho) = \frac{1}{2}\rho$ in L^1 (see the proof of Proposition 4.1), and so

$$\liminf_{\varepsilon\to 0} \int_\Omega |\nabla F(\phi^\varepsilon)| \, dx \geq \frac{1}{2} \int_\Omega |\nabla\rho|.$$

On the other hand, the assumption of convergence of the energy, (5.7), implies

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon^2 + v_\varepsilon^2 \, dx = 2\mathscr{I}_0(\rho) = \frac{1}{2} \int |\nabla\rho|.$$

Together, these inequalities imply that there is equality in (5.14) when $\varepsilon \to 0$, which means that $u_\varepsilon^2 + v_\varepsilon^2 - |\nabla F(\phi^\varepsilon)| \to 0$ in L^1 and that $\lim_{\varepsilon\to 0} \int_\Omega |\nabla F(\phi^\varepsilon)| \, dx = \frac{1}{2} \int |\nabla\rho|$. A Classical result (see Proposition A.1) now implies that

$$\lim_{\varepsilon\to 0} \int_\Omega [u_\varepsilon^2 + v_\varepsilon^2]\operatorname{div}\xi \, dx = \lim_{\varepsilon\to 0} \int_\Omega |\nabla F(\phi^\varepsilon)|\operatorname{div}\xi \, dx = \frac{1}{2} \int_\Omega |\nabla\rho|\operatorname{div}\xi.$$

To pass to the limit in the last term in (5.12), we note that the (asymptotic) equality in Young's inequality in (5.14) also implies that $u_\varepsilon - v_\varepsilon \to 0$ in L^2 and so $2\varepsilon^{-1}|\nabla\phi^\varepsilon|^2 = 2v_\varepsilon^2 \sim u_\varepsilon^2 + v_\varepsilon^2 \sim |\nabla F(\phi^\varepsilon)|$ which proves the convergence of $2\varepsilon^{-1}|\nabla\phi^\varepsilon|^2$. A simple (if somewhat technical) result (see Proposition A.3) then shows that the convergence of $\varepsilon^{-1}|\nabla\phi^\varepsilon|^2$ implies that of $\varepsilon^{-1}\nabla\phi^\varepsilon \otimes \nabla\phi^\varepsilon$.

Additional care will be needed to take care of the boundary condition when $\alpha \neq 0$, which is why we will first give the detailed proof for Neumann boundary conditions, then Dirichlet conditions (which requires extending ϕ^ε to $\mathbb{R}^d$ by 0) and finally general Robin boundary conditions (which combine the Neumann and Dirichlet case).

Proof of Proposition 5.2. We note that (5.13) together with (2.1) immediately imply (5.8).

The difficult part of the proof is, of course, to establish the limit (5.9), and we will first give the proof in the simpler case of **Neumann boundary conditions** ($\alpha = 0$). We have to pass to the limit in (5.12). As above, we use the function F such that $F'(\phi^\varepsilon) = 2 \min(\phi^\varepsilon, 1 - \phi^\varepsilon)$ (see (4.2)) and note that

$$F'(\phi^\varepsilon) \leq 2 \left[(1 - \rho)(\phi^\varepsilon)^2 + \rho(1 - \phi^\varepsilon)^2 \right]^{1/2}$$

when $0 \leq \rho \leq 1$. We thus have $|\nabla F(\phi^\varepsilon)| \leq 2|\nabla \phi^\varepsilon| \left[(1 - \rho^\varepsilon)(\phi^\varepsilon)^2 + \rho^\varepsilon(1 - \phi^\varepsilon)^2 \right]^{1/2}$ and so

$$(5.15) \qquad\qquad |\nabla F(\phi^\varepsilon)| \leq 2u_\varepsilon v_\varepsilon \leq u_\varepsilon^2 + v_\varepsilon^2 - (u_\varepsilon - v_\varepsilon)^2,$$

where

$$u_\varepsilon := \varepsilon^{-1/2} \left[(1 - \rho^\varepsilon)(\phi^\varepsilon)^2 + \rho^\varepsilon(1 - \phi^\varepsilon)^2 \right]^{1/2} \quad \text{and} \quad v_\varepsilon = \varepsilon^{1/2} |\nabla \phi^\varepsilon|.$$

Next, the strong convergence of ρ^ε and (4.6) imply that $F(\phi^\varepsilon)$ converges to $F(\rho)$ strongly in L^1. Since ρ is a characteristic function and $F(0) = 0$, $F(1) = 1/2$, we deduce $F(\phi^\varepsilon) \to F(\rho) = \frac{1}{2}\rho$ strongly in L^1 and so

$$\liminf_{\varepsilon \to 0} \int_\Omega |\nabla F(\phi^\varepsilon)| \, dx \geq \frac{1}{2} \int_\Omega |\nabla \rho|.$$

On the other hand, the convergence assumption (5.7) implies

$$\int_\Omega u_\varepsilon^2 + v_\varepsilon^2 \, dx = 2 \mathscr{J}_\varepsilon(\rho^\varepsilon) \to \frac{1}{2} \int_\Omega |\nabla \rho|.$$

Inequality (5.15) thus implies:

$$(5.16) \qquad\qquad u_\varepsilon^2 + v_\varepsilon^2 - |\nabla F(\phi^\varepsilon)| \to 0 \quad \text{in } L^1(\Omega),$$

$$(5.17) \qquad\qquad \int_\Omega |\nabla F(\phi^\varepsilon)| \, dx \to \frac{1}{2} \int_\Omega |\nabla \rho|,$$

$$(5.18) \qquad\qquad u_\varepsilon - v_\varepsilon \to 0 \quad \text{in } L^2(\Omega).$$

These facts allow us to pass to the limit in the first two terms of (5.12). Indeed, using first the definition of u^ε and v^ε, then the limit (5.16) and finally (5.17) (together with Proposition A.1), we can write:

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} \int_\Omega \left[(1 - \rho^\varepsilon)(\phi^\varepsilon)^2 + \rho^\varepsilon(1 - \phi^\varepsilon)^2 \right] \operatorname{div} \xi \, dx + \varepsilon \int_\Omega |\nabla \phi^\varepsilon|^2 \operatorname{div} \xi \, dx = \lim_{\varepsilon \to 0} \int_\Omega (u_\varepsilon^2 + v_\varepsilon^2) \operatorname{div} \xi \, dx$$

$$= \lim_{\varepsilon \to 0} \int_\Omega |\nabla F(\phi^\varepsilon)| \operatorname{div} \xi \, dx$$

$$(5.19) \qquad\qquad = \frac{1}{2} \int_\Omega \operatorname{div} \xi |\nabla \rho|.$$

Furthermore, (5.16) and (5.18) yields:

$$(5.20) \qquad\qquad 2v_\varepsilon^2 - |\nabla F(\phi^\varepsilon)| \to 0 \quad \text{in } L^1(\Omega),$$

which we use to pass to the limit in the term involving $\nabla \phi^\varepsilon \otimes \nabla \phi^\varepsilon$. Indeed, we can write

$$2\varepsilon \int_\Omega \partial_i \phi^\varepsilon \partial_j \phi^\varepsilon \partial_i \xi_j \, dx = 2\varepsilon \int_\Omega \frac{\partial_i \phi^\varepsilon}{|\nabla \phi^\varepsilon|} \frac{\partial_j \phi^\varepsilon}{|\nabla \phi^\varepsilon|} \partial_i \xi_j |\nabla \phi^\varepsilon|^2 dx = \int_\Omega \frac{\partial_i \phi^\varepsilon}{|\nabla \phi^\varepsilon|} \frac{\partial_j \phi^\varepsilon}{|\nabla \phi^\varepsilon|} \partial_i \xi_j \, 2v_\varepsilon^2 dx,$$

and since $\frac{\partial_i \phi^\varepsilon}{|\nabla \phi^\varepsilon|} \frac{\partial_j \phi^\varepsilon}{|\nabla \phi^\varepsilon|} \partial_i \xi_j$ is bounded in L^∞, (5.20) implies that

$$\lim_{\varepsilon \to 0} 2\varepsilon \int_\Omega \partial_i \phi^\varepsilon \partial_j \phi^\varepsilon \partial_i \xi_j \, dx = \lim_{\varepsilon \to 0} \int_\Omega \frac{\partial_i \phi^\varepsilon}{|\nabla \phi^\varepsilon|} \frac{\partial_j \phi^\varepsilon}{|\nabla \phi^\varepsilon|} \partial_i \xi_j |\nabla F(\phi^\varepsilon)| dx.$$

Using the fact that $F'(\phi^\varepsilon) \geq 0$, we can also write

$$(5.21) \qquad \lim_{\varepsilon \to 0} 2\varepsilon \int_\Omega \partial_i \phi^\varepsilon \partial_j \phi^\varepsilon \partial_i \xi_j \, dx = \lim_{\varepsilon \to 0} \int_\Omega \frac{\partial_i F(\phi^\varepsilon)}{|\nabla F(\phi^\varepsilon)|} \frac{\partial_j F(\phi^\varepsilon)}{|\nabla F(\phi^\varepsilon)|} \partial_i \xi_j \, |\nabla F(\phi^\varepsilon)| dx$$

and using (5.17) and Proposition A.3 we deduce

$$(5.22) \qquad \lim_{\varepsilon \to 0} 2\varepsilon \int_\Omega \partial_i \phi^\varepsilon \partial_j \phi^\varepsilon \partial_i \xi_j \, dx = \frac{1}{2} \int_\Omega \frac{\partial_i \rho}{|\nabla \rho|} \frac{\partial_j \rho}{|\nabla \rho|} \partial_i \xi_j \, |\nabla \rho|.$$

Lastly (5.12) together with (5.19) and (5.22) yields

$$\lim_{\varepsilon \to 0} -\varepsilon^{-1} \int_\Omega (1 - 2\phi^\varepsilon) \xi \cdot \nabla \rho = \frac{1}{2} \int_\Omega \operatorname{div} \xi |\nabla \rho| - \frac{1}{2} \int_\Omega \frac{\partial_i \rho}{|\nabla \rho|} \frac{\partial_j \rho}{|\nabla \rho|} \partial_i \xi_j \, |\nabla \rho|,$$

and we obtain (5.9) in the case of Neumann boundary condition $\alpha = 0$ (and when $\sigma = 1$)

For the case of **Dirichlet conditions** $(\beta = 0)$, we proceed similarly, but we first extend ϕ^ε to $\mathbb{R}^d$ by setting it equal to 0 in $\mathbb{R}^d \setminus \Omega$. Denoting $\bar{\phi}^\varepsilon$ this extension, we find (using inequality (5.15) and the Dirichlet boundary condition for ϕ^ε):

$$\int_{\mathbb{R}^d} |\nabla F(\bar{\phi}^\varepsilon)| \, dx = \int_\Omega |\nabla F(\phi^\varepsilon)| \, dx \leq \int_\Omega u_\varepsilon^2 + v_\varepsilon^2 \, dx = 2 \mathscr{J}_\varepsilon(\rho^\varepsilon).$$

The lower semicontinuity of the BV norm and assumption (5.7) give

$$\liminf_{\varepsilon \to 0} \int_{\mathbb{R}^d} |\nabla(F(\bar{\phi}^\varepsilon)| \, dx \geq \int_{\mathbb{R}^d} |\nabla F(\bar{\rho})| = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla \bar{\rho}| = \lim_{\varepsilon \to 0} 2 \mathscr{J}_\varepsilon(\rho^\varepsilon),$$

where $\bar{\rho}$ denotes the extension of ρ to $\mathbb{R}^d$ by 0. Indeed, when $\beta = 0$ (and $\sigma = 1$), we have

$$\mathscr{J}_0(\rho) = \frac{1}{4} \left[\int_\Omega |\nabla \rho| + \int_{\partial \Omega} \rho(x) d\mathcal{H}^{n-1}(x) \right] = \frac{1}{4} \int_{\mathbb{R}^d} |\nabla \bar{\rho}|.$$

We can thus proceed as before to show that

$$u_\varepsilon^2 + v_\varepsilon^2 - |\nabla F(\phi^\varepsilon)| \to 0 \quad \text{in } L^1(\Omega),$$

$$\int_{\mathbb{R}^d} |\nabla F(\bar{\phi}^\varepsilon)| \, dx \to \frac{1}{2} \int_{\mathbb{R}^d} |\nabla \bar{\rho}|,$$

$$u_\varepsilon - v_\varepsilon \to 0 \quad \text{in } L^2(\Omega).$$

These are the same convergences as (5.16)-(5.18), except for (5.17) which involves the extension $\bar{\phi}^\varepsilon$. We can now write:

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} \int_\Omega \left[(1 - \rho^\varepsilon)(\phi^\varepsilon)^2 + \rho^\varepsilon (1 - \phi^\varepsilon)^2 \right] \operatorname{div} \xi \, dx + \varepsilon \int_\Omega |\nabla \phi^\varepsilon|^2 \operatorname{div} \xi \, dx$$

$$= \lim_{\varepsilon \to 0} \int_\Omega \operatorname{div} \xi (u_\varepsilon^2 + v_\varepsilon^2) \, dx$$

$$= \lim_{\varepsilon \to 0} \int_\Omega \operatorname{div} \xi |\nabla F(\phi^\varepsilon)| \, dx$$

$$= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \operatorname{div} \xi |\nabla F(\bar{\phi}^\varepsilon)| \, dx$$

$$= \frac{1}{2} \int_{\mathbb{R}^d} \operatorname{div} \xi |\nabla \bar{\rho}| = \frac{1}{2} \left(\int_\Omega \operatorname{div} \xi |\nabla \rho| + \int_{\partial \Omega} \rho \operatorname{div} \xi \, d\mathcal{H}^{n-1}(x) \right).$$

Similarly, we can show (as before)

$$2\varepsilon \int_\Omega \nabla\phi^\varepsilon \otimes \nabla\phi^\varepsilon : D\xi \, dx \longrightarrow \frac{1}{2} \int_{\mathbb{R}^d} \left(\frac{\nabla\bar\rho}{|\nabla\bar\rho|} \otimes \frac{\nabla\bar\rho}{|\nabla\bar\rho|} : D\xi \right) |\nabla\bar\rho|,$$

where (using (A.2))

$$\int_{\mathbb{R}^d} \left(\frac{\nabla\bar\rho}{|\nabla\bar\rho|} \otimes \frac{\nabla\bar\rho}{|\nabla\bar\rho|} : D\xi \right) |\nabla\bar\rho| = \int_\Omega (\nu \otimes \nu : D\xi) |\nabla\rho| + \int_{\partial\Omega} (n \otimes n : D\xi) \rho \, d\mathcal{H}^{n-1}(x).$$

This gives the result when $\beta = 0$ (Dirichlet boundary conditions).

For the case of **Robin conditions**, we need to combine the two cases above. Let $\varphi : \mathbb{R}^d \to [0,1]$ be a continuous function such that

$$\tag{5.23} \int_{\partial\Omega} |\rho - \varphi| \, d\mathcal{H}^{n-1} \leq \delta.$$

Introducing the function $G(t) := \frac{\alpha}{\beta} t^2 - F(t)$, we then write:

$$\int_{\mathbb{R}^d} |\nabla F(\overline{\phi^\varepsilon})| \varphi \, dx + \int_{\partial\Omega} G(\phi^\varepsilon) \varphi d\mathcal{H}^{n-1} = \int_\Omega |\nabla F(\phi^\varepsilon)| \varphi \, dx + \int_{\partial\Omega} F(\phi^\varepsilon) \varphi d\mathcal{H}^{n-1} + \int_{\partial\Omega} G(\phi^\varepsilon) \varphi d\mathcal{H}^{n-1}$$

$$= \int_\Omega |\nabla F(\phi^\varepsilon)| \varphi \, dx + \int_{\partial\Omega} \frac{\alpha}{\beta} |\phi^\varepsilon|^2 \varphi d\mathcal{H}^{n-1},$$

which leads to

$$\int_\Omega |\nabla F(\phi^\varepsilon)|(1-\varphi) \, dx + \int_{\mathbb{R}^d} |\nabla F(\overline{\phi^\varepsilon})| \varphi \, dx + \int_{\partial\Omega} G(\phi^\varepsilon) \varphi d\mathcal{H}^{n-1} + \int_{\partial\Omega} \frac{\alpha}{\beta} |\phi^\varepsilon|^2 (1-\varphi) d\mathcal{H}^{n-1}$$

$$\tag{5.24} = \int_\Omega |\nabla F(\phi^\varepsilon)| \, dx + \int_{\partial\Omega} \frac{\alpha}{\beta} |\phi^\varepsilon|^2 d\mathcal{H}^{n-1}.$$

The right hand side satisfies (with the same notations as above):

$$\int_\Omega |\nabla F(\phi^\varepsilon)| \, dx + \int_{\partial\Omega} \frac{\alpha}{\beta} |\phi^\varepsilon|^2 d\mathcal{H}^{n-1} \leq \int_\Omega u_\varepsilon^2 + v_\varepsilon^2 - (u_\varepsilon - v_\varepsilon)^2 \, dx + \int_{\partial\Omega} \frac{\alpha}{\beta} |\phi^\varepsilon|^2 d\mathcal{H}^{n-1}$$

$$\tag{5.25} = 2 \mathscr{J}_\varepsilon(\rho^\varepsilon) - \int_\Omega (u_\varepsilon - v_\varepsilon)^2 \, dx,$$

so in order to proceed as before, we need to show that the $\liminf$ of the left hand side is greater than $2\mathscr{J}_0(\rho)$. For this, we notice that the function

$$G(t) = \frac{\alpha}{\beta} t^2 - F(t) = \begin{cases} (\frac{\alpha}{\beta} - 1)t^2 & \text{for } 0 \leq t \leq 1/2; \\ (\frac{\alpha}{\beta} + 1)t^2 - 2t + \frac{1}{2} & \text{for } 1/2 \leq t \leq 1, \end{cases}$$

satisfies $G(t) \geq \min\{0, \frac{\alpha}{\alpha+\beta} - \frac{1}{2}\} = \min\{0, \frac{\alpha-\beta}{2(\alpha+\beta)}\}$ for all $t \in [0,1]$. (This can be checked by a direct computation. We can also note that $F(t) \leq t^2$ (see (4.2)) and so $G(t) \geq 0$ for all t whenever $\alpha \geq \beta$. When $\beta > \alpha$, $t \mapsto G(t)$ is decreasing on $[0, 1/2]$ so the minimum value of G is attained at $t = \frac{\beta}{\alpha+\beta} \in (1/2, 1)$. We can then compute that $G(\frac{\beta}{\alpha+\beta}) = \frac{\alpha-\beta}{2(\alpha+\beta)} < 0$).

Hence we can write

$$\liminf_{\varepsilon \to 0} \int_\Omega |\nabla F(\phi^\varepsilon)|(1-\varphi)\,dx + \int_{\mathbb{R}^d} |\nabla F(\overline{\phi^\varepsilon})|\varphi\,dx + \int_{\partial\Omega} G(\phi^\varepsilon)\varphi d\mathcal{H}^{n-1} + \int_{\partial\Omega} \frac{\alpha}{\beta}|\phi^\varepsilon|^2(1-\varphi)d\mathcal{H}^{n-1}$$

$$\geq \int_\Omega |\nabla F(\rho)|(1-\varphi)\,dx + \int_{\mathbb{R}^d} |\nabla F(\bar\rho)|\varphi\,dx + \int_{\partial\Omega} \min\left\{0, \frac{\alpha}{\alpha+\beta} - \frac{1}{2}\right\}\varphi d\mathcal{H}^{n-1} + 0$$

$$\geq \int_\Omega \frac{1}{2}|\nabla\rho|\,dx + \int_{\partial\Omega} \frac{1}{2}\rho\varphi\,d\mathcal{H}^{n-1} + \int_{\partial\Omega} \min\left\{0, \frac{\alpha}{\alpha+\beta} - \frac{1}{2}\right\}\varphi d\mathcal{H}^{n-1}$$

$$\geq \int_\Omega \frac{1}{2}|\nabla\rho|\,dx + \int_{\partial\Omega} \min\left\{\frac{1}{2}, \frac{\alpha}{\alpha+\beta}\right\}\rho d\mathcal{H}^{n-1} - C\delta$$

(5.26)

$$= 2\mathscr{J}_0(\rho) - C\delta,$$

where we used the fact that φ is an approximation of ρ on $\partial\Omega$, see (5.23). Going back to (5.24), we see that we just showed that

$$\liminf_{\varepsilon \to 0} \int_\Omega |\nabla F(\phi^\varepsilon)|\,dx + \int_{\partial\Omega} \frac{\alpha}{\beta}|\phi^\varepsilon|^2 d\mathcal{H}^{n-1} \geq 2\mathscr{J}_0(\rho) - C\delta,$$

and since this holds for any $\delta > 0$, we get

$$\liminf_{\varepsilon \to 0} \int_\Omega |\nabla F(\phi^\varepsilon)|\,dx + \int_{\partial\Omega} \frac{\alpha}{\beta}|\phi^\varepsilon|^2 d\mathcal{H}^{n-1} \geq 2\mathscr{J}_0(\rho).$$

We can now conclude as in the previous cases: Using (5.25) and the assumption that $\lim_{\varepsilon \to 0} \mathscr{J}_\varepsilon(\rho^\varepsilon) = \mathscr{J}_0(\rho)$ to conclude that

(5.27) $$u_\varepsilon^2 + v_\varepsilon^2 - |\nabla F(\phi^\varepsilon)| \to 0 \quad \text{in } L^1(\Omega),$$

(5.28) $$u_\varepsilon - v_\varepsilon \to 0 \quad \text{in } L^2(\Omega).$$

Furthermore, using (5.26), we also get

(5.29)
$$\begin{cases}
\limsup_{\varepsilon \to 0} \left| \int_\Omega |\nabla F(\phi^\varepsilon)|(1-\varphi)\,dx - \frac{1}{2}\int_\Omega |\nabla\rho|(1-\varphi)\right| \leq C\delta, \\[2mm]
\limsup_{\varepsilon \to 0} \left| \int_{\mathbb{R}^d} |\nabla F(\overline{\phi^\varepsilon})|\varphi\,dx - \frac{1}{2}\int_{\mathbb{R}^d} |\nabla\bar\rho|\varphi\right| \leq C\delta, \\[2mm]
\limsup_{\varepsilon \to 0} \left| \int_{\partial\Omega} G(\phi^\varepsilon)\varphi d\mathcal{H}^{n-1} - \int_{\partial\Omega} \min\left\{0, \frac{\alpha}{\alpha+\beta} - \frac{1}{2}\right\}\varphi d\mathcal{H}^{n-1}\right| \leq C\delta, \\[2mm]
\limsup_{\varepsilon \to 0} \int_{\partial\Omega} \frac{\alpha}{\beta}|\phi^\varepsilon|^2(1-\varphi)d\mathcal{H}^{n-1} \leq C\delta.
\end{cases}$$

We then write (using (5.27))

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} \int_\Omega [(1-\rho^\varepsilon)(\phi^\varepsilon)^2 + \rho^\varepsilon(1-\phi^\varepsilon)^2]\text{div}\,\xi\,dx + \varepsilon \int_\Omega |\nabla\phi^\varepsilon|^2 \text{div}\,\xi\,dx + \int_{\partial\Omega} \frac{\alpha}{\beta}|\phi^\varepsilon|^2\text{div}\,\xi\,d\mathcal{H}^{n-1}(x)$$

$$= \lim_{\varepsilon \to 0} \int_\Omega (u_\varepsilon^2 + v_\varepsilon^2)\text{div}\,\xi\,dx + \int_{\partial\Omega} \frac{\alpha}{\beta}|\phi^\varepsilon|^2\text{div}\,\xi\,d\mathcal{H}^{n-1}(x)$$

(5.30)

$$= \lim_{\varepsilon \to 0} \int_\Omega |\nabla F(\phi^\varepsilon)|\text{div}\,\xi\,dx + \int_{\partial\Omega} \frac{\alpha}{\beta}|\phi^\varepsilon|^2\text{div}\,\xi\,d\mathcal{H}^{n-1}(x).$$

Using the same function φ as above, we decompose the integral in (5.30) as follows:

$$
\begin{aligned}
I_\varepsilon := & \int_\Omega |\nabla F(\phi^\varepsilon)| \mathrm{div}\,\xi\, dx + \int_{\partial\Omega} \frac{\alpha}{\beta}|\phi^\varepsilon|^2 \mathrm{div}\,\xi\, d\mathcal{H}^{n-1}(x) \\
= & \int_\Omega |\nabla F(\phi^\varepsilon)|\varphi\mathrm{div}\,\xi\, dx + \int_{\partial\Omega} \frac{\alpha}{\beta}|\phi^\varepsilon|^2\varphi\mathrm{div}\,\xi\, d\mathcal{H}^{n-1}(x) \\
& + \int_\Omega |\nabla F(\phi^\varepsilon)|(1-\varphi)\mathrm{div}\,\xi\, dx + \int_{\partial\Omega} \frac{\alpha}{\beta}|\phi^\varepsilon|^2(1-\varphi)\mathrm{div}\,\xi\, d\mathcal{H}^{n-1}(x) \\
= & \int_{\mathbb{R}^d} |\nabla F(\overline{\phi^\varepsilon})|\varphi\mathrm{div}\,\xi\, dx + \int_{\partial\Omega} G(\phi^\varepsilon)\varphi\mathrm{div}\,\xi\, d\mathcal{H}^{n-1}(x) \\
& + \int_\Omega |\nabla F(\phi^\varepsilon)|(1-\varphi)\mathrm{div}\,\xi\, dx + \int_{\partial\Omega} \frac{\alpha}{\beta}|\phi^\varepsilon|^2(1-\varphi)\mathrm{div}\,\xi\, d\mathcal{H}^{n-1}(x).
\end{aligned}
$$
(5.31)

Hence (note that the inequalities (5.29) can be localized to include the $\mathrm{div}\,\xi$, see Remark A.2)

$$
\limsup_{\varepsilon\to 0}\left|\int_\Omega I_\varepsilon - \frac{1}{2}\int_\Omega |\nabla\rho|(1-\varphi)\mathrm{div}\,\xi - \frac{1}{2}\int_{\mathbb{R}^d}|\nabla\bar\rho|\varphi\mathrm{div}\,\xi - \int_{\partial\Omega}\min\left\{0, \frac{\alpha}{\alpha+\beta}-\frac{1}{2}\right\}\varphi\mathrm{div}\,\xi d\mathcal{H}^{n-1}\right| \le C\delta,
$$

which implies (using our particular choice of φ, see (5.23)):

$$
\limsup_{\varepsilon\to 0}\left|I_\varepsilon - \frac{1}{2}\int_\Omega |\nabla\rho|\mathrm{div}\,\xi - \int_{\partial\Omega}\min\left\{\frac{1}{2}, \frac{\alpha}{\alpha+\beta}\right\}\rho\,\mathrm{div}\,\xi d\mathcal{H}^{n-1}\right| \le C\delta.
$$

Since the left hand side is now independent of φ, we can take $\delta \to 0$ and use (5.30) to get

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\varepsilon^{-1}\int_\Omega [(1-\rho^\varepsilon)(\phi^\varepsilon)^2 + \rho^\varepsilon(1-\phi^\varepsilon)^2]\mathrm{div}\,\xi\, dx + \varepsilon\int_\Omega |\nabla\phi^\varepsilon|^2\mathrm{div}\,\xi\, dx + \int_{\partial\Omega}\frac{\alpha}{\beta}|\phi^\varepsilon|^2\mathrm{div}\,\xi\, d\mathcal{H}^{n-1}(x) \\
&= \frac{1}{2}\int_\Omega |\nabla\rho|\mathrm{div}\,\xi + \int_{\partial\Omega}\min\left\{\frac{1}{2}, \frac{\alpha}{\alpha+\beta}\right\}\rho\,\mathrm{div}\,\xi d\mathcal{H}^{n-1}.
\end{aligned}
$$

It then only remains to show that

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} 2\varepsilon\int_\Omega \nabla\phi^\varepsilon \otimes \nabla\phi^\varepsilon : D\xi\, dx + \int_{\partial\Omega}\frac{\alpha}{\beta}|\phi^\varepsilon|^2 n\otimes n : D\xi\, d\mathcal{H}^{n-1}(x) \\
&= \frac{1}{2}\int_\Omega \nu\otimes\nu : D\xi|\nabla\rho| + \min\left(\frac{1}{2}, \frac{\alpha}{\alpha+\beta}\right)\int_{\partial\Omega} n\otimes n : D\xi\,\rho\, d\mathcal{H}^{n-1}(x).
\end{aligned}
$$

This can be proven by combining the arguments above with how that term was handled in the case of Dirichlet boundary conditions. Indeed, using (5.27)-(5.28) and (A.2), we can write (see the derivation of (5.21) above)

$$
\begin{aligned}
2\varepsilon\int_\Omega \nabla\phi^\varepsilon \otimes \nabla\phi^\varepsilon : D\xi\, dx \sim & \int_\Omega \frac{\nabla F(\phi^\varepsilon)}{|\nabla F(\phi^\varepsilon)|} \otimes \frac{\nabla F(\phi^\varepsilon)}{|\nabla F(\phi^\varepsilon)|} : D\xi\,|\nabla F(\phi^\varepsilon)| dx \\
= & \int_\Omega \frac{\nabla F(\phi^\varepsilon)}{|\nabla F(\phi^\varepsilon)|} \otimes \frac{\nabla F(\phi^\varepsilon)}{|\nabla F(\phi^\varepsilon)|} : D\xi\,|\nabla F(\phi^\varepsilon)|\,(1-\varphi)\, dx \\
& + \int_{\mathbb{R}^d} \frac{\nabla F(\overline{\phi^\varepsilon})}{|\nabla F(\overline{\phi^\varepsilon})|} \otimes \frac{\nabla F(\overline{\phi^\varepsilon})}{|\nabla F(\overline{\phi^\varepsilon})|} : D\xi\,|\nabla F(\overline{\phi^\varepsilon})|\,\varphi\, dx \\
& - \int_{\partial\Omega} F(\phi^\varepsilon) n\otimes n : D\xi\,\varphi\, d\mathcal{H}^{n-1}(x),
\end{aligned}
$$

and so

$$2\varepsilon \int_\Omega \nabla\phi^\varepsilon \otimes \nabla\phi^\varepsilon : D\xi \, dx + \int_{\partial\Omega} \frac{\alpha}{\beta} |\phi^\varepsilon|^2 n \otimes n : D\xi \, d\mathcal{H}^{n-1}(x)$$

$$\sim \int_\Omega \frac{\nabla F(\phi^\varepsilon)}{|\nabla F(\phi^\varepsilon)|} \otimes \frac{\nabla F(\phi^\varepsilon)}{|\nabla F(\phi^\varepsilon)|} : D\xi \, |\nabla F(\phi^\varepsilon)| \, (1-\varphi) \, dx + \int_{\mathbb{R}^d} \frac{\nabla F(\overline{\phi^\varepsilon})}{|\nabla F(\overline{\phi^\varepsilon})|} \otimes \frac{\nabla F(\overline{\phi^\varepsilon})}{|\nabla F(\overline{\phi^\varepsilon})|} : D\xi \, |\nabla F(\overline{\phi^\varepsilon})| \, \varphi \, dx$$

$$+ \int_{\partial\Omega} G(\phi^\varepsilon) n \otimes n : D\xi \, \varphi \, d\mathcal{H}^{n-1}(x) + \int_{\partial\Omega} \frac{\alpha}{\beta} |\phi^\varepsilon|^2 n \otimes n : D\xi \, (1-\varphi) \, d\mathcal{H}^{n-1}(x).$$

This term has the same structure as (5.31) and we can thus proceed as above, using (5.29) and Corollary A.4 to show that

$$\lim_{\varepsilon\to 0} 2\varepsilon \int_\Omega \nabla\phi^\varepsilon \otimes \nabla\phi^\varepsilon : D\xi \, dx + \int_{\partial\Omega} \frac{\alpha}{\beta} |\phi^\varepsilon|^2 n \otimes n : D\xi \, d\mathcal{H}^{n-1}(x)$$

$$= \frac{1}{2} \int_\Omega \frac{\nabla\rho}{|\nabla\rho|} \otimes \frac{\nabla\rho}{|\nabla\rho|} : D\xi \, |\nabla\rho| \, (1-\varphi) \, dx + \frac{1}{2} \int_{\mathbb{R}^d} \frac{\nabla\bar\rho}{|\nabla\bar\rho|} \otimes \frac{\nabla\bar\rho}{|\nabla\bar\rho|} : D\xi \, |\nabla\bar\rho| \, \varphi \, dx$$

$$+ \int_{\partial\Omega} \min\left\{0, \frac{\alpha}{\alpha+\beta} - \frac{1}{2}\right\} n \otimes n : D\xi \, \varphi \, d\mathcal{H}^{n-1}(x) + \mathcal{O}(\delta)$$

$$= \frac{1}{2} \int_\Omega \frac{\nabla\rho}{|\nabla\rho|} \otimes \frac{\nabla\rho}{|\nabla\rho|} : D\xi \, |\nabla\rho| \, dx$$

$$+ \int_{\partial\Omega} \frac{1}{2} n \otimes n : D\xi \, \rho \, \varphi \, d\mathcal{H}^{n-1}(x) + \int_{\partial\Omega} \min\left\{0, \frac{\alpha}{\alpha+\beta} - \frac{1}{2}\right\} n \otimes n : D\xi \, \varphi \, d\mathcal{H}^{n-1}(x) + \mathcal{O}(\delta),$$

and we conclude thanks to (5.23) by letting $\delta \to 0$.

$\square$

Proof of Lemma 5.4. We write (using (1.4)):

$$\varepsilon^{-1} \int_\Omega \mathrm{div}\,((1-2\phi^\varepsilon)\xi)\,\rho = \varepsilon^{-1} \int_\Omega \rho(1-2\phi^\varepsilon)\mathrm{div}\,\xi \, dx - 2\varepsilon^{-1} \int_\Omega \rho\nabla\phi^\varepsilon \cdot \xi \, dx$$

$$= \varepsilon^{-1} \int_\Omega \rho(1-2\phi^\varepsilon)\mathrm{div}\,\xi \, dx - 2\varepsilon^{-1} \int_\Omega \phi^\varepsilon \nabla\phi^\varepsilon \cdot \xi \, dx + 2\varepsilon \int_\Omega \Delta\phi^\varepsilon \nabla\phi^\varepsilon \cdot \xi \, dx$$

$$= \varepsilon^{-1} \int_\Omega \rho(1-2\phi^\varepsilon)\mathrm{div}\,\xi \, dx + \varepsilon^{-1} \int_\Omega \phi^{\varepsilon 2}\mathrm{div}\,\xi \, dx + 2\varepsilon \int_\Omega \Delta\phi^\varepsilon \nabla\phi^\varepsilon \cdot \xi \, dx$$

$$= \varepsilon^{-1} \int_\Omega [\rho - 2\rho\phi^\varepsilon + (\phi^\varepsilon)^2]\mathrm{div}\,\xi \, dx + 2\varepsilon \int_\Omega \Delta\phi^\varepsilon \nabla\phi^\varepsilon \cdot \xi \, dx.$$

For the first term, we write $\rho - 2\rho\phi^\varepsilon + (\phi^\varepsilon)^2 = (1-\rho)\phi^{\varepsilon 2} + \rho(1-\phi^\varepsilon)^2$. For the second term, we note that

$$\int_\Omega \Delta\phi^\varepsilon \nabla\phi^\varepsilon \cdot \xi \, dx = -\int_\Omega \partial_i\phi^\varepsilon \partial_{ij}\phi^\varepsilon \xi_j \, dx - \int_\Omega \partial_i\phi^\varepsilon \partial_j\phi^\varepsilon \partial_i\xi_j \, dx + \int_{\partial\Omega} \partial_i\phi^\varepsilon \partial_j\phi^\varepsilon \xi_j \nu_i \, d\mathcal{H}^{n-1}(x)$$

$$= -\int_\Omega \nabla\left(\frac{|\nabla\phi^\varepsilon|^2}{2}\right) \cdot \xi \, dx - \int_\Omega \nabla\phi^\varepsilon \otimes \nabla\phi^\varepsilon : D\xi \, dx + \int_{\partial\Omega} \nabla\phi^\varepsilon \cdot n\nabla\phi^\varepsilon \cdot \xi \, d\mathcal{H}^{n-1}(x)$$

$$= \frac{1}{2} \int_\Omega |\nabla\phi^\varepsilon|^2\mathrm{div}\,\xi \, dx - \int_\Omega \nabla\phi^\varepsilon \otimes \nabla\phi^\varepsilon : D\xi \, dx + \int_{\partial\Omega} \nabla\phi^\varepsilon \cdot n\nabla\phi^\varepsilon \cdot \xi \, d\mathcal{H}^{n-1}(x).$$

We deduce:

$$\varepsilon^{-1} \int_\Omega \operatorname{div}\left((1-2\phi^\varepsilon)\xi\right)\rho = \varepsilon^{-1} \int_\Omega [(1-\rho)\phi^{\varepsilon 2} + \rho(1-\phi^\varepsilon)^2]\operatorname{div}\xi\, dx + \varepsilon \int_\Omega |\nabla\phi^\varepsilon|^2 \operatorname{div}\xi\, dx$$
$$- 2\varepsilon \int_\Omega \nabla\phi^\varepsilon \otimes \nabla\phi^\varepsilon : D\xi\, dx + 2\varepsilon \int_{\partial\Omega} \nabla\phi^\varepsilon \cdot n\nabla\phi^\varepsilon \cdot \xi\, d\mathcal{H}^{n-1}(x).$$

When $\alpha = 0$, we have $\nabla\phi^\varepsilon \cdot n = 0$ on $\partial\Omega$ so the last term vanishes. If $\beta = 0$, then $\phi^\varepsilon = 0$ on $\partial\Omega$ and since ξ is tangential to $\partial\Omega$, we have $\nabla\phi^\varepsilon \cdot \xi = 0$ on $\partial\Omega$ and the last term vanish also. In both cases, we get formula (5.12).

In the case of general Robin boundary conditions (with in particular $\beta \neq 0$), we can write the last term as:

$$2\varepsilon \int_{\partial\Omega} \nabla\phi^\varepsilon \cdot n\nabla\phi^\varepsilon \cdot \xi\, d\mathcal{H}^{n-1}(x) = -2\frac{\alpha}{\beta} \int_{\partial\Omega} \phi^\varepsilon \nabla\phi^\varepsilon \cdot \xi\, d\mathcal{H}^{n-1}(x)$$
$$= -\frac{\alpha}{\beta} \int_{\partial\Omega} \nabla|\phi^\varepsilon|^2 \cdot \xi\, d\mathcal{H}^{n-1}(x).$$

Formula (1.16) applied to $\Sigma = \partial\Omega$ (since $\xi \cdot n = 0$) gives

$$\int_{\partial\Omega} \operatorname{div}(\phi^{\varepsilon 2}\xi)\, d\mathcal{H}^{n-1}(x) = \int_{\partial\Omega} n \otimes n : D((\phi^\varepsilon)^2\xi)\, d\mathcal{H}^{n-1}(x)$$
$$= \int_{\partial\Omega} (\phi^\varepsilon)^2 n \otimes n : D\xi\, d\mathcal{H}^{n-1}(x).$$

Writing $\operatorname{div}(\phi^{\varepsilon 2}\xi) = \nabla(|\phi^\varepsilon|^2) \cdot \xi + |\phi^\varepsilon|^2 \operatorname{div}\xi$, we deduce

$$2\varepsilon \int_{\partial\Omega} \nabla\phi^\varepsilon \cdot n\nabla\phi^\varepsilon \cdot \xi\, d\mathcal{H}^{n-1}(x) = -\frac{\alpha}{\beta} \int_{\partial\Omega} |\phi^\varepsilon|^2 n \otimes n : D\xi\, d\mathcal{H}^{n-1}(x) + \frac{\alpha}{\beta} \int_{\partial\Omega} |\phi^\varepsilon|^2 \operatorname{div}\xi\, d\mathcal{H}^{n-1}(x),$$

and so

$$\varepsilon^{-1} \int_\Omega \operatorname{div}\left((1-2\phi^\varepsilon)\xi\right)\rho = \varepsilon^{-1} \int_\Omega [(1-\rho)(\phi^\varepsilon)^2 + \rho(1-\phi^\varepsilon)^2]\operatorname{div}\xi\, dx + \varepsilon \int_\Omega |\nabla\phi^\varepsilon|^2 \operatorname{div}\xi\, dx$$
$$+ \frac{\alpha}{\beta} \int_{\partial\Omega} |\phi^\varepsilon|^2 \operatorname{div}\xi\, d\mathcal{H}^{n-1}(x) - 2\varepsilon \int_\Omega \nabla\phi^\varepsilon \otimes \nabla\phi^\varepsilon : D\xi\, dx$$
$$- \frac{\alpha}{\beta} \int_{\partial\Omega} \phi^{\varepsilon 2} n \otimes n : D\xi\, d\mathcal{H}^{n-1}(x),$$

which is (5.13). $\qquad\square$

6. JKO Scheme and convergence of the discrete time approximation

The main result of this paper can also be proved at the level of the discrete time approximation constructed in [15]. Such a result can be relevant to some numerical applications so we will state it here.

We briefly recall the construction of the JKO scheme. For simplicity, we assume that $\int_\Omega \rho_{in}(x)\, dx = 1$. We denote by $\mathcal{P}(\Omega)$ the set of probability measures on Ω, and we define the domain of densities as

(6.1) $$K := \{\rho \in \mathcal{P}(\Omega),\ \rho(x) \leq 1 \text{ a.e. in } \Omega\}.$$

In particular all $\rho \in K$ are absolutely continuous with respect to the Lebesgue measure and we can identify the measure with its density. The set $\mathcal{P}(\Omega)$ is equipped with the usual Wasserstein distance, defined by

$$W_2^2(\rho_1, \rho_2) := \inf_{\pi \in \Pi(\rho_1, \rho_2)} \int_{\Omega \times \Omega} |x - y|^2 d\pi(x, y),$$

where $\Pi(\rho_1, \rho_2)$ denotes the set of all probability measures $\pi \in \mathcal{P}(\Omega \times \Omega)$ with marginals ρ_1 and ρ_2.

The idea of the JKO scheme is to construct a time-discrete approximation of the solution by successive applications of a minimization problem: For a given initial data $\rho_{in} \in K$, we fix a time step $\tau > 0$ (destined to go to zero) and define the sequence ρ^n by:

$$(6.2) \qquad \rho^0 = \rho_{in}, \qquad \rho^n \in \text{argmin}\left\{\frac{1}{2\tau}W_2^2(\rho, \rho^{n-1}) + \mathscr{F}_\varepsilon(\rho)\,;\, \rho \in K\right\} \qquad \forall n \geq 1.$$

The fact that this problem has a minimizer is proved in [15, Proposition 2.1]. Furthermore, if T^n is the unique optimal transport map from ρ^n to ρ^{n-1} (that is $T^n \# \rho^n = \rho^{n-1}$ and $W_2^2(\rho^n, \rho^{n-1}) = \int_\Omega |x - T^n(x)|^2 d\rho^n$), we define the velocity

$$v^n(x) := \frac{x - T^n(x)}{\tau}$$

and the pressure variable $p^n(x)$ such that

$$\rho^n v^n = \varepsilon^{-1}\rho^n \nabla(\phi^{\varepsilon n} - p^n), \qquad p^n \in H_{\rho^n}^1,$$

with $\phi^{\varepsilon n}$ solution of (1.4) with ρ^n. The existence of p^n is shown in [15, Proposition 2.6].

We can then define the piecewise constant function $\rho^{\tau, \varepsilon}, p^{\tau, \varepsilon} : [0, T] \mapsto P(\Omega)$ by

$$(6.3) \qquad \begin{aligned} \rho^{\tau, \varepsilon}(t) &:= \rho^{n+1} \quad \text{for all } t \in [n\tau, (n+1)\tau), \\ p^{\tau, \varepsilon}(t) &:= p^{n+1} \quad \text{for all } t \in [n\tau, (n+1)\tau). \end{aligned}$$

The main result of [15] is the convergence of $(\rho^{\tau, \varepsilon}, p^{\tau, \varepsilon})$ when $\tau \to 0$ with $\varepsilon > 0$ fixed to a weak solution of (1.3)-(1.4). The proof of Theorem 1.6 can easily be adapted to establish the convergence of $(\rho^{\tau, \varepsilon}, p^{\tau, \varepsilon})$ to a weak solution of (1.13) when τ and ε both go to zero:

Theorem 6.1 (Convergence when $\varepsilon, \tau \to 0$). *Given $T > 0$, consider an initial data $\rho_{in} = \chi_{E_{in}} \in BV(\Omega; \{0, 1\})$ and $\mu \geq 0$. Consider a subsequence (ε_n, τ_n) with $\max\{\varepsilon_n, \tau_n\} \to 0$. Then along a subsequence (still denoted (ε_n, τ_n)), the discrete time approximation $\rho^{\varepsilon_n, \tau_n}$ converges to ρ strongly in $L^\infty((0, T); L^1(\Omega))$ and $q^{\varepsilon_n, \tau_n}$ converges to q weakly-$*$ in $L^2((0, T); (C^s(\Omega))^*)$ (for any $s > 0$).*

Furthermore, if the following energy convergence assumption holds:

$$\lim_{n \to \infty} \int_0^T \mathscr{I}_{\varepsilon_n}(\rho^{\varepsilon_n, \tau_n}(t))\, dt = \int_0^T \mathscr{I}_0(\rho(t))\, dt,$$

then (ρ, q) is a weak solution of (1.13) in the sense of Definition 1.3 with initial condition $\chi_{E_{in}}$ and contact angle

$$\gamma = -\min\left(1, \frac{2\alpha}{\alpha + \sqrt{\sigma}\beta}\right).$$

We will not provide the details of the proof of this result which is a straightforward adaption of the arguments presented in this paper to prove Theorem 1.6. The key is to recall that the discrete approximations $\rho^{\varepsilon, \tau}$ and $p^{\varepsilon, \tau}$ satisfy some approximation of equations (1.10)-(1.11). Indeed, in addition to $\rho^{\varepsilon, \tau}$ and $p^{\varepsilon, \tau}$, we can define the piecewise constant interpolations $\rho^{\tau, \varepsilon}(x, t)$, $p^{\tau, \varepsilon}(x, t)$, $v^{\tau, \varepsilon}(x, t)$ and $\phi^{\tau, \varepsilon}(x, t)$ by

$$(6.4) \qquad \begin{aligned} v^{\tau, \varepsilon}(t) &:= v^{n+1} \qquad \text{for all} \quad t \in [n\tau, (n+1)\tau), \\ \phi^{\tau, \varepsilon}(t) &:= \phi^{\varepsilon(n+1)} \quad \text{for all} \quad t \in [n\tau, (n+1)\tau). \end{aligned}$$

and the momentum

$$E^{\varepsilon, \tau}(x, t) = \rho^{\tau, \varepsilon}(x, t)v^{\tau, \varepsilon}(x, t).$$

Then we have (see [15]):

Proposition 6.2. *For any $\zeta \in C_c^\infty(\Omega \times [0,T))$ and N such that $N\tau \geq T$, there holds:*

$$\int_0^\infty \int_\Omega E^{\varepsilon,\tau} \cdot \nabla\zeta \, dx \, dt = -\int_\Omega \rho_{in}(x)\zeta(x,0) \, dx - \int_0^\infty \int_\Omega \rho^{\varepsilon,\tau}(x,t)\partial_t\zeta(x,t) \, dx \, dt$$

$$(6.5) \qquad\qquad\qquad + \mathcal{O}\left(\|D^2\zeta\|_{L^\infty(\Omega \times \mathbb{R}_+)} \sum_{k=0}^N W_2^2(\rho_\varepsilon^n, \rho_\varepsilon^{n-1}) + \tau\|\partial_t\zeta\|_\infty + \tau T\|\partial_t^2\zeta\|_\infty \right).$$

For any $\xi \in C^\infty(\bar\Omega \times (0,T); \mathbb{R}^d)$ satisfying $\xi \cdot n = 0$ on $\partial\Omega$, there holds:

$$(6.6) \qquad\quad \int_0^\infty \int_\Omega E^{\varepsilon,\tau} \cdot \xi \, dx \, dt = \int_0^\infty \int_\Omega (\varepsilon^{-1}\rho^{\varepsilon,\tau}\nabla\phi^{\varepsilon,\tau} \cdot \xi + \mu\rho^{\varepsilon,\tau}\mathrm{div}\,\xi + p^{\varepsilon,\tau}\mathrm{div}\,\xi) \, dx \, dt.$$

Passing to the limit in the continuity equation (6.5) can be done exactly as in the case $\tau \to 0$ with $\varepsilon > 0$ fixed (see [15]), while equation (6.6) is exactly the same as our equation (1.11), so we can adapt the proof presented in Section 5 of the present paper to pass to the limit in (6.6) and prove Theorem 6.1.

Appendix A. A few facts about BV functions

We recall here some essential facts about BV functions which we use in our proof (we refer the reader to [1] or [7] for details). First we have the classical result:

Proposition A.1. *Let f_k be a sequence of functions such that $f_k \to f$ in $L^1(\Omega)$ when $k \to \infty$. Then*

$$\liminf_{k\to\infty} \int_\Omega \zeta(x)|\nabla f_k| \geq \int_\Omega \zeta(x)|\nabla f|$$

for all $\zeta \in C(\Omega)$ with $\zeta \geq 0$. Furthermore, if

$$\int_\Omega |\nabla f_k| \to \int_\Omega |\nabla f|,$$

then

$$\lim_{k\to\infty} \int_\Omega \zeta(x)|\nabla f_k| = \int_\Omega \zeta(x)|\nabla f| \qquad \text{for all } \zeta \in C(\Omega).$$

Remark A.2. *If instead of assuming $\int_\Omega |\nabla f_k| \to \int_\Omega |\nabla f|$, we only assume that $\int_\Omega \varphi(x)|\nabla f_k| \to \int_\Omega \varphi(x)|\nabla f|$ for some $\varphi \in C(\Omega)$, $\varphi \geq 0$, then we have*

$$(A.1) \qquad\quad \lim_{k\to\infty} \int_\Omega \zeta(x)\varphi(x)|\nabla f_k| = \int_\Omega \zeta(x)\varphi(x)|\nabla f| \qquad \text{for all } \zeta \in C(\Omega).$$

Indeed, since we can split ζ into its positive and negative part, we can assume that $\zeta \geq 0$ and after dividing by $\|\zeta\|_\infty$, we can assume that $0 \leq \zeta(x) \leq 1$. We then have

$$0 = \lim_{k\to\infty} \int_\Omega \varphi(x)|\nabla f_k| - \int_\Omega \varphi(x)|\nabla f|$$

$$= \lim_{k\to\infty} \left(\int_\Omega \zeta(x)\varphi(x)|\nabla f_k| + \int_\Omega (1 - \zeta(x))\varphi(x)|\nabla f_k| - \int_\Omega \varphi(x)|\nabla f_k| \right)$$

$$\geq \liminf_{k\to\infty} \left(\int_\Omega \zeta(x)\varphi(x)|\nabla f_k| - \int_\Omega \zeta(x)\varphi(x)|\nabla f| \right)$$

$$+ \liminf_{k\to\infty} \left(\int_\Omega (1 - \zeta(x))\varphi(x)|\nabla f_k| - \int_\Omega (1 - \zeta(x))\varphi(x)|\nabla f| \right).$$

Since these two $\liminf$ are non-negative, they must both be zero. In particular we have $\liminf_{k\to\infty} \int_\Omega \zeta(x)\varphi(x)|\nabla f_k| = \int_\Omega \zeta(x)\varphi(x)|\nabla f|$ and since this is true along any subsequence as well, (A.1) follows. Finally, we point

out that if we only assume that

$$\limsup_{k\to\infty} \int_\Omega \varphi(x)|\nabla f_k| - \int_\Omega \varphi(x)|\nabla f| \le C\delta$$

then the same argument implies

$$\limsup_{k\to\infty} \int_\Omega \zeta(x)\varphi(x)|\nabla f_k| - \int_\Omega \zeta(x)\varphi(x)|\nabla f| \le C\delta$$

for all $\zeta \in C(\Omega)$.

When the domain Ω has Lipschitz boundary any function $f \in \mathrm{BV}(\Omega)$ has a well defined trace $f|_{\partial\Omega} \in L^1(\partial\Omega, \mathcal{H}^{d-1})$ and the extension $\bar{f}$ of f to $\mathbb{R}^d$ by 0 belongs to $\mathrm{BV}(\mathbb{R}^d)$. Viewing ∇f as a measure on $\mathbb{R}^d$ concentrated on Ω, we have

$$\nabla\bar{f} = \nabla f + f\, n\, \mathcal{H}^{d-1}|_{\partial\Omega}$$

and so

$$\int_{\mathbb{R}^d} |\nabla\bar{f}| = \int_\Omega |\nabla f| + \int_{\partial\Omega} |f| d\mathcal{H}^{d-1}$$

and (for all continuous function $h : \Omega \times \mathbb{S}^{d-1} \to \mathbb{R}$)

$$(A.2) \qquad \int_{\mathbb{R}^d} h\left(x, \frac{\nabla\bar{f}}{|\nabla\bar{f}|}\right)|\nabla\bar{f}| = \int_\Omega h\left(x, \frac{\nabla f}{|\nabla f|}\right)|\nabla f| + \int_{\partial\Omega} h(x,n)|f| d\mathcal{H}^{d-1}.$$

We also need the following particular case of Reshetnyak's continuity theorem (see [1] Theorem 2.39)

Proposition A.3. *Let f_k be a sequence of function such that $f_k \to f$ in $L^1(\Omega)$ and $\int_\Omega |\nabla f_k| dx \to \int_\Omega |\nabla f|$. Then*

$$\lim_{k\to\infty} \int_\Omega \zeta(x) \frac{\partial_i f_k}{|\nabla f_k|} \frac{\partial_j f_k}{|\nabla f_k|} |\nabla f_k| = \int_\Omega \zeta(x) \frac{\partial_i f}{|\nabla f|} \frac{\partial_j f}{|\nabla f|} |\nabla f|$$

for all $\zeta \in C(\overline{\Omega})$.

A proof of this result can be found for example in [1]. We provide below a short proof in our particular case, using arguments which can be found in [27]. This proof can be adapted to give the following localization of Reshetnyak's theorem:

Corollary A.4. *Let f_k be a sequence of function such that $f_k \to f$ in $L^1(\Omega)$ and $\int_\Omega \varphi(x)|\nabla f_k| dx \to \int_\Omega \varphi(x)|\nabla f|$ for some $\varphi \in C^1(\Omega)$, $\varphi \ge 0$. Then we have*

$$\lim_{k\to\infty} \int_\Omega \zeta(x)\varphi(x) \frac{\partial_i f_k}{|\nabla f_k|} \frac{\partial_j f_k}{|\nabla f_k|} |\nabla f_k| = \int_\Omega \zeta(x)\varphi(x) \frac{\partial_i f}{|\nabla f|} \frac{\partial_j f}{|\nabla f|} |\nabla f|$$

for all $\zeta \in C^1(\overline{\Omega})$.

Proof of Proposition A.3. We denote $\nu_i^k = \frac{\partial_i f_k}{|\nabla f_k|} \in L^1(\Omega, |\nabla f_k|)$ (the Radon-Nikodym derivative of $\partial_i f_k$ with respect to the measure $|\nabla f_k|$). Similarly, we note $\nu_i = \frac{\partial_i f}{|\nabla f|}$. First, we note that it is enough to prove that under the conditions of the proposition we have

$$(A.3) \qquad \lim_{k\to\infty} \int_\Omega \zeta(x)|\nu_i^k|^2 |\nabla f_k| = \int_\Omega \zeta(x)|\nu_i|^2 |\nabla f| \qquad i = 1, \dots, n$$

for all $\zeta \in C(\overline{\Omega})$ with $\zeta \ge 0$ since we can write $\nu_i \nu_j = \frac{1}{2}[(\nu_i + \nu_j)^2 - \nu_i^2 - \nu_j^2]$.

Given a vector field $g = (g_1, \ldots g_n) \in C_0^1(\Omega, \mathbb{R}^d)$ such that $|g(x)| \leq 1$ for all x, we write

$$\left| \int_\Omega \zeta(x) |\nu_i^k|^2 |\nabla f_k| - \int_\Omega \zeta(x) |\nu_i|^2 |\nabla f| \right| \leq \left| \int_\Omega \zeta(x) |\nu_i^k|^2 |\nabla f_k| - \int_\Omega \zeta(x) |g_i|^2 |\nabla f_k| \right|$$

$$+ \left| \int_\Omega \zeta(x) |g_i|^2 |\nabla f_k| - \int_\Omega \zeta(x) |g_i|^2 |\nabla f| \right|$$

$$+ \left| \int_\Omega \zeta(x) |g_i|^2 |\nabla f| - \int_\Omega \zeta(x) |\nu_i|^2 |\nabla f| \right|.$$

Since $|g_i|, |\nu_i| \leq 1$, we have

$$\left| |g_i|^2 - |\nu_i|^2 \right| = |g_i - \nu_i| \, |g_i + \nu_i| \leq 2|g_i - \nu_i|$$

and so for any $\delta > 0$ there exists C_δ such that

$$\left| |g_i|^2 - |\nu_i|^2 \right| \leq \delta + C_\delta (g_i - \nu_i)^2 \leq \delta + C_\delta |g - \nu|^2 \leq \delta + 2C_\delta (1 - g \cdot \nu)$$

where we used the fact that $|g| \leq 1$ and $|\nu| = 1$. We deduce

$$\left| \int_\Omega \zeta(x) |\nu_i^k|^2 |\nabla f_k| - \int_\Omega \zeta(x) |\nu_i|^2 |\nabla f| \right| \leq \delta \|\zeta\|_\infty (|\nabla f_k|(\Omega) + |\nabla f|(\Omega)|)$$

$$+ 2C_\delta \int_\Omega \zeta(x)(1 - g \cdot \nu_k)|\nabla f_k| + 2C_\delta \int_\Omega \zeta(x)(1 - g \cdot \nu)|\nabla f|$$

$$+ \left| \int_\Omega \zeta(x) |g_i|^2 |\nabla f_k| - \int_\Omega \zeta(x) |g_i|^2 |\nabla f| \right|.$$

We now pass to the limit $k \to \infty$. We note that the last term goes to zero thanks to Proposition A.1 and we can pass to the limit in the other terms using Proposition A.1 and the fact that

$$\int_\Omega \zeta(x) g \cdot \nu_k |\nabla f_k| = \int_\Omega \zeta(x) g \cdot \nabla f_k = -\int_\Omega \mathrm{div}\,(\zeta g) f_k \to -\int_\Omega \mathrm{div}\,(\zeta g) f.$$

We deduce

$$\limsup_{k \to \infty} \left| \int_\Omega \zeta(x) |\nu_i^k|^2 |\nabla f_k| - \int_\Omega \zeta(x) |\nu_i|^2 |\nabla f| \right| \leq 2\delta \|\zeta\|_\infty |\nabla f|(\Omega)| + 4C_\delta \|\zeta\|_\infty \int_\Omega (1 - g \cdot \nu)|\nabla f|$$

$$\leq 2\delta \|\zeta\|_\infty |\nabla f|(\Omega)| + 4C_\delta \|\zeta\|_\infty \left(\int_\Omega |\nabla f| - \int_\Omega g \cdot \nabla f \right)$$

We now take the infimum over all functions $g \in C_0^1(\Omega, \mathbb{R}^d)$ such that $|g(x)| \leq 1$: the definition of the BV norm implies that the last term goes to zero. We then let $\delta \to 0$ to get the result. $\square$

APPENDIX B. A LIONS-AUBIN COMPACTNESS RESULT

The following result is a simple adaptation of some standard result. We provide a proof for the sake of completeness.

Lemma B.1. *Let u_n be a sequence of function bounded in $L^\infty(0, T; L^1(\Omega))$ such that u_n is bounded in $L^\infty((0, T); \mathrm{BV}(\Omega))$ and $u_n \to u$ in $L^\infty((0, T); H^{-1}(\Omega))$ Then*

$$\sup_{t \in [0,T]} \|u_n(t) - u(t)\|_{L^1(\Omega)} \to 0$$

Proof. Since $\int |\nabla u_n(t)| \leq C$ and $u_n(t)$ converges to $u(t)$ in $H^{-1}(\Omega)$, we can show that $u(t) \in BV(\Omega)$ and $\int |\nabla u(t)| \leq C$.

Next we claim that for any $\delta > 0$, there exists C_δ such that

(B.1) $$\|v\|_{L^1(\Omega)} \leq \delta\|v\|_{BV(\Omega)} + C_\delta\|v\|_{H^{-1}(\Omega)}$$

for all $v \in BV(\Omega)$. This inequality can be proved via a compactness argument: If not, there exists $\delta_0 > 0$ such that for all $k \in \mathbb{N}$ there exists $v_k \in BV(\Omega)$ such that

$$\|v_k\|_{L^1(\Omega)} \geq \delta_0\|v_k\|_{BV(\Omega)} + k\|v_k\|_{H^{-1}(\Omega)}.$$

We can always normalize v_k so that $\|v_k\|_{L^1(\Omega)} = 1$. The inequality above then implies that v_k is bounded in $BV(\Omega)$ (and thus precompact in $L^1(\Omega)$) and converges to zero (strongly) in $H^{-1}(\Omega)$. It follows that v_k converges strongly to zero in $L^1(\Omega)$, which contradicts the normalization $\|v_k\|_{L^1(\Omega)} = 1$.

Inequality (B.1) now gives:

$$\|u_n(t) - u(t)\|_{L^1(\Omega)} \leq \delta\|u_n(t) - u(t)\|_{BV(\Omega)} + C_\delta\|u_n(t) - u(t)\|_{H^{-1}(\Omega)}$$

and the BV bound, together with the strong convergence in $H^{-1}(\Omega)$ implies

$$\limsup \|u_n(t) - u(t)\|_{L^\infty((0,T);L^1(\Omega))} \leq C\delta.$$

Since this holds for all $\delta > 0$, we deduce

$$\limsup \|u_n - u\|_{L^\infty((0,T);L^1(\Omega))} = 0$$

and the result follows.

$\square$

APPENDIX C. Γ-CONVERGENCE OF $\mathscr{J}_\varepsilon$

We wish to prove the following proposition which gives the Γ convergence of $\mathscr{J}_\varepsilon$ to $\mathscr{J}_0$:

Proposition C.1. *The following holds:*
(i) *For any family $\{\rho^\varepsilon\}_{\varepsilon>0}$ that converges to ρ in $L^1(\Omega)$,*

$$\liminf_{\varepsilon \to 0} \mathscr{J}_\varepsilon(\rho^\varepsilon) \geq \mathscr{J}_0(\rho).$$

(ii) *Given $\rho \in L^1(\Omega)$, there exists a sequence $\{\rho^\varepsilon\}_{\varepsilon>0}$ that converges to ρ in $L^1(\Omega)$ such that*

$$\limsup_{\varepsilon \to 0} \mathscr{J}_\varepsilon(\rho^\varepsilon) \leq \mathscr{J}_0(\rho).$$

We recall that this proposition is proved in [21] (Proposition 5.3) when $\mathscr{J}_\varepsilon$ is restricted to characteristic functions. We show below how the proof can be adapted to our more general case.

Proof of Proposition C.1. First, we note that the limsup properties (part (ii)) follows from the corresponding result in [21, Proposition 5.3] . Indeed, if $\rho \notin BV(\Omega; \{0,1\})$, then $\mathscr{J}_0(\rho) = \infty$ and there is nothing to prove, while if $\rho \in BV(\Omega; \{0,1\})$ then $\rho = \chi_E$ for some E satisfying $P(E) < \infty$, so [21, Proposition 5.3] applies.

To prove the liminf property (part (i)), we need to slightly modify the proof of [21, Proposition 5.3] by using the formula (2.2) instead of (2.1) for $\mathscr{J}_\varepsilon$. We only provide details in the case of Neumann and Dirichlet conditions (the Robin boundary condition is then proved combining both arguments, as in [21]).

Neumann boundary conditions ($\alpha = 0$). If $\liminf_{\varepsilon \to 0} \mathscr{J}_\varepsilon(\rho^\varepsilon) = \infty$, there is nothing to prove, so we can assume (up to a subsequence) that $\mathscr{J}_\varepsilon(\rho^\varepsilon) \leq C$ and $\liminf_{\varepsilon \to 0} \mathscr{J}_\varepsilon(\rho^\varepsilon) < \infty$. Up to another subsequence, we can also assume that $\rho^\varepsilon \to \rho$ a.e. in Ω.

Next, we recall that (2.2) gives

(C.1) $$\mathscr{I}_\varepsilon(\rho) = \frac{1}{2\sigma\varepsilon}\int_\Omega (1-\rho)(\sigma\phi^\varepsilon)^2 + \rho(1-\sigma\phi^\varepsilon)^2\,dx + \frac{\varepsilon}{2}\int_\Omega |\nabla\phi^\varepsilon|^2\,dx$$

and introducing the functions (both defined for $t \in [0,1]$)

$$f(t) = 2\min(t, 1-t), \qquad F(t) = \int_0^t f(\tau)\,d\tau = \begin{cases} t^2 & \text{for } 0 \le t \le 1/2 \\ 2t - t^2 - \frac{1}{2} & \text{for } 1/2 \le t \le 1 \end{cases}$$

we find (see (4.3)):

(C.2) $$\frac{1}{2\sigma^{3/2}}\int_\Omega |\nabla F(\sigma\phi^\varepsilon)|\,dx \le \mathscr{I}_\varepsilon(\rho^\varepsilon).$$

Furthermore, (2.1) give

$$\mathscr{I}_\varepsilon(\rho) = \frac{1}{2\sigma\varepsilon}\int_\Omega \rho(1-\rho)\,dx + \frac{1}{2\sigma\varepsilon}\int_\Omega (\rho - \sigma\phi^\varepsilon)^2\,dx + \frac{\varepsilon}{2}\int_\Omega |\nabla\phi^\varepsilon|^2\,dx$$

which implies

$$\int_\Omega (\rho^\varepsilon - \sigma\phi^\varepsilon)^2\,dx \le 2\sigma\varepsilon\,\mathscr{I}_\varepsilon(\rho^\varepsilon) \le C\varepsilon$$

and

$$\int_\Omega \rho^\varepsilon(1-\rho^\varepsilon)\,dx \le 2\sigma\varepsilon\,\mathscr{I}_\varepsilon(\rho^\varepsilon) \le C\varepsilon.$$

The first inequality implies that $\sigma\phi^\varepsilon$ converges in L^2 to ρ. The second inequality implies that that $\rho = 0$ or 1 a.e. in Ω.

We deduce that $F(\sigma\phi^\varepsilon)$ converges (strongly in L^1 for example) to $F(\rho) = \frac{1}{2}\rho$ (since $F(0) = 0$ and $F(1) = 1/2$), and (C.2) gives

$$\liminf_{\varepsilon \to 0} \mathscr{I}_\varepsilon(\rho^\varepsilon) \ge \frac{1}{2\sigma^{3/2}}\int_\Omega |\nabla F(\rho)|\,dx = \frac{1}{4\sigma^{3/2}}\int_\Omega |\nabla\rho|\,dx = \mathscr{I}_0(\rho).$$

Dirichlet boundary conditions. We still have (C.1) and thus (C.2) in this case, but since $\phi^\varepsilon = 0$ on $\partial\Omega$, we can extend the function ϕ^ε by zero outside Ω. Denoting by $\overline{\phi^\varepsilon}$ this extension, we find

$$\mathscr{I}_\varepsilon(\rho^\varepsilon) \ge \frac{1}{2\sigma^{3/2}}\int_\Omega |\nabla F(\phi^\varepsilon)|\,dx = \frac{1}{2\sigma^{3/2}}\int_{\mathbb{R}^n} |\nabla F(\overline{\phi^\varepsilon})|\,dx$$

and so (proceeding as above)

$$\begin{aligned}
\liminf_{\varepsilon \to 0} \mathscr{I}_\varepsilon(\rho^\varepsilon) &\ge \frac{1}{2\sigma^{3/2}}\int_{\mathbb{R}^n} |\nabla F(\rho)|\,dx = \frac{1}{4\sigma^{3/2}}\int_{\mathbb{R}^n} |\nabla\rho|\,dx \\
&= \frac{1}{4\sigma^{3/2}}\left[\int_\Omega |\nabla\rho|\,dx + \int_{\partial\Omega} \rho\,dH^{n-1}(x)\right] \\
&= \mathscr{I}_0(\rho).
\end{aligned}$$

$\square$

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