INCOMPRESSIBLE LIMIT OF A POROUS MEDIA EQUATION WITH BISTABLE AND MONOSTABLE REACTION TERM

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ABSTRACT. We study the incompressible limit of the porous medium equation with a reaction term that is non-monotone with respect to the pressure variable. More specifically we consider reaction terms that are either bistable or monostable. We show that this type of reaction term generates many interesting differences in the qualitative behavior of solutions, in contrast to the problem with monotone reaction terms that have been extensively studied in recent literature. After characterizing the limit problem, we embark on a comprehensive study of the problem in one space dimension, to illustrate the delicate nature of the problem, including the generic nature of non-uniqueness and instability. For compactly supported initial data, we show that the density can either perish or thrive, even if it starts from the same initial data, depending on its initial pressure configuration. When the initial pressure is a characteristic function, we establish the existence of the sharp threshold separating the two behaviors. Lastly we present a detailed analysis of the behavior of traveling waves in this incompressible limit. We study the existence of traveling waves for the limiting model and prove convergence results in the incompressible limit (depending on the reaction term).

1. INTRODUCTION

1.1. **A degenerate reaction diffusion equation and its incompressible limit.** We consider the following porous media equation with a reaction term:

$$\begin{cases} \partial_t\rho - \operatorname{div}(\rho\nabla p) = \rho f(p), \quad p = P_m(\rho) := \frac{m}{m-1}\rho^{m-1} & \text{in } \mathbb{R}^n\times\mathbb{R}_+ \\ \rho(x,0) = \rho_{in,m}(x) & \text{in } \mathbb{R}^n \end{cases} \tag{1.1}$$

where f is a Lipschitz continuous function. This is a very classical equation which arises in many applications, and it has been studied extensively in recent years as a mechanical model for tumor growth, see for instance [19], [20] and references therein. In this context several papers have investigated the asymptotic behavior of the solution of (1.1) in the so-called incompressible regime, corresponding to the limit $m\to\infty$ [1, 6, 7, 12, 13, 19, 18, 8] In the context of tumor growth, it is typically assumed that $p\mapsto f(p)$ is monotone decreasing and satisfies

$$f(p)\le 0 \quad \text{for } p\ge p_M, \tag{1.2}$$

Under such a monotonicity assumption, the nonlinearity $g(\rho) := \rho f(p) = \rho f(\frac{m}{m-1}\rho^{m-1})$ is a Fisher-KPP nonlinearity [4, 14]. Indeed, we trivially have $g(\rho)>0$ in $(0,\rho_M)$, $g(\rho)<0$ in $(-\infty,0)\cup(\rho_M,+\infty)$ and the condition $g(\rho)\le g'(0)\rho$ is equivalent to $f(p)\le f(0)$ for all $p\ge 0$.

The goal of this paper is to understand the dynamic of this problem, still in the regime $m\gg 1$, but when we **remove** the monotonicity assumption on f (but still assume (1.2) to have uniform bounds on the pressure). A simple example that we have in mind is the following quadratic function:

$$f(p) = (1-p)(p-\alpha), \quad \alpha\in[0,1), \quad \int_0^1 f(s)\,ds > 0, \tag{1.3}$$

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although our analysis will not be restricted to this case. The reaction-diffusion equation (1.1)-(1.3) can be used to model the propagation of a biological population whose tendency to disperse depends on the population density when the nonlinearity (1.3) takes into account a weak ($\alpha = 0$) or strong ($\alpha > 0$) Allee effect (which assumes a reduced, or negative, growth rate at low density [9]). When $\alpha \in (0, 1)$, the function $g(\rho) = \rho f(p) = \rho f\left(\frac{m}{m-1}\rho^{m-1}\right)$ is a **bistable** reaction term while when $\alpha = 0$, g is a **monostable** nonlinearity.

In the context of tumor growth, such a function f can be used to model the competition between drugs invading the tumor and the contact inhibition of tumor cells. Indeed, in classical tumor models, drugs are injected far from the tumor and diffuse toward the tumor (see [5]). They therefore act mostly on the cells in the regions with lower pressure, namely near the periphery of the tumor, while deep inside the tumor the natural growth due to cell division dominates the dynamic. Naturally one could ask whether this problem yields a stable evolution of the tumor zone. We will prove that it is not the case and that non-uniqueness may occur in the incompressible limit. Furthermore we will see that the tumor region may disappear or thrive with the same initial configuration, depending on its initial pressure. This is in contrast to the case with monotone f, where the evolution of the tumor is uniquely determined by its initial configuration.

Note that we could easily extend our result to more general settings, in particular to inhomogeneous f that depends also on time and space. In the context of tumor growth, this could include models that take into account spatial in-homogeneities or dependence of f on the concentration of nutrient. We will keep the framework simple to focus on the non-monotonicity of f with respect to the pressure variable and its consequences.

1.2. **Known results when $p \mapsto f(p)$ is decreasing.** Before stating our main results, we briefly recall here the most important aspects of the analysis when $p \mapsto f(p)$ is monotone decreasing (and satisfies (1.2)): For appropriate initial data $\rho_{in,m}$, satisfying in particular $p_{in,m} \le p_M$ and $\rho_{in,m} \to \rho_{in}$, the following basic properties have been proved

(1) The functions $\rho_m(x,t)$ and $p_m(x,t)$ are bounded in $L^\infty(\mathbb{R}^n \times \mathbb{R}_+)$ and in $BV(\mathbb{R}^n \times (0,T))$ and converge (strongly in L^1_{loc} and a.e.) to $\rho_\infty(x,t)$ and $p_\infty(x,t)$.

(2) The pair (ρ_∞, p_∞) is solution (in the sense of distribution) of

$$\begin{cases} \partial_t \rho_\infty = \Delta p_\infty + \rho_\infty f(p_\infty) & \text{in } \mathbb{R}^n \times \mathbb{R}_+ \\ \rho_\infty(x,0) = \rho_{in}(x) & \text{in } \mathbb{R}^n \end{cases} \tag{1.4}$$

and satisfies the Hele-Shaw graph condition

$$p_\infty \in P_\infty(\rho_\infty) := \begin{cases} 0 & \text{if } \rho_\infty < 1 \\ [0,\infty) & \text{if } \rho_\infty = 1 \\ \infty & \text{if } \rho_\infty > 1. \end{cases} \tag{1.5}$$

Furthermore, the problem (1.4)-(1.5) has a unique weak solution in $L^\infty((0,\infty); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, with $\rho \in C([0,\infty); L^1(\mathbb{R}^n))$.

(3) For all $t > 0$, $p_\infty(\cdot, t^+)$ (defined as the trace of the BV function p_∞) is the unique solution p^* of the obstacle problem

$$\begin{cases} p^* \ge 0, \quad \Delta p^* + f(p^*) \le 0, & \text{in } \{\rho_\infty(\cdot,t) = 1\} \\ \Delta p^* + f(p^*) = 0 \text{ in } \{p^* > 0\}. \end{cases} \tag{1.6}$$

The first two points are proved in [19], while the characterization of $p_\infty(\cdot, t^+)$ as the solution of an elliptic obstacle problem (1.6) for all $t > 0$ is proved in [12]. We can interpret (1.4)-(1.5) as a Hele-Shaw type free boundary problem in which the saturated set $\Omega(t) = \{\rho_\infty(\cdot,t) = 1\}$ expands with normal velocity proportional to the pressure gradient (see [12] for a rigorous statement). We note that these results hold if f depends on (x,t) as well, provided the function $p \mapsto f(x,t,p)$ satisfies (1.2) for all (x,t). But the monotonicity of $p \mapsto f(p)$ plays a crucial role in the proof of the results above. Without it, we will see that neither the obstacle problem (1.6) nor the asymptotic equation (1.4)-(1.5) can be expected to have a unique solution. Even the derivation of the BV bounds, crucial to derive strong convergence of the density variable, appears to require this assumption.

1.3. **Main results: General case.** We will first state the result that one can show for general function f satisfying the condition (1.2) (but no monotonicity). We start with the following proposition:

Proposition 1.1. *Assume that f is a Lipschitz function satisfying (1.2) and that the initial condition $\rho_{in,m}$ is a non-negative function such that*

$$\rho_{in,m} \in L^1(\mathbb{R}^n), \qquad p_{in,m} = P_m(\rho_{in,m}) \le p_M \text{ a.e. in } \mathbb{R}^n. \tag{1.7}$$

Then, the solution (ρ_m, p_m) of (1.1) satisfies the following:

(i) *$\rho_m(x,t)$ and $p_m(x,t)$ are uniformly bounded in $L^\infty(\mathbb{R}^n \times \mathbb{R}_+) \cap L^\infty(0,T;L^1(\mathbb{R}^n))$ for all $T>0$. Up to a subsequence, ρ_m and p_m converge weakly-$*$ in L^∞ to ρ_∞ and p_∞.*

(ii) *p_m is uniformly bounded in $L^2(0,T;H^1(\mathbb{R}^n))$ and ρ_m is uniformly bounded in $C^{1/2}(0,T;H^{-1}(\mathbb{R}^n))$ for all $T>0$.*

(iii) *Any accumulation point (ρ_∞, p_∞) of $\{(\rho_m,p_m)\}_{m\ge1}$ satisfies the Hele-Shaw graph condition (1.5) a.e. in $\mathbb{R}^n \times \mathbb{R}_+$.*

However, the lack of BV estimates, or more precisely the lack of strong convergence and almost everywhere convergence of p_m, is hindering further characterization of the limit. In particular, passing to the limit in the equation (1.1) to show that (ρ_∞, p_∞) solves (1.4) is very delicate since the nonlinear term $f(p_m)$ might not converge to $f(p)$. Nevertheless, we can observe that $f(p_m)$ is bounded in $L^\infty(\mathbb{R}^n \times \mathbb{R}_+) \cap L^2(0,T;H^1(\mathbb{R}^n))$. We can thus assume, up to another subsequence, that there exists $g \in L^\infty(\mathbb{R}^n \times \mathbb{R}_+) \cap L^2(0,T;H^1(\mathbb{R}^n))$ such that

$$f(p_m) \rightharpoonup g \qquad \text{weak-* in } L^\infty(\mathbb{R}^n \times \mathbb{R}_+). \tag{1.8}$$

With this weak limit we can still characterize the limit pressure:

Proposition 1.2. *Under the assumption of Proposition 1.1 and for a subsequence such that (1.8) holds, the limit (ρ_∞, p_∞) satisfies, in the sense of distribution,*

$$\begin{cases} \partial_t \rho_\infty = \Delta p_\infty + \rho_\infty g, \quad p_\infty(1-\rho_\infty) = 0 & \text{in } \mathbb{R}^n \times \mathbb{R}_+ \\ \rho_\infty(x,0) = \rho_{in}(x) & \text{in } \mathbb{R}^n. \end{cases} \tag{1.9}$$

Furthermore, for a.e. $t>0$ the function $q = p_\infty(\cdot,t)$ is in $H^1(\mathbb{R}^n)$ and satisfies

$$\int_{\mathbb{R}^n} \nabla q \cdot \nabla \phi - g\phi \, dx = 0 \tag{1.10}$$

for all $\phi \in H^1(\mathbb{R}^n)$ such that $\phi(x)(1-\rho_\infty(x,t)) = 0$ a.e. $x \in \mathbb{R}^n$.
Finally, if f is concave, then $g \le f(p_\infty)$ and if f is convex then $g \ge f(p_\infty)$: in particular, if f is linear, then $g = f(p_\infty)$.

Since p_m is bounded in $L^2(0,T;H^1(\mathbb{R}^n))$, we see that the lack of strong convergence of p_m is due to the possibility of oscillating behavior in time. While we do not yet know if such oscillations actually happen, it is plausible given the non-uniqueness of the expected limiting pressure equation $-\Delta p = f(p)$ (see the discussion below). This issue can be avoided if we can guarantee that the solution of (1.1) is monotone in time which is true for some particular initial data. Indeed, we have:

Proposition 1.3. *Assume that the initial pressure $p_{in,m}$ satisfy*

$$\Delta p_{in,m} + f(p_{in,m})\chi_{\{p_{in,m}>0\}} \geq 0 \text{ in } \mathbb{R}^n$$

for $m \geq m_0$. Then p_m and ρ_m are non-decreasing in time, and, up to a subsequence, p_m converges to p_∞ strongly in L^1_{loc} and a.e. in $\mathbb{R}^n \times \mathbb{R}_+$. In particular, we can assume that

$$g = f(p_\infty)$$

in Proposition 1.2. In other words, (ρ_∞, p_∞) solves (1.4)-(1.5).

In the cases in which we are able to prove that $g = f(p_\infty)$ (e.g. for linear f or for monotone-in-time solutions), it is natural to ask whether (1.4)-(1.5) actually identifies the limit (ρ_∞, p_∞) uniquely. This would imply the convergence of the whole sequence (ρ_m, p_m). Such a result, which was proved in [19] with monotone decreasing f, does not hold in general. As we will prove below in the simple one dimensional framework, such uniqueness is false even for relatively simple f, such as the quadratic function given by (1.3). This is not so surprising. Indeed, equation (1.9) typically identifies a unique pressure function $p_\infty(\cdot, t)$ as the solution of the elliptic problem (1.10). But without the monotonicity of f this problem is not well-posed. Indeed, given a set Ω, it is well known [2, 3, 15] that the Dirichlet problem

$$\begin{cases} -\Delta q = f(q) & \text{in } \Omega \\ q = 0 & \text{on } \partial\Omega \end{cases} \tag{1.11}$$

is well posed if f is monotone decreasing, or if $\mathrm{Lip} f \leq \lambda_\Omega$ where λ_Ω denotes the first eigenvalue of the laplacian in Ω. But in general, even if f is linear, this boundary value problem might have no solution or multiple solutions.

1.4. **Monostable and bistable reaction term in dimension 1.** For any given $m > 1$, equation (1.1) is well posed, but as $m \to \infty$, the discussion above suggests it is asymptotically close to a potentially ill-posed problem. This suggests a relatively unstable dynamic when $m \gg 1$. In order to illustrate the non-uniqueness of the limit problem and to better understand the dynamic of (ρ_m, p_m) when $m \gg 1$, we will now focus on the simpler one-dimensional framework. Throughout this section, we assume that f satisfies one of the following conditions:

Either

$$f(p) > 0 \text{ in } (\alpha, p_M), \quad f(p) < 0 \text{ in } [0, \alpha) \cup (p_M, \infty) \text{ for some } \alpha \in (0, p_M), \tag{1.12}$$

or (satisfied with f given by (1.3) with $\alpha = 0$)

$$f(0) = 0, \quad f(p) > 0 \text{ in } (0, p_M), \quad f(p) < 0 \text{ in } (p_M, \infty). \tag{1.13}$$

Above conditions are satisfied with f given by (1.3), (1.12) for $\alpha \in (0, p_M]$ or (1.13) for $\alpha = 0$. It is easy to check that the reaction term $\rho \mapsto \rho f(p) = \rho f\left(\frac{m}{m-1}\rho^{m-1}\right)$ in (1.1) is a bistable nonlinearity in case (1.12) and a monostable nonlinearity in case (1.13). In the sequel we will thus refer to (1.12) and (1.13) respectively as the bistable and the monostable case.

For the standard bistable reaction-diffusion equation (with linear diffusion), the dynamic strongly depends on the sign of the integral of the reaction term. For the nonlinear diffusion equation (1.1), the relevant integral (see for example Theorem 1.7 below) is

$$\int_0^{\rho_M} \rho^m f\left(\frac{m}{m-1}\rho^{m-1}\right) d\rho = \int_0^{p_M} \frac{1}{m}\left(\frac{m-1}{m}p\right)^{\frac{2}{m-1}} f(p)\, dp.$$

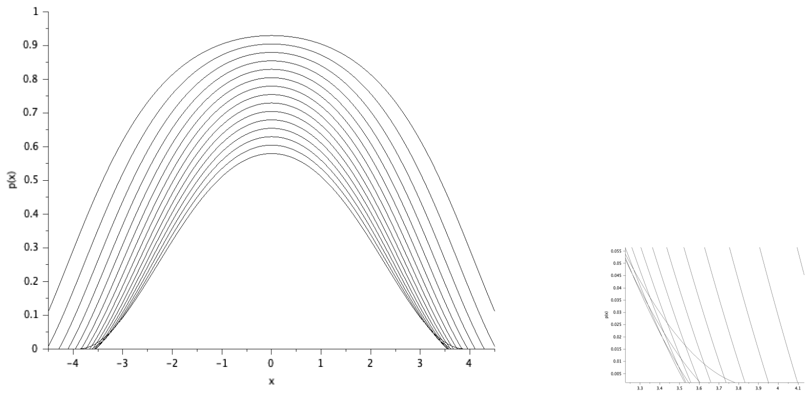

FIGURE 1. Solutions of equation (1.15) on $[-L, L]$ for various values of L when $f(p) = (1-p)(p-1/4)$. The picture on the right, which is a close up near the boundary $x = L$, clearly shows that two solutions exists for some values of L.

In the regime $m \gg 1$, many results below will thus depend on the sign of the integral $\int_0^{p_M} f(p)\,dp$. We also introduce the constant K defined by

$$K := \sup_{p>0} \frac{f(p)}{p} < \infty \tag{1.14}$$

(since $f(0) \leq 0$, we have in particular $K \leq \mathrm{Lip}(f)$).

1.5. **The pressure equation.** The first step is to study the semi-linear boundary value problem (1.11) which determines the limiting pressure. In one dimension, we have:

$$\begin{cases} -u'' = f(u), \quad u \geq 0 \quad \text{in } (0, L) \\ u(0) = u(L) = 0. \end{cases} \tag{1.15}$$

When f satisfies (1.13) ($\alpha = 0$), then (1.15) has a solution $u = 0$, which we referred to as the trivial solution below. Even in that case, it may also have non-trivial solution, as shown by the following result:

Proposition 1.4. *Assume that f is a Lipschitz function satisfying (1.12) or (1.13).*
If $\int_0^{p_M} f(s)\,ds < 0$ then (1.15) has no solution.
If $\int_0^{p_M} f(s)\,ds > 0$, then there exists $L_0 \geq \frac{\pi}{\sqrt{K}}$ (with K given by (1.14)) such that

(i) *If $L < L_0$ then (1.15) has no non-trivial solution.*
(ii) *If $L > L_0$ then (1.15) has at least one non-trivial solution with support $(0, L)$.*

We note that $u = 0$ is a solution of (1.15) in the monostable case (1.13) but not in the bistable case (1.12), so (1.15) does not have any solution in the bistable case (1.12) when $L < L_0$. We will also show that $L_0 > \frac{\pi}{\sqrt{K}}$ except in the degenerate case where $f(p) = Kp$ for small $p > 0$. Numerical computations in the case $\alpha > 0$ (see Figure 1) show that, at least for some $L > L_0$, (1.15) might have two non-trivial solutions. A rigorous verification of this non-uniqueness is given in Remark 4.2.

This proposition proves that the characterization of p_∞ given in Proposition 1.2 does not necessarily provide a unique characterization for the limit pressure. In addition we will see below that, unlike the case studied in [19, 12] and others, the behavior of (ρ_∞, p_∞) is not uniquely determined by the initial condition ρ_{in}. Note that even if we are given both ρ_{in} and p_{in}, it is not obvious that there is a unique solution for the limiting problem, but we do not address this issue here.

1.6. **Propagation vs extinction for compactly supported initial data.** We establish this fact by studying the classical question of the long-term survival or extinction of bistable populations in this incompressible setting (in the framework of tumor growth, the question can be recast as: *who wins? the tumor or the drugs?*).

First we show that if the normalized initial pressure $p_{in,m}/p_M$ is a characteristic function, then a sharp threshold appears depending on the length of the support of $\rho_{in,m}$ when f is in addition assumed to be concave (for instance when f is given by (1.3))

Theorem 1.5. *Assume that f is a Lipschitz function satisfying (1.12) or (1.13) and that it is* **concave.** *Assume further that the initial condition is such that*

$$p_{in,m} = p_M \chi_{(0,L)} \qquad \textit{for some } L > 0 \tag{1.16}$$

and let (ρ_∞, p_∞) be a limit point of (ρ_m, p_m) with $L^\infty - *$ topology. Then the following holds:

1. *If $\int_0^{p_M} f(s)\,ds < 0$, then $p_\infty(x,t) = 0$ a.e. in $\mathbb{R}\times\mathbb{R}_+$ and $\rho_\infty(t,x) = \rho_{in}(x)e^{f(0)t}$ for all $t > 0$.*
2. *If $\int_0^{p_M} f(s)\,ds > 0$, then, with L_0 as given in Proposition 1.4,*
   (i) *If $L < L_0$ then $p_\infty(x,t) = 0$ a.e. in $\mathbb{R}\times\mathbb{R}_+$ and $\rho_\infty(t,x) = \rho_{in}(x)e^{f(0)t}$ for all $t > 0$.*
   (ii) *If $L > L_0$, then $\lim_{t\to\infty} \rho_\infty(t,x) = 1$ for all $x \in \mathbb{R}$.*

We point out that when $\int_0^{p_M} f(s)\,ds < 0$ or $\int_0^{p_M} f(s)\,ds > 0$ and $L < L_0$, the limit $(\rho_\infty, p_\infty) = (\rho_{in}(x)e^{f(0)t}, 0)$ is unique and therefore the whole sequence (ρ_m, p_m) converges. When $\alpha = 0$, we then have $\rho_\infty(t) = \rho_{in}$ for all $t > 0$, while when $\alpha > 0$, we find $\lim_{t\to\infty}\rho_\infty(t,x) = 0$ for all $x \in \mathbb{R}$. In the last case (ii), we can characterize the long time behavior of ρ_∞ (for any accumulation point), but we do not prove that the limit is unique.

This sharp transition between extinction and propagation is reminiscent of a similar result for the reaction-diffusion equation with ignition nonlinearity (see [23]).

When (1.16) holds, the initial density satisfies

$$\rho_{in,m} = \left(\frac{m}{m-1}p_M\right)^{\frac{1}{m-1}} \chi_{(0,L)} \to \chi_{(0,L)}.$$

However, we can have many different sequences of initial pressure $p_{in,m}$ (which do not satisfy (1.16)) for which we also have $\rho_{in,m} \to \chi_{(0,L)}$ and it is easy to see that the behavior of ρ_∞ is not uniquely determined by $\rho_{in} = \lim \rho_{in,m}$, but also depends on $p_{in} = \lim p_{in,m}$. For example, when $\int_0^{p_M} f(s)\,ds > 0$ and $L > L_0$, we can have (1.16) and thus $\lim_{t\to\infty}\rho_\infty(t,x) = 1$ as in Theorem 1.5. But we can also have $\rho_{in,m} \to \chi_{(0,L)}$ with $p_{in,m}(x) \le \alpha - \delta$. Since $\alpha - \delta$ is a supersolution for (1.1), we can easily show that any an accumulation point of $\{(\rho_m, p_m)\}_{m\ge1}$ is such that $p_\infty(x,t) = 0$ a.e. in $\mathbb{R}\times\mathbb{R}_+$ and $\lim_{t\to\infty}\rho_\infty(t,x) = 0$ for all $x \in \mathbb{R}$.

This illustrates the unstable nature of the dynamic of (1.1) when $m \gg 1$.

Finally, we note that Theorem 1.5 requires the assumption that f is concave. When f is not concave, it is not clear that we still have a sharp threshold, but we can prove the following:

Proposition 1.6. *Assume that f is a Lipschitz function satisfying (1.12) or (1.13) and that the initial condition satisfies (1.7) and*

$$p_{in,m} \le p_M \chi_{(0,L)} \quad \textit{for some } L < L^* := \frac{\pi}{\sqrt{K}}$$

(with K defined by (1.14)). Then any accumulation point (ρ_∞, p_∞) of (ρ_m, p_m) (for the $L^\infty - *$ topology) satisfies

$$p_\infty(x,t) = 0 \quad \text{and} \quad \rho_\infty(x,t) = \rho_{in}(x)e^{f(0)t} \ \text{a.e. in } \mathbb{R}\times\mathbb{R}_+.$$

1.7. **Traveling Wave Solutions.** Finally, we address a classical question with any reaction-diffusion equation: the existence and behavior of traveling wave solutions joining the two equilibrium $\rho = 0$ (and $p = 0$) and $\rho_m^+ := \left(\frac{m-1}{m}p_M\right)^{\frac{1}{m-1}}$ (corresponding to $p = p_M$). Traveling wave solutions of the reaction-diffusion equation (1.1) are global in time solutions of the form $\rho(x,t) = \bar\rho(x-ct)$ with $\bar\rho \in C(\mathbb{R})$ satisfying

$$-c\bar\rho' = (\bar\rho\bar p')' + \bar\rho f(\bar p), \qquad \text{where } \bar p = \frac{m}{m-1}\bar\rho^{m-1} \qquad \text{in } \mathbb{R} \tag{1.17}$$

together with the limit conditions

$$\begin{cases} \lim_{y\to-\infty}\bar\rho(y) = \rho_m^+ \\ \lim_{y\to+\infty}\bar\rho(y) = 0. \end{cases} \tag{1.18}$$

Existence and uniqueness of traveling waves for this degenerate diffusion-reaction equation have been extensively studied, in particular, by Gilding and Kersner in [11]. The results depend strongly on the nonlinearity $\rho \mapsto \rho f\left(\frac{m}{m-1}\rho^{m-1}\right)$ and we summarize in the two theorems below the properties that are relevant to our analysis:

Theorem 1.7. *[Traveling wave of the m-equation - Bistable case] Assume that f satisfies (1.12). For all $m \geq 1$, there exists a unique (up to translation) solution $(\bar\rho_m, c_m^*)$ of (1.17)-(1.18). $\bar\rho_m$ is monotone decreasing and the velocity c_m^* has the same sign as*

$$\int_0^{\rho_m^+} \rho^m f\left(\frac{m}{m-1}\rho^{m-1}\right) d\rho.$$

Furthermore $\rho_m^+ - \bar\rho_m(x)$ and its derivative up to order 2 decay exponentially fast at $-\infty$ and
(i) *If $c_m^* < 0$ then $\operatorname{Supp}\bar\rho_m = \mathbb{R}$ and $\bar\rho_m(x)$ and its derivative up to order 2 decay exponentially fast at $+\infty$.*
(ii) *If $c_m^* \geq 0$ (and $m > 1$), then (up to a translation) $\operatorname{Supp}\bar\rho_m = (-\infty, 0)$ and when $c_m^* > 0$, we have $\lim_{x\to0^-}\bar p_m'(x) = -c_m^*$.*

We refer to [11] for the proof of this result. In particular, the existence of a unique traveling wave in our framework follows from Theorem 41 and Corollary 41.1. The support properties of the traveling wave follow from Theorem 45 and the limit $\lim_{x\to0^-}\bar p_m'(x) = -c_m^*$ is proved in Theorem 46. We point out that

$$\int_0^{\rho_m^+} \rho^m f\left(\frac{m}{m-1}\rho^{m-1}\right) d\rho = \int_0^{p_M} \left(\frac{m-1}{m}p\right)^{\frac{2}{m-1}} f(p)\,\frac{1}{m}dp$$

which has the same sign, for m large enough, as $\int_0^{p_M} f(p)\,dp$.

Theorem 1.8. *[Traveling wave of the m-equation - Monostable case] Assume that f satisfies (1.13). For all $m \geq 1$ there exists c_m^* such that (1.17)-(1.18) has a unique traveling wave with speed c for all $c \geq c_m^*$, and no traveling waves exist when $c < c_m^*$. For $c \geq c_m^*$, $\bar\rho_m$ is monotone decreasing and $\rho_m^+ - \bar\rho_m(x)$ and its derivative up to order 2 decay exponentially fast at $-\infty$. Moreover*
(i) *If $c = c_m^*$, then $\operatorname{Supp}\bar\rho_m = (-\infty, 0)$ and $\lim_{x\to0^-}\bar p_m'(x) = -c_m^*$.*
(ii) *If $c > c_m^*$, then $\operatorname{Supp}\bar\rho_m = \mathbb{R}$.*

The existence of a unique traveling wave for $c \geq c^*$ follows from [11, Theorem 33]. The support property of the traveling waves (points (i) and (ii)) follows from Theorem 38 in that same paper.

Next, we consider the limit equation (1.4)-(1.5). As before, traveling waves are solutions of (1.4)-(1.5) of the form $\rho(x,t) = \bar\rho(x-ct)$. In particular, $\bar\rho$ must solve

$$-c\bar\rho' = \bar p'' + \bar\rho f(\bar p), \quad \bar p \in P_\infty(\bar\rho),$$

Furthermore, a traveling wave should connect equilibrium points, that is solutions of $\bar\rho f(\bar p) = 0$, $\bar p \in P_\infty(\bar\rho)$. We note that

$$\bar\rho f(\bar p) = \begin{cases} f(\bar p) & \text{if } \bar\rho = 1 \\ \bar\rho f(0) & \text{if } \bar\rho < 1 \end{cases}$$

When f satisfies (1.12) (bistable), the only solutions are $(\bar\rho, \bar p) = (0,0)$ and $(\bar\rho, \bar p) = (1,\alpha)$ or $(1, p_M)$. Since $(1,\alpha)$ is unstable, we will be looking for traveling waves connecting $(0,0)$ and $(1,p_M)$. When f satisfies (1.13) (monostable), we have $f(0) = 0$ so any $(\bar\rho, \bar p) = (\ell, 0)$ is an equilibrium point for all $\ell \in [0,1)$. We will be looking for traveling waves connecting $(\ell, 0)$ and $(1, p_M)$. Summarizing, we adopt the following definition:

Definition 1.9. *We say that $(\bar\rho(x), \bar p(x))$ is a traveling wave of (1.4)-(1.5) if $x \mapsto \bar p(x)$ and $x \mapsto \bar\rho(x)$ are non-increasing and there exists c such that*

$$-c\bar\rho' = \bar p'' + \bar\rho f(\bar p), \quad \bar p \in P_\infty(\bar\rho) \quad \textit{in the sense of distribution} \tag{1.19}$$

with the boundary conditions

$$\begin{cases} \lim_{x\to-\infty} \bar\rho(x) = 1 \\ \lim_{x\to+\infty} \bar\rho(x) = \ell \end{cases} \quad \textit{and} \quad \begin{cases} \lim_{x\to-\infty} \bar p(x) = p_M \\ \lim_{x\to+\infty} \bar p(x) = 0 \end{cases} \tag{1.20}$$

with $\ell = 1$ in the bistable case and $\ell \in [0,1)$ in the monostable case.

Remark 1.10. *It is likely unnecessary to assume that $\bar p$ and $\bar\rho$ are monotone to get uniqueness of the traveling waves. We choose to assume this given our context: since we are interested in the limits of the traveling waves given by Theorems 1.7 and 1.8, the monotonicity will be an immediate consequence of the monotonicity of $\bar\rho_m$.*

Let us point out that the limiting problem has some interesting degeneracies (see Proposition 5.1). For example, if f satisfies (1.12) and $\int_0^{p_M} f(p)\,dp \geq 0$, then there exists a solution of (1.19) for all $c < 0$ with periodic $\bar p(x)$ in $(-\infty, 0)$ and monotone $\bar\rho$. This solution does not satisfy $\lim_{x\to-\infty} \bar p(x) = p_M$. Note also that when $f(0) = 0$ (monostable case) and $c = 0$, the equation (1.19) has infinitively many solutions found by taking $\bar p \equiv 0$ and any function $\bar\rho < 1$.

With this definition, we can prove the following proposition (to be compared with the classical result for the semilinear reaction-diffusion equation [10]):

Proposition 1.11. *We can classify the travelling waves of (1.4)-(1.5) as follows:*

A. *Suppose f is bistable, i.e. that it satisfies (1.12).*
   (i) *If $\int_0^{p_M} f(p)\,dp < 0$, then there exist no traveling waves of (1.4)-(1.5) (for any $c \in \mathbb{R}$).*
   (ii) *If $\int_0^{p_M} f(p)\,dp = 0$, then there exist a traveling wave of (1.4)-(1.5) for all $c \leq 0$. If $c < 0$ then $\operatorname{Supp} \bar\rho = \mathbb{R}$, while for $c = 0$ we have $\operatorname{Supp} \bar\rho = (-\infty, 0)$. In both cases, $\ell = 0$.*
   (iii) *If $\int_0^{p_M} f(p)\,dp > 0$, then there exists a unique (up to translation) traveling wave of (1.4)-(1.5). It satisfies (1.20) with $\ell = 0$ and its speed is given by*

$$c^* := \left(2\int_0^{p_M} f(p)\,dp\right)^{1/2}.$$

B. *Suppose f is monostable, i.e. that it satisfies (1.13). Then for all $\ell \in [0,1)$ there exists a unique (up to translation) traveling wave with non-zero speed satisfying (1.20). Its speed is given by*

$$c(\ell) = \frac{1}{1-\ell}\left(2\int_0^{p_M} f(p)\,dp\right)^{1/2} \geq c^*.$$

We will also show that, up to translation, the traveling waves are given by $\bar\rho = \chi_{(-\infty,0)} + \ell\chi_{(0,+\infty)}$ and $\bar p$ as a solution h of

$$h'' + f(h) = 0 \text{ in } (-\infty,0), \quad h(-\infty) = p_M, \quad h(0) = 0. \tag{1.21}$$

Lastly, we consider the traveling wave $(\bar\rho_m, c_m^*)$ of (1.1) given either by Theorem 1.7 or by Theorem 1.8 with the minimal speed. We then have the following result:

Theorem 1.12.
(i) *If f satisfies (1.12) and $\int_0^{p_M} f(p)\,dp < 0$, then there exists m_0 and $\eta_0 > 0$ such that*

$$c_m^* \leq -\eta_0 m \qquad \textit{for all } m \geq m_0. \tag{1.22}$$

In particular $c_m^* \to -\infty$ and $p_m \to 0$ as $m \to \infty$.
(ii) *If f satisfies (1.12) and $\int_0^{p_M} f(p)\,dp > 0$, or if f satisfies (1.13), then $(\bar\rho_m, \bar p_m, c_m^*)$ converges to the unique traveling wave of (1.4)-(1.5) with $\ell = 0$. In particular*

$$\lim_{m\to\infty} c_m^* = \left(2\int_0^{p_M} f(p)\,dp\right)^{1/2}.$$

This theorem does not consider the case $\int_0^{p_M} f(p)\,dp = 0$. One issue in that case is that the sign of $\int_0^{\rho_m^+} \rho^m f\left(\frac{m}{m-1}\rho^{m-1}\right) d\rho$ might change as $m \to \infty$. Also missing in the statement is the behavior of the traveling waves with speed $c > c_m^*$ in the monostable case. We recall that for a given $c > c^*$, there exists a traveling wave $\bar\rho_m$ of (1.1) (for m large enough). The expectation is that, after translation, this traveling wave converges to the traveling wave of (1.4)-(1.5) with ℓ determined by the condition $c = \frac{1}{1-\ell}c^*$ (that is $\ell = 1 - \frac{c^*}{c}$ given by Proposition 1.11-B.

Outline for the rest of the paper: In the next section we briefly recall and prove some well-known fundamental properties of the porous media equation (1.1). Section 3 is then devoted to the proof of our results in the general case (namely all the theorems and propositions presented in Section 1.3 above) while sections 4 and 5 focus on the particular one-dimensional case: In Section 4 we establish the asymptotic behavior of the solution for compactly supported initial data, while Section 5 is devoted to traveling waves solutions.

2. Preliminaries: The porous media equation (1.1)

We recall here some important properties of the porous media equation (1.1) which we will use throughout the paper. We refer to [22] for the proofs of these results (and much more). First, the degenerate diffusion - reaction equation (1.1) is well-posed. Indeed we have:

(1) Let ρ_{in}^k be a sequence of smooth approximations of $\rho_{in,m}$ in $L^1(\mathbb{R}^n)$, satisfying in particular $\rho_{in}^k \geq \frac{1}{k}$. Then (1.1) has a unique smooth solution ρ^k with initial condition ρ_{in}^k.
(2) When $k \to \infty$, ρ^k converge to a solution of (1.1) locally uniformly in $L^\infty(\mathbb{R}^n \times \mathbb{R}_+)$.
(3) The solution obtained this way is in fact the unique solution of (1.1).

This last point follows from the following lemma which we prove below for the reader's sake:

Lemma 2.1. *Given $\rho_1(x,t)$ and $\rho_2(x,t)$ two solutions of (1.1), we have*

$$\int_{\mathbb{R}^n} (\rho_1-\rho_2)_+(x,t)dx \le e^{C(m+1)t}\int_{\mathbb{R}^n}(\rho_1-\rho_2)_+(x,0)dx \qquad \forall t>0,$$

where C depends on $\sup f$, $\sup f'_+$ and $\|\rho_i\|_{L^\infty}$ but is independent of m. In particular, if $\rho_1(x,0)\le \rho_2(x,0)$ then $\rho_1(x,t)\le\rho_2(x,t)$ for all $t>0$.

This lemma implies in particular the uniqueness, comparison property and L^1 stability of (1.1) for any $m>1$. Of course, when $m\gg 1$, the stability estimate blows up, which is not surprising since we will prove that when $m\to\infty$ the limit equation exhibits non-uniqueness in some cases. Nevertheless, Lemma 2.1 implies that if we consider solutions with ordered initial data, they will be ordered in the limit $m\to\infty$. We also recall that stability results that are uniform in m can be obtained when $p\mapsto f(p)$ is monotone decreasing (see [19]).

Proof. The proof is classical (see [22]): let $\beta_\delta(s)$ be a smooth approximation of $\mathrm{sign}^+(s)$ such that $\beta_\delta(s)=0$ if $s<0$, $0\le\beta_\delta(s)\le 1$ and $\beta'_\delta(s)\ge 0$ for all s. Denoting $w:=\rho_1^m-\rho_2^m$, we can write:

$$\begin{aligned}\int_{\mathbb{R}^n}(\rho_1-\rho_2)_t\beta_\delta(w)dx &= \int_{\mathbb{R}^n}\Delta w\beta_\delta(w)dx+\int_{\mathbb{R}^n}(f(p_1)\rho_1-f(p_2)\rho_2)\beta_\delta(w)dx\\ &= -\int_{\mathbb{R}^n}|\nabla w|^2\beta'_\delta(w)dx+\int_{\mathbb{R}^n}f(p_1)(\rho_1-\rho_2)\beta_\delta(w)dx+\int_{\mathbb{R}^n}(f(p_1)-f(p_2))\rho_2\beta_\delta(w)dx\\ &\le (\sup f)\int_{\mathbb{R}^n}(\rho_1-\rho_2)_+dx+\|\rho_2\|_{L^\infty}(\sup f')\int_{\mathbb{R}^n}(p_1-p_2)_+dx\\ &\le [\sup f+m\|\rho_2\|_{L^\infty}\|\rho_1\|^{m-2}_{L^\infty}(\sup f')]\int_{\mathbb{R}^n}(\rho_1-\rho_2)_+dx.\end{aligned}$$

Passing to the limit $\delta\to 0$ and using the fact that $\mathrm{sign}^+(\rho_1^m-\rho_2^m)=\mathrm{sign}^+(\rho_1-\rho_2)$, we deduce

$$\frac{d}{dt}\int_{\mathbb{R}^n}(\rho_1-\rho_2)_+dx\le C(m+1)\int_{\mathbb{R}^n}(\rho_1-\rho_2)_+dx,$$

and the result follows. $\square$

3. General nonlinearity: Proofs of the main results

3.1. **Proof of Proposition 1.1.** We start with the following lemma, which is a consequence of the comparison principle and Assumption (1.2):

Lemma 3.1. *Under the assumptions of Proposition 1.1, the functions ρ_m and p_m are uniformly bounded in $L^\infty(\mathbb{R}_+\times\mathbb{R}^n)$ and $L^\infty((0,T);L^1(\mathbb{R}^n))$ with respect to m.*

Proof. We recall that the pressure p_m solves

$$\partial_t p=(m-1)p(\Delta p+f(p))+|\nabla p|^2 \tag{3.1}$$

so the comparison principle and (1.7) gives $0\le p_m(x,t)\le p_M$ and $0\le\rho_m(x,t)\le p_M^{\frac{1}{m-1}}$. Integrating (1.1) gives

$$\frac{d}{dt}\int_{\mathbb{R}^n}\rho_m(x,t)\,dx\le\left(\sup_{p\in[0,p_M]}f(p)\right)\int_{\mathbb{R}^n}\rho_m(x,t)\,dx$$

which implies the bound for ρ_m in $L^\infty((0,T);L^1(\mathbb{R}^n))$. Since $p_m\le\frac{m}{m-1}p_M^{\frac{m-2}{m-1}}\rho_m$, the corresponding bound for p_m follows.

$\square$

Lemma 3.2. *Under the assumptions of Proposition 1.1, the pressure p_m satisfies (for all $T > 0$):*

$$\|\nabla p_m\|^2_{L^2(Q_T)} \leq \frac{m-1}{m-2}\left(\sup_{p\in[0,p_M]} f(p)\right)\|p_m\|_{L^1(Q_T)} + \frac{1}{m-2}\|p_{in,m}\|_{L^1(\mathbb{R}^n)}, \tag{3.2}$$

where $Q_T := \mathbb{R}^n \times [0,T]$, and the density ρ_m is uniformly bonded in $C^{1/2}(0,T;H^{-1}(\mathbb{R}^n))$.

Proof. Integrating the pressure equation (3.1) gives

$$\frac{d}{dt}\int_{\mathbb{R}^n} p_m(x,t)\,dx = -(m-2)\int_{\mathbb{R}^n} |\nabla p_m|^2\,dx + (m-1)\int_{\mathbb{R}^n} p_m f(p_m)\,dx$$

and the bound on p_m follows by integrating with respect to $t \in [0,T]$. Next, the density equation (1.1) gives, for any $\varphi \in C_c^\infty(\mathbb{R}^n)$

$$\frac{d}{dt}\int_{\mathbb{R}^n} \rho_m\varphi dx = \int_{\mathbb{R}^n} \rho_m \nabla p_m\cdot\nabla\varphi dx + \int_{\mathbb{R}^n} f(p_m)\rho_m\varphi dx.$$

It follows that for any $0 \leq a < b \leq T$, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^n}[\rho_m(\cdot,b)-\rho_m(\cdot,a)]\varphi dx\right| &\leq \|\rho_m\|_{L^\infty(Q_T)}\|\nabla p_m\|_{L^2(Q_T)}(b-a)^{1/2}\|\nabla\varphi\|_{L^2(\mathbb{R}^n)}\\ &\quad + (b-a)^{1/2}\|f(p_m)\rho_m^{1/2}\|_{L^\infty(Q_T)}\|\rho_m\|^{1/2}_{L^1(Q_T)}\|\varphi\|_{L^2(\mathbb{R}^n)},\\ &\leq C|b-a|^{1/2}\|\varphi\|_{H^1(\mathbb{R}^n)},\end{aligned}$$

and the result follows. □

Lemma 3.1 and 3.2 immediately imply the statements (i) and (ii) in Proposition 1.1. It only remains to prove (iii), namely the fact that any limit (ρ_∞, p_∞) satisfies the **Hele-Shaw graph condition**

$$p_\infty \in P_\infty(\rho_\infty).$$

Classically this condition is obtained by the relation $p_m(1-\rho_m) = p_m - \left(\frac{m-1}{m}\right)^{\frac{1}{m-1}} p_m^{\frac{m}{m-1}}$, since the right hand side converges to zero as $m \to \infty$. But passing to the limit in the product $p_m\rho_m$ requires some extra care. In [19, 12] this was easily done since the BV bounds provided some strong convergence. We do not have such strong convergence here, but we note that Lemma 3.2 implies some time regularity for ρ_m and space regularity for p_m. This means that some space-time compensated compactness type argument could yield the convergence of $p_m\rho_m$ to $p_\infty\rho_\infty$ in $\mathcal{D}'$. We present here a short proof which was first proposed in [21]:

Lemma 3.3. *Under the conditions of Proposition 1.1, any accumulation point (ρ_∞, p_∞) of (ρ_m, p_m) satisfies $0 \leq \rho_\infty \leq 1$ and*

$$\int_{\mathbb{R}^n} p_\infty(x,t)(1-\rho_\infty(x,t))dx = 0 \qquad \textit{for a.e. } t > 0.$$

Proof. Throughout the proof, we fix a subsequence such that (ρ_m, p_m) converge to (ρ, p) weak-* in L^∞. Since $p_m \leq p_M$ we have

$$\liminf_{m\to\infty}\int_{\mathbb{R}^n} p_m(1-\rho_m)\,dx \geq \liminf_{m\to\infty}\int_{\mathbb{R}^n} p_m\left(1-\left(\frac{m-1}{m}p_M\right)^{\frac{1}{m-1}}\right)dx \geq 0$$

and using the fact that $x^{\frac{m}{m-1}} \geq \frac{m}{m-1}x - \frac{1}{m-1}$, we can see that

$$\limsup_{m\to\infty}\int_{\mathbb{R}^n} p_m(1-\rho_m)\,dx = \limsup_{m\to\infty}\int_{\mathbb{R}^n}\left[p_m - \left(\frac{m-1}{m}\right)^{\frac{1}{m-1}} p_m^{\frac{m}{m-1}}\right]dx \leq 0.$$

Hence

$$\int_{\mathbb{R}^n} p_m(1-\rho_m)\,dx \to 0 \qquad \text{as } m\to\infty$$

so the proof requires us to show that $p_m\rho_m$ converges (at least in the sense of distribution) to $p\rho$. We now define the time-averaged functions

$$p_m^{a,b}(x) := \frac{1}{b-a}\int_a^b p_m(x,t)dt \qquad \text{and} \qquad p^{a,b}(x) := \frac{1}{b-a}\int_a^b p(x,t)dt$$

and note that the bound (3.2) implies that $p_m^{a,b}$ converges to $p^{a,b}$ strongly in $L^2(\mathbb{R}^n)$. We then write

$$\begin{aligned}\frac{1}{a-b}\int_a^b\int_{\mathbb{R}^n} p_m(1-\rho_m)dxdt &= \frac{1}{b-a}\int_a^b\int_{\mathbb{R}^n} p_m(x,t)(1-\rho_m(x,a))dxdt\\ &\quad+\frac{1}{b-a}\int_a^b\int_{\mathbb{R}^n} p_m(x,t)(\rho_m(x,a)-\rho_m(x,t))dxdt.\end{aligned}$$

The first integral in the right side can also be written as $\int p_m^{a,b}(x)(1-\rho_m(x,a))dx$ and thus converges to $\int p^{a,b}(x)(1-\rho(x,a))dx$ when $m\to\infty$.. As for the second integral, we use the uniform density bound in $C^{1/2}([0,T];H^{-1}(\mathbb{R}^n))$:

$$\begin{aligned}\int_a^b\int_{\mathbb{R}^n} p_m(x,t)(\rho_m(x,a)-\rho_m(x,t))dx\,dt &\leq \int_a^b \|\nabla p_m(\cdot,t)\|_{L^2(\mathbb{R}^n)}\|\rho_m(\cdot,a)-\rho_m(\cdot,t)\|_{H^{-1}(\mathbb{R}^n)}dt\\ &\leq \left(\int_a^b \|\nabla p_m(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}dt\right)^{1/2}\left(\int_a^b \|\rho_m(\cdot,a)-\rho_m(\cdot,t)\|^2_{H^{-1}(\mathbb{R}^n)}\,dt\right)^{1/2}\\ &\leq C\left(\int_a^b \|\nabla p_m(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}dt\right)^{1/2}\left(\int_a^b (t-a)\,dt\right)^{1/2}\\ &\leq C\left(\int_a^b \|\nabla p_m(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}dt\right)^{1/2}(b-a)\end{aligned}$$

where C is independent of m. Since $f_m(t) := \|\nabla p_m(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}$ is uniformly bounded in $L^1([0,T])$, along a subsequence $f_m(t)$ weakly converges in L^1 to a measure μ such that $\mu([0,T])<\infty$. Hence

$$\begin{aligned}\limsup_{m\to\infty}\frac{1}{b-a}\int_a^b\int_{\mathbb{R}^n} p_m(x,t)(\rho_m(x,a)-\rho_m(x,t))dxdt &\leq C\lim_{m\to\infty}\left(\int_a^b \|\nabla p_m(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}dt\right)^{1/2}\\ &\leq C(\mu[a,b])^{1/2}.\end{aligned}$$

We have thus proved:

$$\begin{aligned}0 &= \lim_{m\to\infty}\frac{1}{a-b}\int_a^b\int_{\mathbb{R}^n} p_m(1-\rho_m)dxdt\\ &= \frac{1}{b-a}\int_{\mathbb{R}^n} p^{a,b}(x,t)(1-\rho(x,a))dxdt + O(\mu[a,b])^{1/2}.\end{aligned}$$

Now we fix a and send b to a. Since $\int_0^T d\mu<\infty$, $\mu[a,b]$ tends to zero as $b\to a$ expect for possibly countably many a. On the other hand we have

$$\int p^{a,b}(x)(1-\rho(x,a))dx \to \int p(x,a)(1-\rho(x,a))dx$$

at Lebesgue points of $F(t) := \int p(x,t)(1-\rho(x,t))dx$. The result follows. $\square$

Remark 3.4. *The proof shows in particular that $\rho_m p_m$ converges to ρp in the sense of distribution. We note that $|\nabla f(p_m)| = |f'(p_m)\nabla p_m| \leq \mathrm{Lip} f |\nabla p_m|$ and so $f(p_m)$ is bounded in $L^2(0,T;H^1(\mathbb{R}^n))$ just like p_m. So if we assume that*

$$f(p_m) \rightharpoonup g \qquad \text{weak-* in } L^\infty(\mathbb{R}^n \times \mathbb{R}_+)$$

then the same argument shows that

$$\rho_m f(p_m) \longrightarrow \rho g \quad \text{ in } \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}_+).$$

3.2. **Proof of Proposition 1.2.** Using Remark 3.4 we can prove the following proposition, which is the first part of Proposition 1.2:

Proposition 3.5. *Under the assumptions of Proposition 1.1 and for a subsequence such that (1.8) holds, the limit (ρ_∞, p_∞) satisfies*

$$\begin{cases} \partial_t \rho_\infty = \Delta p_\infty + \rho_\infty g & \text{in } \mathbb{R}^n \times \mathbb{R}_+ \\ \rho_\infty(x,0) = \rho_{in}(x) & \text{in } \mathbb{R}^n \end{cases} \tag{3.3}$$

in the sense of distribution.

Proof. We can rewrite equation (1.1) as

$$\partial_t \rho_m = \Delta \rho_m^m + \rho_m f(p_m)$$

where, due to $\rho_m^m = \left(\frac{m-1}{m} p_m\right)^{\frac{m}{m-1}}$ converges to p_∞ in $L^\infty(\mathbb{R}^n \times \mathbb{R}^+)$ (since p_m is bounded uniformly by p_M). Remark 3.4 gives the convergence of $\rho_m f(p_m)$ to $\rho_\infty g$ and the result follows. □

The fact that $g \geq f(p_\infty)$ if f is convex (and the opposite inequality when f is concave) is classical: We write $f(p) = \sup_{a\in\mathbb{R}}\{ap - f^*(a)\}$ with f^* the Legendre transform of f. This implies $f(p_m(x,t)) \geq a p_m(x,t) - f^*(a)$ and so $g(x,t) \geq a p_\infty(x,t) - f^*(a)$ for all $a \in \mathbb{R}$. The result follows by taking the supremum over $a \in \mathbb{R}$. In particular, when $f(p) = ap + b$ is linear we get that any accumulation point (ρ_∞, p_∞) is a solution of

$$\partial_t \rho = \Delta p + \rho f(p), \quad \rho \leq 1, \ \int p(\cdot,t)(1-\rho(\cdot,t))dx = 0 \text{ a.e. } t.$$

Thanks to Lemma 3.3, to complete the proof of Proposition 1.2, it remains to show that $p_\infty(\cdot,t)$ solves (1.10) for a.e. $t > 0$. With the lack of BV estimates on ρ_∞ we cannot proceed as in [12] to characterize pressure at all times as solutions of obstacle problems. But still it is possible to show that the pressure equation holds for a.e. time by proceeding as in [16]:

Proposition 3.6. *Under the assumptions of Proposition 3.5, for a e. $t > 0$, $q = p_\infty(\cdot,t)$ solves*

$$\int_{\mathbb{R}^n} \nabla q \cdot \nabla \phi - g\phi \, dx = 0$$

for all $\phi \in H^1(\mathbb{R}^n)$ such that $\phi(x)(1-\rho_\infty(t,x)) = 0$ a.e. $x \in \mathbb{R}^n$. In particular, p_∞ satisfies

$$-\Delta p_\infty = g \qquad \text{in} \quad \mathrm{Int}(\{\rho_\infty(\cdot,t) = 1\}) \quad \text{for a.e. } t > 0. \tag{3.4}$$

Proof. We write (p,ρ) for (p_∞, ρ_∞) in this proof. For given $t_0 > 0$, let us define the functional space

$$H^1_{\rho(t_0)} := \{\phi \in H^1(\mathbb{R}^n), \phi(1-\rho(\cdot,t_0)) = 0 \text{ a.e. in } \mathbb{R}^m\}.$$

We choose $\phi \in H^1_{\rho(t_0)}$ such that $\phi \geq 0$. Since (ρ, p) solves $\rho_t - \Delta p = g\rho$ in the weak sense with $p, g \in L^2([0,T]; H^1_0(\mathbb{R}^n))$ for any $T > 0$,

$$\int_{t_0}^{t_0+h} \int \nabla\phi(x) \cdot \nabla p(x,t) - g(x,t)\rho(x,t)\, dx\, dt = \int [\rho(x,t_0) - \rho(x,t_0+h)]\phi(x) dx \text{ for any } h > 0.$$

Since $\phi \in H^1_{\rho(t_0)}$, we have $\phi(x)\rho(x,t_0) = \phi(x)$ a.e. and since $\rho(x,t) \leq 1$ and $\phi(x) \geq 0$ a.e., we deduce

$$\frac{1}{h} \int_{t_0}^{t_0+h} \int \nabla\phi(x) \cdot \nabla p(x,t) - g(x,t)\rho(x,t)\, dx\, dt \leq 0 \text{ for any } h > 0.$$

Finally, we note that since $H^1(\mathbb{R}^n)$ is separable, Lebesgue's differentiation theorem implies that a.e. $t_0 > 0$ is a differentiation point for the function $t \mapsto \int \nabla\phi(x) \cdot \nabla p(x,t)$ for all $\phi \in H^1(\mathbb{R}^n)$. We deduce that for a.e. $t_0 > 0$ (independent of ϕ) we have

$$\int \nabla\phi(x) \cdot \nabla p(x,t_0) dx \leq \int g(x,t_0)\rho(x,t_0) dx. \tag{3.5}$$

Hence we have $-\Delta p \leq g$ in the interior of the set $\{\rho_\infty(\cdot,t) = 1\})$ a.e. t.

We prove the reverse inequality (still with $\phi \geq 0$) by arguing similarly by integrating from $t_0 - h$ to t_0. Once we have equality in (3.5) for $\phi \geq 0$, it is easy to show that it holds for all ϕ.

□

3.3. **Proof of Proposition 1.3.** For general f, it is still possible to recover $g = f(p_\infty)$ for solutions with time monotonicity. This is the result of Proposition 1.3 which we prove now:

Proof of Proposition 1.3. Suppose initial pressure $p_{in,m}$ satisfy

$$\Delta p_{in,m} + f(p_{in,m})\chi_{\{p_{in,m}>0\}} \geq 0$$

for sufficiently large m. Then $h(x,t) := p_{in,m}(x)$ is a subsolution of the pressure equation (3.1), so the comparison principle yields $p_m(\cdot,\varepsilon) \geq p_{in,m}(x) = p_m(\cdot,0)$ for any $\varepsilon > 0$. Using the comparison principle again, we deduce that for any $\varepsilon > 0$ we have

$$p_m(\cdot, t+\varepsilon) \geq p_m(\cdot,t) \text{ for } t \geq 0,$$

or, in other words, p_m (and thus ρ_m) are non-decreasing functions. This monotonicity in time yields, together with the previous estimates obtained above, that p_m is uniformly bounded in $W^{1,1}_{loc}(\mathbb{R}^n \times (0,T))$. Indeed, since $\partial_t p_m \geq 0$ we can write

$$\int_0^T \int_{\mathbb{R}^n} |\partial_t p_m|\, dx\, dt = \int_0^T \int_{\mathbb{R}^n} \partial_t p_m\, dx\, dt \leq \int_{\mathbb{R}^n} p_m(x,T)\, dx$$

and Lemma 3.1 implies that $\partial_t p_m$ (and similarly $\partial_t \rho_m$) is bounded in $L^1(\mathbb{R}^n \times (0,T))$. Using Lemma 3.2, we deduce that p_m is uniformly bounded in $W^{1,1}_{loc}(\mathbb{R}^n \times (0,T))$.

Hence it follows that along subsequences we have strong convergence of p_m in $L^1_{loc}(\mathbb{R}^n \times \mathbb{R}_+)$ and almost everywhere convergence. In particular, we can assume that $g = f(p_\infty)$ in Proposition 3.5.

□

4. The bistable case in dimension 1: Invasion vs. Extinction

For the remainder of the paper, we assume that $n = 1$ and that the nonlinearity f satisfies (1.12) or (1.13). We denote

$$F(p) = \int_0^p f(s)\, ds.$$

4.1. **Preliminaries: Proof of Proposition 1.4.** In this section, we investigate the properties of the elliptic equation (1.15), which we recall here for convenience:

$$
\begin{cases} -u'' = f(u), \quad u \geq 0 \quad \text{ in } (0,L) \\ u(0) = u(L) = 0. \end{cases} \tag{4.1}
$$

Proof of Proposition 1.4. We first make a couple of simple and classical remarks about (4.1): Multiplying (4.1) by u' and integrating, we immediately see that any solution of (4.1) satisfies:

$$
\frac{1}{2}|u'(x)|^2 + F(u(x)) = \frac{1}{2}|u'(0)|^2, \quad \forall x \in [0,L]. \tag{4.2}
$$

If u is a nontrivial solution of (4.1), there exists x_0 such that $u(x_0) = \sup_{x\in(0,L)} u(x) > 0$. We have $-u''(x_0) = f(u(x_0)) \geq 0$ which gives in particular $u(x_0) \in [\alpha, p_M]$ and $u'(x_0) = 0$ which implies $\frac{1}{2}|u'(0)|^2 = F(u(x_0)) \leq F(p_M)$. When $\alpha > 0$, we cannot have $u(x_0) = \alpha$ since $F(\alpha) < 0$ and if $u(x_0) = p_M$, then $u'(x_0) = u''(x_0) = 0$ so (4.1) (and the fact that f is Lipschitz) implies that $u \equiv p_M$, a contradiction. We deduce

$$
u(x_0) \in (\alpha, p_M), \qquad u'(0) \in (0, \sqrt{2F(p_M)}).
$$

Finally, when $\alpha > 0$, we observe that if $x_1 \in (0,L)$ is such that $u(x_1) = 0$, then (4.1) implies $-u''(x_1) = f(0) < 0$ and so x_1 is a strict minimum of u. In particular, while we cannot claim that $u > 0$ in $(0,L)$, we see that the zeroes of u are isolated in $(0,L)$ (when $\alpha = 0$, the fact that f is Lipschitz implies that either $u \equiv 0$ in $(0,L)$ or $u > 0$ in $(0,L)$).

When $F(p_M) < 0$, we have $F(p) < 0$ for all $p \in (0, p_M]$ so (4.2) gives $\frac{1}{2}|u'(x)|^2 \geq -F(u(x)) > 0$ for all x, which contradicts the existence of x_0 such that $u'(x_0) = 0$. Hence no solution exists in that case.

Next, we show that when $F(p_M) > 0$, a non-trivial solution exists at least for some L: Given $\gamma \in (0, \sqrt{2F(p_M)})$, let

$$
G(s) := \int_0^s \frac{1}{\sqrt{\gamma^2 - 2F(t)}}\, dt, \qquad s \in (0, s_0)
$$

where $s_0 \in (\alpha, p_M)$ is such that $F(s_0) = \frac{1}{2}\gamma^2$. We see that G is a monotone increasing function such that

$$
G(0) = 0, \quad G(s_0) = b_0 \in (0,\infty), \quad G'(0) = \frac{1}{\gamma}, \quad G'(s_0) = +\infty
$$

(the fact that $b_0 < \infty$ follows from the fact that $f(s_0) \neq 0$ and $\gamma^2 - 2F(t) = 2(F(s_0) - F(t) = 2f(s_0)(s_0 - t) + o(s_0 - t))$. and

$$
G'(s) > \frac{1}{\sqrt{2(F(p_M) - F(\alpha))}} \qquad \text{for } s \in (0, s_0).
$$

The function

$$
u(x) := \begin{cases} G^{-1}(x) & \text{for } x \in [0, b_0] \\ G^{-1}(2b_0 - x) & \text{for } x \in (b_0, 2b_0] \end{cases}
$$

then solves (4.1) with $L = 2b_0$. We now introduce the set

$$
S = \{L\,;\ (4.1) \text{ has at least one non-trivial solution in } (0,L)\}.
$$

The construction above shows that S is non empty and Proposition 1.4 will follow if we show that S is an interval of the form (L_0, ∞) or $[L_0, \infty)$ with $L_0 > 0$.

Given $L_1 \in S$ and u_1 solving (4.1) in $(0, L_1)$, we construct a subsolution of (4.1) in $(0, L)$ for $L > L_1$ by using the translation invariance of the equation. First, we note that the extension of u_1 by 0 (still denoted u_1) solves

$$-u_1'' = f(u_1) \qquad \text{in } \{u_1 > 0\}$$

and so do its translations $u_1(x - y)$ for all $y \in \mathbb{R}$. We now define

$$v_1(x) := \sup\{u_1(x - y)\,;\, y \in [0, L - L_1]\} \tag{4.3}$$

which satisfies

$$-v_1'' \le f(v_1) \qquad \text{in } \{v_1 > 0\}$$

and $\{v_1 > 0\} = (0, L)$ because u_1 is non-negative in $(0, L_1)$ with isolated zeroes.

Next, consider v_2 such that

$$-v_2'' = M, \text{ in } (0, L), v_2(0) = v_2(L) = 0.$$

It is a supersolution for (4.1) if $M \ge \sup f$ and satisfies $v_2 \ge v_1$ if M is large enough. The existence of a non trivial solution of (4.1) satisfying $v_1 \le u \le v_2$ now follows by Perron's principle.

Finally, it only remains to show that $L_0 = \inf\{L\,;\, L \in S\}$ satisfies $L_0 > 0$. For this, we assume that u solves (4.1) in $(0, L)$ for some $L > 0$. Then

$$\int_0^L |u'|^2\,dx = -\int_0^L uu''\,dx = \int_0^L uf(u)\,dx \le K\int_0^L u^2 dx$$

with $K = \sup_{p>0} \frac{f(p)}{p} < \infty$. Poincaré inequality gives

$$\int_0^L u^2 dx \le \left(\frac{L}{\pi}\right)^2 \int_0^L |u'|^2\,dx.$$

Together with the previous inequality, we get

$$\int_0^L |u'|^2\,dx \le \left(\frac{L}{\pi}\right)^2 K\int_0^L |u'|^2\,dx.$$

If u is not the trivial solution, we must have $\left(\frac{L}{\pi}\right)^2 K \ge 1$, that is $L \ge \frac{\pi}{\sqrt{K}}$. Furthermore, if $\left(\frac{L}{\pi}\right)^2 K = 1$ then the inequalities above must be equalities and so $f(u(x)) = K(u(x))$ for all x and so $f(p) = Kp$ for $p \in [0, \sup u]$.

$\square$

Remark 4.1. *The non-existence parts of this result can be extended to subsolutions, that is to functions satisfying*

$$\begin{cases} -u'' \le f(u) & \text{in } (0, L) \\ u(0) = u(L) = 0. \end{cases}$$

Indeed in that case (4.2) becomes $\frac{1}{2}|u'(x)|^2 \ge -F(u(x)) + \frac{1}{2}|u'(0)|^2$ which is always positive if $F(p_M) < 0$, and this contradicts the existence of x_0 such that $u(x_0) = \sup_{(0,L)} u$.

When $F(p_M) > 0$ and $L < L_0$, if a subsolution u exists in $(0, L)$ then we can use the supersolution v_2 as in the proof above (with M very large so that $v_2 > u$) and find a solution w of (4.1) in $(0, L)$ satisfying $u \le w \le v_2$, which contradiction the definition of L_0.

Remark 4.2. *When f is a bistable nonlinearity ($\alpha > 0$), (4.1) can have a solution with $u'(0) = u'(L) = 0$. This solution can be found by solving*

$$-u'' = f(u), \quad u(0) = 0, \quad u'(0) = 0.$$

Since f is Lipschitz, this second order initial value problem has a unique solution, which satisfies $\frac{1}{2}|u'(x)|^2 + F(u(x)) = 0$. It is not difficult to show that u will be increasing until it reaches the value β such that $F(\beta) = 0$ and then decreasing until it goes back to zero and repeats that process periodically. If we denote by L_c the first time that u reaches back to zero, we reduce that (4.1) has solutions satisfying $u'(0) = u'(L) = 0$ if and only if $L = kL_c$ for some $k \in \mathbb{N}$. On the other hand, the construction given in the proof of Proposition 1.4 above provides a solution u with nonzero slope on $\partial(0, L)$. Indeed, u_1, and thus v_1 given by (4.3), has nonzero slope at $x = 0$, L and since $v_1 \leq u$ with the same support $(0, L)$, the same property holds for u. It follows that there are at least two solutions of (4.1) when $L = kL_c$ for $k \in \mathbb{N}$ sufficiently large.

4.2. **Proof of Theorem 1.5 (for f concave). Extinction:** First, we prove Theorem 1.5 in the cases where we observe extinction.

We recall that $p_\infty \in L^2(0, T, H^1(\mathbb{R}))$, and so $x \to p_\infty(x, t)$ is in $H^1(\mathbb{R})$ and therefore continuous for a.e. $t > 0$. For such t, we assume that the open set $\{p_\infty(\cdot, t) > 0\}$ is not empty. It can then be written as the union of its connected components (a_i, b_i) and Proposition 1.2 together with the assumption that f is concave implies that (for a.e. $t > 0$)

$$-p''_\infty = g \leq f(p_\infty) \text{ in } (a_i, b_i), \qquad p_\infty > 0 \text{ in } (a_i, b_i).$$

When $\int_0^{p_M} f(s)\, ds < 0$, Proposition 1.4 gives a contradiction (see Remark 4.1). It follows that $p_\infty(x, t) = 0$ for all $x \in \mathbb{R}$ and a.e. $t > 0$. Equation (1.9) and the concavity of f then implies

$$\partial_t \rho_\infty = \rho_\infty g \leq \rho_\infty f(p_\infty) = \rho_\infty f(0)$$

and so $\rho_\infty(x, t) \leq \rho_{in} e^{f(0)t} \to 0$ as $t \to \infty$.

When $\int_0^{p_M} f(s)\, ds > 0$ and $\rho_{in} = \chi_{(0,L)}$ with $L < L_0$, we will also get a contradiction from Proposition 1.4 if we can show that $|b_i - a_i| < L_0$ (using Remark 4.1). To show that $|b_i - a_i| < L_0$, we note that

$$\frac{d}{dt} \int_{\mathbb{R}} \rho_m(x, t)\, dx = \int_{\mathbb{R}} \rho_m f(p_m)\, dx \leq C \int_{\mathbb{R}} \rho_m(x, t)\, dx$$

and we deduce that

$$\int_{\mathbb{R}} \rho(x, t)\, dx \leq \int_{\mathbb{R}} \rho_{in}(x)\, dx + Ct < L_0 + Ct \qquad \text{a.e } t > 0.$$

In particular there exists $\eta > 0$ such that

$$\int_{\mathbb{R}} \rho(t, x)\, dx < L_0 \qquad \text{a.e. } t \in (0, \eta). \tag{4.4}$$

In view of Lemma 3.3, we can always assume that $\int p_\infty(x, t)(1 - \rho_\infty(x, t))dx = 0$ so that $\rho_\infty(x, t) = 1$ a.e. in (a_i, b_i) and (4.4) gives $|b_i - a_i| < L_0$ (note that this argument only requires $\int_{\mathbb{R}} \rho_{in}(x)\, dx < L_0$). and Proposition 1.4 again gives that $p_\infty(x, t) = 0$ $\forall x \in \mathbb{R}$, a.e. $t \in (0, \eta)$. As above, it follows that

$$\partial_t \rho_\infty = \rho_\infty g \leq \rho_\infty f(p_\infty) = \rho_\infty f(0) \text{ in } \mathbb{R} \times (0, \eta).$$

and so $\int_{\mathbb{R}} \rho(x, \eta)\, dx \leq \int_{\mathbb{R}} \rho_{in}(x)\, dx$. We can then use the same argument to prove that $p_\infty(\cdot, t) = 0$ a.e. $t \in (0, 2\eta)$. Successive iterations yield the result.

Invasion: We now prove the last part of Theorem 1.5. Given $L > L_0$, we consider $q(x)$, a solution of (4.1), which we extend to $\mathbb{R}$ by 0. Assumption (1.16) implies that $p_{in,m}(x) \geq q(x)$, so we can compare $p_m(x, t)$ with the solution of the porous media equation with initial condition $q(x)$, which is better behaved than $p_m(x, t)$ thanks to Proposition 1.3. We note that we can always assume that $q'(0) > 0$.

Indeed, (4.1) has solution with $q'(0) = 0$ only for discrete values of L (see Remark 4.2). If L happens to be one of those values, we can always take a solution $q(x)$ of (4.1) with a smaller value of L (still greater than L_0) so that $q'(0) > 0$.

We recall that q satisfies $\frac{1}{2}|q'(x)|^2 + F(q(x)) = F(M)$ with

$$M := \sup_{x\in(0,L)} q(x)$$

and $|q'(0)| = |q'(L)| = \sqrt{2F(M)} > 0$.

With this function $q(x)$, we have the following lemma, which immediately implies the last part of Theorem 1.5:

Lemma 4.3. *Let $\tilde{\rho}_m$ be the solution of (1.1) with $\tilde{p}_m(x,0) = q(x)$. Then*
(i) *For all m the functions $\tilde{\rho}_m$ and $\tilde{p}_m$ are monotone increasing in time.*
(ii) *The limit $(\tilde{\rho}_\infty, \tilde{p}_\infty)$ of $(\tilde{\rho}_m, \tilde{p}_m)$ along any convergent subsequence satisfies*

$$\partial_t \tilde{\rho}_\infty = \partial_{xx}\tilde{p}_\infty + \tilde{\rho}_\infty f(\tilde{p}_\infty), \qquad \tilde{p}_\infty \in P_\infty(\tilde{\rho}_\infty). \tag{4.5}$$

(iii) *For all $t > 0$, we have $\tilde{\rho}_\infty(\cdot,t) = \chi_{\Omega(t)}$ for some open set $\Omega(t)$ satisfying*

$$\Omega(t) \supset (-\sqrt{2F(M)}t, L + \sqrt{2F(M)}t).$$

Proof. (i) and (ii) immediately follow from Proposition 1.3 since q satisfies

$$q'' + f(q)\chi_{\{q>0\}} \geq 0 \qquad \text{in } \mathbb{R}.$$

Furthermore, since the functions are monotone increasing in time, we can use the argument in [17] to prove that $\tilde{\rho}_\infty$ is a characteristic function: let $w(x,t) := \int_0^t e^{-f(0)s}\tilde{p}_\infty(x,s)\,ds$, which solves (using (4.5))

$$\partial_{xx}w(t) = e^{-f(0)t}\tilde{\rho}_\infty(t) - \tilde{\rho}_{in} + \int_0^t e^{-f(0)s}\tilde{\rho}_\infty(s)[f(0) - f(\tilde{p}_\infty(s)]\,ds \text{ a.e. } x \text{ in } \mathbb{R}.$$

Since the right hand side is bounded in $L^\infty(\mathbb{R})$, we see that for all $t > 0$, $w(\cdot,t) \in W^{2,\infty}(\mathbb{R})$. As a consequence, the set $\Omega(t) := \{w(t) > 0\}$ is open and $\partial_{xx}w = 0$ a.e. in $\{w(t) = 0\} = \mathbb{R}\setminus\Omega(t)$ (see Remark 4.4 below). Finally we check that $\tilde{\rho}_\infty(\cdot,t) = \chi_{\Omega(t)}$. Since $\Omega(t) \subset \{\tilde{p}_\infty(t) > 0\}$, we have $\tilde{\rho}_\infty(x,t) = 1$ a.e. in $\Omega(t)$. On the other hand, for $x \in \mathbb{R}\setminus\Omega(t)$, we have $\tilde{p}_\infty(x,s) = 0$ a.e. $s \in (0,t)$ by definition of w and thus

$$0 = e^{-f(0)t}\tilde{\rho}_\infty(t) - \tilde{\rho}_{in} \text{ a.e. in } \mathbb{R}\setminus\Omega(t).$$

Due to the choice of initial data we have $\tilde{\rho}_{in} = 0$ in $\{w(t) = 0\} \subset \{\tilde{p}_{in} = 0\}$, and so it follows that $\tilde{\rho}_\infty(x,t) = 0$ a.e. in $\mathbb{R}\setminus\Omega(t)$. Summarizing, we showed that

$$\tilde{\rho}_\infty(x,t) = \chi_{\Omega(t)}, \quad \Omega(t) := \{w(t) > 0\} \tag{4.6}$$

From the Hele-Shaw condition we have $\tilde{p}_\infty(t) = 0$ a.e. in $\mathbb{R}\setminus\Omega(t)$, and thus we also have

$$|\Omega(t)\Delta\{\tilde{p}_\infty(t) > 0\}| = 0.$$

Equation (4.5), together with (4.6) gives, for all smooth test function $\phi(x)$:

$$\begin{aligned}\frac{d}{dt}\int_{\Omega(t)}\phi(x)\,dx = \frac{d}{dt}\int_{\mathbb{R}}\tilde{\rho}_\infty(x,t)\phi(x)\,dx &= \int_{\mathbb{R}}\partial_t\tilde{\rho}_\infty(x,t)\phi(x)\,dx \\ &= \int_{\mathbb{R}}\tilde{p}_\infty\partial_{xx}\phi + \tilde{\rho}_\infty f(\tilde{p}_\infty)\phi\,dx \\ &= \int_{\Omega(t)}\tilde{p}_\infty\partial_{xx}\phi + f(\tilde{p}_\infty)\phi\,dx.\end{aligned}$$

As mentioned before, since we are in one space dimension. Proposition 1.3 implies that $\tilde{p}_\infty(\cdot,t)$ solves $-\partial_{xx}\tilde{p}_\infty = f(\tilde{p}_\infty)$ on any connected components of $\Omega(t)$ and so

$$\int_{\Omega(t)} \tilde{p}_\infty \partial_{xx}\phi + f(\tilde{p}_\infty)\phi \, dx = \int_{\partial\Omega(t)} |\partial_x \tilde{p}_\infty|\phi \tag{4.7}$$

Furthermore, if $(a(t), b(t))$ is the connected component of $\{\tilde{p}_\infty(\cdot,t) > 0\}$ containing $(0, L)$ (recall that $\tilde{p}_\infty(x,t) \geq q(x)$ and so $\tilde{p}_\infty(\cdot,t) > 0$ in $(0, L)$ for all $t \geq 0$), we can write

$$\frac{1}{2}|\partial_x \tilde{p}_\infty(x)|^2 + F(\tilde{p}_\infty(x)) = F\left(\sup_{(a,b)} \tilde{p}_\infty\right) \geq F\left(\sup_{(a,b)} q\right) = F(M) \text{ for all } x \in [a, b]$$

and so

$$|\partial_x \tilde{p}_\infty(a(t))| = |\partial_x \tilde{p}_\infty(b(t))| \geq \sqrt{2F(M)}.$$

We thus have

$$\frac{d}{dt}\int_{\Omega(t)} \phi(x)\, dx \geq \sqrt{2F(M)}[\phi(a(t)) + \phi(b(t)] \qquad \text{in } \mathcal{D}'(0,\infty)$$

which implies

$$\int_{\partial\Omega(t)} V\phi(x) d\mathcal{H}^0(x) \geq \int_{\partial(a(t),b(t))} \sqrt{2F(M)}\phi(x) d\mathcal{H}^0(x)$$

where V denotes the normal velocity of $\partial\Omega(t)$ (which is a countable set of points). Since this holds for any test function ϕ, we deduce that $V \geq 0$ on $\partial\Omega(t)$ and $V \geq \sqrt{2F(M)}$ on $\partial(a(t), b(t))$. The result follows.

□

Remark 4.4. *We recall that if $u \in W^{1,1}_{loc}(\Omega)$ and if we denote $E_\alpha = \{u = \alpha\}$ for any $\alpha \in \mathbb{R}$, then $\nabla u(x) = 0$ for almost every $x \in E_\alpha$. In our setting, this classical result implies that $\partial_x w = 0$ a.e. in $\{w = 0\}$ and that $\partial_{xx} w = 0$ a.e. in $\{\partial_x w = 0\}$. The fact that $\partial_{xx} w = 0$ a.e. in $\{w = 0\}$ follows. Importantly, this argument does not require any particular properties for the boundary $\partial\{w > 0\}$.*

4.3. **Proof of Proposition 1.6 (general f).** We now assume that $p_{in,m}(x) \leq p_M \chi_{(0,L)}(x)$ with L such that

$$L < L^* = \frac{\pi}{\sqrt{K}}. \tag{4.8}$$

(with K defined by (1.14)).

The proof is similar to the first part of the proof of Theorem 1.5. First we remark that (1.14) implies $f(p_m) \leq K p_m$, and so we have $g \leq K p_\infty$, with the notations of Proposition 1.2.

Recall that $p_\infty(\cdot,t)$ is continuous for a.e. $t > 0$. For such t, we assume that the open set $\{p_\infty(\cdot,t) > 0\}$ is not empty. It can then be written as the union of its connected components (a_i, b_i) and Proposition 1.2 together with the remark above implies that, for a.e. $t > 0$, $p_\infty(\cdot,t)$ solves

$$-p_\infty'' = g \leq K p_\infty \text{ in } (a_i, b_i), \qquad p_\infty > 0 \text{ in } (a_i, b_i).$$

We can now show (as in the proof Theorem 1.5 in the extinction case) that $|b_i - a_i| < L^*$ for some small time $t \in (0, \eta)$. Proceeding as in the last part of proof for Proposition 1.4, we see that Poincaré inequality yields $p_\infty(x,t) = 0$ in (a_i, b_i). Since this holds for any connected components of $\{p_\infty(\cdot,t) > 0\}$ it follows that $p_\infty(x,t) = 0$ for a.e. $t \in (0, \eta)$.

This in turn implies that $\rho_\infty(x,t) = \rho_{in}(x)e^{f(0)t}$ for $t < \eta$, so that the support of $p_\infty(x,\cdot)$ has not grown and we can iterate to show that $p_\infty(x,t) = 0$ for all $t > 0$.

5. Traveling wave solutions

First we study the existence of traveling waves for the limiting problem and prove Proposition 1.11). We start with the pressure equation: Recalling that $F(p)=\int_0^p f(s)\,ds$, we have:

Proposition 5.1.
(i) *if f satisfies (1.13) (monostable), then the equation*

$$h''+f(h)=0,\qquad h\geq 0 \text{ in } (-\infty,0),\quad h(0)=0 \tag{5.1}$$

has a unique nonzero, bounded solution which is decreasing. This solution satisfies $h'(0)=-\sqrt{2F(p_M)}$ and $\lim_{x\to-\infty}h=p_M$.
(ii) *If f satisfies (1.12) (bistable) then*

- *if $F(p_M)>0$, then the equation (5.1) has two bounded solutions: One monotone solution as in (i) and one periodic solution with $h'(0)=0$.*
- *if $F(p_M)=0$, then (5.1) has a unique bounded solution which is decreasing. This solution satisfies $h'(0)=0$ and $\lim_{-\infty}h=p_M$.*
- *if $F(p_M)<0$, then (5.1) has no bounded solution.*

Proof of Proposition 5.1. Assume that h is a bounded solution of (5.1).

First, we note that if there is $x_0<0$ such that $h'(x_0)=0$ then uniqueness principle for second order ODE on $[0,x_0]$ with $h(x_0)$ and $h'(x_0)=0$ fixed yields that $h(x_0+s)=h(x_0-s)$ for all $s\in[0,x_0]$. This implies that $h(2x_0)=h(0)=0$ and $h'(2x_0)=-h'(0)$. Since $h\geq 0$ in $(-\infty,0)$, we must then have $h'(0)=0$. From the uniqueness principle again, it follows that $h(2x_0+s)=h(s)$, namely h is periodic. When $h'(0)\neq 0$, we deduce that h' cannot vanish in $(-\infty,0)$ and is thus always negative.

Next we show that $h<p_M$. Note that $h''\geq -f(h)>0$ if $h>p_M$, and thus h is strictly convex when it is above p_M. Hence if $h(x_1)>p_M$ for some $x_1<0$, then h must be unbounded in $(-\infty,0)$, a contradiction. If $h(x_1)=p_M$, then either h is unbounded or $h'(x_1)=0$, but in this last case, we would then have $h(x)=p_M$ for all x which contradicts the fact that $h(0)=0$.

We thus have that h is either periodic or monotone decreasing and that $0\leq h<p_M$ in $(-\infty,0)$. When h is nontrivial and monotone, $\lim_{x\to-\infty}h(x)=\ell\in(0,p_M]$ exists and satisfies $f(\ell)=0$, that is $\ell=\alpha$ or p_M.

Next let us show that h can be nontrivial and periodic only when $F(p_M)>0$ and $f(0)<0$. Multiplying (5.1) by h' and integrating, we find

$$\frac{1}{2}|h'|^2+F(h)=\frac{1}{2}|h'(0)|^2 \text{ in } (-\infty,0). \tag{5.2}$$

When h is periodic, we have $h'(0)=h'(x_0)=0$. Hence it follows that $F(h(z_0))=0$ for some $h(z_0)\in(0,p_M)$. This situation can only happen when $f(0)<0$ and $F(p_M)>0$.

Lastly, when h is monotone, (5.2) implies $F(\ell)=\frac{1}{2}|h'(0)|^2$. If $F(p_M)<0$, then this is impossible and no bounded solution exist, thus proving the last part of the proposition. If $F(p_M)\geq 0$, and since $F(\alpha)<0$, we deduce $\ell=p_M$ or $\ell=0$. The latter case yields a trivial h when $f(0)=0$. In the former case we have

$$h'(0)=-\sqrt{2F(p_M)}.$$

The uniqueness of h follows since the initial value problem $h''+f(h)=0$ with $h(0)=0$ and $h'(0)=-\sqrt{2F(p_M)}$ has a unique solution.

The existence of h when $F(p_M)\geq 0$ can now be obtained by solving

$$h''+f(h)=0 \text{ in } (-\infty,0),\quad h(0)=0 \text{ and } h'(0)=-\sqrt{2F(p_M)}.$$

To check that h is bounded, we will use (5.2). Observe that $h < p_M$: if $h(x_0) = p_M$ for some $x_0 < 0$ then $h'(x_0) = 0$, and the uniqueness principle for the ODE yields $h(x) \equiv p_M$ for all $x < 0$, a contradiction. From (5.2) and the fact that F has a maximum at p_M in $[0, p_M]$ it follows that $h'(x)$ cannot vanish, and so h is monotone decreasing and bounded above and therefore positive and bounded in $(-\infty, 0)$. □

We can now prove the existence and uniqueness of traveling wave for the limiting equation:

Proof of Proposition 1.11. We start with the following remark: Since $\bar\rho$ is non-increasing and $\bar\rho(+\infty) = \ell < 1$, there must exist $a \in [-\infty, +\infty)$ such that $\bar\rho(x) = 1$ for $x \in (-\infty, a)$ and $\bar\rho(x) < 1$ in $(a, +\infty)$. If $a = -\infty$, then $\bar\rho < 1$ and $\bar p = 0$ in $\mathbb{R}$ which contradicts (1.20). We can thus always assume that $a \in (-\infty, +\infty)$ and up to translation we will take $a = 0$.

Let us first classify possible pressure profiles. Up to a translation, any traveling wave pressure must satisfy $\bar p = 0$ in $(0, +\infty)$ and $\bar p'' + f(\bar p) = 0$ in $(-\infty, 0)$. In particular, $\bar p$ coincides on $(-\infty, 0)$ with a monotone decreasing solution of (5.1). Using Proposition 5.1, we deduce that there is no such $\bar p$ when $F(p_M) < 0$ which implies (i). On the other hand, if $F(p_M) \geq 0$ then again Proposition 5.1 yields that $\bar p$ is uniquely determined and satisfies $\bar p'(0) = -\sqrt{2F(p_M)}$.

Next we consider possible travelling wave densities. To this end, we observe that (1.19) is equivalent to

$$-c\bar\rho' = -\bar p'(0)\delta_{x=0} + \bar\rho f(0)\chi_{(0,\infty)} = \sqrt{2F(p_M)}\delta_{x=0} + \bar\rho f(0)\chi_{(0,\infty)}. \tag{5.3}$$

When $F(p_M) = 0$ and $f(0) < 0$, (5.3) yields a continuous solution

$$\bar\rho(x) = \chi_{(-\infty,0)}(x) + e^{-\frac{f(0)}{c}x}\chi_{(0,+\infty)}(x),$$

which satisfies the boundary conditions with $\ell = 0$ for all $c < 0$ and

$$\bar\rho(x) = \chi_{(-\infty,0)}(x)$$

with $c = 0$. This implies (iii).

When $F(p_M) > 0$, c is positive due to (5.3) at $x = 0$ with our assumption $\bar\rho' \leq 0$. So (5.3) again gives $\bar\rho f(0) \geq 0$ in $(0, \infty)$. When $f(0) < 0$, it follows that $\bar\rho = 0$ in $(0, +\infty)$, yielding $\ell = 0$, concluding (ii).

Lastly let us consider the monostable case $f(0) = 0$. In this case (5.3) implies $-c\bar\rho' = 0$ in $(0, +\infty)$, which yields $\bar\rho = \ell$ in $(0, +\infty)$. Since $\bar\rho = 1$ in $(-\infty, 0)$, it follows from (5.3) that $c(1-\ell) = \sqrt{2F(p_M)}$. We now conclude.

The existence of the traveling wave when $F(p_M) > 0$ is proved by taking $\bar p = h$ as the unique monotone solution of (5.1), $c = \frac{\sqrt{2F(p_M)}}{1-\ell}$ and $\rho = \chi_{(-\infty,0)} + \ell\chi_{(0,+\infty)}$ and checking that this solves (1.19) for appropriate values of ℓ. □

Proof of Theorem 1.12 (i) (Receding Traveling Waves). Let us fix $m > 1$ and denote by $(\bar\rho(x), c)$ the traveling wave of (1.1) given by Theorem 1.7, and by $\bar p(x)$ the corresponding pressure given by $\bar p(x) = \frac{m}{m-1}\bar\rho(x)^{m-1}$. Since we assume that $F(p) = \int_0^p f(u)\,du$ satisfies $F(p_M) < 0$, we have $c < 0$ for large m.

Recall from Theorem 1.7 that $\bar\rho$ and thus $\bar p$ is monotone decreasing and that $0 \leq \bar p(x) \leq p_M$. Furthermore, we claim that there exists a constant $C > 0$ independent of m such that

$$\bar p'(x)^2 \leq C\bar p(x) \leq Cp_M. \tag{5.4}$$

Indeed, since $\partial_t(\bar\rho(x-ct)) = -c\bar\rho' \leq 0$, (1.1) gives $\bar\rho'\bar p' + \bar\rho\bar p'' \leq -\bar\rho f(p)$. Since $\rho' p' \geq 0$ and $\bar\rho$ is positive, $p''(x) \leq \sup_{0\leq p\leq p_M} -f(p) \leq C$. Multiplying by $\bar p'(x)$ and integrating over (x, ∞), we find (5.4).

Next, set $\eta := -F(p_M)/(2p_M)$ and consider the function

$$g(x) = \frac{1}{2}\bar{p}'(x)^2 + F(\bar{p}(x)) + \eta\bar{p}(x).$$

We claim that there exists $\bar{x} \in \mathbb{R}$ such that

$$\begin{cases} g(\bar{x}) \leq \frac{F(p_M)}{4} \\ g'(\bar{x}) \geq 0 \\ \bar{p}(\bar{x}) \geq \eta_1 \end{cases} \tag{5.5}$$

for some small $\eta_1 > 0$. Indeed, we have $g(-\infty) = F(p_M) + \eta p_M = \frac{F(p_M)}{2} < 0$ and $g(+\infty) = 0$, which implies the existence of $\bar{x}$ satisfying the first two conditions. The first condition implies in particular

$$F(\bar{p}(\bar{x})) \leq \frac{F(p_M)}{4} - \frac{1}{2}\bar{p}'(\bar{x})^2 - \eta\bar{p}(\bar{x}) \leq \frac{F(p_M)}{4} < 0$$

which gives the third condition in (5.5). Since $g'(x) = \bar{p}'(x)(\bar{p}''(x)+f(\bar{p}(x))+\eta)$, the condition $g'(\bar{x}) \geq 0$ implies

$$\bar{p}''(\bar{x}) + f(\bar{p}(\bar{x})) \leq -\eta$$

and so

$$\bar{p}(\bar{x})(\bar{p}''(\bar{x}) + f(\bar{p}(\bar{x}))) \leq -\eta\eta_1.$$

The pressure equation now gives

$$-c\bar{p}'(\bar{x}) = (m-1)\bar{p}(\bar{x})(\bar{p}''(\bar{x}) + f(\bar{p}(\bar{x}))) + \bar{p}'(\bar{x})^2 \leq -(m-1)\eta\eta_1 + \bar{p}'(\bar{x})^2$$

that is (recall that $0 \leq -\bar{p}' \leq C$)

$$c \leq -(m-1)\frac{\eta\eta_1}{C} + C$$

and (1.22) follows.

□

Proof of Theorem 1.12 (ii) (Advancing traveling Waves). We now assume that $F(p_M) = \int_0^{p_M} f(p)\,dp > 0$. As before we denote by $\bar{\rho}_m$, $\bar{p}_m$, $c_m^* > 0$ the traveling wave given by Theorems 1.7 and 1.8 and its associated pressure and velocity for fixed m. For m large enough, we have $\int_0^{\rho_m^+} \rho^m f(p_m))\,d\rho > 0$. We also denote by $\bar{\rho}^*, \bar{p}^*, c^* > 0$ the unique traveling wave of the limiting problem given by Proposition 1.11. Finally, we write $p_m(x,t) := \bar{p}_m(x - c_m^* t)$.

Since we have uniqueness of the traveling waves for the limiting problem, it will be easy to conclude if we can show that the limit of $\bar{\rho}_m, \bar{p}_m$ solves (1.19) and are supported in $(-\infty, 0)$. In order to get (1.19), the main step is to show that c_m^* converges which requires an upper bound on c_m^*. To show that $\bar{\rho}_m, \bar{p}_m$ do not degenerate and that their limits are positive in $(-\infty, 0)$, we will need to know that c_m^* is uniformly bounded away from 0 when m is large.

We thus start the proof by proving that $c_m^* \to c^*$, by constructing appropriate barriers for the pressure equation. We recall the pressure equation, solved by $p_m(x,t)$:

$$p_t = (m-1)p(\partial_{xx}p + f(p)) + |\partial_x p|^2. \tag{5.6}$$

Set $f_\varepsilon(p) := f(p) + \varepsilon$. Since $f(p) < 0$ for $p > p_M$, if ε is small then there exists $p_M^\varepsilon > p_M$ with $p_M^\varepsilon = p_M + o(1)$ such that $f_\varepsilon(p) < 0$ for $p > p_M^\varepsilon$. Let h_ε be the monotone decreasing solution of (1.21) with f_ε instead of f, that is:

$$\begin{cases} h_\varepsilon''(x) = f_\varepsilon(h_\varepsilon(x)) & \text{for } x \in (-\infty, 0), \\ h_\varepsilon(-\infty) = p_M^\varepsilon, \quad h_\varepsilon(x) = 0 \text{ in } [0, \infty) \end{cases}$$

(we can proceed as in Proposition 5.1 to show that such an h_ε exists). We then claim that the function $p_\varepsilon(x,t) = h_\varepsilon(x - c_\varepsilon t)$ with $c_\varepsilon = |h'_\varepsilon|(0) + \varepsilon$ is a supersolution for (5.6). Indeed, we have

$$p_\varepsilon(\partial_{xx}p_\varepsilon + f(p_\varepsilon)) = p_\varepsilon(-f_\varepsilon(p_\varepsilon) + f(p_\varepsilon)) = -\varepsilon p_\varepsilon \qquad \text{in } \{p_\varepsilon > 0\} = \{x - c_\varepsilon t < 0\}$$

and

$$\begin{aligned}\partial_t p_\varepsilon(x,t) - |\partial_x p_\varepsilon|^2 &= c_\varepsilon |h'_\varepsilon|(x - c_\varepsilon t) - |h'_\varepsilon|^2(x - c_\varepsilon t)\\ &= (|h'_\varepsilon|(0)| + \varepsilon)|h'_\varepsilon|(x - c_\varepsilon t) - |h'_\varepsilon|^2(x - c_\varepsilon t)\\ &\geq 0 \qquad \text{when } -\delta \leq x - c_\varepsilon t \leq 0\end{aligned}$$

for some small δ. We thus have

$$\partial_t p_\varepsilon - (m-1)p_\varepsilon(\partial_{xx}p_\varepsilon + f(p_\varepsilon)) - |\partial_x p_\varepsilon|^2 \geq \begin{cases} 0 & \text{if } x - c_\varepsilon t \geq -\delta \\ c_\varepsilon \partial_x p_\varepsilon - |\partial_x p_\varepsilon|^2 + \varepsilon(m-1)p_\varepsilon & \text{if } x - c_\varepsilon t \leq -\delta \end{cases}$$

Since p_ε is bounded below when $x - c_\varepsilon t < -\delta$, this last term is non-negative of m is large enough.

Having verified the supersolution property of p_ε, let us now compare p_m with p_ε. Let now x_0 be such that $p_\varepsilon(x,0) \geq p_M$ for $x \leq -x_0$. Since $p_m(x,0) = 0$ for $x > 0$ (see Theorem 1.7), we have $p_m(x,0) \leq p_\varepsilon(x + x_0, 0)$ and the comparison principle implies that $p_m(x,t) \leq p_\varepsilon(x + x_0, t)$ for all $t > 0$. We deduce that

$$c_m \leq c_\varepsilon$$

for sufficiently large m and so $\limsup_{m\to\infty} c_m \leq c_\varepsilon$. Furthermore, it is easy to check that

$$c_\varepsilon = \sqrt{2F_\varepsilon(p^\varepsilon_M)} + \varepsilon \to c^* \text{ as } \varepsilon \to 0$$

hence

$$\limsup_{m\to\infty} c_m \leq c^*.$$

Similarly, we can construct a subsolution of (5.6) by using the function $f_\varepsilon(p) = f(p) - \varepsilon$ and $c_\varepsilon = |h'_\varepsilon|(0) - \varepsilon$. We then have:

$$\partial_t p_\varepsilon - (m-1)p_\varepsilon(\partial_{xx}p_\varepsilon + f(p_\varepsilon)) - |\partial_x p_\varepsilon|^2 = (|h'_\varepsilon|(0) - \varepsilon)|h'_\varepsilon|(x - c_\varepsilon t) - |h'_\varepsilon|^2(x - c_\varepsilon t) - \varepsilon(m-1)h_\varepsilon(x - c_\varepsilon t)$$

which is negative when $-\delta \leq x - c_\varepsilon t \leq 0$ if δ is such that $|h'_\varepsilon(0)| - \varepsilon \leq |h'_\varepsilon(y)|$ for $y \in (-\delta, 0)$ and negative for $x - c_\varepsilon t \leq -\delta$, provided m is large enough depending on ε and δ. Proceeding as above, we deduce that:

$$\liminf c_m \geq c^*.$$

We have thus showed that $c_m \to c^*$ and it remains to show that $\bar\rho_m \to \chi_{(0,\infty)}$ and $\bar p_m \to h$, the monotone solution of (5.1).

Since $\bar\rho_m$ and $\bar p_m$ are monotone decreasing and uniformly bounded, they are uniformly bounded in $BV(\mathbb{R})$. Hence along a subsequence $\bar\rho_m$ and $\bar p$ converge to monotone functions $\bar\rho_\infty$ and $\bar p_\infty$ strongly in L^1_{loc} and almost everywhere. Proceeding as in the proof of Proposition 1.3 we get that $\bar\rho_\infty$ and $\bar p_\infty$ solve (1.19) (with $c = c^*$). In order to conclude, it remains to show that $\bar\rho_\infty$ and $\bar p_\infty$ satisfy(1.20).

Let us first show that $p_\infty(x) > 0$ in $(-\infty, 0)$. We use the fact that $c^*_m \to c^* > 0$ and so $c^*_m \geq c^*/2 > 0$ for m large enough and that $\bar p_m(x)$ solves

$$c^*_m(-\bar p'_m) - |\bar p'_m|^2 = (m-1)\bar p_m(\bar p''_m + f(\bar p_m)) \tag{5.7}$$

In particular, we must have $(\bar p_m)'' + f(\bar p_m) \geq 0$ whenever $\bar p'_m \in (-c^*_m, 0)$, and thus $\bar p''_m > -f(\bar p_m) \geq -K\bar p_m$ for this range of $\bar p'_m$. Finally, we recall that $\bar p_m(0) = 0$ with $\bar p'_m(0^-) = -c^*_m < c^*/2$. It follows that $\bar{p}_m' < -c^*/4$ on $[-\eta, 0]$, for some η independent of m and thus $\bar{p}_m(x) > \frac{c^* x}{4}$ for $x \in [-\eta, 0)$. From the monotonicity of $\bar{p}_\infty$ implies that $\bar{p}_\infty$ is positive in $(-\infty, 0)$.

Since $\operatorname{Supp} \rho_m = (-\infty, 0)$, we have $\bar{\rho}_\infty \leq \chi_{(0,\infty)}$ and $\bar{p}_\infty = 0$ on $(0, \infty)$. Since $\bar{p}_\infty(1 - \bar{\rho}_\infty) = 0$, we deduce that $\bar{\rho}_\infty = \chi_{(0,\infty)}$. Now Proposition 5.1 implies $\bar{p}_\infty = h$, and we can conclude.

□

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