A PHASE FIELD MODEL FOR THE MOTION OF PRISMATIC DISLOCATION LOOPS BY BOTH CLIMB AND SELF-CLIMB

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Abstract. We study the sharp interface limit and the existence of weak solutions of a phase field model for climb and self-climb of prismatic dislocation loops in periodic settings. The model is set up in a Cahn-Hilliard/Allen-Cahn framework featured with degenerate phase-dependent diffusion mobility with an additional stabilizing function. Moreover, a nonlocal climb force is added to the chemical potential. We introduce a notion of weak solutions for the nonlinear model. The existence result is obtained by approximations of the proposed model with nondegenerate mobilities. Lastly, the numerical simulations are performed to validate the phase field model and the simulation results show a big difference for the prismatic dislocation loops in the evolution time and the pattern with and without self-climb contribution.

1. Introduction. Dislocations are line defects in crystalline materials. Dislocations climb is the motion of dislocations out of their slip planes with the assistance of vacancy diffusion over the bulk of the materials, and it is an important mechanism in the plastic properties of crystalline materials at high temperatures (e.g., in dislocation creep) [19, 15, 38, 39, 3, 28, 23, 16, 14, 42, 4]. The self-climb of dislocations also plays important roles in the properties of irradiated materials [19]. The self-climb motion is driven by pipe diffusion of vacancies along the dislocations, and is the dominant mechanism of prismatic loop motion and coalescence at not very high temperatures [25, 21, 10, 37, 33, 30, 29, 17, 27, 31, 4]. Both dislocation climb by vacancy bulk diffusion and self climb by vacancy pipe diffusion have been studied extensively in the literature over the years. Most of the work in the past focuses on one type of motion, either pure dislocation climb by vacancy bulk diffusion or self climb motion by vacancy pipe diffusion. Recently, there have been growing interest on the cooperative effects of the two mechanisms of climb by vacancy bulk diffusion and self-climb by vacancy pipe diffusion and [17, 4, 24].

Phase field models (e.g., of the Cahn-Hilliard type [6] or the Allen-Cahn type [1]) are a class of numerical simulation models that evolve the moving interfaces (e.g. curves in two dimensions) implicitly. Compared with the front-tracking methods that directly track the locations of the interfaces, the phase field models have the advantages of being able to handle topological changes of the interfaces automatically with simple numerical implementation on a uniform mesh of the simulation domain. Phase field models for dislocation climb by vacancy bulk diffusion have been developed in the literature [20, 22, 13, 14]. In a previous work, the author Niu and collaborators proposed a phase field model for self-climb of prismatic dislocation loops [31]. To the best of our knowledge, a phase field model that accounts for the combined effect of these two mechanisms and yields correct sharp interface limit is not available in the literature.

In this paper, we present a phase field model for the motion of prismatic dislocation loops by both climb and self-climb. We shall study the sharp interface limit and derive the existence of weak solutions (defined in section 3) for our proposed model. In addition, we present numerical results validating our phase field model. The model

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that we propose is the following modified Cahn-Hilliard Allen-Cahn type equation:

$$g(u)(\partial_t u+\beta\mu) = \nabla\cdot(M(u)\nabla\frac{\mu}{g(u)}) \text{ for } x\in\mathbb{T}^2\subset\mathbb{R}^2, t\in[0,\infty), \tag{1.1}$$

$$\mu = -\Delta u+\frac{1}{\varepsilon^2}q'(u)+\frac{1}{\varepsilon}h(u)f_{cl}. \tag{1.2}$$

In this model, the dislocation climb by vacancy bulk diffusion is incorporated by the $\beta\mu$ term. Without the $\beta\mu$ term on the left-hand side, it describes the self-climb of prismatic dislocation loops, and Here $\beta>0$ is a constant that enables a correct dislocation climb velocity, $M(u)=M_0(1-u^2)^2$, $M_0>0$, is the diffusion mobility, $q(u)=2(1-u^2)^2$ is the double well potential which takes minimums at ±1, $g(u)=(1-u^2)^2$ is the stabilizing function which guarantees correct asymptotics in the sharp interface limit, and ε is a small parameter controlling the width of the dislocation core.

In this model, we assume prismatic dislocation loops lie and evolve by self-climb in the xy plane and all dislocation loops have the same perpendicular Burgers vector $=(0,0,b)$. The local dislocation line direction is given by $(\mathbf{b}/b)\times(\nabla u/|\nabla u|)$. The last contribution f_{cl} in the chemical potential μ in (1.2) is the total climb force, with

$$f_{cl}=f_{cl}^d+f_{cl}^{app}, \tag{1.3}$$

where f_{cl}^d is the climb force generated by all the dislocations:

$$f_{cl}^d(x,y,u)=\frac{Gb^2}{8\pi(1-\nu)}\int_{\mathbb{T}^2}\left(\frac{x-\overline{x}}{R^3}u_{\overline{x}}+\frac{y-\overline{y}}{R^3}u_{\overline{y}}\right)d\overline{x}d\overline{y} \tag{1.4}$$

with G being the shear modulus, ν the Poisson ratio, and $R=\sqrt{(x-\overline{x})^2+(y-\overline{y})^2}$, and f_{cl}^{app} is the applied force. The smooth cutoff factor $h(u)=H_0(1-u^2)^2$ is to guarantee the climb force acts only on the dislocations. The constant $H_0>0$ is chosen such that the phase field model generates accurate climb force of the dislocations [31] (c.f. Sec. 2).

The chemical potential μ comes from variations of the classical Cahn-Hilliard energy and the elastic energy due to dislocations, i.e.

$$\mu=\frac{\delta E_{CH}}{\delta u}+\frac{1}{\varepsilon}h(u)\frac{\delta E_{el}}{\delta u},$$

where

$$E_{CH}(u)=\int_{\mathbb{T}^2}\left(\frac{1}{2}|\nabla u|^2+q(u)\right)dx,$$

is the classical Cahn-Hilliard energy, and

$$E_{el}=\int_{\mathbb{T}^2}\left(\frac{1}{2}uf_{cl}^d+uf_{cl}^{app}\right)dx$$

is the elastic energy. The climb force generated by the dislocations can be expressed as

$$f_{cl}^d(x,y,u)=\frac{Gb^2}{4(1-\nu)}(-\Delta)^{\frac{1}{2}}u. \tag{1.5}$$

Here $(-\Delta)^s u$ is the fractional operator defined by

$$\mathcal{F}((-\Delta)^s f) = (\xi_1^2 + \xi_2^2)^{\frac{s}{2}} \mathcal{F}(f)(\xi),$$

where ξ is the frequency.

Taking $g \equiv 1$ and $\beta = 0$ in (1.1), and omitting the nonlocal term $\frac{1}{\varepsilon} h(u) f_{cl}$ from (1.2), we arrive at the classical degenerate Cahn-Hilliard equation. The degenerate Cahn-Hilliard equation has been widely studied as a diffuse-interface model for phase separation in binary system [6, 5, 7, 8, 9, 11]. There is a critical issue in modeling surface diffusion by the degenerate Cahn-Hilliard model [18, 34] though, due to the presence of incompatibility in the asymptotic matching between the outer and inner expansions. Rätz, Ribalta, and Voigt (RRV) [34] fixed this incompatibility by introducing a singular factor $1/g(u)$ in front of the chemical potential μ to force it to vanish in the far field. In our earlier work [31], we proposed a phase field model for the self-climb of dislocation loops by adding a stabilizing factor to a modified Cahn-Hilliard type model. The model we proposed here is obtained by incorporating the dislocation climb motion into the phase field model for the self-climb motion of prismatic dislocation loops that we proposed earlier [31]. We point out that the presence of the stabilizing factor on the left side of the equation(1.1) is mainly for the proof of existence of weak solutions, and without it, the results of dislocation velocity given by the sharp interface limit (see the remark at the end of Sec. 2) and numerical simulations are similar. When $g \equiv 1$ and the climb force f_{cl} is omitted, the model reduces to the Cahn-Hilliard/Allen-Cahn equation with degenerate mobility. Such models have attracted lots of attentions in recent years [40].

Our main results in this paper concerns the sharp interface limit and the existence of weak solutions for (1.1)-(1.2). We remark that the uniqueness of such solutions is not known. Numerical simulations are also performed using the obtained phase field model.

We first derive a sharp interface limit equation for (1.1) and (1.2) via formal asymptotic analysis. The following sharp interface equation is obtained as $\varepsilon \to 0$,

$$v = -\lambda \partial_{ss} \left(\alpha\kappa - H_0 f_{cl}^{(0)}(s) \right) + \eta \left(\alpha\kappa - H_0 f_{cl}^{(0)}(s) \right). \tag{1.6}$$

Here λ, α and η are positive constants whose exact forms can be found in section 2.

For the existence of weak solutions of (1.1)-(1.2), we consider the following modified problem in a periodic setting. Set $\Omega = \mathbb{T}^2$, we consider

$$g(u)(\partial_t u + \beta\mu) = \nabla \cdot \left(M(u) \nabla \frac{\mu}{g(u)} \right), \qquad \text{for } x \in \mathbb{T}^2, t \in [0, \infty) \tag{1.7}$$

$$\mu = -\Delta u + q'(u) + (-\Delta)^{\frac{1}{2}} u. \tag{1.8}$$

Here $g(u) = |1 - u^2|^m$ for $2 \le m < \infty$, $M(u) = M_0 g(u)$ for some constant $M_0 > 0$, $q(u) \in C^2(\mathbb{R}, \mathbb{R})$ and there exist constants $C_i > 0$, $i = 1, \cdots, 10$, and $1 \le r < \infty$, such that for all $u \in \mathbb{R}$,

$$C_1 |u|^{r+1} - C_2 \le q(u) \le C_3 |u|^{r+1} + C_4, \tag{1.9}$$

$$|q'(u)| \le C_5 |u|^r + C_6 \tag{1.10}$$

$$C_7 |u|^{r-1} - C_8 \le q''(u) \le C_9 |u|^{r-1} + C_{10}. \tag{1.11}$$

We see that the classical double well potential $q(u) = (1 - u^2)^2$ satisfies (1.9)-(1.11) with $r = 3$.

In the proof, we consider approximations of the proposed model (1.7)-(1.8) with positive mobilities. Given any $\theta > 0$, we define

$$g_\theta(u) := \begin{cases} |1-u^2|^m & \text{if } |1-u^2| > \theta, \\ \theta^m & \text{if } |1-u^2| \le \theta, \end{cases} \tag{1.12}$$

and

$$M_\theta(u) := M_0 g_\theta(u). \tag{1.13}$$

Our first step is to find a sufficiently regular solution for (1.7)-(1.8) with mobility $M_\theta(u)$ and stabilizing function $g_\theta(u)$ together with a smooth potential $q(u)$. This result is summarized in the following Proposition.

PROPOSITION 1.1. *Let M_θ, g_θ be defined by (1.13) and (1.12), under the assumptions (1.9)-(1.11), for any $u_0 \in H^1(\mathbb{T}^2)$ and any $T > 0$, there exists a function u_θ such that*

a) $u_\theta \in L^\infty(0,T;H^1(\mathbb{T}^2)) \cap C([0,T];L^p(\mathbb{T}^2)) \cap L^2(0,T;W^{3,s}(\mathbb{T}^2))$, where $1 \le p < \infty$, $1 \le s < 2$,
b) $\partial_t u_\theta \in L^2(0,T;(W^{1,q}(\mathbb{T}^2))')$ for $q > 2$,
c) $u_\theta(x,0) = u_0(x)$ for all $x \in \mathbb{T}^2$,

which satisfies (1.7)-(1.8) in the following weak sense

$$\begin{aligned} &\int_0^T < \partial_t u_\theta, \phi >_{((W^{1,q}(\mathbb{T}^2))', W^{1,q}(\mathbb{T}^2))} dt \\ = \; & -\int_0^T \int_{\mathbb{T}^2} M_\theta(u_\theta) \nabla \frac{-\Delta u_\theta + q'(u_\theta) + (-\Delta)^{\frac{1}{2}} u_\theta}{g_\theta(u_\theta)} \cdot \nabla \frac{\phi}{g_\theta(u_\theta)} dx dt \\ & -\int_0^T \int_{\mathbb{T}^2} \beta(-\Delta u_\theta + q'(u_\theta) + (-\Delta)^{\frac{1}{2}} u_\theta) \phi dx dt \end{aligned} \tag{1.14}$$

for all $\phi \in L^2(0,T;W^{1,q}(\mathbb{T}^2))$ with $q > 2$. In addition, the following energy inequality holds for all $t > 0$.

$$\begin{aligned} & \int_{\mathbb{T}^2} \left(\frac{1}{2} |\nabla u_\theta(x,t)|^2 + q(u_\theta(x,t)) + u_\theta(x,t)(-\Delta)^{\frac{1}{2}} u_\theta(x,t) \right) dx \\ & + \int_0^t \int_{\mathbb{T}^2} M_\theta(u_\theta(x,\tau)) \left| \nabla \frac{-\Delta u_\theta(x,\tau) + q'(u_\theta(x,\tau)) + (-\Delta)^{\frac{1}{2}} u_\theta(x,\tau)}{g_\theta(u_\theta(x,\tau))} \right|^2 dx d\tau \\ & + \int_0^t \int_{\mathbb{T}^2} \beta \left(-\Delta u_\theta(x,\tau) + q'(u_\theta(x,\tau)) + (-\Delta)^{\frac{1}{2}} u_\theta(x,\tau) \right)^2 dx d\tau \\ \le & \int_{\mathbb{T}^2} \left(\frac{1}{2} |\nabla u_0(x)|^2 + q(u_0(x)) + u_0(x)(-\Delta)^{\frac{1}{2}} u_0 \right) dx. \end{aligned} \tag{1.15}$$

Proposition 1.1 is proved via Galerkin approximations. Due to the presence of the stabilizing function g_θ, it is not obvious how to pass to the limit in the nonlinear term of the Galerkin approximations. Our main observation in this step is the strong convergence of ∇u^N in $L^2(\mathbb{T}^2 \times [0,T])$ which allows us to pass to the limit.

In order to obtain a weak solution to (1.7)-(1.8), we consider the limit of u_θ as $\theta \to 0$. Our definition of the weak solutions to (1.7)-(1.8) follows the formulation of weak solutions in [9] for degenerate Cahn-Hilliard equations (corresponding to choosing $\beta = 0$ and $g(u) \equiv 1$ in (1.7) and omitting the nonlocal term from (1.8)). The main difficulty is how to pass to the limit in the nonlinear term in the approximation equation. For degenerate Cahn-Hilliard equations on $\Omega = \mathbb{T}^n$, the authors in [9] proved the existence of weak solutions by the following idea. The estimates for the positive mobility approximations yield uniform bounds for $\partial_t u_{\theta_i}$ in $L^2(0,T;(H^2(\mathbb{T}^n))')$, and uniform bounds on u_{θ_i} in $L^\infty(0,T;H^1(\mathbb{T}^n))$. Those uniform bounds imply the strong convergence of $\sqrt{M_i(u_{\theta_i})}$ in $C(0,T;L^n(\mathbb{T}^n))$. By this and the weak convergence of $\sqrt{M_i(u_{\theta_i})}\nabla\mu_{\theta_i}$ in $L^2(\Omega_T)$, authors in [9] showed (up to a subsequence) that $M_{\theta_i}(u_{\theta_i})\nabla\mu_{\theta_i} \rightharpoonup \sqrt{M(u)}\xi$ weakly in $L^2(0,T;L^{\frac{2n}{n+2}}(\mathbb{T}^n))$ where ξ is the weak limit of $\sqrt{M_i(u_{\theta_i})}\nabla\mu_{\theta_i}$. The main task left is to show $\sqrt{M(u)}\xi = M(u)(-\nabla\Delta u + q''(u)\nabla u)$ and the limit equation becomes a weak form Cahn-Hilliard equation. Authors in [9] proved that this is almost true in the set where $u \neq \pm 1$. For small numbers δ_j monotonically decreasing to 0, they consider the limit in a subset B_j of Ω_T where approximate solutions converges uniformly and $|\Omega_T \backslash B_j| < \delta_j$. By decomposing $B_j = D_j \cup \tilde{D}_j$ where mobility is bounded from below uniformly in D_j and controlled above in $\tilde{D}_j$ by suitable multiples of δ_j, they obtain the weak form equation for the limit function by passing to the limit of u_{θ_i} on D_j then letting j goes to ∞. Under further regularity assumptions on $\nabla\Delta u$, they obtained the explicit expression for ξ in the weak form of the equation.

Due to the existence of the stabilizing function $g(u)$ in our model, it is much more delicate to carry out a similar analysis. Unlike the degenerate Cahn-Hilliard case in [9], where there is uniform bound on $\partial_t u_{\theta_i}$ in $L^2(0,T;(H^2(\mathbb{T}^n))')$, the first obstacle for our model is that the bound estimate on $\partial_t u_{\theta_i}$ blows up when θ_i goes to zero. Secondly, it is more complicated to derive an explicit expression of the weak limit of $M_i(u_{\theta_i})\nabla\frac{\mu_{\theta_i}}{g_{\theta_i}(u_{\theta_i})}$ in terms of u in the limit equation. We shall follow ideas in a recent work by the authors [32] by which we derive convergence of $g_{\theta_i}(u_{\theta_i})$ (consequently $M_{\theta_i}(u_{\theta_i})$) from convergence of $G_i(u) = \int_0^u g_{\theta_i}(s)ds$. We then follow the idea in [9] to pass to the limit on the right hand side of the approximation equation. Below is our main theorem. Here and throughout the paper, we use the notation $\Omega_t = \mathbb{T}^2 \times [0,t]$ for any $t > 0$.

THEOREM 1.2. *For any $u_0 \in H^1(\mathbb{T}^2)$ and $T > 0$, there exists a function $u : \Omega_T \to \mathbb{R}$ satisfying*

i) $u \in L^\infty(0,T;H^1(\mathbb{T}^2)) \cap C([0,T];L^s(\mathbb{T}^2)) \cap L^2(0,T;H^2(\mathbb{T}^2))$, *where* $1 \le s < \infty$,

ii) $g(u)\partial_t u \in L^p(0,T;(W^{1,q}(\mathbb{T}^2))')$ *for* $1 \le p < 2$ *and* $q > 2$.

iii) $u(x,0) = u_0(x)$ *for all* $x \in \mathbb{T}^2$,

which solves (1.7)-(1.8) *in the following weak sense*

a) There exists a set $B \subset \Omega_T$ *with* $|\Omega_T \backslash B| = 0$ *and a function* $\zeta : \Omega_T \to \mathbb{R}^n$ *satisfying* $\chi_{B\cap P} M(u)\zeta \in L^{\frac{p}{p-1}}(0,T;L^{\frac{q}{q-1}}(\mathbb{T}^2,\mathbb{R}^2))$ *such that*

$$\begin{aligned}(1.16)\quad &\int_0^T < g(u)\partial_t u, \phi >_{(W^{1,q}(\mathbb{T}^2))', W^{1,q}(\mathbb{T}^2)}\, dt \\ &= \int_{B\cap P} M(u)\zeta\cdot\nabla\phi dxdt - \int_{\Omega_T} \beta\left[\nabla u\cdot\nabla\phi + q'(u)\phi + (-\Delta)^{\frac{1}{2}}(u)\phi\right] dxdt\end{aligned}$$

for all $\phi \in L^p(0,T;W^{1,q}(\mathbb{T}^2))$ *with* $p, q > 2$. *Here* $P := \{(x,t) \in \Omega_T : |1-u^2| \neq 0\}$ *is the set where* $M(u), g(u)$ *are nondegenerate and* $\chi_{B\cap P}$ *is the characteristic* *function of set $B \cap P$.*

b) Assume $u \in L^2(0,T;H^2(\mathbb{T}^2))$. For any open set $U \in \Omega_T$ on which $g(u) > 0$ and $\nabla\Delta u \in L^p(U)$ for some $p > 1$, we have

(1.17)
$$\zeta = \frac{-\nabla\Delta u + q''(u)\nabla u + \nabla(-\Delta)^{\frac{1}{2}}u}{g(u)} - \frac{g'(u)}{g^2(u)}\left(-\Delta u + q'(u) + (-\Delta)^{\frac{1}{2}}u\right)\nabla u.$$

a.e. in U.

Moreover, the following energy inequality holds for all $t > 0$

$$\begin{aligned}
&\int_{\mathbb{T}^2}\left(\frac{1}{2}|\nabla u(x,t)|^2 + q(u(x,t)) + u(x,t)(-\Delta)^{\frac{1}{2}}u(x,t)\right)dx \\
&+\int_{\Omega_r\cap B\cap P} M(u(x,\tau))|\zeta(x,\tau)|^2 dxd\tau \\
&+\int_{\Omega_r\cap B\cap P} \beta\left(-\Delta u + q'(u) + (-\Delta)^{\frac{1}{2}}u\right)^2 dxd\tau \\
\le &\int_{\mathbb{T}^2}\left(\frac{1}{2}|\nabla u_0(x)|^2 + q(u_0(x))\right)dx.
\end{aligned} \tag{1.18}$$

Remark 1.3. Our definition of weak solutions follows the formulation of weak solutions in [9] for degenerate Cahn-Hilliard equations. Further study is needed to explore the regularity of such weak solutions. It is also not clear if such solution is unique. When passing to the limit from the regularized solution u_θ and the corresponding chemical potential $\mu_\theta = -\Delta u_\theta + q'(u_\theta) + (-\Delta)^{\frac{1}{2}}u_\theta$, a key challenge is the convergence of nonlinear term $M_\theta(u_\theta)\nabla\frac{\mu_\theta}{g_\theta(u_\theta)}$. Up to a subsequence, we can show that

$$M_\theta(u_\theta)\nabla\frac{\mu_\theta}{g_\theta(u_\theta)} \rightharpoonup \sqrt{M(u)}\xi \text{ weakly in } L^p(0,T;L^s(\mathbb{T}^2))$$

for some $\xi \in L^2(\Omega_T)$. Thus we have, for any $\phi \in L^p(0,T;W^{1,q}(\mathbb{T}^2))$ with $p, q > 2$,

$$\int_0^T \langle g(u)\partial_t u + \beta u, \phi\rangle_{((W^{1,q}(\mathbb{T}^2))', W^{1,q}(\mathbb{T}^2))} dt = -\int_0^T\int_{\mathbb{T}^2}\sqrt{M(u)}\xi\cdot\nabla\phi dxdt. \tag{1.19}$$

Ideally we want

$$\sqrt{M(u)}\xi = M(u)\nabla\frac{-\Delta u + q'(u) + (-\Delta)^{\frac{1}{2}}u}{g(u)},$$

under which (1.19) becomes a weak form of (1.1)-(1.2). In general this is too much to ask for due to the degeneracy in the set where $u = \pm 1$. We show that this is almost true in the set $u \ne \pm 1$. More precisely, assuming $u \in L^2(0,T;H^2(\mathbb{T}^2))$, let P be the set where $M(u)$ is non-degenerate, then there exists a set B with $|\Omega_T \backslash B| = 0$, a sequence of increasing sets D_j whose limit is $B \cap P$, and a function ζ satisfying $\chi_{B\cap P}M(u)\zeta \in L^{\frac{p}{p-1}}(0,T;L^{\frac{q}{q-1}}(\mathbb{T}^2,\mathbb{R}^2))$ such that (1.16) holds for all $\phi \in L^p(0,T;W^{1,q}(\mathbb{T}^2))$ with $p, q > 2$; and

$$\zeta = \frac{-\nabla\Delta u + q''(u)\nabla u + \nabla(-\Delta)^{\frac{1}{2}}u}{g(u)} - \frac{g'(u)}{g^2(u)}\left(-\Delta u + q'(u) + (-\Delta)^{\frac{1}{2}}u\right)\nabla u \text{ a.e. in } U$$

on every open set $U \subset \Omega_T$ on which $g(u) > 0$ and $\nabla\Delta u \in L^s(U)$ for some $s > 1$. In addition, for any $\Psi \in L^{\frac{p}{p-1}}(0,T;L^{\frac{q}{q-1}}(\mathbb{T}^2;\mathbb{R}^2))$,

$$\int_0^T \int_{\mathbb{T}^2} \sqrt{M(u)}\xi \cdot \nabla\phi dxdt = \int_{B\cap P} M(u)\zeta \cdot \Psi dxdt.$$

Consequently, for any $\phi \in L^p(0,T;W^{1,q}(\mathbb{T}^2))$ with $p,q > 2$,

$$\int_0^T \langle g(u)\partial_t u, \phi\rangle_{((W^{1,q}(\mathbb{T}^2))',W^{1,q}(\mathbb{T}^2))}\, dt = -\int_{B\cap P} M(u)\zeta \cdot \nabla\phi dxdt.$$

Remark 1.4. Our proof for Proposition 1.1 and Theorem 1.2 also works for the 1D case. One important fact we use here is $H^1(\mathbb{T}^n) \subset L^p(\mathbb{T}^n)$ for any $p \geq 1$ when $n = 1,2$. For dimension ≥ 3, Sobolev embedding theorem only gives $H^1(\mathbb{T}^n) \subset L^p(\mathbb{T}^n)$ for $p \leq \frac{2n}{n-2}$, which is not enough integrability to handle the nonlinear term in the limiting step for Galerkin approximation or when letting mobility goes to zero. Different approach needs to be explored for higher dimensional cases.

Lastly, we perform numerical simulations to validate our model. Using the proposed phase field model, we did simulations of evolution of an elliptic prismatic loop and interactions between two circular prismatic loops under the combined effect of self-climb and non-conservative climb. Our numerical results indicate that the self-climb effect slows down the shrinking of loop for the evolution of an elliptic prismatic loop. For interaction between two circular loops, the patterns in the two shrinking process are quite different with or without the self-climb effect .

The paper is organized as follows. We shall derive sharp interface limit for (1.1) and (1.2) through formal asymptotic expansions in section 2. Section 3.1 is devoted to the proof of Proposition 1.1 and Theorem1.2 is proved in section 3.2. Numerical simulations are presented in section 4.

2. Sharp interface limit via asymptotic expansions.

In this section, we perform a formal asymptotic analysis to obtain the dislocation self-climb velocity of the proposed phase field model (1.1) and (1.2) in the sharp interface limit $\varepsilon \to 0$.

2.1. Outer expansions.

We first perform expansion in the region far from the dislocations. Assume the expansion for u is

$$u(x,y,t) = u^{(0)}(x,y,t) + u^{(1)}(x,y,t)\varepsilon + u^{(2)}(x,y,t)\varepsilon^2 + \cdots \tag{2.1}$$

Correspondingly, we have

$$\begin{aligned}
M(u) &= M(u^{(0)}) + M'(u^{(0)})u^{(1)}\varepsilon + \left(M'(u^{(0)})u^{(2)} + \frac{1}{2}M''\left(u^{(0)}\right)\left(u^{(1)}\right)^2\right)\varepsilon^2 + \cdots,\\
g(u) &= g(u^{(0)}) + g'(u^{(0)})u^{(1)}\varepsilon + \left(g'(u^{(0)})u^{(2)} + \frac{1}{2}g''(u^{(0)})\left(u^{(1)}\right)^2\right)\varepsilon^2 + \cdots,\\
q'(u) &= q'(u^{(0)}) + q''(u^{(0)})u^{(1)}\varepsilon + \left(q''(u^{(0)})u^{(2)} + \frac{1}{2}q^{(3)}(u^{(0)})\left(u^{(1)}\right)^2\right)\varepsilon^2 + \cdots,\\
f_{cl}^d(x,y,u) &= f_{cl}^d(x,y,u^{(0)}) + f_{cl}^d(x,y,u^{(1)})\varepsilon + f_{cl}^d(x,y,u^{(2)})\varepsilon^2 + \cdots.
\end{aligned}$$

We also expand the chemical potential μ as

$$\mu = \frac{1}{\varepsilon^2}\left(\mu^{(0)} + \mu^{(1)}\varepsilon + \mu^{(2)}\varepsilon^2 + \cdots\right). \tag{2.2}$$

Rewrite equation (1.1) as

$$g(u)(\partial_t u+\beta\mu)=M_0\nabla\cdot(\nabla\mu-\mu\frac{g'(u)}{g(u)}\nabla u), \tag{2.3}$$

and set

$$w=-\mu\frac{g'(u)}{g(u)}=\frac{1}{\varepsilon^2}\left(w^{(0)}+w^{(1)}\varepsilon+w^{(2)}\varepsilon^2+\cdots\right). \tag{2.4}$$

Plugging the expansions into (2.3) and (1.2) and matching the coefficients of ε powers in both equations, the $O(\frac{1}{\varepsilon^2})$ equations of (2.3) and (1.2) yield

$$\beta g(u^{(0)})\mu^{(0)}=M_0\nabla\cdot\left(\nabla\mu^{(0)}+w^{(0)}\nabla u^{(0)}\right), \tag{2.5}$$

and

$$\mu^{(0)}=q'(u^{(0)}). \tag{2.6}$$

Since

$$w^{(0)}=\mu^{(0)}\frac{g'(u^{(0)})}{g(u^{(0)})},$$

then $u^{(0)}=1$ or $u^{(0)}=-1$ satisfies equations (2.5)-(2.6). In particular, such choice of $u^{(0)}$ implies $\mu^{(0)}=0$.

The $O(\frac{1}{\varepsilon})$ equations of (2.3) and (1.2) yield

$$\begin{aligned} & \beta\left(g(u^{(0)})\mu^{(1)}+g'(u^{(0)})u^{(1)}\mu^{(0)}\right) \\ = \; & M_0\nabla\cdot\left(\nabla\mu^{(1)}+w^{(0)}\nabla u^{(1)}+w^{(1)}\nabla u^{(0)}\right), \end{aligned} \tag{2.7}$$

and

$$\mu^{(1)} = q''(u^{(0)})u^{(1)}+h(u^{(0)})f_{cl}^d(x,y,u^{(0)}). \tag{2.8}$$

Since $u^{(0)}=1$ or -1, $u^{(1)}=0$ satisfies (2.7)-(2.8). Moreover, such choice of $u^{(1)}$ guarantees $\mu^{(1)}=0$.

The $O(1)$ equations of (2.3) and (1.2) are

$$\begin{aligned} & u_t^{(0)}g(u^{(0)})+\beta\left(g(u^{(0)})\mu^{(2)}+g'(u^{(0)})u^{(1)}\mu^{(1)}+\mu^{(0)}\left(g'(u^{(0)})u^{(2)}+\frac{1}{2}g''(u^{(0)})\left(u^{(1)}\right)^2\right)\right) \\ = \; & M_0\nabla\cdot\left(\nabla\mu^{(2)}+w^{(0)}\nabla u^{(2)}+w^{(1)}\nabla u^{(1)}+w^{(2)}\nabla u^{(0)}\right), \end{aligned}$$

and

$$\begin{aligned} \mu^{(2)} = \; & -\Delta u^{(0)}+q''(u^{(0)})u^{(2)}+\frac{1}{2}q^{(3)}(u^{(0)})(u^{(1)})^2 \\ & +h(u^{(0)})f_{cl}^d(x,y,u^{(1)})+h'(u^{(0)})u^{(1)}f_{cl}^d(x,y,u^{(0)}). \end{aligned}$$

Taking into account of the fact $u^{(0)}=\pm 1$, $u^{(1)}=\mu^{(0)}=\mu^{(1)}=0$, the equations above reduce to

$$0 = \nabla\cdot\left(\nabla\mu^{(2)}+w^{(0)}\nabla u^{(2)}\right), \tag{2.9}$$

and

$$\mu^{(2)} = q''(u^{(0)})u^{(2)} + h(u^{(0)})f_{cl}^{d}(x,y,u^{(1)}). \tag{2.10}$$

Thus $u^{(2)} = 0$ satisfies (2.9)-(2.10). Moreover, such choice of $u^{(2)}$ guarantees $\mu^{(2)} = 0$.

In general, if $u^{(0)} = \pm 1$, $u^{(1)} = u^{(2)} = \cdots = u^{(k+1)} = 0$, the $O(\varepsilon^k)$ of the $k \geq 1$ equations of (2.3) and (1.2) yield

$$0 = \nabla \cdot \left(\nabla \mu^{(k+2)} + w^{(0)} \nabla u^{(k+2)}\right), \tag{2.11}$$

and

$$\mu^{(k+2)} = q''(u^{(0)})u^{(k+2)} + h(u^{(0)})f_{cl}^{d}(x,y,u^{(k+1)}). \tag{2.12}$$

Thus $u^{(k+2)} = 0$ satisfies (2.11) and (2.12).

In summary, we have $u = 1$ or $u = -1$ in the outer region.

2.2. Inner expansions. For the small inner regions near the dislocations, we introduce local coordinates near the dislocations. Considering a dislocation C parameterized by its arc length parameter s. We denote a point on the dislocation by $\mathbf{r}_0(s)$ with tangent unit vector $\mathbf{t}(s)$ and inward normal vector $\mathbf{n}(s)$. A point near the dislocation C is expressed as

$$\mathbf{r}(s,d) = \mathbf{r}_0(s) + d\mathbf{n}(s), \tag{2.13}$$

where d is the signed distance from point $\mathbf{r}$ to the dislocation. Since the gradient fields are of order $O(\frac{1}{\varepsilon})$, we introduce the variable $\rho = \frac{d}{\varepsilon}$ and use coordinates (s,ρ) in the inner region. Under this setting, we write $u(x,y,t) = U(s,\rho,t)$ and equation (1.1)-(1.2) can be written as

$$\begin{aligned} g(U)\left(\partial_t U - \frac{1}{\varepsilon} v_n \partial_\rho U + \beta\mu\right) = {} & \frac{M_0}{1-\varepsilon\rho\kappa}\partial_s\left(\frac{1}{1-\varepsilon\rho\kappa}\left(\partial_s\mu - \mu\frac{g'(U)}{g(U)}\partial_s U\right)\right) \\ & + \frac{1}{\varepsilon^2}\frac{M_0}{1-\varepsilon\rho\kappa}\partial_\rho\left((1-\varepsilon\rho\kappa)\left(\partial_\rho\mu - \mu\frac{g'(U)}{g(U)}\partial_\rho U\right)\right), \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} \mu = {} & -\frac{1}{1-\varepsilon\rho\kappa}\partial_s\left(\frac{1}{1-\varepsilon\rho\kappa}\partial_s U\right) - \frac{1}{\varepsilon^2}\frac{1}{1-\varepsilon\rho\kappa}\partial_\rho\left((1-\varepsilon\rho\kappa)\partial_\rho U\right) \\ & + \frac{1}{\varepsilon^2}q'(U) + \frac{1}{\varepsilon}h(U)f_{cl}(s,\rho,U). \end{aligned} \tag{2.15}$$

Assume that μ takes the same form expansion as (2.2). The following expansions hold for U and the climb force f_{cl} within the dislocation core region:

$$U(s,\rho,t) = U^{(0)}(\rho) + \varepsilon U^{(1)}(s,\rho,t) + \varepsilon^2 U^{(2)}(s,\rho,t) + \cdots, \tag{2.16}$$

and

$$f_{cl}(s,\rho,U) = \frac{1}{\varepsilon}f_{cl}^{(-1)}(\rho,U) + f_{cl}^{(0)}(s) + O(\varepsilon), \tag{2.17}$$

where

$$f_{cl}^{(-1)}(\rho,U) = \frac{Gb^2}{4\pi(1-\nu)}\int_{-\infty}^{+\infty}\frac{\partial_\rho U(\tau)}{\rho-\tau}d\tau, \tag{2.18}$$

$$f_{cl}^{(0)}(s) = f_{cl}^{d}(s) + f_{cl}^{app}(s), \tag{2.19}$$

$$f_{cl}^{d}(s) = \frac{Gb^2}{4\pi(1-\nu)}\kappa\ln\varepsilon + O(1). \tag{2.20}$$

Here we assume the leading order solution $U^{(0)}$, which describe the dislocation core profile, remains the same at all points on the dislocation at any time. The term $\frac{1}{\varepsilon}f_{cl}^{(-1)}(\rho,U)$ in the climb force expansion is due to the singular stress field near the dislocation and vanishes on the dislocation (i.e. $f_{cl}^{(-1)}(\rho,U^{(0)}) = 0$). The climb force $f_{cl}^{d}(s)$ is generated by the dislocations and has asymptotic expansions (2.20). This asymptotic expansion of climb force f_{cl} in the phase field model was obtained in [31] based on the dislocation theories [19, 12, 41].

Letting

$$W = \mu\frac{g'(U)}{g(U)} = \frac{1}{\varepsilon^2}\left(W^{(0)} + W^{(1)}\varepsilon + W^{(2)}\varepsilon^2 + \cdots\right), \tag{2.21}$$

the leading orders of equations (2.14) and (2.15) are $O(\frac{1}{\varepsilon^4})$ and $O(\frac{1}{\varepsilon^2})$, respectively, which yield

$$0 = \partial_\rho\left(\partial_\rho\mu^{(0)} - W^{(0)}\partial_\rho U^{(0)}\right), \tag{2.22}$$

and

$$\mu^{(0)} = -\partial_{\rho\rho}U^{(0)} + q'(U^{(0)}) + h(U^{(0)})f_{cl}^{(-1)}(\rho,U^{(0)}). \tag{2.23}$$

Integrating Eq. (2.22), we have

$$\partial_\rho\mu^{(0)} - W^{(0)}\partial_\rho U^{(0)} = C_0(s). \tag{2.24}$$

Since $\mu^{(0)} = 0$, $u^{(0)} = 1$ or -1 in the outer region, we must have $\mu^{(0)} \to 0$ and $\partial_\rho U^{(0)} \to 0$ as $\rho \to \pm\infty$. Therefore $C_0(s) = 0$. Dividing (2.24) by $\mu^{(0)}$ and taking integration, using $W^{(0)} = \mu^{(0)}\frac{g'(U^{(0)})}{g(U^{(0)})}$, we have $\mu^{(0)} = \tilde{C}_0(s)g(U^{(0)})$. Since $\mu^{(0)}/g(u^{(0)})$ is 0 in the outer region, we must have $\tilde{C}_0(s) = 0$. Thus

$$\mu^{(0)} = -\partial_{\rho\rho}U^{(0)} + q'(U^{(0)}) + h(U^{(0)})f_{cl}^{(-1)}(\rho,U^{(0)}) = 0. \tag{2.25}$$

Solution $U^{(0)}$ to (2.25) subject to far field condition $U^{(0)}(+\infty) = -1$ and $U^{(0)}(-\infty) = 1$ can be found numerically (see [31]). In particular, $\partial_\rho U^{(0)} < 0$ for all ρ.

Next, the $O(\frac{1}{\varepsilon^3})$ equation of (2.14) and $O(\frac{1}{\varepsilon})$ equation of (2.15) yield, using $\mu^{(0)} = 0$, that

$$0 = \partial_\rho\left(\partial_\rho\mu^{(1)} - W^{(1)}\partial_\rho U^{(0)}\right), \tag{2.26}$$

and

$$\begin{aligned}\mu^{(1)} &= -\partial_{\rho\rho}U^{(1)} + \kappa\partial_\rho U^{(0)} + q''(U^{(0)})U^{(1)} + h'(U^{(0)})f^{(-1)}(\rho,U^{(0)})U^{(1)} \\ &\quad + h(U^{(0)})f^{-1}(\rho,U^{(1)}) + h(U^{(0)})f_{cl}^{(0)}(s).\end{aligned} \tag{2.27}$$

Similar to the calculation from Eq. (2.22) to Eq. (2.24) given above, by matching with the outer solutions, we have $\partial_\rho \mu^{(1)} - W^{(1)}\partial_\rho U^{(0)} = 0$. Since $\mu^{(0)} = 0$, we have $W^{(1)} = \mu^{(1)}\frac{g'(U^{(0)})}{g(U^{(0)})}$, the obtained equation becomes

$$\partial_\rho \mu^{(1)} - \mu^{(1)}\partial_\rho \ln g(U^{(0)}) = 0. \tag{2.28}$$

Dividing (2.28) by $\mu^{(1)}$ and integrating, we have $\mu^{(1)} = \tilde{C}_1(s)g(U^{(0)})$. Thus equation (2.27) can be rewritten as

$$LU^{(1)} = -\kappa\partial_\rho U^{(0)} - h(U^{(0)})f_{cl}^{(0)}(s) + \tilde{C}_1(s)g(U^{(0)}), \tag{2.29}$$

where $L = -\partial_{\rho\rho} + q''(U^{(0)}) + h'(U^{(0)})f^{(-1)}(\rho, U^{(0)}) + h(U^{(0)})f^{-1}(\rho,\cdot)$ is a linear operator whose kernal is span$\{\partial_\rho U^{(0)}\}$. Multiplying both sides of Eq. (2.29) by $\partial_\rho U^{(0)}$ and integrate with respect to ρ over $(-\infty, +\infty)$, we have

$$\int_{-\infty}^{+\infty}\left(-\kappa\partial_\rho U^{(0)} - h(U^{(0)})f_{cl}^{(0)}(s) + \tilde{C}_1(s)g(U^{(0)})\right)\partial_\rho U^{(0)} d\rho = 0.$$

From this, we conclude

$$\tilde{C}_1(s) = -\alpha\kappa + H_0 f_{cl}^{(0)}(s),$$

where $\alpha > 0$ is given by

$$\alpha = -\frac{\int_{-\infty}^{+\infty}\left(\partial_\rho U^{(0)}\right)^2 d\rho}{\int_{-\infty}^{+\infty} g(U^{(0)})\partial_\rho U^{(0)} d\rho}.$$

Therefore

$$\mu^{(1)} = g(U^{(0)})\left(-\alpha\kappa + H_0 f_{cl}^{(0)}(s)\right). \tag{2.30}$$

Letting $\overline{\mu} = \frac{\mu}{g(U)}$, (2.14) can be written as

$$\begin{aligned} & g(U)\left(\partial_t U - \frac{1}{\varepsilon}v_n\partial_\rho U + \beta\mu\right) \\ = \; & \frac{M_0}{1-\varepsilon\rho\kappa}\partial_s\left(\frac{g(U)}{1-\varepsilon\rho\kappa}\left(\partial_s\overline{\mu}\right)\right) + \frac{1}{\varepsilon^2}\frac{M_0}{1-\varepsilon\rho\kappa}\partial_\rho\left(\left(1-\varepsilon\rho\kappa\right)g(U)\partial_\rho\overline{\mu}\right) \end{aligned} \tag{2.31}$$

Using $\mu^{(0)} = 0$, $\partial_\rho\overline{\mu}^{(1)} = \partial_\rho\frac{\mu^{(1)}}{g(U^{(0)})} = 0$, the $O(\frac{1}{\varepsilon^2})$ order equation of (2.31) reduces to

$$\partial_\rho\left(g(U^{(0)})\partial_\rho\overline{\mu}^{(2)}\right) = 0.$$

Integrating with respect to ρ, we have $g(U^{(0)})\partial_\rho\overline{\mu}^{(2)} = C_2(s)$. Matching with outer solutions, we must have $C_2(s) = 0$. Thus $\partial_\rho\overline{\mu}^{(2)} = 0$ which gives $\overline{\mu}^{(2)} = \tilde{C}_2(s)$.

Next we look at the $O(\frac{1}{\varepsilon})$ equation of (2.31). Using $\mu^{(0)} = 0$, $\partial_\rho\overline{\mu}^{(1)} = 0$ and $\partial_\rho\overline{\mu}^{(2)} = 0$, we have

$$g(U^{(0)})(-v_n\partial_\rho U^{(0)} + \beta\mu^{(1)}) = M_0\partial_s\left(g(U^{(0)})\partial_s\overline{\mu}^{(1)}\right) + M_0\partial_\rho\left(g(U^{(0)})\partial_\rho\overline{\mu}^{(3)}\right).$$

Integrating this equation with respect to ρ and matching with outer solutions yields

$$v_n = \lambda\partial_{ss}\overline{\mu}^{(1)} - \eta\overline{\mu}^{(1)} \tag{2.32}$$

where we used the fact that $g(U^{(0)})$ is independent of s, $\overline{\mu}^{(1)} = -\alpha\kappa + H_0 f_{cl}^{(0)}(s)$ by (2.30), and

$$\lambda = -\frac{M_0 \int_{-\infty}^{+\infty} g(U^{(0)}) d\rho}{\int_{-\infty}^{+\infty} g(U^{(0)}) \partial_\rho U^{(0)} d\rho} > 0, \quad \eta = -\frac{\beta \int_{-\infty}^{+\infty} g(U^{(0)}) d\rho}{\int_{-\infty}^{+\infty} g(U^{(0)}) \partial_\rho U^{(0)} d\rho} > 0. \tag{2.33}$$

Substitute $\overline{\mu}^{(1)} = -\alpha\kappa + H_0 f_{cl}^{(0)}(s)$ into (2.32), the sharp interface limit equation is

$$v_n = -\lambda \partial_{ss} \left(\alpha\kappa - H_0 f_{cl}^{(0)}(s) \right) + \eta \left(\alpha\kappa - H_0 f_{cl}^{(0)}(s) \right). \tag{2.34}$$

Remark 2.1. The velocity in the obtained sharp interface limit equation (2.34) is a combination of the dislocation self-climb velocity [30, 29, 31] (the first term), and the dislocation climb velocity by mobility law [38, 39, 2] (the second term). The coefficients of these two contributions are determined through Eq. (2.33) by the parameters M_0 and β, respectively, in the phase field model in (1.1) based on the physics. Note that the curvature term in both contributions is a correction to the dislocation self-force to fix the problem of larger numerical dislocation core size in the phase field model than the actual dislocation core size [31]. We remark that the factor $g(U)$ on the left-hand side in Eq. (1.1) is mainly for the proof of existence of weak solutions, and without it, the dislocation velocity given by the sharp interface limit is similar, with $\lambda = \frac{M_0}{2} \int_{-\infty}^{+\infty} g(U^{(0)}) d\rho$ and $\eta = \frac{\beta}{2} \int_{-\infty}^{+\infty} g(U^{(0)}) d\rho$.

3. Weak solution for the phase field model.

3.1. Weak solution for the phase field model with positive mobilities.

In this subsection, we prove the existence of weak solutions for phase field model with positive mobilities summarized in Proposition 1.1.

Let $\mathbb{Z}_+$ be the set of nonnegative integers and we choose an orthonormal basis for $L^2(\mathbb{T}^2)$ as

$$\{\phi_j : j = 1, 2, \cdots\} = \left\{ (2\pi)^{-1}, \mathrm{Re}\left(\pi^{-1} e^{i\xi\cdot x}\right), \mathrm{Im}\left(\pi^{-1} e^{i\xi\cdot x}\right) : \xi \in \mathbb{Z}^2{}_+ \backslash \{0, \cdots, 0\} \right\}.$$

Observe that $\{\phi_j\}$ is also orthogonal in $H^k(\mathbb{T}^2)$ for any $k \geq 1$.

3.1.1. Galerkin approximations.

Define

$$u^N(x,t) = \sum_{j=1}^N c_j^N(t) \phi_j(x), \qquad \mu^N(x,t) = \sum_{j=1}^N d_j^N(t) \phi_j(x),$$

where $\{c_j^N, d_j^N\}$ satisfy

$$\int_{\mathbb{T}^2} \partial_t u^N \phi_j dx = -\int_{\mathbb{T}^2} M_\theta(u^N) \nabla \frac{\mu^N}{g_\theta(u^N)} \cdot \nabla \frac{\phi_j}{g_\theta(u^N)} dx - \beta \int_{\mathbb{T}^2} \mu^N \phi_j dx \tag{3.1}$$

$$\int_{\mathbb{T}^2} \mu^N \phi_j dx = \int_{\mathbb{T}^2} \left(\nabla u^N \cdot \nabla \phi_j + q'(u^N) \phi_j + \phi_j (-\Delta)^{\frac{1}{2}} u^N \right) dx, \tag{3.2}$$

$$u^N(x,0) = \sum_{j=1}^N \left(\int_{\mathbb{T}^2} u_0 \phi_j dx \right) \phi_j(x). \tag{3.3}$$

(3.1)-(3.3) is an initial value problem for a system of ordinary equations for $\{c_j^N(t)\}$. Since right hand side of (3.1) is continuous in c_j^N, the system has a local solution.

Define energy functional

$$E(u) = \int_{\mathbb{T}^2} \left\{ \frac{1}{2}|\nabla u|^2 + q(u) + |(-\Delta)^{\frac{1}{4}} u|^2 \right\} dx.$$

Direct calculation using (3.1) and (3.2) yields

$$\frac{d}{dt} E(u^N(x,t)) = -\int_{\mathbb{T}^2} M_\theta(u^N) \left| \nabla \frac{\mu^N}{g_\theta(u^N)} \right|^2 dx - \beta \int_{\mathbb{T}^2} \left(\mu^N\right)^2 dx,$$

integration over t gives the following energy identity.

$$\begin{aligned}
& \int_{\mathbb{T}^2} \left(\frac{1}{2}|\nabla u^N(x,t)|^2 + q(u^N(x,t)) + u^N (-\Delta)^{\frac{1}{2}} u^N \right) dx \\
& + \int_0^t \int_{\mathbb{T}^2} \left[M_\theta(u^N(x,\tau)) \left| \nabla \frac{\mu^N(x,\tau)}{g_\theta(u^N(x,\tau))} \right|^2 + \beta \left(\mu^N\right)^2 \right] dx d\tau \\
= & \int_{\mathbb{T}^2} \left(\frac{1}{2}|\nabla u^N(x,0)|^2 + q(u^N(x,0)) + u^N(x,0)(-\Delta)^{\frac{1}{2}} u^N(x,0) \right) dx \qquad (3.4) \\
\le & \ \|\nabla u_0\|^2_{L^2(\mathbb{T}^2)} + C\left(\|u_0\|^{r+1}_{H^1(\mathbb{T}^2)} + |\mathbb{T}^2| \right) + \frac{1}{2} \|u_0\|^2_{L^2(\mathbb{T}^2)} \le C < \infty
\end{aligned}$$

Here and throughout the paper, C represents a generic constant possibly depending only on β, T, $\mathbb{T}^2$, u_0 but not on θ. Since $\mathbb{T}^2$ is bounded region, by growth assumption (1.9) and Poincare's inequality, the energy identity (3.4) implies $u^N \in L^\infty(0,T;H^1(\mathbb{T}^2))$ and $\mu^N \in L^2(\Omega_T)$ with

$$\left\|\mu^N\right\|_{L^2(\Omega_T)}, \left\|u^N\right\|_{L^\infty(0,T;H^1(\mathbb{T}^2))} \le C \text{ for all } N, \qquad (3.5)$$

and

$$\left\| \sqrt{M_\theta(u^N)} \nabla \frac{\mu^N}{g_\theta(u^N)} \right\|_{L^2(\Omega_T)} \le C \text{ for all } N. \qquad (3.6)$$

By (3.5), the coefficients $\{c_j^N(t)\}$ are bounded in time, thus the system (3.1)-(3.3) has a global solution. In addition, by Sobolev embedding theorem and growth assumption (1.10) on $q'(u)$, we have

$$q'(u^N) \in L^\infty(0,T;L^p(\mathbb{T}^2)), \qquad M_\theta(u^N) \in L^\infty(0,T;L^p(\mathbb{T}^2))$$

for any $1 \le p < \infty$ with

$$\left\|q'(u^N)\right\|_{L^\infty(0,T;L^p(\mathbb{T}^2))} \le C \text{ for all } N, \qquad (3.7)$$

$$\left\|M_\theta(u^N)\right\|_{L^\infty(0,T;L^p(\mathbb{T}^2))} \le C \text{ for all } N. \qquad (3.8)$$

3.1.2. Convergence of u^N. Given $q > 2$ and any $\phi \in L^2(0,T;W^{1,q}(\mathbb{T}^2))$, let $\Pi_N \phi(x,t) = \sum_{j=1}^N \left(\int_{\mathbb{T}^2} \phi(x,t)\phi_j(x) dx \right) \phi_j(x)$ be the orthogonal projection of ϕ onto span$\{\phi_j\}_{j=1}^N$. Then

$$
\begin{aligned}
&\left|\int_{\mathbb{T}^2} \partial_t u^N \phi dx\right| = \left|\int_{\mathbb{T}^2} \partial_t u^N \Pi_N \phi dx\right| \\
= & \left|\int_{\mathbb{T}^2} \left[M_\theta(u^N)\nabla \frac{\mu^N}{g_\theta(u^N)} \cdot \nabla \frac{\Pi_N\phi}{g_\theta(u^N)} - \beta\mu^N \Pi_N\phi\right] dx\right| \\
\le & \left\|\sqrt{M_\theta(u^N)}\nabla \frac{\mu^N}{g_\theta(u^N)}\right\|_{L^2(\mathbb{T}^2)} \left\|\sqrt{M_\theta(u^N)}\nabla \frac{\Pi_N\phi}{g_\theta(u^N)}\right\|_{L^2(\mathbb{T}^2)} + \beta\|\mu^N\|_{L^2(\mathbb{T}^2)}\|\phi\|_{L^2(\mathbb{T}^2)}.
\end{aligned}
$$

Since

$$
\nabla \frac{\Pi_N\phi}{g_\theta(u^N)} = \frac{1}{g_\theta(u^N)}\nabla\Pi_N\phi - \Pi_N\phi\frac{g_\theta'(u^N)}{g_\theta^2(u^N)}\nabla u^N,
$$

we have

$$
\begin{aligned}
&\int_{\mathbb{T}^2} M_\theta(u^N)\left|\nabla\frac{\Pi_N\phi}{g_\theta(u^N)}\right|^2 dx \\
\le\ & 2M_0\int_{\mathbb{T}^2}\left(\frac{1}{g_\theta(u^N)}\left|\nabla\Pi_N\phi\right|^2 + \frac{|g_\theta'(u^N)|^2}{g_\theta^3(u^N)}|\Pi_N\phi|^2|\nabla u^N|^2\right)dx \\
\le\ & C(M_0,\theta)\left(\|\nabla\Pi_N\phi\|_{L^2(\mathbb{T}^2)}^2 + \|\Pi_N\phi\|_{L^\infty(\mathbb{T}^2)}^2\left\|\nabla u^N\right\|_{L^2(\mathbb{T}^2)}^2\right) \\
\le\ & C(M_0,\theta)\left(\|\Pi_N\phi\|_{W^{1,q}(\mathbb{T}^2)}^2\right) \le C(M_0,\theta)\,\|\phi\|_{W^{1,q}(\mathbb{T}^2)}^2.
\end{aligned}
$$

Therefore

$$
\left\|\partial_t u^N\right\|_{L^2(0,T;(W^{1,q}(\mathbb{T}^2))')} \le C(M_0,\theta) \text{ for all } N. \tag{3.9}
$$

For $1 \le s < \infty$, by Sobolev embedding theorem and Aubin-Lions Lemma (see [36] and Remark 3.1) , the following embeddings are compact :

$$
\left\{f \in L^2(0,T;H^1(\mathbb{T}^2)) : \partial_t f \in L^2(0,T;(W^{1,q}(\mathbb{T}^2))')\right\} \hookrightarrow L^2(0,T;L^s(\mathbb{T}^2)),
$$

and

$$
\left\{f \in L^\infty(0,T;H^1(\mathbb{T}^2)) : \partial_t f \in L^2(0,T;(W^{1,q}(\mathbb{T}^2))')\right\} \hookrightarrow C([0,T];L^s(\mathbb{T}^2)).
$$

From this and the boundedness of $\{u^N\}$ and $\{\partial_t u^N\}$, we can find a subsequence, and $u_\theta \in L^\infty(0,T;H^1(\mathbb{T}^2))$ such that as $N \to \infty$, for $1 \le s < \infty$.

$$
u^N \rightharpoonup u_\theta \text{ weakly-* in } L^\infty(0,T;H^1(\mathbb{T}^2)), \tag{3.10}
$$

$$
u^N \to u_\theta \text{ strongly in } C([0,T];L^s(\mathbb{T}^2)), \tag{3.11}
$$

$$
u^N \to u_\theta \text{ strongly in } L^2(0,T;L^s(\mathbb{T}^2)) \text{ and a.e. in } \Omega_T, \tag{3.12}
$$

$$
\partial_t u^N \rightharpoonup \partial_t u_\theta \text{ weakly in } L^2(0,T;(W^{1,q}(\mathbb{T}^2))'). \tag{3.13}
$$

In addition

$$
\|u_\theta\|_{L^\infty(0,T;H^1(\mathbb{T}^2))} \le C, \qquad \left\|\partial_t u_\theta\right\|_{L^2(0,T;(W^{1,q}(\mathbb{T}^2))')} \le C(M_0,\theta).
$$

By (3.11), growth assumption (1.10) on $q'(u^N)$, and generalized dominated convergence theorem (see Remark 3.2), we have

$$M_\theta(u^N) \to M_\theta(u_\theta) \text{ strongly in } C([0,T];L^s(\mathbb{T}^2)) \tag{3.14}$$

$$\sqrt{M_\theta(u^N)} \to \sqrt{M_\theta(u_\theta)} \text{ strongly in } C([0,T];L^s(\mathbb{T}^2)) \tag{3.15}$$

$$q'(u^N) \to q'(u_\theta) \text{ strongly in } C([0,T];L^s(\mathbb{T}^2)) \tag{3.16}$$

for $1 \le s < \infty$. By (3.7) and (3.16), we have

$$q'(u^N) \rightharpoonup q'(u_\theta) \text{ weakly-* in } L^\infty([0,T];L^s(\mathbb{T}^2)). \tag{3.17}$$

Remark 3.1. Let X, Y, Z be Banach spaces with compact embedding $X \hookrightarrow Y$ and continuous embedding $Y \hookrightarrow Z$. Then the embeddings

$$\{f \in L^p(0,T;X); \partial_t f \in L^1(0,T;Z)\} \hookrightarrow L^p(0,T;Y) \tag{3.18}$$

and

$$\{f \in L^\infty(0,T;X); \partial_t f \in L^r(0,T;Z)\} \hookrightarrow C([0,T];Y) \tag{3.19}$$

are compact for any $1 \le p < \infty$ and $r > 1$ (Corollary 4, [36], see also [26]) . For convergence of u^N, we apply this for $p = 2 = r$ with $X = H^1(\mathbb{T}^2)$, $Y = L^s(\mathbb{T}^2)$ for $1 \le s < \infty$ and $Z = W^{1,q}(\mathbb{T}^2)'$.

3.1.3. Weak solution. By (3.2), we have

$$\int_{\mathbb{T}^2} \mu^N u^N dx = \int_{\mathbb{T}^2} \left(|\nabla u^N|^2 dx + q'(u^N)u^N + u^N(-\Delta)^{\frac{1}{2}} u^N \right) dx.$$

Integration with respect to t from 0 to T gives

$$\begin{aligned} &\int_{\Omega_T} \mu^N(x,\tau)u^N(x,\tau)dxd\tau \\ = &\int_{\Omega_T} \left(\nabla u^N(x,\tau)|^2 dx + q'(u^N(x,\tau))u^N(x,\tau) + u^N(-\Delta)^{\frac{1}{2}} u^N \right) dxd\tau. \end{aligned}$$

By (3.5). there exists a subsequence of μ^N, not relabeled, converges weakly to $\mu_\theta \in L^2(\Omega_T)$. Passing to the limit in the equation above, by (3.12), (3.16), we have

$$\begin{aligned} \int_{\Omega_T} \mu_\theta u_\theta dxd\tau \quad = \quad & \lim_{N\to\infty} \int_{\Omega_T} |\nabla u^N|^2 dxd\tau + \int_{\Omega_T} q'(u_\theta)u_\theta dxd\tau \\ & + \int_{\Omega_T} u_\theta(-\Delta)^{\frac{1}{2}} u_\theta dxd\tau \end{aligned} \tag{3.20}$$

On the other hand,

$$\begin{aligned} &\int_{\Omega_T} \mu^N(x,\tau)u_\theta(x,\tau)dxd\tau = \int_{\Omega_T} \mu^N(x,\tau)\Pi_N u_\theta(x,\tau)dxd\tau \\ = &\int_{\Omega_T} \left(\nabla u^N \cdot \nabla \Pi_N u_\theta(x,\tau) + q'(u^N)\Pi_N u_\theta(x,\tau) + \Pi_N u_\theta(x,\tau)(-\Delta)^{\frac{1}{2}} u^N \right) dxd\tau \\ = &\int_{\Omega_T} \left(\nabla u^N \cdot \nabla u_\theta(x,\tau) + q'(u^N)\Pi_N u_\theta(x,\tau) + u_\theta(-\Delta)^{\frac{1}{2}} u^N \right) dxd\tau. \end{aligned} \tag{3.21}$$

Since $\Pi_N u_\theta \to u_\theta$ strongly in $L^2(\Omega_T)$, $\mu^N \rightharpoonup \mu_\theta$ in $L^2(\Omega_T)$, by (3.10),(3.17), passing to the limit in (3.21) yields

$$\int_{\Omega_T} \mu_\theta u_\theta dx d\tau = \int_{\Omega_T} \left(|\nabla u_\theta|^2 + q'(u_\theta)) u_\theta + u_\theta (-\Delta)^{\frac{1}{2}} u_\theta \right) dx d\tau. \tag{3.22}$$

(3.20) and (3.22) gives

$$\lim_{N\to\infty} \int_{\Omega_T} |\nabla u^N|^2 dx d\tau = \int_{\Omega_T} |\nabla u_\theta|^2 dx d\tau. \tag{3.23}$$

By (3.5), $\nabla u^N \rightharpoonup \nabla u_\theta$ weakly in $L^2(\Omega_T)$, thus (3.23) implies

$$\nabla u^N \to \nabla u_\theta \text{ strongly in } L^2(\Omega_T). \tag{3.24}$$

By (3.6) and the lower bound on M_θ, we have

$$\left\| \nabla \frac{\mu^N}{g_\theta(u^N)} \right\|_{L^2(\Omega_T)} \le C\theta^{-\frac{m}{2}}.$$

By (3.2), (3.5) and (3.7) we have for $\phi_1 = (2\pi)^{-1}$,

$$\begin{aligned}
&\left| \int_{\mathbb{T}^2} \frac{\mu^N \phi_1}{g_\theta(u_N)} \right| dx = \left| \int_{\mathbb{T}^2} \mu^N \Pi_N \left(\frac{\phi_1}{g_\theta(u^N)} \right) \right| dx \\
\le\ & \left| \int_{\mathbb{T}^2} \nabla u^N \cdot \nabla \Pi_N \left(\frac{\phi_1}{g_\theta(u^N)} \right) dx + \int_{\mathbb{T}^2} q'(u^N) \Pi_N \left(\frac{\phi_1}{g_\theta(u^N)} \right) dx \right| \\
& + \left| \int_{\mathbb{T}^2} (-\Delta)^{\frac{1}{2}} u^N \Pi_N \left(\frac{\phi_1}{g_\theta(u^N)} \right) dx \right| \\
=\ & \left| \int_{\mathbb{T}^2} \nabla u^N \cdot \nabla \left(\frac{\phi_1}{g_\theta(u^N)} \right) dx + \int_{\mathbb{T}^2} q'(u^N) \Pi_N \left(\frac{\phi_1}{g_\theta(u^N)} \right) dx \right| \\
& + \left| \int_{\mathbb{T}^2} (-\Delta)^{\frac{1}{2}} u^N \left(\frac{\phi_1}{g_\theta(u^N)} \right) dx \right| \\
\le\ & C\theta^{-m-1} \left\| \nabla u^N \right\|^2_{L^2(\mathbb{T}^2)} + C\theta^{-m} \left\| q'(u^N) \right\|_{L^2(\mathbb{T}^2)} \|\phi_1\|_{L^2(\mathbb{T}^2)} \\
& + C\theta^{-m} \left\| \nabla u^N \right\|_{L^2(\mathbb{T}^2)} \|\phi_1\|_{L^2(\mathbb{T}^2)} \\
\le\ & C\theta^{-m-1}.
\end{aligned} \tag{3.25}$$

Poincare's inequality yields

$$\left\| \frac{\mu^N}{g_\theta(u^N)} \right\|_{L^2(0,T;H^1(\mathbb{T}^2))} \le C(\theta^{-m-1} + 1).$$

Thus there exists a $w_\theta \in L^2(0,T;H^1(\mathbb{T}^2))$ and a subsequence of $\frac{\mu^N}{g_\theta(u^N)}$, not relabeled, such that

$$\frac{\mu^N}{g_\theta(u^N)} \rightharpoonup w_\theta \text{ weakly in } L^2(0,T;H^1(\mathbb{T}^2)). \tag{3.26}$$

Therefore by (3.14), (3.26) and Sobolev embedding theorem, we have

$$\mu^N = g_\theta(u^N) \cdot \frac{\mu^N}{g_\theta(u^N)} \rightharpoonup \mu_\theta = g_\theta(u_\theta) w_\theta \text{ weakly in } L^2(0,T;W^{1,s}(\mathbb{T}^2)) \tag{3.27}$$

for any $1 \le s < 2$. Combining (3.15), (3.26)and (3.27), we have

$$\sqrt{M_\theta(u^N)}\nabla\frac{\mu^N}{g_\theta(u^N)} \rightharpoonup \sqrt{M_\theta(u_\theta)}\nabla\frac{\mu_\theta}{g_\theta(u_\theta)} \text{ weakly in } L^2(0,T;L^q(\mathbb{T}^2)) \tag{3.28}$$

for any $1 \le q < 2$. By (3.6), we can improve this convergence to

$$\sqrt{M_\theta(u^N)}\nabla\frac{\mu^N}{g_\theta(u^N)} \rightharpoonup \sqrt{M_\theta(u_\theta)}\nabla\frac{\mu_\theta}{g_\theta(u_\theta)} \text{ weakly in } L^2(0,T;L^2(\mathbb{T}^2)). \tag{3.29}$$

Since $g_\theta \ge \theta^m$, (3.12) implies

$$\frac{g_\theta'(u^N)}{g_\theta^{\frac{3}{2}}(u^N)} \to \frac{g_\theta'(u_\theta)}{g_\theta^{\frac{3}{2}}(u_\theta)} \text{ a.e in } \Omega_T. \tag{3.30}$$

In addition, $\frac{g_\theta'(u^N)}{g_\theta^{\frac{3}{2}}(u^N)}$ is bounded by

$$\left|\frac{g_\theta'(u^N)}{g_\theta^{\frac{3}{2}}(u^N)}\right| \le C\theta^{-1-\frac{m}{2}}. \tag{3.31}$$

It follows from (3.24), (3.30), (3.31) and generalized dominated convergence theorem (see Remark 3.2) that

$$\frac{g_\theta'(u^N)}{g_\theta^{\frac{3}{2}}(u^N)}\nabla u^N \to \frac{g_\theta'(u_\theta)}{g_\theta^{\frac{3}{2}}(u_\theta)}\nabla u_\theta \text{ strongly in } L^2(\Omega_T). \tag{3.32}$$

Let

$$f^N(t) = \left\|\frac{g_\theta'(u^N(x,t))}{g_\theta^{\frac{3}{2}}(u^N(x,t))}\nabla u^N(x,t) - \frac{g_\theta'(u_\theta(x,t))}{g_\theta^{\frac{3}{2}}(u_\theta(x,t))}\nabla u_\theta(x,t)\right\|_{L^2(\mathbb{T}^2)},$$

by (3.32), we can extract a subsequence of f^N, not relabeled, such that $f^N(t) \to 0$ a.e. in (0,T). By Egorov's theorem, for any given $\delta > 0$, there exists $T_\delta \subset [0,T]$ with $|T_\delta| < \delta$ such that $f^N(t)$ converges to 0 uniformly on $[0,T]\backslash T_\delta$.

Given $\alpha(t) \in L^2(0,T)$, for any $\varepsilon > 0$, there exists $T_\delta \subset [0,T]$ with $|T_\delta| < \delta$ such that

$$\int_{T_\delta} \alpha^2(t)dt < \epsilon. \tag{3.33}$$

Multiplying (3.1) by $\alpha(t)$ and integrating in time yield

$$\begin{aligned} &\int_0^T \alpha(t)\int_{\mathbb{T}^2} \partial_t u^N \phi_j dxdt \\ = \; & -\beta\int_{\Omega_T}\alpha(t)\mu^N\phi_j dxdt - \int_{\Omega_T}\alpha(t)M_\theta(u^N)\nabla\frac{\mu^N}{g_\theta(u^N)}\cdot\nabla\frac{\phi_j}{g_\theta(u^N)}dxdt \\ = \; & -\beta\int_{\Omega_T}\mu^N\alpha(t)\phi_j dxdt - \int_{\Omega_T}M_0\alpha(t)\nabla\frac{\mu^N}{g_\theta(u^N)}\cdot\nabla\phi_j dxdt \\ & + \int_{\Omega_T}\alpha(t)\sqrt{M_0}\phi_j\frac{g_\theta'(u^N)}{g_\theta^{\frac{3}{2}}(u^N)}\nabla u^N\cdot\sqrt{M_\theta(u^N)}\nabla\frac{\mu^N}{g_\theta(u^N)}dxdt \\ & = -A^N - I^N + II^N. \end{aligned} \tag{3.34}$$

Since $\alpha(t)\phi_j \in L^2(0,T;H^1(\mathbb{T}^2))$, by (3.26) and (3.27), we have

$$A^N = \beta \int_{\Omega_T} \mu^N \alpha(t)\phi_j dxdt \to \beta \int_{\Omega_T} \mu_\theta \alpha(t)\phi_j dxdt, \tag{3.35}$$

and

$$I^N = \int_{\Omega_T} M_0 \alpha(t) \nabla \frac{\mu^N}{g_\theta(u^N)} \cdot \nabla \phi_j dxdt \to \int_{\Omega_T} M_0 \alpha(t) \nabla \frac{\mu_\theta}{g_\theta(u_\theta)} \cdot \nabla \phi_j dxdt. \tag{3.36}$$

To find the limit of II^N, since

(3.37)

$$\begin{aligned}
&\int_{\Omega_T} \alpha(t)\phi_j \left(\frac{g_\theta'(u^N)}{g_\theta^{\frac{3}{2}}(u^N)} \nabla u^N \sqrt{M_\theta(u^N)} \nabla \frac{\mu^N}{g_\theta(u^N)} - \frac{g_\theta'(u_\theta)}{g_\theta^{\frac{3}{2}}(u_\theta)} \nabla u_\theta \sqrt{M_\theta(u_\theta)} \nabla \frac{\mu_\theta}{g_\theta(u_\theta)} \right) \\
= &\int_{\Omega_T} \alpha(t)\phi_j \left(\frac{g_\theta'(u^N)}{g_\theta^{\frac{3}{2}}(u^N)} \nabla u^N - \frac{g_\theta'(u_\theta)}{g_\theta^{\frac{3}{2}}(u_\theta)} \nabla u_\theta \right) \cdot \sqrt{M_\theta(u^N)} \nabla \frac{\mu^N}{g_\theta(u^N)} dxdt \\
&+ \int_{\Omega_T} \alpha(t)\phi_j \frac{g_\theta'(u_\theta)}{g_\theta^{\frac{3}{2}}(u_\theta)} \nabla u_\theta \cdot \left(\sqrt{M_\theta(u^N)} \nabla \frac{\mu^N}{g_\theta(u^N)} - \sqrt{M_\theta(u_\theta)} \nabla \frac{\mu_\theta}{g_\theta(u_\theta)} \right) dxdt \\
= &\ II_1^N + II_2^N
\end{aligned}$$

From bound

$$\begin{aligned}
&\int_{\Omega_T} \left| \alpha(t)\phi_j \frac{g_\theta'(u_\theta)}{g_\theta^{\frac{3}{2}}(u_\theta)} \nabla u_\theta \right|^2 dxdt \\
\leq &\ C\theta^{-2-m} \|\nabla u_\theta\|^2_{L^\infty(0,T;L^2(\mathbb{T}^2))} \int_0^T \alpha^2(t)^2 dt,
\end{aligned}$$

we conclude that $\alpha(t)\phi_j \frac{g_\theta'(u_\theta)}{g_\theta^{\frac{3}{2}}(u_\theta)} \nabla u_\theta \in L^2(\Omega_T)$. By (3.29), we can pass to the limit in II_2^N and conclude

$$II_2^N = \int_{\Omega_T} \alpha(t)\phi_j \frac{g_\theta'(u_\theta)}{g_\theta^{\frac{3}{2}}(u_\theta)} \nabla u_\theta \cdot \left(\sqrt{M_\theta(u^N)} \nabla \frac{\mu^N}{g_\theta(u^N)} - \sqrt{M_\theta(u_\theta)} \nabla \frac{\mu_\theta}{g_\theta(u_\theta)} \right) dxdt \to 0.$$

To pass to the limit in II_1^N, we write

$$\begin{aligned}
II_1^N &= \int_{\Omega_T} \alpha(t)\phi_j \left(\frac{g_\theta'(u^N)}{g_\theta^{\frac{3}{2}}(u^N)} \nabla u^N - \frac{g_\theta'(u_\theta)}{g_\theta^{\frac{3}{2}}(u_\theta)} \nabla u_\theta \right) \cdot \sqrt{M_\theta(u^N)} \nabla \frac{\mu^N}{g_\theta(u^N)} dxdt \\
&= \int_{T_\delta} \int_{\mathbb{T}^2} \alpha(t)\phi_j \left(\frac{g_\theta'(u^N)}{g_\theta^{\frac{3}{2}}(u^N)} \nabla u^N - \frac{g_\theta'(u_\theta)}{g_\theta^{\frac{3}{2}}(u_\theta)} \nabla u_\theta \right) \cdot \sqrt{M_\theta(u^N)} \nabla \frac{\mu^N}{g_\theta(u^N)} dxdt \\
&\quad + \int_{[0,T]\setminus T_\delta} \int_{\mathbb{T}^2} \alpha(t)\phi_j \left(\frac{g_\theta'(u^N)}{g_\theta^{\frac{3}{2}}(u^N)} \nabla u^N - \frac{g_\theta'(u_\theta)}{g_\theta^{\frac{3}{2}}(u_\theta)} \nabla u_\theta \right) \cdot \sqrt{M_\theta(u^N)} \nabla \frac{\mu^N}{g_\theta(u^N)} dxdt \\
&= II_{11}^N + II_{12}^N.
\end{aligned}$$

We bound II_{11}^N by

$$
\begin{aligned}
|II_{11}^N| &\le \int_{T_\delta} |\alpha(t)| \left\| \frac{g_\theta'(u^N)}{g_\theta^{\frac{3}{2}}(u^N)} \nabla u^N - \frac{g_\theta'(u_\theta)}{g_\theta^{\frac{3}{2}}(u_\theta)} \nabla u_\theta \right\|_{L^2(\mathbb{T}^2)} \left\| \sqrt{M_\theta(u^N)} \nabla \frac{\mu^N}{g_\theta(u^N)} \right\|_{L^2(\mathbb{T}^2)} dt \\
&\le \|\alpha(t)\|_{L^2(T_\delta)} \left\| \sqrt{M_\theta(u^N)} \nabla \frac{\mu^N}{g_\theta(u^N)} \right\|_{L^2(\Omega_T)} \left\| \frac{g_\theta'(u^N)}{g_\theta^{\frac{3}{2}}(u^N)} \nabla u^N - \frac{g_\theta'(u_\theta)}{g_\theta^{\frac{3}{2}}(u_\theta)} \nabla u_\theta \right\|_{L^\infty(0,T;L^2(\mathbb{T}^2))} \\
&\le C(\theta)\varepsilon.
\end{aligned}
$$

For II_{12}^N, we have

$$
\begin{aligned}
|II_{12}^N| &\le \int_{[0,T]\backslash T_\delta} |\alpha(t)| \left\| \frac{g_\theta'(u^N)}{g_\theta^{\frac{3}{2}}(u^N)} \nabla u^N - \frac{g_\theta'(u_\theta)}{g_\theta^{\frac{3}{2}}(u_\theta)} \nabla u_\theta \right\|_{L^2(\mathbb{T}^2)} \left\| \sqrt{M_\theta(u^N)} \nabla \frac{\mu^N}{g_\theta(u^N)} \right\|_{L^2(\mathbb{T}^2)} dt \\
&= \int_{[0,T]\backslash T_\delta} |\alpha(t)| f^N(t) | \left\| \sqrt{M_\theta(u^N)} \nabla \frac{\mu^N}{g_\theta(u^N)} \right\|_{L^2(\mathbb{T}^2)} dt.
\end{aligned}
$$

Since $f^N(t)$ converges to 0 uniformly, $\alpha(t) \in L^2(0,T)$ and $\left\| \sqrt{M_\theta(u^N)} \nabla \frac{\mu^N}{g_\theta(u^N)} \right\|_{L^2(\Omega_T)} \le C$, letting $N \to \infty$ in II_{12}^N yields $II_{12}^N \to 0$. Letting $\varepsilon \to 0$, we conclude $II_1^N \to 0$ as $N \to \infty$. Passing to the limit in (3.34), we have

$$
\begin{aligned}
&\int_0^T \alpha(t) \int_{\mathbb{T}^2} \langle \partial_t u_\theta, \phi_j \rangle_{(W^{1,q}(\mathbb{T}^2))', W^{1,q}(\mathbb{T}^2))} dt \\
= \; & -\beta \int_{\Omega_T} \alpha(t) \mu_\theta \phi_j dx dt - \int_{\Omega_T} \alpha(t) M_\theta(u_\theta) \nabla \frac{\mu_\theta}{g_\theta(u_\theta)} \cdot \nabla \frac{\phi_j}{g_\theta(u_\theta)} dx dt. \qquad (3.38)
\end{aligned}
$$

Fix $q > 2$, given any $\phi \in L^2(0,T;W^{1,q}(\mathbb{T}^2))$, its Fourier series $\sum_{j=1}^\infty a_j(t)\phi_j(x)$ converges strongly to ϕ in $L^2(0,T;W^{1,q}(\mathbb{T}^2))$. Hence

$$
\begin{aligned}
&\int_{\Omega_T} M_\theta(u_\theta) \nabla \frac{\mu_\theta}{g_\theta(u_\theta)} \cdot \nabla \frac{\phi - \Pi_N \phi}{g_\theta(u_\theta)} dx dt \\
= \; & \int_{\Omega_T} M_0 \nabla \frac{\mu_\theta}{g_\theta(u_\theta)} \cdot \nabla (\phi - \Pi_N \phi) dx dt \\
& - \int_{\Omega_T} (\phi - \Pi_N \phi) \sqrt{M_0} \frac{g_\theta'(u_\theta)}{g_\theta^{\frac{3}{2}}(u_\theta)} \nabla u_\theta \cdot \sqrt{M_\theta(u_\theta)} \nabla \frac{\mu_\theta}{g_\theta(u_\theta)} dx dt \\
= \; & J_1^N - J_2^N,
\end{aligned}
$$

where by(3.26), (3.27) and strong convergence of $\Pi_N \phi$ to ϕ in $L^2(0,T;H^1(\mathbb{T}^2))$, we conclude

$$
J_1^N = \int_{\Omega_T} M_0 \nabla \frac{\mu_\theta}{g_\theta(u_\theta)} \cdot \nabla (\phi - \Pi_N \phi) dx dt \to 0
$$

We can bound J_2^N by

$$\begin{aligned}
|J_2^N| &= \left|\int_{\Omega_T}(\phi-\Pi_N\phi)\sqrt{M_0}\frac{g'_\theta(u_\theta)}{g_\theta^{3/2}(u_\theta)}\nabla u_\theta\cdot\sqrt{M_\theta(u_\theta)}\nabla\frac{\mu_\theta}{g_\theta(u_\theta)}dxdt\right| \\
&\le \sqrt{M_0}\int_0^T\|\phi-\Pi_N\phi\|_{L^\infty(\mathbb{T}^2)}\left\|\frac{g'_\theta(u_\theta)}{g_\theta^{3/2}(u_\theta)}\nabla u_\theta\right\|_{L^2(\mathbb{T}^2)}\left\|\sqrt{M_\theta(u_\theta)}\nabla\frac{\mu_\theta}{g_\theta(u_\theta)}\right\|_{L^2(\mathbb{T}^2)} \\
&\le \sqrt{M_0}\left\|\frac{g'_\theta(u_\theta)}{g_\theta^{3/2}(u_\theta)}\nabla u_\theta\right\|_{L^\infty(0,T;L^2(\mathbb{T}^2))}\left\|\sqrt{M_\theta(u_\theta)}\nabla\frac{\mu_\theta}{g_\theta(u_\theta)}\right\|_{L^2(\Omega_T)}\|\phi-\Pi_N\phi\|_{L^2(0,T;W^{1,q}(\mathbb{T}^2))} \\
&\to 0 \text{ as } N\to\infty.
\end{aligned}$$

Consequently (3.38) implies

$$\begin{aligned}
&\int_0^T\langle\partial_t u_\theta,\phi\rangle_{(W^{1,q}(\mathbb{T}^2))',W^{1,q}(\mathbb{T}^2))}\,dt \\
&= -\beta\int_{\Omega_T}\mu_\theta\phi dxdt-\int_{\Omega_T}M_\theta(u_\theta)\nabla\frac{\mu_\theta}{g_\theta(u_\theta)}\cdot\nabla\frac{\phi}{g_\theta(u_\theta)}dxdt
\end{aligned} \tag{3.39}$$

for all $\phi\in L^2(0,T;W^{1,q}(\mathbb{T}^2))$ with $q>2$. Moreover, since $u^N(x,0)=\Pi_N u_0(x)\to u_0(x)$ in $H^1(\mathbb{T}^2)$, we see that $u_\theta(x,0)=u_0(x)$ by (3.11).

Remark 3.2. (Generalized dominated convergence theorem See, e.g. Theorem 17 of section 4.4 of [35], p 92) Assume $E\subset\mathbb{R}^n$ is measurable. $g_n\to g$ strongly in $L^q(E)$ for $1\le q<\infty$ and $f_n,\ f\colon E\to\mathbb{R}^n$ are measurable functions satisfying

$$f_n\to f \text{ a.e. in } E;\quad |f_n|^p\le|g_n|^q \text{ a.e. in } E$$

with $1\le p<\infty$, then $f_n\to f$ in $L^p(E)$.

3.1.4. Regularity of u_θ. We now consider the regularity of u_θ. Given any $a_j(t)\in L^2(0,T)$, $a_j(t)\phi_j\in L^2(0,T;C(\mathbb{T}^2))$. Integrating (3.2) from 0 to T, by (3.17),(3.27) and (3.24), we have

$$\begin{aligned}
&\int_{\Omega_T}\mu_\theta(x,t)a_j(t)\phi_j(x)dxdt \\
&= \int_{\Omega_T}\left(\nabla u_\theta\cdot a_j(t)\nabla\phi_j+q'(u_\theta)a_j(t)\phi_j+a_j(t)\phi_j(-\Delta)^{\frac12}u_\theta\right)dxdt
\end{aligned}$$

for all $j\in\mathbf{N}$. Given any $\phi\in L^2(0,T;H^1(\mathbb{T}^2))$, its Fourier series strongly converges to ϕ in $L^2(0,T;H^1(\mathbb{T}^2))$, therefore

$$\int_{\Omega_T}\mu_\theta(x,t)\phi(x)dxdt=\int_{\Omega_T}\left(\nabla u_\theta\cdot\nabla\phi+q'(u_\theta)\phi+\phi(-\Delta)^{\frac12}u_\theta\right)dxdt. \tag{3.40}$$

Recall $\mu_\theta\in L^2(0,T;L^p(\mathbb{T}^2))$ and $q'(u_\theta)\in L^\infty(0,T;L^p(\mathbb{T}^2))$ for any $1\le p<\infty$, regularity theory implies $u_\theta\in L^2(0,T;H^2(\mathbb{T}^2))$. Hence

$$\mu_\theta=-\Delta u_\theta+q'(u_\theta)+(-\Delta)^{\frac12}u_\theta \text{ a.e. in } \Omega_T. \tag{3.41}$$

By Sobolev embedding theorem, $u_\theta\in L^\infty(0,T;H^1(\mathbb{T}^2))\hookrightarrow L^\infty(0,T;L^p(\mathbb{T}^2))$ for any $1\le p<\infty$. Since growth assumption on q implies $|q''(u)|\le C(1+|u|^{r-1})$, pick $p>2$, we have

$$
\begin{aligned}
\int_{\mathbb{T}^2} |\nabla q'(u_\theta)|^2 dx &= \int_{\mathbb{T}^2} |q''(u_\theta)|^2 |\nabla u_\theta|^2 dx \\
\le\ & \|q''(u_\theta)\|^2_{L^{\frac{2p}{p-2}}(\mathbb{T}^2)} \|\nabla u_\theta\|^2_{L^p(\mathbb{T}^2)} \\
\le\ & C\left(1 + \|u_\theta\|^{2(r-1)}_{L^{\frac{2p}{p-2}(r-1)}(\mathbb{T}^2)}\right) \|\nabla u_\theta\|^2_{L^p(\mathbb{T}^2)} \\
\le\ & C\left(1 + \|u_\theta\|^{2(r-1)}_{L^\infty(0,T;H^1(\mathbb{T}^2))}\right) \|\nabla u_\theta\|^2_{L^p(\mathbb{T}^2)} \\
\le\ & C\left(1 + \|u_\theta\|^{2(r-1)}_{L^\infty(0,T;H^1(\mathbb{T}^2))}\right) \|\nabla u_\theta\|^2_{H^1(\mathbb{T}^2)}.
\end{aligned}
$$

Therefore $\nabla q'(u_\theta) = q''(u_\theta)\nabla u_\theta \in L^2(\Omega_T)$ with

$$
\int_{\Omega_T} |\nabla q'(u_\theta)|^2 dxdt \le C\left(1 + \|u_\theta\|^{2(r-1)}_{L^\infty(0,T;H^1(\mathbb{T}^2))}\right) \|\nabla u_\theta\|^2_{L^2(0,T;H^1(\mathbb{T}^2))}.
$$

Hence $q'(u_\theta) \in L^2(0,T;H^1(\mathbb{T}^2))$, combined with $\mu_\theta \in L^2(0,T;W^{1,s}(\mathbb{T}^2))$ for any $1 \le s < 2$ and $(-\Delta)^{\frac{1}{2}} u_\theta \in L^2(0,T;H^1(\mathbb{T}^2))$, we have $u_\theta \in L^2(0,T;W^{3,s}(\mathbb{T}^2))$ and

$$
\nabla \mu_\theta = -\nabla\Delta u_\theta + q''(u_\theta)\nabla u_\theta + \nabla(-\Delta)^{\frac{1}{2}} u_\theta \text{ a.e. in } \Omega_T. \tag{3.42}
$$

Regularity of u_θ implies $\nabla u_\theta \in L^\infty(0,T;L^2(\mathbb{T}^2)) \cap L^2(0,T;L^\infty(\mathbb{T}^2))$. A simple interpolation shows $\nabla u_\theta \in L^{\frac{2s}{s-2}}(0,T;L^s(\mathbb{T}^2))$ for any $s > 2$. Given any $\phi \in L^p(0,T;W^{1,q}(\mathbb{T}^2))$ with $p > 2$ and $q > 2$, we have $g_\theta(u_\theta)\phi \in L^2(0,T;W^{1.r}(\mathbb{T}^2))$ for any $r < q$. Picking $g_\theta(u_\theta)\phi$ as a test function in (3.39), we have

$$
\int_{\Omega_T} \partial_t u_\theta g_\theta(u_\theta)\phi dxdt = -\beta \int_{\Omega_T} g_\theta(u_\theta)\mu_\theta \phi dxdt - \int_{\Omega_T} M_\theta(u_\theta)\nabla \frac{\mu_\theta}{g_\theta(u_\theta)} \cdot \nabla\phi dxdt \tag{3.43}
$$

for any $\phi \in L^p(0,T;W^{1,q}(\mathbb{T}^2))$ with $p, q > 2$.

Remark 3.3. In fact, since $M_\theta(u_\theta) \in L^\infty(0,T;L^p(\mathbb{T}^2))$ for $1 \le p < \infty$, the right hand side of (3.43) is well defined for any $\phi \in L^2(0,T,W^{1,q}(\mathbb{T}^2))$ and we can extend (3.43) to hold for all $\phi \in L^2(0,T,W^{1,q}(\mathbb{T}^2))$.

3.1.5. Energy Inequality. Since u^N and μ^N satisfies energy identity (3.4), passing to the limit as $N \to \infty$ and using the weak convergence of u^N, $q'(u^N)$ and $\sqrt{M_\theta(u^N)}\nabla\frac{\mu^N}{g_\theta(u^N)}$, the energy inequality (1.15) follows.

This finishes the proof of Proposition 1.1.

3.2. Phase field model with degenerate mobility. In this subsection, we prove Theorem 1.2.

Fix initial data $u_0 \in H^1(\mathbb{T}^2)$. We pick a monotone decreasing positive sequence θ_i with $\lim\limits_{i\to\infty} \theta_i = 0$. By Proposition 1.1 and (3.43), for each θ_i, there exists

$$
u_i \in L^\infty(0,T;H^1(\mathbb{T}^2)) \cap L^2(0,T;W^{3,s}(\mathbb{T}^2)) \cap C([0,T];L^p(\mathbb{T}^2))
$$

with weak derivative

$$
\partial_t u_i \in L^2(0,T;(W^{1,q}(\mathbb{T}^2))'),
$$

where $1 \le p < \infty$, $1 \le s < 2$, $q > 2$ such that $u_{\theta_i}(x,0) = u_0(x)$ and for all $\phi \in L^2(0,T;W^{1,q}(\mathbb{T}^2))$,

$$
\int_{\Omega_T} \partial_t u_i \phi dxdt = -\beta \int_{\Omega_T} \mu_i \phi dxdt - \int_{\Omega_T} M_i(u_i)\nabla\frac{\mu_i}{g_i(u_i)}\nabla\frac{\phi}{g_i(u_i)} dxdt, \tag{3.44}
$$

$$
\mu_i = -\Delta u_i + q'(u_i) + (-\Delta)^{\frac{1}{2}} u_i. \tag{3.45}
$$

Moreover, for all $\phi \in L^p(0,T;W^{1,q}(\mathbb{T}^2))$ with $p,q>2$, the following holds:

$$\int_{\Omega_T} g_i(u_i)\partial_t u_i \phi dxdt = -\beta \int_{\Omega_T} g_i(u_i)\mu_i\phi dxdt - \int_{\Omega_T} M_i(u_i)\nabla \frac{\mu_i}{g_i(u_i)}\nabla\phi dxdt. \tag{3.46}$$

Here we write $u_i = u_{\theta_i}$, $M_i(u_i) = M_{\theta_i}(u_{\theta_i})$, $g_i(u_i) = g_{\theta_i}(u_{\theta_i})$ for simplicity of notations. Noticing the bounds in (3.5) and (3.6) depend only on u_0, we can find a constant C, independent of θ_i such that

$$\|\mu_i\|_{L^2(\Omega_T)}, \|u_i\|_{L^\infty(0,T;H^1(\mathbb{T}^2))} \le C, \tag{3.47}$$

$$\left\| \sqrt{M_i(u_i)}\nabla \frac{\mu_i}{g_i(u_i)} \right\|_{L^2(\Omega_T)} \le C. \tag{3.48}$$

Growth condition on q', and Sobolev embedding theorem gives

$$\|q'(u_i)\|_{L^\infty(0,T;L^p(\mathbb{T}^2))} \le C,$$
$$\|M_i(u_i)\|_{L^\infty(0,T;L^p(\mathbb{T}^2))} \le C$$

for any $1 \le p < \infty$. By (3.46), for any $\phi \in L^p(0,T;W^{1,q}(\mathbb{T}^2))$ with $p,q>2$,

$$\begin{aligned}
&\left|\int_{\Omega_T} g_i(u_i)\partial_t u_i\phi dxdt\right| = \left|\int_{\Omega_T}\left[\beta g_i(u_i)\mu_i\phi + M_i(u_i)\nabla\frac{\mu_i}{g_i(u_i)}\nabla\phi\right]dxdt\right| \\
\le\ & \beta\int_0^T \left(\|\mu_i\|_{L^2(\mathbb{T}^2)}\|g_i(u_i)\|_{L^{\frac{2q}{q-2}}(\mathbb{T}^2)}\|\phi\|_{L^q(\mathbb{T}^2)}\right)dt \\
&+\int_0^T\left(\left\|\sqrt{M_i(u_i)}\nabla\frac{\mu_i}{g_i(u_i)}\right\|_{L^2(\mathbb{T}^2)}\left\|\sqrt{M_i(u_i)}\right\|_{L^{\frac{2q}{q-2}}(\mathbb{T}^2)}\|\nabla\phi\|_{L^q(\mathbb{T}^2)}\right)dt \\
\le\ & \beta\|g_i(u_i)\|_{L^{\frac{2p}{p-2}}(0,T;L^{\frac{2q}{q-2}}(\mathbb{T}^2))}\|\mu_i\|_{L^2(\Omega_T)}\|\phi\|_{L^p(0,T;L^q(\mathbb{T}^2))} \\
&+\|M_i(u_i)\|^{\frac12}_{L^{\frac{p}{p-2}}(0,T;L^{\frac{q}{q-2}}(\mathbb{T}^2))}\left\|\sqrt{M_i(u_i)}\nabla\frac{\mu_i}{g_i(u_i)}\right\|_{L^2(\Omega_T)}\|\nabla\phi\|_{L^p(0,T;L^q(\mathbb{T}^2))} \\
\le\ & C\|\phi\|_{L^p(0,T;W^{1,q}(\mathbb{T}^2))}.
\end{aligned}$$

Let

$$G_i(u_i) = \int_0^{u_i} g_i(a)da. \tag{3.49}$$

Then $\partial_t G_i(u_i) = g_i(u_i)\partial_t u_i \in L^{p'}(0,T;(W^{1,q}(\mathbb{T}^2))')$ with $p' = \frac{p}{p-1}$ and

$$\|\partial_t G_i(u_i)\|_{L^{p'}(0,T;(W^{1,q}(\mathbb{T}^2))')} \le C \text{ for all } i. \tag{3.50}$$

Moreover, by growth assumption on g and estimates on u_i, we have

$$\|G_i(u_i)\|_{L^\infty(0,T;W^{1,s}(\mathbb{T}^2))} \le C. \tag{3.51}$$

for $1 \le s < 2$. By (3.47), (3.48)-(3.51) and Remark 3.1 we can find a subsequence, not relabeled, a function $u \in L^\infty(0,T;H^1(\mathbb{T}^2))$, a function $\mu \in L^2(\Omega_T)$, a function

$\xi \in L^2(\Omega_T)$ and a function $\eta \in L^\infty(0,T;W^{1,s}(\mathbb{T}^2))$ such that as $i \to \infty$,

(3.52) $$u_i \rightharpoonup u \text{ weakly-* in } L^\infty(0,T;H^1(\mathbb{T}^2)),$$

(3.53) $$\mu_i \rightharpoonup \mu \text{ weakly in } L^2(\Omega_T),$$

(3.54) $$\sqrt{M_i(u_i)}\nabla \frac{\mu_i}{g_i(u_i)} \rightharpoonup \xi \text{ weakly in } L^2(\Omega_T),$$

(3.55) $$G_i(u_i) \rightharpoonup \eta \text{ weakly-* in } L^\infty(0,T;W^{1,s}(\mathbb{T}^2))$$

(3.56) $$G_i(u_i) \to \eta \text{ strongly in } L^\alpha(0,T;L^\beta(\mathbb{T}^2)) \text{ and a.e. in } \Omega_T,$$

(3.57) $$G_i(u_i) \to \eta \text{ strongly in } C(0,T;L^\beta(\mathbb{T}^2)),$$

(3.58) $$\partial_t G_i(u_i) \rightharpoonup \partial_t \eta \text{ weakly in } L^{p'}(0,T;(W^{1,q}\mathbb{T}^2))').$$

where $1 \le \alpha, \beta < \infty$. By (3.57) and (3.66) from Remark 3.4, we have

$$\|G_i(u_i(x,t+h)) - G_i(u_i(x,t))\|_{C([0,T];L^\beta(\mathbb{T}^2))} \to 0 \text{ uniformly in } i \text{ as } h \to 0.$$

Thus given any $\varepsilon > 0$, there exists $h_\varepsilon > 0$ such that for all $0 < h < h_\varepsilon$ and all i,

$$\|G_i(u_i(x,t+h)) - G_i(u_i(x,t))\|^\beta_{C([0,T];L^\beta(\mathbb{T}^2))} < \varepsilon.$$

Given any $\delta > 0$, let $I_\delta = (1-\delta, 1+\delta) \cup (-1-\delta, -1+\delta)$. Consider the interval having $u_i(x,t)$ and $u_i(x,t+h)$ as end points. Denote this interval by $J_i(x,t;h)$. We consider three cases.

Case I: $J_i(x,t;h) \cap I_\delta = \varnothing$**.**

In this case, $g_i(s) \ge \max\{\theta_i^m, \delta^m\}$ for any $s \in J_i(x,t;h)$ and

$$|G_i(u_i(x,t+h)) - G_i(u_i(x,t))| = \left|\int_{u_i(x,t)}^{u_i(x,t+h)} g_i(s)ds\right| \ge \delta^m |u_i(x,t+h) - u_i(x,t)|.$$

Case II: $J_i(x,t;h) \cap I_\delta \neq \varnothing$ **and** $|u_i(x,t+h) - u_i(x,t)| \ge 3\delta$ **.**

In this case, we have

$$|J_i(x,t;h) \cap I_\delta^c| \ge \frac{1}{3}|J_i(x,t;h)|$$

and

$$\begin{aligned} |G_i(u_i(x,t+h)) - G_i(u_i(x,t))| &\ge \left|\int_{J_i(x,t;h)\cap I_\delta^c} g_i(s)ds\right| \\ &\ge \frac{\delta^m}{3}|u_i(x,t+h) - u_i(x,t)|. \end{aligned}$$

Case III: $J_i(x,t;h) \cap I_\delta \neq \varnothing$ **and** $|u_i(x,t+h) - u_i(x,t)| < 3\delta$

In this case, we have

$$g_i(s) \le \max\{(8\delta + 16\delta^2)^m, \theta_i^m\} \text{ for any } s \in J_i(x,t;h).$$

Thus

$$|G_i(u_i(x,t+h)) - G_i(u_i(x,t))| \le 3\delta \max\{(8\delta + 16\delta^2)^m, \theta_i^m\}.$$

Pick $\delta = \varepsilon^{\frac{1}{2m\beta}}$ and fix t. Let

$$\Omega_i = \{x \in \mathbb{T}^2 : J_i(x,t:h) \text{ satisfies case I or II}\}.$$

Then

$$\begin{aligned}
&\int_{\mathbb{T}^2} |u_i(x,t+h) - u_i(x,t)|^\beta \, dx \\
= \; & \int_{\Omega_i} |u_i(x,t+h) - u_i(x,t)|^\beta \, dx + \int_{\mathbb{T}^2 \backslash \Omega_i} |u_i(x,t+h) - u_i(x,t)|^\beta \, dx \\
\le \; & 3^\beta \varepsilon^{-\frac{1}{2}} \int_{\Omega_i} |G_i(u_i(x,t+h)) - G_i(u_i(x,t))|^\beta \, dx + \int_{\mathbb{T}^2 \backslash \Omega_i} |u_i(x,t+h) - u_i(x,t)|^\beta \, dx \\
\le \; & 3^\beta \varepsilon^{\frac{1}{2}} + C\varepsilon^{\frac{1}{2m}}
\end{aligned}$$

Taking maximum over $t \in [0,T]$ on the left side, we have for all i, any $h < h_\varepsilon$,

$$\|(u_i(x,t+h) - u_i(x,t)\|^\beta_{C([0,T];L^\beta(\mathbb{T}^2))} < \varepsilon^{\frac{1}{2}} + C\varepsilon^{\frac{1}{2m}}.$$

Thus

$$\|u_i(x,t+h) - u_i(x,t)\|^\beta_{C([0,T];L^\beta(\mathbb{T}^2))} \to 0 \text{ uniformly as } h \to 0.$$

In addition, for any $0 < t_1 < t_2 < T$, (3.47) implies

$$\int_{t_1}^{t_2} u_i(x,t) dt \text{ is relatively compact in } L^\beta(\mathbb{T}^2).$$

Therefore we conclude from Remark 3.4 that

$$u_i \to u(x,t) \text{ strongly in } C([0,T];L^\beta(\mathbb{T}^2)) \text{ for } 1 \le \beta < \infty. \tag{3.59}$$

Similarly. we can prove

$$u_i \to u(x,t) \text{ strongly in } L^\alpha(0,T;L^\beta(\mathbb{T}^2)) \text{ for } 1 \le \alpha, \beta < \infty \text{ and a.e. in } \Omega_T. \tag{3.60}$$

Growth condition on $M(u)$ and (3.59), (3.60) yield

$$M_i(u_i) \to M(u) \text{ strongly in } C([0,T];L^\beta(\mathbb{T}^2)) \text{ for } 1 \le \beta < \infty, \tag{3.61}$$

$$M_i(u_i) \to M(u) \text{ strongly in } L^\alpha(0,T;L^\beta(\mathbb{T}^2)) \text{ for } 1 \le \alpha, \beta < \infty, \tag{3.62}$$

$$\sqrt{M_i(u_i)} \to \sqrt{M(u)} \text{ strongly in } C([0,T];L^\gamma(\mathbb{T}^2)) \text{ for } 1 \le \gamma < \infty. \tag{3.63}$$

Hence $G_i(u_i)$ converges to $G(u)$ a.e. in Ω_T and $\eta = G(u)$. Passing to the limit in (3.46), by (3.47), (3.54), (3.58), (3.61) and (3.63), we have

$$\begin{aligned}
&\int_0^T \langle g(u)\partial_t u, \phi \rangle_{((W^{1,q}(\mathbb{T}^2))', W^{1,q}(\mathbb{T}^2))} \, dt \\
&= \; -\beta \int_{\Omega_T} g(u)\mu\phi dx dt - \int_{\Omega_T} \sqrt{M(u)}\xi \cdot \nabla\phi dx dt
\end{aligned} \tag{3.64}$$

for any $\phi \in L^p(0,T;W^{1,q}(\mathbb{T}^2))$ with $p,q > 2$.

Remark 3.4. (Compactness in $L^p(0,T;B)$ Theorem 1 in [36]) Assume B is a Banach space and $F \subset L^p(0,T;B)$. F is relatively compact in $L^p(0,T;B)$ for $1 \le p < \infty$, or in $C([0,T],B)$ for $p = \infty$ if and only if

$$\left\{ \int_{t_1}^{t_2} f(t) dt : f \in F \right\} \text{ is relatively compact in } B, \forall 0 < t_1 < t_2 < T \tag{3.65}$$

$$\|\tau_h f - f\|_{L^p(0,T;B)} \to 0 \text{ as } h \to 0 \text{ uniformly for } f \in F. \tag{3.66}$$

Here $\tau_h f(t) = f(t+h)$ for $h > 0$ is defined on $[-h, T-h]$.

3.2.1. Weak convergence of $\nabla\frac{\mu_i}{g_i(u_i)}$. We now look for relation between ξ and u. Following the idea in [9], we decompose Ω_T as follows. Let δ_j be a positive sequence monotonically decreasing to 0. By (3.54) and Egorov's theorem, for every $\delta_j > 0$, there exists $B_j \subset \Omega_T$ satisfying $|\Omega_T \backslash B_j| < \delta_j$ such that

$$u_i \to u \text{ uniformly in } B_j. \tag{3.67}$$

We can pick

$$B_1 \subset B_2 \subset \cdots \subset B_j \subset B_{j+1} \subset \cdots \subset \Omega_T. \tag{3.68}$$

Define

$$P_j := \{(x,t) \in \Omega_T : |1-u^2| > \delta_j\}.$$

Then

$$P_1 \subset P_2 \subset \cdots \subset P_j \subset P_{j+1} \subset \cdots \subset \Omega_T. \tag{3.69}$$

Let $B = \cup_{j=1}^{\infty} B_j$ and $P = \cup_{j=1}^{\infty} P_j$. Then $|\Omega_T \backslash B| = 0$ and each B_j can be split into two parts:

$$\begin{aligned} D_j &= B_j \cap P_j, \text{ where } |1-u^2| > \delta_j, \text{ and } u_i \to u \text{ uniformly},\\ \tilde{D}_j &= B_j \backslash P_j, \text{ where } |1-u^2| \le \delta_j, \text{ and } u_i \to u \text{ uniformly }. \end{aligned}$$

(3.68) and (3.69) imply

$$D_1 \subset D_2 \subset \cdots \subset D_j \subset D_{j+1} \subset \cdots \subset D := B \cap P. \tag{3.70}$$

For any $\Psi \in L^p(0,T;L^q(\mathbb{T}^2,\mathbb{R}^2))$ with $p,q > 2$, we have

$$\begin{aligned} &\int_{\Omega_T} M_i(u_i)\nabla\frac{\mu_i}{g_i(u_i)}\cdot\Psi dxdt\\ =\ &\int_{\Omega_T\backslash B_j} M_i(u_i)\nabla\frac{\mu_i}{g_i(u_i)}\cdot\Psi dxdt + \int_{D_j} M_i(u_i)\nabla\frac{\mu_i}{g_i(u_i)}\cdot\Psi dxdt\\ &+\int_{\tilde{D}_j} M_i(u_i)\nabla\frac{\mu_i}{g_i(u_i)}\cdot\Psi dxdt \end{aligned} \tag{3.71}$$

The left hand side of (3.71) converges to $\int_{\Omega_T}\sqrt{M(u)}\xi\cdot\Psi dxdt$. We analyze the three terms on the right hand side separately. To estimate the first term on the right hand side of (3.71), noticing $|\Omega_T \backslash B_j| \to 0$ and

$$\lim_{i\to\infty}\int_{\Omega_T\backslash B_j} M_i(u_i)\nabla\frac{\mu_i}{g_i(u_i)}\cdot\Psi dxdt = \int_{\Omega_T\backslash B_j}\sqrt{M(u)}\xi\cdot\Psi dxdt,$$

we have

$$\lim_{j\to\infty}\lim_{i\to\infty}\int_{\Omega_T\backslash B_j} M_i(u_i)\nabla\frac{\mu_i}{g_i(u_i)}\cdot\Psi dxdt = 0.$$

By uniform convergence of u_i to u in B_j, we introduce subsequence $u_{j,k}$ such that $u_{j,k} \to u$ uniformly in B_j and there exists N_j such that for all $k \ge N_j$,

$$|1-u_{j,k}^2| > \frac{\delta_j}{2} \text{ in } D_j, \qquad |1-u_{j,k}^2| \le 2\delta_j \text{ in } \tilde{D}_j. \tag{3.72}$$

Thus the third term on the right hand side of (3.71) can be estimated by

$$\begin{aligned}
&\lim_{j\to\infty}\lim_{k\to\infty}\left|\int_{\tilde{D}_j} M_{j,k}(u_{j,k})\nabla\frac{\mu_{j,k}}{g_{j,k}(u_{j,k})}\cdot\Psi dxdt\right| \\
\le\ & \lim_{j\to\infty}\lim_{k\to\infty}\left\{\left(\sup_{\tilde{D}_j}\sqrt{M_{j,k}(u_{j,k})}\right)\|\Psi\|_{L^2(\tilde{D}_j)}\left\|\sqrt{M_{j,k}(u_{j,k})}\nabla\frac{\mu_{j,k}}{g_{j,k}(u_{j,k})}\right\|_{L^2(\tilde{D}_j)}\right\} \\
\le\ & \left(\sup_{\tilde{D}_j}\sqrt{M_{j,k}(u_{j,k})}\right)|\mathbb{T}^2|^{\frac{q-2}{2q}}\|\Psi\|_{L^2(0,T;L^q(\mathbb{T}^2))}\left\|\sqrt{M_{j,k}(u_{j,k})}\nabla\frac{\mu_{j,k}}{g_{j,k}(u_{j,k})}\right\|_{L^2(\tilde{D}_j)} \\
&\le C\lim_{j\to\infty}\lim_{k\to\infty}\max\left\{(2\delta_j)^{m/2},\theta_{j,k}^{m/2}\right\} \\
=\ & 0.
\end{aligned}$$

For the second term, we see that

$$\begin{aligned}
&\left(\frac{\delta_j}{2}\right)^m\int_{D_j}|\nabla\frac{\mu_{j,k}}{g_{j,k}(u_{j,k})}|^2dxdt \\
\le\ & \int_{D_j}M_{j,k}(u_{j,k})|\nabla\frac{\mu_{j,k}}{g_{j,k}(u_{j,k})}|^2dxdt \\
\le\ & \int_{\Omega_T}M_{j,k}(u_{j,k})|\nabla\frac{\mu_{j,k}}{g_{j,k}(u_{j,k})}|^2dxdt\le C.
\end{aligned}$$

Therefore $\nabla\frac{\mu_{j,k}}{g_{j,k}(u_{j,k})}$ is bounded in $L^2(D_j)$ and we can extract a further subsequence, not relabeled, which converges weakly to some $\xi_j\in L^2(D_j)$. Since D_j is an increasing sequence of sets with $\lim_{j\to\infty}D_j=D$, we have $\xi_j=\xi_{j-1}$ a.e. in D_{j-1}. By setting $\xi_j=0$ outside D_j, we can extend ξ_j to a L^2 function $\tilde{\xi}_j$ defined in D. Therefore for a.e. $x\in D$, there exists a limit of $\tilde{\xi}_j(x)$ as $j\to\infty$. Let $\xi(x)=\lim_{j\to\infty}\tilde{\xi}_j(x)$, we see that $\xi(x)=\xi_j(x)$ for a.e $x\in D_j$ and for all j.

By a standard diagonal argument, we can extract a subsequnce such that

$$\nabla\frac{\mu_{k,N_k}}{g_{k,N_k}(u_{k,N_k})}\rightharpoonup\zeta \text{ weakly in } L^2(D_j) \text{ for all } j. \tag{3.73}$$

By strong convergence of $\sqrt{M_i(u_i)}$ to $\sqrt{M(u)}$ in $C([0,T];L^\beta(\mathbb{T}^2))$ for $1\le\beta<\infty$, we obtain

$$\chi_{D_j}\sqrt{M_{k,N_k}(u_{k,N_k})}\nabla\frac{\mu_{k,N_k}}{g_{k,N_k}(u_{k,N_k})}\rightharpoonup\chi_{D_j}\sqrt{M(u)}\zeta$$

weakly in $L^2(0,T;L^q(\mathbb{T}^2))$ for $1\le q<2$ and all j. Recall $\sqrt{M_i(u_i)}\nabla\frac{\mu_i}{g_i(u_i)}\to\xi$ weakly in $L^2(\Omega_T)$, we have $\xi=\sqrt{M(u)}\zeta$ in D_j for all j. Hence $\xi=\sqrt{M(u)}\zeta$ in D and consequently

$$\chi_D M_{k,N_k}(u_{k,N_k})\nabla\frac{\mu_{k,N_k}}{g_{k,N_k}(u_{k,N_k})}\rightharpoonup\chi_D M(u)\zeta$$

weakly in $L^2(0,T;L^q(\mathbb{T}^2))$ for $1\le q<2$.

Replacing u_i by subsequence u_{k,N_k} in (3.71) and letting $k\to\infty$ then $j\to\infty$, we have

$$\begin{aligned}
\int_{\Omega_T}\sqrt{M(u)}\xi\cdot\Psi dxdt &= \lim_{j\to\infty}\int_{D_j}M(u)\zeta\cdot\Psi dxdt \\
&= \int_D M(u)\zeta\cdot\Psi dxdt.
\end{aligned} \tag{3.74}$$

It follows from (3.64) and (3.74) that

$$\begin{aligned}(3.75)\qquad &\int_0^T \langle g(u)\partial_t u, \phi\rangle_{((W^{1,q}(\mathbb{T}^2))', W^{1,q}(\mathbb{T}^2))}\, dt \\ = \; & -\beta \int_{\Omega_T} g(u)\mu\phi dxdt - \int_D M(u)\zeta\cdot\nabla\phi dxdt\end{aligned}$$

for all $\phi \in L^p(0,T;W^{1,q}(\mathbb{T}^2))$ where $p,q > 2$.

3.2.2. Relation between ζ and u. The desired relation between ζ and u is

$$\zeta = \frac{1}{g(u)}\nabla\mu - \mu\frac{g'(u)}{g^2(u)}\nabla u \tag{3.76}$$

$$\mu = -\Delta u + q'(u) + (-\Delta)^{\frac{1}{2}}u. \tag{3.77}$$

Given the known regularity $u \in L^\infty(0,T;H^1(\mathbb{T}^2))$ and degeneracy of $g(u)$, the right hand side of (3.76) might not be defined as a function. We can, however, under suitable assumptions on integrability of $\nabla\Delta u$, find an explicit expression of ζ in the form of (3.76)-(3.77) in suitable subset of Ω_T.

Claim I: If for some j, the interior of D_j, denoted by $(D_j)^\circ$, is not empty, then

$$\nabla\Delta u \in L^1((D_j)^\circ)$$

and

$$\zeta = \frac{-\nabla\Delta u + q''(u)\nabla u + \nabla(-\Delta)^{\frac{1}{2}}u}{g(u)} - \frac{g'(u)}{g^2(u)}\left(-\Delta u + q'(u) + (-\Delta)^{\frac{1}{2}}u\right)\nabla u$$

a.e. in $(D_j)^\circ$.

Proof of the claim I. Since

$$\mu_{k,N_k} = -\Delta u_{k,N_k} + q'(u_{k,N_k}) + (-\Delta)^{\frac{1}{2}}u_{k,N_k} \text{ in } \Omega_T, \tag{3.78}$$

The right hand side of (3.78) converges to $-\Delta u + q'(u) + (-\Delta)^{\frac{1}{2}}u$ in distributional sense while the left side converges weakly to μ in $L^2(\Omega_T)$. Hence

$$\mu = -\Delta u + q'(u) + (-\Delta)^{\frac{1}{2}}u \text{ in } L^2(\Omega_T).$$

Therefore $u \in L^2(0,T;H^2(\mathbb{T}^2))$. On the other hand, using u_{k,N_k} and u as test functions in (3.40) yield

$$\int_{\Omega_T}\mu_{k,N_k}u_{k,N_k}dxdt = \int_{\Omega_T}\left(|\nabla u_{k,N_k}|^2 + q'(u_{k,N_k})u_{k,N_k} + u_{k,N_k}(-\Delta)^{\frac{1}{2}}u_{k,N_k}\right)dxdt$$
$$\int_{\Omega_T}\mu_{k,N_k}udxdt = \int_{\Omega_T}\left(\nabla u_{k,N_k}\cdot\nabla u + q'(u_{k,N_k})u + u(-\Delta)^{\frac{1}{2}}u_{k,N_k}\right)dxdt.$$

Passing to the limit, by (3.60), growth assumptions on q' and (3.53), we have

$$\lim_{k\to\infty}\int_{\Omega_T}|\nabla u_{k,N_k}|^2 = \int_{\Omega_T}|\nabla u|^2.$$

Therefore

$$\nabla u_{k,N_k} \to \nabla u \text{ strongly in } L^2(\Omega_T).$$

Since $u_{k,N_k} \in L^2(0,T;W^{3,s}(\mathbb{T}^2))$, we can differentiate (3.78) and get

$$\nabla\mu_{k,N_k} = -\nabla\Delta u_{k,N_k} + q''(u_{k,N_k})\nabla u_{k,N_k} + \nabla(-\Delta)^{\frac{1}{2}}u_{k,N_k}, \tag{3.79}$$

and

$$\nabla\frac{\mu_{k,N_k}}{g_{k,N_k}(u_{k,N_k})} = \frac{1}{g_{k,N_k}(u_{k,N_k})}\nabla\mu_{k,N_k} - \mu_{k,N_k}\frac{g'_{k,N_k}(u_{k,N_k})}{g^2_{k,N_k}(u_{k,N_k})}\nabla u_{k,N_k} \tag{3.80}$$

on D_j°. Thus

$$\nabla\mu_{k,N_k} = g_{k,N_k}(u_{k,N_k})\nabla\frac{\mu_{k,N_k}}{g_{k,N_k}(u_{k,N_k})} + \frac{\mu_{k,N_k}}{g_{k,N_k}(u_{k,N_k})}g'_{k,N_k}(u_{k,N_k})\nabla u_{k,N_k}. \tag{3.81}$$

Since

$$\begin{aligned}
&g_{k,N_k}(u_{k,N_k}) \to g(u) \text{ uniformly in } D_j^\circ,\\
&\frac{g'_{k,N_k}(u_{k,N_k})}{g_{k,N_k}(u_{k,N_k})} \to \frac{g'(u)}{g(u)} \text{ uniformly in } D_j^\circ,\\
&\nabla\frac{\mu_{k,N_k}}{g_{k,N_k}(u_{k,N_k})} \rightharpoonup \zeta \text{ weakly in } L^2(D_j^\circ),\\
&\mu_{k,N_k} \rightharpoonup \mu \text{ weakly in } L^2(\Omega_T),\\
&\nabla u_{k,N_k} \to \nabla u \text{ strongly in } L^2(\Omega_T),
\end{aligned}$$

we have, for any $\phi \in L^\infty(D_j^\circ)$,

$$\begin{aligned}
&\int_{D_j^\circ}\phi\left(g_{k,N_k}(u_{k,N_k})\nabla\frac{\mu_{k,N_k}}{g_{k,N_k}(u_{k,N_k})} + \frac{\mu_{k,N_k}}{g_{k,N_k}(u_{k,N_k})}g'_{k,N_k}(u_{k,N_k})\nabla u_{k,N_k}\right)dxdt\\
&\to \int_{D_j^\circ}\phi\left(g(u)\zeta + \frac{g'(u)}{g(u)}\mu\nabla u\right)dxdt,
\end{aligned}$$

i.e.

$$\nabla\mu_{k,N_k} \rightharpoonup \eta := g(u)\zeta + \frac{g'(u)}{g(u)}\mu\nabla u \quad \text{weakly in } L^1(D_j^\circ).$$

Passing to the limit in (3.79), we obtain, in the sense of distribution, that

$$\eta = -\nabla\Delta u + q''(u)\nabla u + \nabla(-\Delta)^{\frac{1}{2}}u.$$

Since $q''(u)\nabla u + \nabla(-\Delta)^{\frac{1}{2}}u \in L^2(\Omega_T)$, we have $-\nabla\Delta u \in L^1(D_j^\circ)$, hence

$$\eta = -\nabla\Delta u + q''(u)\nabla u + \nabla(-\Delta)^{\frac{1}{2}}u \text{ a.e. in } D_j^\circ \tag{3.82}$$

Since $\frac{1}{g_{k,N_k}(u_{k,N_k})} \to \frac{1}{g(u)}$ uniformly in D_j, we have

$$\frac{1}{g_{k,N_k}(u_{k,N_k})}\nabla\mu_{k,N_k} \rightharpoonup \frac{1}{g(u)}\eta \text{ weakly in } L^1(D_j^\circ).$$

Since $\frac{g'_{k,N_k}(u_{k,N_k})}{g^2_{k,N_k}(u_{k,N_k})} \to \frac{g'(u)}{g^2(u)}$ uniformly in D_j, we have

$$\frac{g'_{k,N_k}(u_{k,N_k})}{g^2_{k,N_k}(u_{k,N_k})}\mu_{k,N_k}\nabla u_{k,N_k} \rightharpoonup \frac{g'(u)}{g^2(u)}\mu\nabla u \text{ weakly in } L^1(D_j^\circ).$$

Passing to the limit in (3.80), we have

$$\begin{aligned}\zeta &= \frac{1}{g(u)}\eta - \mu\frac{g'(u)}{g^2(u)}\nabla u\\ &= \frac{-\nabla\Delta u + q''(u)\nabla u + \nabla(-\Delta)^{\frac{1}{2}}u}{g(u)} - \frac{g'(u)}{g^2(u)}\left(-\Delta u + q'(u) + (-\Delta)^{\frac{1}{2}}u\right)\nabla u\end{aligned}$$

on $(D_j)^{\circ}$. Noticing the value of ζ on $\Omega_T\backslash D$ doesn't matter since it does not appear on the right hand side of (3.74).

Claim II: For any open set $U \in \Omega_T$ in which $\nabla\Delta u \in L^p(U)$ for some $p > 1$ and $g(u) > 0$, we have

$$\zeta = \frac{-\nabla\Delta u + q''(u)\nabla u + \nabla(-\Delta)^{\frac{1}{2}}u}{g(u)} - \frac{g'(u)}{g^2(u)}\left(-\Delta u + q'(u) + (-\Delta)^{\frac{1}{2}}u\right)\nabla u. \tag{3.83}$$

in U.

To prove this, since

$$\nabla\mu_{k,N_k} = -\nabla\Delta u_{k,N_k} + q''(u_{k,N_k})\nabla u_{k,N_k} + \nabla(-\Delta)^{\frac{1}{2}}u_{k,N_k} \text{ in } \Omega_T \tag{3.84}$$

and

$$\nabla\frac{\mu_{k,N_k}}{g_{k,N_k}(u_{k,N_k})} = \frac{1}{g_{k,N_k}(u_{k,N_k})}\nabla\mu_{k,N_k} + \mu_{k,N_k}\cdot\nabla\frac{1}{g_{k,N_k}(u_{k,N_k})} \text{ on } D_j. \tag{3.85}$$

The right hand side of (3.84) converges weakly to $-\nabla\Delta u + q''(u)\nabla u + \nabla(-\Delta)^{\frac{1}{2}}u$ in $L^q(U)$ for $q = \min\{p, 2\} > 1$. Hence

$$\nabla\mu_{k,N_k} \rightharpoonup \eta = -\nabla\Delta u + q''(u)\nabla u + \nabla(-\Delta)^{\frac{1}{2}}u \text{ weakly in } L^q(U).$$

The right hand side of (3.85) converges weakly to

$$\frac{\eta}{g(u)} - \frac{g'(u)}{g^2(u)}\mu\cdot\nabla u$$

in $L^1(U \cap D_j)$ for each j and therefore

$$\zeta = \frac{-\nabla\Delta u + q''(u)\nabla u + \nabla(-\Delta)^{\frac{1}{2}}u}{g(u)} - \frac{g'(u)}{g^2(u)}\left(-\Delta u + q'(u) + (-\Delta)^{\frac{1}{2}}u\right)\nabla u$$

a.e. in $U \cap D$. and the definition of ζ can be extended to $U\backslash D$ by our integrability assumption on u. Define

$$\tilde{\Omega}_T = \{U \subset \Omega_T : \nabla\Delta u \in L^p(U) \text{ for some } p > 1; g(u) > 0 \text{ on } U \text{ depending on } U\}.$$

Then $\tilde{\Omega}_T$ is open and ζ is defined by (3.83) on $\tilde{\Omega}_T$. Since $|\Omega_T\backslash B| = 0$, $M(u) = 0$ on $\Omega_T\backslash P$ and

$$\Omega_T\backslash\{D \cup \tilde{\Omega}_T\} \subset \{\Omega_T\backslash B\} \cup \{\Omega_T\backslash P\},$$

we can take the value of ζ to be zero outside $D \cup \Omega_T$, sand it won't affect the integral on the right side of (1.16).

Lastly the energy inequality (1.18) follows by taking limit in the energy inequality for u_{k,N_k}.

This finishes the proof of Theorem 1.2.

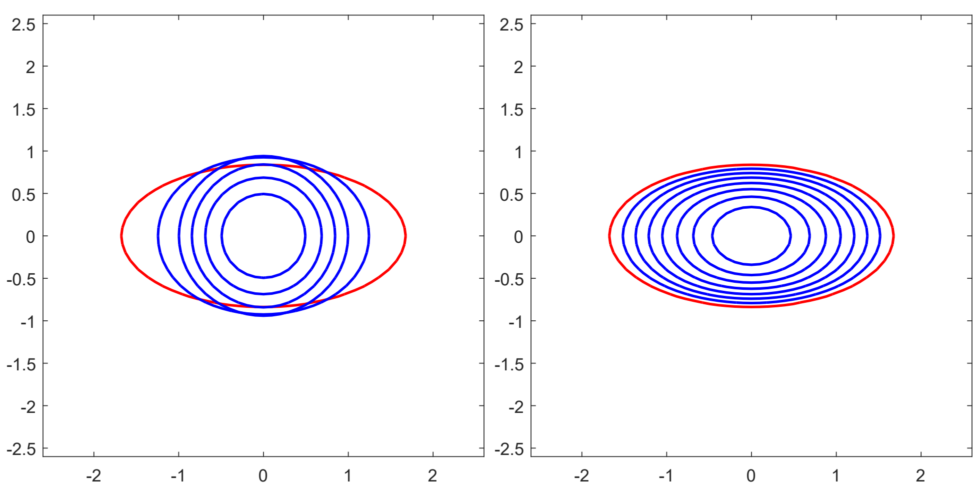
4. Simulations. In this section, we use the proposed phase field model to simulate the climb motions of prismatic dislocation loops, incorporating the conservative motion and nonconservative motion. We use the evolution equation in Eqs. (1.1) without the factor $g(u)$ on the right-hand side, i.e.,

$$\partial_t u + \beta\mu = \nabla \cdot \left(M(u)\nabla \frac{\mu}{g(u)} \right), \tag{4.1}$$

together with Eqs. (1.2) and (1.4). Recall that the non conservative climb motion will result into the shrinking and growing of the dislocation loops [19], whereas the self-climb is a conservative motion, which will keep the enclosed area of a prismatic loop unchanged [25, 30, 29].

In the simulations, we choose the simulation domain $\mathbb{T}^2 = [-\pi, \pi]^2$ and mesh size $dx = dy = 2\pi/M$ with $M = 64$. Periodic boundary conditions are used for the simulation domain. The small parameter in the phase field model $\varepsilon = dx$. The simulation domain corresponds to a physical domain of size $(300b)^2$, i.e., $b = 2\pi/300$. Under this setting, the parameter H_0 in the phase field model calculated in the paper [31] is $H_0 = 52.65\left(2(1-\nu)/\mu b^2\right)$. The prismatic loops are in the counterclockwise direction meaning vacancy loops, unless otherwise specified.

In the numerical simulations, we use the pseudo spectral method: All the spatial partial derivatives are calculated in the Fourier space using FFT. For the time discretization, we use the forward Euler method. The climb force generated by dislocations $f_{\mathrm{cl}}^{\mathrm{d}}$ is calculated by FFT using Eq. (1.4). We regularize the function $g(u)$ in the denominator in Eq. (4.1) as $\sqrt{g(u)^2 + e_0^2}$ with small parameter $e_0 = 0.005$. In the initial configuration of a simulation, ϕ in the dislocation core region is set to be a tanh function with width 3ε. The location of the dislocation loop is identified by the contour line of $u = 0$.



(a) (b)

FIG. 1. *shrinking of an elliptic prismatic loops by climb with/without self-climb.*

4.1. Evolution of an elliptic prismatic loop under the combined climb effect. In the first numerical example, we simulate evolution of an elliptic prismatic

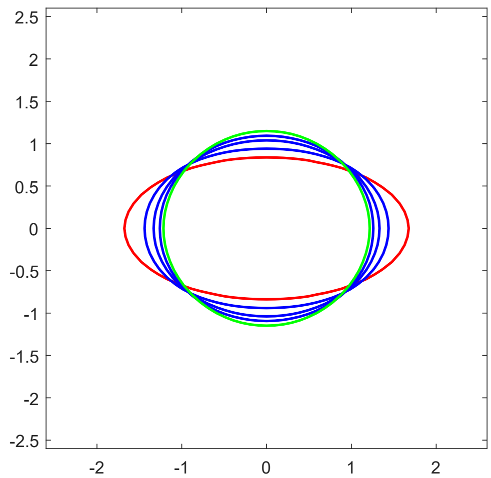


Fig. 2. *Evolution of an elliptic prismatic loop only by self-climb effect using the phase field model. Red ellipse is the initial state, and green circle is the final state.*

loop using the phase field model, see Fig. 1 and Fig. 2. The two axes of the initial elliptic profile are $l_1 = 80b$ and $l_2 = 40b$. Fig. 1(a) shows the elliptic prismatic loop will not directly shrink, due to the self-climb effect, and there is a trend to evolve to a circle in the shrinking process. Fig. 1(b) shows that without the self-climb effect, the elliptic loop directly shrink until vanishing. The shrinking of loop with self-climb takes much longer time than the case without self-climb. The shapes are also totally different in the process. These will influence the pattern of the interactions of two loops, see details in the simulations; see Sec.(4.2). Moreover, we show the evolution of an elliptic prismatic loop only by self-climb using the phase field model, seeing Fig. 2, to illustrate the effect of the self-climb effect. Red ellipse is the initial state, and the loop converges to the equilibrium shape of a circle (green circle) under its self-stress. The area enclosed by a prismatic loop is conserved during the self-climb motion. More simulation information about the self-climb effect can be found in our previous papers [30, 29, 31].

4.2. The interactions between the circular prismatic loops under the combined climb effect. In this subsection, we use our phase field model to simulate the interaction of two circular prismatic loops for three conditions: with self-climb, without self-climb and only with self-climb. The detailed shrinking process obtained by our simulations are shown in Fig. 3, Fig. 4 and Fig. 5. The two loops are attracted to each other by self-climb under the elastic interaction between them for all these three conditions, but the later change of the shapes are totally different. For the simulation of dislocation climb with the self-climb effect, firstly, the two loops are attracted to each other by self-climb. When the two loops meet, they quickly combine into a single loop; see Fig. 3(a-b). The combined single loop eventually evolves into a circular shape; see Fig. 3(c)-(e). Finally the circular loops shrink and vanish; see Fig. 3(f). For the simulation of dislocation climb without the self-climb effect, see Fig. 4. Firstly, the two loops are attracted to each other under self-stress; see Fig. 4(a)-(c), but quickly they separate due to the non-conservative climb effect; see Fig. 4(d). The small loop vanishes first in the shrinking process; see Fig. 4(e). Finally the larger loop shrinks and vanishes; see Fig. 4(f). Comparing these two climb interaction processes with and without self-climb effect, we conclude that even though both loops will vanish eventually, the processes are quite different. With the effect of self-climb, these two close loop will coalescence first when they shrink. Without the self-climb, these two loops will shrink directly and simply after the quick connecting and separation. The total time for the shrinking of these two loops differs greatly. It takes longer time for the loops to shrink with the self-climb effect than without the self-climb effect. Fig. 3 and Fig. 4 give details of the patterns in these two shrinking process and show the great difference, which will help us to understand the formation process of the patterns and predict the stable state of the patterns in the physics experiments. Moreover, to understand the self-climb effect in the interactions of the two loop, we show the detailed coalescence process driven by self-climb only in Fig. 5. Firstly, the two loops are attracted to each other by self-climb under the elastic interaction. They quickly combine into a single loop after meeting; see Fig. 5(a-c). The combined single loop eventually evolves into a stable, circular shape; see Fig. 5(d)-(f). It is noteworthy that the area of the final circle are equal to the total area of the initial two circles theoretically, and these two areas also agree well in numerical simulation. More simulation information about the self-climb effect of the interactions of loops can be found in our previous papers [30, 29, 31].

5. Conclusions and discussion. In this paper, we have presented a phase field model for the motion of prismatic dislocation loops by both climb and self-climb, based on a Cahn-Hilliard/Allen-Cahn framework with degenerate mobility and an additional stabilizing factor. This phase field model provides an efficient simulation tool compared with those available front-tracking based methods adopted in most of the available dislocation dynamics simulations with climb motions. The proposed phase field model is validated by asymptotic analysis, weak solution existence analysis, and numerical simulations.

Physically, self-climb by vacancy pipe diffusion is the dominant dislocation climb mechanism at a not very high temperature in irradiated materials, while at a high temperature, dislocation climb by vacancy bulk diffusion also becomes important. These two types of motions have essential different nature, as can be seen in our simulation results. The contributions of the two types of climb motions can be adjusted by the parameter β in our phase field model depending on the physical settings and materials properties. Our phase field model can be employed to obtain further the materials properties associated with dislocation climb motions, e.g. for irradiated materials.

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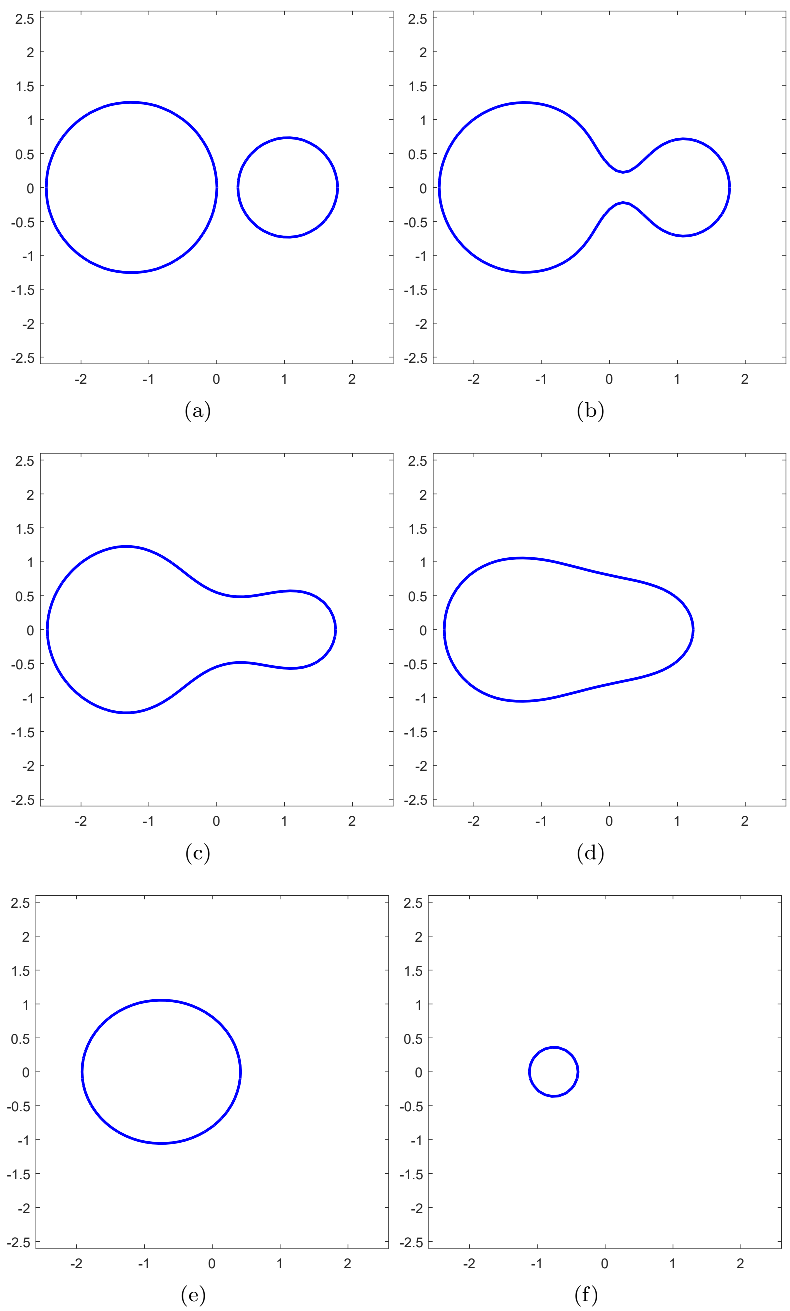

Fig. 3. *The interaction of two circular prismatic loops by climb with self-climb.*

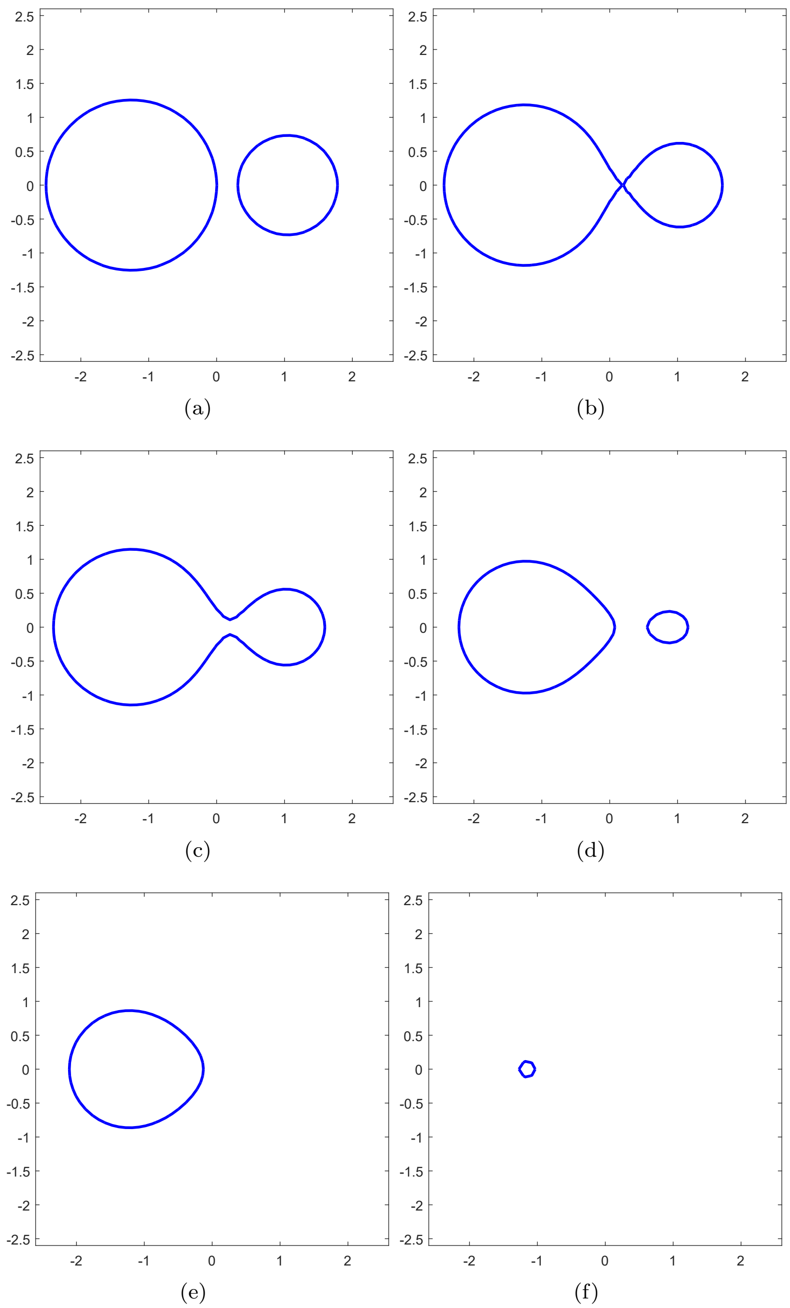

Fig. 4. *The interaction of two circular prismatic loops by climb without self-climb.*

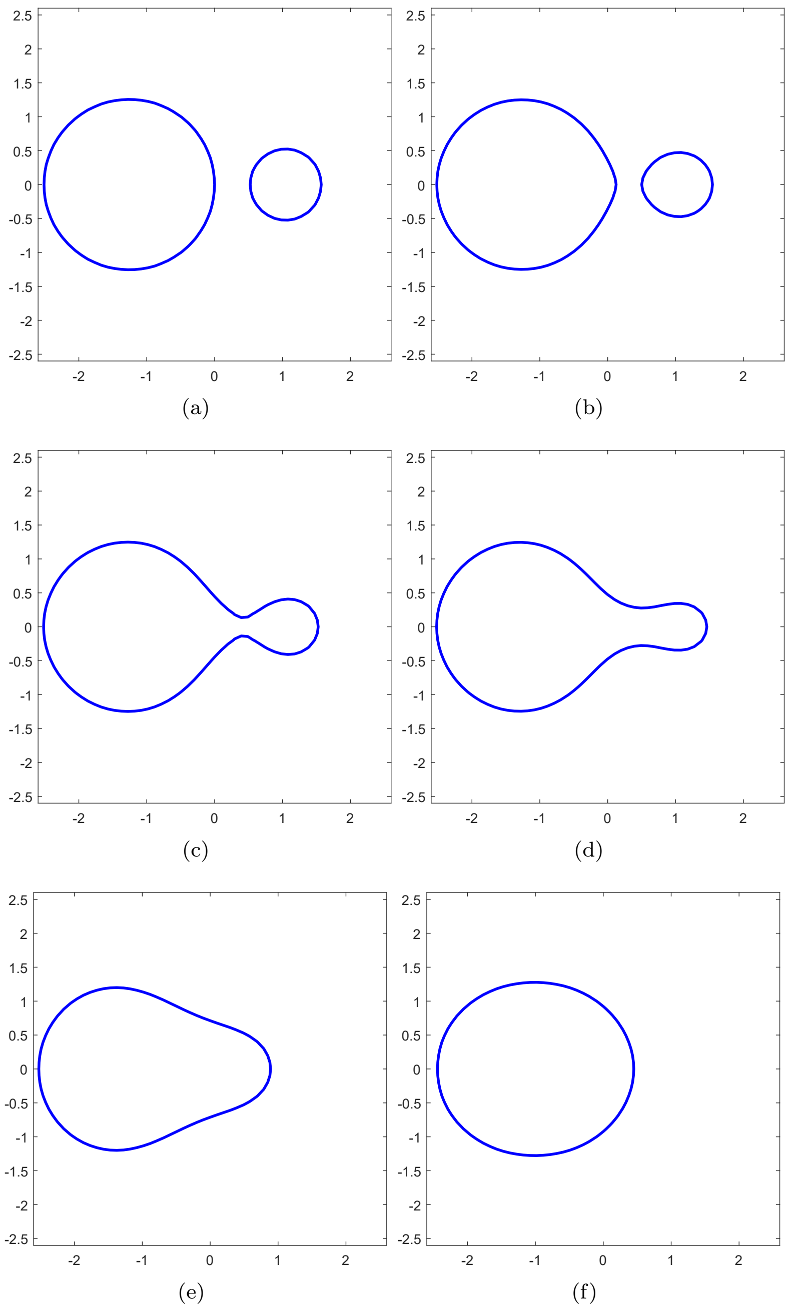

Fig. 5. *Coalescence of two prismatic loops only by self-climb under their elastic interaction obtained by the phase field model.*

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