PURE and APPLIED
ANALYSIS

PAA

GONG CHEN AND JASON MURPHY

STABILITY ESTIMATES FOR THE RECOVERY OF THE NONLINEARITY FROM SCATTERING DATA

msp

vol. 6 no. 1 2024

PURE and APPLIED
ANALYSIS

Vol. 6, No. 1, 2024
https://doi.org/10.2140/paa.2024.6.305

STABILITY ESTIMATES FOR THE RECOVERY OF THE NONLINEARITY FROM SCATTERING DATA

GONG CHEN AND JASON MURPHY

We prove stability estimates for the problem of recovering the nonlinearity from scattering data. We focus our attention on nonlinear Schrödinger equations of the form

$$(i\partial_t + \Delta)u = a(x)|u|^p u$$

in three space dimensions, with $p \in \left[\frac{4}{3}, 4\right]$ and $a \in W^{1,\infty}$.

1. Introduction

We consider the problem of determining an unknown nonlinearity from the small-data scattering behavior of solutions in the setting of nonlinear Schrödinger equations of the form

$$(i\partial_t + \Delta)u = a(x)|u|^p u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d. \tag{1-1}$$

We focus on the three-dimensional *intercritical* setting, i.e., $d = 3$ and $p \in \left[\frac{4}{3}, 4\right]$. In this setting, equation (1-1) admits a small-data scattering theory in H^1 for any $a \in W^{1,\infty}$ (see Theorem 1.1). In particular, given sufficiently small $u_- \in H^1$, there exists a global-in-time solution u to (1-1) that scatters backward in time to u_- and forward in time to some $u_+ \in H^1$, that is,

$$\lim_{t\to\pm\infty} \|u(t) - e^{it\Delta}u_\pm\|_{H^1} = 0.$$

One can therefore define the *scattering map* $S_a : u_- \mapsto u_+$ on some ball in H^1.

As it turns out, the scattering map encodes all of the information about the nonlinearity in (1-1), in the sense that the map $a \mapsto S_a$ is injective (see Theorem 1.2). In fact, knowledge of S_a suffices to reconstruct the inhomogeneity a pointwise.

In this work, we consider the closely related problem of stability. That is, if two scattering maps S_a and S_b are close in some sense, must the corresponding inhomogeneities a and b necessarily be close? Our main result (Theorem 1.3) provides an estimate of this type. It is essentially a quantitative version of Theorem 1.2.

We first state the small data scattering result for (1-1). The proof utilizes Strichartz estimates and a standard contraction mapping argument. For completeness, we provide the proof in Section 3.

MSC2020: 35Q55.
Keywords: NLS, scattering, inverse scattering, stability.

Theorem 1.1 (small data scattering). *Let $a \in W^{1,\infty}(\mathbb{R}^3)$ and $p \in \left[\frac{4}{3}, 4\right]$. There exists $\eta > 0$ sufficiently small so that for any $u_- \in H^1$ satisfying $\|u_-\|_{H^1} < \eta$, there exists a unique global solution u to (1-1) satisfying*

$$\lim_{t\to\pm\infty} \|u(t) - e^{it\Delta}u_\pm\|_{H^1} = 0,$$

where u_+ satisfies the formula

$$u_+ = u_- - i\int_{\mathbb{R}} e^{-it\Delta} a|u|^p u(t)\,dt. \tag{1-2}$$

Using Theorem 1.1, we define the *scattering map* $S_a : B \to H^1$ via $S_a(u_-) = u_+$, where B is a ball in H^1 and $u_\pm$ are as in the statement of the theorem. This map uniquely determines the function a (see, e.g., [Strauss 1974; Murphy 2023]):

Theorem 1.2. *Let $p \in \left[\frac{4}{3}, 4\right]$ and let $a, b \in W^{1,\infty}(\mathbb{R}^3)$. Let S_a, S_b denote the corresponding scattering maps for (1-1) with nonlinearities $a|u|^p u$ and $b|u|^p u$, respectively. If S_a is equal to S_b on their common domain, then $a = b$.*

Our main result is essentially a quantitative version of Theorem 1.2. To measure the difference between two scattering maps, we use the Lipschitz constant at 0. In particular, we define

$$\|S_a - S_b\| := \sup\left\{\frac{\|S_a(\varphi) - S_b(\varphi)\|_{H^1}}{\|\varphi\|_{H^1}} : \varphi \in B\backslash\{0\}\right\},$$

where $B \subset H^1$ is the common domain of S_a and S_b.

Theorem 1.3 (stability estimate). *Let $p \in \left[\frac{4}{3}, 4\right]$. Let $a, b \in W^{1,\infty}$, and let S_a, S_b denote the corresponding scattering maps for (1-1) with nonlinearities $a|u|^p u$ and $b|u|^p u$, respectively. Then*

$$\|a - b\|_{L^\infty} \lesssim \{\|a\|_{W^{1,\infty}} + \|b\|_{W^{1,\infty}}\}^{\frac{8}{9}} \|S_a - S_b\|^{\frac{1}{9}} + \{\|a\|_{W^{1,\infty}} + \|b\|_{W^{1,\infty}}\}^{\frac{10}{9}} \|S_a - S_b\|^{\frac{8}{9}}.$$

Remark 1.4. If we assume a priori bounds of the form

$$\|a\|_{W^{1,\infty}}, \|b\|_{W^{1,\infty}} \lesssim M \quad \text{and} \quad \|S_a - S_b\| \ll 1,$$

then the estimate in Theorem 1.3 reduces to the following Hölder estimate:

$$\|a - b\|_{L^\infty} \lesssim_M \|S_a - S_b\|^{\frac{1}{9}}. \tag{1-3}$$

The precise powers appearing in these estimates do not have any special meaning. Indeed, they arise from some ad hoc choices made in the argument in order to treat the range $p \in \left[\frac{4}{3}, 4\right]$ uniformly. By refining the arguments, one could improve the estimate (1-3) to

$$\|a - b\|_{L^\infty} \lesssim_{M,\varepsilon} \|S_a - S_b\|^{\frac{3p-2}{9p-2} - \varepsilon},$$

but even in this case there seems to be no special meaning to this exponent.

The problem of recovering an unknown nonlinearity from scattering data (or other data) is a well-studied problem. For results of this type in the setting of nonlinear dispersive equations (particularly nonlinear Schrödinger equations), we refer the reader to [Sá Barreto et al. 2022; Sá Barreto and Stefanov 2022; Carles and Gallagher 2009; Chen and Murphy 2023; Enss and Weder 1995; Hogan et al. 2023;

Killip et al. 2023; Lee and Yu 2023; Morawetz and Strauss 1973; Murphy 2023; Pausader and Strauss 2009; Sasaki 2007; 2008; Sasaki and Watanabe 2005; Watanabe 2001; 2018; Weder 1997; 2000; 2001a; 2001b; 2002]. To the best of our knowledge, the problem of stability has not yet been investigated in this particular setting; however, we refer the reader to [Lassas et al. 2022] to some stability estimates related to recovering an unknown coefficient in a semilinear wave equation from the Dirichlet-to-Neumann map.

Our main result, Theorem 1.3, provides a stability estimate in the intercritical setting for nonlinearities of the form $a(x)|u|^p u$ in three space dimensions. Killip et al. [2023] proved an analogue of Theorem 1.2 for a more general class of nonlinearities in two dimensions; however, the results presented here do not suffice to establish a stability estimate in this more general setting. In the case that modified scattering holds, we [Chen and Murphy 2023] also showed that the small-data modified scattering behavior also suffices to determine the inhomogeneity present in the nonlinearity. A stability estimate in this setting would also require some new ideas compared to what is presented here.

The strategy of the proof of Theorem 1.3 builds on the one used to prove Theorem 1.2 (see, e.g., [Strauss 1974; Murphy 2023]). The starting point is the implicit formula for the scattering map appearing in (1-2), which implies that

$$\langle S_a(u_-) - u_-, u_- \rangle = -i \int_{\mathbb{R}\times\mathbb{R}^3} a(x)|u(t,x)|^p u(t,x)\, \overline{e^{it\Delta}u_-(x)}\, dx\, dt,$$

where u is the solution to (1-1) that scatters backward in time to u_-. We then approximate the full solution $u(t)$ by $e^{it\Delta}u_-$ (the Born approximation), using the Duhamel formula for (1-1) to express the difference (see (3-2)). The difference contains the nonlinearity and hence is smaller than the main term, which is given by

$$\int_{\mathbb{R}\times\mathbb{R}^3} a(x)|e^{it\Delta}u_-(x)|^{p+2}\, dx\, dt. \tag{1-4}$$

The next step is to specialize to Gaussian data of the form

$$u_-(x) = \exp\Big\{-\frac{|x-x_0|^2}{4\sigma^2}\Big\},$$

which is small in H^1 for $0 < \sigma \ll 1$. We then rely on the fact that the free evolution of a Gaussian may be computed explicitly (and is still Gaussian), a fact that has already been exploited in the related works [Killip et al. 2023; Chen and Murphy 2023; Murphy 2023]. Using the scaling symmetry for the linear Schrödinger equation, we can therefore express the main term (1-4) in the form $F_\sigma * a(x_0)$, where $c^{-1}\sigma^{-5}F_\sigma$ forms a family of approximate identities as $\sigma \to 0$ for suitable $c > 0$. Using the explicit form of F_σ, we can estimate the difference

$$|c^{-1}\sigma^{-5}F_\sigma * a(x_0) - a(x_0)|$$

quantitatively in terms of σ (see Proposition 2.2). Carrying out the same estimates with S_b ultimately leads to a bound of the form

$$\|a-b\|_{L^\infty} \lesssim \sigma^{-2}\|S_a - S_b\| + \mathcal{O}\{\sigma^{\frac14} + \sigma^2\},$$

where $\sigma^{\frac14}$ arises from the approximate identity estimate and σ^2 arises from the Born approximation. Optimizing with respect to σ leads to the estimate appearing in Theorem 1.3.

Theorem 1.3 concerns the comparison of nonlinearities of the form $a|u|^p u$ and $b|u|^p u$; in particular, the power of each nonlinearity is a priori assumed to be equal. In fact, the result Theorem 1.2 (the determination of the nonlinearity from the scattering map) can be extended to allow nonlinearities of the form $a|u|^p u$ without assuming that p is already known. In particular, one can show that if nonlinearities $a(x)|u|^p u$ and $b(x)|u|^\ell u$ have the same scattering map, then $p=\ell$ and $a\equiv b$ (see, e.g., [Murphy 2023; Watanabe 2018]). Thus it is also natural to ask whether one can bound $|p-\ell|$ in terms of the difference between the scattering maps.

In this paper we also take the preliminary step of estimating $|p-\ell|$ in terms of the difference between the scattering maps corresponding to the pure power-type nonlinearities $|u|^p u$ and $|u|^\ell u$.

Theorem 1.5. *Suppose $p,\ell\in\left[\frac{4}{3},4\right]$. Let S_p and S_ℓ denote the scattering maps for (1-1) corresponding to nonlinearities $|u|^p u$ and $|u|^\ell u$. Then*

$$|p-\ell|\lesssim \|S_p-S_\ell\|^{\frac{1}{9}}.$$

The proof of Theorem 1.5 begins along similar lines to the proof of Theorem 1.3. In the present setting, one needs to analyze the normalizing constant $\lambda(p)$ arising in the approximate identity argument mentioned above (see Proposition 2.2). In particular, we derive an upper bound on $|\lambda(p)-\lambda(\ell)|$ in terms of $\|S_p-S_\ell\|$, and then establish a lower bound of the form

$$|\lambda(p)-\lambda(\ell)|\gtrsim |p-\ell|.$$

Combining the arguments used to prove Theorems 1.3 and 1.5, one can also obtain an estimate of the form

$$\|\lambda(p)a-\lambda(\ell)b\|_{L^\infty}\lesssim \|S_1-S_2\|^{\frac{1}{9}},$$

where S_1, S_2 are the scattering maps corresponding to (1-1) with nonlinearities $a|u|^p u$ and $b|u|^\ell u$, respectively. While this estimate is harder to interpret directly, it can still be used to prove that if $S_1=S_2$ then $p=\ell$ and $a\equiv b$ (recovering results of [Watanabe 2018; Murphy 2023]).

The rest of this paper is organized as follows: In Section 2, we collect some preliminary results. We also prove the approximate identity result Proposition 2.2. In Section 3, we prove the small-data scattering result for (1-1). In Section 4, we prove the main result, Theorem 1.3. Finally, in Section 5 we prove Theorem 1.5.

2. Preliminaries

We write $A\lesssim B$ to denote $A\le CB$ for some $C>0$, with dependence on parameters indicated by subscripts. We write $W^{1,\infty}$ for the Sobolev space with norm

$$\|a\|_{W^{1,\infty}}=\|a\|_{L^\infty}+\|\nabla a\|_{L^\infty}.$$

For $1<r<\infty$ we write $H^{s,r}$ for the Sobolev space with norm

$$\|u\|_{H^{s,r}}=\|\langle\nabla\rangle^s u\|_{L^r},$$

where $\langle\nabla\rangle=\sqrt{1-\Delta}$. We write q' for the Hölder dual of an exponent q, i.e., the solution to $\frac{1}{q}+\frac{1}{q'}=1$.

We write $e^{it\Delta}$ for the Schrödinger group $e^{it\Delta}=\mathcal{F}^{-1}e^{-it|\xi|^2}\mathcal{F}$, where $\mathcal{F}$ denotes the Fourier transform.

We utilize the following Strichartz estimates in three space dimensions.

Proposition 2.1 [Ginibre and Velo 1992; Keel and Tao 1998; Strichartz 1977]. *For any* $2 \le q, \tilde{q}, r, \tilde{r} \le \infty$ *satisfying*

$$\frac{2}{q} + \frac{3}{r} = \frac{2}{\tilde{q}} + \frac{3}{\tilde{r}} = \frac{3}{2},$$

we have

$$\|e^{it\Delta}\varphi\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|\varphi\|_{L^2},$$
$$\left\|\int_{-\infty}^t e^{i(t-s)\Delta} F(s)\,ds\right\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(\mathbb{R}\times\mathbb{R}^3)}.$$

The following approximate identity estimate plays a key role in both Theorems 1.3 and 1.5. It is based on the explicit computation of the solution to the linear Schrödinger equation with Gaussian data. We present the result in the setting of general dimensions and short-range powers.

Proposition 2.2 (approximate identity estimate). *Let* $d \ge 1$ *and* $p > \frac{2}{d}$. *Given* $x_0 \in \mathbb{R}^d$ *and* $\sigma > 0$, *define*

$$\varphi_{\sigma,x_0}(x) = \exp\left\{-\frac{|x-x_0|^2}{4\sigma^2}\right\}$$

and

$$\lambda(d,p) := \pi^{\frac{d}{2}+1}\left[\frac{4}{p+2}\right]^{\frac{d}{2}} \frac{\Gamma\left(\frac{dp}{4}-\frac{1}{2}\right)}{\Gamma\left(\frac{dp}{4}\right)}. \tag{2-1}$$

Given $a \in W^{1,\infty}(\mathbb{R}^d)$, *we have*

$$\left|\iint_{\mathbb{R}\times\mathbb{R}^d} |e^{it\Delta}\varphi_{\sigma,x_0}(x)|^{p+2} a(x)\,dx\,dt - \sigma^{d+2}\lambda(d,p)a(x_0)\right| \le c_s \sigma^{d+2+s}\|a\|_{W^{1,\infty}}$$

for any $0 < s < 1 - \frac{2}{dp}$, *where* $c_s \to \infty$ *as* $s \to 1 - \frac{2}{dp}$.

Proof. We have

$$e^{it\Delta}\varphi_{\sigma,x_0}(x) = \left[\frac{\sigma^2}{\sigma^2+it}\right]^{\frac{d}{2}} \exp\left\{-\frac{|x-x_0|^2}{4(\sigma^2+it)}\right\}$$

(see [Vişan 2014]), so that

$$\begin{aligned} |e^{it\Delta}\varphi_{\sigma,x_0}(x)|^{p+2} &= \left[\frac{\sigma^4}{\sigma^4+t^2}\right]^{\frac{d(p+2)}{4}} \exp\left\{-\frac{\sigma^2|x-x_0|^2(p+2)}{4(\sigma^4+t^2)}\right\} \\ &= K\left(\frac{t}{\sigma^2}, \frac{x-x_0}{\sigma}\right), \end{aligned}$$

where

$$K(t,x) := \left[\frac{1}{1+t^2}\right]^{\frac{d(p+2)}{4}} \exp\left\{-\frac{|x|^2(p+2)}{4(1+t^2)}\right\}.$$

We now show that $\int K(t,x)\,dx\,dt = \lambda(d,p)$. To this end, we first recall the Gaussian integral

$$\int_{\mathbb{R}} \exp\{-cy^2\}\,dy = \left(\frac{\pi}{c}\right)^{\frac{1}{2}}.$$

We next use the change of variables $u=(1+t^2)^{-1}$ to obtain

$$\begin{aligned}\int_{\mathbb{R}}(1+t^2)^{-c}\,dt &= 2\int_0^\infty (1+t^2)^{-c}\,dt\\ &= \int_0^1 u^{c-\frac{3}{2}}(1-u)^{-\frac{1}{2}}\,du\\ &= B\big(\tfrac{1}{2}, c-\tfrac{1}{2}\big) = \frac{\Gamma\big(\frac{1}{2}\big)\Gamma\big(c-\frac{1}{2}\big)}{\Gamma(c)} = \pi^{\frac{1}{2}}\frac{\Gamma\big(c-\frac{1}{2}\big)}{\Gamma(c)}\end{aligned}$$

for $c>\frac{1}{2}$, where B is the Euler beta function. Thus

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{R}^d} K(t,x)\,dx\,dt &= \left[\frac{4\pi}{p+2}\right]^{\frac{d}{2}}\int_{\mathbb{R}}(1+t^2)^{-\frac{dp}{4}}\,dt\\ &= \pi^{\frac{d}{2}+1}\left[\frac{4}{p+2}\right]^{\frac{d}{2}}\frac{\Gamma\big(\frac{dp}{4}-\frac{1}{2}\big)}{\Gamma\big(\frac{dp}{4}\big)} = \lambda(d,p),\end{aligned}$$

where we have used the fact that $p>\frac{2}{d}$.

We also observe that for any $R>0$ and any $0<s<\frac{dp}{2}-1$, we may estimate

$$\begin{aligned}\int_{\mathbb{R}}\int_{|x|>R} K(t,x)\,dx\,dt &\lesssim R^{-s}\iint |x|^s K(t,x)\,dx\,dt\\ &\lesssim R^{-s}\int (1+t^2)^{-\frac{dp}{4}+\frac{s}{2}}\,dt \lesssim_s R^{-s}.\end{aligned} \tag{2-2}$$

By a change of variables, we have

$$\iint_{\mathbb{R}\times\mathbb{R}^d} K\left(\frac{t}{\sigma^2},\frac{x}{\sigma}\right)dx\,dt = \sigma^{d+2}\lambda(d,p). \tag{2-3}$$

Thus we can write

$$\begin{aligned}&\left|\iint_{\mathbb{R}\times\mathbb{R}^d}|e^{it\Delta}\varphi_{\sigma,x_0}(x)|^{p+2}a(x)\,dx\,dt - \sigma^{d+2}\lambda(d,p)a(x_0)\right|\\ &\qquad = \left|\iint_{\mathbb{R}\times\mathbb{R}^d} K\left(\frac{t}{\sigma^2},\frac{x}{\sigma}\right)[a(x_0-x)-a(x_0)]\,dx\,dt\right|\\ &\qquad \le \int_{\mathbb{R}}\int_{|x|\le\delta} K\left(\frac{t}{\sigma^2},\frac{x}{\sigma}\right)|a(x_0-x)-a(x_0)|\,dx\,dt \qquad (2\text{-}4)\\ &\qquad\quad + \int_{\mathbb{R}}\int_{|x|>\delta} K\left(\frac{t}{\sigma^2},\frac{x}{\sigma}\right)|a(x_0-x)-a(x_0)|\,dx\,dt, \qquad (2\text{-}5)\end{aligned}$$

where $\delta>0$ will be determined below.

By the fundamental theorem of calculus and (2-3), we first estimate

$$(2\text{-}4) \lesssim \delta\sigma^{d+2}\|\nabla a\|_{L^\infty}.$$

Next, we use (2-2) to obtain

$$(2\text{-}5) \lesssim \sigma^{d+2}\|a\|_{L^\infty}\int_{\mathbb{R}}\int_{|y|>\frac{\delta}{\sigma}} K(t,y)\,dy\,dt \lesssim_s \left[\frac{\sigma}{\delta}\right]^s \sigma^{d+2}\|a\|_{L^\infty}$$

for $0 < s < \frac{dp}{2} - 1$. Choosing $\delta = \sigma^{\frac{s}{1+s}}$ leads to

$$\left| \iint_{\mathbb{R}\times\mathbb{R}^d} |e^{it\Delta}\varphi_{\sigma,x_0}(x)|^{p+2} a(x)\,dx\,dt - \sigma^{d+2}\lambda(d,p)a(x_0) \right| \lesssim_s \sigma^{\frac{s}{1+s}} \sigma^{d+2} \|a\|_{W^{1,\infty}}$$

for any $0 < s < \frac{dp}{2} - 1$, which yields the result. □

3. Small-data scattering

In this section we prove the following small-data scattering result.

Theorem 3.1. *Let $a \in W^{1,\infty}(\mathbb{R}^3)$ and $p \in \left[\frac{4}{3}, 4\right]$. Define*

$$(q,r) = \left(p+2, \frac{6(p+2)}{3(p+2)-4}\right) \quad \text{and} \quad s_c = \frac{3}{2} - \frac{2}{p}. \tag{3-1}$$

There exists $\eta > 0$ sufficiently small so that for any $u_- \in H^1$ satisfying $\|u_-\|_{H^1} < \eta$, there exists a unique global solution u to (1-1) and $u_+ \in H^1$ satisfying the following:

$$\|u\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|u_-\|_{L^2},$$
$$\|\nabla u\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|u_-\|_{H^1},$$
$$\||\nabla|^{s_c} u\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|u_-\|_{\dot{H}^{s_c}},$$

and

$$\lim_{t\to\pm\infty} \|u(t) - e^{it\Delta}u_\pm\|_{H^1} = 0.$$

Proof. We construct u to satisfy the Duhamel formula

$$u(t) = \Phi u(t) := e^{it\Delta}u_- - i\int_{-\infty}^t e^{i(t-s)\Delta} a(x)|u|^p u(s)\,ds, \tag{3-2}$$

where $\|u_-\|_{H^1} \le \eta \ll 1$. It suffices to prove that Φ is a contraction on a suitable complete metric space. To this end, we fix $u_- \in H^1$ and define X to be the set of functions $u : \mathbb{R}\times\mathbb{R}^d \to \mathbb{C}$ satisfying the bounds

$$\|u\|_{L_t^q L_x^r} \le 4C\|u_-\|_{L^2}, \quad \|\nabla u\|_{L_t^q L_x^r} \le 4C\|u_-\|_{H^1}, \quad \||\nabla|^{s_c} u\|_{L_t^q L_x^r} \le 4C\|u_-\|_{\dot{H}^{s_c}},$$

where q, r are defined in (3-1), all space-time norms are over $\mathbb{R}\times\mathbb{R}^3$, and C encodes implicit constants in inequalities such as Strichartz and Sobolev embedding. We equip X with the metric

$$d(u,v) = \|u - v\|_{L_t^q L_x^r}$$

and we define

$$r_c = \frac{3p(p+2)}{4}, \quad \text{so that} \quad \dot{H}^{s_c,r} \hookrightarrow L^{r_c}. \tag{3-3}$$

For $u \in X$, we use Strichartz and Hölder, to estimate

$$\begin{aligned} \|\Phi u\|_{L_t^q L_x^r} &\lesssim \|u_-\|_{L^2} + \|a|u|^p u\|_{L_t^{q'} L_x^{r'}} \\ &\lesssim \|u_-\|_{L^2} + \|a\|_{L^\infty} \|u\|^p_{L_t^q L_x^{r_c}} \|u\|_{L_t^q L_x^r} \\ &\lesssim \|u_-\|_{L^2} + \eta^p \|a\|_{L^\infty} \|u_-\|_{L^2} \end{aligned}$$

Similarly, using the product rule and Sobolev embedding as well,

$$\begin{aligned}\|\nabla\Phi u\|_{L_t^q L_x^r} &\lesssim \|u_-\|_{\dot H^1} + \|a\|_{L^\infty}\|u\|_{L_t^q L_x^{r_c}}^p \|\nabla u\|_{L_t^q L_x^r} + \|\nabla a\|_{L^\infty}\|u\|_{L_t^q L_x^{r_c}}^{p-1}\|u\|_{L_t^q L_x^r}\|u\|_{L_t^q L_x^{r_c}} \\ &\lesssim \|u_-\|_{\dot H^1} + \eta^p\|a\|_{L^\infty}\|u_-\|_{\dot H^1} + \eta^p\|\nabla a\|_{L^\infty}\||\nabla|^{s_c}u\|_{L_t^q L_x^r} \\ &\lesssim \|u_-\|_{\dot H^1} + \eta^p\|a\|_{W^{1,\infty}}\|u_-\|_{H^1}.\end{aligned}$$

Finally, we have

$$\begin{aligned}\||\nabla|^{\frac12}\Phi u\|_{L_t^q L_x^r} &\lesssim \|u_-\|_{\dot H^{s_c}} + \|a|u|^p u\|_{L_t^{q'} H_x^{1,r'}} \\ &\lesssim \|u_-\|_{\dot H^{s_c}} + \|a\|_{W^{1,\infty}}\|u\|_{L_t^q L_x^{r_c}}^{p-1}\|u\|_{L_t^q H_x^{1,r}}\|u\|_{L_t^q L_x^{r_c}} \\ &\lesssim \|u_-\|_{\dot H^{s_c}} + \|a\|_{W^{1,\infty}}\|u\|_{L_t^q L_x^{r_c}}^{p-1}\|u\|_{L_t^q H_x^{1,r}}\||\nabla|^{s_c}u\|_{L_t^q L_x^r} \\ &\lesssim \|u_-\|_{\dot H^{s_c}} + \eta^p\|a\|_{W^{1,\infty}}\|u_-\|_{\dot H^{s_c}}.\end{aligned}$$

It follows that for η sufficiently small, $\Phi : X \to X$.

To see that Φ is a contraction, we use Strichartz and Hölder to estimate as follows: for $u, v \in X$,

$$\begin{aligned}\|\Phi u - \Phi v\|_{L_t^q L_x^r} &\lesssim \|a[u-v]\|_{L_t^{q'} L_x^{r'}} \\ &\lesssim \|a\|_{L^\infty}[\|u\|_{L_t^q L_x^{r_c}}^p + \|v\|_{L_t^q L_x^{r_c}}^p]\|u-v\|_{L_t^q L_x^r} \\ &\lesssim \eta^p\|a\|_{L^\infty}\|u-v\|_{L_t^q L_x^r},\end{aligned}$$

which shows that Φ is a contraction for η sufficiently small.

It follows that Φ has a unique fixed point $u \in X$, which is our desired solution.

It is not difficult to show that u scatters backward in time to u_-, and hence it remains to show that $e^{-it\Delta}u(t)$ has a limit in H^1 as $t \to \infty$. To this end, we fix $t > s > 0$ and use the estimates above to obtain

$$\begin{aligned}\|e^{-it\Delta}u(t) - e^{-is\Delta}u(s)\|_{H^1} &\lesssim \|a|u|^p u\|_{L_t^{q'} L_x^{r'}((s,t)\times\mathbb{R}^3)} \\ &\lesssim \|a\|_{W^{1,\infty}}\|u\|_{L_t^q L_x^{r_c}((s,t)\times\mathbb{R}^3)}^2\|u\|_{L_t^q H_x^{1,r}((s,t)\times\mathbb{R}^3)} \\ &\to 0 \quad \text{as} \quad s, t \to \infty.\end{aligned}$$

Thus $\{e^{-it\Delta}u(t)\}$ is Cauchy in H^1 as $\to \infty$ and hence has some limit $u_+ \in H^1$ as $t \to \infty$. In fact, from the Duhamel formula (3-2) we can obtain the implicit formula

$$u_+ = \lim_{t\to\infty} e^{-it\Delta}u(t) = u_- - i\int_{\mathbb{R}} e^{-is\Delta}a|u|^p u(s)\,ds. \tag{3-4}$$

for the final state u_+. □

Remark 3.2. By introducing some additional space-time norms into the argument, one can upgrade the estimate

$$\|\nabla u\|_{L_t^q L_x^r} \lesssim \|u_-\|_{H^1} \quad \text{to} \quad \|\nabla u\|_{L_t^q L_x^r} \lesssim \|u_-\|_{\dot H^1}.$$

However, this refinement is not needed in what follows, so we have opted to keep the argument as simple as possible above.

4. Proof of Theorem 1.3

Proof of Theorem 1.3. We let $p \in \left[\frac{4}{3}, 4\right]$ and $a, b \in W^{1,\infty}$. Let S_a, S_b denote the scattering maps for (1-1) with nonlinearities $a(x)|u|^p u$ and $b(x)|u|^p u$, respectively. Given $\sigma > 0$ and $x_0 \in \mathbb{R}^3$, we define

$$\varphi_{\sigma,x_0}(x) = \exp\left\{-\frac{|x-x_0|^2}{4\sigma^2}\right\}. \tag{4-1}$$

As

$$\|\varphi_{\sigma,x_0}\|_{\dot{H}^s(\mathbb{R}^3)} \lesssim \sigma^{\frac{3}{2}-s} \quad \text{for } s \in \mathbb{R}, \tag{4-2}$$

we have that φ_{σ,x_0} belongs to the common domain of S_a and S_b for all σ sufficiently small. Using (3-4), we write

$$\begin{aligned} S_a(\varphi_{\sigma,x_0}) = \varphi_{\sigma,x_0} &- i \iint_{\mathbb{R}} e^{-it\Delta}\left\{a\, |e^{it\Delta}\varphi_{\sigma,x_0}|^p e^{it\Delta}\varphi_{\sigma,x_0}\right\} dt \\ &- i \int_{\mathbb{R}} e^{-it\Delta}\left\{a\left[|u|^p u - |e^{it\Delta}\varphi_{\sigma,x_0}|^p e^{it\Delta}\varphi_{\sigma,x_0}\right]\right\} dt, \end{aligned}$$

where u is the solution to (1-1) that scatters to φ_{σ,x_0} as $t \to -\infty$ (cf. Theorem 1.1). Similarly,

$$\begin{aligned} S_b(\varphi_{\sigma,x_0}) = \varphi_{\sigma,x_0} &- i \int_{\mathbb{R}} e^{-it\Delta}\left\{b\, |e^{it\Delta}\varphi_{\sigma,x_0}|^p e^{it\Delta}\varphi_{\sigma,x_0}\right\} dt \\ &- i \int_{\mathbb{R}} e^{-it\Delta}\left\{b\left[|v|^p v - |e^{it\Delta}\varphi_{\sigma,x_0}|^p e^{it\Delta}\varphi_{\sigma,x_0}\right]\right\} dt, \end{aligned}$$

where v is the solution to the NLS (with nonlinearity $b|v|^p v$) that scatters to φ_{σ,x_0} as $t \to -\infty$. Thus

$$\langle S_a(\varphi_{\sigma,x_0}) - S_b(\varphi_{\sigma,x_0}), \varphi_{\sigma,x_0}\rangle \tag{4-3}$$

$$= -i \iint_{\mathbb{R}\times\mathbb{R}^3} [a(x) - b(x)]|e^{it\Delta}\varphi_{\sigma,x_0}|^{p+2}\, dx\, dt \tag{4-4}$$

$$- i \iint_{\mathbb{R}\times\mathbb{R}^3} a(x)[|u|^p u - |e^{it\Delta}\varphi_{\sigma,x_0}|^p e^{it\Delta}\varphi_{\sigma,x_0}]\overline{e^{it\Delta}\varphi_{\sigma,x_0}}\, dx\, dt \tag{4-5}$$

$$- i \iint_{\mathbb{R}\times\mathbb{R}^3} b(x)[|v|^p v - |e^{it\Delta}\varphi_{\sigma,x_0}|^p e^{it\Delta}\varphi_{\sigma,x_0}]\overline{e^{it\Delta}\varphi_{\sigma,x_0}}\, dx\, dt. \tag{4-6}$$

The terms (4-5) and (4-6) are estimated as in the proof of Theorem 3.1 (see (3-1) and (3-3) for the definitions of q, r, r_c). We use Hölder, Strichartz, the Duhamel formula (3-2), Sobolev embedding, Theorem 3.1, and (4-2) to obtain

$$\begin{aligned} &\|a[|u|^p u - |e^{it\Delta}\varphi_{\sigma,x_0}|^p e^{it\Delta}\varphi_{\sigma,x_0}]e^{it\Delta}\varphi_{\sigma,x_0}\|_{L^1_{t,x}} \\ &\quad\lesssim \|a\|_{L^\infty}\|e^{it\Delta}\varphi_{\sigma,x_0}\|_{L^q_t L^r_x}\||u|^p + |e^{it\Delta}\varphi_{\sigma,x_0}|^p\|_{L_t^{\frac{q}{p}} L_x^{\frac{r_c}{p}}}\|u(t) - e^{it\Delta}\varphi_{\sigma,x_0}\|_{L^q_t L^r_x} \\ &\quad\lesssim \|a\|_{L^\infty}\|\varphi_{\sigma,x_0}\|_{L^2}\{\|u\|^p_{L^q_t L^{r_c}_x} + \|e^{it\Delta}\varphi_{\sigma,x_0}\|^p_{L^q_t L^{r_c}_x}\}\left\|\int_{-\infty}^t e^{i(t-s)\Delta}a(x)|u|^p u\, ds\right\|_{L^q_t L^r_x} \\ &\quad\lesssim \|a\|_{L^\infty}\|\varphi_{\sigma,x_0}\|_{L^2}\|\varphi_{\sigma,x_0}\|^p_{\dot{H}^{s_c}}\|a|u|^p u\|_{L^{q'}_t L^{r'}_x} \\ &\quad\lesssim \|a\|^2_{L^\infty}\|\varphi_{\sigma,x_0}\|_{L^2}\|\varphi_{\sigma,x_0}\|^p_{\dot{H}^{s_c}}\|u\|^p_{L^q_t L^{r_c}_x}\|u\|_{L^q_t L^r_x} \\ &\quad\lesssim \|a\|^2_{L^\infty}\|\varphi_{\sigma,x_0}\|^2_{L^2}\|\varphi_{\sigma,x_0}\|^{2p}_{\dot{H}^{s_c}} \\ &\quad\lesssim \sigma^7\|a\|^2_{L^\infty}. \end{aligned}$$

Similarly,

$$\| b[|v|^p v - |e^{it\Delta}\varphi_{\sigma,x_0}|^p e^{it\Delta}\varphi_{\sigma,x_0}] \|_{L^1_{t,x}} \lesssim \sigma^7 \|b\|^2_{L^\infty}.$$

For (4-3), we use Cauchy–Schwarz and (4-2) to obtain

$$\begin{aligned} |\langle S_a(\varphi_{\sigma,x_0}) - S_b(\varphi_{\sigma,x_0}), \varphi_{\sigma,x_0}\rangle| &\le \|S_a(\varphi) - S_b(\varphi)\|_{\dot H^1} \|\varphi_{\sigma,x_0}\|_{\dot H^{-1}} \\ &\lesssim \|S_a - S_b\| \, \|\varphi_{\sigma,x_0}\|_{H^1} \|\varphi_{\sigma,x_0}\|_{\dot H^{-1}} \\ &\lesssim \sigma^3 \|S_a - S_b\| \end{aligned}$$

For (4-4), we make use of Proposition 2.2 with $d = 3$ and $s = \frac{1}{4}$. Using the fact that $p \ge \frac{4}{3}$, this proposition implies that

$$\left| \iint [a-b] |e^{it\Delta}\varphi_{\sigma,x_0}|^{p+2} \, dx \, dt - c\sigma^5 [a(x_0) - b(x_0)] \right| \lesssim \sigma^{5+\frac{1}{4}} [\|a\|_{W^{1,\infty}} + \|b\|_{W^{1,\infty}}],$$

where $c = \lambda(3, p)$. Combining this with the estimates for (4-5)–(4-6), we deduce

$$|a(x_0) - b(x_0)| \lesssim \sigma^{-2} \|S_a - S_b\| + \sigma^{\frac{1}{4}} \{\|a\|_{W^{1,\infty}} + \|b\|_{W^{1,\infty}}\} + \sigma^2 \{\|a\|^2_{L^\infty} + \|b\|^2_{L^\infty}\}.$$

If we now choose

$$\sigma = \varepsilon \cdot \left[\frac{\|S_a - S_b\|}{\|a\|_{W^{1,\infty}} + \|b\|_{W^{1,\infty}}} \right]^{\frac{4}{9}}$$

for sufficiently small $\varepsilon > 0$, then we obtain

$$|a(x_0) - b(x_0)| \lesssim \{\|a\|_{W^{1,\infty}} + \|b\|_{W^{1,\infty}}\}^{\frac{8}{9}} \|S_a - S_b\|^{\frac{1}{9}} + \{\|a\|_{W^{1,\infty}} + \|b\|_{W^{1,\infty}}\}^{\frac{10}{9}} \|S_a - S_b\|^{\frac{8}{9}}.$$

Taking the supremum over $x_0 \in \mathbb{R}^3$ now yields the result. □

5. Proof of Theorem 1.5

Proof of Theorem 1.5. The proof begins similarly to the proof of Theorem 1.3.

Let S_p and S_ℓ denote the scattering maps corresponding to (1-1) with nonlinearities $|u|^p u$ and $|u|^\ell u$, respectively, and define φ_σ as in (4-1) with $x_0 = 0$. We let u, v denote the solutions to (1-1) with nonlinearities $|u|^p u$ and $|v|^\ell v$ that scatter backward in time to φ_σ. Arguing as in the proof of Theorem 1.3, we can write

$$\iint_{\mathbb{R}\times\mathbb{R}^3} \big[|e^{it\Delta}\varphi_\sigma|^{p+2} - |e^{it\Delta}\varphi_\sigma|^{\ell+2} \big] \, dx \, dt \tag{5-1}$$

$$= i \langle S_p(\varphi_\sigma) - S_\ell(\varphi_\sigma), \varphi_\sigma \rangle \tag{5-2}$$

$$+ \iint_{\mathbb{R}\times\mathbb{R}^3} \big[|u|^p u - |e^{it\Delta}\varphi_\sigma|^p e^{it\Delta}\varphi_\sigma \big] \overline{e^{it\Delta}\varphi_\sigma} \, dx \, dt \tag{5-3}$$

$$+ \iint_{\mathbb{R}\times\mathbb{R}^3} \big[|v|^\ell v - |e^{it\Delta}\varphi_\sigma|^\ell e^{it\Delta}\varphi_\sigma \big] \overline{e^{it\Delta}\varphi_\sigma} \, dx \, dt. \tag{5-4}$$

The estimates of (4-5)–(4-6) in the proof of Theorem 1.3 apply to (5-3)–(5-4), so that

$$|(5\text{-}3)| + |(5\text{-}4)| \lesssim \sigma^7.$$

Similarly, estimating as we did for (4-4), we have

$$|(5\text{-}2)| \lesssim \sigma^2 \|S_p - S_\ell\|$$

For (5-1), we use Proposition 2.2 with $d = 3$ and $s = \frac{1}{4}$, which shows that

$$\big|(5\text{-}1) - \sigma^5[\lambda(p) - \lambda(\ell)]\big| \lesssim \sigma^{5+\frac{1}{4}},$$

where we abbreviate $\lambda(3, p)$ and $\lambda(3, \ell)$ by $\lambda(p)$ and $\lambda(\ell)$, respectively. It follows that

$$\begin{aligned} |\lambda(p) - \lambda(\ell)| &\lesssim \sigma^{-2}\|S_p - S_\ell\| + \sigma^{\frac{1}{4}} + \sigma^2 \\ &\lesssim \sigma^{-2}\|S_p - S_\ell\| + \sigma^{\frac{1}{4}}. \end{aligned}$$

Optimizing in σ implies that

$$|\lambda(p) - \lambda(\ell)| \lesssim \|S_p - S_\ell\|^{\frac{1}{9}},$$

and thus the proof reduces to proving that

$$|\lambda(p) - \lambda(\ell)| \gtrsim |p - \ell|. \tag{5-5}$$

In fact, recalling the definition of λ in (2-1), a direct calculation shows that

$$\lambda'(p) = -c\frac{1}{(p+2)^{\frac{3}{2}}} \frac{\Gamma\big(\frac{3p}{4} - \frac{1}{2}\big)}{\Gamma\big(\frac{3p}{4}\big)} \left\{ \frac{3}{2(p+2)} + \frac{3}{4}\Big[\psi\Big(\frac{3p}{4}\Big) - \psi\Big(\frac{3p}{4} - \frac{1}{2}\Big)\Big]\right\},$$

where ψ is the digamma function, i.e., $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$.

By Gautschi's inequality (see, e.g., [Rademacher 1973, Theorem A, p. 68]), we have

$$\frac{\Gamma\big(\frac{3p}{4} - \frac{1}{2}\big)}{\Gamma\big(\frac{3p}{4}\big)} > \Big(\frac{3p}{4}\Big)^{-\frac{1}{2}}.$$

Using the fact that ψ is increasing on $(0, \infty)$, it follows that

$$|\lambda'(p)| \geq \frac{3c}{2}(p+2)^{-\frac{5}{2}}\Big(\frac{3p}{4}\Big)^{-\frac{1}{2}} \gtrsim 1 \quad \text{uniformly for } p \in \big[\tfrac{4}{3}, 4\big].$$

This implies (5-5) and completes the proof of Theorem 1.5. □

Acknowledgements

Murphy was supported by NSF grant DMS-2137217. We are grateful to Gunther Uhlmann for suggesting that we consider stability estimates in this setting, as well as to John Singler for some helpful suggestions. We would also like to thank the referee for helpful comments.

References

[Sá Barreto and Stefanov 2022] A. Sá Barreto and P. Stefanov, "Recovery of a cubic non-linearity in the wave equation in the weakly non-linear regime", *Comm. Math. Phys.* **392**:1 (2022), 25–53. MR Zbl

[Sá Barreto et al. 2022] A. Sá Barreto, G. Uhlmann, and Y. Wang, "Inverse scattering for critical semilinear wave equations", *Pure Appl. Anal.* **4**:2 (2022), 191–223. MR Zbl

[Carles and Gallagher 2009] R. Carles and I. Gallagher, "Analyticity of the scattering operator for semilinear dispersive equations", *Comm. Math. Phys.* **286**:3 (2009), 1181–1209. MR Zbl

[Chen and Murphy 2023] G. Chen and J. Murphy, "Recovery of the nonlinearity from the modified scattering map", 2023. arXiv 2304.01455

[Enss and Weder 1995] V. Enss and R. Weder, "The geometrical approach to multidimensional inverse scattering", *J. Math. Phys.* **36**:8 (1995), 3902–3921. MR Zbl

[Ginibre and Velo 1992] J. Ginibre and G. Velo, "Smoothing properties and retarded estimates for some dispersive evolution equations", *Comm. Math. Phys.* **144**:1 (1992), 163–188. MR Zbl

[Hogan et al. 2023] C. Hogan, J. Murphy, and D. Grow, "Recovery of a cubic nonlinearity for the nonlinear Schrödinger equation", *J. Math. Anal. Appl.* **522**:1 (2023), art. id. 127016. MR Zbl

[Keel and Tao 1998] M. Keel and T. Tao, "Endpoint Strichartz estimates", *Amer. J. Math.* **120**:5 (1998), 955–980. MR Zbl

[Killip et al. 2023] R. Killip, J. Murphy, and M. Vişan, "The scattering map determines the nonlinearity", *Proc. Amer. Math. Soc.* **151**:6 (2023), 2543–2557. MR Zbl

[Lassas et al. 2022] M. Lassas, T. Liimatainen, L. Potenciano-Machado, and T. Tyni, "Uniqueness, reconstruction and stability for an inverse problem of a semi-linear wave equation", *J. Differential Equations* **337** (2022), 395–435. MR Zbl

[Lee and Yu 2023] Z. Lee and X. Yu, "On recovering the nonlinearity for generalized higher-order Schrödinger equations", 2023. Zbl arXiv 2303.06312

[Morawetz and Strauss 1973] C. S. Morawetz and W. A. Strauss, "On a nonlinear scattering operator", *Comm. Pure Appl. Math.* **26** (1973), 47–54. MR Zbl

[Murphy 2023] J. Murphy, "Recovery of a spatially-dependent coefficient from the NLS scattering map", *Comm. Partial Differential Equations* **48**:7-8 (2023), 991–1007. MR Zbl

[Pausader and Strauss 2009] B. Pausader and W. A. Strauss, "Analyticity of the nonlinear scattering operator", *Discrete Contin. Dyn. Syst.* **25**:2 (2009), 617–626. MR Zbl

[Rademacher 1973] H. Rademacher, *Topics in analytic number theory*, edited by E. Grosswald et al., Grundl. Math. Wissen. Band **169**, Springer, 1973. MR Zbl

[Sasaki 2007] H. Sasaki, "The inverse scattering problem for Schrödinger and Klein–Gordon equations with a nonlocal nonlinearity", *Nonlinear Anal.* **66**:8 (2007), 1770–1781. MR

[Sasaki 2008] H. Sasaki, "Inverse scattering for the nonlinear Schrödinger equation with the Yukawa potential", *Comm. Partial Differential Equations* **33**:7-9 (2008), 1175–1197. MR

[Sasaki and Watanabe 2005] H. Sasaki and M. Watanabe, "Uniqueness on identification of cubic convolution nonlinearity", *J. Math. Anal. Appl.* **309**:1 (2005), 294–306. MR

[Strauss 1974] W. A. Strauss, "Nonlinear scattering theory", pp. 53–78 in *Scattering Theory in Mathematical Physics*, edited by J. A. Lavita and J. P. Marchand, Springer, 1974.

[Strichartz 1977] R. S. Strichartz, "Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations", *Duke Math. J.* **44**:3 (1977), 705–714. MR Zbl

[Vişan 2014] M. Vişan, "Dispersive equations", pp. 223–308 in *Dispersive equations and nonlinear waves*, vol. 45, Oberwolfach Seminars, Birkhäuser/Springer, Basel, 2014. MR

[Watanabe 2001] M. Watanabe, "Inverse scattering for the nonlinear Schrödinger equation with cubic convolution nonlinearity", *Tokyo J. Math.* **24**:1 (2001), 59–67. MR Zbl

[Watanabe 2018] M. Watanabe, "Time-dependent method for non-linear Schrödinger equations in inverse scattering problems", *J. Math. Anal. Appl.* **459**:2 (2018), 932–944. MR Zbl

[Weder 1997] R. Weder, "Inverse scattering for the nonlinear Schrödinger equation", *Comm. Partial Differential Equations* **22**:11-12 (1997), 2089–2103. MR Zbl

[Weder 2000] R. Weder, "L^p-$L^{\dot{p}}$ estimates for the Schrödinger equation on the line and inverse scattering for the nonlinear Schrödinger equation with a potential", *J. Funct. Anal.* **170**:1 (2000), 37–68. MR Zbl

[Weder 2001a] R. Weder, "Inverse scattering for the non-linear Schrödinger equation: reconstruction of the potential and the non-linearity", *Math. Methods Appl. Sci.* **24**:4 (2001), 245–254. MR Zbl

[Weder 2001b] R. Weder, "Inverse scattering for the nonlinear Schrödinger equation, II: Reconstruction of the potential and the nonlinearity in the multidimensional case", *Proc. Amer. Math. Soc.* **129**:12 (2001), 3637–3645. MR

[Weder 2002] R. Weder, "Multidimensional inverse scattering for the nonlinear Klein–Gordon equation with a potential", *J. Differential Equations* **184**:1 (2002), 62–77. MR

Received 10 May 2023. Revised 23 Jul 2023. Accepted 30 Oct 2023.

GONG CHEN: gc@math.gatech.edu
Georgia Institute of Technology, Atlanta, GA, United States

JASON MURPHY: jason.murphy@mst.edu
Missouri University of Science & Technology, Rolla, MO, United States

PURE and APPLIED ANALYSIS

msp.org/paa

Cover image: The figure shows the outgoing scattered field produced by scattering a plane wave, coming from the northwest, off of the (stylized) letters P A A. The total field satisfies the homogeneous Dirichlet condition on the boundary of the letters. It is based on a numerical computation by Mike O'Neil of the Courant Institute.

See inside back cover or msp.org/paa for submission instructions.

The subscription price for 2024 is US $625/year for the electronic version, and $700/year (+$30, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Pure and Applied Analysis (ISSN 2578-5885 electronic, 2578-5893 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online.

PAA peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PURE and APPLIED ANALYSIS

vol. 6 no. 1 2024