BERTINI THEOREMS FOR DIFFERENTIAL ALGEBRAIC GEOMETRY

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Abstract. We study intersection theory for differential algebraic varieties. Particularly, we study families of differential hypersurface sections of arbitrary affine differential algebraic varieties over a differential field. We prove the differential analogue of Bertini's theorem, namely that for an arbitrary geometrically irreducible differential algebraic variety which is not an algebraic curve, generic hypersurface sections are geometrically irreducible and codimension one. Surprisingly, we prove a stronger result in the case that the order of the differential hypersurface is at least one; namely that the generic differential hypersurface sections of an irreducible differential algebraic variety are irreducible and codimension one. We also calculate the Kolchin polynomials of the intersections and prove several other results regarding intersections of differential algebraic varieties.
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1. Introduction

Consider the following theorem from algebraic geometry:

Theorem. [7, 7.1, page 48] *Let Y, Z be irreducible algebraic varieties of dimensions r, s in $\mathbb{A}^n$ Then every irreducible component W of $Y \cap Z$ has dimension greater than or equal to $r+s-n$.*

This theorem fails for differential algebraic varieties embedded in affine space, as the following examples show.

Example 1.1. (Ritt's example). We work in $\mathbb{A}^3$ over an ordinary differential field, k. Let $V = V(f)$ be the zero set of the single differential polynomial:

$$f(x,y,z) = x^5 - y^5 + z(x\delta y - y\delta x)^2$$

Though V is the zero set of an absolutely irreducible differential polynomial, it is not irreducible in the Kolchin topology. V has six components. Let μ_5 denote the set of fifth roots of unity. For each $\zeta \in \mu_5$, $x - \zeta y$ cuts out a subvariety of V. Note that

$$f = (x-\zeta y)\left(\prod_{\eta \in \mu_5 \setminus \{\zeta\}} (x - \eta y) + z(\delta y(x-\zeta y) - y\delta(x-\zeta y))^2\right)$$

is a preparation eqution for f with respect to $x - \zeta y$ [11, see chapter 4, section 13]. Further, one obtains the preparation congruence, $f = (x-\zeta y)\left(\prod_{\eta\in\mu_5\setminus\{\zeta\}}(x-\eta y)\right)$ modulo $[x-\zeta y]^2$, so by the Low Power Theorem [18, chapter 7] or [11, chapter 4, section 15], $[x-\zeta]$ is the ideal of a component of V. The general component is given by the saturation by the separant (with respect to some ranking) of $[f]$. For instance, one possible choice of ranking would yield $[f] : \frac{\partial f}{\partial \delta x}^{\infty} = \{g \mid \left(\frac{\partial f}{\partial \delta x}\right)^n g \in [f], \text{for some } n \in \mathbb{N}\}$ as the ideal of the general component. By the component theorem [11, Theorem 5, page 185], these are the only components. Now we consider the differential

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algebraic variety W, which is the general component of V. Establishing that $(0,0,0) \in W$ can be done by noting that f is differentially homogeneous [1] and applying [10, Proposition 2]. In fact, one can prove that $W \cap H$ consists of precisely $(0,0,0)$ (see [19] for a nice exposition of the proof).

Example 1.2. Let X be a projective curve of genus at least two over a differentially closed field K which does not descend to the constants. Buium [1] proves that X may be embedded as a differential algebraic variety in projective space over K such that X lies outside of some unique hyperplane. Passing to the affine cone of the projective space again gives an intersection with $r+s-n=1$, but the intersection has dimension 0 (consisting of the origin).

The motivating questions of the paper come from the examples of Ritt and Buium:

Question 1.3. *In the space of differential hypersurfaces of a particular order and degree, what is the set of coefficients on which the intersection theorem fails for a given arbitrary differential algebraic variety?*

The main thrust of this paper is to provide an answer to this question by proving the differential algebraic analogue of Bertini's theorem. Roughly, we prove that the intersection of an irreducible differential algebraic variety of dimension d and a generic hyperplane is a irreducible differential algebraic variety of dimension $d-1$.

Specifically, if V is given by the differential ideal $\mathfrak{I} \subseteq K\{\bar{y}\}$ and $\bar{u}$ is a suitably long tuple of independent differential transcendentals over K, then we analyze the properties of the ideal $[\mathfrak{I}, f_{\bar{u}}] \subseteq K\langle\bar{u}\rangle\{y_1, \ldots, y_n\}$ where $f_{\bar{u}}$ is the differential polynomial given by $u_0 + \sum_i u_i m_i(\bar{y})$ where the m_i are all of the differential monomials of order and degree bounded by some pair of natural numbers. This analysis was already performed in the ordinary case [21]. In particular, we show that $[\mathfrak{I}, f_{\bar{u}}]$ is a prime differential ideal of dimension one less than $\mathfrak{I}$.

This result is the differential algebraic analogue of [8, Theorem 2 page 54], which is an algebraic precursor to Bertini's theorem. The main problem with applying this result for making inductive arguments in differential algebraic geometry is that the primality of the ideal holds only in the differential polynomial ring over $K\langle\bar{u}\rangle$. For various applications, and inductive arguments, one would wish to take the coefficients $\bar{u}$ in some ambient large differentially closed field and establish the primality of $[\mathfrak{I}, f_{\bar{u}}]$ in the polynomial ring over the differentially closed field. This is too much to ask, because the corresponding theorem is not even true for algebraic varieties unless the dimension of V is at least 2; the intersection of a degree d curve with a generic hyperplane consists d points. Surprisingly, we show that algebraic curves are the *only* obstruction to the theorem. This portion of the argument uses a differential lying-over theorem [20], a geometric or model theoretic argument about ranks and the algebraic results established for $[\mathfrak{I}, f_{\bar{u}}]$. In other words, we prove *geometric irreducibility* results:

Definition 1.4. An affine differential algebraic variety, V over k, is geometrically irreducible if $I(V) \otimes_k k'$ is a prime differential ideal for *any* k', a differential field extension of k.

Theorem 1.5. *($|\Delta| = m$) Let V be a geometrically irreducible affine differential algebraic variety over a Δ-field K. Let H be a generic differential hypersurface over K. Assume that the order of H is greater than zero or that the Kolchin polynomial of V is greater than $\binom{t+m}{m}$. Then $V \cap H$ is a geometrically irreducible differential algebraic variety, which is nonempty just in case $dim(V) > 0$. In that case, $V \cap H$ has Kolchin polynomial:*

$$\omega_{V/K}(t) - \binom{t+m-h}{m}.$$

[1] This is an observation of Phyllis Cassidy.

Our analysis draws inspiration from [21], but we also use arguments of a more geometric and model theoretic nature. In particular, we employ the theory of prolongation spaces, in the sense of Moosa and Scanlon [15] and arguments which use several properties of Lascar rank.

Following the proof of our main theorem, we give a result regarding intersections of differential algebraic varieties with generic differential hypersurfaces passing through a given point. Specifically, we prove that if V is a d dimensional differential algebraic variety and $H_1, H_2, \dots H_{d+1}$ are generic differential hypersurfaces which contain $\bar{a}$ and $V \cap \bigcap_{i=1}^{d+1} H_i \neq \emptyset$, then $\bar{a} \in V$.

The special case in which the H_i are hyperplanes in the case of one derivation was proved in [21]; our proof of the generalization is much shorter, owing to using stability theoretic tools (e.g. Lascar's symmetry lemma). This also generalizes [17, Theorem 1.7].

One expects the main theorem of paper to be useful for proofs by induction on the differential transcendence degree of a differential algebraic variety. For instance, in [2], the theorem is used to study completeness in the Kolchin topology and [5] uses the results of this paper to construct differential Chow varieties.

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2. Setting and definitions

We will very briefly review *some* of the developments from model theory and differential algebra necessary for our results; more complete expositions can be found in various sources, which we cite below. We use standard model-theoretic notation, following [12] and differential algebraic notation, following [11]. In this paper, we will take k to be a differential field of characteristic zero in m commuting derivations, $\Delta = \{\delta_1, \dots, \delta_m\}$. In this setting, we have a *model companion*, the theory of Δ-closed fields, denoted $DCF_{0,m}$. One may work entirely within a saturated model $\mathcal{U} \models DCF_{0,m}$ for this paper, taking all differential field extensions therein. However, the results of this paper require care with respect to the field we work over. We do not consider abstract differential algebraic varieties or differential schemes; we only consider affine differential algebraic varieties over a differential field. One can easily extend many of the results of this paper to the projective case, but we do not address this directly.

The *type* (in the sense of model theory) of a finite tuple of $\mathcal{U}$ over k is the collection of all first order formulae with parameters from k which hold of the tuple; for a given tuple $\bar{a}$, we write $tp(\bar{a}/k)$. A realization of a type p over k (we write $p \in S(k)$) is a tuple from a field extension satisfying all of the first order formulae in the type. As $DCF_{0,m}$ has *quantifier elimination*, we have a correspondence between types and prime differential ideals and differential algebraic varieties. Given a type $p \in S(k)$, we have the corresponding (prime) differential ideal via

$$p \mapsto I_p = \{f \in k\{y\} \mid f(y) = 0 \in p\}.$$

The corresponding variety is the zero set of $I_p = \{f \in k\{y\} \mid f(y) = 0 \in p\}$ in $\mathcal{U}^n$ where n is the length of the tuple $\bar{y}$.

For the reader not acquainted with the language of model theory, the type of a tuple b over k corresponds to the isomorphism type of the differential field extension $k\langle b\rangle/k$ *with specified generators.*

Let Θ be the free commutative monoid generated by Δ. For $\theta \in \Theta$, if $\theta = \delta_1^{\alpha_1} \dots \delta_m^{\alpha_m}$, then $ord(\theta) = \alpha_1 + \dots + \dots + \alpha_m$. The order gives a grading on the monoid Θ. We let

$$\Theta(s) = \{\theta \in \Theta : ord(\theta) \leq s\}.$$

When R is a Δ-ring and $S \subseteq R$, by $[S]$ we mean the Δ-ideal generated by S in R. When f is an element in a differential polynomial ring, by $V(f)$, we mean the zero set of the differential polynomial. Let f be a differential polynomial of order h in $K\{x_1, \dots, x_n\}$. Then f is a polynomial in the ring $K[\theta x_i : i = 1, \dots, n, \theta \in \Theta(h)]$. When we think of f as a polynomial in $K[\theta x_i : i = 1, \dots, n, \theta \in \Theta(h)]$, we will write $Z(f)$ for the corresponding Zariski closed subset of $\mathbb{A}^{n \cdot \binom{m+h}{h}}$. [2]

Analagous notation applies to ideals or sets of elements in (differential) rings.

Theorem 2.1. *(Theorem 6, page 115, [11]) Let $\eta = (\eta_1, \dots, \eta_n)$ be a finite family of elements in some extension of k. There is a numerical polynomial $\omega_{\eta/k}(t)$ with the following properties.*

(1) *For sufficiently large $t \in \mathbb{N}$, the transcendence degree of $k((\theta\eta_j)_{\theta \in \Theta(t),\, 1 \leq j \leq n})$ over k is equal to $\omega_{\eta/k}(t)$.*
(2) *$deg(\omega_{\eta/k}(t)) \leq m$*
(3) *One can write*

$$\omega_{\eta/k}(t) = \sum_{o \leq i \leq m} a_i \binom{t+i}{i}$$

In this case, a_m is the differential transcendence degree *of $k\langle\eta\rangle$ over k.*

Definition 2.2. When V is a differential algebraic variety over k and $\bar{b} \in V$ is such that

$$\left\{f \in k\{\bar{x}\} \,|\, f(\bar{b}) = 0\right\} = \left\{f \in k\{\bar{x}\} \,|\, \forall \bar{a} \in V,\ f(\bar{a}) = 0\right\},$$

we call $\bar{b}$ a generic point on V over k. When V is a differential algebraic variety over k and $\bar{b} \in V$ is a generic point over k, we define $dim(V)$ to be the Δ-transcendence degree of $k\langle\bar{b}\rangle$ over k.

Definition 2.3. The polynomial from the theorem is called the *Kolchin polynomial* or the *differential dimension polynomial*. Let $p \in S(k)$. Then $\omega_p(t) := \omega_{b/k}(t)$ where b is any realization of the type p over k.

Suppose that p and q are types such that q extends p. This means $p \in S(k)$ for some differential field k and $q \in S(K)$ where K is a differential field extension of k; further, as a sets of first order formulae $p \subset q$.

Definition 2.4. Let q extend p. We say q is a *nonforking extension* of p if

$$\omega_p(t) = \omega_q(t).$$

Note that the Kolchin polynomial on the left is being calculated over k and the Kolchin polynomial on the right is being calculated over K. When $w_p(t) \neq \omega_q(t)$ (in which case $w_p(t) > \omega_q(t)$) we say that q is a *forking* extension of p.

If v_1, v_2 are tuples in a differential field extension of K and $tp(v_1/K\langle v_2\rangle)$ is a nonforking extension of $tp(v_1/K)$ then we say that v_1 and v_2 are *independent over* K (or simply independent if K is clear from context).

Note that $tp(v_1/K\langle v_2\rangle)$ being a nonforking extension of $tp(v_1/K)$ actually implies that $tp(v_2/K\langle v_1\rangle)$ is a nonforking extension of $tp(v_2/K)$, so this notion is symmetric.

Definition 2.5. Let p be a type. Then,

- $RU(p) \geq 0$ just in case p is consistent.

[2] It is well-known that $|\Theta(h)| = \binom{m+h}{h}$.

- $RU(p) \geq \beta$, where β is a limit just in case $RU(p) \geq \alpha$ for all $\alpha < \beta$.
- $RU(p) \geq \alpha + 1$ just in case there is a forking extension q of p such that $RU(q) \geq \alpha$.

The ordinal $RU(p)$ is called the *Lascar rank* of p. When $\bar{a} \in \mathcal{U}^n$, we will write $RU(\bar{a}/K)$ for $RU(tp(\bar{a}/K))$.

Remark 2.6. The last three definitions are specific instances of model theoretic notation in the setting of differential algebra; for the more general definitions, see [16]. Our development of these notions is rather nonstandard; normally, forking is defined in a much more general manner. That forking specializes to the above notion in differential algebra requires proof, but is a natural consequence of the basic model theory of differential fields, see [13, section4]. A forking extension of the type of $\bar{a}$ over k is given by some differential field extension $K \supset k$ such that $\{f \in K\{\bar{x}\} \mid f(a) = 0\}$ properly contains the ideal $I(V) \otimes_k K$.

Specific instances of calculations of Lascar rank in this setting are often quite involved and can be found in various sources [4, 3].

Let U be a definable set over k and let $\bar{U}$ be the closure of V in the Kolchin topology over k. Suppose that $\bar{U}$ is irreducible over k. We will set $RU(V)$ to be equal to the supremum of the collection $\{RU(p) \mid p \in U\}$. We will be using Lascar rank at various points, and remind the reader of the following result, which we use throughout the paper:

Proposition 2.7. [13, 5.2.2] *Let b be a tuple in a differential field extension of k. Then*

$$dim(b/k) = n \text{ if and only if } \omega^m \cdot n \leq RU(tp(b/k)) < \omega^m \cdot (n+1)$$

We will also require a differential notion of specializations:

Definition 2.8. Let $\Delta = \{\delta_1, \dots, \delta_m\}$. Let $\Delta' = \{\delta'_1, \dots, \delta'_m\}$. A homomorphism ϕ from Δ-ring (R, Δ) to Δ'-ring (S, Δ') is called a differential homomorphism if for each i, $\phi \circ \delta_i = \delta'_i \circ \phi$. When R is an integral domain and S is a field, then such a map is called a Δ-*specialization.*

The following proposition is proved in a constructive manner in [21, Theorem 2.16]; and an analogous proof works in the case of several commuting derivations.

Proposition 2.9. *Let $\bar{u} = (u_1, \dots, u_r) \subset \mathcal{U}$ be a set of Δ-K independent differential transcendental elements. Let $\bar{y} = (y_1, \dots, y_n)$ be a set of differential indeterminates. Let $P_i(\bar{u}, \bar{y}) \in K\{\bar{u}, \bar{y}\}$ for $i = 1, \dots, n_1$. Suppose $\phi : K\{\bar{y}\} \to \mathcal{U}$ be a differential specialization into $\mathcal{U}$ such that $\bar{u}$ is a set of Δ-transcendentals over $K\langle\phi(\bar{y})\rangle$. Suppose that $P_i(\bar{u}, \phi(\bar{y}))$ are (as a collection), Δ-dependent over $K\langle\bar{u}\rangle$. Then let ψ be a differential specialization from $K\langle\bar{u}\rangle \to K$. The collection $\{P_i(\psi(\bar{u}), \phi(\bar{y}))\}_{i=1,\dots,n_1}$ are Δ-dependent over K.*

3. Intersections

In this section we develop an intersection theory for differential algebraic varieties with generic Δ-polynomials. The influence of [21] for proving statements about irreducibility over specific differential fields is obvious; we have adapted their techniques to the setting of several commuting derivations.

Definition 3.1. In $\mathbb{A}^n$, the differential hypersurfaces are the zeros of a Δ-polynomial of the form

$$a_0 + \sum a_i m_i$$

where m_i are differential monomials in $\mathcal{F}\{y_1, \dots, y_n\}$. For convenience, in the following discussion, we do not consider 1 to be a monomial. A *generic Δ-polynomial* of order s and degree r over K is a Δ-polynomial

$$f = a_0 + \sum a_i m_i$$

where m_i ranges over all differential monomials of order less than or equal to s and degree less than or equal to r (i.e. monomials of $\Theta(s)(\bar{y})$ of degree at most r) and $(a_0, a_1, \dots, a_{n_1})$ is a tuple of independent Δ-transcendentals over K. A *generic* Δ*-hypersurface* of order s and degree r is the zero set of a generic Δ-polynomial of order s and degree r. When f is given as above, we let a_f be the tuple of coefficients of f. Throughout, we adopt the notation $\bar{a}_f = a_f \backslash \{a_0\}$.

By a generic differential polynomial (hypersurface), we mean a generic differential polynomial (hypersurface) of some order s and some $r \geq 1$.

The next lemma is proved in the ordinary case in [21, Lemma 3.5]. The proof in this case works similarly, assuming that one sets the stage with the proper reduction theory in the case of several commuting derivations. One might notice that Lemma 3.5 of [21] has a second portion. For now, we will concentrate only on the irreducibility of the intersection. Necessary and sufficient conditions for the intersection to be nonempty will be given later.

Lemma 3.2. *Let $\bar{y} = (y_1, \dots, y_n)$ and y_0 be another indeterminate. Let $\mathfrak{I}$ be a prime Δ-ideal in $K\{\bar{y}\}$ and let $f = y_0 + \sum_i a_i m_i$ where the sum ranges over all monomials of $\Theta(h)\bar{y}$ of degree d_1 and $\bar{a}$ is a tuple of independent Δ-transcendentals over K. Then $\mathfrak{I}_0 = [\mathfrak{I}, f]$ is a prime Δ-ideal of $K\langle \bar{a}_f \rangle \{\bar{y}, y_0\}$. Further, $\mathfrak{I}_0 \cap K\langle \bar{a}_f \rangle \{y_0\} \neq 0$ if and only if V has dimension zero.*

Proof. Let $\bar{b} = (b_1, \dots, b_n)$ be a generic point of $V = V(\mathfrak{I})$ over K such that $\bar{b}$ is independent from $\bar{a}$ over K (see Definition 2.4). Consider the tuple $(b_1, \dots, b_n, -\sum_i a_i m_i(\bar{b}))$. We show irreducibility of the variety $V(\mathfrak{I}_0)$ in $\mathbb{A}^{n+1}$ via showing that it is the Kolchin closure of $(b_1, \dots, b_n, -\sum_i a_i m_i(\bar{b}))$ over K, from which the result follows because the ideal of differential polynomials in $K\{\bar{y}, y_0\}$ which vanish at a given point is a prime ideal.

Suppose g is a Δ-polynomial in $K\langle \bar{a}_f \rangle \{\bar{y}, y_0\}$ which vanishes at $(b_1, \dots, b_n, -\sum_i a_i m_i(\bar{b}))$. Fix a ranking so that y_0 is the leader of f. Then the partial remainder of g with respect to f [11, page 77] gives some g_0 (which is equal to g modulo the differential ideal generated by f). This g_0 must be in $K\langle \bar{a}_f \rangle \{y_1, \dots, y_n\}$. Since $\bar{b}$ is generic for $\mathfrak{I}$, we must have that $g_0 \in K\langle \bar{a}_f \rangle \cdot \mathfrak{I}$. But then $g \in \mathfrak{I}_0$ and the claim follows.

We have seen that $(b_1, \dots, b_n, -\sum_i a_i m_i(\bar{b}))$ is generic on $V(\mathfrak{I}_0)$. If $K\langle b_1, \dots, b_n \rangle$ over K has differential transcendence degree zero, then $K\langle \bar{a}, -\sum_i a_i m_i(\bar{b}) \rangle$ over $K\langle \bar{a} \rangle$ has differential transcendence degree zero, so $\mathfrak{I}_0 \cap K\langle \bar{a} \rangle \{y_0\}$ is not the zero ideal. Conversely, if $K\langle b_1, \dots, b_n \rangle$ over K has differential transcendence degree greater than zero, then since $\bar{a}$ is independent from $\bar{b}$ over K, $\sum_i a_i m_i(\bar{b})$ is a differential transcendental over $K\langle \bar{a} \rangle$, and so $\mathfrak{I}_0 \cap K\langle \bar{a}_f \rangle \{y_0\}$ is the zero ideal. □

Now we turn towards establishing necessary and sufficient conditions for the intersection to be nonempty when we relax the sorts of intersections under consideration. In the case that the intersection is nonempty, we calculate the differential transcendence degree.

This next lemma was originally proved in [17, Theorem 1.7] in the ordinary case, and was reproved in [21] in the ordinary case. The proof in the case of several commuting derivations can be found in [3, Proposition 4.1].

Lemma 3.3. *Suppose that V is a differential algebraic variety such that $RU(V/K) < \omega^m$. Then $V \cap V(f(\bar{x})) = \emptyset$ for any generic differential polynomial $f(\bar{x})$.*

Lemma 3.4. *Suppose that V is a differential algebraic variety embedded in $\mathbb{A}^n$ and that V is of dimension $d \geq 1$. If $f(\bar{x})$ is a generic differential polynomial, then $V \cap V(f) \neq \emptyset$ and $dim(V \cap V(f)) = d - 1$.*

Proof. Let $V = V(\mathfrak{I})$; we will be following the general notation of 3.2, with $f = y_0 + \sum_i a_i m_i(\bar{y})$. Define $f_0 := \sum_i a_i m_i(\bar{y})$ and let $\bar{a}_f$ be the tuple of a_i coefficients which appear in f. Define

$\mathfrak{I}_0 = [\mathfrak{I}, f] \subseteq K\langle \bar{a}_f \rangle\{\bar{y}, y_0\}$ as in 3.2. Here y_0 appears as an indeterminate in the differential polynomial ring. Define $\mathfrak{I}_1 = [\mathfrak{I}, f] \subseteq K\langle \bar{a}_f, y_0\rangle\{\bar{y}\}$.

Let $\bar{b}$ be a realization of the generic type of V over K. Reorder the coordinates if necessary so that $b_1, \dots, b_d$ are a Δ-transcendence basis for the Δ-field extension generated by $\bar{b}$ over K. Because $V \subset \mathbb{A}^n$ is isomorphic to $V(\mathfrak{I}_0) \subset \mathbb{A}^{n+1}$, $dim(V(\mathfrak{I}_0)) = d$.

Since $\mathfrak{I} \subset \mathfrak{I}_1$, each of y_i for $i > d$ is Δ-dependent with $y_1, \dots, y_d$ modulo $\mathfrak{I}$ (and thus $\mathfrak{I}_1$). So, $y_0, y_1, \dots, y_d$ are Δ-dependent modulo $\mathfrak{I}_0$ and so $y_1, \dots, y_d$ are Δ-dependent modulo $\mathfrak{I}_1$. Thus $dim(V(\mathfrak{I}_1)) \leq d-1$.

Now supopose that $y_1, \dots, y_{d-1}$ are Δ-dependent modulo $\mathfrak{I}_1$; then there is some nonzero element $p(y_1, \dots, y_{d-1}) \in \mathfrak{I}_1$. By clearing denominators one, can take $p \in K\{\bar{a}_f, y_1, \dots, y_{d-1}, y_0\}$. Then $p(\bar{a}_f, b_1, \dots, b_{d-1}, -f_0(\bar{b})) = 0$.

Now specialize a_d, the coefficient of y_d in the generic differential hypersurface to -1 and specialize all other $a_i \in \bar{a}_f$ to 0. But then $b_1, \dots, b_d$ are dependent over K by 2.9, a contradiction to the assumption that V has dimension d.

□

Lemma 3.5. *Let $\mathfrak{I}$ be a prime Δ-ideal in $K\{y_1, \dots, y_n\}$. Let $f = y_0 + \sum_i a_i m_i(\bar{y})$ be a generic differential polynomial. Then $\mathfrak{I}_1 = [\mathfrak{I}, f]$ is a prime Δ-ideal in $K\langle y_0, a_f\rangle\{y_1, \dots, y_n\}$.*

Proof. First, suppose that the dimension of V is at least one. Then by Lemma 3.4, $V(I) \cap V(f) \neq \emptyset$. Recall the notation of $\mathfrak{I}_0$ from Lemma 3.2. We will show that $\mathfrak{I}_1 \cap K\langle a_f\rangle\{y_1, \dots, y_n, y_0\} = \mathfrak{I}_0$. Suppose that we have $g, h \in K\langle a_f, y_0\rangle\{y_1, \dots, y_n\}$ such that $g \cdot h \in \mathfrak{I}_1$. Since we are taking a field extension over K, the coefficients of the differential polynomials might involve differential rational functions in a_f, y_0 over K. Clearing denominators, by multiplying by suitable differential polynomials in y_0, a_f over K, we obtain $g, h \in K\{a_f, y_0, y_1, \dots, y_n\}$ such that $g \cdot h \in \mathfrak{I}_0$. But, $\mathfrak{I}_0$ is prime by Lemma 3.2. So, we have a contradiction and $\mathfrak{I}_1$ is prime. Further, we can see (again, simply by clearing denominators) that $\mathfrak{I}_1$ lies over $\mathfrak{I}_0$, when we regard $\mathfrak{I}_0$ as an ideal of $R\{y_1, \dots, y_n\}$ where $R = K\langle a_f\rangle\{y_0\}$.

In the case that $dim(V) = 0$, $\mathfrak{I}_0 \cap K\langle a_f\rangle\{y_0\} \neq 0$ by Lemma 3.2, and so $\mathfrak{I}_1$ must be the unit ideal. □

Lemma 3.6. *Let $\mathfrak{I}$ be a prime differential ideal of $K\{y_1, \dots, y_n\}$ and let $V = V(\mathfrak{I})$ be the corresponding irreducible differential variety. Suppose that the dimension d of V is greater than zero. Let $f = a_0 + \sum_i a_i m_i(\bar{y})$ be generic of order $h \geq 0$ and degree $d_1 > 0$. Let $\mathfrak{I}_1 = [\mathfrak{I}, f] \subseteq K\langle a_f\rangle\{y_1, \dots, y_n\}$. Then,*

$$\omega_{V(\mathfrak{I}_1)/K\langle a_0, a_1, \dots, a_n\rangle}(t) = \omega_{V(\mathfrak{I})/K}(t) - \binom{t+m-h}{t}.$$

Proof. In this proof, we associate a differential algebraic variety V naturally with its *prolongation sequence* (for complete details, see [14]). Briefly, recall, the data of a prolongation sequence is the sequence of algebraic varieties:

$$V_l = \{(\theta \bar{x}) \,|\, x \in X(\mathfrak{U}),\, \theta \in \Theta(l)\}^{cl} \subseteq \mathbb{A}^{n \cdot \binom{l+m}{m}},$$

where $(-)^{cl}$ denotes Zariski closure and the coordinates are ordered by the canonical orderly ranking induced by taking $\delta_i < \delta_j$ when $i < j$. It is a fact that the sequence V_l determines V; by Noetherianity of the Kolchin topology, a finite subsequence determines V. Given a sequence of algebraic varieties $(V_l)_{l \in \mathbb{N}}$ with $V_l \subseteq \mathbb{A}^{n \cdot \binom{l+m}{m}}$, we call the sequence a prolongation sequence if for all $l \in \mathbb{N}$, the projection map $V_l \to V_l$ (to the first $n \cdot \binom{l+m}{m}$ corrdinates) is dominant and the variety V_{l+1} satisfies the differential relations forced by V_l. In the notation of [14, page 7, preceeding Proposition 2.5] this second condition is formally expressed by saying that after embedding $\tau_\ell(\mathbb{A}^n)$ in $\tau^\ell(\mathbb{A}^n)$ and $\tau_{\ell+1}(\mathbb{A}^n)$ in $\tau^{\ell+1}(\mathbb{A}^n)$, we have that V_{l+1} is a closed subvariety of $\tau(V_l)$.[3]

There is a bijective correspondence between differential algebraic varieties and prolongation sequences. V is irreducible over K if and only if the all of the varieties in the corresponding prolongation sequence are irreducible over K. The ideal corresponding to V_ℓ given by the ideal of differential polynomials in $K\{y_1, \ldots, y_n\}$ of order at most ℓ which vanish on V.

For large enough values of l, the dimension of V_l is given by the value of the Kolchin polynomial of V. We will show that

$$V_l \cap \bigcap_{\theta \in \Theta(l-h)} (Z(\theta(f)))$$

is a prolongation sequence. Once this fact is established, it is clear that it must be the prolongation sequence corresponding to $V \cap V(f)$ (since each the given algebraic relations clearly hold on $V \cap V(f)$). Consider the differential algebraic variety $W \subseteq \mathbb{A}^{n+1}$ given by $\mathfrak{I}_0$ as in Lemma 3.2 above. As differential algebraic varieties, W and V are isomorphic, by the obvious maps. The prolongation sequence associated with W is given by $W_l \cong V_l \times \mathbb{A}^1_l$ when $l < h$ and

$$W_l \cong (V_l \times \mathbb{A}^1_l) \cap \bigcap_{\theta \in \Theta(l-h)} V(\theta(f_{y_0})),$$

when $l \geq h$, where f_{y_0} is the differential polynomial f with y_0 in place of a_0. To verify that this is a prolongation sequence, we need only determine that the maps $W_{l+1} \to W_l$ are dominant (since the second condition is obvious from the definition of the sequence). Then since the relation $f_{y_0} = 0$ holds on W, by the bijective correspondence between prolongation sequences and differential algebraic varieties, the sequence must be the prolongation sequence associated with W. When $l+1 < h$, this follows simply from the fact that V_l forms a prolongation sequence, noting that $W_l \cong V_l \times \mathbb{A}^1_l$.

In what follows, we reorder the coordinates of each W_l so that $W_l \subseteq V_l \times \mathbb{A}^1_l$ for each $l \in \mathbb{N}$. When $l \geq h$, W_l is a subvariety of $V_l \times \mathbb{A}^1_l$ determined by the zero set of $\theta(f_{y_0}) = 0$ for each $\theta \in \Theta(l-h)$. So, W_l is a subvariety of

$$V_l \times \mathbb{A}^1_l \cong V_l \times \mathbb{A}^{\binom{l-h+m}{m}} \times A^{\binom{l+m}{m} - \binom{l-h+m}{m}}$$

where the the copy of $\mathbb{A}^{\binom{l-h+m}{m}}$ in the above equation corresponds to the coordinates $\theta(y_0)$ with $\theta \in \Theta(l-h)$ and the copy of $A^{\binom{l+m}{m} - \binom{l-h+m}{m}}$ corresponds to the coordinates $\theta(y_0)$ with $\theta \in \Theta(l) \setminus \Theta(l-h)$. Since the order of $\theta(f_{y_0})$ is the order of θ plus h, one can see that W_l is given

$$W_l = \left(\left(V_l \times \mathbb{A}^{\binom{l-h+m}{m}} \right) \cap W'_l \right) \times A^{\binom{l+m}{m} - \binom{l-h+m}{m}},$$

where

$$W'_l \subseteq V_l \times \mathbb{A}^1_{l-h}$$

is given the vanishing of $\theta(f_{y_0}) = 0$ for each $\theta \in \Theta(l-h)$. Since $\theta(f_{y_0})$ is linear in $\theta(y_0)$, W'_l is the graph of a function $V_l \to \mathbb{A}^{\binom{l-h+m}{m}}$.

By our assumption $dim(V) > 1$, for some j, $\{\theta \circ \delta_m^h y_j \mid \theta \text{ of order } l-h\}$ are independent transcendentals over $K\langle \bar{a}_f \rangle(\{\eta y_i \mid \eta \in \Theta(l),\, i \neq j\} \cup \{\eta y_j \mid \eta < \delta_1^{l-h} \delta_m^h\})$ where we note that

$$\{\eta y_j \mid \eta < \delta_1^{l-h} \delta_m^h\} = \{\eta y_j \mid \eta \in \Theta(l)\} \setminus \{\theta \circ \delta_m^n y_j \mid \theta \text{ of order } l-h\}.$$

But, on W_{l+1}, for θ of order $l+1-h$, θy_0 is linearly dependent with $\theta \delta_m^h(y_j)$ over $K\langle \bar{a}_f \rangle(\{\eta y_i \mid \eta \in \Theta(l),\, i \neq j\} \cup \{\eta y_j \mid \eta < \delta_1^{l-h} \delta_m^h\})$.

[3] This is condition b in [14] in the paragraphs before Proposition 2.5.

Thus on W_{l+1}, $\{\theta(y_0)\,|\,\theta$ of order $l+1-h\}$ are transcendental over

$$K\langle \bar{a}_f\rangle(\{\eta y_i\,|\,\eta \in \Theta(l+1),\, i \neq j\} \cup (\{\eta y_j\,|\,\eta \in \Theta(l+1)\} \setminus \{\theta \circ \delta_m^n y_j\,|\,\theta \text{ of order } l+1-h\})).$$

Thus, onto the coordinates indexed by θy_0 for θ of order $l+1-h$, the image of W_{l+1} in W_l is dense. Thus the map $W_{l+1} \to W_l$ must be dominant.

The sequence of varieties

$$S_l := V_l \cap \bigcap_{\theta \in \Theta(l-h)} (Z(\theta(f)))$$

is the fiber of the family W_l when $y_0 = a_0$. In order to verify the dominance of the projection maps in the sequence, it suffices to consider only bounded subsequences. Verifying the dominance of the projection maps in a sequence $(S_l)_l$ for l bounded by some s is a constructible condition on the coefficients given by a_0 and its derivatives of order at most $s-h$.

That is, the condition that these finitely many maps be dominant is constructible in $\{\theta(a_0)\,|\,\theta \in \Theta(s-h)\}$ over $K\langle \bar{a}_f\rangle$ and since that dominance holds for $(W)_l$, it must hold for some Zariski-open subset. By the genericity of a_0 over $K\langle \bar{a}_f\rangle$, our finitely many maps must be dominant and so $(S_l)_l$ is a prolongation sequence.

For the calculation of the Kolchin polynomial, note that both $V_l \cap \bigcap_{\theta\in\Theta(l-h)} Z(\theta(f))$ and W_l are sequences of irreducible varieties and the surjective map $W_l \to V_l \cap \bigcap_{\theta\in\Theta(l-h)} Z(\theta(f))$ has fiber dimension $\binom{l+m}{m}$.

The dimension of W_l is given by $dim(V_l) + \binom{l+m}{m} - \binom{l+m-h}{m}$ because above the coordinate y_0, $\binom{l+m}{m}$ is the number of coordinates $\Theta(l)(y_0)$, and $\binom{l+m-h}{m}$ is the number of equations which give the graph of the function in the above definition of W_l'.

□

Putting together the previous results of the section, and restating the theorem in the form we will apply it later, we have:

Theorem 3.7. *Let V be a Kolchin-closed (over K) subset of $\mathbb{A}^n$ with differential transcendence degree d. Let H be a generic (with respect to K) differential hypersurface of some degree and order h with coefficients given by $\bar{a}$. Then $V \cap H$ is irreducible over $K\langle\bar{a}\rangle$. In the case that $d = 0$, $V \cap H = \emptyset$. If $d > 0$, then the Kolchin polynomial of $V \cap H$ is given by*

$$\omega_{V\cap H/K\langle a_H\rangle}(t) = \omega_{V/K}(t) - \binom{t+m-h}{m}.$$

One key point to notice is that

$$\binom{t+m-h}{m} = \binom{t+m}{m} - \sum_{i=0}^{h-1} \binom{t+m-1-i}{m-1}$$

as long as $h > 0$ and t is sufficiently large, and that under these circumstances,

$$\sum_{i=0}^{h-1} \binom{t+m-1-i}{m-1}$$

is a *positive* integer. In the special case that $m = 1$, this integer is h; meaning the previous theorem is a generalization of the following theorem, proved in [21, Theorem 3.13] when K is an ordinary differential field:

Theorem 3.8. *Let $\mathfrak{I}$ be a prime δ-ideal in $K\{\bar{y}\}$ with Kolchin polynomial $(t+1)d + c$. Let f be a generic δ-polynomial of order h and degree d. Then $\mathfrak{I}_1 = [\mathfrak{I}, f]$ is a prime δ-ideal in $K\langle a_f\rangle\{\bar{y}\}$ with Kolchin polynomial $(t+1)(d-1) + c + h$*

Remark 3.9. Note that in the results of this section, we are considering the Kolchin topology over a *specific field* (in many of the above statements, the field was $K\langle a_f \rangle$ or $K\langle a_f, y_0\rangle$). For various applications, it is useful to take the elements a_f and y_0 as above from some particular differential field K_1 extending K (for instance, K_1 might be a differentially closed field). For this, we would need to establish the irreducibility of various above intersections over K_1, which extends $K\langle a_f, y_0\rangle$.

We have not proved any statements about irreducibility over field extensions of $K\langle a_f, y_0\rangle$, nor do the authors of [21] in the ordinary setting. In fact, at least one additional hypothesis is necessary for results of this nature: if the hypothesis were *purely in terms of dimension, d,*, we would have to restrict to the situation $d \geq 2$. For instance, take any degree $d_1 > 1$ plane curve. This curve meets the generic hypersurface of degree d_2 in precisely $d_1 \cdot d_2$ points, so the intersection is not irreducible over any algebraically closed field (so in particular $K\langle a_f, y_0\rangle^{alg}$).

4. Geometric irreducibility

Before discussing geometric irreducibility, we will require some results about the Kolchin polynomials of prime differential ideals lying over a fixed prime differential ideal in extensions. The next result follows from [11, page 131, Proposition 3, part b], which is more general, allowing for positive characteristic.

Proposition 4.1. *Let $\mathfrak{p}$ be a prime differential ideal in $K\{y_1, \ldots, y_n\}$ and let $\mathfrak{G}$ be a differential field extension of K. Then $\mathfrak{G}\mathfrak{p}$ has finitely many prime components in $\mathfrak{G}\{y_1, \ldots, y_n\}$. If $\mathfrak{q}$ is any one of the components, then $\mathfrak{q} \cap K\{y_1, \ldots, y_n\}$ and we have equality of the Kolchin polynomials:*

$$\omega_{\mathfrak{q}} = \omega_{\mathfrak{p}}.$$

Remark 4.2. In model theoretic terms, the generic types of the components $V(\mathfrak{p}_1), \ldots, V(\mathfrak{p}_n)$ of $V(\mathfrak{p})$ are each nonforking extensions of the generic type of $V(\mathfrak{p})$. Assuming that the base field K is algebraically closed would ensure that the generic type of $V(\mathfrak{p})$ is stationary; consequently $\mathfrak{G}\mathfrak{p}$ is a prime differential ideal for any field extension $\mathfrak{G}$ of K.

Recall the following definition given in the introduction:

Definition 4.3. An affine differential algebraic variety, V over K, is geometrically irreducible if $I(V/K') = I(V) \otimes_K K'$ is a prime differential ideal for *any* K', a differential field extension of K.

Remark 4.4. Remark 4.2 shows that for differential varieties over a field K, enough to consider irreducibility over K^{alg}, the algebraic closure of K.

Theorem 4.5. *Let V be a geometrically irreducible Kolchin-closed over K subset of $\mathbb{A}^n$ with Kolchin polynomial $\omega_V(t) > \binom{t+m}{m}$. Let H be a generic hypersurface. Then $V \cap H$ is geometrically irreducible and $\omega_{H\cap V}(t) = \omega_{V/K}(t) - \binom{t+m}{m}$.*

Proof. Let d_1 be the degree of H. Consider the the differential algebraic variety $W = \{(v_1, v_2, \beta) \,|\, v_i \in V,\, v_i \in H_\beta\} \subseteq V \times V \times \mathbb{A}^{\binom{n+d_1}{d_1}-1}$ where H_β is the hypersurface given by $\sum \beta_i m_i = 1$, where the sum ranges over all monomials in $\bar{x}$ of degree bounded by d_1. Note tuple β is of length $(\binom{n+d_1}{d_1} - 1)$, the number of monomials of degree bounded by d_1 in n variables, excluding 1.

Consider $V \cap H_\beta$. When β is generic over K, we know that $V \cap H_\beta$ is irreducible over $K\langle\beta\rangle$, so by the Proposition 4.1, all of the components of V over the algebraic closure of $K\langle\beta\rangle$ have Kolchin polynomial equal to $\omega_{V\cap H_\beta/K\langle\beta\rangle}(t)$. If $V \cap H_\beta$ has more than one component, then W has more than one component with Kolchin polynomial at least

$$2 \cdot \omega_{V/K}(t) + \left(\binom{n+d_1}{d_1} - 3\right) \cdot \binom{m+t}{t}.$$

To see this, again note that the length of the tuple β is $(\binom{n+d_1}{d_1}-1)$, the number of monomials of degree bounded by d_1 in n variables, excluding 1. The Kolchin polynomial of a generic β over K is given by $\left(\binom{n+d_1}{d_1}-1\right)\cdot\binom{m+t}{t}$. The Kolchin polynomial of two independent (see Definition 2.4) generic points (v_1,v_2) on $V\cap H_\beta$ is given by $2\left(\omega_{V/K}(t)-\binom{t+m}{m}\right)$. Thus by Sit's lemma [4, Lemma 2.9], the Kolchin polynomial of the tuple (v_1,v_2,β) is at least

$$2\cdot\omega_{V/K}(t)+\left(\binom{n+d_1}{d_1}-3\right)\cdot\binom{m+t}{t}.$$

Suppose there is more than one component of $V\cap H_\beta$ over $K\langle\beta\rangle^{alg}$. Then there are elements $v_1,v_2\in V\cap H_\beta$ over $K\langle\beta\rangle^{alg}$ with Kolchin polynomial $\omega_{v_i/K\langle\beta\rangle}(t)=\omega_{V/K}(t)-\binom{t+m}{m}$ for $i=1,2$ and $tp(v_1/K\langle\beta\rangle)\neq tp(v_2/K\langle\beta\rangle)$. So there is more than one type of the triples (v_1,v_2,β) with v_1,v_2 generic and forking independent on $V\cap H_\beta$ over $K\langle\beta\rangle$, depending on if v_1 and v_2 are in the same component of $V\cap H_\beta$ over $K\langle\beta\rangle^{alg}$. Now, we consider components of W with Kolchin polynomial at least $2\cdot\omega_{V/K}(t)+\left(\binom{n+d_1}{d_1}-3\right)\cdot\binom{m+t}{t}$.

Suppose v_1 and v_2 are points on V, and β is generic subject to the condition that H_β contains v_1,v_2. If $v_1\neq v_2$, then we claim that

$$\omega_{v_1,v_2,\beta/K}(t)=\omega_{v_1,v_2/K}(t)+\left(\binom{n+d_1}{d_1}-3\right)\cdot\binom{m+t}{t}.$$

To see this, simply note that for $v_1\neq v_2$, we get two independent linear conditions on β.

The only way that

$$\omega_{v_1,v_2/K}(t)+\left(\binom{n+d_1}{d_1}-3\right)\cdot\binom{m+t}{t}\geq 2\cdot\omega_{V/K}(t)+\left(\binom{n+d_1}{d_1}-3\right)\cdot\binom{m+t}{t}$$

is for v_1,v_2 to be independent generic points on V, in which case, equality holds.

By similar analysis, if $v_1=v_2$, then $\omega_{v_1,v_2,\beta/K}(t)=\omega_{v_1/K}(t)+\left(\binom{n+d_1}{d_1}-2\right)\cdot\binom{m+t}{t}$.

Since $\omega_V(t)>\binom{t+m}{m}$,

$$\omega_{v_1/K}(t)+\left(\binom{n+d_1}{d_1}-2\right)\cdot\binom{m+t}{t}<2\cdot\omega_{V/K}(t)+\left(\binom{n+d_1}{d_1}-3\right)\cdot\binom{m+t}{t}.$$

So there is a unique type on W of rank $2\cdot\omega_{V/K}(t)+\left(\binom{n+d_1}{d_1}-3\right)\cdot\binom{m+t}{t}$.

By our earlier arguments, there is a unique component of $V\cap H_\beta$ over $K\langle\beta\rangle^{alg}$ with Kolchin polynomial $\omega_V-\binom{t+m}{m}$. But, by Proposition 4.1, we know any of component of $V\cap H_\beta$ must have Kolchin polynomial $\omega_V-\binom{t+m}{m}$. So, $V\cap H_\beta$ is geometrically irreducible. □

In the proof of the previous theorem, we can weaken the assumption that $\omega_V(t)>\binom{t+m}{m}$ to $\omega_V(t)\geq\binom{t+m}{m}$ in the case that the order of the differential hypersurface we consider is greater than 0. That is, in this case, V might be an algebraic curve:

Theorem 4.6. *Let V be a geometrically irreducible Kolchin-closed over K subset of $\mathbb{A}^n$ with Kolchin polynomial $\omega_V(t)\geq\binom{t+m}{m}$. Let H be a generic differential hypersurface of order $h>0$. Then $V\cap H$ is geometrically irreducible and*

$$\omega_{H\cap V}(t)=\omega_V(t)-\binom{t+m-h}{m}.$$

Proof. Let d_1 be the degree of H. We define $W = \{(v_1, v_2, \beta) \,|\, v_i \in V,\, v_i \in H_\beta\} \subseteq V \times V \times \mathbb{A}^{n_1}$ where H_β is the differential hypersurface given by $\sum \beta_i m_i = 1$, where the sum ranges over all monomials in $\bar{x}$ of order bounded by h and degree bounded by d_1. Note that

$$n_1 = \binom{n \cdot \binom{t+m}{m} + d_1}{d_1} - 1.$$

Fixing two independent generic points on $V \cap H_\beta$, v_1 and v_2, choose coefficients of the generic differential hypersurface, relative to the condition that H_β contains v_1 and v_2. That is, $v_i \in H_\beta$ for $i = 1, 2$ imposes a linear condition on the coefficients β of the differential polynomial whose zero set is H_β; we choose the coefficients, β, generically in this linear subspace (over $K\langle v_1, v_2\rangle$). This gives a tuple (v_1, v_2, β) with Kolchin polynomial

$$2\omega_V(t) - 2\binom{t+m-h}{m} + n_1\binom{t+m}{m}.$$

If there is more than one type on $V \cap H_\beta$ with Kolchin polynomial $\omega_V(t) - \binom{t+m-h}{m}$, then there is more than one type on W with Kolchin polynomial $2\omega_V(t) - 2\binom{t+m-h}{m} + n_1\binom{t+m}{m}$.

In general, for any (possibly) non-generic choice of $v_1, v_2 \in V$, if $v_1 \neq v_2$, the Kolchin polynomial of (v_1, v_2, β) is bounded by $\omega_{v_1,v_2/K}(t) + (n_1)\binom{t+m}{m} - 2\binom{t+m-h}{m}$. To see this, note that any β such that (v_1, v_2, β) has the property that the coordinates β satisfy to linearly independent linear equations in $K\langle v_1, v_2\rangle$ which are of order h in v_1 and v_2, respectively.

Thus, the only way for

$$\omega_{v_1,v_2/K}(t) + n_1\binom{t+m}{m} - 2\binom{t+m-h}{m} = 2\omega_V(t) - 2\binom{t+m-h}{m} + n_1\binom{t+m}{m}$$

is to choose v_1, v_2 independent generics on V.

If $v_1 = v_2$ then noting that since $h > 0$ implies $\omega_V(t) - \binom{t+m-h}{m} > 0$, the Kolchin polynomial of (v_1, v_2, β) is bounded by

$$\omega_{v_1/K}(t) + n_1\binom{t+m}{m} - \binom{t+m-h}{m} < 2\omega_V(t) - 2\binom{t+m-h}{m} + n_1\binom{t+m}{m}.$$

Thus W has a unique type of maximal Kolchin polynomial. If $V \cap H_\beta$ has more than one component over $K\langle\beta\rangle$, then there are two distinct types on $V \cap H$ with Kolchin polynomial $\omega_V(t) - \binom{t+m-h}{m}$ and thus at least two types on W with Kolchin polynomial $2\omega_V(t) - 2\binom{t+m-h}{m} + n_1\binom{t+m}{m}$, a contradiction. □

Combining Theorems 4.5 and 4.6, we obtain our main Theorem 1.5 which we restate here:

Theorem. *Let V be a geometrically irreducible affine differential algebraic variety over a Δ-field K. Let H be a generic differential hypersurface over K. Assume that the order of H is greater than zero or that the Kolchin polynomial of V is greater than $\binom{t+m}{m}$. Then $V \cap H$ is a geometrically irreducible differential algebraic variety, which is nonempty just in case $dim(V) > 0$. In that case, $V \cap H$ has Kolchin polynomial:*

$$\omega_{V/K}(t) - \binom{t+m-h}{m}.$$

Remark 4.7. There are several notions of smoothness in the context of differential algebraic geometry, coming from the differential arc spaces considered in [14, 15] and from Kolchin's differential tangent spaces [11]. One can prove that generic intersections preserve any of these notions of smoothness (for more details, see [6]). For other notions of smoothness, see [9] and [20].

5. Generic differential hypersurfaces through a given point

In this section we prove a generalization of [21, Theorem 4.42]. The proof of the special case in [21] uses differential specializations. Our approach here is rather different, though a proof by suitably generalized methods of [21] is possible. Such an approach would avoid any machinery of stability theory (e.g. Lascar rank), but the use of this machinery allows for a quick proof.

Theorem 5.1. *Let V be a differential algebraic variety of dimension d. If the set of $d+1$ independent generic differential hypersurfaces of order h and degree d_1 through $\bar{a}$ intersects V, then $\bar{a} \in V$.*

Proof. We note that the fact that V is of dimension d implies that $\omega^m \cdot d \leq RU(V/K) < \omega^m \cdot (d+1)$ (see Proposition 2.7). We will prove the result by induction on d.

Let $\bar{a} \notin V$. First, we will argue the result in the case that $\bar{a} = (0, \dots, 0)$. Any hypersurface through the origin is of the the form $\sum c_i m_i(\bar{y}) = 0$, where the sum ranges over all monomials of order less than or equal to h and degree less than or equal to d_1. We assume that the c_i are independent differential transcendentals over K. We denote this differential hypersurface by $H_{\bar{c}}$. Suppose that $V \cap H_{\bar{c}} \neq \emptyset$ and $\bar{b}$ is a generic point on one of the irreducible components of $V \cap H_{\bar{c}}$ over $K\langle \bar{c} \rangle$. We may suppose that $\bar{b} \neq (0, \dots, 0)$, since the Theorem holds in this case.

Now since $\bar{b} \in H_{\bar{c}} \cap V$, we know that $\sum c_i m_i(\bar{b}) = 0$. Note that this is a nontrivial linear relation for $\bar{c}$ over $K\langle \bar{b} \rangle$. Since over K, $\bar{c}$ is a tuple of independent differential transcendentals, using Proposition 2.7,

$$RU(\bar{c}/K\langle \bar{b} \rangle) + \omega^m \leq RU(\bar{c}/K).$$

But, then by Lascar's symmetry lemma [16, chapter 19]

$$RU(\bar{b}/K\langle \bar{c} \rangle) + \omega^m \leq RU(\bar{b}/K). \tag{$\mathcal{A}$}$$

Thus, by Proposition 2.7, the differential transcendence degree of $\bar{b}/K\langle \bar{c} \rangle$ is at least one less than that of $\bar{b}/K$. In the case that $RU(V/K) < \omega^m$, the above argument using Lascar's symmetry lemma shows that $V \cap H_{\bar{c}} = \emptyset$. Note that by Proposition 2.7, if $d = 0$, then $RU(\bar{b}/K) < \omega^m$, and so equation $\mathcal{A}$ can not hold; thus our assumption that $V \cap H_{\bar{c}} \neq \emptyset$ must have been incorrect, establishing the case $d = 0$.

By Theorem 1.5 and Proposition 2.7,

$$\omega^m \cdot (d-1) \leq RU(V \cap H_{\bar{c}}/K\langle \bar{c} \rangle) < \omega^m \cdot d.$$

Now the result follows (again in the case $\bar{a}$ is the origin) by the induction hypothesis since $V \cap H_{\bar{c}}$ has dimension $d-1$.

Now, suppose that $\bar{a}$ is some point besides $(0, \dots, 0)$. If so, adjoin $\bar{a}$ to the field K and consider the V over $K\langle \bar{a} \rangle$. It is possible that V is not irreducible over $K\langle \bar{a} \rangle$; suppose that V has components $V_1, \dots, V_k$ over $K\langle \bar{a} \rangle$. By Proposition 4.1 each of the components has the same Kolchin polynomial as V. Fix some component V_i for $i = 1, \dots, k$. By the translation $\phi_{\bar{a}} : \mathbb{A}^n \to \mathbb{A}^n$, defined by $\bar{x} \mapsto \bar{x} - \bar{a}$, we replace V_i with $\phi_{\bar{a}}(V_i)$ and $\bar{a}$ with $(0, \dots, 0)$. Because the map $\phi_{\bar{a}}$ sends generic differential hypersurfaces over $K\langle \bar{a} \rangle$ to generic differential hypersurfaces over $K\langle \bar{a} \rangle$, in order to establish the result for the point $\bar{a}$ and variety V_i, it suffices to prove the result for the point $(0, \dots, 0)$ and the variety $\phi_{\bar{a}}(V_i)$. Our above argument applies to establish the result for the point $(0, \dots, 0)$ and the variety $\phi_{\bar{a}}(V_i)$. The component V_i was chosen arbitrarily, so we have established the result for the entire variety V. □

Notice that this result, in the case that $d = 0$, generalizes Lemma 3.3, which is itself a generalization of [17, Theorem 1.7]. We also note that the previous result and various results of this paper can also be seen to hold under the weaker hypothesis of quasi-genericity [21, page 4592 for the definition] of the differential hypersurfaces.

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