Positive scalar curvature on manifolds with boundary and their doubles

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Dedicated to Blaine Lawson on his 80th birthday, with appreciation and admiration

Abstract: This paper is about positive scalar curvature on a compact manifold X with non-empty boundary ∂X. In some cases, we completely answer the question of when X has a positive scalar curvature metric which is a product metric near ∂X, or when X has a positive scalar curvature metric with positive mean curvature on the boundary, and more generally, we study the relationship between boundary conditions on ∂X for positive scalar curvature metrics on X and the positive scalar curvature problem for the double $M = \mathrm{Dbl}(X, \partial X)$.

Keywords: Positive scalar curvature, mean curvature, surgery, bordism, K-theory, index.

1. Introduction

This paper is motivated by two important theorems of Blaine Lawson (one with Misha Gromov and one with Marie-Louise Michelsohn) relating curvature properties of a compact manifold with non-empty boundary to mean curvature of the boundary:

Theorem 1.1 ([14, Theorem 5.7])**.** *Let X be a compact manifold with boundary, of dimension n, and let $M = \mathrm{Dbl}(X, \partial X)$ denote the double of X along the boundary ∂X. If X admits a metric of positive scalar curvature with positive mean curvature $H > 0$ along the boundary ∂X (with respect to the outward-pointing normal), then M admits a metric of positive scalar curvature.*

Theorem 1.2 ([16, Theorem 1.1]). *If X is a compact manifold with non-empty boundary, of dimension $n \neq 4$, then X admits a Riemannian metric with positive sectional curvature and positive mean curvature along the boundary ∂X (with respect to the outward-pointing normal) if and only if $\pi_1(X, \partial X) = *$. Furthermore ([16, Theorem 1.2]), if X is parallelizable, then one can replace positive sectional curvature in this result by* **constant** *positive sectional curvature.*

Theorem 1.2, in turn, is a consequence of an intermediate result which we will also need:

Theorem 1.3 ([16, Theorem 3.1]). *If M is a (normally oriented) hypersurface of positive mean curvature in a Riemannian manifold Ω of dimension n, and if $M' \subset \Omega$ is obtained from M by attaching a p-handle to the positive side (the side of increasing area) of M, then if $n - p \geq 2$, one can arrange (without changing the metric on Ω) for M' also to have positive mean curvature, to be as close as one wants to M, and to agree with M away from a small neighborhood of the $S^{p-1} \hookrightarrow M$ where the handle is attached.*

Remark 1.4. One should note that there are two different (and conflicting!) sign conventions in use for mean curvature, so that the condition $H > 0$ along the boundary ∂X with respect to the outward-pointing normal in Theorems 1.1 and 1.2 is what is called in other papers (such as [8, 1]) $H > 0$ with respect to the *inward*-pointing normal. We will stick with the terminology in [14], which means that close hypersurfaces parallel to the boundary in the interior have *smaller* volume than the boundary, and that volume grows as one moves *outward*. This condition is sometime called "strictly mean convex boundary."

Other proofs of Theorem 1.1 can be found in [8, Theorem 1.1] and in [1, Corollary 34]. These two references make it clear that one can weaken the condition $H > 0$ to $H \geq 0$. However, as Christian Bär pointed out to us, the theorem fails if one replaces $H > 0$ by $H < 0$. He kindly provided us with the following simple counterexample. Let X be S^2 with three open disks removed, where each disk fits within a single hemisphere. (See Figure 1.) With the restriction of the standard metric on S^2, X has positive curvature (in fact $K = 1$) and the mean curvature of each boundary circle is *negative*, as parallel circles slightly inward from the boundary components have *bigger* length. But the double of X is a surface of genus 2, which cannot have nonnegative scalar curvature (by Gauss-Bonnet). The example can be jacked up to any higher dimension by crossing with a torus.
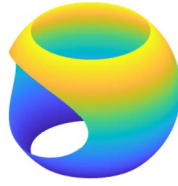
Figure 1: A 3-holed sphere.

Many years ago, we began studying whether there might be a sort of converse to Theorem 1.1. Recently, Christian Bär and Bernhard Hanke [1] have made a comprehensive study of boundary conditions (mostly involving mean curvature) for scalar curvature properties of compact manifolds with non-empty boundary, and this makes it possible to reexamine the question of a possible converse to Theorem 1.1, which we have formulated as Conjecture 7.1, the "Doubling Conjecture." That is the principal subject of this paper. Our main results on this question are Theorems 3.1, 3.3, 5.4, and 5.5.

Another closely related problem which we also study (in Section 4) is the question of when a compact manifold with boundary admits a positive scalar curvature metric which is a product metric in a neighborhood of the boundary. We show that in optimal situations (depending on the fundamental groups and whether or not things are spin and of high enough dimension) it is possible to give necessary and sufficient conditions for this to happen.

Since the proof of Theorem 1.1 in [14] is a bit sketchy and it seems some of the formulas there are not completely correct, we have redone the proof in Section 6.

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2. The relatively 1-connected case

Because of Theorem 1.2, the simplest case to deal with is the one where X and ∂X are connected and the inclusion $\partial X \hookrightarrow X$ induces an isomorphism on π_1. This case should philosophically be viewed as an analogue of Wall's "π-π Theorem" [27, Theorem 3.3], which says that relative surgery problems are unobstructed if X and ∂X are connected and the inclusion $\partial X \hookrightarrow X$ induces an isomorphism on π_1.

Theorem 2.1. *Let X be a connected compact spin manifold with boundary, of dimension $n \geq 6$, with connected boundary ∂X, and such that the inclusion $\partial X \hookrightarrow X$ induces an isomorphism on π_1. Let $M = \mathrm{Dbl}(X, \partial X)$ denote the*

double of X along its boundary, which is a closed spin manifold of dimension n. Then the following statements are always true:

(1) *M admits a metric of positive scalar curvature.*
(2) *∂X admits a metric of positive scalar curvature.*
(3) *X admits a positive scalar curvature metric which is a product metric in a collar neighborhood of ∂X.*
(4) *X admits a positive scalar curvature metric which gives ∂X positive mean curvature with respect to the interior normal.*
(5) *X admits a positive scalar curvature metric for which ∂X is minimal (i.e., has vanishing mean curvature).*
(6) *X admits a positive scalar curvature metric which gives ∂X positive mean curvature with respect to the outward normal.*

Proof. (6) is the conclusion of Theorem 1.2, and (1) then follows by Theorem 1.1. To prove (2), observe that if $\pi = \pi_1(\partial X)$, then the classifying map $c \colon \partial X \to B\pi$ extends to a classifying map $\bar{c}$ for X because of the fundamental group assumption, and so $(\partial X, c)$ bounds $(X, \bar{c})$ and the class of $c \colon \partial X \to B\pi$ in $\Omega_{n-1}^{\mathsf{spin}}(B\pi)$ vanishes. Then by the Bordism Theorem ([18, Proposition 2.3] or Theorems 4.1 and 4.11 in [21]), (2) holds. In fact, the proof of the Bordism Theorem also yields (3), because the bordism over $B\pi$ from S^{n-1} to ∂X obtained by punching out a disk from X can be decomposed into surgeries of codimension ≥ 3, and then the Surgery Theorem [13, Theorem A] makes it possible to "push" a standard positive scalar curvature metric on the n-disk that is a product metric near the boundary across the bordism to a positive scalar curvature metric on X that is a product metric near ∂X.

By the construction in Theorem 1.1, either in [14] or in Section 6, M has a positive scalar curvature metric which is symmetric with respect to reflection across ∂X. So this metric is what Bär and Hanke call a "doubling metric" in [1]. This metric necessarily has vanishing second fundamental form on ∂X, so (5) holds. (Alternatively, (5) trivially follows from (3).) By [1, Corollary 34], (4) holds as well. So we have shown that all the conditions hold. $\qquad\square$

If one keeps the condition that ∂X is connected and the condition that $\pi_1(\partial X) \to \pi_1(X)$ is surjective, but drops the condition that $\pi_1(\partial X) \to \pi_1(X)$ is injective, then the theorem has to be modified as follows.

Theorem 2.2. *Let X be a connected compact manifold with boundary, of dimension $n \geq 6$, with connected boundary ∂X, and such that the inclusion $\partial X \hookrightarrow X$ induces a surjection on π_1. Let $M = \mathrm{Dbl}(X, \partial X)$ denote the double of X along its boundary, which is a closed manifold of dimension n. Then the following statements are always true:*

(1) *M admits a metric of positive scalar curvature.*
(2) *X admits a positive scalar curvature metric which gives ∂X positive mean curvature with respect to the outward normal.*
(3) *X admits a positive scalar curvature metric for which ∂X is minimal (i.e., has vanishing mean curvature).*
(4) *X admits a positive scalar curvature metric for which ∂X is totally geodesic (i.e., has vanishing second fundamental form).*

Proof. Again, (2) follows from Theorem 1.2 and then (1) follows from Theorem 1.1. [1, Corollary 34] shows that (3) and (4) then follow. □

Note that under the hypotheses of Theorem 2.2, it is not necessary true that ∂X admits positive scalar curvature, and so in general X does not have a positive scalar curvature metric which is a product metric in a neighborhood of the boundary. A counterexample is given at the beginning of Section 5.

For the results of Theorem 2.1, the spin restriction is not really necessary, but the case where X is not spin but has a spin cover gets messy. For this reason, it's convenient to make the following definition.

Definition 2.3. If X is a manifold (with or without boundary), we say it is *totally non-spin* if the second Stiefel-Whitney class w_2 of X is non-zero on the image of the Hurewicz map $\pi_2(X) \to H_2(X, \mathbb{Z})$. This is equivalent to saying that the universal cover of X does not admit a spin structure.

The following result is an example of a modification to the totally non-spin case.

Theorem 2.4. *Let X be a connected compact oriented manifold with boundary, of dimension $n \geq 6$, with connected boundary ∂X, such that X and ∂X are totally non-spin, and such that the inclusion $\partial X \hookrightarrow X$ induces an isomorphism on π_1. Let $M = \mathrm{Dbl}(X, \partial X)$ denote the double of X along its boundary, which is a closed oriented manifold of dimension n. Then the following statements are always true:*

(1) *M admits a metric of positive scalar curvature.*
(2) *∂X admits a metric of positive scalar curvature.*
(3) *X admits a positive scalar curvature metric which is a product metric in a collar neighborhood of ∂X.*
(4) *X admits a positive scalar curvature metric which gives ∂X positive mean curvature with respect to the interior normal.*
(5) *X admits a positive scalar curvature metric for which ∂X is minimal (i.e., has vanishing mean curvature).*

(6) *X admits a positive scalar curvature metric which gives ∂X positive mean curvature with respect to the outward normal.*

Proof. This is exactly like the proof of Theorem 2.1, with spin bordism replaced by oriented bordism and with [18, Proposition 2.3] replaced by [18, Theorem 2.13]. □
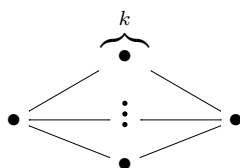
3. The simply connected case

Next, we consider the case where X and all the components of ∂X are simply connected, but there can be multiple boundary components.
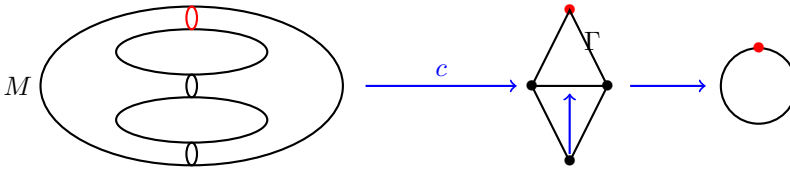
Theorem 3.1. *Let X be a simply connected compact spin manifold with non-empty boundary, with $n = \dim X \geq 6$. Note that ∂X can have any number k of boundary components, $\partial_1 X, \ldots, \partial_k X$. Suppose all components of ∂X are simply connected. Let $M = \mathrm{Dbl}(X, \partial X)$ denote the double of X along its boundary, which is a closed spin manifold of dimension n. Then the following are equivalent:*

(1) *M admits a metric of positive scalar curvature.*
(2) *All components $\partial_j X$ of ∂X admit metrics of positive scalar curvature.*
(3) *For each $j = 1, \ldots, k$, the α-invariant $\alpha(\partial_j X) \in ko_{n-1}$ vanishes.*
(4) *X admits a positive scalar curvature metric which is a product metric in a collar neighborhood of ∂X.*
(5) *X admits a positive scalar curvature metric for which ∂X is minimal (i.e., has vanishing mean curvature).*
(6) *X admits a positive scalar curvature metric which gives ∂X positive mean curvature with respect to the outward normal.*

Proof. If the number k of boundary components is 1, then the pair $(X, \partial X)$ is relatively 1-connected and all six conditions hold by Theorem 2.1. Thus for most of the proof we can restrict to the case $k \geq 2$. We begin by observing that since X and all components of ∂X are simply connected, Van Kampen's Theorem implies that the fundamental group of M is the same as for the

graph Γ given by   •——•——•   if $k = 1$,   if $k = 2$,

   for larger k, and is thus the free group F_{k-1} on $k-1$

Figure 2: Computing the α_Γ-invariant.

generators. This is a group for which [21, Theorem 4.13] applies, and thus M has a Riemannian metric of positive scalar curvature if and only if the obstruction class $\alpha_\Gamma(M)$ (the image of the *ko*-fundamental class of the spin manifold M under the classifying map $c\colon M \to BF_{k-1} \simeq \Gamma$) vanishes in $ko_n(\Gamma) \cong ko_n \oplus ko_{n-1}^{k-1}$. The component of $\alpha_\Gamma(M)$ in ko_n is just the ordinary α-invariant of M, i.e., the image of the fundamental class $[M] \in ko_n(M)$ under the "collapse" map $M \to \mathrm{pt}$. This always vanishes since M, being a double of a compact spin manifold with boundary, is always a spin boundary (the boundary of the manifold obtained by rounding the corners of $X \times I$). Hence condition (1) holds if and only if the component of $\alpha_\Gamma(M)$ in ko_{n-1}^{k-1} vanishes. It is easy to see that this is the same as the vanishing of all the α-invariants $\alpha(\partial_j X)$ for all the boundary components $\partial_1 X, \ldots, \partial_k X$. Indeed, the sum of these k α-invariants in ko_{n-1} is just $\alpha(\partial X) = 0$, since ∂X is a spin boundary and α is a spin bordism invariant. (In other words, the summand ko_{n-1}^{k-1} in $ko_n(\Gamma)$ is better described as the set of elements in ko_{n-1}^k that sum to 0.) And we can compute the piece of $\alpha_\Gamma(M)$ in one summand of ko_{n-1} by taking the α-invariant of the transverse inverse image of a point under the map $M \to S^1$ obtained by collapsing $k-2$ of the loops in $\Gamma \simeq \bigvee_{k-1} S^1$ and composing with the classifying map c as in Figure 2. Thus (1) is equivalent to (3). Applying Stolz's Theorem [23] and using the assumption that $n - 1 \geq 5$, we obtain the equivalence of (2) and (3), and thus (1), (2), and (3) are all equivalent.

The equivalence of (2) and (4) is related to Chernysh's Theorem ([7, Theorem 1.1] and [10, Theorem 1.1] — see also [28, Corollary D]). Let's explain this in more detail, since Chernysh proves that the map $\mathcal{R}^+(X, \partial X) \to \mathcal{R}^+(\partial X)$ is a Serre fibration,[1] but not that $\mathcal{R}^+(X, \partial X) \neq \emptyset$. (In fact, even in nice situations such as $X = D^8$ and $\partial X = S^7$, the map $\mathcal{R}^+(X, \partial X) \to \mathcal{R}^+(\partial X)$ is known not to be surjective.) Obviously (4) implies (2). If (2) holds, we need to show that for some choice of positive scalar curvature metrics on

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[1] Here $\mathcal{R}^+(\partial X)$ is the space of positive scalar curvature metrics on ∂X, and $\mathcal{R}^+(X, \partial X)$ is the space of positive scalar curvature metrics on X that restrict to a product metric on a collar neighborhood of the boundary.

$\partial_j X$, the metric on ∂X extends to a positive scalar curvature metric on X which is a product metric in a collar neighborhood of the boundary. For this we use the assumption that $k \geq 2$, and we fix metrics of positive scalar curvature on the boundary components $\partial_j X$, $1 \leq j \leq k - 1$. Then we can view X as a spin cobordism from $\partial_1 X \coprod \cdots \coprod \partial_{k-1} X$ to $-\partial_k X$. The Gromov-Lawson surgery theorem [13, Theorem A], in the variant found in [7, 10, 28], shows that for *some* spin cobordism W between these manifolds, obtained from X by doing surgeries on embedded 2-spheres with trivial normal bundles, away from the boundary, to make the pair $(W, \partial_k X)$ 2-connected (the pair $(X, \partial_k X)$ is already 1-connected by hypothesis), we can push the positive scalar curvature metric on $\partial_1 X \coprod \cdots \coprod \partial_{k-1} X$ across the cobordism W to get a positive scalar curvature metric on W restricting to product metrics on collar neighborhoods of the boundary components. If $W = X$, we have shown that $(2) \Rightarrow (4)$. In general X and W will not be the same, but we can go back from W to X by doing surgery on embedded $(n-3)$-spheres away from the boundary, so we can use the surgery theorem again to carry the metric of positive scalar curvature over from W to X.

Now we need to check equivalence of the conditions (1)–(4) with (5) and (6). But by [1, Theorem 33], existence of a positive scalar curvature metric on X with $H \geq 0$ (H denotes the mean curvature of ∂X with respect to the outward normal) is equivalent to existence of a positive scalar curvature metric with $H > 0$. Thus (5) implies (6). It is also trivial that $(4) \Rightarrow (5)$, so (1)–(4) implies (5) and (6). Theorem 1.1 shows that (6) implies (1) and thus all the other conditions. Thus all six conditions are equivalent. □

Remark 3.2. If the boundary of X is empty, then $\mathrm{Dbl}(X, \partial X) = X \coprod -X$, which clearly has a metric of positive scalar curvature if and only if X does. So in this case (2) and (3) always hold, (4)–(6) amount to saying that X has a metric of positive scalar curvature, and these conditions are equivalent to (1).

Once again, one can easily modify Theorem 3.1 to the non-spin case as follows:

Theorem 3.3. *Let X be a simply connected compact manifold with non-empty boundary, with $n = \dim X \geq 6$. Note that ∂X can have any number k of boundary components, $\partial_1 X, \ldots, \partial_k X$. Suppose all components of ∂X are simply connected and that **none** of X and the $\partial_j X$ admit a spin structure. Let $M = \mathrm{Dbl}(X, \partial X)$ denote the double of X along its boundary, which is a closed oriented manifold of dimension n. Then the following are true:*

(1) *M admits a metric of positive scalar curvature.*
(2) *All components $\partial_j X$ of ∂X admit metrics of positive scalar curvature.*

(3) *X admits a positive scalar curvature metric which is a product metric in a collar neighborhood of ∂X.*

(4) *X admits a positive scalar curvature metric for which ∂X is minimal (i.e., has vanishing mean curvature).*

(5) *X admits a positive scalar curvature metric which gives ∂X positive mean curvature with respect to the outward normal.*

Proof. Assertion (2) follows immediately from [13, Corollary C]. As in the proof of Theorem 3.1, $\pi_1(M) \cong F_{k-1}$, a free group. When $k = 1$, we can apply Theorem 2.4. So we can assume $k \geq 2$. To prove (1), we can apply [21, Theorem 4.11] or [12, Theorem 1.2], which says that it suffices to show that there is a manifold with positive scalar curvature representing the same class as M in $H_n(BF_{k-1}, \mathbb{Z}) = H_n(\bigvee_1^{k-1} S^1, \mathbb{Z}) = 0$ (since $n > 1$). So this is automatic. The proof of the remaining conditions is exactly the same as for Theorem 3.1. □

4. Obstruction theory for the relative problem

In this section we extend some of the results of Theorem 2.1 and Theorem 3.1 to get a general obstruction theory and a conjectured answer for the following problem:

Question 4.1. Suppose that X is a connected compact spin manifold of dimension n with non-empty boundary ∂X (which could be disconnected). Assume that ∂X admits a metric of positive scalar curvature. (When the fundamental groups of the components $\partial_1 X, \ldots, \partial_k X$ of ∂X are nice enough and $n \geq 6$, the Gromov-Lawson-Rosenberg Conjecture holds and one has a necessary and sufficient condition for this in terms of KO-theoretic "α-invariants" $\alpha_{\pi_1(\partial_j X)}(\partial_j X)$. See Theorem 4.2 below.) When does X admit a metric of positive scalar curvature which is a product metric in a collar neighborhood of ∂X?

To explain our approach to this, we first establish some notation. The fundamental groupoid Λ of ∂X is equivalent to $\coprod_j \Lambda_j$, the disjoint union of the groups $\Lambda_j = \pi_1(\partial_j X)$, where $\partial_1 X, \ldots, \partial_k X$ are the components of ∂X and we pick a basepoint in each component. The classifying space $B\Lambda$ is homotopy equivalent to $\coprod_j B\Lambda_j$. There is a classifying map $c^\Lambda \colon \partial X \to B\Lambda$, unique up to homotopy equivalence, which is an isomorphism on fundamental groups on each component. The spin structure of X determines spin structures on each $\partial_j X$ and thus *ko*-fundamental classes $[\partial_j X] \in ko_{n-1}(\partial_j X)$. By [21, Theorem 4.11] (or if you prefer, [12, Theorem 1.2]), the question of whether

or not $\partial_j X$ admits a metric of positive scalar curvature only depends on $c_*^\Lambda([\partial_j X]) \in ko_{n-1}(B\Lambda_j)$, and so the question of whether or not ∂X admits a metric of positive scalar curvature only depends on $c_*^\Lambda([\partial X]) \in ko_{n-1}(B\Lambda)$. Furthermore, the image of $c_*^\Lambda([\partial X])$ under "periodization" (inverting the Bott element) per: $ko_{n-1}(B\Lambda) \to KO_{n-1}(B\Lambda)$, followed by the KO-assembly map $A \colon KO_{n-1}(B\Lambda) \to KO_{n-1}(C_{r,\mathbb{R}}^*(\Lambda))$ (or one could use the full C^*-algebra here), is an obstruction to positive scalar curvature on ∂X. Thus if for each j, periodization and assembly are injective for Λ_j, we obtain the *Gromov-Lawson-Rosenberg Conjecture*:

Theorem 4.2 ([21, Theorem 4.13])**.** *Suppose* $(X, \partial X)$ *is a connected compact spin manifold of dimension n with non-empty boundary* ∂X. *With the above notation, if* ∂X *admits a metric of positive scalar curvature, then* $A \circ \mathrm{per}(c_*^\Lambda([\partial X])) = 0$ *in* $KO_{n-1}(C_{r,\mathbb{R}}^*(\Lambda))$. *If* $n \geq 6$ *and if* A *and* per *are injective for* Λ, *then vanishing of* $c_*^\Lambda([\partial X]) \in ko_{n-1}(B\Lambda)$ *is necessary and sufficient for* ∂X *to admit a metric of positive scalar curvature.*

Now in the situation of Theorem 4.2, one also has a ko-fundamental class $[X, \partial X] \in ko_n(X, \partial X)$, which maps under the boundary map of the long exact sequence of the pair $(X, \partial X)$ to $[\partial X] \in ko_{n-1}(\partial X)$. Let $\Gamma = \pi(X)$ and $\Lambda = \pi(\partial X)$ as above. (Note that we are using the fundamental groupoids to avoid having to make changes of basepoint. But there is a natural map of topological groupoids $\Lambda \to \Gamma$.) We have the following commuting diagram of ko-groups, coming from the long exact sequences of the pairs $(X, \partial X)$ and $(B\Gamma, B\Lambda)$:

(1)
$$[X, \partial X] \xmapsto{\quad \partial \quad} [\partial X]$$

$$
\begin{array}{ccccc}
& & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
ko_n(X) \longrightarrow & ko_n(X, \partial X) & \xrightarrow{\ \partial\ } & ko_{n-1}(\partial X) \\
\Big\downarrow{\scriptstyle c_*^\Gamma} & \Big\downarrow{\scriptstyle c_*^{\Gamma,\Lambda}} & & \Big\downarrow{\scriptstyle c_*^\Lambda} \\
ko_n(B\Gamma) \longrightarrow & ko_n(B\Gamma, B\Lambda) & \xrightarrow{\ \partial\ } & ko_{n-1}(B\Lambda).
\end{array}
$$

One of course gets similar diagrams with ko replaced by H (ordinary homology), Ω^{spin}, and KO (periodic K-homology). If ∂X admits a metric of positive scalar curvature, then Question 4.1 is meaningful. Here is our first major result on Question 4.1.

Theorem 4.3. *Suppose* $(X, \partial X)$ *is a connected compact spin manifold of dimension* $n \geq 6$ *with non-empty boundary* ∂X. *Let* $\Gamma = \pi(X)$ *and* $\Lambda = \pi(\partial X)$
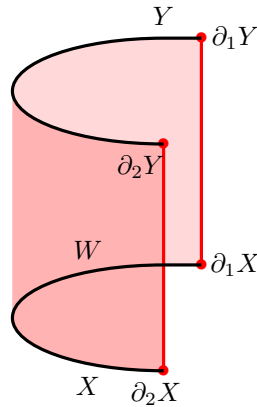
Figure 3: The spin bordism W from $(Y, \partial Y)$ to $(X, \partial X)$.

be the fundamental groupoids. As explained above, note that $(X, \partial X)$ defines a class $c_*^{\Gamma,\Lambda}([X, \partial X]) \in \Omega_n^{\mathsf{spin}}(B\Gamma, B\Lambda)$. Suppose there is another compact spin manifold $(Y, \partial Y)$ of the same dimension, defining the same class in $\Omega_n^{\mathsf{spin}}(B\Gamma, B\Lambda)$. If Y admits a metric of positive scalar curvature which is a product metric in a neighborhood of the boundary, then so does X.

Remark 4.4. Note that Theorem 4.3 includes the assertion that if ∂X is connected and $\Gamma = \Lambda$, then since in this case the relative groups for $(B\Gamma, B\Lambda)$ vanish for any homology theory, we conclude that X always has a metric of positive scalar curvature which is a product metric in a neighborhood of the boundary. We already know this from Theorem 2.1.

Proof. Note that the condition of the theorem implies allows for Y to be disconnected or to have more boundary components than X. But the condition implies that there is a spin manifold W with corners, of dimension $n + 1$, giving a spin bordism from $(Y, \partial Y)$ to $(X, \partial X)$, which restricts on the boundary to a spin bordism of closed manifolds from ∂Y to ∂X. Furthermore, this bordism is "over" $B\Lambda$ on the boundary and $B\Gamma$ on the interior. We can number the boundary components of X as $\partial_1 X, \dots, \partial_k X$, so that $\partial Y = \coprod_j \partial_j Y$ and W gives a spin bordism from $\partial_j Y$ to $\partial_j X$ over $B\Lambda_j$. See Figure 3, which illustrates the case $k = 2$.

Now we proceed as in the proof of the Gromov-Lawson bordism theorem [13, Theorem B], or more exactly of the generalization of this to the non-simply connected case [21, Theorem 4.2]. Start with a metric of positive scalar curvature on Y that is a product metric in a neighborhood of the boundary. We can do surgeries by attaching handles of dimensions 1 and 2 to adjust ∂Y

so that it has the same number of components as ∂X and so that $\partial_j Y$ has the same fundamental group Λ_j as $\partial_j X$. These surgeries are in codimension 3 or more since $\dim \partial_j Y \geq 5$, so we can do this preserving positive scalar curvature on the successive transformations of $\partial_j Y$. We always extend the metric to a collar neighborhood in W so as to be a product metric near this component of the boundary. Similarly, we can then do surgeries on the interior of Y and then on the interior of W so that these have the same fundamental group Γ as X. Then the bordism from Y to X can be decomposed into a sequence of surgeries (over $B\Lambda$ on the bordism from ∂Y to ∂X and over $B\Gamma$ in the interior) in codimension 3 or more so that we can carry the metric across the bordism, preserving the positive scalar curvature conditions. The spin condition is used to know that whenever we want to do surgery on an embedded 1-sphere or 2-sphere, it has trivial normal bundle and thus the surgery is possible. □

The work of Stolz and Jung leading to [21, Theorem 4.11] can be used to give a substantial improvement to Theorem 4.3. (Note that Jung's work was never published, but that his results were reproved by Führing in [12].)

Theorem 4.5. *Suppose* $(X, \partial X)$ *is a connected compact spin manifold of dimension* $n \geq 6$ *with non-empty boundary* ∂X. *Let* $\Gamma = \pi(X)$ *and* $\Lambda = \pi(\partial X)$ (*and remember that* Λ *can have multiple connected components*). *As explained above, note that* $(X, \partial X)$ *defines a class* $c_*^{\Gamma, \Lambda}([X, \partial X]) \in ko_n(B\Gamma, B\Lambda)$. *If this class vanishes in this relative ko-homology group, then* X *admits a metric of positive scalar curvature which is a product metric in a neighborhood of the boundary.*

Proof. First of all, note that by Theorem 4.3, there is a subgroup $\Omega_n^{\mathsf{spin},+}(B\Gamma, B\Lambda)$ of $\Omega_n^{\mathsf{spin}}(B\Gamma, B\Lambda)$ such that X admits a metric of positive scalar curvature which is a product metric in a neighborhood of the boundary if and only if $c_*^{\Gamma, \Lambda}([X, \partial X])$ lies in this subgroup. (This sub*set* is a sub*group* since the condition is clearly stable under disjoint union and orientation reversal, which represent addition and inversion in the spin bordism group.) So we just need to show that the kernel of $\alpha_{(\Gamma, \Lambda)} \colon \Omega_n^{\mathsf{spin}}(B\Gamma, B\Lambda) \to ko_n(B\Gamma, B\Lambda)$ lies in $\Omega_n^{\mathsf{spin},+}(B\Gamma, B\Lambda)$. In addition, we can localize and check this separately after localizing at 2 and after inverting 2. After localizing at 2, we can invoke [24, Theorem B], which says that after localizing at 2, the kernel of $\Omega_n^{\mathsf{spin}}(Z, W) \to ko_n(Z, W)$ is, for any Z and W, generated by the image of a transfer map associated to $\mathbb{HP}^2$-bundles. What this means geometrically is that there is a generating set for the kernel consisting of pairs $(X, \partial X)$ (mapping to (Z, W)) which can be taken (up to bordism) to be fiber bundles

$\mathbb{HP}^2 \to (X, \partial X) \to (Y, \partial Y)$, with structure group the isometry group of $\mathbb{HP}^2$, where $\dim Y = n - 8$. If X is of this form, just give Y a metric which is a product metric near the boundary and lift it to a metric on X which on the fibers is the usual metric on $\mathbb{HP}^2$, but rescaled to have very large curvature. This will have positive scalar curvature and still be a product metric near the boundary.

Now we just have to study the kernel of α after inverting 2. For this we can use [12, Corollary 3.2], which says that for any space Z, the kernel of $\alpha \colon \Omega_*^{\mathsf{spin}}(Z)[\frac{1}{2}] \to ko_*(Z)[\frac{1}{2}]$ is generated by manifolds which carry a positive scalar curvature metric. Almost exactly the same argument works in the relative case. $\square$

Incidentally, there is an exact counterpart to Theorems 4.3 and 4.5 in the totally non-spin case (Definition 2.3).

Theorem 4.6. *Suppose $(X, \partial X)$ is a connected compact oriented manifold of dimension $n \geq 6$ with non-empty boundary ∂X. Suppose that X and each component of ∂X is totally non-spin. Let $\Gamma = \pi(X)$ and $\Lambda = \pi(\partial X)$. Suppose there is another compact oriented manifold $(Y, \partial Y)$ of the same dimension, defining the same class in $\Omega_n(B\Gamma, B\Lambda)$. If Y admits a metric of positive scalar curvature which is a product metric in a neighborhood of the boundary, then so does X.*

Proof. This is exactly like the proof of Theorem 4.3 except for a few points regarding the totally non-spin condition. Start with an oriented bordism W from Y to X, as in Figure 3 (except that it can't be spin since X is not). As in the proof of Theorem 4.3, start by doing surgeries on the boundary so that ∂Y has the same number of components as ∂X, and number them so that ∂W gives a bordism from $\partial_j Y$ to $\partial_j X$ over Λ_j and so that the appropriate piece $\partial_j W$ of ∂W has fundamental group Λ_j. Recall that w_2 restricted to the image of the Hurewicz map gives the obstruction to triviality of the normal bundle for embedded 2-spheres. Since $w_2(\partial_j X)$ is non-zero on the image of the Hurewicz map, we consider the commutative diagram

$$\pi_2(\partial_j X) \xrightarrow{\hspace{3cm}} \pi_2(\partial_j W)$$
$$\searrow{\scriptstyle w_2} \qquad \swarrow{\scriptstyle w_2}$$
$$\mathbb{Z}/2$$

(where the horizontal map on the top is induced by the inclusion $\partial_j X \hookrightarrow \partial_j W$) and do surgeries on 2-spheres (which have to have trivial normal bundles)

representing generators of the kernel of w_2 on $\pi_2(\partial_j W)$, in order to make $(\partial_j W, \partial_j X)$ 2-connected. Similarly, we can make (W, X) 2-connected by the same argument. The rest of the proof is as before. $\square$

Theorem 4.7. *Suppose $(X, \partial X)$ is a connected compact oriented manifold of dimension $n \geq 6$ with non-empty boundary ∂X. Assume that X and each component of ∂X is totally non-spin. Let $\Gamma = \pi(X)$ and $\Lambda = \pi(\partial X)$ and note that $(X, \partial X)$ defines a class $c_*^{\Gamma,\Lambda}([X, \partial X]) \in H_n(B\Gamma, B\Lambda; \mathbb{Z})$. If this class vanishes in this relative homology group, then X admits a metric of positive scalar curvature which is a product metric in a neighborhood of the boundary.*

Proof. By Theorem 4.6, there is a subgroup $\Omega_n^+(B\Gamma, B\Lambda)$ such that X admits a metric of positive scalar curvature which is a product metric in a neighborhood of the boundary if and only if the bordism class $c_*^{\Gamma,\Lambda}([X, \partial X])$ lies in this subgroup. (Again, this is a subgroup and not just a subset, for the same reasons as in the proof of Theorem 4.5.) We just need to show that $\Omega_n^+(B\Gamma, B\Lambda)$ contains the kernel of $\Omega_n(B\Gamma, B\Lambda) \to H_n(B\Gamma, B\Lambda; \mathbb{Z})$. As in the spin case, it suffices to do the calculation separately first after localizing at 2 and then after inverting 2. If we localize at 2, then it is known[2] that MSO becomes an Eilenberg-MacLane spectrum, or in other words, the Atiyah-Hirzebruch spectral sequence $H_p(Z, Y; \Omega_q) \Rightarrow \Omega_{p+q}(Z, Y)$ always collapses after localizing at 2, and $\Omega_n(Z, Y)_{(2)} = \bigoplus_{p+q=n} H_p(Z, Y; \Omega_{(2),q})$ for any Z and Y. Apply this with $Z = B\Gamma$ and $Y = B\Lambda$. Since Gromov and Lawson showed [13] that Ω_* has a set of additive generators which are closed oriented manifolds of positive scalar curvature, the result localized at 2 clearly follows. The result after inverting 2 follows from [12], just as in the spin case. $\square$

Now we can give a complete answer to Question 4.1 when the relevant fundamental groups are nice enough. First we need to discuss an obstruction to a positive answer to Question 4.1. This involves C^*-algebraic K-theory. We always work over $\mathbb{R}$ instead of over $\mathbb{C}$, though this makes very little difference in the formal structure of the argument. C^*-algebras of fundamental *groupoids* may be unfamiliar to many readers, but note from the theory of groupoid C^*-algebras [17] that for X a nice path-connected locally compact space such as a connected manifold, $C^*(\pi(X))$ is strongly Morita equivalent to the group C^*-algebra $C^*(\pi_1(X, x_0))$, for any choice of a basepoint x_0 in X. In fact $C^*(\pi(X)) \cong C^*(\pi_1(X, x_0)) \otimes \mathcal{K}$, where $\mathcal{K}$ is the algebra of compact operators.

The following result is really due to Chang, Weinberger and Yu [6, Theorem 2.18] and to Schick and Seyedhosseini [22, Theorem 5.2]; the following is just a slight repackaging of their results in our current language.

---

[2]This is really due to Wall [26], but one can find it more explicitly in [25].

Theorem 4.8 (Obstruction to PSC with Product Structure on the Boundary). *Suppose $(X, \partial X)$ is a connected compact spin manifold of dimension n with non-empty boundary ∂X. Let $\Gamma = \pi(X)$ and $\Lambda = \pi(\partial X)$. Suppose that Γ and Λ both satisfy injectivity of the KO-assembly map $A_\Gamma\colon KO_*(B\Gamma) \to KO_*(C^*(\Gamma))$ and injectivity of the periodization map $\mathrm{per}_\Gamma\colon ko_*(B\Gamma) \to KO_*(B\Gamma)$ (and similarly for Λ). Also assume injectivity of the periodization map $\mathrm{per}_{\Gamma,\Lambda}\colon ko_n(B\Gamma, B\Lambda) \to KO_n(B\Gamma, B\Lambda)$. Then vanishing of the class $c_*^{\Gamma,\Lambda}([X, \partial X]) \in ko_n(B\Gamma, B\Lambda)$ is necessary for X to admit a Riemannian metric of positive scalar curvature which is a product metric in a neighborhood of the boundary.*

Proof. Note that by diagram (1), $c_*^{\Gamma,\Lambda}([X, \partial X])$ maps to $c_*^\Lambda([\partial X]) \in ko_{n-1}(B\Lambda)$, which since A_Λ and per_Λ are injective, must vanish by Theorem 4.2 for ∂X to admit a positive scalar curvature metric. Thus, at a minimum, we know that $c_*^{\Gamma,\Lambda}([X, \partial X])$ must be in the kernel of the boundary map, and thus must be the image of a class in $ko_n(B\Gamma)$. The rest of the proof can really be found in [6] and [22], but we'll just restate the argument in slightly different form. Suppose that X has been given a Riemannian metric g_X which is a positive scalar curvature product metric $dt^2 + g_{\partial X}$ on a neighborhood of the boundary. We can attach a metric cylinder $\partial X \times [0, \infty)$ to X along ∂X, using the metric $dt^2 + g_{\partial X}$ on the cylindrical end. The noncompact manifold $\widehat{X} = X \cup_{\partial X} \partial X \times [0, \infty)$, equipped with the complete metric $g_{\widehat{X}}$ obtained by patching together g_X and the product metric $dt^2 + g_{\partial X}$ on the cylindrical end, has uniformly positive scalar curvature off a compact set. Therefore $\slashed{\partial}_{\widehat{X}}$ is invertible off a compact set and thus Fredholm. Since $\widehat{X}$ is diffeomorphic to $\mathring{X} = X \setminus \partial X$ (it is just $\mathring{X}$ with a collar attached to the boundary), $\slashed{\partial}_{\widehat{X}}$ together with the multiplication action of $C_0^{\mathbb{R}}(\widehat{X})$ define a class in $KO_n(X, \partial X) \cong KKO(C_0^{\mathbb{R}}(\widehat{X}), \mathcal{C}\ell_n)$, where $\mathcal{C}\ell_n$ is the real Clifford algebra acting on the spinor bundle, and this class is an analytic representative for the class $\mathrm{per}_{X,\partial X}[X, \partial X] \in KO_n(X, \partial X)$. For future use, note that the inclusion $\widehat{X} \hookrightarrow X \cup_{\partial X} \partial X \times [0, \infty] \simeq X$ (where the final $\simeq$ denotes "is homotopy equivalent to") gives us a class $\widehat{c}_X \in KKO(C_0^{\mathbb{R}}(\widehat{X}), C(B\Gamma))$. (If there is no finite model for $B\Gamma$, this is interpreted as $\varinjlim KKO(C_0^{\mathbb{R}}(\widehat{X}), C(Z))$, as Z runs over finite subcomplexes of $B\Gamma$.)

Now assume in addition that g_X has positive scalar curvature everywhere, and consider the Mishchenko-Fomenko index problem for $\slashed{\partial}_{\widehat{X}}$ with coefficients in the Mishchenko-Fomenko flat $C^*(\Gamma)$-bundle over $\widehat{X}$. Positivity of the scalar curvature and flatness of the bundle guarantee that the operator has vanishing index in the sense of $C^*(\Gamma)$-linear elliptic operators. By the Mishchenko-Fomenko Index Theorem, this C^*-index is the image of $[\slashed{\partial}_{\widehat{X}}] \in KO_n(X, \partial X)$

under $\widehat{c}_X \colon KO_*(X, \partial X) \to KO_*(B\Gamma)$, followed by the assembly map A_Γ (see for example [20]). Since the assembly map is injective, we have $\widehat{c}_X([\bar{\partial}_{\widehat{X}}]) = 0$ in $KO_n(B\Gamma)$. Recall that we are trying to show that $c_*^{\Gamma,\Lambda}([X, \partial X]) = 0$ in $ko_n(B\Gamma, B\Lambda)$. Since we are assuming $\mathrm{per}_{\Gamma,\Lambda}$ is injective, it suffices to show that $\mathrm{per}_{\Gamma,\Lambda} \circ c_*^{\Gamma,\Lambda}([X, \partial X]) = 0$ in $KO_n(B\Gamma, B\Lambda)$. Now chase the commutative diagram:

$$(2)$$

The class whose vanishing we are trying to show lies in the group in the lower right, and is the same as $c_*^{\Gamma,\Lambda}([\bar{\partial}_X])$, where $[\bar{\partial}_X] \in KO_n(X, \partial X)$. Since this is the image of $\widehat{c}_X([\bar{\partial}_X]) = 0$ in $KO_n(B\Gamma)$, this class vanishes, as desired. $\qquad\square$

Remark 4.9. It might be useful to remark that the injectivity assumptions in Theorem 4.8 on the periodization and assembly maps don't necessarily have to hold in all degrees, just in the degrees where the relevant indices appear (n for X and $n - 1$ for ∂X).

One can also repackage the argument in the proof using not just group or groupoid C^*-algebras but also certain relative C^*-algebras. A complication in doing that is that the reduced group C^*-algebra C_r^* is functorial for injective group homomorphisms, but not for surjective group homomorphisms. Indeed, for some discrete groups G, $C_r^*(G)$ is known to be simple (see [15] for a characterization of when this happens), hence there cannot be a morphism $C_r^*(G) \to \mathbb{R}$ corresponding to the map of groups $G \twoheadrightarrow \{1\}$. We can get around this problem by defining $C^*(\Gamma, \Lambda)$ to be the mapping cone (see [19, §1] of the map on full C^*-algebras $C^*(\Lambda) = \bigoplus_j C^*(\Lambda_j) \to C^*(\Gamma)$ induced by the morphism of groupoids $\Lambda \to \Gamma$ coming from the inclusion of ∂X into X. Note that we are using the *maximal* groupoid C^*-algebra here, and are working with *real* C^*-algebras throughout. Thus there is a canonical short

exact sequence

$$0 \to C_0(\mathbb{R}) \otimes C^*(\Gamma) \to C^*(\Gamma, \Lambda) \to C^*(\Lambda) \to 0,$$

making the KO-spectrum of $C^*(\Gamma, \Lambda)$ into the homotopy fiber of the map of KO-spectra associated to $C^*(\Lambda) \to C^*(\Gamma)$. Then we can think of the idea of the proof as dealing with vanishing of a "relative index" in $KO_n(C^*(\Gamma, \Lambda))$. Note that injectivity of the KO-assembly map for the maximal C^*-algebra follows from injectivity of the KO-assembly map for the reduced C^*-algebra, and thus is automatic for torsion-free groups satisfying the Baum-Connes Conjecture. Injectivity of the KO-assembly map for the relative C^*-algebra is a more obscure condition, but it holds by diagram chasing if Γ and the Λ_j are K-amenable and the assembly maps for each of them are isomorphisms (for example if they are free, free abelian, or surface groups).

Putting Theorems 4.5 and 4.8 together, we obtain:

Corollary 4.10. *Suppose $(X, \partial X)$ is a connected compact spin manifold of dimension $n \geq 6$ with non-empty boundary ∂X. Let $\Gamma = \pi(X)$ and $\Lambda = \pi(\partial X)$. Suppose that Γ, the Λ_j, and the pair (Γ, Λ) satisfy both injectivity of the KO-assembly map $A_\Gamma \colon KO_*(B\Gamma) \to KO_*(C^*(\Gamma))$ and injectivity of the periodization map $\mathrm{per}_\Gamma \colon ko_*(B\Gamma) \to KO_*(B\Gamma)$ (and similarly for the Λ_j and the periodization map for the pair). Then vanishing of the class $c_*([X, \partial X]) \in ko_n(B\Gamma, B\Lambda)$ is necessary and sufficient for X to admit a Riemannian metric of positive scalar curvature which is a product metric in a neighborhood of the boundary.*

Proof. This is simply the amalgamation of Theorems 4.5 and 4.8.   □

Remark 4.11. One might wonder how generally the hypotheses of Theorem 4.8 are valid. The injectivity of the periodization map usually has to be checked by an *ad hoc* comparison of the Atiyah-Hirzebruch spectral sequences for ko_* and KO_*. It usually fails in general for groups with torsion, but sometimes it may hold in certain special dimensions.

For the hypothesis about injectivity of the assembly map, more general techniques often apply for the groupoids Γ and Λ_j, and then for the relative groups one can apply the following simple lemma.

Lemma 4.12. *Suppose one has compatible splitting maps for A_Γ and A_Λ, i.e., there are splitting maps $s_\Lambda \colon KO_*(C^*(\Lambda)) \to KO_*(B\Lambda)$ for A_Λ and*

$s_\Gamma \colon KO_*(C^*(\Gamma)) \to KO_*(B\Gamma)$ *for* A_Γ *such that*

$$
\begin{array}{ccc}
KO_*(B\Lambda) & \longrightarrow & KO_*(B\Gamma) \\
\big\uparrow{\scriptstyle s_\Lambda} & & \big\uparrow{\scriptstyle s_\Gamma} \\
KO_*(C^*(\Lambda)) & \longrightarrow & KO_*(C^*(\Gamma))
\end{array}
$$

commutes. Then the relative assembly map $A_{(\Gamma,\Lambda)}$ *is injective.*

Proof. Chase the diagram

$$
\begin{array}{ccccccccc}
KO_n(B\Lambda) & \longrightarrow & KO_n(B\Gamma) & \longrightarrow & KO_n(B\Gamma, B\Lambda) & \xrightarrow{\partial} & KO_{n-1}(B\Lambda) & \longrightarrow & \cdots \\
{\scriptstyle s_\Lambda}\big\updownarrow\big\updownarrow{\scriptstyle A_\Lambda} & & {\scriptstyle s_\Gamma}\big\updownarrow\big\updownarrow{\scriptstyle A_\Gamma} & & \big\downarrow{\scriptstyle A_{\Gamma,\Lambda}} & & {\scriptstyle s_\Lambda}\big\updownarrow\big\updownarrow{\scriptstyle A_\Lambda} & & \\
KO_n(C^*(\Lambda)) & \rightarrowtail & KO_n(C^*(\Gamma)) & \rightarrowtail & KO_n(C^*(\Gamma,\Lambda)) & \xrightarrow{\partial} & KO_{n-1}(C^*(\Lambda)) & \rightarrowtail & \cdots .
\end{array}
$$

If $x \in KO_n(B\Gamma, B\Lambda)$ goes to 0 under $A_{\Gamma,\Lambda}$, then by commutativity of the right square and injectivity of A_Λ, it maps under ∂ to 0 in $KO_{n-1}(B\Lambda)$, and hence comes from a class $y \in KO_n(B\Gamma)$. But $y = s_\Gamma \circ A_\Gamma(y)$, and $A_\Gamma(y)$ maps to 0, so $A_\Gamma(y)$ is the image of some $z \in KO_n(C^*(\Lambda))$. By commutativity of the diagram in the statement of the Lemma, y is the image of $s_\Lambda(z)$, and thus x, the image of y, must vanish. $\qquad\square$

One can apply Lemma 4.12 in the following context. Suppose all the fundamental groups belong to a class of groups for which one can prove split injectivity of the KO-theory assembly map in a functorial way. There are many such classes, using "dual Dirac" methods or embeddings into Hilbert spaces. One may also have to assume injectivity of the map $\Lambda \to \Gamma$ (at least if one just assumes that Γ is coarsely embeddable into a Hilbert space). Then the hypothesis of Lemma 4.12 holds and one gets injectivity of the relative assembly map as well.

5. More complicated cases with non-trivial fundamental groups

Now we move on to the more complicated situation, where ∂X is not necessarily connected, or ∂X is connected but the inclusion $\partial X \hookrightarrow X$ does not induce an isomorphism on fundamental groups. In the latter case, the conclusions of Theorems 2.1, 3.1, and 4.5 have to fail in some situations, as one can see from the following simple example (mentioned also in [1]). Let $X = T^{n-2} \times D^2$, which has boundary the $(n-1)$-torus $\partial X = T^{n-1}$. The

double of X is $M = \mathrm{Dbl}(X, \partial X) = T^{n-2} \times S^2$, which obviously has a metric of positive scalar curvature (because of the S^2 factor). On the other hand, ∂X cannot have a positive scalar curvature metric. It is also clear (since one can give T^{n-2} a flat metric and give D^2 the metric corresponding to a spherical cap around the north pole of a standard 2-sphere, either close to the pole or extending beyond the equator) that one can arrange for X to have a positive scalar curvature metric so that the mean curvature H of ∂X is either strictly positive or strictly negative. So for this example, of the six conditions in Theorem 3.1, (1), (5), and (6) hold, and (2) and (4) fail. ((3) holds but does not say much, since the α-invariant gives insufficient information in the non-simply connected case.)

Definition 5.1. Extending standard terminology from the case of 3-manifolds, we say that X has *incompressible boundary* if for each component $\partial_j X$ of ∂X, the map $\pi_1(\partial_j X) \to \pi_1(X)$ induced by the inclusion is injective.

The incompressible case is the easiest case and also the one in which we get the strongest conclusion.

Theorem 5.2. *Let X be a connected compact spin manifold of dimension $n \geq 6$ with non-empty incompressible boundary ∂X. X can have any number k of boundary components, $\partial_1 X, \ldots, \partial_k X$. Let $M = \mathrm{Dbl}(X, \partial X)$. Pick basepoints in each boundary component of X and let $\Gamma = \pi_1(X)$ (this can be with respect to any choice of basepoint), $\Lambda_j = \pi_1(\partial_j X)$, and $\Lambda = \coprod_j \Lambda_j$. Suppose that injectivity of the periodization map* per *and of the KO-assembly map A both hold for the fundamental group $\Gamma *_\Lambda \Gamma$ of M. Then the following conditions are equivalent:*

(1) *M admits a metric of positive scalar curvature.*
(2) *X admits a positive scalar curvature metric which is a product metric in a neighborhood of the boundary ∂X.*
(3) *X admits a positive scalar curvature metric for which ∂X is minimal (i.e., has vanishing mean curvature).*
(4) *X admits a positive scalar curvature metric which gives ∂X positive mean curvature with respect to the outward normal.*

Proof. As we have already mentioned, (2) $\Rightarrow$ (3) is trivial, (4) $\Rightarrow$ (1) is Theorem 1.1 and (4) $\iff$ (3) $\Rightarrow$ (1) is part of [1, Corollary 34]. So it suffices to show that (1) $\Rightarrow$ (2), and we can try to apply Theorem 4.5. Let $[M]$ be the ko-fundamental class of M in $ko_n(M)$. This restricts (since $M = X \cup_{\partial X} (-X)$ and thus int X is an open submanifold of M) to the ko-fundamental class $[X, \partial X]$ of X in $ko_n(X, \partial X) = ko_n(M, -X)$ (by excision). This, in turn, maps under

the *ko*-homology boundary map to the *ko*-fundamental class $[\partial X]$ of ∂X in $ko_{n-1}(\partial X)$. We get a commutative diagram

(3)
$$
\begin{array}{ccccc}
[M] & \longmapsto & [X, \partial X] & \longmapsto & [\partial X] \\
\Uparrow & & \Uparrow & & \Uparrow \\
ko_n(M) & \longrightarrow & ko_n(X, \partial X) & \overset{\partial}{\longrightarrow} & ko_{n-1}(\partial X) \\
\downarrow {\scriptstyle c_M} & & \downarrow {\scriptstyle c_X} & & \downarrow {\scriptstyle c_{\partial X}} \\
ko_n(B(\Gamma *_\Lambda \Gamma)) & \longrightarrow & ko_n(B\Gamma, B\Lambda) & \overset{\partial}{\longrightarrow} & ko_{n-1}(B\Lambda).
\end{array}
$$

The commutativity of the square on the right comes from naturality of the boundary maps, but the incompressibility assumption is needed to get commutativity of the square on the left. Indeed, without this assumption, the commutativity must fail; just think of the simple example where Γ is the trivial group and $\Lambda = \mathbb{Z}$, and $X = D^2$ with boundary $\partial X = S^1 = B\Lambda$, $M = S^2$. Then $B(\Gamma *_\Lambda \Gamma) = *$, and $[M] \mapsto 0$ in $ko_2(*)$ (since M admits positive scalar curvature), while $[X, \partial X]$ is a generator of $ko_2(D^2, S^1) \cong ko_2(\mathbb{R}^2) \cong ko_0(*) = \mathbb{Z}$ and $c_X \colon ko_2(D^2, S^1) \to ko_2(*, S^1)$ is an isomorphism.

So we need to explain why the left-hand square in (3) commutes when we assume incompressibility. The explanation is that $\pi_1(M)$ is the pushout or colimit of the diagram $\Gamma \longleftarrow \Lambda \longrightarrow \Gamma$ (in the category of groupoids), so $B\pi_1(M)$ is the *homotopy pushout* (hocolim) of $B\Gamma \longleftarrow B\Lambda \longrightarrow B\Gamma$ in spaces. This is usually *not* the same as the ordinary pushout since the kernel of $\Lambda_j \to \Gamma$ is invisible in the homotopy colimit, but it *is* the same when each Λ_j injects into Γ, i.e., ∂X is incompressible. (See [9, §10] for an explanation.) So in that case we get the commutativity via excision for *ko*, since $B(\Gamma *_\Lambda \Gamma)$ with $B(\Lambda *_\Lambda \Gamma)$ collapsed is the the same as $B\Gamma$ with $B\Lambda$ collapsed, and hence the relative groups $ko_n(B(\Gamma *_\Lambda \Gamma), B(\Lambda *_\Lambda \Gamma))$ and $ko_n(B\Gamma, B\Lambda)$ are the same in this case.

Alternatively, following the point of view in [3, Appendix to Chapter II], note that $\pi_1(M)$ is associated to the graph of groups shown in Figure 4. More precisely, the universal cover of this graph is a tree on which $\pi_1(M)$ acts, with Γ stabilizing the vertices and with the edges stabilized by the Λ_j. By Bass-Serre theory, this action on a tree corresponds to the decomposition of $\pi_1(M)$ as $\Gamma *_\Lambda \Gamma$, and displays $\pi_1(M)$ as an extension of Γ by a free group. From this picture we can also read off $B(\Gamma *_\Lambda \Gamma)$ as the ordinary pushout of $B\Gamma \longleftarrow B\Lambda \longrightarrow B\Gamma$.

Now because of the hypothesis on $\pi_1(M)$ and the assumption that M admits a metric of positive scalar curvature, we can apply [21, Theorem 4.13],
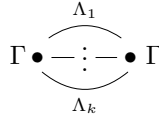
Figure 4: The graph of groups for computing $\pi_1(M)$.

and deduce that the image of $[M]$ under c_M (the leftmost downward arrow in (3)) vanishes in $ko_n(B(\Gamma *_\Lambda \Gamma))$. Chasing the diagram, we see that $(c_X)_*([X, \partial X])$ and $(c_{\partial X})_*([\partial X])$ also vanish, so by Theorem 4.5, X admits a positive scalar curvature metric which is a product metric in a neighborhood of ∂X.   □

We don't quite get the same sort of theorem in the totally non-spin case since we don't know what all the obstructions are (if any) in this situation where there is no Dirac operator. However, we can at least prove the following.

Theorem 5.3. *Let X be a connected compact totally non-spin manifold of dimension $n \geq 6$ with non-empty incompressible boundary ∂X, and let $M = \mathrm{Dbl}(X, \partial X)$. There can be any number k of boundary components $\partial_1 X, \ldots, \partial_k X$, but assume that each of these is also totally non-spin. Pick basepoints in each boundary component and let $\Gamma = \pi_1(X)$ (this can be with respect to any choice of basepoint), $\Lambda_j = \pi_1(\partial_j X)$. Suppose that Γ has finite homological dimension and that $n \geq \mathrm{hom}\dim \Gamma + 2$. Then the following conditions all hold:*

(1) *M admits a metric of positive scalar curvature.*
(2) *X admits a positive scalar curvature metric which is a product metric in a neighborhood of the boundary ∂X.*
(3) *X admits a positive scalar curvature metric for which ∂X is minimal (i.e., has vanishing mean curvature).*
(4) *X admits a positive scalar curvature metric which gives ∂X positive mean curvature with respect to the outward normal.*

Proof. The description of $\pi_1(M)$ is just as in the proof of Theorem 5.2, so $\Gamma *_\Lambda \Gamma$ splits as $F \rtimes \Gamma$, where F is a free group. That means $\mathrm{hom}\dim \pi_1(M) \leq \mathrm{hom}\dim \Gamma + 1$. Since the Λ_j's are subgroups of Γ, they also have finite homological dimension bounded by $\mathrm{hom}\dim \Gamma$. Also, by the long exact sequence of the pair, $\mathrm{hom}\dim(B\Gamma, B\Lambda) \leq \max(\mathrm{hom}\dim \Gamma, \mathrm{hom}\dim \Lambda + 1) < n$. So by our assumption on n, $c_*[M] = 0$ in $H_n(B\pi_1(M); \mathbb{Z})$, $c_*([\partial_j X]) = 0$ in

$H_{n-1}(B\Lambda; \mathbb{Z})$, and $c_*([X, \partial X]) = 0$ in $H_n(B\Gamma, B\Lambda; \mathbb{Z})$. Since M and the $\partial_j X$ are all totally-non-spin, we conclude by [21, Theorem 4.11] or by [12, Theorem 1.2] that all of them admit metrics of positive scalar curvature. Then we obtain (2), and thus also (3) and (4), by Theorem 4.7. □

Now we go on to the non-incompressible case, as well as to the case where X is not spin but has spin boundary. In the latter case, even if X and ∂X are both simply connected, we cannot expect X to have a positive scalar curvature metric which is a product metric near the boundary. For example, ∂X could be an exotic sphere in dimension $\equiv 1$ or 2 mod 8 with non-zero α-invariant. Then ∂X is an oriented boundary but not a spin boundary, and thus there is an oriented manifold (which can be chosen simply connected) having ∂X as its boundary. The double of this manifold is simply connected and non-spin, so it admits positive scalar curvature by [13, Corollary C]. In this case, by Theorem 1.2, X has a positive scalar curvature metric which has positive mean curvature on the boundary, even though it can't have a positive scalar curvature metric which is a product metric near the boundary, since ∂X does not admit positive scalar curvature. We will see other generalizations of this later. First we consider the non-incompressible spin case of the converse to Theorem 1.1.

Theorem 5.4. *Let X be a connected compact manifold of dimension $n \geq 6$ with non-empty boundary ∂X. There can be any number k of boundary components $\partial_1 X, \dots, \partial_k X$. Assume that X is spin and that $\Gamma = \pi_1(X)$ has finite homological dimension less than n and satisfies the Baum-Connes Conjecture with coefficients and the Gromov-Lawson-Rosenberg conjecture. (As an example, Γ could be free abelian, free, or a surface group.) Let $M = \mathrm{Dbl}(X, \partial X)$ be the double of X along ∂X, a closed n-manifold. Then the following conditions are equivalent:*

(1) *M admits a metric of positive scalar curvature.*
(2) *X admits a positive scalar curvature metric for which ∂X is minimal (i.e., has vanishing mean curvature).*
(3) *X admits a positive scalar curvature metric which gives ∂X positive mean curvature with respect to the outward normal.*

Proof. We have $\pi_1(M) = \Gamma *_\Lambda \Gamma$ by Van Kampen's Theorem. (This means the colimit of the diagram $\Gamma \longleftarrow \Lambda \longrightarrow \Gamma$.) We have the "folding map" $f: M \twoheadrightarrow X$, sending each point $x \in X$ and the corresponding point $\bar{x} \in -X$ both back to x, which is split by the inclusion of X into M, and thus an induced split surjection of groups $f_*: \Gamma *_\Lambda \Gamma \to \Gamma$. By lifting to the universal cover of X and arguing as in the proof of Theorem 3.1, we see that $\Gamma *_\Lambda \Gamma$
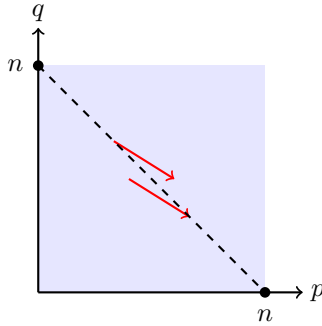
Figure 5: The Atiyah-Hirzebruch spectral sequence for computing $ko_n(B\pi_1(M))$. The corresponding spectral sequence for $KO_n(B\pi_1(M))$ is similar except that it extends into the fourth quadrant, and the map from this spectral sequence to that one preserves differentials affecting the line $p+q = n$. Typical differentials are shown in color.

splits as $F \rtimes \Gamma$, where F is a free group. Thus if we know Baum-Connes with coefficients for Γ, the same follows for $\pi_1(M) = \Gamma *_\Lambda \Gamma$. Furthermore, the homological dimension of $\pi_1(M)$ is at most one more than the homological dimension of Γ, and is thus $\leq n$. This implies injectivity of the periodization maps $ko_n(B\Gamma) \to KO_n(B\Gamma)$ and $ko_n(B\pi_1(M)) \to KO_n(B\pi_1(M))$, because any differentials in the Atiyah-Hirzebruch spectral sequences $H_p(B\Gamma, ko_q) \Rightarrow ko_{p+q}(B\Gamma)$ and $H_p(B\Gamma, KO_q) \Rightarrow KO_{p+q}(B\Gamma)$ affecting the line $p+q = n$ have to be the same (and similarly for $B\pi_1(M)$), because H_p vanishes for $p > n$. (See Figure 5.) Since Γ and $\pi_1(M)$ both have finite homological dimension, they are torsion-free and Baum-Connes implies that the KO assembly maps for both are injective.

As before, it suffices to prove that $(1) \Rightarrow (3)$. So assume M has a metric of positive scalar curvature. It will be convenient to denote by Λ' the image of Λ in Γ, so that also $\pi_1(M) = \Gamma *_{\Lambda'} \Gamma$. Start by choosing a collar neighborhood (diffeomorphic to $\partial X \times [0,1]$) of ∂X inside X and let $\partial' X = \coprod \partial'_j X$ be a "parallel copy" of ∂X inside this collar. First assume that for each j, $K_j = \ker(\Lambda_j \twoheadrightarrow \Lambda'_j)$ is finitely generated. Then we modify each $\partial'_j X$ to form a new hypersurface Z_j, by doing a finite number of surgeries on embedded circles to kill off the kernel K_j, inserting the necessary 2-handles inward in X, as in Figure 6. Note that the fact that K_j dies in $\pi_1(X)$ is what guarantees that we can build the necessary 2-handles inside X.

After this initial surgery step, we have a new manifold Y with boundary, of codimension 0 in X, with $\partial Y = Z = \coprod Z_j$ incompressible in X. Thus

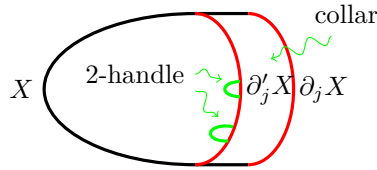Figure 6: First step in the proof of Theorem 5.4. Z_j is the result of attaching 2-handles to $\partial'_j X$ to reduce the fundamental group to Λ'_j.

positive scalar curvature for M implies via the proof of Theorem 5.2 that Y has a metric of positive scalar curvature which is a product metric in neighborhood of ∂Y. We double the region between ∂Y (which we recall is $\partial'X$ modified by surgeries) and ∂X across ∂X, obtaining a manifold W with boundary of codimension 0 in M, whose boundary looks like two copies of ∂Y, and is thus incompressible. (See Figure 7.) Since the fundamental group of ∂Y is, by construction, a quotient of that of ∂X, Van Kampen's Theorem gives that $\pi_1(\partial Y) \cong \pi_1(W)$, with the isomorphism induced by the inclusion. Thus W is a π_1-preserving spin cobordism over $B\pi_1(\partial Y)$ from ∂Y to its reflection across ∂X. (See Figure 7 again.) By the surgery theorem for positive scalar curvature metrics ([13] and [10, Corollary 6.2]), we can extend the metric on Y to a positive scalar curvature metric on $Q = Y \cup_{\partial Y} W$ which is a product metric in a neighborhood of ∂Q (the reflection of the original ∂Y across ∂X). At this point, ∂Y is now a two-sided totally geodesic hypersurface in Q. By a slight deformation, using [1, Proposition 28], we can preserve positive scalar curvature on Q but arrange for ∂Y to have positive mean curvature. Now observe that $\partial X \subset Q$ is obtained by adding handles back to ∂Y, the duals of the surgeries used to construct Y from X, and these are in codimension 2. So we can apply Theorem 1.3 to deform ∂X so that it has positive mean curvature. Since X is inside Q, which has positive scalar curvature, we have proven that $(1) \Rightarrow (3)$.

There is just one more step if one or more of the kernels K_j is not finitely generated. The problem now is that it appears we need to do infinitely many surgeries to get from X to Y and back again. However, we can construct a sequence of modifications of $\partial_j X$ (each obtained from the previous one by attaching more 2-handles into the interior of X), say $Z_j^{(k)}$, $k = 1, 2 \ldots$, so that $\pi_1(Z_j^{(k)}) \to \Lambda'_j$ as $k \to \infty$. We get a sequence $X^{(k)}$ of manifolds with boundary, $\partial X^{(k)} = \coprod_j Z_j^{(k)}$, and ko-homology classes, the images of $[X, \partial X]$ in $ko_n(X^{(k)}, \partial X^{(k)})$ which tend to 0 in the limit. By the behavior of homology under inductive limits, there must be some finite stage at which $[X^{(k)}, \partial X^{(k)}]$
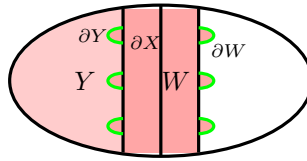
Figure 7: Second step in the proof of Theorem 5.4. First we construct a positive scalar curvature metric on Y (light color), then push it across W (dark color), and then get back to ∂X from ∂Y by doing outward-pointing surgeries.

vanishes, and thus $X^{(k)}$ has a positive scalar curvature metric which is a product metric near the boundary. The proof is now concluded as before. $\qquad\square$

The analogous theorem in the totally non-spin case is this:

Theorem 5.5. *Let X be a connected compact manifold of dimension $n \geq 6$ with non-empty boundary ∂X. There can be any number k of boundary components $\partial_1 X, \ldots, \partial_k X$. Assume that X is totally non-spin. Pick basepoints in each boundary component and let $\Gamma = \pi_1(X)$ (this can be with respect to any choice of basepoint), $\Lambda_j = \pi_1(\partial_j X)$. Assume that for each j, the image of Λ_j in Γ is finitely presented. Suppose that Γ has finite homological dimension and that $n \geq \hom \dim \Gamma + 2$. Let $M = \mathrm{Dbl}(X, \partial X)$ be the double of X along ∂X, a closed n-manifold. Then the following conditions hold:*

(1) *M admits a metric of positive scalar curvature.*
(2) *X admits a positive scalar curvature metric for which ∂X is minimal (i.e., has vanishing mean curvature).*
(3) *X admits a positive scalar curvature metric which gives ∂X positive mean curvature with respect to the outward normal.*

Proof. Note that M is totally non-spin and its dimension n exceeds the homological dimension of $\pi_1(M) = \Gamma *_\Lambda \Gamma$, which is no more than $\hom \dim \Gamma + 1$, as in the proof of Theorem 5.3. So by [21, Theorem 4.11] or [12, Theorem 1.2], M admits a metric of positive scalar curvature, verifying (1). To check (2) and (3), we apply the same argument as in Theorem 5.4; i.e., we first do surgeries into the interior to kill off the kernel of $\Lambda_j \to \Gamma$ for each j, ending up with a connected totally non-spin manifold Y of codimension 0 in X with incompressible boundary. Now recall that X is totally non-spin, and so is Y, but some of the boundary components $\partial_j Y$ might "accidentally" fail to be totally non-spin. If this is the case for some j, do an additional surgery to

take a connected sum of $\partial_j Y$ with a tubular neighborhood of an embedded 2-sphere in the interior of Y whose normal bundle is non-trivial (such 2-spheres exist since Y is totally non-spin). After doing this, all the $\partial_j Y$ will also be totally non-spin, and hence will admit metrics of positive scalar curvature for the same reason as M (since the condition on the homological dimension of Γ inherits to subgroups). Now we can apply Theorem 5.3 to Y, and then apply Theorem 1.3 to get positive mean curvature on the boundary of X.   □

Remark 5.6. By the same reasoning as in the proof of Theorem 5.4, one can dispense with the finite presentation hypothesis on the images of Λ_j in Γ. Even though, when one of these images is not finitely presented, it won't be possible to modify Y in finitely many steps so as to have incompressible boundary, $c_*([Y, \partial Y])$ vanishes in $H_n(B\Gamma, B\operatorname{image}(\Lambda); \mathbb{Z}) = 0$, and thus will vanish in $H_n(B\pi_1(Y), B\pi_1(\partial Y); \mathbb{Z})$ for Y "close enough" to having incompressible boundary, and then we can proceed as before.
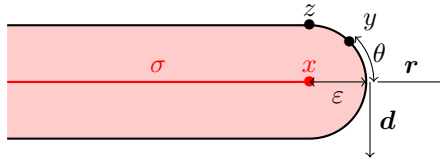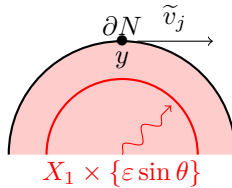
6. The Gromov-Lawson doubling theorem

Since the proof of Theorem 1.1 in [14] is somewhat sketchy, we include for the convenience of the reader a more detailed proof. We should mention that the paper [1] also gives another approach to "doubling." Indeed, [1, Corollary 34] gives a homotopy equivalence between $\mathcal{R}^+(X)_{H>0}$ (positive scalar curvature metrics on X with positive mean curvature on ∂X) and $\mathcal{R}^+(X)_{\mathrm{Dbl}}$ (positive scalar curvature metrics on X with vanishing second fundamental form on the boundary, that extend to reflection-invariant positive scalar curvature metrics on $\mathrm{Dbl}(X, \partial X) = M$).

Proof of Theorem 1.1. Following the proof in [14], let $I = [-1, 1]$, give $X \times I$ the product metric, and identify X with $X \times \{0\}$. Let

$$N = \{(x, t) \in X \times I : \operatorname{dist}((x, t), X_1) \le \varepsilon\} \subset X \times \mathbb{R},$$

where X_1 is the complement in X of a small open collar around ∂X. Then the double M is obtained by smoothing the C^1 manifold ∂N. We give N the metric inherited from $X \times I$. The portions of ∂N given by $X_1 \times \{\varepsilon\}$ and $X_1 \times \{-\varepsilon\}$ are smooth and isometric to X_1, but the second derivative of the metric is discontinuous at points in $\partial X_1 \times \{\pm\varepsilon\}$ like z in Figure 8. As in [14], choose $x \in \partial X_1$ and let σ be a small geodesic segment in X passing through x and orthogonal to the boundary of X_1. Then $(\sigma \times I) \cap N$ is flat and totally geodesic in N, and a local picture of it near x looks like Figure 8. In this plane we have the unit vector field $\boldsymbol{r}$ ('r' for right) which along $\sigma \times \{t\}$ is the

Figure 8: Slice of N normal to X_1.

Figure 9: Slice of N through y parallel to X_1.

unit tangent vector pointing away from the interior of X_1. We also have the unit vector field $\boldsymbol{d}$ ('d' for down) pointing downward along the lines $\{m\} \times \mathbb{R}$, $m \in X$. If we instead take a slice of N parallel to X_1 through a point y at angle θ from X_1 (see Figure 8 and Figure 8 in [14]), we get a picture like Figure 9.

Choose an orthonormal frame $v_1, \ldots, v_{n-1}$ for $T_x(\partial X_1)$ that diagonalizes the second fundamental form for ∂X_1 in X_1 with respect to the outward-pointing normal vector vector field $\boldsymbol{n}$, which at x coincides with $\boldsymbol{r}$. Thus we can assume that the shape operator has the form $S_{\partial X_1} v_j = \mu_j v_j$, and the mean curvature $H_{\partial X_1}$ of ∂X_1 is $\sum_j \mu_j$. The assumption of the theorem implies that this is positive. (Positive mean curvature of ∂X in X implies positivity of $H_{\partial X_1}$ if X_1 is close enough to X.) At a point $y \in \partial N$ at distance ε from $x \in \partial X_1$ as in Figure 8, the outward-pointing normal vector to N is given by $\boldsymbol{n} = \cos\theta\,\boldsymbol{r} - \sin\theta\,\boldsymbol{d}$. There is an orthonormal frame $w_0, w_1, \ldots, w_{n-1}$ for $T_y(\partial N)$ with $w_0 = \cos\theta\,\boldsymbol{d} + \sin\theta\,\boldsymbol{r}$, and with w_j close to the parallel transport $\widetilde{v}_j$ of v_j. Since $\sigma \times \mathbb{R}$ is totally geodesic in $X \times \mathbb{R}$ and $\partial N \cap (\sigma \times \mathbb{R})$, shown in Figure 8, consists of $\sigma \times \{\pm\varepsilon\}$ joined together by a semicircular arc of radius ε, it follows that w_0, which lies in this 2-plane, is an eigenvector for the shape operator of ∂N, with eigenvalue $\frac{1}{\varepsilon}$. On the other hand, for $j = 1, \ldots, n-1$, $\nabla_{w_j}\boldsymbol{d} = 0$ and so

$$S_{\partial N}(w_j) = -\nabla_{w_j}\boldsymbol{n} \approx \cos\theta\,\mu_j w_j$$

and so by the Gauss curvature formula, the scalar curvature $\kappa_{\partial N}$ of ∂N at y

works out to

$$\kappa_{\partial N} \approx \kappa_X + \sum_{j=1}^{n-1} \frac{2}{\varepsilon} \cos\theta\, \mu_j + \sum_{1 \le j \ne k \le n-1} \cos^2\theta\, \mu_j \mu_k$$

$$= \kappa_X + \frac{2\cos\theta}{\varepsilon}\, H_{\partial X_1} + \cos^2\theta\, H_{\partial X_1}^2 - \sum_{j=1}^{n-1} \cos^2\theta\, \mu_j^2.$$

Since κ_X and $H_{\partial X_1}$ are positive, this is positive and tends to κ_X as $\theta \to \pm\frac{\pi}{2}$. So we can round the corners at these points keeping positivity of the scalar curvature. (Note that the formula obtained here is slightly different from the one in [14], but this doesn't affect the conclusion.) $\qquad\square$

7. Open problems

We have left several open problems in our discussion. In this section, we list a few of these and say something about where they stand.

(1) The most obvious question is how generally a converse to Theorem 1.1 is valid. One can state this as

Conjecture 7.1 ("Doubling Conjecture"). If X is a compact manifold with boundary and $M = \mathrm{Dbl}(X, \partial X)$ admits a metric of positive scalar curvature, then X admits a metric of positive scalar curvature with positive mean curvature on ∂X.

At the moment we do not know of any counterexamples, nor do we know of any technology that could be used to disprove this in general. Conjecture 7.1 has now been proved in dimension 3 by Carlotto and Li [5, 4]. Note by the way that [1, §2.3] gives an obstruction to a manifold X with boundary admitting a positive scalar curvature metric with $H > 0$ on the boundary, but it can't give any counterexamples to the Doubling Conjecture since in all cases where the obstruction applies, it applies to the double as well.

(2) Another question is how generally Corollary 4.10 and Theorem 4.6 can hold. One cannot get rid of the dimension restriction, since Theorem 4.7 fails when $\dim X = 5$, as one can see from the following counterexample. Let Y be a smooth non-spin simply connected projective algebraic surface (over $\mathbb{C}$) of general type, for example a hypersurface of degree d in $\mathbb{CP}^3$ of even degree $d \ge 4$. Then $b_2^+(Y) = 1 + \frac{(d-1)(d-2)(d-3)}{3} > 1$ and Y has non-trivial Seiberg-Witten invariants, hence does not admit

a metric of positive scalar curvature. Let $X = Y \times [0, 1]$, which is a compact 5-manifold with boundary. The double M of X along its boundary is $Y \times S^1$, which is a closed totally non-spin 5-manifold with fundamental group $\mathbb{Z}$, and which thus admits positive scalar curvature as a consequence of the theorem of Stolz and Jung ([21, Theorem 4.11] or [12, Theorem 1.2]). However X cannot admit a positive scalar curvature metric which is a product metric near the boundary, since $\partial X = Y \sqcup -Y$ does not admit a metric of positive scalar curvature. Yet the relative homology group $H_5(B*, B(* \sqcup *); \mathbb{Z})$ vanishes.

One can modify this counterexample so that it is even a counterexample to the weaker Theorem 4.6. Let Y' be a connected sum of copies of $\overline{\mathbb{CP}^2}$ with the same signature $-\frac{d}{3}(d^2 - 4)$ as Y, and let $X' = Y' \times [0, 1]$. Then X' obviously has a product metric of positive scalar curvature, while X does not; yet they represent the same class in $\Omega_5(B*, B(* \sqcup *); \mathbb{Z}) \cong \Omega_4 \cong \mathbb{Z}$ since that class is detected just by the signature.

(3) Suppose X is a compact manifold with boundary, of dimension $n \geq 6$ so that our theorems apply. A major problem is to try to determine the homotopy type of $\mathcal{R}^+(X)_{H>0}$ (positive scalar curvature metrics on X with positive mean curvature on ∂X) when this space is non-empty, or at least to give nontrivial lower bounds on the homotopy groups. As we mentioned before, [1, Corollary 34] shows that this space is homotopy equivalent to $\mathcal{R}^+(M)^{\mathbb{Z}/2}$, the reflection-invariant positive scalar curvature metrics on the double $M = \mathrm{Dbl}(X, \partial X)$. And [1, Corollary 40] shows that when ∂X admits a positive scalar curvature metric, then considering the space of positive scalar curvature metrics with product boundary conditions would give us the same homotopy type. In this case (when there is a positive scalar curvature metric which is a product metric near the boundary) and when everything is spin, [2, 11] give lower bounds on the homotopy groups of $\mathcal{R}^+(X)_{H>0}$ in terms of the KO-groups of the C^*-algebra of the fundamental group. It is possible that a similar analysis, using APS methods as in [1, §2.1], would also work with the weaker boundary condition $H > 0$.

The references just cited, and other similar ones, also say something about the homotopy type of $\mathcal{R}^+(M)$ (the positive scalar curvature metrics with no equivariance condition), but it's not immediately clear how this translates into information about $\mathcal{R}^+(M)^{\mathbb{Z}/2}$. An example worked out in [1, Corollary 45 and Remark 46] does give a case where $\mathcal{R}^+(X)_{H>0}$ has infinitely many path components and infinite homotopy groups in all dimensions, but the construction is somewhat *ad hoc*.

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