A WALL-CROSSING FORMULA FOR GROMOV-WITTEN INVARIANTS UNDER VARIATION OF GIT QUOTIENT

EDUARDO GONZALEZ AND CHRIS T. WOODWARD

Abstract. We prove a quantum version of a wall-crossing formula of Kalkman [37], [44] that compares intersection pairings on geometric invariant theory (git) quotients related by a change in polarization. Each expression in the classical formula is quantized in the sense that it is replaced by an integral over moduli spaces of certain stable maps; in particular, the wall-crossing terms are gauged Gromov-Witten invariants with smaller structure group. As an application, we show that the genus zero graph Gromov-Witten potentials of quotients related by wall-crossings of crepant type are equivalent up to a distribution in one of the quantum parameters that is almost everywhere zero. This is a version of the crepant transformation conjecture of Li-Ruan [46], Bryan-Graber [10], Coates-Ruan [15] etc. in cases where the crepant transformation is obtained by variation of git.

Contents

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1. Introduction

1.1. Kalkman's wall-crossing formula. According to the geometric invariant theory introduced by Mumford [50], the git quotient $X/\!/G$ of an action of a complex reductive group G on a projective variety X equipped with a polarization (ample equivariant line bundle) $L \to X$ has coordinate ring equal to the G-invariant part of the coordinate ring on X. Geometrically $X/\!/G$ is the quotient of an open semistable locus $X^{ss} \subset X$ by an equivalence relation, where a point $x \in X$ is semistable if there is a non-constant invariant section of a tensor power of the polarization that is non-zero at the point. If the action of G on the semistable locus has only finite stabilizers, then $X/\!/G$ is the quotient of X^{ss} by the action of G, by which we mean here the stack-theoretic quotient, see for example [21]. If X is in addition smooth, then $X/\!/G$ is a smooth proper Deligne-Mumford stack with projective coarse moduli space. In Kempf-Ness [39], see also Mumford et al [50], the coarse moduli space of the git quotient is identified with the symplectic quotient of X by a maximal compact subgroup of G.

The question of how the git quotient depends on the polarization, or equivalently, choice of moment map is studied in a series of papers by Guillemin-Sternberg [32], Brion-Procesi [7], Dolgachev-Hu [23], and Thaddeus [59]. Under suitable stable=semistable and smoothness conditions, the git quotient $X/\!/G$ undergoes a sequence of blow-ups and blow-downs. The class of birational equivalences which appear via variation of git is reasonably large. In fact, for a class of so-called Mori dream spaces, any birational equivalence can be written as a composition of birational equivalences induced by variation of git [35, 49].

The question of how the cohomology of the quotient depends on the polarization is studied in Kalkman [37], which proves a wall-crossing formula for the intersection pairings under variation of symplectic quotient. Similar results, in the context of Donaldson theory, are given in Ellingsrud-Göttsche [25]. Let X be a smooth projective G-variety as above such that G acts locally freely on the semistable locus (that is, with finite stabilizers) and $H_G(X)$ its equivariant cohomology with rational coefficients. There is a natural map

$$\kappa_X^G : H_G(X) \to H(X/\!/G)$$

studied in Kirwan's thesis [41], given by restriction to the semistable locus $H_G(X) \to H_G(X^{ss})$ and then descent under the quotient $H_G(X^{ss}) = H(X/\!/G)$. Consider the simplest case $G = \mathbb{C}^*$. Let $X/\!/_{\pm}G$ denote the git quotients corresponding to polarizations $L_{\pm} \to X$. Let

$$\kappa_{X,\pm}^G : H_G(X) \to H(X/\!/_{\pm}G)$$

be the Kirwan maps and let

$$\tau_{X/\!/G} : H(X/\!/G) \to \mathbb{Q}, \quad \alpha \mapsto \int_{[X/\!/G]} \alpha$$

denote integration over $X/\!/G$. Kalkman's formula expresses the difference between the integrals $\tau_{X/\!/G}\kappa_{X,\pm}$ as a sum of fixed point contributions from components $X^{G,t} \subset X^G$ that are semistable for elements in the rational Picard group $\mathrm{Pic}^G_{\mathbb{Q}}(X)$ interpolating between L_- and L_+,

$$[L_t := L_-^{(1-t)/2} \otimes L_+^{(1+t)/2}] \in \mathrm{Pic}^G_{\mathbb{Q}}(X)$$

for some rational $t \in (-1, 1)$. Each such fixed point has a contribution to the localization formula for the integral of a class $\alpha \in H_G(X)$ over X given as follows. Let $\nu_{X^{G,t}} \to X^{G,t}$ denote the normal bundle of the inclusion $X^{G,t} \to X$, and $\mathrm{Eul}_G(\nu_{X^{G,t}}) \in H_G(X^{G,t})$ its G-equivariant Euler class, or equivalently, equivariant Chern class of degree equal to the real rank. We identify $H(BG)$ with the polynomial ring $\mathbb{Q}[\xi]$ in a single element ξ of degree 2, representing the hyperplane class in the cohomology of $BG = \mathbb{C}P^\infty$. If $m_t = \mathrm{codim}(X^{G,t})$ and

$$\nu_{X^{G,t}} = \bigoplus_{i=1}^{m_t} \nu_{X^{G,t},i} \tag{1}$$

is a decomposition into line bundles with weights $\mu_i \in \mathbb{Z}$ then

$$\mathrm{Eul}_G(\nu_{X^{G,t}}) = \prod_{i=1}^{m_t} \mathrm{Eul}_G(\nu_{X^{G,t},i}) = \prod_{i=1}^{m_t} (c_1(\nu_{X^{G,t},i}) + \mu_i \xi).$$

Since G acts with no non-trivial fixed vectors on $\nu_{X^{G,t}}$, the Euler class has an inverse

$$\mathrm{Eul}_G(\nu_{X^{G,t}})^{-1} \in H(X^{G,t})[\xi, \xi^{-1}]$$

after inverting the equivariant parameter ξ. If one has a splitting as in (1) then the inverted class admits an expansion

$$\begin{aligned}
\mathrm{Eul}_G(\nu_{X^{G,t}})^{-1} &= \prod_{i=1}^{m_t} (\mu_i \xi)^{-1} \left(1 + \frac{c_1(\nu_{X^{G,t},i})}{\mu_i \xi}\right)^{-1} \\
&= \prod_{i=1}^{m_t} (\mu_i \xi)^{-1} \left(1 - \frac{c_1(\nu_{X^{G,t},i})}{\mu_i \xi} + \left(\frac{c_1(\nu_{X^{G,t},i})}{\mu_i \xi}\right)^2 - \ldots\right).
\end{aligned}$$

The contribution from a fixed point component $X^{G,t} \subset X^G$ is the integral of the restriction $\alpha|_{X^{G,t}}$ times the inverted Euler class, denoted

$$\tau_{X^{G,t}} : H_G(X) \to \mathbb{Q}[\xi, \xi^{-1}], \quad \alpha \mapsto \int_{[X^{G,t}]} \alpha|_{X^{G,t}} \cup \mathrm{Eul}_G(\nu_{X^{G,t}})^{-1}.$$

Let Resid denote the residue, that is, the coecient of ^{1}:

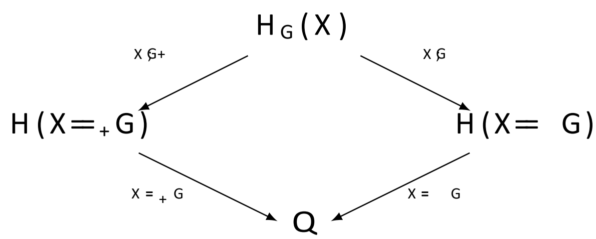
$$\text{Resid} : Q[; ^{1}] ! \quad Q; \quad \sum_{n2Z} a_n{}^{n} ! \quad a_{1}:$$

(More invariantly, the residue should be a map Q[; ^{1}]d ! Q, but we omit the one-form from the notation.)

Theorem 1.1 (Kalkman wall-crossing formula, circle group case). Let G = C and let X be a smooth projective G-variety equipped with polarizations L ! X . Suppose that stable=semistable for the G-action on P(L$_{+}$ L). Then

(2) $$_{X=_{+}G} \quad _{X;+}{}^{G} \qquad _{X= \ G} \quad _{X;}{}^{G} = \sum_{t2(\ 1;1)} \text{Resid} \ _{X^{G;t}}:$$

In other words, failure of the following square to commute is measured by an explicit sum of wall-crossing terms:



The formula (2) also holds for certain quasi-projective varieties, such as vector spaces whose weights are contained in an open half-space, see the more general Theorem 2.5 below.

Example 1.2. (Integration over projective space) The following simple example illustrates the notation involved. let G = C acting on X = C^{k} by scalar multiplication,

$$g[z_1; : : : ; z_n] = [gz_1; : : : ; gz_n]$$

so that $H_G(X)$ = Q[] where is the equivariant parameter representing the hyperplane class under the isomorphism $H_G(X) = H(BG) = H(CP^{1})$: Suppose that polarizations L correspond to the characters 1. Invariant sections are then mono-mials of positive resp. negative degree, hence the semistable locus is $X^{ss;+}$ for L_{+} and the emptyset for $X^{ss;}$. Thus

$$X= \ G = ;; \quad X=_{+}G = P^{k \ 1}$$

and the two chambers are separated by the value t = 0 so that 0 2 X is semistable for $L^{(1 \ \ t)=2}_{+}$ $L^{(1+t)=2}$. The Kirwan map $^{G}_{X}$: $H_G(X)$! $H(X=_{+}G)$ sends the generator 2 $H_G(X)$ to the hyperplane class ! 2 $H^{2}(X=_{+}G)$. We compute the integrals

$\int_{P^{k-1}} \omega^a$ for $a \in \mathbb{Z}_{\geq 0}$ via wall-crossing. In the negative chamber, the integral is zero, since the quotient is empty. By the Kalkman formula (2)

$$\int_{P^{k-1}} \omega^a = \operatorname{Resid} \int_{[0]} \xi^a \cup \operatorname{Eul}_G(\mathbb{C}^k)^{-1}$$
$$= \operatorname{Resid} \xi^a / \xi^k = \begin{cases} 1 & a = k-1 \\ 0 & \text{otherwise} \end{cases}$$

confirming that ω^{k-1} is the dual of the fundamental class. This ends the example.

1.2. **Quantum Kirwan map and adiabatic limit theorem.** The main result of this paper is a generalization of Theorem 1.1 to the setting of genus-zero Gromov-Witten theory, that is, quantum cohomology. The Kirwan map, trace, and fixed point contributions become quantized in the sense that each is replaced by a formal map depending on a formal variable q whose specialization to $q = 0$ gives the classical version above. First we recall the definition of quantum cohomology of a smooth polarized projective G-variety X. The equivariant Novikov field

$$\Lambda_X^G \subset \operatorname{Map}(H_2^G(X;\mathbb{Q}); \mathbb{Q})$$

associated to a G-variety X with polarization $L \to X$ consists of linear combinations of delta-functions q^d at $d \in H_2^G(X;\mathbb{Q})$ satisfying a finiteness condition:

$$\Lambda_X^G := \left\{ \sum_{d \in H_2^G(X;\mathbb{Q})} c_d q^d, \quad \forall e > 0, \#\{c_d \mid \langle d, c_1^G(L)\rangle < e\} < \infty \right\}.$$

The equivariant quantum cohomology is the tensor product

$$QH_G(X) := H_G(X) \otimes \Lambda_X^G.$$

We write $\Lambda^G_{X,L}$ resp. $QH_G(X;L)$ if we wish to emphasize the dependence on L. A more standard definition in algebraic geometry would use the cone of effective curve classes, but the definition we give here has better invariance properties, for example, under Hamiltonian perturbation. Below we will need several variations on this definition. Let $\Lambda_X^{G,\mathrm{fin}} \subset \Lambda_X^G$ denote the space of finite linear combinations of the symbols q^d. Denote by $QH_{G,\mathrm{fin}}(X) := H_G(X) \otimes \Lambda_X^{G,\mathrm{fin}} \subset QH_G(X)$ the subspace with only finitely many non-vanishing exponents of q non-vanishing. Let $\Lambda_X^{G,0} \subset \Lambda_X^G$ be the space consisting of expressions involving only powers q^d with positive pairing $\langle d, c^G(L)\rangle \geq 0$. Denote by $QH_{G,0}(X) \subset QH_G(X)$ the subspace $H_G(X) \hat{\otimes} \Lambda_X^{G,0}$.

The quantum cohomology $QH_G(X)$ has the structure of a Frobenius manifold, in particular, it is equipped with a family of products

$$\star_\alpha : TQH_G(X)^2 \to TQH_G(X), \quad \alpha \in QH_G(X)$$

dened by equivariant virtual enumeration of genus zero stable maps, that is, maps u : C ! X from projective nodal curves C of arithmetic genus zero to X with no innitesimal automorphisms. The moduli stack $\overline{M}_{g;n}(X; d)$ of stable n-marked maps of homology class d 2 $H_2(X; Z)$ and arithmetic genus g is a proper Deligne-Mumford stack equipped with a perfect relative obstruction theory and evaluation map

$$ev = (ev_1; : : : ; ev_n) : \overline{M}_{g;n}(X; d) ! \quad X^n :$$

The action of G on X induces an action of G on $\overline{M}_{g;n}(X; d)$ by translation; then the evaluation maps induce a map

$$ev : H_G(X)^n ! \quad H_G(\overline{M}_{g;n}(X; d)):$$

The quantum product at 2 $QH_G(X)$ is dened by restricting to g = 0 and dening for ; 2 $H_G(X)$ $TQH_G(X)$

$$? := \sum_{n 0} \sum_{d 2 H_2(X;Z)} \frac{q^d}{n!} ev_{n+3;} ev_1 [: : : [ev_n [ev_{n+1} [ev_{n+2}$$

where the push-forwards are dened using the Behrend-Fantechi virtual fundamental classes [4], [3]. The denition is extended to $TQH_G(X) = QH_G(X)$ by linearity over $_X$. Note that we take the virtual fundamental classes to lie in the equivariant homology $H_G(\overline{M}_{g;n}(X; d))$ of the coarse moduli space, rather than in the Chow ring as in [31]. The quantum cohomology $QH_G(X)$ can also be dened using the smaller ring $^{G;0}_X$ but not $^{G;n}_X$, because of the innite sums.

The orbifold quantum cohomology $QH(X=G)$ of the quotient $X=G$ is dened by virtual enumeration of stable maps from orbifold curves, as follows. For any element g 2 G let Z_g be the centralizer of g 2 G, [g] G its conjugacy class, and hgi Z_g the subgroup generated by g. Denote by $\overline{I}_{X=G}$ the rigidied inertia stack from Abramovich-Graber-Vistoli [1], given by

$$\overline{I}_{X=G} = \bigcup_{[g]} X^{g;ss}=(Z_g=hgi):$$

Denote by

$$QH_G(X=G) = H(\overline{I}_{X=G}) \quad X$$

the quantum cohomology of the git quotient $X=G$ dened using the same Novikov eld ^G_X; this larger ring contains $_{X=G}$ by virtue of Kirwan's injection $H_2(X=G) ! \quad H^G_2(X)$. Virtual enumeration of representable morphisms from orbifold curves sat-isfying certain conditions to $X=G$ denes a family of products

$$? : TQH(X=G)^2 ! \quad TQH(X=G); \qquad 2 \ QH(X=G):$$

A quantum version of the Kirwan map

$$^G_X: QH_G(X) ! \quad QH(X=G)$$

(we keep the same notation as in the classical case) was constructed by the second author in [62]. The map ${}^{G}_{X}$ is a formal, non-linear map with the property that each linearization

$$D_X{}^{G} : TQH_G(X) \; ! \quad T_{X()}QH(X=G); \qquad 2\ QH_G(X)$$

is a ?-homomorphism, dened by virtual enumeration of ane gauged maps. Such a map is by denition a representable morphism $u : P(1;r)\ !\ \ X=G$ from a weighted projective line $P(1;r); r > 0$ to the quotient stack $X=G$ mapping the stacky point at innity $P(r)\ \ P(1;r)$ to the semistable locus $X=G\ \ X=G$. (More precisely, the domain is a smooth Deligne-Mumford stack of dimension one with a single stacky point with automorphism group of order r.) These are the algebro-geometric analogs of the vortex bubbles considered in Gaio-Salamon [26]. The compactied moduli stack $\overline{M}^{G}_{n;1}(C;X;d)$ of ane gauged maps of homology class $d\ 2\ H^G_2(X;Q)$ is, if stable=semistable for X, a proper smooth Deligne-Mumford stack with a perfect relative obstruction theory over the complexication of Stashe's multiplihedron [62]. It has evaluation maps

$$ev\ ev_1 : \overline{M}^{G}_{n;1}(C;X)\ !\ \ (X=G)^n\ \ \overline{I}_{X=G};$$

at the markings and the point at innity. The quantum Kirwan map is dened for $2\ H_G(X)\ \ QH_G(X)$ and a sequence of classes ${}_n\ 2\ H(\overline{M}_{n;1}(C))$ by

$$^G_X() := \sum_{(X;Q)} \sum_{G} \frac{q^d}{n!} ev_{1;} ev(;:::;) [\ f_n : {}^{n0}{}_{d2H_2}$$

We denote by ${}^{G;n}_{X}$ its n-th Taylor coecient in . The map ${}^{G}_{X}$ is dened over the smaller equivariant Novikov ring ${}^{G;0}_{X}$, but one obtains good surjectivity properties only using the Novikov eld ${}^{G}_{X}$, see [29]. The map ${}^{G}_{X}$ is a quantization of Kirwan's in the sense that $D_0{}^{G}_{X}\ j_{q=0}$ is the map studied in [41]. It admits a natural C-equivariant generalization from $QH_G(X)$ to $QH_C(X=G)$, induced by the natural action of C on $P(1;r)$.

A quantization of the classical integration map over $X=G$ is given by the graph potential in Givental [27]

$$_{X=G} : QH(X=G)\ !\ \ {}^{G}_{X}$$

dened by virtual enumeration of genus zero orbifold stable maps to $C\ (X=G)$, for $C = P$, of homology class $(1;d)$ for some $d\ 2\ H_2(X=G;Q)$. Let

$$\overline{M}_n(C;X=G;d) := \overline{M}_{0;n}(C\ \ (X=G);(1;d))$$

denote the stack of such maps of class $d\ 2\ H_2(X=G;Q)$, which we view as an element of $H^G_2(X;Q)$ via the inclusion of the semistable locus. It has evaluation and forgetful maps

(3) $$ev : \overline{M}_n(C; X{=}G; d) \; ! \; (X{=}G)^n; \quad f : \overline{M}_n(C; X{=}G; d) \; ! \; \overline{M}_n(C)$$

where $\overline{M}_n(C)$ is the moduli space of stable maps to C of class [C]. The graph potential is dened for 2 $\overline{H}(I_{X=G})$ QH(X=G) and a sequence of classes $_n$ 2 $H(\overline{M}_n(C))$ by

(4) $$_{X=G}() := \sum_{d \in H_2(X=G;Q)} \sum \frac{q^d}{n!} \int_{[\overline{M}_n(C;X=G;d)]} ev(; : : : ;) \; [\; f_n : {}_{n0}$$

Again $_{X=G}$ is dened over the equivariant Novikov ring $^{G;0}_X$. The potential $_{X=G}$ admits a natural C-equivariant extension $QH_C(X{=}G)$! ^G_X induced by identify-ing $QH_C(X{=}G)$ = QH(X=G)[] where is the equivariant parameter.

The graph potential $_{X=G}$ is related via the quantum Kirwan map to a gauged Gromov-Witten potential ^G_X: $QH_G(X)$! ^G_X dened by virtual enumeration of gauged maps, by which we mean morphisms from C to the quotient stack X=G, sat-isfying a Mundet stability condition [51] generalizing semistability for vector bundles on curves:

Denition 1.3. (Mundet semistability) A gauged map from a smooth projective curve C to the quotient stack X=G is a morphism v : C ! X=G, consisting of a pair

$$(p : P \; ! \; C; u : C \; ! \; P(X) := P \times_G X)$$

see [21]. We suppose that the Lie algebra g of G is the complexication of a unitary form g_R. Let g_R be equipped with an inner product invariant under the action of $G_R = \exp(g_R)$, inducing an identication g_R ! $g_R^\vee$.

(a) (Projections on Levi subgroups) Let R G be a parabolic subgroup. A Levi subgroup is a maximal reductive subgroup L R. The quotient U = R=L admits an embedding in G as a unipotent subgroup. Denote the corresponding Lie algebras l; u r. The group R admits a decomposition R = LU and the projection

(5) $$'^L : R \; ! \; L$$

is a group homomorphism which may also be dened as follows: Let 2 r be an element acting positively on the roots of r=l. Then for z 2 C the automorphism

$$r \; ! \; r; \quad r \; ! \; Ad(z)r$$

acts on the -weight space u by the scalar $z^{h;i}$. The corresponding Lie group automorphism

$$R \; ! \; R; \quad r \; ! \; zrz$$

has a limit as z ! 0 which is the projection R ! L. For example, if R is the group of upper triangular matrices in $G = GL(r)$ then U is the unipotent group of upper triangular matrices with 1's along the diagonal and Ad(z) conjugation by z acts on the ij-th entry in the matrix by the scalar $z^{i \quad j}$; the latter tends to zero for $i < j$ if $_i <$ $_j$.

(b) (Associated graded bundle) Given a reduction of P to a parabolic subgroup R G given by a section : C ! $P=R$ and a element in the Lie algebra r of R acting positively on u and commuting with l, there is an associated graded morphism given by a bundle

$$Gr(P) \; ! \quad C$$

whose structure group is the Levi L and whose transition maps are obtained by composing the transition maps for P with the projection ' ^{L} of (5). Thus in particular $Gr(P)$ is the central ber in a family of bundles

(6) $$\tilde{P} \; ! \quad C \quad C$$

whose ber over z is the bundle P_z obtained by conjugating the transition maps of P with z. The bundle $Gr(P)$ has a natural automorphism C ! $Gr(P)$ generated by z, since the structure group of $Gr(P)$ reduces to L and commutes with L.

(c) (Associated graded morphism) Consider the associated bundle associated to the family of bundles (6)

$$\tilde{P}(X) := (\tilde{P} \quad X)=G$$

A section α of $\tilde{P}(X)$ is given by a collection of maps $\alpha_i : U_i$! X in lo-cal trivializations of $P(X)$, satisfying $\alpha_j = \, _{ji}\alpha_i$ where $_{ji}$ are the transition maps of the bundle. Therefore a section of $P(X)$ over C C is given by ($;z$) ! $zu_i()$, where u_i : U_i ! X are the local maps dening u. By Gro-mov compactness (the bundle $P(X)$ is quasiprojective) the section α extends uniquely over the central ber $Gr(P)$! C as a stable map

$$Gr(u) : \hat{C} \; ! \quad (Gr(P))(X)$$

with domain $\hat{C}$. Here $\hat{C}$ is a projective nodal curve with some components possibly mapping into the bers of $(Gr(P))(X)$ and a distinguished principal component C_0 $\hat{C}$ mapping isomorphically onto C via composition with the projection $(Gr(P))(X)$! C.

(d) (Hilbert-Mumford weight) Since Gr(u) is a limit of the section u under the automorphism z of $\tilde{P}(X)$, the section Gr(u) is automatically xed (up to automorphism) by z for z 2 C. It follows there exists family of automorphisms

$$(z) : C \hat{!} \quad C; \hat{z} \; 2 \; C$$

such that

$$Gr(u)((z)) = z\, Gr(u); \qquad 8z \; 2 \; C:$$

Since C_0 maps isomorphically onto C, the map (z) must be trivial on C_0 and so $Gr(u)jC_0$ the associated graded section Gr(u) takes values in the xed point set P (X) where X denotes the xed point set of the automorphism z : X ! X . The Hilbert-Mumford weight

(7) $$_H(;) \; 2 \; Z$$

(with notation dened in (b)) determined by the polarization L is the usual Hilbert-Mumford weight, that is, the weight of the C-action

$$P(L) \; ! \quad P(L); \qquad l \; ! \quad zl$$

over any point u(z) 2 P (X) in the image of C_0 under Gr(u).

(e) (Ramanathan weight) Assume that 2 r = r– is a weight of R. The Ramanathan weight of $_R(;)$ of (P; u) with respect to (;) is given by the rst Chern number of the line bundle determined by via the associated bundle construction: If P denotes the principal R-bundle associated to as in [54] and C is the one-dimensional representation of R with weight then the line bundle is

$$P \;_R\; C \; ! \quad C:$$

The Ramanathan weight is

(8) $$_R(;) = \int_{[C]} c_1(P \;_R\; C) \; 2 \; Z:$$

(f) (Mundet weight) The Mundet weight is the sum of the Hilbert-Mumford and Ramanathan weights:

$$_M(;) := {}_H(;) + {}_R(;):$$

(g) (Mundet semistability) The morphism (P; u) is Mundet semistable if

(9) $$_M(;) \quad 0$$

for all such pairs (;) [51]. The morphism is Mundet stable if the above inequalities (9) are satised strictly.

Remark 1.4. We explain how this notation compares with that in Schmitt [56]. Any one-parameter subgroup : C ! G determines a parabolic subgroup

$$Q := \left\{ g \; 2 \; G \quad 9l \; 2 \; G; \lim_{z ! 0} (z)g(z) \quad ^{1} = l: \right\}$$

Let be such a subgroup and

$$: C \; ! \quad P = Q$$

a parabolic reduction of P to the parabolic subgroup Q determined by , and Schmitt [56] denotes by (E(); ()) the weighted ltration of E = P(V) deter-mined by of length say s, with the coecients of the coweight generating . Schmitt [56] denes

$$M(E(); ()) = \sum_{i=1}^{s} {}_{i}(\deg(E)\, rk(E_i) \qquad \deg(E_i)\, rk(E)) \; 2 \; Q:$$

which agrees with minus Ramanathan weight of (8) by a standard computation involving Chern classes, and

$$(E; ; ') \; 2 \; Q$$

the Hilbert-Mumford weight (7), see [56, p. 139] (opposite to our convention). The Mundet weight for stability parameter 2 Q is then minus

$$M(E;) + (E; ; ') \; 2 \; Q$$

and a bundle with map is semistable i this quantity is non-negative for all pairs (;). This ends the Remark.

To obtain a proper moduli stack with a perfect obstruction theory, we allow bub-bling in the bers.

Denition 1.5. A nodal n-marked gauged map over a scheme S of homology class d consists of an n-marked prestable curve $(C; \hat{\underline{z}})$ over S and a morphism

$$u : \hat{C} \; ! \quad C \quad X = G; \qquad u[\hat{C}] = (1; d) \; 2 \; H_2(C; Z) \quad H_2^G(X; Z)$$

By the condition on the homology class, over each point s 2 S there exists a prin-cipal component $C_0 \quad \hat{C}_s$ which maps under u and projection to the rst factor isomorphically to C, and a collection of bubble components

$$C_1; : : : ; C_k \quad \hat{C}; \qquad \dim(u(C_i)) = 0; \; i = 1; : : : k$$

that map to points $p_i = {}_1(u(C_i))$ in C, ${}_1$ being the projection C X=G ! C. A marked gauged map $(\hat{C}; u; \underline{z})$ over a point is Mundet semistable if the following two conditions hold:

the restriction $u|_{C_0} : C_0 \; ! \quad X = G$ is Mundet semistable, and each bubble component $C_i, i = 1,\ldots,k$ on which u is given by a trivial G-bundle with constant section has at least three special (marked or nodal) points.

We introduce the following notation. Let $\overline{M}_n^G(C, X, d)$ denote the stack of n-marked gauged maps and $\overline{M}_n^G(C, X, L, d)$ (or $\overline{M}_n^G(C, X, d)$ for short if the polarization is understood) the substack consisting of Mundet semistable gauged maps for the polarization L. Taking X and G to be points, and d to be trivial, one obtains the moduli stacks

$$\overline{\mathfrak{M}}_n(C) := \overline{M}_{\mathrm{genus}(C),n}(C,[C]); \quad \overline{M}_n(C) := \overline{M}_{\mathrm{genus}(C),n}(C,[C])$$

of prestable resp. stable maps to C of class $[C] \in H_2(C,\mathbb{Z})$. The category of Mundet-semistable gauged maps from C to X/G of homology class d and n markings forms an Artin stack $\overline{M}_n^G(C, X, L, d)$, which if all automorphism groups are finite is a proper Deligne-Mumford stack with evaluation map and forgetful morphisms [28], [30, Theorem 1.1]

Example 1.6. (Toric Case) Suppose that G is a torus acting on a vector space X with weight decomposition $\bigoplus_{i=1}^k X_i$ so that G acts on the one-dimensional representation X_i with weights $\mu_i \in \mathfrak{g}^\vee$. Suppose that X is equipped with a polarization given by a trivial line bundle with character $\theta \in \mathfrak{g}^\vee$. Let C be a curve of genus 0. Then

$$\begin{aligned}\overline{M}_0^G(C, X, d) &= M_0^G(C, X, d) = H^0(\mathcal{O}_C(d) \times_G X)/\!/G \\ &= \bigoplus_{i=1}^k X_i^{\oplus(\langle\mu_i, d\rangle+1)}/\!/G\end{aligned}$$

where the polarization on $H^0(\mathcal{O}_C(d) \times_G X)$ is the trivial line bundle with character θ, see [62]. For example, if $G = \mathbb{C}^*$ acts on $X = \mathbb{C}^k$ by scalar multiplication then $\overline{M}_0^G(C, X, d) = \mathbb{P}^{(d+1)k-1}$ for $d \geq 0$ and positive character θ.

Remark 1.7. In [62, Example 6.4(e)] a relative obstruction theory for $\overline{M}_n^G(C, X, d)$ is constructed by the method in Behrend [3], with a small extension to the case of quotient stacks in Olsson [52, Theorem 1.5]. The complex in the relative obstruction theory is $(Rp_* e^* T_{X/G})^\vee$, and comes equipped with a morphism to the relative cotangent complex L. A morphism to the shifted cotangent complex $L_{\overline{\mathfrak{M}}_n(C)}[1]$ is obtained by composing with $L \to L_{\overline{\mathfrak{M}}_n(C)}[1]$. Completing the diagram in the derived category gives rise to an absolute deformation theory as in [31, Appendix A]. If all automorphism groups are finite then this obstruction theory is perfect.

The moduli stack admits evaluation and forgetful maps

$$ev : \overline{M}_n^G(C, X, d) \to (X/G)^n, \quad f : \overline{M}_n^G(C, X, d) \to \overline{M}_n(C).$$

Definition 1.8. (Gauged Gromov-Witten potential) Suppose that all automorphism groups are finite for Mundet-semistable gauged maps. The gauged potential ^G_X for a smooth projective curve C is the formal map defined by for $\in H_G(X)$ and a sequence of classes $_n \in H(\overline{M_n}(C))$

(10) $$^G_X(;) = \sum_{n \geq 0; d \in H_2^G(X;\mathbb{Z})} \frac{q^d}{n!} \int_{[\overline{M}_n^G(C;X;d)]} ev^*(;\ldots;) \cup f^*_n$$

extended to $QH_G(X)$ by linearity.

The gauged potential and the graph potential of the quotient are related by the adiabatic limit theorem of [62] (which is a generalization of an earlier result of Gaio-Salamon [26]): Let be a rational stability parameter; we consider Mundet stability with respect to the polarization L with $\to \infty$.

Definition 1.9. Let $\overline{M_{n;1}}(C)$ be the moduli stack of scaled n-marked maps from [62]; a generic element is an n-marked map $: C \to \hat{C}$ with a relative differential $\in H^0(C;\hat{!})$. It contains a prime divisor $M_n(\overline{C})$ corresponding to maps with zero differential $= 0$, and for any partition $I_1 \cup \ldots \cup I_r = \{1;\ldots;n\}$ a prime divisor isomorphic to $\overline{M}_r \times \prod_{j=1}^r \overline{M}_{|I_j|}(C))$ whose generic element is a curve $\hat{C}$ with infinite differential $= 1$ on the one unmarked component $C_0 = C$ and finite differentials on the remaining components $C_1;\ldots;C_r$.

Theorem 1.10 (Adiabatic limit theorem). If $X/\!/G$ is a locally free quotient then all automorphism groups are finite for sufficiently large (more precisely, for any class $d \in H^G_2(X;\mathbb{Z})$ there exists an $r > 0$ such that $> r$ implies that all automorphism groups are finite) and

$$_{X/\!/G}\ _X = \lim_{\to \infty}\ ^G_X$$

in the following sense of Taylor coefficients: For any class $\in \overline{M_{n;1}}(C)$ let

$$\sum_{k=1}^{l} \sum_{I_1 \cup \ldots \cup I_r = \{1;\ldots;n\}} \prod \ ^k_1 \ldots ^k_r;\ ^k_0$$

be its restrictions to

$$H\left(\overline{M}_r(C) \times \prod_{j=1}^r \overline{M}_{|I_j|}(C)\right); \quad \text{resp. } H(\overline{M}_n(C))$$

respectively. Then

$$\sum^l_{k=1} {}_{X/\!/G}(;_1)\ ^{G;|I_j|}_{k}(;_X) = \lim_{\to\infty} \ ^{G;n}_k(;_0):$$

In other words, the diagram

$$
\begin{array}{ccc}
QH_G(X) & \xrightarrow{\ \ X\,G\ \ } & QH_C(X{=}G) \\
& {\scriptstyle X\,G}\searrow \quad \swarrow{\scriptstyle X=G} & \\
& X\ G &
\end{array}
\tag{11}
$$

commutes in the limit ! 1 .

Remark 1.11. We often take in examples as insertion the class 2 $H^6(\overline{M}_{3;1}(C))$ pulled back from the point class $_0$ 2 $H^6(\overline{M}_3(C))$ under the forgetful map : $\overline{M}_{3;1}(C)$! $\overline{M}(C)$, in which case the class $_{3;1}$ is also the point class which restricts to the point class $_0$ 2 $H^6(M_3(\overline{C}))$ in which case the classes $_1$ is also the point class and the classes $_j$ trivial, since ^{1}(pt) contains a single curve with innite scaling, consisting of one innitely-scaled components and three nitely-scaled components each with a single marking. For C = P the projective line the result is a compari-son between the three-point Gromov-Witten invariants in X=G and the three-point gauged Gromov-Witten invariants in X . See Example 1.16 below.

1.3. Quantum Kalkman formula. In order to study the dependence of the Gromov-Witten graph potential of the quotient on the choice of polarization, suppose that L ! X are two polarizations of X and

$$L_t := L^{(1\ \ t)=2} \qquad {}_{+} \qquad L^{(1+t)=2}$$

is the family of rational polarizations given by interpolation. Let X=G denote the git quotients,

$$_{X;}^{G} : QH_G(X;L)\ !\ \ QH_C(X{=}G)$$

the quantum Kirwan maps (note that we do not introduce new notation for the quantum version; the classical Kirwan map is obtained by setting q = 0) for the two polarizations and

$$_{X=G} : QH_C(X{=}G)\ !\ \ _{X;L}\,\text{the}^{G}$$

graph potentials. Denote by

$$QH_G^n(X)\ \ QH_G(X;L\ \ )\setminus QH_G(X;L_+)$$

the subset of the quantum cohomology of nite sums in the Novikov variable. The main result of this note is a formula for the dierence

$$_{X=_+G}\ _{X;+}^{\ \ G} \qquad _{X=\ \ G}\ _{X;}^{\ \ G} : QH_G^n(X)\ !\ \ _X^{\ \ G}$$

as a sum of xed point contributions given by gauged Gromov-Witten invariants with smaller structure group. Namely, for any non-zero 2 g generating a one-parameter subgroup C denote by X the xed point set of C generated by and G the centralizer of . The adjoint action of on g is semisimple and the Lie algebra g of G is then

$$g = f x \; 2 \; g \; j \; [x;] = 0g:$$

It follows that G is reductive.

Denote by $X^{;t} = X$ the locus that is semistable with respect to L_t. For any t 2 (1; 1) such that $X^{;t}$ is non-empty, we introduce in Denition 3.16 a stack $\overline{M}_n^G(C; X; L_t;)$ of reducible n-marked gauged maps from C to X=G consisting of a principal component $C_0 = C$ mapping to $X^{;t}$=G and bubbles $C_1; : : : ; C_k$ C mapping to X=G. This stack admits a perfect equivariant obstruction theory whose relative part is the cone on the map $(RpeT_{X=G})^{-}$! C^{-} given by the innitesimal action, see Remark 1.7 above. (The complex $(RpeT_{X=G})^{-}$ is not perfect because of the C-automorphisms; taking the cone has the eect of cancelling this additional automorphism.) Denote by $_t$ the virtual normal complex, given as the moving part of the obstruction theory on ~~$M_n^G(C; X; L_t)$ pulled back to M_n^G~~$(C; X; L_t;)$ and by

$$Eul(_t) \; 2 \; H(\overline{M}_n^G(C; X; L_t;))[; \; ^1]$$

its (invertible) Euler class, where is the equivariant parameter. The Mundet stability condition in general depends on a choice of equivariant Kähler class; here we are interested in the large area limit in which the gauged Gromov-Witten invariants are related to the Gromov-Witten invariants of the git quotient. Denote by

$$\tilde{}_X^G := Map(H_2^G(X; Z); Q); \qquad \tilde{}_X^G[; \; ^1] := Map(H_2^G(X; Q); Q[; \; ^1])$$

the space of Q resp. Q[; ^{1}]-valued functions on $H_2^G(X; Z)$. Note that $\tilde{}_X^G$ has no ring structure extending that on $_X^G$. The space $\tilde{}_X^G$ can be viewed as the space of dis-tributions in the quantum parameter q, and we use it as a master space interpolating Novikov parameters for the quotients as t varies.

Denition 1.12 (Fixed point potential). Let X; G; L be as above, and 2 g; t 2 (1; 1) such that $X^{;t}$ is non-empty. The xed point potential associated to this data is the map

$$(12) \quad _{X;G;t;} : QH_{G;n}(X) \; ! \; \tilde{}_X^G[; \; ^1]$$
$$! \sum_{d \in H_2^G(X;Z)} \sum_{n \geq 0} \int_{[\overline{M}_n^G(C;X;L_t;;d)]} \frac{q^d}{n!} ev(; : : : ;) \cup Eul(_t)^{\; 1} \cup f_n$$

for 2 $H_G(X)$, extended to $QH_{G;n}(X)$ by linearity.

We may now state the main result of the paper, in the case of torus actions.

Theorem 1.13 (Quantum Kalkman formula, abelian case). Suppose that G is a torus and X is a smooth projective G-variety equipped with polarizations $L_{\pm} \to X$, and all automorphism groups are finite for the polarization L. Then

$$(13) \qquad {}_{X=_+G} \; {}_{X;+}{}^{G} \quad {}_{X=\,G} \; {}_{X;}{}^{G} = \sum_{t \in (\;1;1)} \sum_{2g=G} \mathrm{Resid}\; {}_{X;G;t;} :$$

In other words, failure of the following square to commute is measured by an explicit sum of wall-crossing terms given by certain gauged Gromov-Witten invariants:

$$\begin{array}{ccccc}
QH_G(X;L\;) & \longleftarrow & QH^n_G(X) & \longrightarrow & QH_G(X;L_+) \\
\downarrow {\scriptstyle X\,G} & & & & \downarrow {\scriptstyle X\,G+} \\
QH_C(X=\;G) & & & & QH_C(X=_+G) \\
\downarrow {\scriptstyle X=\;G} & & & & \downarrow {\scriptstyle X=_+G} \\
{}_{X;}{}^{G} & \longrightarrow & {}_{X}^{G} & \longleftarrow & {}_{X;L_+}^{G} :
\end{array}$$

The diagram is somewhat more complicated than in the classical case in (11), because the maps ${}_{X=G}\;{}_{X;}^{G}$ are defined using different Novikov rings ${}_{X;L}^{G;0}$. If a symbol q^d appears in one Novikov ring ${}_{X;L}^{G;0}$ but not the other then the correspond-ing contribution to ${}_{X=\;G}\;{}_{X;}^{G}$ must be equal to the q^d-term in the wall-crossing contribution on the right. See Example 1.16.

The wall-crossing formula for Gromov-Witten invariants Theorem 1.13 should be considered mirror to various results in on the behavior of the derived category of bounded complexes of coherent sheaves under variation of git quotient appearing recently in Segal [57], Halpern-Leistner [33] and Ballard-Favero-Katzarkov [2], and more generally for crepant birational transformations, earlier in [38].

Remark 1.14. The fixed point potential ${}_{X;G;t}^{G}$ quantizes the fixed point contributions in Kalkman's formula (2), in the following sense: Let C be a genus zero curve and consider the n-th Taylor coefficient ${}_{X;t;}^{G;n}$. Consider the integral with insertion of the class ${}_n \in H(\overline{M}_n(C))$ given by

$${}_{X;t;}^{G;n}(;\,{}_n) := \sum q^d \int_{[M_n(C;X;L_t;;d)]^G} ev_n^*[\; f_n^*[\; \mathrm{Eul}(\,{}_t)^{-1}: {}_d$$

If ${}_1 \in H(\overline{M}_1(C)) = H(C)$ is the point class then

$$(14) \qquad {}_{X;t;}^{G;1}(;\,{}_1)|_{q=0} = \int_{[X;t]} [\;\mathrm{Eul}_G(\,{}_{X;t})^{-1}$$

which is the contribution from the xed point components appearing in Kalkman's wall-crossing formula. Indeed any map from the genus zero curve C to $X=G$ of homology class 0 consists of a trivial bundle and constant section. It follows that

$$\overline{M}_n^G(C; X; L_t; ; 0) = X^{;t} \; M_n(\overline{C}): \text{In}$$

particular, for $n = 1$ we obtain

$$\overline{M}_1^G(C; X; L_t; ; 0) = X^{;t} \; C$$

which implies the claim. Furthermore, the Euler class $Eul(_t)$ is the Euler class of the virtual normal complex to $X^{;t}$.

Remark 1.15. We have stated the formula (13) in its simplest form; there are various extensions which include:

(a) (Twistings by Euler classes) One can introduce twisted gauged Gromov-Witten invariants as follows. The universal curve

$$p : \overline{C}_n^G(C; X) \; ! \; \overline{M}_n^G(C; X)$$

admits a universal gauged map

$$e : \overline{C}_n^G(C; X) \; ! \; X=G:$$

For any complex of G-equivariant vector bundles $E \; ! \; X$ denote by

$$\text{(15)} \qquad Ind(E) := Rp e E$$

the index of the complex $E=G \; ! \; X=G$. The complex $Ind(E)$ is an object in the bounded derived category of $\overline{M}_n^G(C; X)$. Indeed, p is a local complete intersection morphism [17, Appendix] and so $Rp e E$ admits a resolution by vector bundles. The Euler class

$$\text{(16)} \qquad (E) := Eul_C(Ind(E)) \; 2 \; H(\overline{M}_n^G(C; X))[; \; ^1]$$

is well-dened $2 \; H^2_C(pt)$ is the equivariant parameter. For any equivariant bundle E on X, inserting the Euler class of (16) gives twisted gauged Gromov-Witten invariants. Introducing similar twistings in the denition of ^G_X, the wall-crossing formula extends to this case as well.

(b) (Wall-crossing for individual Gromov-Witten invariants) Although we have written the formula (13) as a dierence of potentials, after unraveling the denitions one obtains wall-crossing formulas for individual Gromov-Witten invariants, or at least nite combinations of them. See Example 1.2 below.

(c) (Wall-crossing for non-convex actions) In some cases, the action of G on X is not convex at innity (e.g. G is a torus acting on a vector space X with weights $_1; : : : ;_k \; 2 \; g_-$ not contained in any open half-space $H \quad g_-$) and the moduli spaces $\overline{M}_n^G(C; X)$ are non-compact. Often, the moduli spaces

$\overline{M}^G_n(C;X)$ admit an auxiliary group action with proper xed point loci, and thus the maps $_{X=G}$; ^G_X etc. can be dened via localization. The wall-crossing formula holds in this case as well, as long as the auxiliary group action extends to the various master spaces involved and the xed point loci on these master spaces is proper (after xing the homology class of the gauged map.) See Example 1.19 below.

Example 1.16. (Three-point Gromov-Witten invariants of projective space) The following simple example illustrates the notation involved in Theorem 1.13. As in Example 1.2, let $G = C$ acting on $X = C^k$ by scalar multiplication, so that $H_G(X) = Q[]$ where is the equivariant parameter. Suppose that polarizations L correspond to the characters 1 and

$$X= G = ;; \quad X=_+G = P^{k \; 1}$$

and the two chambers are separated by the value $t = 0$ so that $0 \; 2 \; X$ is semistable for $L^{(1 \quad t)=2}$ $L^{(1+t)=2}_+$. Let $! \; 2 \; H^2(X=_+G)$ denote the hyperplane class. For integers $a;b;c \quad 0$ we compute genus 0, $d = 1$, $n = 3$ invariants $h!^a;!^b;!^c i_{0;1}$ of $P^{k \; 1}$ via wall-crossing; this was already covered in Cieliebak-Salamon [13]. Of course from the elementary computation of quantum cohomology of projective space one knows that

$$h!^a;!^b;!^c i_{0;1} = \begin{cases} 1 & a+b+c = 2k \quad 1 \\ 0 & \text{otherwise:} \end{cases}$$

First we relate the above three-point invariant to a gauged invariant. Since $c_1^G(X) = k$, the minimal Chern number of X is k. For dimensional reasons, there are no quantum corrections in $D_0{}^G_X(^i)$; $i \quad k \quad 1$. Thus

$$D_0{}^G_X(^i) = !^i; \; i \quad k \quad 1:$$

The adiabatic limit Theorem 1.10 with insertions implies that the genus zero, three-point invariants $h!^a;!^b;!^c i_{0;d}$ of class $d \; 2 \; H_2(X=G) = Z$ in the quotient $X=G$ equal gauged Gromov-Witten invariants:

$$h!^a;!^b;!^c i_{0;d} = \int_{[\overline{M}^G(P;X;d)]} ev_1^a \,[\; ev_2^b \,[\; ev_3^c \,[\; f_3 \,[M_3$$

where P is the projective line and $_3 \; 2 \; H^6(\overline{M_3}(P))$ is the point class, that is, the class xing the location of the three marked points.

We apply the wall-crossing formula Theorem 1.13 to compute the gauged invariant. There are no holomorphic spheres in X, so the moduli stack $\overline{M}^G_0(P;X;L_t;d)^G$ is a point, for $t = 0$, consisting of the bundle P with rst Chern class $c_1(P) = d \; 2$

$H^2_G(X;Z) = Z$ with constant section equal to zero. Thus the xed point stack is

$$\overline{M}^G_3(P;X;L_{t=0};;d) = \overline{M}_3(P):$$

for any non-zero 2 g. The index bundle of TX at the xed point for $d = 1$ is

$$\mathrm{Ind}(T_{X=G})j_0 = H^0(P;O_P(1)_C C^k) = C^{2k}:$$

It has Euler class $\mathrm{Eul}(_{t=0}) = {}^{2k}$: By the wall-crossing formula 1.13

$$h!^a;!^b;!^c i_{0;1} = \mathrm{Resid} \int_{[\overline{M}^G_3(P;X;L_0;;1)]} \frac{ev_1^a [\; ev_2^b [\; ev_3^c [\; f_3}{\mathrm{Eul}(\;_{t=0})}$$
$$= \mathrm{Resid}^{\;a+b+c=2k}$$
$$= \begin{cases} 1 & a+b+c = 2k \quad 1 \\ 0 & \text{otherwise:} \end{cases}$$

Thus

$$!^a \; ? \; !^b = q!^{a+b \quad k}; \quad k \quad a+b \quad 2k \qquad 1$$

as is well-known. For example, taking $a = b = k \quad 1$ we obtain that there is a unique line in projective space passing through two generic points and a generic hyperplane. We give another Fano example in Example 1.2, where we compute the change in a coecient in the fth quantum power of the rst Chern class for a blow-up of the projective plane. This ends the example.

The most interesting case of the wall-crossing formula Theorem 1.13 is the crepant case by which we mean that the sum of the weights at any xed point vanishes (Denition 4.2); the term crepant was introduced by Reid [55] as the opposite of discrepant in the context of the minimal model program. In the last section we give a proof of a version of the crepant transformation conjecture of Li-Ruan [46], Bryan-Graber [10], Coates-Corti-Iritani-Tseng [16], Coates-Ruan [15] on equivalence of Gromov-Witten theories in this case: We say that two elements of $\mathrm{Map}(H_2(X=G;Q);Z)$ are equal almost everywhere (a.e.) in the quantum parameter q if their dierence is an element of the form $\sum f()q$ with $f(+d); d \; 2 \; Z$ polynomial in d. If, as distributions, these functions are tempered then by Fourier transform the dierence is supported on a set of measure zero.

Theorem 1.17 (Wall-crossing for crepant birational transformations of git type). Suppose that $X;G$ are as in Theorem 1.13, and C has genus zero. If all the wall-crossings are crepant then

$$X = \; G \quad X; \quad \overset{G}{=}_{\,a:e} \; X =_+ G \quad X;_+ : \qquad G$$

Remark 1.18. (a) Theorem 1.17 implies many of the special cases already known in the literature, although actually computing the transformations ${}^{G}{}_{X;}$ relating the graph potentials ${}_{X= \; G}$ can be a dicult task. Note that Iwao-Lee-Lin-Wang [36], Lee-Lin-Wang [43] extend the invariance to cases not necessarily related by variation of git, while the results here are more general than that of [43], [36] since we allow \weighted ops". More recently Coates-Iritani-Jian [19] have proved a version of the crepant transformation conjecture for toric complete intersections, which overlaps in many cases with the results here.

(b) The results here are for the graph potential, whereas the results in Coates-Iritani-Jiang [19] are for the fundamental solution. One natural expects the results here to extend to the case of localized graph potentials (fundamental solutions) using the results of Halpern-Leistner [33] and Ballard-Faver-Katzarkov [2]. We hope to return to this in future work.

(c) Almost everywhere equality in the formal parameter q means the following: considering both sides as elements in $\mathrm{Map}(H_2^G(X;Z)=\mathrm{torsion};Q)$, the dierence is a polynomial in at least one direction. In particular, in the case of a single quantum parameter the dierence is tempered distribution its Fourier transform in that direction has support of measure zero, see Section 4.

(d) The proof of Theorem 1.17 uses an action of the Picard stack of the curve on the xed point stacks, see Lemma 4.5. In the crepant case the (almost) invariance under this action implies that, after summing over degrees, the wall-crossing term is a sum of derivatives of delta-functions in the quantum parameter.

Example 1.19. (Simple three-fold op, cf. Li-Ruan [46], Iwao-Lee-Lin-Wang [43], Lee-Lin-Wang [36]) The following simple example may help to explain the notion of \almost everywhere vanishing" of the wall-crossing contributions in the quantum parameter. Let $G = C$ acting on $X = C^4$ with weights 1 each of multiplicity 2. Let L be the trivial polarization and L_t the trivial bundle shifted by tensoring with a representation of weight t. The invariant sections of L_t are spanned by monomials

$$z_1^{d_1} z_2^{d_2} z_3^{d_3} z_4^{d_4} \; 2 \; H^0(L_t)^G; \qquad d_1 + d_2 \quad d_3 \quad d_4 = \quad t:$$

Thus the semistable locus $X^{ss;t}$ is either $(z_1; z_2) = 0$ or $(z_3; z_4) = 0$ depending on the sign of t, and the git quotients $X{=}G$ factor over $P = (C^2 \quad f0g)=C$ giving identications

$$X{=}G = O_P(\quad 1)^2;$$

where P is the projective line and we abuse notation by denoting by $O_P(\quad 1)^2$ the total space of two copies of the tautological line bundle $O_P(\quad 1)$. The quotients $X{=}_+G$ and $X{=} \quad G$ are isomorphic but the birational transformation relating them, induced by the variation of git quotient, is a simple op.

We consider the wall-crossing formula corresponding to the three-point invariants with each insertion given by the hyperplane class. In order to make sense of the non-compact integration in the d = 0 case one must take an equivariant extension with respect to the action of C acting by scalar multiplication, and use localization for the residual C-action. Thus if is the equivariant parameter for the auxiliary C and 2 $H^2_G(X)$ is the hyperplane class,

$$_X^G(+) = ! +$$

where ! 2 $H^2(X=G)$ is the symplectic class integrating to 1 on the zero-section P $O_P($ $1)^2$. The xed point set of the residual C action is the zero section of X=G, which acts with weights 2 (with multiplicity two) on the normal bundle. The degree zero moduli space is X=G itself, and integration of = ^{3} over this non-compact space can be dened via localization as

$$\begin{aligned} I \quad := & \int_P {}^G_X() := \int_P (! +)^3 \, [\; Eul(O_P(\; 1)^2) \; ^1{}_{X=G} \\ = & \int_P \frac{(! +)^3}{(\; ! \; 2)^2} \\ = & \int_P 3!^2(2) \; ^2 + {}^3(2) \; ^2(1 \qquad !=(2) + ::::)^2 \\ = & \int_P \frac{}{4} \; \frac{3!}{4} \; \frac{!}{4} \\ = & \; 3=4 \;\; 1=4 = \;\; 1=2: \end{aligned}$$

The auxiliary circle action naturally extends to the various master spaces involved, and the wall-crossing formula holds in this case as well, dening each contribution via localization. There is a unique xed point 0 2 X, with normal bundle isomorphic to X. Thus the wall-crossing term is

$$\text{Resid} \; ^3=(^2(\quad)^2) = \; 1:$$

Thus Kalkman's wall-crossing formula reduces to

$$I_+ \quad I \quad = (1=2) \quad (\quad 1=2) = \; 1:$$

We now study wall-crossing for invariants of positive degree. For each d 2 $H^G_2(X;Z)$ = Z, there is a unique gauged map u : C ! X=G of class d mapping C to the xed point 0 2 X. Its normal complex RpeT X has weight 1 with multiplicity 2 + 2d and weight 1 with multiplicity 2 2d, by Riemann-Roch. Thus the wall-crossing term for class d is

$$\text{Resid} \; ^3=(^{2+2d}(\quad)^{2 \quad 2d}) = \; 1:$$

The class d occurs in the expression $_{X=GX;}$ i d 0. Indeed, the contribution of q^d is the contribution of q^d to the gauged Gromov-Witten potential in Theorem 1.10, which by denition is an integral over Mundet stable maps $\overline{M}^G(C;X;d)$, which in this case is the git quotient of

$$\overline{M}^G(C;X;d) = H^0(O_C(d)\ _C\ (C^2\ \ C_1^2))=C\ \ _1$$

by the C action corresponding to the polarization L, where C_1 denote the one-dimensional representations with weight 1. For d = 0, only one factor

$$H^0(O_C(d)\ _C\ (C^2)) =_1 C_1^{2(jdj+1)}$$

is non-vanishing. It follows easily that the moduli space of gauged maps of class d is empty for the polarization L corresponding to the character 1. Each $q^d; d = 0$ appears in only one of the Novikov rings $^{G;0}_{X;}$. Thus the higher degree integrals for L_+ are 1 for d > 0 and 0 for d < 0 resp. for L are 1 for d < 0 and 0 for d > 0, each corresponding to an integral over multiple covers of the zero section. Summing over classes d the wall-crossing formula for gauged invariants in Theorem 1.13 becomes

$$\left(\frac{1}{2} + \sum_{d>0} q^d\right) \quad \left(\frac{1}{2} \sum_{d<0} q^d\right) = \sum_{d 2 Z} q^d \underset{a:e:}{=} 0:$$

The reader may compare with the treatment of simple ops in [46], [43, Corollary 3.2] which contains essentially the same computation. Note that here we have not given an explicit description of the maps $^G\ _{X;}$ which relate the two graph potentials. This ends the example.

2. Kalkman's wall-crossing formula

In this section we give a proof of Kalkman's formula 1.1 rst for circle group actions and then for the general case.

2.1. The wall-crossing formula for circle actions. The wall-crossing formula is somewhat simpler for the case of a circle group, so we begin with that case. Let G = C and X a smooth projective G-variety as above, equipped with polarizations L ! X. The proof of the wall-crossing formula is by localization on a proper Deligne-Mumford stack $\tilde{X}$ whose xed points include X=G and the xed point components $X^{G;t}$ with t 2 (1; 1). From the point of view of symplectic geometry, this is the symplectic cut construction in Lerman [44], but we need the algebraic approach here given in Thaddeus [59, Section 3].

Recall from the introduction the notation for variation of git quotient. Let

$$L_t := L^{(1\ \ t)=2}\ _+\ L^{(1+t)=2}; t 2 (\ \ 1; 1) \setminus Q$$

denotes the family of rational polarizations interpolating between L. Denote by $X/\!/_t G$ the corresponding git quotients, by which we mean the stack-theoretic quotient $X^{ss;L_t}/G$ of the semistable locus $X^{ss;L_t}$ for L_t by the action of G. (Most authors would enclose a stack-theoretic quotient by square brackets, which we omit since we always mean stack-theoretic quotient unless otherwise stated.) In symplectic geometric terms, this means that if

$$: X \; ! \; \mathfrak{g}_R^{\;-}$$

are moment maps for action of a maximal compact G_R of G on L with respect to a unitary connection then

$$_t := \frac{1 \quad t}{2} + \frac{1 + t}{2}_+ : X \; ! \; \mathfrak{g}_R^{\;-}$$

is a moment map for the action of G on L_t. Even more concretely, if are equal up to a constant $c = \; _+$ then the maps

$$_t = \frac{+ \; _+}{2} + \frac{t}{2}c$$

are all equal up to a constant $\frac{t}{2}c$ depending on t. For the following see Thaddeus [59, Section 3].

Lemma 2.1 (Existence of a master space). Suppose that G acts with nite stabilizers on the semistable loci $X^{ss;}$ and for any $t \; 2 \; (\; 1;1)$ and any t-semistable point $x \; 2 \; X$, G_x acts with nite stabilizer on the ber $(L_+ \; L^{\;1})_x$. There exists a smooth proper Deligne-Mumford C-stack $\tilde{X}$ equipped with a line bundle ample for the coarse moduli space whose git quotients $\tilde{X}/\!/_t C$ are isomorphic to those $X/\!/_t G$ of X by the action of G with respect to the polarization L_t and whose xed point set $\tilde{X}^C$ is given by the union

$$\tilde{X}^C = (X/\!/ \; G) \; [\; (X/\!/_+ G) \; [\; @ \bigcup_{t2(\; 1;1)} X^{G;t} A$$

where $X^{G;t}$ is the component of X^G that is semistable for parameter t. Furthermore, the normal bundle of $_{G;t}$ of $X^{G;t}$ in $\tilde{X}$ is isomorphic to the normal bundle of $X^{G;t}$ in X, equivariantly after the identication $G = C$.

Proof. The projectivization $P(L \;\; L_+)$ of the direct sum $L \;\; L_+$ of the polariza-tions L $! \; X$ has a natural polarization given by the relative hyperplane bundle $O_{P(L \;\; L_+)}(1)$ having bers

(17) $$O_{P(L \;\; L_+)}(1)_{[l \;\; ;l_+]} = \operatorname{span}(l \;\; + l_+):$$

Let

$$: P(L \;\; L_+) \; ! \; X$$

denote the projection. (A word of warning: the notation π will be used for a number of different projections in this paper.) The group $\mathbb{C}^\times$ acts on $\mathbb{P}(L_- \oplus L_+)$ by

$$w[l_-; l_+] = [l_-; w l_+], \quad w \in \mathbb{C}^\times, l_\pm \in L_\pm .$$

For $k \geq 0$ the space of sections of $\mathcal{O}_{\mathbb{P}(L_- \oplus L_+)}(k)$ has a decomposition

$$H^0(\mathbb{P}(L_- \oplus L_+); \mathcal{O}_{\mathbb{P}(L_- \oplus L_+)}(k)) = \bigoplus_{k_- + k_+ = k} H^0(X; L_-^{k_-} \otimes L_+^{k_+}) \tag{18}$$

under the natural $\mathbb{C}^\times$-action with eigenspaces given by the sections of

$$L_-^{k_-} \otimes L_+^{k_+}, \quad k_- + k_+ = k, k \geq 0.$$

The G-semistable locus in $\mathbb{P}(L_- \oplus L_+)$ is the union of non-vanishing loci of non-zero invariant eigensections. Hence

$$\mathbb{P}(L_- \oplus L_+)^{ss} = X^{ss,-} \cup X^{ss,+} \cup \bigcup_{t \in [-1,1]} \pi^{-1}(X^{ss,t}) \cap (\mathbb{P}(L_- \oplus L_+) - \mathbb{P}(L_-) - \mathbb{P}(L_+)) \tag{19}$$

where $X^{ss,t} \subset X$ is the semistable locus for L_t and $X^{ss,\pm}$ are considered subsets of $\mathbb{P}(L_- \oplus L_+)$ via the isomorphisms $X \to \mathbb{P}(L_\pm)$. Let

$$\tilde{X} := \mathbb{P}(L_- \oplus L_+) /\!/ G$$

denote the geometric invariant theory quotient, by which we mean the stack-theoretic quotient of the semistable locus. The assumption on the action of the stabilizers in Lemma 2.1 implies that the action of G on the semistable locus in $\mathbb{P}(L_- \oplus L_+)$ has only finite stabilizers. It follows that $\tilde{X}$ is a proper smooth Deligne-Mumford stack. The quotient $\tilde{X}$ contains the quotients of $\mathbb{P}(L_\pm) = X$ with respect to the polarizations $L_\pm$, that is, $X_\pm /\!/ G$.

Next we describe the fixed point set. The fixed points for the action of $\mathbb{C}^\times$ on $\tilde{X}$ are represented by pairs $[l_-; l_+]$ with a positive dimensional stabilizer under the action of $G \times \mathbb{C}^\times$. Necessarily either $l_- = 0, l_+ = 0$, or l_-, l_+ are both non-zero but the projection to X is G-fixed. In the latter case, semistability implies that there exists an invariant section of $L_t = L_-^{(1-t)/2} \otimes L_+^{(1+t)/2}$, $t \in [-1,1]$, non-vanishing at x. Since $l_\pm$ are both non-zero, the weight t of $\mathbb{C}^\times$ on the fiber cannot be in $\{-1, 1\}$, hence x is t-semistable for some $t \in (-1,1)$. The normal bundle of $X^{\mathbb{C}^\times}$ lifts to the normal bundle $\tilde{\nu}$ for the fixed point set some one-parameter subgroup in $\mathbb{P}(L_- \oplus L_+)$, which projects to the normal bundle in $X^{\mathbb{C}^\times}$. Both projections have trivial fiber, hence the claim on normal bundles.

Proof of the classical Kalkman's wall-crossing Theorem 1.1. First note that the integrand in the wall-crossing formula lifts to the master space in a natural way: The projection $P(L \quad L_+)$! X is G-equivariant and C-invariant. Composing the pull-back map

$$H_G(X) \; ! \; H_{GC}(P(L \quad L_+))$$

with the Kirwan map

$$H_{GC}(P(L \quad L_+)) \; ! \; H_C(X)$$ one

obtains a canonical map

$$: H_G(X) \; ! \; H_C(X):$$

The composition of with the Kirwan map

$$\sim^C_{X;t} : H_C(X) \; ! \; H(X=_tC) = H(X=_tG) \text{ is}$$

pull-back to the t-semistable locus and so equal to

$$\sim^C_{X;t} = {}_{X;t}{}^G : H_G(X) \; ! \; H(X=_tG):$$

In particular, () 2 $H_C(X)$ restricts to $_{X;}$ on the two distinguished xed point components $X=G \quad X^C$.

Taking the residue of the localization formula for the circle action on the master space gives Theorem 1.1, see Lerman [44]. Indeed for any equivariant class 2 $H_G(X)$ of top degree, its pullback to $P(L \quad L_+)$ descends to a class () 2 $H_C(X)$ whose restriction to $X=G$ is $^G_{X;}$ (), and whose restriction to $X^{G;t} \quad X^G$ is $_{X^{G;t}}$. Since

$$\deg(()) = \deg() = \dim(X) \qquad 2;$$

the integral of () over X vanishes. On the other hand, by localization the integral of () is

$$(20) \qquad \int_{[X]} () = \int_{[X=G]} \frac{{}^G_{X;}()}{\mathrm{Eul}_G(\;)} + \int_{[X=_+G]} \frac{{}^G_{X;+}()}{\mathrm{Eul}_G(_+)} + \sum_{t\in(1;1)} \int_{[X^{G;t}]} \frac{X^{G;t}}{\mathrm{Eul}_G(X^{G;t})}$$

where ! $X=G$ are the normal bundles to $X=G$ in X. Since the normal bundle of the "sections at zero and innity" $P(L)$ in $P(L \quad L_+)$ may be canonically identied with $(L_+ \quad L^1)^1$, the group C acts on with weights 1. Hence the inverted Euler classes are

$$\mathrm{Eul}_G()^{\;1} = (\; + c_1())^{\;1} = {}^1(1 + c_1()()^{\;1} + :::):$$

Taking residues on both sides of (20) one obtains

$$0 = {}_{X=_+G}\,{}_{X;+}{}^G + {}_{X=\;G}\,{}_{X;} + \mathrm{Resid} \sum_{t\in(1;1)} \int_{[X^{G;t}]} \frac{X^{G;t}}{\mathrm{Eul}_G(X^{G;t})}$$

as claimed.

2.2. **Kalkman wall-crossing for actions of non-abelian groups.** Kalkman's wall-crossing formula can be used to study the intersection pairings for variation of git for the action of an arbitrary connected complex reductive group G on a smooth polarized projective variety X as follows. As above, let

$$\tilde{X} = P(L \quad L_+) = G:$$

We examine the structure of the xed point set of C on X . For any 2 g, denote by G G the stabilizer of the line C under the adjoint action of G. Recall from the introduction that

$$X = fx\ 2\ X \ j\ zx = x;\ 8z\ 2\ C\ g$$

is the xed point set of the one-parameter subgroup C . Since G commutes with C , it acts on X . The t-semistable locus

$$X^{;t} \quad X$$

has, by assumption, the property that the action of G=C has nite stabilizers, that is, g_x = C for all x 2 $X^{;t}$: It follows the owout of the semistable locus is

$$GX^{;t} = G \ _G \ X^{;t}:$$

In particular, there exists a canonical action of g=g (considered as an abelian group) ber-wise on the normal bundle $_{X^{;t}}$. Denote by $_{X^{;t}}$=(g=g) the quotient by the action.

Lemma 2.2 (Structure of the xed point components). *Suppose that stable=semistable for the G-action on P(L L₊), so that X is a smooth proper Deligne-Mumford stack with C action, constructed in the proof of Lemma 2.1.*

(a) For any xed point $\tilde{x}$ 2 X^{C} with $\tilde{x}$ = [l] for some l 2 P(L L₊), there exists 2 g such that

$$8z\ 2\ C; \qquad zl = zl:$$

(b) Any xed point $\tilde{x}$ 2 $\tilde{X}^{C}$ is equal to [l] for some l 2 P(L L₊) in the ber over x 2 X that is t-semistable for some t 2 (1; 1) and has stabilizer generated by 2 g, with the property that the weight of the one-parameter subgroup generated by on $(L^{(1 \quad t)=2}_{+} \ L^{(1+t)=2})_x$ vanishes:

$$zl = l; \qquad 8l\ 2\ (L^{(1 \quad t)=2}_{+} \ L^{(1+t)=2})_x:$$

(c) Denote by $X{=}_t(G{=}C\ )$ the git quotient of X by the group G=C with respect to the polarization determined by the restriction of $L^{(1 \quad t)=2}_{+} \ L^{(1+t)=2}$. For each 2 g generating a one-parameter subgroup C there is a morphism

$$: X{=}_t(G{=}C\ )\ ! \quad \tilde{X}^{C}:$$

The images of all the cover X^{C}, disjointly after passing to conjugacy classes of one-parameter subgroups C .

(d) For any 2 $H_G(X)$, the pull-back of ~()jX^C under is equal to image of under the restriction map $H_G(X)$! $H_C(X=_t(G=C))$.

(e) The pull-back of the normal bundle $_{X^C}$ of X^C under is isomorphic to the quotient of $_{X;t}=(g=g)$ by a fractional action of G=C, via an isomorphism that intertwines the action of C on $(_{X;t}=(g=g))=(G=C)$ with the action of C on $_{X^C}$.

Proof. (a) Denote by P(L L₊) the complement of P(L) [P(L₊) in P(L L₊). If [l] 2 X^C, with l 2 P(L L₊) then [l] 2 X , where is a generator of the Lie algebra of C and X is the zero set of the vector eld ~ generated by . Since X is the quotient of P(L L₊) by G, if $_L$ denotes the vector eld on P(L L₊) generated by then $_L(l)$ = $_L(l)$ for some 2 g. Since G acts locally freely must be unique. Integrating gives z l = zl for all z 2 C, hence (a). (b) is a consequence of (a) and the denition of semistability in terms of invariant sections. Item (c) is a consequence of items (a) and (b). Item (d) follows from the fact that ~ is pullback to $X=_t(G=C)$. For (e), the normal bundle $_{X^C}$ of X^C restricted to the image of is isomorphic to the quotient of the normal bundle of G_G P(L L₊) by G, which in turn is isomorphic to the quotient of the normal bundle of P(L L₊) by g=g. The projection to X identies this normal bundle with the normal bundle $_X$ to X quotiented by g=g. Thus $_{X^C}$ is isomorphic to a quotient of $_X$ =(g=g) by G=C .

Remark 2.3. An anonymous referee has pointed out the following alternative perspective on the xed point loci. Consider the G-equivariant bundle

$$\tilde{} := {}_{X;t}=(g=g) \,!\, X^{;t}:$$

Taking quotients gives a vector bundle ~=G ! $X^{;t}$=G over the Artin stack $X^{;t}$=G. On the other hand, we have a C-bundle

$$P := (L \; L^{1}) \,!\, X^{;t}$$

which is a sub-bundle of P(L L₊)j $_{;t}$, and where superscript denotes removal of the zero section. The map P ! $X^{;t}$ is equivariant for the G-action and the Deligne-Mumford stack P=G gives a component of X^C. These spaces t into a diagram (quotients being stack-theoretic)

$$\begin{array}{ccc} P=G & \hookrightarrow & X^{\sim C}\ p \\ \downarrow & & \\ X^{;t}=G: & & \end{array}$$

The pull-back of the vector bundle ~=G ! $X^{;t}$=G by p is isomorphic to the restriction of the normal bundle $_{X\,C\sim}$ to the component P=G:

$$p[\sim=G] = {}_{X\,C}\, j_{P=G}:$$

Here the left-hand side p[~=G] is naturally a C-equivariant bundle over P=G (since p~ is a G C-equivariant bundle over P) and this isomorphism intertwines the C-actions. The various pull-backs t into a commutative diagram of maps in equivariant cohomology

$$\begin{array}{ccccccc} H_G(X) & & \xrightarrow{\quad\sim\quad} & & & & H_C(X)^{\sim} \\ \downarrow & \searrow & & & & & \downarrow \\ H_G(X^{;t}) & \xrightarrow{p} & H_{GC}(P) & = & H_C(P=G) & \longleftarrow & H_C(X^{C})^{\sim} \end{array}$$

It follows that the xed point contribution from $X^{;t}$ in the wall-crossing formula is given by

$$\text{Resid} \int \sim() \,[\ p\,\text{Eul}_G\,(\sim)^{1}:_{P=G}$$

One may re-write this as an integral over $X^{;t}$=(G =C) as follows. Let k 2 Z f0g be the weight of the C-action on bers of P ! $X^{;t}$; by changing the choice of one-parameter subgroup by a sign we may assume that k is positive. The projection P ! $X^{;t}$ denes a B(Z=kZ)-bundle

$$: P=G\ !\ X^{;t}=(G=C\,)$$

Choose a splitting of the Lie algebra

$$g = C\ \ (g=C):$$

Let

$$2\ H_C^2(pt); \qquad \sim 2\ H_C^2(pt)$$

denote the standard generators. Consider the commutative diagram

$$\begin{array}{ccc} H_C(P=G) = H_{GC}(P) = H_G(X^{;t}) & \xrightarrow{=} & H(X^{;t}=(G=C)) \; C(\sim) \\ \uparrow & & \uparrow \\ H(P=G) & \longleftarrow & H(X^{;t}=(G=C)) \end{array} :$$

Note that the identication of the C -equivariant parameter ~ in $H_{G\ C}(P)$ re-quires the choice of a splitting. We claim that the pullback of to $H(P=G)$ corresponds to a class k~ + ! 2 $H_C(X^{;t}=(G=C))$ C(~) for some nilpotent ele-ment ! 2 $H^2(X^{;t}=(G=C))$ where k is the ber weight above. Indeed let

$$K = f(z; z^k) 2 C \quad Cg \text{ be}$$

the subgroup that acts trivially on P. Then

$$k\sim 2 \ H^2_{GC}(P) \text{ lies}$$

in the image of

$$H^2_{GC}(pt) \ ! \ H^2_{(GC)=K}(P) \ ! \ H_{GC}(P): \ ^2$$

Since (G C)=K acts on P locally freely, k~ is nilpotent. This proves the claim. The splitting also denes action of (G=C) C on ~, so that

$$Eul_{GC}(p\sim) = Eul_C(\sim=(G=C)) \text{ under}$$

the isomorphism

$$H_{GC}(P) = H(X^{;t}=(G=C)) \; C[\sim]:$$

The xed point contribution can be rewritten as

$$\text{Resid} \ \frac{1}{k} \int_{X^{;t}=(G=C)} j_{X^{;t}} \ [\ Eul_C(\sim=(G=C))^{\ 1} \quad {}_{\sim=(\ !)=k}$$

where we regard $j_{X^{;t}}$ as an element of $H_G(X^{;t}) = H(X^{;t}=(G=C))$ using the splitting and the factor 1=k arises as the degree of

$$: P=G \ ! \ X^{;t}=(G=C):$$

Under the substitution ~ = (!)=k we have for i 2 Z_+

$$\sim^i \ ! \ \frac{k^i}{\ !)^i} =^i k \ \frac{1}{{}^{i+1}} + \frac{i\,!}{2^{i+2}} + \frac{i(i+1)!^2}{} \quad (+ :::$$

and thus

$$\mathrm{Resid}\ \frac{1}{\tilde{}_i} j_{\tilde{}=(\ !=k)} = \begin{cases} k & i = 1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore the above residue equals

$$\mathrm{Resid}_{\tilde{}} \int_{X;t} j_{X;t} [\ \mathrm{Eul} \ (\tilde{}=(G=C)) \ ^{1}; \ _{C} \ X;t=(G=C \ )$$

This ends the Remark.

We introduce the following notation for xed point contributions.

Denition 2.4. For any xed point component $X^{;t}$ X that is t-semistable, denote by $_{X;t}$ the normal bundle of $X^{;t}$ as above. Let

$$_{X;;t} : H_G(X) \ ! \ \ Q[; \ ^{1}]; \qquad ! \int_{[X^{;t}=(G=C)]} \mathrm{Eul}_C((_{X^{;t}}=(g=g))=(G=C \ ))$$

where we have omitted the restriction-and-quotient map $H_G(X) \ ! \ \ H_C(X^{;t}=(G \ =C))$ to simplify notation, and is the equivariant parameter for C .

We dene an equivalence class on rational elements of the Lie algebra as follows. Recall that an element 2 g is rational if C is the Lie algebra of a one-dimensional subgroup C. We declare two rational elements $_0;_1$ to be equivalent if the one-dimensional subgroups are conjugate:

$$(21) \qquad (_0 \ \ _1) \ (\ ) \qquad 9g \ 2 \ G; \quad C_0 = gC_1g^{\ 1}$$

or equivalently, C_1 is related to C_0 by the adjoint action. Denote by [] the equivalent class of a rational element 2 g.

Theorem 2.5 (Kalkman wall-crossing). Let X be a smooth projective G-variety and L ! X polarizations with stable=semistable for the G action on $P(L \ \ L_+)$. Then

$$(22) \qquad _{X=_+GX;+}^{G} \qquad _{X= \ GX;}^{G} = \sum_{1;1);[]} \mathrm{Resid} \ _{X;;t} \ t2($$

where the sum is over equivalence classes [].

Proof. This is an immediate consequence of Kalkman's result for circle actions, Theorem 1.1, applied to the master space constructed in Lemma 2.1, using the identication of the xed point components and normal bundles in Lemma 2.2.

Example 2.6. (a) (Blow-up of the projective plane as a quotient of a product of projective lines by a circle action) Let $X = (P)^3$ with polarization

$$L = O_P(a) \ O_P(b) \ O_P(c); \qquad a \ b \ c:$$

We let G = C acting on each projective line P by

$$g[z_0; z_1] = [z_0; gz_1]; \quad g\ 2\ C; z_0; z_2 C:$$

Let G act on $O_P(n)$ so that the weights at the xed points [1; 0], [0; 1] are n=2; +n=2. Let G act diagonally on the factors in X = $(P)^3$. Consider the family of polarizations L_t = L C_t obtained by shifting L by a trivial line bundle with weight t.

The chamber structure for the various git quotients is governed by the weights of the action on the polarizing line bundle on the xed points, given by (a b c)=2. Thus there are nine chambers, of which two have empty git quotients and seven non-empty chambers. In the rst and last chamber, we have $X{=}_t G$ = P (C^3) resp. P $((C^3)_-)$, while the six wall-crossings represent three blow-ups and three blow-downs involved in the Cremona transformation.

We study the application of the Kalkman formula to the square of the rst Chern class. That is, let

$$= c_1^G(X)^2\ 2\ H^4(X):$$

Since T X is isomorphic to the pull-back of T $(X{=}_t G)$ plus a trivial line bundle with ber g, we have

$$_{X;}^{G}(c_1^G(X)) = c_1(X{=}_t G):$$

Consider the wall-crossing from the chamber t < a b c to the rst non-empty chamber t 2 (a b c; a + b c), corresponding to the wall-crossing over the \lowest" xed point x = ([1; 0]; [1; 0]; [1; 0]) 2 X . Since all weights of the action on the tangent bundle at this xed point x 2 X are 1, we have

$$c_1^G(X)jx = 3\ 2\ H_G(fxg):$$

Hence

$$_{X=_+G\,X;+}^{G}() \qquad _{X=\ G\,X;}^{G}() = \text{Resid}\ \frac{(3)^2}{} = \frac{9}{3}:$$

Indeed, $c_1(P^2)$ is three times the generator of $H^2(P^2)$, so

$$_{X;}^{G}\ _{X;}^{G}() = \int_{[P^2]} c_1(P^2)^2 = 9:$$

Consider next the wall-crossing from the chamber with quotient X= G = P^2 to the chamber with quotient $X{=}_+G$ = $Bl(P^2)$, where $Bl(P^2)$ is the blow-up of P^2 at a point. Letting : $Bl(P^2)$! P^2 denote the blow-down map we have

$c_1(\mathrm{Bl}\,P^2) = 3H \quad E$ where $H; E$ are the hyperplane class and class of the exceptional divisor respectively. Hence

$$(23) \qquad {}_{X=_+G} \, {}^{G}_{X;+}(\;) \;=\; (3H \quad E)^2$$

$$(24) \qquad = (3H)^2 \quad 6HE + E^2 = 9 \quad 0 \quad 1 = 8:$$

To compare this result with the wall-crossing formula, note that the xed point set $X^{G;t}$ consists of a unique point $([0;1];[1;0];[1;0])$ which is semistable exactly for $t = a \quad b \quad c$, with tangent weights $1; 1; 1$. It follows that the rst Chern class squared and Euler classes are

$$c_1(TX|X^{G;t})^2 = (\quad + \quad + \;)^2 = {}^2; \qquad \mathrm{Eul}_G(TX|X^{G;t}) = {}^3:$$

Hence the wall-crossing term is

$$\mathrm{Resid} \int_{[X^{G;t}]} {}^{G;t}_{X} [\; \mathrm{Eul}_G(_{X^{G;t}})^{\;1} \quad = \mathrm{Resid} \; \frac{2}{3} \; = \; 1:$$

The wall-crossing formula reduces to

$$_{X=_+G}(;) \qquad _{X= \;\; G}(;) = 8 \quad 9 = \quad 1:$$

This matches the well-known fact that each blow-up of P^2 lowers c_1^2 by 1.

(b) (Blow-up of the projective plane as a quotient of ane four-space by a two-torus) Let us do the same example in a dierent way, namely as a quo-tient of an ane space. Let $X = C^4$ with $G = (C)^2$ acting with weights $(1;0);(1;0);(1;1);(0;1)$. Consider the path L_t of polarizations whose rst Chern classes $H^2_G(X;Q) = Q^2$ are the line segment from $(1;2)$ to $(2;1)$. The chamber structure is determined by the rays generated by the weights, so that the \negative" chamber is spanned by $(0;1);(1;1)$ and the \positive chamber" by $(1;1);(1;0)$. The quotient in the negative chamber $X= \; G$ is isomorphic to P^2 via the map

$$X= \;\; G \;!\;\; P^2; \quad [a;b;c;d] = [(a;b;cd^{\;\;1};1)] \;! \quad [a;b;cd^{\;\;1}]:$$

On the other hand $X=_+G$ is isomorphic to the blow-up of P^2 with the map to P^2 blowing down the exceptional divisor given by

$$X=_+G \;!\;\; P^2; \quad [a;b;c;d] \;! \quad [a;b;cd^{\;\;1}]:$$

As one moves in a line from say $(\quad 1;2)$ to $(2; \quad 1)$ the symplectic quotients are in order $;; P^2; \mathrm{Bl}(P^2);;$ where the initial and nal contractions are projective bundles over the xed point sets pt resp. P for the residual action of C on the quotient of X by the diagonal action. See Figure 1, where the two chambers for the possible quotients are shown together with the moment polytopes of the quotients in each chamber.

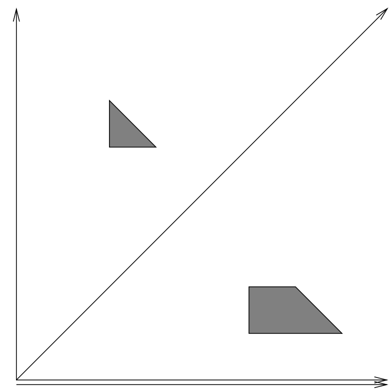


Figure 1. Chambers for $(C)^2$ action on C^4

Consider rst the wall-crossing from the empty chamber to the negative chamber in Figure 1. We have $c_1^G(X) = 3_1 + 2_2$. Hence the wall-crossing term is

$$\mathrm{Resid}_{=1} \; \frac{(3_1 + 2_2)^2}{2(1_1 + 2)} \Big|_{2=0} = 9$$

hence $\int_{[P^2]} c_1(P^2)^2 = 9$ as expected.

The wall-crossing term for passing from the negative to positive chamber is the residue:

$$\mathrm{Resid} \; \frac{(3_1 + 2_2)^2}{{}_1^2 \; {}_2} \Big|_{=1=2} = 1:$$

We obtain from the wall-crossing formula that $\int_{[Bl(P^2)]} c_1(P^2)^2 = 8$ as we already computed in item (a) above.

Consider next wall-crossing from the positive chamber to the empty chamber. The xed point locus contributing to the wall-crossing term is

$$X^{;t}=(G=C) = P$$

since the multiplicity of the weight $(1;0)$ is 2. The wall-crossing term is

$$\mathrm{Resid} \int_{[P]} {}_X^C \; \frac{(3_1 + 2_2)^2}{2(1 + 2)}$$

where ${}_X^C : H_G(X) ! \; H_{G=C}(X=C)$ is the Kirwan map for the quotient by the rst factor of C, so that $X=C$ has C-quotients $X=G$. Under ${}^C_X$ we have that ${}_1$ maps to the generator ! of $H^2(P)$ while ${}_2$ maps to the parameter for the residual C-action. The wall-crossing term is therefore

$$\mathrm{Resid} \int_{[P]} \frac{(3! + 2)^3}{(! +)} = \mathrm{Resid} \int_{[P]} \frac{(3! \; 2)^2}{2} \; 1 \; \frac{!}{} + \frac{!}{2}^2 \; ::: $$

$$= \mathrm{Resid} \int_{[P]} 12\frac{!}{} \; 4\frac{!}{} = 8:$$

Thus as expected the wall-crossing formula reduces to $\int_{;} c_1(;) = 8 \quad 8 = 0:$

(c) (Resolution of a crepant singularity) The following example illustrates the application of the wall-crossing to orbifolds, in which the integrals become rational, and also the non-compact case, in which the integrals must be dened via localization. Suppose that $X = C^{k+1}$ and $G = C$ acts with weights 1 with multiplicity k and k with multiplicity 1. Take L to be the trivial polarization, and L_t the family obtained by shifting by a trivial line bundle with weight t. Thus invariant sections are spanned by monomials.

$$z_0^{d_0} z_1^{d_1} ::: z_k^{d_k}; \qquad kd_0 + d_1 + ::: + d_k = t:$$

The latter requires $d_0 = 0$ for $t < 0$ resp. $(d_1; ::: ; d_k) = 0$ for $t > 0$. It follows that the semistable locus for $t < 0$ is $z_0 = 0$ and for $t > 0$ the locus where $(z_1; ::: ; z_k) = 0$. The git quotients are then $X =_+ G = C^k = Z_k$ (by our conventions, a stack with trivial canonical bundle) while $X = G$ is isomorphic to the total space of $O_{P^{k-1}}(k) ! \; P^{k-1}$.

We apply the wall-crossing formula to the Euler class of the quotients. Let $= c^G_k(C^{k+1})$ (the next-to-highest Chern class of C^{k+1}) so that ${}^G_{;}()_X$ is the top Chern class of $X = G$. Then

$$_{X =_+ G}({}^G_{X;+}()) = \int_{[C^k = Z_k]} \mathrm{Eul}(C^k = Z_k) = 1 = k:$$

Indeed, interpreting this integral via localization using the C-action given by scalar multiplication, there is a unique xed point with stabilizer of order k which contributes $\mathrm{Eul}(C^k) = (k\,\mathrm{Eul}(C^k)) = 1 = k$ to the integral. On the other side of the wall,

$$_{X = G}({}^G_{X;}()) = \int_{[O_{P^{k-1}}(k) ! P^{k-1}]} \mathrm{Eul}(P^{k-1}) = k:$$

The wall-crossing term is

$$\mathrm{Resid} \frac{(1 +)^k (1 \quad k)}{^k(\quad k)} = k \quad 1 = k$$

as expected.

(d) (A del Pezzo surface as a quotient of a product of projective lines by a non-abelian group) Let $X = (P)^5$ with the diagonal action of $G = SL(2; C)$. The action of $SL(2; C)$ on P lifts uniquely to an action on the hyperplane bundle $O(1)$ and we take as polarization the exterior tensor product $O(1)^t \; O(1)^4$ varying on the rst factor only. The stability condition for a tuple $(v_1; :::; v_5)$ 2 X is

$$\sum_{v_i = w} {}_i v_i \quad \sum_{v_i = w} {}_i v_i ; \qquad {}_1 = t; {}_i = 1; i = 1; \qquad 8w \; 2 \; P$$

The values of t for which there are semistable points with innite stabilizers are $t = 2; 4$; these points are given by

$$v_1 = v_i = v_j = v_k = v_l; \quad v_1 = v_2 = v_3 = v_4 = v_5:$$

Thus the chamber structure for t is $(0; 2); (2; 4); (4; 1)$. Since the quotient in the last chamber is empty, and all the weights are one, the quotient $X{=}_t G$ for the second chamber must be P^2. Passing from the second to rst chamber the quotient $X{=}_t G$ undergoes a blow-up at 4 points. Consider the wall-crossing given by passing between chambers so that

$$X{=} \; G = P^2; \quad X{=}_+ G = Bl^4(P^2):$$

We compute the square of the rst Chern class by wall-crossing: Let

$$= c_1^G(X)^2; \qquad \text{so that} \quad {}_{X;}() = c_1(X{=}G)^2:$$

The xed points of C action for the singular value $t = 2$ are the 4 (up to the action of the disconnected group $G = N(T)$) congurations with $v_1 = v_i = v_j = v_k = v_l$. Consider the case that $x = (x_1; :::; x_5)$ 2 X is xed by the maximal torus action; the tangent bundle of this point has weights $1; 1; 1; 1;$ 1 hence $c_1^G(X)j_x = {}_1$. The tangent bundle $T_x X$ modulo $g{=}g$ has weights $1; 1; 1$. Thus

$$\int_{[X{=}_+ G]} c_1(X{=}_+ G)^2 = \int_{[X{=} \; G]} c_1(X{=} \; G)^2 + 4 \, \mathrm{Resid} \, \frac{2}{3}$$
$$= 9 \quad 4 = 5:$$

This again matches the fact that each blow-up of P^2 lowers c_1^2 by 1.

2.3. **The virtual wall-crossing formula.** Our main result will be derived from a virtual extension of wall-crossing formulas, similar to Kiem-Li [40] using the virtual localization formula of Graber-Pandharipande [31]; we assume that the reader is familiar with the concepts of equivariant perfect obstruction theories etc. from those papers. In this section we explain the virtual extension and give an application to a simple complete intersection.

We rst need a simple lemma on obstruction theories on quotients. Let X be a Deligne-Mumford G-stack equipped with a G-equivariant perfect obstruction theory which admits a global resolution by vector bundles. This means that X is equipped with an object E of the bounded derived category of G-equivariant coherent sheaves, together with a morphism from E to the cotangent complex, satisfying certain axioms, see [4], [31]; a typical example is an invariant complete intersection as in Example 2.9 below. Let g^- denote the sheaf of sections of the trivial bundle with ber g^-. The G-action on X induces a canonical morphism $a^- : L_X \to g^-$ that we call the innitesimal action. The obstruction complex E comes equipped with a lift $\tilde{a}^- :$ $E \to g^-$ of the innitesimal action $a^- : L_X \to g^-$. If X^{ss} is the semistable locus for some polarization and stable=semistable, denote $E^{ss}; L_{X^{ss}}$ etc. the restrictions to the semistable locus. The following lemma is probably well-known:

Lemma 2.7. If X^{ss} is the semistable locus for some polarization and stable=semistable, then the perfect obstruction theory $E^{ss} \to L_{X^{ss}} := L_X | X^{ss}$ descends to a perfect obstruction theory on the quotient X^{ss}/G.

Proof. From the bration $: X^{ss} \to X^{ss}/G$ one obtains an exact triangle of cotangent complexes

$$L_{X^{ss}/G} \to L_{X^{ss}} \to \underline{g^-} \to (L_{X^{ss}/G})[1]: \tag{25}$$

Let Cone($\tilde{a}^-$) denote the mapping cone on the lift of the innitesimal action $\tilde{a}^-$. The exact triangle

$$\mathrm{Cone}(\tilde{a}^-) \to E \to \underline{g}^- \to \mathrm{Cone}(\tilde{a}^-)[1]$$

admits a morphism to (25), in particular making $\mathrm{Cone}(\tilde{a}^-) \to L_{X^{ss}/G}$ into an obstruction theory with support in $[\ 1; 1]$. By the assumption on the stabilizers, this obstruction theory is perfect.

We now study the virtual normal complexes of the xed point stacks. If $\ 2\ g$ an element generating a one-parameter subgroup then the xed point stacks X also have equivariant perfect obstruction theories compatible with that on X. We choose a splitting of Lie algebras

$$g = C \quad (g/C)$$

inducing a splitting of Lie groups after passing to a nite cover $\tilde{G} \to G$

$$\tilde{G} = G/C \quad C : \text{Denote}$$

by $_X$ the conormal complex for the embedding

$$X/(G/C) \to X/(G/C):$$

Denote by $X^{;t}$ the locus of X semistable for L_t and by $_{;t}$ the restriction of $_X$ to $X^{;t}$. The Euler class $\mathrm{Eul}_C(_{X;t})$ is well-dened in the equivariant cohomology $H(X^{;t})[; ^{1}]$, after inverting the equivariant parameter . From the virtual localization formula in Chang-Kiem-Li [12, Theorem 3.4], improving Graber-Pandharipande [31], one obtains the following virtual version of the wall-crossing Theorem 2.5, similar to results of Kiem-Li [40]:

Theorem 2.8 (Virtual Kalkman wall-crossing). Let X be a proper Deligne-Mumford G-stack equipped with a G-equivariant perfect obstruction theory which admits a global resolution by vector bundles. Let L ! X be G-line bundles that are am-ple for the coarse moduli spaces, so that stable=semistable for $P(L \; L_+)$. Let $_{X=G}$ resp. $_{X;;t}$ denote integration resp. equivariant integration over $X=G$ resp. $X^{;t}$ times $\mathrm{Eul}_C(_{X;t})^{1}$. Then

$$(26) \qquad {}_{X=_+G}\; {}^{G}_{X;+} \qquad {}_{X=\;G}\; {}^{G}_{X;} = \sum_{t2(\;1;1);[]} \mathrm{Resid}\; {}_{X;;t}$$

where the sum is over [] from (21).

Example 2.9. (Wall-crossing over a nodal xed point) The following simple example may help illustrate the notation. Suppose that $X = P^1 [_{1\;1} P^1$ is a nodal projective line with a single node 1, equipped with the standard C-action on each component, so that the weights of the action on the tangent spaces at the node 1 are 1. We equip X with a polarization so that the weights are 1 at the smooth xed points $0\;2\;P^1$, and 0 at the nodal point. Then $X=_tG$ is a point for $t\;2\;(\;1;1)$, and is singular for $t = 0$. Since X is a complete intersection, X has a perfect obstruction theory [4, Example before Remark 5.4] and the virtual wall-crossing formula of Theorem 2.8 applies. We examine the wall-crossing for the trivial class $= 1$ at the singular value $t = 0$. The virtual normal complex at the nodal point is the quotient of $C_1 \; C_{\;1}$, the sum of one-dimensional representations with weights $1; \; 1$, modulo their tensor product $C_1 \; C_{\;1}$, which has weight zero. Hence the normal complex has inverted Euler class

$$\mathrm{Eul}_G(_{X;G;t})^{\;1} = \frac{}{(\;)^p} = 0:$$

The integrals on the left and right hand sides are 1 (being the integrals over points) while the wall-crossing term is

$$\begin{aligned} 1 \quad 1 &= {}_{X=_+G}\; {}^{G}_{X;+} \qquad {}_{X=\;G}\; {}^{G}_{X;} \\ &= \mathrm{Resid}\; {}^{G}_{X;;0} = \mathrm{Resid} \int_{[pt]} \mathrm{Eul}_G(_{X;G;t})^{\;1} = \mathrm{Resid}\; 0 = 0 \end{aligned}$$

as desired.

We begin the proof of Theorem 2.8 by construction of a master space.

Lemma 2.10. Let X be a Deligne-Mumford G-stack equipped with a G-equivariant perfect obstruction theory which admits a global resolution by vector bundles as well as an embedding in a smooth Deligne-Mumford G-stack. Let $L \to X$ be polarizations (G-line bundles with ample coarse moduli spaces) such that stable=semistable for L and for any $t \in (1;1)$ and any t-semistable point $x \in X^G$, G_x acts with nite stabilizer on the 1 ber $(L_+ L_{\sim})_x$. There exists a proper Deligne-Mumford C-stack $\tilde{X}$ equipped with a line bundle ample for the coarse moduli space whose git quotients $\tilde{X} /\!/_t C$ are isomorphic to those $X /\!/_t G$ of X by the action of G with respect to the polarization L_t and whose xed point set $\tilde{X}^C$ is given by the union

$$\tilde{X}^C = (X /\!/ G) \cup (X /\!/_+ G) \cup \bigcup_{[]} (X /\!/_t (G/C))$$

where $[]$ is as in (21) and is the natural map to X as in Lemma 2.2. Furthermore, $\tilde{X}$ has a perfect obstruction theory admitting a global resolution by vector bundles with the property that the virtual normal complex of $X /\!/_t (G/C)$ is isomorphic to the image of $_X = (g/g)$ under the quotient map $X \to X /\!/ (G/C)$, by an isomorphism that intertwines the action of C on $(_X = (g/g))/(G/C)$ with the action of C on $_{\tilde{X}^C}$.

Proof. The construction of the master space is the same as in 2.1, that is, the master space is the stack-theoretic quotient $\tilde{X} = P(L \oplus L_+) /\!/ G$: The assumption on the action of the stabilizers implies that the action of G on the semistable locus in $P(L \oplus L_+)$ is locally free, so that stable=semistable for $P(L \oplus L_+)$. It follows that $\tilde{X}$ is a proper Deligne-Mumford stack, and by Lemma 2.7 has a perfect obstruction theory induced from the natural obstruction theory on $P(L \oplus L_+)$ given by considering it as a bundle over X. The quotient $\tilde{X}$ contains the quotients of $P(L) = X$ with respect to the polarizations L, that is, $X /\!/ G$.

The same argument in Lemma 2.2 describes the xed point loci: they correspond to xed point loci in $P(L \oplus L_+)$ for one-parameter subgroups of $C \times G$. Given such a locus $P(L \oplus L_+)$, the pull-back of the virtual normal complex is by denition the moving part of $\mathrm{Cone}(\tilde{a}_{P(L \oplus L_+)})$, where

$$\tilde{a}_{P(L \oplus L_+)} : E_{P(L \oplus L_+)} \to g$$

is the lift of the innitesimal action of G. Consider the bration $: P(L \oplus L_+) \to X$. By denition $E_{P(L \oplus L_+)}$ ts into an exact triangle

$$E_X \to E_{P(L \oplus L_+)} \to L \to E_X[1]:$$

Over the complement $P(L \; L_+) \; P(L \; L_+)$ of the sections at zero and innity we may identify $L = \underline{C}$ using the C-action on the bers, by the assumption on the weights of the C action on the ber. The projection to X identies the mapping cones

$$\mathrm{Cone}(\tilde{a}_{P(L \; L_+)} j_{P(L \; L_+)} \;) ! \quad \mathrm{Cone}(\tilde{a}_{\overline{X}}) \text{ where}$$

$$\tilde{a}_{\overline{X}} : E_X jX \; ! \quad \underline{(g=C)}$$

is the lift of the innitesimal action of g=C. Now the virtual normal complex is by denition the C-moving part of the perfect obstruction theory; the Lemma follows.

Proof of Theorem 2.8. The proof of 2.8 is similar to that of Theorem 2.5. Namely we take the residue of the virtual localization formula applied to $\tilde{X}$: For any equivariant class 2 $H_G(X)$ of top degree, its pullback to $P(L \; L_+)$ descends to a class ~ 2 $H_G(X)$ whose restriction to X =G is ^{G} (), and whose pullback under $X^{;t}$! X^C is $_{X;t}$. By virtual localization the integral is

$$\int_{[\tilde{X}]} = \int_{[X = \; G]} \frac{{}^{G}_{X;} ()}{\mathrm{Eul}_G(\;)} + \int_{[X =_+ G]} \frac{{}^{G}_{X;+} ()}{\mathrm{Eul}_G(_+)} + \sum_{t2(\;\; 1;1 \;\; ;[])} \int_{[X^{;t}=(G=C)]} \frac{{}^{;t}_{X}}{\mathrm{Eul}_G(_{X;t})} :$$

Taking residues and using Lemma 2.10 to identify the last term with

$$\sum_{[X^{G;t}]} \mathrm{Resid} \int_{\sim} \frac{{}^{G;t}}{\mathrm{Eul}^X_G(\; \tilde{}_{X^{G;t}})} = \sum_{t2(\;\; 1;1);[]} \mathrm{Resid}_{X;;t} \; t2(\;\; 1;1)$$

gives the formula in the Theorem.

3. Wall-crossing for Gromov-Witten invariants

In this section, we prove a quantum generalization of Kalkman's wall-crossing formula Theorem 2.5. In the rst two subsections, we dene the wall-crossing terms as integrals over moduli spaces of gauged maps xed by a central subgroup. The last two subsections contain a construction of a master space for moduli spaces of gauged maps, and a proof of the wall-crossing formula via virtual localization on the master space. The construction of the master space is obtained from one for a dierent compactication of gauged maps introduced by Schmitt [56], pulled back under a relative version of Givental's morphism from stable maps to the quot scheme. Schmitt's compactication has the advantage that it is constructed by git methods so the classical techniques apply.

3.1. Construction of a master space. The proof of the wall-crossing formula 3.25 depends on the construction of master space in the sense of 2.1 whose quotients are the moduli spaces of Mundet stable gauged maps.

Proposition 3.1 (Existence of a master space). Under suitable stable=semistable conditions, there exists a proper Deligne-Mumford C-stack $\overline{M}^G_n(C; X; L_-; L_+)$ with the following properties:

(a) $\overline{M}^G_n(C; X; L_-; L_+)$ admits a perfect C-equivariant relative obstruction theory

(b) the git quotients of $\overline{M}^G_n(C; X; L_-; L_+)$ are the moduli stacks

$$\overline{M}^G_n(C; X; L_-; L_+) /\!\!/_t C = \overline{M}^G_n(C; X; L_t)$$

for parameter $t \in (-1; 1)$;

(c) the C-xed substack includes $\overline{M}^G_n(C; X; L_-; d)$ and $\overline{M}^G_n(C; X; L_+; d)$;

(d) $\overline{M}^G_n(C; X; L_-; L_+; d)$ admits an embedding in a non-singular Deligne-Mumford stack.

The proof will be given after several constructions. First recall the quot-scheme compactication of Mundet semistable morphisms from C to $X{=}G$ constructed by Schmitt [56, Theorem 2.7.1.4]:

Denition 3.2. (Bundles with maps)

(a) Let $X = P^{r-1}$ and $G = GL(r)$. A projective bundle with map over a smooth projective curve C over a point $S = fptg$ is a datum $(E; L; ')$ consisting of a vector bundle $E \to C$ of rank r; a line bundle $L \to C$; and a surjective morphism $' : E \to L$; to obtain a compactication one allows this morphism to be non-zero rather than surjective. On the locus $' = 0$ we obtain a section of the associated projective bundle $Fr(E^-) \times_G X = P(E^-)$,

$$\{' = 0\} \to Fr(E^-) \times_G X; \qquad z \to \operatorname{im} '_{\bar z}: \tag{27}$$

For more general schemes S, a projective bundle with map over a curve $C \times S \to S$ of degree l is a datum $(E; L; ; ')$ where is a morphism from S to the Jacobian $Jac^l(C)$ of degree l line bundles, $N()$ the corresponding line bundle on C dened by pulling back a Poincare bundle over $C \times Jac^l(C)$, and $' : E \to N() \otimes L$ is a non-zero map. For G a product of groups $GL(r_i); k = i; : : : ; k$, a projective bundle with map is collection of bundles E_i of rank r_i, a line bundle $L \to C$ and a surjective morphism $' : \bigoplus_{i=1}^k E_i \to L$.

(b) For a reductive subgroup G of a product $GL(\underline{r}) = \prod_{i=1}^k GL(r_i)$ let $: G \to GL(\underline{r})$ be a faithful homogeneous representation, that is, so that the central subgroup of G maps to the center $(C^\times)^k$ of $GL(\underline{r})$. A projective bundle with map is a projective $GL(\underline{r})$-bundle with map $(E; L; ')$ and a reduction $: Fr(E) \to Fr(E){=}G$ of the frame bundle to G.

(c) A G-bundle with map $(E; L; ';)$ is Mundet semistable if it satises the inequality of Denition 1.3 for every pair $(;)$ consisting of a parabolic reduc-tion and antidominant coweight .

We introduce the following notation the moduli stacks of bundles with maps with given numerical invariants.

Denition 3.3. Given a G-module V and integers $d_E; d_L$ let $M^{G;quot}(C; V; d_E; d_L)$ denote the stack of Mundet semistable G-bundles with maps

$$(P; L \quad E := P(V)_{\_}; ' : E \; ! \quad L)$$

whose bundles $(E; L)$ have rst Chern classes

$$c_1(E) = d_E; \quad c_1(L) = d_L \; 2 \; H_2(C) = Z:$$

Schmitt [56, Theorem 2.7.1.4] proves using a git construction that $\overline{M}^{G;quot}(C; V; d_E; d_L)$ has projective coarse moduli space. We will need some details regarding the git construction which involves level structures, dened as follows.

Denition 3.4. Let $n \quad 1$ be an integer. An n-twisted level structure for a projec-tive bundle with map $(E; L; ')$ is a collection of sections $s_1; : : : ; s_l$ generating E: a surjective map

$$s : O_C^l(\quad n) \; ! \quad E:$$

An isomorphism between two projective bundles with maps and level structures $(E^k; L^k; '^k; s^k); k \; 2 \; f1; 2g$ is a pair of isomorphisms $E^1 \; ! \quad E^2; L^1 \; ! \quad L^2$ intertwining the maps $'_k$ and level structures s_k. In the case of $G \quad GL(\underline{r})$ bundles, a level structure is a level structure for each factor $E_i \quad E; i = 1; : : : ; k$.

Denote by $\overline{M}^{G;quot;lev}(C; V; d_E; d_L)$ the compactied stack of projective bundles with maps and level structures. The group $GL(\underline{r})$ acts on $\overline{M}^{G;quot;lev}(C; P^{r \quad 1}; d_E; d_L)$ by changing the level structure:

$$g(E; L; '; s) \; ! \quad (E; L; '; gs):$$

Schmitt [56, Section 2.7] constructs a line bundle $D(L)$ (the pull-back of an ample line bundle on a suitable quot scheme by a nite morphism) such that the quotient of the inverse image of the semistable locus in $M^{G;quot;lev}(C; P^{r \quad 1}; d_E; d_L)$ by $GL(\underline{r})$ is $M^{G;quot}(C; P^{r \quad 1}; d_E; d_L)$. (The notation stands roughly speaking for determinant line bundle.)

A well-known construction of Givental [27] provides a morphism from the Kontsevich-style compactication to the quot-scheme compactication. Let X be a smooth projective G-variety. Given $d \; 2 \; H_2^G(X; Z)$ let $d_E \; 2 \; Z$ denote the image of d under

$$H_2^G(X; Z) \; ! \quad H(BG; Z) \; ! \quad H(BGL(r); Z) = Z$$

and $d_L \in \mathbb{Z}$ the image of d under

$$H_2^G(X;\mathbb{Z}) = H_2(BG) \oplus H_2(X) = \mathbb{Z}.$$

Lemma 3.5. There is a proper morphism of Artin stacks

$$\overline{M}^G(C;X;L;d) \to \overline{M}^{G,\mathrm{quot}}(C;\mathbb{P}^{r-1};L;d_E;d_L) \tag{28}$$

which maps

$$u : \hat{C} \to E := \mathbb{P}(V^-)$$

with principal component $u_0 : C \to \mathbb{P}(V^-)$ to the pair $(L;\varphi)$ where the line bundle L is defined by

$$L := (u_0)^* E^\vee \otimes \mathcal{O}\Big(\sum_{i=1}^k d_i p_i\Big) \tag{29}$$

where d_i is the degree of the i-th bubble component

$$u|C_i : C_i \to \mathbb{P}(E^-);\ i = 1,\ldots,k$$

and $p_i \in C$ is its projection $\pi(u(C_i))$ onto the principal component C; and if φ_0 is the quotient corresponding to u_0 then φ is defined by tensoring with a section of $\mathcal{O}(\sum d_i p_i)$, so that in a local coordinate z

$$\varphi(z) := \varphi_0(z)(z - p_i)^{d_i}. \tag{30}$$

Proof. In the setting of families of stable maps this is an application of Popa-Roth [53, Theorem 7.1], see also Marian-Oprea-Pandharipande [48, Section 5.2] for a similar construction. The stack $\overline{M}^{G,\mathrm{lev}}(C;X;L;d)$ admits a universal bundle

$$E \to C \times \overline{M}^{G,\mathrm{lev}}(C;X;L;d).$$

Letting

$$\overline{C}^{G,\mathrm{lev}}(C;X;L;d) \to \overline{M}^{G,\mathrm{lev}}(C;X;L;d)$$

denote the universal curve, we have a universal map

$$\overline{C}^{G,\mathrm{lev}}(C;X;L;d) \to \mathbb{P}(E), \quad (u : \hat{C} \to C \times X/G;\ z \in \hat{C};\ s) \mapsto u(z)$$

where s denotes the level structure from above. The morphism in [53, Theorem 7.1] maps this datum to the morphism $E^- \to L$ for the line bundle $L \to C \times \overline{M}^{G,\mathrm{lev}}(C;X;L;d)$ defined by (29); at least locally. Since the construction in [53, Theorem 7.1] is functorial, the local constructions patch together to the required morphism, even though $\overline{M}^{G,\mathrm{lev}}(C;X;L;d)$ is a priori a stack of possibly infinite type. The map (30) giving a projective bundle with map and level structure in the sense of Schmitt [56, Section 2.7]. Taking the quotient by the action of $GL(\underline{r})$ gives the desired morphism $\overline{M}^G(C;X;L;d) \to \overline{M}^{G,\mathrm{quot}}(C;\mathbb{P}^{r-1};L;d_E;d_L)$.

Corollary 3.6. $\overline{M}_n^G(C;X;d)$ is proper for each $d \in H_2^G(X;\mathbb{Z})$.

Proof. Forgetting the markings and stabilizing the stable section denes a morphism

$$\overline{M}_n^G(C;X;d) \to \overline{M}^G(C;X;d);$$

this is the composition of morphisms forgetting a single marking, each of which is isomorphic to a universal marked curve. Combining these we obtain a proper morphism

$$\overline{M}_n^G(C;X;d) \to \overline{M}^{G;\mathrm{quot}}(C;\mathbb{P}^{r-1};d_E;d_L).$$

Since the composition of proper morphisms is proper, $\overline{M}_n^G(C;X;d)$ is proper.

A master space for stable gauged maps is constructed by considering bundles with pairs of maps. Associated to each map is a determinant line bundle, and a repeat of the construction in Thaddeus [59] will create a master space for the variation of stability condition. First we introduce a suitable moduli space of bundles with pairs of maps.

Denition 3.7. (Stack of bundles with pairs of maps)

(a) Suppose $L_{\pm} \to X$ are polarizations. Given tuples $\underline{r}_-;\underline{r}_+ > 0$ a class $d_G \in H^2(BG)$ and integers $d_{\underline{L}} = (d_{L_-};d_{L_+})$ and G-modules $V_-;V_+$ a bundle with pair is a tuple

$$(P;L_-;L_+;'_-;'_+) \qquad '_{\pm}: P(V_{\pm}) =: E_{\pm} \to L_{\pm}$$

consisting of a G-bundle P with rst Chern class d_G, line bundles $L_-;L_+$ of degrees $d_{\underline{L}}$ and non-zero maps $'_-;'_+$ from the associated vector bundles $E_{\pm} := P(V_{\pm})$.

(b) A stability condition on bundles with pairs is given by combining the Ramanathan and Hilbert-Mumford weights in (7), (8): For weights $;_+ > 0$ a parabolic reduction and Lie algebra element generating a one-parameter subgroup we dene

$$_{;+}(;) = _R(;) + _{H;}(;) + _{+H;+}(;)$$

where $_{H;}(;)$ is the weight of the one-parameter subgroup on the asso-ciated graded for the map $'$. A datum $(P;L_-;L_+;'_-;'_+)$ is semistable i

$$_{;+} \le 0 \qquad \forall(;):$$

Let $\overline{M}^{G;\mathrm{quot}}(C;V_-;V_+;d_G;\underline{d_L})$ denote the moduli stack of $(;_+)$-semistable data $(P;L_-;L_+;'_-;'_+)$.

A very similar construction appears in [56, Section 2.8.1] under the name of twisted ane bumps, but with a dierent stability condition.

Lemma 3.8. For suciently large twisting in Denition 3.4, there exists a projective $GL(\underline{r}_-) \times GL(\underline{r}_+)$-scheme $M^{G;quot;lev}(C; V_-; V_+; d_G; \underline{d}_L)$ such that for any $_+;$, the stack $M^{G;quot}(C; V_-; V_+; d_G; \underline{d}_L)$ has coarse moduli space that is the good quotient of an open subset of semistable points $M^{G;quot;lev}(C; V_-; V_+; d_G; \underline{d}_L)$

$$\overline{M}^{G;quot}(C; V_-; V_+; d_G; \underline{d}_L) = \overline{M}^{G;quot;lev}(C; V_-; V_+; d_G; \underline{d}_L)^{ss} / GL(\underline{r}_-) \times GL(\underline{r}_+):$$

Furthermore, the semistable locus is a git semistable locus in the sense that there exists a nite injective equivariant morphism

$$\overline{M}^{G;quot;lev}(C; V_-; V_+; d_G; \underline{d}_L) \to \overline{Q}^{G;lev}(C; V_-; V_+; d_G; \underline{d}_L)$$

to a $GL(\underline{r}_-) \times GL(\underline{r}_+)$-scheme $Q^{G;lev}(C; V_-; V_+; d_G; \underline{d}_L)$ and a line bundle

$$D(L_-; L_+) \to \overline{Q}^{G;lev}(C; V_-; V_+; d_G; \underline{d}_L)$$

so that the following holds: A bundle with pair $(P; L_-; L_+; ' _-; '_+)$ is semistable i its image in $Q^{G;lev}(C; V_-; V_+; d_G; \underline{d}_L)$ is semistable, that is, there exists a non-trivial invariant section of $D(L_-; L_+)$ non-vanishing at $(P; L_-; L_+; '_-; '_+)$.

Proof. We will embed the given moduli stack into a larger moduli stack via a tensor product construction. Let $\overline{M}^{G;quot;lev}(C; V_-; V_+; \underline{d}_L)$ denote the moduli stack of objects that are tuples $(P; '_- : E_- \to L_-; '_+ : E_+ \to L_+; s_+; s_-)$. where the bundles $E_-; E_+$ are equipped with level structures $s_+; s_-$ acted on by $GL(\underline{r})$. The tensor product map

$$(' _-; : E_- \to L_-; '_+ : E_+ \to L_+) \to (' _- \otimes '_+ : E_- \otimes E_+; L_- \otimes L_+)$$

induces an embedding

$$\overline{M}^{G;lev}(C; V_-; V_+; \underline{d}_L) \to \overline{M}^{G;lev}(C; V_- \otimes V_+; d_{L_-} d_{L_+}); \tag{31}$$

$$(P; '_-; '_+; s) \to (P; '_- \otimes '_+; s):$$

Because the Hilbert-Mumford weights are additive under tensor products (the weights on the hyperplane bundles are additive, by construction) the morphism (31) preserves the semistability conditions. By the construction on [56, p. 277], there exists an injective nite morphism

$$: \overline{M}^{G;lev}(C; V_- \otimes V_+; d_G; d_{L_-} d_{L_+}) \to Q^{G;lev}(C; V_- \otimes V_+; d_G; d_{L_-} d_{L_+})$$

to a projective scheme, denoted $Q^{G;lev}(C; V_- \otimes V_+; d_G; d_{L_-} d_{L_+})$ and line bun-dle $D(_-; _+)$ on the codomain $Q^{G;lev}(C; V_- \otimes V_+; d_G; d_{L_-} d_{L_+})$ so that a datum

$(P; E; L; '; s)$ is semistable i $(P; E; L; '; s)$ is git-stable with respect to $D(\; ;_+)$. Let

$$\underline{Q}^{G;lev}(C; V \; ; V_+; d_G; \underline{d}_L) \quad = \quad (\underline{M}^{G;lev}(C; V \; ; V_+; d_G; \underline{d}_L))$$
$$\underline{Q}^{G;lev}(C; V$$
$$\overline{V}^+; d_G; d_L \;\; d_{L_+}) \;\; _+$$

and $D(L \; ; L_+)$ the pull-back of $D(L)$. Then $\overline{Q}^{G;lev}(C; V \; ; V_+; d_G; \underline{d}_L)$ is projective and the morphism $\overline{M}^{G;lev}(C; V \; ; V_+; d_G; \underline{d}_L)$ to $\overline{Q}^{G;lev}(C; V \; ; V_+; d_G; \underline{d}_L)$ is nite and injective, since it is the restriction of a nite and injective morphism. The claim on the semistable locus follows by restriction.

The constructions above give moduli stacks of bundles with sections of projectivizations. We extend this to sections of associated bundles with arbitrary smooth projective bers as follows:

Denition 3.9. (Moduli stack of bundles with pairs of maps) Given the projective G-variety X and G-equivariant embeddings

$$: X \; ! \;\; P(V)$$

let $\overline{M}^{G;quot}(C; X; V \; ; V_+; d_G; \underline{d}_L)$ denote the substack of $\overline{M}^{G;quot}(C; V \; ; V_+; d_G; \underline{d}_L)$ consisting of data

$$(P; ' \;\; : E \;\; ! \;\; L \;\; ; '_+ : E_+ \; ! \;\; L_+)$$

so that

$$([' \;\; (z)]; ['_+(z)]) \; 2 \; (\quad _+)(X) \;\; P(V \;\;) \;\; P(V_+)$$

for generic $z \; 2 \; C$. Including level structures for the bundles $E \;\; ; E_+$ into the data gives a $GL(\underline{r} \;\;) \; GL(\underline{r}_+)$-stack

$$\overline{M}^{G;quot;lev}(C; X; V \; ; V_+; d_G; \underline{d}_L) \qquad \overline{M}^{G;quot;lev}(C; V \; ; V_+; d_G; \underline{d}_L)$$

Let $\overline{Q}^{G;lev}(C; X; d_G; \underline{d}_L)$ denote its image in $\overline{Q}^{G;lev}(C; V \; ; V_+; d_G; \underline{d}_L)$. Denote the quotient stack

$$\overline{M}^{G;quot}(C; V \; ; V_+; d_G; \underline{d}_L) = \overline{M}^{G;quot;lev}(C; V \; ; V_+; d_G; \underline{d}_L)=(GL(\underline{r} \;\;) \; GL(\underline{r}_+)):$$

Denition 3.10. (Master space for quot scheme compactications) Let $L \; ! \;\; X$ denote the pull-backs of the hyperplane bundle under the embeddings and

$$D(L) \; ! \;\; \overline{Q}^{G;lev}(C; X; d_G; \underline{d}_L)$$

denote the pull-backs of the line bundles in the Lemma above. Then

$$P(D(L \;\;) \; D(L_+)) \; ! \;\; \overline{Q}^{G;lev}(C; X; d_G; \underline{d}_L)$$

is a P^1 bundle, equipped with a natural $GL(\underline{r})$-action and a polarization

$$O_{P(D(L\ \ )D(L_+))}(1) \to \ P(D(L\ \ )\ D(L_+))$$

considered in (17). Let

$$\overline{Q}^G(C; X; L\ \ ; L_+; d_G; \underline{d_L}) = P(D(L\ \ )\ D(L_+))/GL(\underline{r})$$

denote its git quotient, that is, the quotient of the semistable locus $P(D(L\ \ )\ D(L_+))^{ss}$ of objects where some non-trivial invariant section $P(D(L\ \ )\ D(L_+)))$ is non-vanishing, by the action of $GL(\underline{r})$. The pull-back

$$P(D(L\ \ )\ D(L_+)) \to \ M^{G;quot;lev}(C; X; d)$$

is a P^1-bundle and admits a nite injective morphism to $Q^G(C; X; L\ \ ; L_+; d_G; \underline{d_L})$. Since $P(D(L\ \ )\ D(L_+)))/(GL(\underline{r}\ \ )\ GL(\underline{r_+}))$ is a good quotient, so is

$$\overline{M}^{quot}(C; X; L\ \ ; L_+; d_G; \underline{d_L}) := P(D(L\ \ )\ D(L_+))/GL(\underline{r}\ \ )\ GL(\underline{r_+}):$$

Proposition 3.11. *If stable=semistable then the stack $\overline{M}^{G;quot}(C; X; L\ \ ; L_+; d_G; \underline{d_L})$ is a proper Deligne-Mumford stack with a projective coarse moduli space. The group C acts naturally on $M^{G;quot}(C; X; L\ \ ; L_+; d_G; \underline{d_L})$ and the quotient of the semistable locus for*

$$D(L_t) := D(L\ \ )^{(1\ \ t)/2} \ D(L_+)^{((1+t)/2}$$

(pulled back from $\overline{Q}^{G;lev}(C; X; L\ \ ; L_+; d_G; \underline{d_L})$) is $\overline{M}^{G;quot}(C; X; L_t; d_G; \underline{d_L})$, the stack of data

$$(P; L\ \ ; L_+; '\ \ ; '_+)$$

that are semistable with respect to the stability condition (

$$\ ;\ _+) = ((1\ \ \ \ t)/2; (1 + t)/2):$$

Proof. The coarse moduli space of $\overline{M}^{G;quot}(C; X; L\ \ ; L_+; d_G; \underline{d_L})$ admits a nite injective morphism to the projective variety $\overline{Q}^G(C; X; L\ \ ; L_+; d_G; \underline{d_L})$. It follows that the stack $\overline{M}^{G;quot}(C; X; L\ \ ; L_+; d_G; \underline{d_L})$ has projective coarse moduli space. By [56, Theorem 2.3.4.1] and [56, Proposition 2.2.3.7] and [56, Corollary 2.2.3.4] after twisting by a suitable tensor power of a positive line bundle on C, every bundle with map $(P; L\ \ ; L_+; '\ \ ; '_+)$ that is semistable for $D(L_t)$ for some $t \in [\ \ 1; 1]$ appears in this quotient construction, and so by construction $\overline{M}^{G;quot}(C; X; L_t; d_G; \underline{d_L})$ is the git quotient for the polarization $D(L_t)$.

The Kontsevich style compactication of the master space is obtained from similar compactications applied to the master space for quot scheme compactications. Let $\overline{M}^G(C; X; V\ \ ; V_+; d)$ denote the substack of $\overline{M}^G(C; X; d)$ consisting of bundles that appear in $\overline{M}^{G;quot}(C; V \; ; V_+; d_G; \underline{d_L})$ together with a stable section of the associated X bundle $P(X) \; ! \; C$; as the twisting in Denition 3.4 goes to innity the union of these loci includes all of $M^G(\overline{C}; X; d)$. The Givental morphism (28) for the pair of embeddings $: X \; ! \; P(V)$ gives a morphism

$$\overline{M}^G(C; X; V \; ; V_+; d) \; ! \; \overline{M}^{G;quot}(C; V \; ; V_+; d_G; \underline{d_L}) \tag{32}$$

which maps the pair $(C; u : C \; ! \; P(X))$ to the line bundles L and maps $' : P(V_-) \; ! \; L$ associated to the morphisms $u : C \; ! \; P(V)$. For the rest of the section we x the degrees $d; d_G; \underline{d_L}$ and omit them to simplify the notation.

Denition 3.12. (Master space for stable gauged maps) Denote by $\overline{M}^G(C; X; L \; ; L_+)$ the ber product of the Givental morphism (32) with the projection from the master space

$$\overline{M}^G(C; X; L \; ; L_+) = \overline{M}^G(C; X; V \; ; V_+) \; {}_{M^{G;quot}(C;V \; ;V_+)} \; M^{G;quot}(C; L \; ; L_+):$$

Proof of Proposition 3.1. Under the assumption that stable=semistable, $d \; 2 \; H_+^G(X; Z)$, $\overline{M}_n^G(C; X; L \; ; L_+)$ is a proper Deligne-Mumford stack. Indeed, the ber product of proper morphisms is proper, and $\overline{M}^{G;quot}(C; X; L \; ; L_+)$ is projective, and so its image in the $\overline{M}^{G;quot}(C; V \; ; V_+)$ has proper coarse moduli spaces. It follows that the coarse moduli space of $\overline{M}^G(C; X; L \; ; L_+)$ is proper, hence (assuming nite stabiliz-ers) $M^G(\overline{C}; X; L \; ; L_+)$ is also proper. Consider the quotients by the circle action. The git quotients are the ber products

$$\begin{aligned}
\overline{M}^G(C; X; L \; ; L_+)=_t C &= \overline{M}^G(C; X; V \; ; V_+) \; {}_{M^{G;quot}(C;X;V \; ;V_+)} \; \overline{M}^{G;quot}(C; X; L \; ; L_+)=_t C \\
&= \overline{M}^G(C; X; V \; ; V_+) \; {}_{M^{G;quot}(C;X;V \; ;V_+)} \; M^{G;quot}(C; X; L_t) \\
&= \overline{M}^G(C; X; L_t)
\end{aligned}$$

which proves (b). Item (c) is similar.

We wish to prove (a): Assuming stable=semistable as above, $\overline{M}_n^G(C; X; L \; ; L_+)$ has an equivariant relative obstruction theory over the moduli stack $\overline{M}_n(C)$ of prestable maps to C of class $[C]$. The restriction of this obstruction theory to $\overline{M}_n^G(C; X; L \; ; L_+)$ is perfect and admits a resolution by vector bundles. To prove this recall the construction of the obstruction theory for $\overline{M}_n^G(C; X)$ from Remark 1.7. The complex in the relative obstruction theory is denoted $E_{\overline{M}_n^G(C;X;L \; ;L_+)}$ in the diagram below, constructed from the following commutative diagram of complexes of coherent sheaves. Let L denote the cotangent complex relative cotangent complex of $: \overline{M}_n^G(C;X;L_-;L_+) \to \overline{M}_n^G(C;X)$. Consider the diagram

$$\begin{array}{ccccccc}
L & \longrightarrow & E_{\overline{M}_n^G(C;X;L_-;L_+)} & \longrightarrow & E_{\overline{M}_n^G(C;X)} & \longrightarrow & L[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
L & \longrightarrow & L_{\overline{M}_n^G(C;X;L_-;L_+)} & \longrightarrow & L_{\overline{M}_n^G(C;X)} & \longrightarrow & L[1]
\end{array}$$

where the horizontal lines are exact triangles and the second vertical arrow exists by the third axiom in the denition of a triangulated category. The map satises the axioms of an obstruction theory by the ve lemma applied to the cohomology of the above diagram. If the automorphisms are nite then the relative obstruction theory is perfect: rst cohomology $H^1(E_{\overline{M}_n^G(C;X;L_-;L_+)})$ is identied with the Lie algebra of the group of automorphisms [52, Theorem 1.5] which vanish by assumption. Existence of a resolution follows from the fact that is a local complete intersection morphism, see [17, Appendix].

It remains to show item (d), that for any $d \in H_2^G(X;\mathbb{Z})$, the moduli stack $\overline{M}_n^G(C;X;L_-;L_+)$ admits an embedding in a non-singular Deligne-Mumford stack. Let

$$\overline{U}^{G;quot;lev}(C;X;L_-;L_+) \to \overline{M}^{G;quot;lev}(C;X;L_-;L_+)$$

denote the universal bundle over the moduli space of bundles with maps and level structures. $\overline{M}^{G;quot;lev}(C;X;L_-;L_+)$, equipped with the action of $GL(\underline{r}_-) \times GL(\underline{r}_+)$ by changing the level structure. Consider the embedding

$$\overline{U}^{G;quot;lev}(C;X;L_-;L_+) \times_G X \to \overline{U}^{quot;lev}(C;X;L_-;L_+) \times_{(GL(\underline{r}_-) \times GL(\underline{r}_+))} P(V_- \oplus V_+).$$ The latter is projective (it is the pull-back of the universal bundle on quot scheme) and so embeds $(GL(\underline{r}_-) \times GL(\underline{r}_+))$-equivariantly in some P^N. Then $\overline{M}_n^G(C;X;L_-;L_+)$ is an embedded substack of $\overline{M}_{0;n}(P^N)/(GL(\underline{r}_-) \times GL(\underline{r}_+))$, with objects given by stable maps that are compositions of stable sections

$$C \to \overline{U}^{G;quot;lev}(C;X;L_-;L_+) \times_G X/(GL(\underline{r}_-) \times GL(\underline{r}_+))$$

with the inclusion into $P^N/(GL(\underline{r}_-) \times GL(\underline{r}_+))$. Since $\overline{M}_{0;n}(P^N)$ is a non-singular Deligne-Mumford stack, the quotient $\overline{M}_{0;n}(P^N)/(GL(\underline{r}_-) \times GL(\underline{r}_+))$ is a non-singular Artin stack. The group $(GL(\underline{r}_-) \times GL(\underline{r}_+))$ acts locally freely on an open subset $\overline{M}_{0;n}(P^N)^{reg} \subset \overline{M}_{0;n}(P^N)$ containing $\overline{M}_n^G(C;X;L_-;L_+)$ by the stable=semistable assumption and the quotient $\overline{M}_{0;n}(P^N)^{reg}/(GL(\underline{r}_-) \times GL(\underline{r}_+))$ is Deligne-Mumford.

3.2. Analysis of the fixed point contributions. In this subsection, we deduce the quantum wall-crossing formula by applying virtual localization to the master space constructed in the previous subsection. By virtual Kalkaman Theorem 2.8 for the $\mathbb{C}^\times$ action on the master space $\overline{M}^G_n(C;X;L_-;L_+;d)$ we obtain the following preliminary version of the wall-crossing formula: For any fixed point component $F \subset \overline{M}^G_n(C;X;L_-;L_+;d)$ denote by ν_F the normal complex, defined as the $\mathbb{C}^\times$-moving part of the perfect obstruction theory in Proposition 3.1.

Proposition 3.13. Suppose that d is such that stable=semistable for $\overline{M}^G_n(C;X;L_\pm;L_+;d)$. Then for any $\alpha \in H_G(X)^n$,

$$\int_{[\overline{M}^G_n(C;X;L_+;d)]} ev^*\alpha - \int_{[\overline{M}^G_n(C;X;L_-;d)]} ev^*\alpha = \operatorname{Resid} \left(\sum_F \int_{[F]} ev_F^*\alpha \cup \operatorname{Eul}_{\mathbb{C}^\times}(\nu_F)^{-1} \right) \tag{33}$$

where F ranges over the fixed point components of $\mathbb{C}^\times$ on $\overline{M}^G_n(C;X;L_-;L_+;d)$ not equal to $\overline{M}^G_n(C;X;L_\pm;d)$.

Next we describe the moduli spaces of circle-fixed gauged maps in terms of gauged maps with smaller structure group. We begin with the following remark on actions of central subgroups on the moduli stacks of gauged maps. To simplify notation we denote $\overline{M}^G_n(C;X) = \overline{M}^G_n(C;X;L_t;d)$ the moduli stack of L-semistable gauged maps of class $d \in H_2^G(X;\mathbb{Z})$.

Proposition 3.14. Let $Z \subset G$ a central subgroup. The action of Z on X induces a natural action of Z on $\overline{M}^G_n(C;X)$.

Proof. For any principal G-bundle $P \to C$, the right action of Z on P induces an action on the associated bundle $P(X)$, and so on the space of sections of $P(X)$. The action of Z on the space of sections of $P(X)$ preserves Mundet semistability (since the parabolic reductions are invariant under the action and the Mundet weights are preserved) and so induces an action of Z on $\overline{M}^G_n(C;X)$. $\square$

The following is similar to the description of fixed point sets in the case of stable maps in Kontsevich [42] and Graber-Pandharipande [31, Section 4].

Proposition 3.15. Let $Z \subset G$ be a central subgroup. The fixed point locus for the action of Z on $\overline{M}^G_n(C;X)$ is the substack whose objects are tuples

$$(p : P \to C;\ u : \hat{C} \to P(X);\ \underline{z})$$

such that

(a) u takes values in $P(X^Z)$ on the principal component C_0;
(b) for any bubble component $C_i \;\; \hat{C}$ mapping to a point in C, u maps C_i to a one-dimensional orbit of Z on $P(X)$; and
(c) any node or marking of $\hat{C}$ maps to the xed point set $P(X^Z)$.

We will identify the following stacks with the xed point sets in the master space.

Denition 3.16. (Fixed point stacks) For any 2 g generating a one-parameter subgroup C G, recall that G denotes the centralizer of C and so contains C as a central subgroup. For each rational 2 g let

$$\overline{M}_n^G(C; X; L_t; ; d) \;\; M_n\overline{(C}^G_; X; L_t; d)$$

denote the stack of L_t-Mundet-semistable morphisms from C to X=G that are C -xed and take values in X on the principal component.

The Mundet semistability condition for xed gauged maps simplies somewhat in the limit of large polarization, see [30, Lemma 6.3] for more details. Let $X^{;t}$ denote the (possibly empty) locus of L_t-semistable points in X .

Lemma 3.17. (Large-area limit of xed gauged maps) For any class d there exists a $_0$ such that for $> _0$, any Mundet-semistable xed map for polarization L_t must consist of a principal component mapping to $X^{;t}=G$ and bubbles mapping to X=G.

Proof. Mundet semistability for L implies that the Hilbert weight $_H(;)$ is at most ^{1} times minus the Ramanathan weight $_R(;)$, for any ;. In particular, this holds = for antidominant and the trivial parabolic reduction, in which case the Ramanathan weight for a pair (P; u) with respect to (;) is simply $hc_1(P); i$ The latter is bounded by a constant c(d)kk depending on d 2 $H^G_2(X; Z)$, since $c_1(P)$ is the projection of d onto $H_2(BG)$. So the Hilbert weight $_M(;)$ is less than ckk where c := c(d) ^{1}. Choose $_0$ suciently large so that for any $> _0$, any point with Hilbert-Mumford weight $_M(;)$ less than ckk for all is semistable; see for example [41, Lemma 3.12]. Then any L_t-semistable pair (P; u), the section u takes values in $P(X^{;t})$.

Proposition 3.18 (Fixed points as reducible gauged maps). Any C-xed compo-nent of $M^G_n(\overline{C; X}; L ; L_+)$ is in the image of $M^G_n(\overline{C; X}; L_t;)$ in $M^G_n(C; \overline{X; L} ; L_+)$ for some t 2 (1; 1) where 2 g is a non-zero element, G is stabilizer, and C G the unparametrized one-parameter subgroup generated by , consisting of maps u : $\hat{C}$! X=G taking values in X=G on the principal component, and X=G on the bubbles.

Proof. Any xed object of C in $\overline{M}^G_n(C; X; L ; L_+)$ not in the xed point components $\overline{M}^G_n(C; X; L)$ is a datum : P ! C; u : $\hat{C}$! P(X) of $\overline{M}^{G;lev}(C; X; L_t)$ for some t 2 (1; 1) with a one-parameter group of automorphisms : P ! P; 2 C and : C ! C trivial on the principal component C_0 and intertwining the section u in the sense that (X) u = u. The innitesimal automorphism corresponding to is a section of the adjoint bundle P(g), given by an element 2 $P(g)_z$ = g at a base point z 2 C. The structure group of P reduces to the centralizer G, and the section u takes values in the xed point set P(X) = P(X) of on the principal component.

Remark 3.19. The xed point locus admits a description in terms of \bubble trees" as follows: There is an isomorphism

$$\overline{M}^G_n(C; X; L_t;) \; ! \bigcup_{r;[I_1;:::;I_r]} M^{G;fr}_r(C; X^{;t})_{(X)^r} \left(\prod_{j=1}^{r} \overline{M}_{j I_j j+1}(X) \right)^C =(G)^r$$

where $I_1 \cup ::: \cup I_r$ f1; : : : ; ng is a disjoint union of subsets describing mark-ings lying on bubble components and $M^{G;fr}(C; X^{;t})$ denotes the moduli stack of gauged maps with framings at the marked points. Indeed, by denition each object of $\overline{M}^G_n(C; X; L_t;)$ consists of a principal component mapping to $X^{;t}=G$ and a collection of bubble trees in X xed (up to reparametrization) by the action of C .

Corollary 3.20. (Obstruction theory for the xed point components) $\overline{M}^G_n(C; X; L_t;)$ is an Artin stack, and if every automorphism group is nite modulo C , each sub-stack with xed homology class d 2 $H^G_2(X; Z)$ is a proper Deligne-Mumford stack with a C-equivariant relatively perfect obstruction theory over $\overline{M}_n(C)$.

Proof. The relatively perfect obstruction theory $\overline{M}^G_n(C; X; L_t;)$ is pulled back from that on the C-xed point set in $\overline{M}^G_n(C; X; L ; L_+)^C$ in Proposition 3.18. The latter is a special case of existence of relatively perfect obstruction theories on xed point loci discussed in [31].

Lemma 3.21. The conormal complex $_t$ of the morphism $\overline{M}^G_n$

$$(C; X; L_t;) \; ! \; \overline{M}^G_n(C; X; L ; L_+)$$

is isomorphic to the C-moving part of the obstruction theory in $\overline{M}^G_n(C; X; L_t)$, whose relative part is $(R p e T_{X=G})$.

Proof. By denition the obstruction theory for $\overline{M}^G_n(C; X; L ; L_+)$ ts into an exact triangle with that of $\overline{M}^G_n(C; X)$ and a trivial factor corresponding to the ber of $P(D(L) \; D(L_+))$. Under projection the normal complex to the xed point component $\overline{M}^G_n(C; L_t; X;)$ of the C-action is isomorphic to the moving part of the obstruction theory on $\overline{M}_n^G(C; L_t; X)$, under the identication of C with C, as in Lemma 2.10.

Virtual integration gives rise to the xed point contributions in the wall-crossing formula. Let $[\overline{M}_n^G(C; X; L_t; ; d)]$ denote the virtual fundamental class in the homology of the coarse moduli space resulting from Corollary 3.20. Integration with respect to these classes yields -xed gauged Gromov-Witten invariants of Deni-tion 3.22. The -xed gauged Gromov-Witten invariants that appear in the wall-crossing formula involve further twists by Euler classes of the virtual normal complex: Recall that we constructed in the previous section a perfect obstruction theory on $\overline{M}_n^G(C; X; L_t;)$, as well as a normal complex for the embedding in $\overline{M}_n^G(C; X; L ; L_+;)$.

Denition 3.22. [Fixed point contributions to wall-crossing for Gromov-Witten invariants] Virtual integration over the stacks $\overline{M}_n^G(C; X; L_t; ; d); d \; 2 \; H_2^G(X; Z)$ denes a \xed point contribution"

$$\text{(34)} \quad {}_{X;;t} : QH_{G;n}(X) \; ! \; {}^G[; \tilde{}^{\,1}_X]; \quad \sum_X \sum_{(X;Z)_G \; n0} \int_{[\overline{M}_n(C;X;L_t;;d)]^G} \frac{q^d}{n!} ev(; : : : ;) \, [\; Eul(_t) \; ^1 [\; f_n \; d2H_2$$

for $2 \; H_G(X)$ and a sequence of classes $_n \; 2 \; H(\overline{M}_n(C))$, extended by (multi)linearity of the integral over ${}^G_X$, and where we omit the restriction map $H_G(X) \; ! \; H_G \; (X)$ to simplify notation.

This completes the construction of the xed point potential in Denition 3.22.

Remark 3.23. The xed point potential ${}_{X;;t}$ takes values in ${}_X^{G}$ rather than in ${}^G_{X;L_t}$. Indeed, the number of possible pairings of classes of gauged maps with $c_1(L_t)$ in the case that a central subgroup C acts trivially can be arbitrarily small, since twisting by a character of C does not change the pairing.

Remark 3.24. The right-hand-side of the formula in Theorem 1.13 can also be re-written using the quantum Kirwan map for

$$QH_{G=C}(X^{;t}) \; ! \; QH_C(X^{;t}=(G=C))$$

using the adiabatic limit theorem for (G=C)-gauged maps. However, in our examples the gauged Gromov-Witten invariants are always easier to compute than the Gromov-Witten invariants of the git quotients, so we have left the formula as written.

3.3. **The wall-crossing formula.** By the adiabatic limit theorem 1.10, to prove the wall-crossing formula 1.13 it suces to prove a formula for the dierence of gauged potentials. The following result is an algebro-geometric generalization of a wall-crossing formula of Cieliebak-Salamon [13] for gauged Gromov-Witten invariants of quotients of vector spaces dened using symplectic geometry. We will deduce our main result Theorem 3.26 by taking the large area limit ! 1 of the following Theorem:

Theorem 3.25 (Wall-crossing for gauged Gromov-Witten potentials). *Let X be a smooth projective G-variety. Suppose that L ! X are polarizations such that semistable=stable for the stack $\overline{M}^G_n(C; X; L ; L_+)$ of 3.1. The gauged Gromov-Witten potentials are related by*

$$(35) \qquad {}^G_{X;+} \quad {}^G_{X;} = \sum_{[];t2(1;1)} \text{Resid}_{X;;t}$$

where the sum is over equivalence classes [] as in (21).

Proof. The statement follows from virtual localization applied to $\overline{M}^G_n(C; X; L ; L_+)$ and the identication of xed point contributions in Proposition 3.18.

Combining Theorem 3.25 with the adiabatic limit Theorem 1.10 implies:

Theorem 3.26 (Quantum Kalkman formula, arbitrary group case). *Suppose that X is equipped with polarizations L so stable=semistable for the action of G on P(L L₊). Then the Gromov-Witten invariants of X=G are related by a sum of twisted gauged Gromov-Witten invariants for subgroups G G*

$$(36) \qquad {}_{X=_+G} \, {}^G_{X;+} \quad {}_{X= G} \, {}^G_{X;} = \lim_{! 1} \sum_{[];t2(1;1)} \text{Resid}_{X;;t};$$

where the sum is over [] in (21).

We already gave a simple Example 1.16 of the formula in Theorem 3.26 in the introduction. We give another Fano example:

Example 3.27. (Quantum powers of the rst Chern class for the blow-up of the projective plane) Suppose that, as in Example 2.6 (b), G = (C)^2 acts on X = C^4 with weights (1; 0); (1; 0); (1; 1); (0; 1) 2 Z^2. Consider the path from (1; 2) to (2; 1) in $H^2_G(X) = Q^2$ crossing through the chambers with git quotients ;; P^2; Bl(P^2); ; as in Example 2.6 (b). Denote by X= G resp. X=₊G the second resp. third quotient. The quantum Kirwan morphism for P^2; Bl(P^2) has no quantum corrections, since these varieties are Fano. Hence

$$(37) \qquad D_{0X;}({}^G c_1^G(X)) = c_1(X=G):$$

We consider the wall-crossing formula for invariants with 5 xed markings, corresponding to the small quantum product

$$c_1(X=G)^{?5} \; 2 \; QH(X=G):$$

In the notation of (4) we wish to compute

$$@^5_{(c_1(X=G);0)X=G}(1; [pt]) \; 2 \; {}_X \quad {}^G$$

where

$$c_1(X=G) \; 2 \; H(X=G); \quad [pt] \; 2 \; H(\overline{M}_{0;5}):$$

By Example 1.6, the moduli space of gauged maps is, after xing the locations of the markings, the quotient of the space of sections $H^0(P; O_C(d) \;_G X)$ by G, with stability condition corresponding to the stability condition for X, see [62]. We take $d = (1; 0)$, so that

$$H^0(P; O_C(d) \;_G X) = C_{(0;1)} \; C_{(1;1)} \; C^2_{(1;0)} \quad {}^4$$

(or C^7 for short) where for any weight , C denotes the one-dimensional representation with weight . We consider the sequence of polarizations L_t corresponding to the vectors

$$(\quad 1; 2); (1; 2); (2; 1); (2; \quad 1) \; 2 \; g_-$$

lying in the path of chambers from left to right in Figure 1. The moduli spaces of gauged maps corresponding to the various chambers are therefore the empty set, P^5, its blow-up along a projective line $Bl_{P^1}(P^5)$, and the empty set again. The stabilizers for the wall-crossing terms are the perpendicular vectors to the walls in Figure 1. The rst wall-crossing term for degree $(1; 0)$ invariants corresponds to the direction $= (1; 0)$, for which there is a unique -xed stable map with normal weights $(1; 0)$ with multiplicity 4 and $(1; 1)$ with multiplicity 2. The wall-crossing term is

$$\mathrm{Resid}_1 \frac{(3 +_1 2 \quad {}_2)^5}{({}^4_1 \quad {}_1 + {}_2)^2} \Big|_{2=0} = 243:$$

Using (37) we obtain that $c_1(P^2)^{?5} \quad 243 q_1[pt]$ which is shorthand for saying that the coecient of $q_1[pt]$ is 243; here $?$ is the small quantum product, as expected since

$$c_1(P^2)^{?5} = (3!)^{?5} = 243!^{?3} \; ? \; !^{?2} = 243 q_1[pt]:$$

The second wall-crossing term corresponds to the change of quotient from P^2 to $Bl(P^2)$ with $= (1; \quad 1)$ is (after xing the ve points on P) an integration over $\overline{M}^G_0(P; X; ; t) \quad P_2$, the quotient of the -xed summand $C_{(1;1)}$ by C,

$$\frac{1}{2} \mathrm{Resid} \int_{C \;\; {}_1 \; {}_2 \; C^7} \frac{(3_1 + \; 2_2)^5 \, [P]}{4}$$

where $\int^{C}_{C^7}$ is the descent map in equivariant cohomology from C^7. Using that $c_{6;C}(\ _1)$ is the generator ! of $H^2(P)$ this gives

$$\frac{1}{2}\,\text{Resid} \int_{[P]} \frac{(5!\ +)^{5}}{(!\ +\ )^4 (!\qquad)} = \frac{1}{2}\,\text{Resid} \int_{[P]} {}^{5}\ 25!^4 + {}^{5}1 \qquad 4\frac{!}{}1 + \ = \frac{!}{} \ 11:$$

It follows that the coecient of [pt] in $c_1(Bl(P^2))^{?5}$ is $232q_1$.

For the transition to the empty chamber, as before $\int^{C}_{C^4}$ maps to $_1$ to the generator ! of $H^2(P^3)$ while $_2$ maps to the parameter for the residual C-action. The wall-crossing term is

$$\text{Resid} \int_{[P^3]} \int^{C}_{C^4} \frac{3\ (\ _1 + 2\ _2)^5}{(\ _1 + \ _2)^2} = \text{Resid} \int_{[P^3]} \frac{(3!\ +\ 2)^5}{(!\ +\ )^2} =$$

$$\text{Resid} \int_{[P^3]} (3!)^3 (2)^2\, 10\ {}^{3} + (3!)^2 (2)^3\, 10\ {}^{3} \quad 2 \quad \frac{!}{}^{2}$$

$$+ (3!)(2)^4\ 5\ {}^{3}3\ {}_2 \frac{!}{}^{2} + (2)^5\ 4\ {}^{3}\ {}_3 \frac{!}{}^{3}$$

$$= 1080 \quad 1440 + 720 \quad 128$$

$$= 232:$$

This is as expected since the quotient $X{=}_tG$ is empty in the last chamber. One can verify that the expansion of $c_1(Bl(P^2))^{?5}$ contains $232q_1$[pt] using the known quantum multiplication table for $Bl(P^2)$ from Crauder-Miranda [20]:

?	e	f	p
e	$p + eq^e + xq^f$	$p \quad eq^e$	fq^f
f		eq^e	xq^{e+f}
p			$(e + f)q^{e+f}$

where $e; f; p; x \ 2\ H(Bl(P^2))$ are the exceptional resp. ber resp. point resp. fundamental classes respectively, $q_1 = q^{e+f}$, and using a little help from Mathematica. (We thank Eric Malm for teaching us how to get Mathematica to compute these coecients.)

4. Invariance under crepant transformations

Ruan and others, see [15], conjectured that crepant resolutions induce equivalences in Gromov-Witten theory. We prove a version Theorem 1.17 of Ruan's conjecture for crepant birational equivalences induced by variation of git.

4.1. Crepant transformations. We consider the birational transformations that are crepant in the following sense:

Denition 4.1. Suppose that Y are smooth proper Deligne-Mumford stacks with projective coarse moduli spaces related by a birational equivalence given by open embeddings

$$Y \longleftarrow Z \underset{+}{\longrightarrow} Y_+ :$$

Such a birational equivalence is called crepant (or a K-equivalence) if extends to morphisms $\tilde{:} Z \,!\tilde{}\, Y$ from a smooth stack $\tilde{Z}$ with projective coarse moduli space such that the pullbacks of the canonical divisors to $\tilde{Z}$ are equal, as in Kawamata [38]. This ends the denition.

A well-known conjecture of Li-Ruan [46], Bryan-Graber [10] and others (perhaps motivated by physics papers such as Witten [61]) that in such a situation (not necessarily arising from geometric invariant theory) the Gromov-Witten theories of Y and Y_+ are equivalent, in a sense to be made precise. Many special cases have been proved, see for example Iwao-Lee-Lin-Wang [36], Lee-Lin-Wang [43], Boissiere-Mann-Perroni [5], Bryan-Gholampour [9], [8], Bryan-Graber-Pandharipande [11], Coates-Corti-Iritani-Tseng [16] and Coates-Iritani-Tseng [18].

We specialize to the case that the birational transformation is obtained by variation of git quotient. Suppose that X is a smooth projective G-variety, and $X=G$ are git quotients obtained from polarizations $L \; ! \; X$. Since the semistable loci are open, the identity on the locus semistable for both polarizations induces a birational transformation from $X= G$ to $X= G$. We call such a birational transformation of git type. Suppose that stable=semistable for $P(L \quad L_+)$ so that the master space $\tilde{X} = P(L \quad L_+)=G$ is a smooth proper Deligne-Mumford stack.

Denition 4.2. A birational transformation of git type $= (\;; _+)$ will be called crepant if the sum of the weights $_i(F)\; 2\; Z$ of C on the normal bundle to any xed point component $F \quad X^{;t}$, counted with multiplicity, vanishes:

$$\sum_{i=1}^{\operatorname{codim}(F)} {}_i(F) = 0; \qquad 8F \quad X^{;t}:$$

Denition 4.3. The denition of crepant transformation of git type is a special case of the denition of crepant transformation (K-equivalence) in Kawamata [38] etc. Indeed, Kempf's descent lemma [22, Theorem 2.3] and the crepant condition together imply that the canonical bundle descends to each singular quotient, from which the canonical bundles on $X=G$ are pulled back. The ber product of these morphism is the required smooth stack in the denition of crepant transformation.

4.2. **The Picard action.** The proof of invariance in Theorem 1.17 uses a symmetry of the xed point contributions under an action of the Picard stack

$$\mathrm{Pic}(C) := \mathrm{Hom}(C; BC)$$

of line bundles on C; a similar action was used in a proof of a generalized Verlinde formula in [58]. The Lie algebra g has a distinguished factor C generated by , and using an invariant metric the weight lattice of g has a distinguished factor Z given by its intersection with the Lie algebra of C. After passing to a nite cover, there exists a splitting $G = G{=}C \; C$.

We dene an action of the Picard group on the moduli stack as follows. Recall that an object of $\overline{M}^G_n(C; X; L_t;)$ consists of a tuple $(P; \hat{C}; u)$ where $P \to \hat{C}$ is a G-bundle and $u : \hat{C} \to P(X)$ is -xed, in particular, the restriction of u to the principal component of C maps into the xed point locus X. Let $C \; G$ denote the subgroup of G generated by $\; 2 \; g$.

Denition 4.4. (Picard action) For $Q \to C$ a line bundle and $(P; \hat{C}; u)$ an object of $\overline{M}^G_n(C; X; L_t;)$ dene

$$Q(P; \hat{C}; u) := (P \times_C Q; \hat{C}; v) \tag{38}$$

where v is dened as follows: We have an isomorphism of associated bundles

$$(P \times_C Q)(X) \; P(X)$$

since the action of C on X is trivial. Hence the principal component of u, which is a section of $P(X)$ induces a corresponding section of $(P \times_C Q)(X)$. Each bubble component of u maps into a ber of $P(X)$, canonically identied with X up to the action of G, and so induces a corresponding map into a ber of $(P \times_C Q)(X)$, well-dened up to isomorphism.

The action of the Picard group preserves semistable loci in the large area limit. Indeed, because the Mundet weights $_M(;)=$ approach $_H(;)$ as $\to 1$, the limiting Mundet weight is unchanged by the shift by Q in the limit $\to 1$ and so Mundet semistability is preserved, see Remark 3.17. It follows that for suciently large the action of an object Q of $\mathrm{Pic}(C)$ induces an isomorphism

$$S : \overline{M}^G_n(C; X; L_t; ; d) \to \overline{M}^G_n(C; X; L_t; ; d +) \tag{39}$$

where $= c_1(Q)$. The action lifts in an obvious way to the universal curves $\overline{C}^G_n(C; X; L_t; ; d) \to \overline{C}^G_n(C; X; L_t; ; d +)$, denoted with the same notation.

Lemma 4.5. The action of $Pic(C)$ in (39) induces isomorphisms of the relative obstruction theories, and so the Behrend-Fantechi virtual fundamental classes. Furthermore, the action preserves the class ev for any 2 $H_G(X)^n$.

Proof. The action of $Pic(C)$ lifts to the universal curves, denoted by the same notation. Since the relative part of the obstruction theory on $\overline{M}^G_n(C; X; L_t; ; d)$ is the C -invariant part of $(Rp_* e^* T_{X=G})^-$ up to the factor C, the isomorphism S preserves the relative obstruction theories on $M^G_n(\overline{C}; X; L_t; ; d)$ and $M^G_n(C; \overline{X}; L_t; ; d +)$ and so the Behrend-Fantechi virtual fundamental classes $[M^G_n(C; X; \overline{L_t; ; d})]$ and $[M^G_n(C; X; L_t ; ; d +)]$. (Note that on the principal component, the obstruction theory is $(Rp_* e^* T X=G)^-$ which is unchanged by the tensor product by C-bundles. On the bubble components $(Rp_* e^* T X=G)^-$ is unchanged by the tensor product since the pull-back of Q to C is trivial.) Since the evaluation map is unchanged by pull-back by S (up to isomorphism given by $\hat{}$ twisting by Q), the class ev is preserved.

Remark 4.6. To interpret the main result we recall the basic denitions from the Schwartz theory of distributions for which the standard reference is Hörmander [34]. We only need the case of distributions on the unit circle S. Denote by $D^0(S)$ the space of continuous linear functionals on the smooth functions on S, and by $E^0(S)$ $D^0(S)$ the space of tempered distributions. Fourier transform denes an isomorphism of $E^0(S)$ with the space of functions on Z with polynomial growth. We view q as a coordinate on the punctured plane C. Any formal power series in $q; q^{1}$ denes a distribution on S, which is tempered if the coecient of q^d has polynomial growth in d. In particular $\sum_{d 2 Z} q^d$ is the delta function at q = 1, and has Fourier transform the constant function with value 1. Any distribution of the form $\sum_{d 2 Z} f(d)q^d$, for f(d) polynomial, is a sum of derivatives of the delta function (since Fourier transform takes multiplication to dierentiation) and so is almost everywhere zero.

4.3. Proof of invariance. In this section we prove Theorem 1.17. We study the dependence of the xed point contributions with respect to the Picard action dened in (38). Suppose that Q is a C -bundle of rst Chern class the generator of $H^2(C)$, after the identication C ! C. Denote the corresponding class in $H_2(X)$ by . Consider the action of the Z-subgroup of $Pic(C)$ generated by Q. The contribution of any component $M^G_n(C; \overline{X}; L_t; ; d)$ of class d 2 $H^G_2(X)$ diers from that from the component induced by acting by Q^r, of class d + r, by the ratio of Euler classes of the virtual normal complex $(Rp_* e^* T(X=G))^+$

$$(40) \qquad \frac{Eul_C((Rp_* e^* T(X=G))^+)}{Eul_C(S^{;}_r (Rp_* e^* T(X=G))^+)} \; 2 \; H(M^G_n(\overline{C}; X; L_t; ; d))$$

which we now compute. Let $X^{;t}$ be the component of the xed point set X which is semistable for t 2 (1; 1). For simplicity, we assume that $X^{;t}$ is connected; in general, one should repeat the following argument for each connected component. Let

$$[_{X^{;t}=G}] = [TX=TX^{;t}] \qquad [g=g]$$

denote the class of the virtual normal complex for $X^{;t}=G$! $X=G$. Consider the decomposition into C-bundles

$$_{X^{;t}=G} = \bigoplus_{i=1}^{m_t} {}_{X^{;t};i}$$

where C acts on $_{X^{;t};i}$ with non-zero weight $_i$ 2 Z and m_t is the codimension of $X^{;t}$, which for simplicity we assume is constant. Then $eT(X=G)$ is canonically isomorphic to $S^r eT(X=G)$ on the bubble components, since the G-bundles are trivial on those components. Because the pull-back complexes are isomorphic on the bubble components, the dierence

$$(eT(X=G))^+ \quad S^{r;}(eT(X=G))^+ \; 2 \; K(\overline{M}_n^G(C; X; L_t; ; d))$$

is the pullback of the dierence of the restrictions to the principal part of the universal curve, that is, the projection on the second factor

$$p_0 : C \; \overline{M}_n^G(C; X; L_t;) \; ! \; \overline{M}_n^G(C; X; L_t;):$$

These restrictions are given by

$$(41) \qquad (eT(X=G))^{+;prin} = \bigoplus_{i=1}^{m_t} e_{X^{;t};i} \qquad g;$$

$$(42) \qquad S^{r;}(eT(X=G))^{+;prin} = \bigoplus_{i=1}^{m_t} \frac{e_{X^{;t};i}}{(eQ_C \; C_{r_i})} \qquad g$$

where e is the map from the universal curve to C. The projection p_0 is a representable morphism of stacks given as global quotients. To compute the dierence in push-forwards we apply Grothendieck-Riemann-Roch for such stacks [60], [24]. The Todd class on the curve is

$$Td_C = 1 + (1 \quad g)!_C$$

so

$$(43) \qquad Td_{CM} = (1 \quad g)!_C + Td_M:$$

Let

$$z : \overline{M}_n^G(C; X; L_t;) \; ! \; C \; \overline{M}_n^G(C; X; L_t;)$$

be a constant section of p_0. Then

$$\begin{aligned} \mathrm{Td}_M \, \mathrm{Ch}(S^{r;} Rp(eT(X=G))^+) &= p_{0;}(\mathrm{Td}_{CM} \, \mathrm{Ch}(S^{r;} eT(X=G))^+) \, (1 \\ &= g) z \, \mathrm{Ch}(S^{r;}(eT(X=G))^+) + \\ & \quad \mathrm{Td}_M \, p_{0;} \, \mathrm{Ch}(S^{r;} eT(X=G)^+) \end{aligned}$$

by Grothendieck-Riemann-Roch and (43). Continuing we have

$$\begin{aligned} ::: &= (1 \quad g) z \, \mathrm{Ch}(e(T(X=G))^+) + \\ & \quad \mathrm{Td}_M \, p_{0;} \sum^{m_t} \mathrm{Ch}(e_{X;t;i}) \, \mathrm{Ch}((eQ_C \quad C_{r_i})) \; {}_{i=1} \\ &= (1 \quad g) z \, \mathrm{Ch}(S^{r;} e(T(X=G))^+) + \quad \mathrm{Td}_M \\ & \quad p_{0;} \sum_{i=1}^{m_t} \mathrm{Ch}(e_{X;t;i})(1 + r_i !_C) \end{aligned}$$

since the bundle Q is trivial on any ber of C M ! M and using (42). Continuing using multiplicativity of the Chern character and Grothendieck-Riemann-Roch again this equals

$$\begin{aligned} ::: &= p_{0;}(\mathrm{Td}_{CM} \, \mathrm{Ch} \left(Rpe(T(X=G))^+ \bigoplus_{i=1}^{m_t} (e_{X;t;i})^{r_i} \right) \\ &= \mathrm{Td}_M \, \mathrm{Ch} \left(Rpe(T(X=G)))^+ \bigoplus_{i=1}^{m_t} (z e_{X;t;i})^{r_i} \right) : \end{aligned}$$

Hence

$$\mathrm{Ch}(S^{r;} Rpe(T(X=G))^+) = \mathrm{Ch} \left(Rpe(T(X=G))^+ \bigoplus_{i=1}^{m_t} (z e_{X;t;i})^{r_i} \right) \tag{44}$$

The equality of Chern characters above implies by injectivity of the Todd map [24] an equality

$$[S^{r;} \, \mathrm{Ind}(T(X=G))^+] = [\mathrm{Ind}(T(X=G))^+ \bigoplus_{i=1}^{m_t} (z e_{X;t;i})^{r_i}] :$$

By the splitting principle we may assume that the ${}_{X;t;i}$ are line bundles. The difference in Euler classes (40) is therefore given by the Euler class of the last summand in (44)

$$\frac{\mathrm{Eul}_{\mathbb{C}}(Rp\ eT(X/G)^+)}{\mathrm{Eul}_{\mathbb{C}}(S^{r;}Rp\ eT(X/G)_+)} = \mathrm{Eul}_{\mathbb{C}}\left(\bigoplus_{i=1}^{m_t} (z e_{X;t;i})_i^{r}\right)$$

$$= \prod_{i=1}^{m_t} (\ _i + c_1(X_{;t;i}))_i^{r}$$

$$= \prod_{i=1}^{m_t} {}_i^{r} \prod_{i=1}^{m_t}\left(+ \frac{c_1(X_{;t;i})}{_i}\right)^{r}.$$

Let

$$ = \sum_{i=1}^{m_t} {}_i \in \mathbb{Z}$$

be the sum of weights of the action of $\mathbb{C}$ at the xed point component $X^{;t}$. Expanding out the product we obtain

(45) $$\prod_{i=1}^{m_t}\left(+ \frac{c_1(X_{;t;i})}{_i}\right)^{r} = {}^{r} + {}^{r-1}\left(r\sum_{i=1}^{k} c_1(X_{;t;i})\right)$$

$$+ {}^{r-2}\left(r^2 \sum_{i=j} c_1(X_{;t;i})c_1(X_{;t;j}) + \sum_{i=1}^{k} r\ r\ \frac{1}{_i}\ c_1(X_{;t;i})^2\right) + \dots$$

and $\dots$ indicates further terms with the property that the coecient of ${}^{r-m}$ is polynomial in r. By the crepant assumption in Denition 4.2, the sum of the weights is $= \sum_{i=1}^{m_t} {}_i = 0$. Write

$$_{X;;t} = \sum _{X;;d;t}\ d$$

where $_{X;;d;t}$ is the contribution from gauged maps of class d. For any singular value $t \in (-1;1)$,

(46) $$\sum_{r \in \mathbb{Z}} q^{d+r}\ _{X;;d+r;t}() = \sum_{r\in\mathbb{Z}} \prod_{i=1}^{k} {}_i^{r}\ q^{d+r} \int_{[\overline{M}_n^G(C;X;L_t;;d)]} \left(1 + {}^{-1}\sum_{i=1}^{k} r c_1(X_{;t;i}) + \dots\right) \mathrm{ev}(;\dots;) \cup \mathrm{Eul}(_t)^{-1} \cup f_n$$

where as before the terms $\ldots$ are polynomial in the r. In the language of distributions, for any polynomial $f(r)$ in r,

$$\sum_{r \in \mathbb{Z}} f(r) \left(\prod_{i=1}^{k} \ldots_i \, q \right)^r =_{a.e.} 0 \tag{47}$$

vanishes almost everywhere in q, being a function times a sum of derivatives of delta functions in q, see Remark 4.6.) Since $\tau_X^{G,+} X/\!/G_+ - \tau_X^{G,-} X/\!/G_-$ is a sum of wall-crossing terms of the form (47), this completes the proof of Theorem 1.17.

Remark 4.7. The standard formulation of the crepant transformation conjecture in Coates-Ruan [15] etc. uses analytic continuation. The above results say nothing about convergence of the gauged Gromov-Witten potentials, so it is rather difficult to put the version above in this language. However, if the potentials $\tau_X^{G,+} X/\!/G_+$ and $\tau_X^{G,-} X/\!/G_-$ have expressions as analytic functions with overlapping regions of definition on the torus $H^2_G(X;\mathbb{Q})/H^2_G(X;\mathbb{Z})$ with coordinate q, then they are equal on that region.

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Department of Mathematics University of Massachusetts Boston 100 William T. Morrissey Boulevard Boston, MA 02125
Email address: eduardo@math.umb.edu

Mathematics-Hill Center, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, U.S.A.
Email address: ctw@math.rutgers.edu