ERRATIC BIRATIONAL BEHAVIOR OF MAPPINGS IN POSITIVE CHARACTERISTIC

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Abstract. Birational properites of generically finite morphisms $X \rightarrow Y$ of algebraic varieties can be understood locally by a valuation of the function field of X. In finite extensions of algebraic local rings in characteristic zero algebraic function fields which are dominated by a valuation there are nice monomial forms of the mapping after blowing up enough, which reflect classical invariants of the valuation. Further, these forms are stable upon suitable further blowing up. In positive characteristic algebraic function fields it is not always possible to find a monomial form after blowing up along a valuation, even in dimension two. In dimension two and positive characteristic, after enough blowing up, there are stable forms of the mapping which hold upon suitable sequences of blowing up. We give examples showing that even within these stable forms, the forms can vary dramatically (erratically) upon further blowing up. We construct these examples in defect Artin-Schreier extensions which can have any prescribed distance.

1. Introduction

Suppose that $\varphi : X \rightarrow Y$ is a morphism of algebraic varieties over a field k. We would like to find resolutions of singularities X_1 of X and Y_1 of Y such that there is an induced morphism $X_1 \rightarrow Y_1$ which has the simplest possible local structure. In [4, Problem 6.2.1], it is asked if, with the assumption that k has characteristic zero, there always exists such an $X_1 \rightarrow Y_1$ which is toroidal (the problem of toroidalization). Certainly no simpler global structure can always be found. This toroidal form can be found if the varieties have characteristic zero and are of dimension ≤ 3 ([10], [11] and a simplified proof in [9]). Stronger local forms than toroidalization are true in complete generality in characteristic zero. To formulate this result, we use valuations.

The use of valuations converts the study of birational properties of morphisms into a problem in local commutative algebra. This approach was initiated and developed by Zariski. Suppose that $\varphi : Y \rightarrow X$ is a dominant morphism of projective (or proper) k-varieties and we have a commutative diagram of morphisms

$$\begin{array}{ccc} Y_1 & \rightarrow & X_1 \\ \downarrow & & \downarrow \\ Y & \rightarrow & X \end{array}$$

where $Y_1 \rightarrow Y$ and $X_1 \rightarrow X$ are birational and projective (or proper). A valuation ω of the function field $k(Y)$ (which is trivial on k) restricts to a valuation ν on $k(X)$. The valuations have centers at points y and x on Y and X respectively and have centers y_1 and x_1 on Y_1 and X_1 respectively. The behavior of the morphisms in the diagram near

Key words and phrases. morphism, valuation, positive characteristic.

Partially supported by NSF grant DMS-2054394.

the centers of the valuations is determined completely by the commutative diagram

$$\begin{array}{ccc} R_1 & \rightarrow & S_1 \\ \uparrow & & \uparrow \\ R & \rightarrow & S \end{array}$$

where R, S, R_1, S_1 are the respective local rings $\mathcal{O}_{X,x}, \mathcal{O}_{Y,y}, \mathcal{O}_{X,x_1}, \mathcal{O}_{Y,y_1}$. Thus, we are able to formulate locally problems in birational geometry in terms of valuation theory and local commutative algebra.

A classical example of this approach is the problem of local uniformization as formulated by Zariski. Suppose that R is a local domain which is essentially of finite type over a field k with quotient field K. Let ν be a valuation of K which dominates R. If $R \rightarrow S$ is an extension of local rings such that the maximal ideal of S contracts to the maximal ideal of R then we say that S dominates R. If S is dominated by the valuation ring $\mathcal{O}_\omega$ of a valuation ω we say that ω dominates S. A local uniformization of R along ν is a birational extension $R \rightarrow R_1$ where R_1 is a regular local ring which is essentially of finite type over R and is dominated by ν. Zariski proved local uniformization in all dimensions and characteristic zero in [36] and established resolution of singularities in characteristic zero from local uniformization in dimension three [37]. Hironaka later proved resolution of singularities in all dimensions and characteristic zero in [22], using a different method. All current proofs of resolution of singularities of 3-folds in positive characteristic are accomplished by first proving local uniformization ([3], A simplification of this proof in [13], Cossart and Piltant [7]). As of this time, the existence of resolution of singularities in dimension ≥ 4 is unknown in positive characteristic. The problem of local uniformization (and resolution of singularities) is closely related to the problem of finding good local forms of mappings along a valuation, as the construction of a resolution of singularities generally starts with a generically finite projection onto a nonsingular variety.

In characteristic zero, there is a very nice local form for morphisms, called local monomialization. This result is a little stronger than what comes immediately from the assumption that toroidalization is possible.

Theorem 1.1. *(local monomialization)(*[8], [11]*) Suppose that k is a field of characteristic zero and $R \rightarrow S$ is an extension of regular local rings such that R and S are essentially of finite type over k and ω is a valuation of the quotient field of S which dominates S and S dominates R. Then there is a commutative diagram*

$$\begin{array}{ccc} R_1 & \rightarrow & S_1 \\ \uparrow & & \uparrow \\ R & \rightarrow & S \end{array}$$

such that ω dominates S_1, S_1 dominates R_1 and the vertical arrows are products of monoidal transforms; that is, these arrows are factored by the local rings of blowups of prime ideals whose quotients are regular local rings. In particular, R_1 and S_1 are regular local rings. Further, $R_1 \rightarrow S_1$ has a locally monomial form; that is, there exist regular parameters $u_1, \ldots, u_m$ in R_1 and $x_1, \ldots, x_n$ in S_1, an $m \times n$ matrix $A = (a_{ij})$ with integral coefficients such that rank$(A) = m$ and units $\delta_i \in S_1$ such that

$$u_i = \delta_i \prod_{j=1}^{n} x_j^{a_{ij}}$$

for $1 \leq i \leq m$.

The difficulty in the proof is to obtain the condition that $\text{rank}(A) = m$. To do this, it is necessary to blow up above both R and S.

In the case when the extension of quotient fields $K \to L$ of the extension $R \to S$ is a finite extension and k has characteristic zero, it is possible to find a local monomialization such that the structure of the matrix of coefficients recovers classical invariants of the extension of valuations in $K \to L$, and this form holds stably along suitable sequences of birational morphisms which generate the respective valuation rings. This form is called strong local monomialization. It is established for rank 1 valuations in [8] and for general valuations in [19]. The case which has the simplest form and will be of interest to us in this paper is when the valuation has rational rank 1. In this case, if $R_1 \to S_1$ is a strong local monomialization, then there exist regular parameters $u_1, \ldots, u_m$ in R_1 and $v_1, \ldots, v_m$ in S_1 , a positive integer a and a unit $\delta \in S_1$ such that

(1) $$u_1 = \delta v_1^a, u_2 = v_2, \ldots, u_m = v_m.$$

The stable forms of mappings in positive characteristic and dimension ≥ 2 are much more complicated. For instance, local monomialization does not always hold. An example is given in [16] where $R \to S$ are local rings of points on nonsingular algebraic surfaces over an algebraically closed field k of positive characteristic p and $k(X) \to k(Y)$ is finite and separable.

This leads to the question of determining the best local forms that are possible along a valuation for a generically finite morphism of surfaces in positive characteristic. In this paper, we will consider this situation, assuming that k is algebraically closed of positive characteristic p.

The obstruction to local monomialization is the defect. The defect $\delta(\omega/\nu)$, which is a power of the residue characteristic p of $\mathcal{O}_\omega$, is defined and its basic properties developed in [39, Chapter VI, Section 11], [25], [19, Section 7.1]. The defect is discussed in Subsection 2.1. We have the following theorem, showing that the defect is the only obstruction to strong local monomialization for maps of surfaces.

Theorem 1.2. *(*[19, Theorem 7.35]*) Suppose that $K \to L$ is a finite, separable extension of algebraic function fields over an algebraically closed field k of characteristic $p > 0$, $R \to S$ is an extension of local domains such that R and S are essentially of finite type over k and the quotient fields of R and S are K and L respectively such that S dominates R. Suppose that ω is valuation of L which dominates S. Let ν be the restriction of ω to K. Suppose that the extension is defectless ($\delta(\omega/\nu) = 1$). Then the conclusions of Theorem 1.1 hold. In particular, $R \to S$ has a local monomialization (and a strong local monomialization) along ω.*

Suppose that $K \to L$ is a Galois extension of fields of characteristic $p > 0$ and ω is a valuation of L, ν is the restriction of ω to K. Then there is a classical tower of fields ([21, page 171])

$$K \to K^s \to K^i \to K^v \to L.$$

where K^v is the ramification field and the extension $K \to K^v$ has no defect. Thus the essential difficulty comes from the extension from K^v to L which could have defect. The extension $K^v \to L$ is a tower of Artin-Schreier extensions, so the Artin-Schreier extension is of fundamental importance in this theory.

Kuhlmann has extensively studied defect in Artin-Schreier extensions in [26]. He separated these extensions into dependent and independent defect Artin-Schreier extensions. This definition is reproduced in Subsection 2.5. Kuhlmann also defined an invariant called the distance to distinguish the natures of Artin-Schreier extensions. This definition is given in Subsections 2.3 and 2.5.

We now specialize to the case of a finite separable extension $K \to L$ of two dimensional algebraic function fields over an algebraically closed field k of characteristic $p > 0$, and suppose that ω is a valuation of L which is trivial on k and ν is the restriction of ω to K. If L/K has defect then ω must have rational rank 1 and be nondiscrete. We will assume that ω has rational rank 1 and is nondiscrete for the remainder of the introduction.

With these restrictions, the distance δ of an Artin-Schreier extension is $\le 0^-$ when the extension has defect. We will define a^- in Subsection 2.2. If the Artin-Schreier extension is a defect extension with $\delta = 0^-$ it is an independent defect extension. If it is a defect extension and the distance is less than 0^- then the extension is a dependent defect extension.

A quadratic transform along a valuation is the center of the valuation at the blow up of a maximal ideal of a regular local ring. There is the sequence of quadratic transforms along ν and ω

$$R \to R_1 \to R_2 \to \cdots \text{ and } S \to S_1 \to S_2 \to \cdots . \tag{2}$$

We have that $\cup_{i=1}^{\infty} R_i = \mathcal{O}_\nu$, the valuation ring of ν, and $\cup_{i=1}^{\infty} S_i = \mathcal{O}_\omega$, the valuation ring of ω. These sequences can be factored by standard quadratic transform sequences (defined in Section 3). It is shown in [19] that given positive integers r_0 and s_0, there exists $r \ge r_0$ and $s \ge s_0$ such that $R_r \to S_s$ has the following form:

$$u = \delta x^a, v = x^b(y^d\gamma + x\Omega) \tag{3}$$

where u, v are regular parameters in R_r, x, y are regular parameters in S_s, γ and τ are units in S_s, $\Omega \in S_s$, a and d are positive integers and b is a non negative integer. If we choose r_0 sufficiently large, then we have that the complexity ad of the extension $R_r \to S_s$ is a constant which depends on the extension of valuations, which we call the stable complexity of (2). When $R_r \to S_s$ has this stable complexity, we call the forms (3) stable forms.

The strongly monomial form is the case when $b = 0$ and $d = 1$; that is, after making a change of variables in y,

$$u = \delta x^a, v = y.$$

As we observed earlier (Theorem 1.2) if the extension $K \to L$ has no defect, then the stable form is the strongly monomial form. If there is defect, then it is possible for the a and d in stable forms along a valuation to vary wildly, even though their product ad is fixed by the extension, as we will see in Theorem 5.4. An interesting open question is if we can always find stable local forms (3) with $b = 0$.

We make an extensive study of the local forms which can occur in an Artin-Schreier extension under a sequence of quadratic transforms in Section 4. In an Artin-Schreier extension, the stable complexity is either 1 or p. If the stable complexity is 1, then a stable form $R_r \to S_s$ is unramified. If the stable complexity is p, then a stable form $R_r \to S_s$ is either of type 1 or of type 2, as defined below, and the type can vary within the sequences (2). The two types are defined as follows. There are regular parameters u_r, v_r in R_r and x_s, y_s in S_s such that if $R_r \to S_s$ has type 1, then

$$u_r = x_s, v_r = y_s^p\gamma + x_s\Sigma \tag{4}$$

where γ is a unit in S_s and $\Sigma \in S_s$. If $R_r \to S_s$ is of type 2, then

$$u_r = \gamma x_s^p, v_r = y_s \tag{5}$$

where γ, τ are units in S_s and $\Omega \in S_s$. The Artin-Schreier extension L/K has no defect if and only if the stable forms $R_r \to S_s$ are of type 2 for $r \gg 0$ (Proposition 5.3). Observe that type 2 is the condition of being strongly monomial.

In Theorem 5.4, it is shown that we can construct defect Artin-Schreier extensions with any prescribed (nonpositive) distance and any prescribed switching between types. Since we are constructing defect extensions, we must impose the condition that the stable forms are not eventually always of type 2.

The construction is such that if

$$R = R_0 \to R_1 \to R_2 \to \cdots \text{ and } S = S_0 \to S_1 \to S_2 \to \cdots \tag{6}$$

are the sequences of standard quadratic transform sequences along the valuation, then $R_i \to S_i$ is a stable form for all i, and if $\Phi : \mathbb{N} \to \{1, 2\}$ is any function (the prescribed switching) which is not equal to 2 for all sufficiently large integers, then $R_i \to S_i$ is of type $\Phi(i)$ for all i.

We use a formula derived in Proposition 7.9 to compute distances from sequences (2) in Artin-Schreier extensions by Piltant and Kuhlmann [27]. For the reader's convenience, we give a proof of this formula in the appendix to this paper. A more general form of the theorem is proven in [28].

We analyze in Section 6 an example in [19], showing failure of strong local monomialization. This example is of the type of the constructions in Theorem 5.4. It is a tower of two defect Artin-Schreier extensions, each of the type of Theorem 5.4. The first extension is of type 1 for even integers and of type 2 for odd integers. The second extension is of type 2 for even integers and of type 1 for odd integers. The composite gives a sequence of extensions of regular local rings $R_i \to S_i$, where R_i has regular parameters u_i, v_i and S_i has regular parameters x_i, y_i such that the stable form is

$$u_i = \gamma x_i^p, v_i = y_i^p \tau + x_i \Omega \tag{7}$$

for all i.

Using the formula of Proposition 7.9, we compute the distances of these two Artin-Schreier extensions, concluding that both extensions have dependent defect. We show that the first Artin-Schreier extension has distance $(-\frac{p^4-2}{p^4-1})^-$ and the second Artin-Schreier extension has distance $(-\frac{cp^3+(c-1)p^2+cp+c}{p^4-1})^-$, where c is a number occuring in the equation defining the second extension. The first of these distances was computed in [23] by a different method.

Suppose that $K \to L$ is a finite extension of fields of positive characteristic and ω is a valuation of L with restriction ν to K. It is known that there is no defect in the extension if and only if there is a finite generating sequence in L for the valuation ω over K ([35], [30]). The calculation of generating sequences for extensions of Noetherian local rings which are dominated by a valuation is extremely difficult. This has been accomplished for two dimensional regular local rings in [33] and [20] and for many hypersurface singularities above a regular local ring of arbitrary dimension in [18].

The nature of a generating sequence in an extension of S over R determines the nature of the mappings in the stable forms. It is shown in [15, Theorem 1] that if $R \to S$ is an extension of two dimensional excellent regular local rings whose quotient fields give a finite extension $K \to L$ and ω is a valuation of L which dominates S then the extension is without defect if and only if there exist sequences of quadratic transform $R \to R_1$ and $S \to S_1$ along ν such that ω has a finite generating sequence in S_1 over R_1. This shows us that we can expect good stable forms (as do hold by Theorem 1.2) if there is no defect, but not otherwise.

I thank Franz-Viktor Kuhlmann and Olivier Piltant for telling me about the beautiful formulas from [27], comparing distance and ramification cuts in an Artin-Schreier extension, and sharing their manuscript [27] with me. For the readers convenience, I give proofs of these formulas in the appendix.

In [5], it is shown that quite generally there are distance bounds for Artin-Schreier extensions, and in particular in the case considered in this paper of two dimensional algebraic function fields over an algebraically closed field, the bound is 4.

I thank the reviewer for their careful reading and helpful comments.

2. Preliminaries

2.1. **Some notation.** Let K be a field with a valuation ν. The valuation ring of ν will be donoted by $\mathcal{O}_\nu$, νK will denote the value group of ν and $K\nu$ will denote the residue field of $\mathcal{O}_\nu$.

The maximal ideal of a local ring A will be denoted by m_A. If $A \to B$ is an extension (inclusion) of local rings such that $m_B \cap A = m_A$ we will say that B dominates A. If a valuation ring $\mathcal{O}_\nu$ dominates A we will say that the valuation ν dominates A.

Suppose that K is an algebraic function field over a field k. An algebraic local ring A of K is a local domain which is a localization of a finite type k-algebra whose quotient field is K. A k-valuation of K is a valuation of K which is trivial on k.

Suppose that $K \to L$ is a finite algebraic extension of fields, ν is a valuation of K and ω is an extension of ν to L. Then the reduced ramification index of the extension is $e(\omega/\nu) = [\omega L : \nu K]$ and the residue degree of the extension is $f(\omega/\nu) = [L\omega : K\nu]$.

The defect $\delta(\omega/\nu)$, which is a power of the residue characteristic p of $\mathcal{O}_\omega$, is defined and its basic properties developed in [39, Chapter VI, Section 11], [25] and [19, Section 7.1]. In the case that L is Galois over K, we have the formula

$$[L : K] = e(\omega/\nu)f(\omega/\nu)\delta(\omega/\nu)g \tag{8}$$

where g is the number of extensions of ν to L. In fact, we have the equation (c.f. [26] or Section 7.1 [19])

$$|G^s(\omega/\nu)| = e(\omega/\nu)f(\omega/\nu)\delta(\omega/\nu),$$

where $G^s(\omega/\nu)$ is the decomposition group of L/K.

If $K \to L$ is a finite Galois extension, then we will denote the Galois group of L/K by $\mathrm{Gal}(L/K)$.

2.2. **Initial and final segments and cuts.** We review some basic material about cuts in totally ordered sets from [26]. Let $(S, <)$ be a totally ordered set. An initial segment of S is a subset Λ of S such that if $\alpha \in \Lambda$ and $\beta < \alpha$ then $\beta \in \Lambda$. A final segment of S is a subset Λ of S such that if $\alpha \in \Lambda$ and $\beta > \alpha$ then $\beta \in \Lambda$. A cut in S is a pair of sets (Λ^L, Λ^R) such that Λ^L is an initial segment of S and Λ^R is a final segment of S satisfying $\Lambda^L \cup \Lambda^R = S$ and $\Lambda^L \cap \Lambda^R = \emptyset$. If Λ_1 and Λ_2 are two cuts in S, write $\Lambda_1 < \Lambda_2$ if $\Lambda_1^L \subsetneq \Lambda_2^L$. Suppose that $S \subset T$ is an order preserving inclusion of ordered sets and $\Lambda = (\Lambda^L, \Lambda^R)$ is a cut in S. Then define the cut induced by $\Lambda = (\Lambda^L, \Lambda^R)$ in T to be the cut $\Lambda \uparrow T = (\Lambda^L \uparrow T, T \setminus (\Lambda^L \uparrow T))$ where $\Lambda^L \uparrow T$ is the least initial segment of T in which Λ^L forms a cofinal subset.

We embed S in the set of all cuts of S by sending $s \in S$ to

$$s^+ = (\{t \in S \mid t \le s\}, \{t \in S \mid t > s\}).$$

we may identify s with the cut s^+. Define

$$s^- = (\{t \in S \mid t < s\}, \{t \in S \mid t \geq s\}).$$

Given a cut $\Lambda = (\Lambda^L, \Lambda^R)$ of an ordered Abelian group S, we define $-\Lambda = (-\Lambda^R, -\Lambda^L)$ where $-\Lambda^L = \{-s \mid s \in \Lambda^L\}$ and $-\Lambda^R = \{-s \mid s \in \Lambda^R\}$. We have that if Λ_1 and Λ_2 are cuts, then $\Lambda_1 < \Lambda_2$ if and only if $-\Lambda_2 < -\Lambda_1$.

2.3. **Distances.** Let $K \to L$ be an extension of fields and ω be a valuation of L with restriction ν to K. Let $\widetilde{\nu K}$ be the divisible hull of νK. Suppose that $z \in L$. Then the distance of z from K is defined in [26, Section 2.3] to be the cut $\mathrm{dist}(z, K)$ of $\widetilde{\nu K}$ in which the initial segment of $\mathrm{dist}(z, K)$ is the least initial segment of $\widetilde{\nu K}$ in which $\omega(z - K)$ is cofinal. That is,

$$\mathrm{dist}(z, K) = (\Lambda^L(z, K), \Lambda^R(z, K)) \uparrow \widetilde{\nu K}$$

where

$$\Lambda^L(z, K) = \{\omega(z - c) \mid c \in K \text{ and } \omega(z - c) \in \nu K\}.$$

The following notion of equivalence is defined in [26, Section 2.3]. If $y, z \in L$, then $z \sim_K y$ if $\omega(z - y) > \mathrm{dist}(z, K)$.

2.4. **Higher ramification groups.** We recall material from pages 78 and 79 of [39]. Suppose that $K \to L$ is a finite Galois extension with Galois group G and ω is a valuation of L. Suppose that $I \subset \mathcal{O}_\omega$ is an ideal. Associate to I higher ramification subgroups of G by

$$G_I = \{s \in G \mid s(x) - x \in I \text{ for every } x \in \mathcal{O}_\omega\},$$
$$H_I = \{s \in G \mid s(x) - x \in Ix \text{ for every } x \in L\}.$$

The group H_I is denoted by G'_I in [27]. We always have that $H_I \subset G_I$. If $I \subset J$ then $G_I \subset G_J$ and $H_I \subset H_J$.

2.5. **Artin-Schreier extensions.** Let $K \to L$ be an Artin-Schreier extension of fields of characteristic $p > 0$ and ω be a valuation of L with restriction ν to K. The field L is Galois over K with Galois group $G \cong \mathbb{Z}_p$, where p is the characteristic of K.

Let $\Theta \in L$ be an Artin-Schreier generator of K; that is, there is an expression

$$\Theta^p - \Theta = a$$

for some $a \in K$. We have that

$$\mathrm{Gal}(L/K) \cong \mathbb{Z}_p = \{\mathrm{id}, \sigma_1, \ldots, \sigma_{p-1}\},$$

where $\sigma_i(\Theta) = \Theta + i$.

Since L is Galois over K, we have that $ge(\omega/\nu)f(\omega/\nu)\delta(\omega/\nu) = p$ where g is the number of extensions of ν to L. So we either have that $g = 1$ or $g = p$. If $g = 1$, then ω is the unique extension of ν to L and either $e(\omega/\nu)f(\omega/\nu) = p$ and $\delta(\omega/\nu) = 1$ or $e(\omega/\nu)f(\omega/\nu) = 1$ and $\delta(\omega/\nu) = p$. In particular, the extension is defect if and only if it is an immediate extension ($e = f = 1$) and ω is the unique extension of ν to L.

For $\alpha \in \omega L$, we have an associated ideal $I_\alpha = \{f \in \mathcal{O}_\omega \mid \omega(f) \geq \alpha\}$. We have that $\beta < \alpha$ implies $I_\alpha \subset I_\beta$. Thus $\{\alpha \in \omega L \mid G_{I_\alpha} = 1\}$ is a final segment of ωL. We define the ramification cut $\mathrm{Ram}(\omega/\nu)$ of the valued field extension L/K to be the cut in $\widetilde{\nu K}$ induced by this final segment; that is, $\mathrm{Ram}(\omega/\nu) = (\mathrm{Ram}(\omega/\nu)^L, \mathrm{Ram}(\omega/\nu)^R)$ where $\mathrm{Ram}(\omega/\nu)^R$ is the smallest final segment of $\widetilde{\nu K}$ in which $\{\alpha \in \omega L \mid G_{I_\alpha} = 1\}$ is coinitial.

From now on in this subsection, suppose that L is a defect extension of K. By [26, Lemma 4.1], the distance $\delta = \mathrm{dist}(\Theta, K)$ does not depend on the choice of Artin-Schreier generator Θ, so δ can be called the distance of the Artin-Schreier extension. Since L/K is an immediate extension, the set $\omega(\Theta - K)$ is an initial segment in νK which has no maximal element by [26, Theorem 2.19].

We have, since the extension is defect, that

$$\delta = \operatorname{dist}(\Theta, K) \le 0^- \tag{9}$$

by [26, Corollary 2.30].

A defect Artin-Schreier extension L is defined in [26, Section 4] to be a dependent defect Artin-Schreier extension if there exists an immediate purely inseparable extension $K(\eta)$ of K of degree p such that $\eta \sim_K \Theta$. Otherwise, L/K is defined to be an independent defect Artin-Schreier defect extension. We have by [26, Proposition 4.2] that for a defect Artin-Schreier extension,

(10) L/K is independent if and only if the distance $\delta = \operatorname{dist}(\Theta, K)$ satisfies $\delta = p\delta$.

2.6. **Extensions of rank 1 valuations in an Artin-Schreier extension.** In this subsection, we suppose that L is an Artin-Schreier extension of a field K of characteristic p, ω is a rank 1 valuation of L and ν is the restriction of ω to K. We suppose that L is a defect extension of K. To simplify notation, we suppose that we have an embedding of ωL in $\mathbb{R}$. Since L has defect over K and L is separable over K, ωL is nondiscrete by the corollary on page 287 of [38], so that ωL is dense in $\mathbb{R}$.

We define a cut in $\mathbb{R}$ by extending the cut $\operatorname{dist}(\Theta, K)$ in $\widetilde{\nu K}$ to a cut of $\mathbb{R}$ by taking the initial segment of the extended cut to be the least initial segment of $\mathbb{R}$ in which the cut $\operatorname{dist}(\Theta, K)$ is cofinal. This cut is then $\operatorname{dist}(\Theta, K) \uparrow \mathbb{R}$. This cut is either s or s^- for some $s \in \mathbb{R}$. If L is a defect extension of K then $\operatorname{dist}(\Theta, K) \uparrow \mathbb{R} = s^-$ where s is a non positive real number by [26, Theorem 2.19] and [26, Corollary 2.30]. We will set $\operatorname{dist}(\omega/\nu)$ to be this real number s, so that

$$\operatorname{dist}(\Theta, K) \uparrow \mathbb{R} = s^- = (\operatorname{dist}(\omega/\nu))^-.$$

The real number $\operatorname{dist}(\omega/\nu)$ is well defined since it is independent of choice of Artin-Schreier generator of L/K by Lemma 4.1 [26].

With the assumptions of this subsection, by (9) and (10), the distance $\delta = \operatorname{dist}(\Theta, K)$ of an Artin-Schreier extension is $\le 0^-$ when the extension has defect. If it is a defect extension with distance equal to 0^- then it is an independent defect extension. If it is a defect extensions and the distance is less than 0^- then the extension is a dependent defect extension. Thus if L/K is a defect extension, we have that $\operatorname{dist}(\omega/\nu) \le 0$ and the defect extension L/K is independent if and only if $\operatorname{dist}(\omega/\nu) = 0$.

3. Extensions of two dimensional regular local rings

Suppose that M is a two dimensional algebraic function field over an algebraically closed field k of characteristic $p > 0$ and μ is a nondiscrete rational rank 1 valuation of M. Suppose that A is an algebraic regular local ring of M such that μ dominates A. A quadratic transform of A is an extension $A \to A_1$ where A_1 is a local ring of the blowup of the maximal ideal of A such that A_1 dominates A and A_1 has dimension two. A quadratic transform $A \to A_1$ is said to be along the valuation μ if μ dominates A_1.

Let

$$A = A_0 \to A_1 \to A_2 \to \cdots$$

be the sequence of quadratic transforms along μ so that the valuation ring $\mathcal{O}_\mu = \cup A_i$ (by [1, Lemma 12]). Let P be a height one prime ideal in A such that A/P is a regular local ring. A_i is said to be free if the radical of PA_i is a height one prime ideal (so that $A_i/\sqrt{PA_i}$ is a regular local ring). In particular, A_0 is free.

Now there exist (as explained in more detail in [19, Definition 7.11]) positive integers r_i' and $\overline{r}_i$ such that $A_0 = A_{r_1'}$ and for all $i \geq 1$, $r_i' \leq \overline{r}_i < r_{i+1}' - 1$ such that if $r_i' \leq j \leq \overline{r}_i$ then A_j is free and if $\overline{r}_i < j < r_{i+1}'$ then A_j is not free. Here we modify [19, Definition 7.5] slightly by definng E_0 on $\mathrm{Spec}(A_0)$ to be $Z(P)$ and E_i on $\mathrm{Spec}(A_i)$ to be $Z(PA_i)$.

Suppose that A_j is free. Then there exists i such that $r_i' \leq j \leq \overline{r}_i$. Let u, v be regular parameters in A_j such that $u = 0$ is a local equation of $Z(PA_j)$; that is, $(u) = \sqrt{PA_j}$. We will say that u, v are admissible parameters in A_i. Let $\overline{u}, \overline{v}$ be defined by

$$u = \overline{u}^m(\overline{v} + \alpha)^{a'}, v = \overline{u}^q(\overline{v} + \alpha)^{b'}$$

where

$$\frac{\nu(u)}{\nu(v)} = \frac{m}{q}$$

with $\gcd(m, q) = 1$ and $0 \neq \alpha \in k$ is such that $\nu(\overline{v}) > 0$.

Then there exists $k > j$ such that $\overline{u}$ and $\overline{v}$ are regular parameters in A_k. Further, A_k is free, and $\overline{u} = 0$ is a local equation of the reduced exceptional divisor of $\mathrm{Spec}(A_k) \rightarrow \mathrm{Spec}(A)$, so that $\overline{u}, \overline{v}$ are admissible parameters in A_k. If $\mu(v) \notin \mu(u)\mathbb{Z}$, then $m > 1$ and $k = r_{i+1}'$. If $\mu(v) \in \mu(u)\mathbb{Z}$ then $m = 1$ and $k \leq \overline{r}_i$.

A regular system of parameters with $\mu(v) \notin \mu(u)\mathbb{Z}$ can always be found from a given regular system of parameters u, v by possibly replacing v with the difference of v and a suitable polynomial $g(u) \in k[u]$ (which necessarily has no constant term).

We will call the sequence $A_0 = A_{r_0'} \rightarrow A_{r_1'} \rightarrow A_{r_2'} \rightarrow \cdots$ the sequence of standard sequences of quadratic transforms along μ. Observe that this sequence depends on the choice of P in A.

Let $K \rightarrow L$ be a finite separable extension of two dimensional algebraic function fields over an algebraically closed field k. Suppose that R is a two dimensional regular algebraic local ring of K and S is a two dimensional regular algebraic local ring of L such that S dominates R. Let P be a height one prime ideal in R such that R/P is a regular local ring and let Q be a height one prime ideal in S such that S/Q is a regular local ring. We do not insist in this definition that the good condition that $Q \cap R = P$ holds.

Let $R \rightarrow R_1 \rightarrow \cdots \rightarrow R_r$ and $S \rightarrow S_1 \rightarrow \cdots \rightarrow S_s$ be sequences of quadratic transforms such that S_s dominates R_r. Let E_i be the divisor $Z(PR_i)$ and F_j be the divisor $Z(QS_j)$.

Definition 3.1. *(*[19, Definition 7.5]*) Suppose that S_s dominates R_r. The map $R_r \rightarrow S_s$ is said to be prepared if both R_r and S_s are free, the critical locus of Spec(S_s) $\rightarrow$ Spec(R_r) is contained in F_s and we have an expression $u = \gamma x^a$, where u is part of a regular system of parameters of R_r such that $u = 0$ is a local equation of E_r, x is part of a regular system of parameters of S_s such that $x = 0$ is a local equation of F_s and γ is a unit in S_s.*

Suppose that $R_r \rightarrow S_s$ is prepared. Let (u, v) and (x, y) be admissible parameters in R_r and S_s respectively; that is, u is part of a regular system of parameters of R_r such that $u = 0$ is a local equation of E_r, x is part of a regular system of parameters of S_s such that $x = 0$ is a local equation of F_s. Further, $u = \gamma x^a$, where γ is a unit in S_s. Then there is an expression

$$u = \gamma x^a, v = x^b f \tag{11}$$

where $\gamma \in S_s$ is a unit, $f \in S_s$ and x does not divide f in S_s.

Definition 3.2. *We will say that $R_r \rightarrow S_s$ is well prepared if f is not a unit in S_s.*

Suppose that $R_r \to S_s$ is well prepared. The complexity of $R_r \to S_s$ is ad where d is the order of the residue of f in the one dimensional regular local ring $S_s/(x)$.

The complexity is defined in [19, Definition 7.9]. It is shown there that the complexity depends only on the extension $R_r \to S_s$.

Proposition 3.3. *(*[19, Proposition 7.2]*) Suppose that S_s dominates R_r, $R_r \to S_s$ is well prepared and R_r and S_s have admissible parameters (u, v) and (x, y) satisfying the equation (11). Let S^* be the local ring which is the localization of the normalization of R_r in L which is dominated by S_s. Then there exists a commutative diagram*

$$\begin{array}{ccc} S^* & \to & S_s \\ \uparrow & & \uparrow \\ R_r & \to & R^* \end{array}$$

where all arrows are dominating maps such that R^* is a two dimensional algebraic normal local ring of K such that S_s is a local ring of the integral closure o R^* in L. We have

$$[QF(\hat{S}_s) : QF(\hat{R}^*)] = ad$$

where d is the order of the residue of f in the one dimensional regular local ring $S_s/(x)$.

Remark 3.4. *Suppose that assumptions are as in Proposition 3.3 with the additional assumption that $K \to L$ is Galois. Then the complexity ad of $R_r \to S_s$ divides the degree $[L : K]$.*

Proof. Let $\overline{S}$ be the integral closure of R^* in L. Let $m_1, \ldots, m_g$ be the maximal ideals of $\overline{S}$. The local ring S_s is one of the localizations $\overline{S}_{m_i}$. The Galois group $G(L/K)$ acts transitively on the local rings $\overline{S}_{m_i}$ so these rings are all isomorphic k-algebras. The m_{R^*}-adic completion of $\overline{S}$ is the direct sum of the complete local rings $\widehat{\overline{S}_{m_i}}$, and one of these local rings is $\hat{S}_s$. We have that

$$[L : K] = \sum_{i=1}^{g} [\mathrm{QF}(\widehat{\overline{S}_{m_i}}) : \mathrm{QF}(\hat{R}^*)]$$

by [2, Proposition 1]. Thus $[L : K] = g[QF(\hat{S}_s) : QF(\hat{R}^*)] = gad$. □

The following remark follows from [1, Theorem 2].

Remark 3.5. *Suppose that ω is a rational rank 1 nondiscrete valuation of the quotient field of K^* with restriction ν to K such that ω dominates S_j for all j and ν dominates R_i for all i. Then given $r_0 > 0$ and $s_0 > 0$ there exist $r \geq r_0$ and $s \geq s_0$ such that S_s dominates R_r and $R_r \to S_s$ is well prepared. This result is true for any initial choice of P in R and Q in S.*

Proposition 3.6. *Let $K \to L$ be a finite separable extension of two dimensional algebraic function fields over an algebraically closed field k of characteristic $p > 0$, ω be a rational rank 1 nondiscrete valuation of L (with residue field k) and ν be the restriction of ω to K.*

Suppose that A is an algebraic local ring of K which is dominated by ν. Then there exists an algebraic regular local ring R' of K which is dominated by ν and dominates A with the following property.

Suppose that R is a regular algebraic local ring of K which dominates R' and S is a regular algebraic local ring of L which is dominated by ω and dominates R such that

1) *There exist regular parameters x, y in S and u, v in R such that $u = \gamma x^a$ and $v = x^b f$ with γ a unit in S and $f \in S$ such that f is not a unit in S, x does not divide f.*
2) *The critical locus of $Spec(S) \to Spec(R)$ is contained in $Z(xS)$.*

Then letting $d = dim_k S/(f, x)$, we have that the complexity $ad = e(\omega/\nu)\delta(\omega/\nu)$.

This is proved in [19, Section 7.9] and [14, Proposition 3.4].

Let $K \to L$ be a finite separable extension of two dimensional algebraic function fields over an algebraically closed field k of characteristic $p > 0$, ω be a rational rank 1 nondiscrete valuation of L (with residue field k) and ν be the restriction of ω to K. Let R be a regular algebraic local ring of K and S be a regular algebraic local ring of L which is dominated by ω and dominates R. Let P be a height one prime ideal in R such that R/P is a regular local ring and let Q be a height one prime ideal in S such that S/Q is a regular local ring. Let

$$R \to R_1 \to \cdots \to \cdots \text{ and } S \to S_1 \to \cdots \to \cdots \tag{12}$$

be the infinite sequences of quadratic transforms along ν and ω respectively. By Remark 3.5 and Proposition 3.6, there exits a positive integer r_0 such that whenever $r \geq r_0$ and $R_r \to S_s$ is well prepared, we have that the complexity ad of this extension is equal to $e(\omega/\nu)\delta(\omega/\nu)$. We will call this the stable complexity of the sequences (12).

Suppose that $K \to L$ is a finite extension of two dimensional algebraic function fields, R is an algebraic regular local ring of K which is dominated by a regular algebraic local ring S of L such that $\dim R = \dim S = 2$. Let x, y be regular parameters in S and u, v be regular parameters in R. Then we can form the Jacobian ideal

$$J(S/R) = \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right).$$

This ideal is independent of choice of regular parameters.

The following proposition is established in [19].

Proposition 3.7. *Suppose that A is an algebraic local ring of K and B is an algebraic local ring of L which is dominated by a rational rank 1 nondiscrete valuation ω of L such that B dominates A. Then there exists a commutative diagram of homomorphisms*

$$\begin{array}{ccc} R & \to & S \\ \uparrow & & \uparrow \\ A & \to & B \end{array}$$

such that R is a regular algebraic local ring of K with regular parameters u, v, S is a regular algebraic local ring of L with regular parameters x, y such that S is dominated by ω, S dominates R, $J(S/R) = (x^{\overline{c}})$ for some non negative integer $\overline{c}$ and there is an expression

$$u = \gamma x^a, v = x^b(y^n\tau + x\Omega)$$

where τ, γ are units in S, $\Omega \in S$ and $n > 0$, $0 \leq b < a$. Thus the quadratic transform of R along ν is not dominated by S.

Proof. By two dimensional local uniformization (or resolution of singularities), [2], [29] or [6], there exists a commutative diagram

$$\begin{array}{ccc} R_0 & \to & S_0 \\ \uparrow & & \uparrow \\ A & \to & B \end{array}$$

such that S_0 is an algebraic local ring of L which dominates B, R_0 is an algebraic regular local ring of K which dominates A, ω dominates S_0 and S_0 dominates R_0. Fix regular parameters u_0, v_0 in R_0.

Let $S_0 \to S$ be a sequence of quadratic transforms along ω such that S is free and S dominates R_1, and the support of $u_0 J(S_0/R_0)S$ is the last exceptional divisor F of the sequence of quadratic transforms factoring $S_0 \to S$ (such an S exists since ω has rank 1 and ω is nondiscrete). Let x, y be regular parameters in S such that $x = 0$ is a local equation of F in $\mathrm{Spec}(S)$. Thus $J(S/R_0) = J(S_0/R_0)J(S/S_0) = (x^f)$ for some positive integer f. Further, $u_0 = \gamma x^l$ for some unit γ in S and positive integer l. Thus there exist $b, n \in \mathbb{N}$, a unit τ in S and $\Omega \in S$ such that $v_0 = x^b(\tau y^n + x\Omega)$. There exists a sequence of quadratic transforms $R_0 \to R_1$ along ν such that S dominates R_1, and after replacing R_0 with R_1 we have that $n > 0$. If $\lfloor \frac{b}{l} \rfloor = 0$ then the quadratic transform of R_0 along ν is not dominated by S and we set $R = R_0$. If $\lfloor \frac{b}{l} \rfloor > 0$, then set $e = \lfloor \frac{b}{l} \rfloor$, and let $R_0 \to R$ be the sequence of e quadratic transforms along ν. Then $R \to S$ satisfies the conclusions of the proposition. $\square$

Remark 3.8. *Let $K \to L$ be an Artin-Schreier extension of two dimensional algebraic function fields over an algebraically closed field k of characteristic $p > 0$. Let ω be a rational rank 1 nondiscrete valuation of L (with residue field k) and ν be the restriction of ω to K. Since L is Galois over K, we have that $g(\omega/\nu)e(\omega/\nu)\delta(\omega/\nu) = p$ where $g = g(\omega/\nu)$ is the number of extensions of ν to L. So we either have that $g = 1$ or $g = p$. If $g = 1$, then ω is the unique extension of ν to L and either $e(\omega/\nu) = p$ and $\delta(\omega/\nu) = 1$ or $e(\omega/\nu) = 1$ and $\delta(\omega/\nu) = p$. If $g = 1$, we have by Proposition 3.6 that the stable complexity of the sequences (12) is $ad = p$. If $g = p$, then $e(\omega/\nu) = 1$ and $\delta(\omega/\nu) = 1$ and the stable complexity of the sequences (12) is $ad = 1$.*

The following proposition is proven in [31].

Proposition 3.9. *Suppose that $K \to L$ is an Artin-Schreier extension of two dimensional algebraic function fields over an algebraically closed field k of characteristic $p > 0$, ω is a rational rank 1 nondiscrete valuation of L with restriction $\nu = \omega|K$. Further suppose that A is an algebraic local ring of K and B is an algebraic local ring of L which is dominated by ω such that B dominates A. Then there exists a commutative diagram of homomorphisms*

$$\begin{array}{ccc} R & \to & S \\ \uparrow & & \uparrow \\ A & \to & B \end{array}$$

such that R is a regular algebraic local ring of K with regular parameters u, v, S is a regular algebraic local ring of L with regular parameters x, y such that S is dominated by ω, S dominates R, $R \to S$ is well prepared with admissible parameters u, v in R and x, y in S (with respect to the prime ideals $P = uR$ in R and $Q = xS$ in S). We further have that $R \to S$ is quasi finite, $J(S/R) = (x^{\overline{c}})$ for some non negative integer $\overline{c}$ and one of the following three cases holds:c

0) *$u = x$, $v = y$ ($R \to S$ is unramified).*
1) *$u = x$, $v = y^p\gamma + x\Sigma$ where γ is a unit in S and $\Sigma \in S$.*
2) *$u = \gamma x^p$, $v = y$ where γ is a unit in S and $\Omega \in S$.*

Proof. By Proposition 3.7 and Remark 3.4, we may construct a diagram

$$\begin{array}{ccc} \overline{R} & \rightarrow & \overline{S} \\ \uparrow & & \uparrow \\ A & \rightarrow & B \end{array}$$

such that all the conclusions of the proposition hold, except possibly $\overline{R} \rightarrow \overline{S}$ is not quasi finite, and we have a form

$$u = \delta x^p, v = x^b y \tag{13}$$

where b is an integer with $0 < b < p$ and δ is a unit in $\overline{S}$. We will show that after one more sequence of blowups, we obtain a map of the form of 0), 1) or 2).

There exists a sequence of quadratic transforms $S \rightarrow S'$ of regular local rings along ω such that S' has regular parameters $\tilde{x}_1, \tilde{y}_1$ defined by

$$x = \tilde{x}_1^{\overline{a}}(\tilde{y}_1 + \alpha)^{\overline{a}'}, y = \tilde{x}_1^{\overline{b}}(\tilde{y}_1 + \alpha)^{\overline{b}'} \tag{14}$$

where $0 \neq \alpha \in k$ and $\overline{a}\overline{b}' - \overline{a}'\overline{b} = 1$. Then we have an expression

$$u = \delta \tilde{x}_1^{d_1}(\tilde{y}_1 + \alpha)^{e_1}, v = \tilde{x}_1^{f_1}(\tilde{y}_1 + \alpha)^{g_1}$$

where $d_1 g_1 - e_1 f_1 = p$. Let $l = \gcd(d_1, f_1)$. The number l must either be 1 or p. We have that

$$\delta = \delta_0 + \tilde{x}_1 \Omega \text{ for some } 0 \neq \delta_0 \in k \text{ and } \Omega \in S'.$$

We have a sequence of quadratic transforms $R \rightarrow R'$ of regular local rings such that S' dominates R' and R' has regular parameters $\tilde{u}$ and $\tilde{v}$ such that

$$\tilde{u} = \tilde{x}_1^l(\tilde{y}_1 + \alpha)^{e_2}\delta^{f_2}, \tilde{v} = (\tilde{y}_1 + \alpha)^{g_2}\delta^{h_2} - \alpha^{g_2}\delta_0^{h_2}$$

We have that $lg_2 = p$ as

$$p = \begin{vmatrix} d_1 & e_1 \\ f_1 & g_1 \end{vmatrix} = \begin{vmatrix} l & e_2 \\ 0 & g_2 \end{vmatrix}.$$

First suppose that $l = p$. Then $g_2 = 1$ and we have an expression of the form of 2). Now suppose that $l \neq p$. Then $l = 1$ and we have an expression of the form of 1). ☐

4. Some calculations in two dimensional Artin-Schreier extensions

Let $K \rightarrow L$ be an Artin-Schreier extension of two dimensional algebraic function fields over an algebraically closed field k of characteristic $p > 0$. Let $R \rightarrow S$ be an extension from a regular algebraic local ring of K to a regular algebraic local ring of L such that S dominates R.

Let u, v be regular parameters in R and x, y be regular parameters in S. We will say that $R \rightarrow S$ is of type 0 with respect to these parameters if

$$\text{Type 0:} \quad u = \gamma x, v = y\tau + x\Omega$$

where γ, τ are units in S and $\Omega \in S$, so that $R \rightarrow S$ is unramified. We will say that $R \rightarrow S$ is of type 1 with respect to these parameters if

$$\text{Type 1:} \quad u = \gamma x, v = y^p\tau + x\Omega$$

where γ, τ are units in S and $\Omega \in S$. We will say that $R \rightarrow S$ is of type 2 with respect to these parameters if

$$\text{Type 2:} \quad u = \gamma x^p, v = y\tau + x\Omega$$

where γ, τ are units in S and $\Omega \in S$.

These definitions are such that if one these types hold, and $\overline{u},\overline{v}$ are regular parameters in R, $\overline{x},\overline{y}$ are regular parameters in S such that $\overline{u}$ is a unit in R times u and $\overline{x}$ is a unit in S times x then $R\rightarrow S$ is of the same type for the new parameters $\overline{u},\overline{v}$ and $\overline{x},\overline{y}$.

If we replace $R\rightarrow S$ with a well prepared map $R_i\rightarrow S_j$ in the sequences (12), we will insist that the above parameters be admissible. We see that the three types are preserved by allowable changes of variables (changes of variables which preserve the condition that the regular parameters are admissible). In particular, we may obtain the respective forms of Proposition 3.9 by an allowable change of variables.

Theorem 4.1. *Suppose that $R\rightarrow S$ is of type 1 with respect to regular parameters x,y in S and u,v in R and that $J(S/R)=(x^{\overline{c}})$. Let $\overline{x}=u$, $\overline{y}=y-g(\overline{x})$ where $g(\overline{x})\in k[\overline{x}]$ is a polynomial with zero constant term, so that $\overline{x},\overline{y}$ are regular parameters in S. Computing the Jacobian determinate $J(S/R)$, we see that*

$$u=\overline{x},v=\overline{y}^p\gamma+\overline{x}^{\overline{c}}\overline{y}\tau+f(\overline{x}) \tag{15}$$

where γ,τ are unit series in $\hat{S}$ and $f(\overline{x})=\sum e_i\overline{x}^i\in k[[\overline{x}]]$. Make the change of variables $\overline{v}=v-\sum e_iu^i$ where the sum is over i such that $i\le\frac{pq}{m}$ so that $u,\overline{v}$ are regular parameters in R.

Suppose that m,q are positive integers with $m>1$ and $gcd(m,q)=1$. Let α be a nonzero element of k. Let $a',b'\in\mathbb{N}$ be such that $mb'-qa'=1$. Then define the sequence of quadratic transforms $S\rightarrow S_1$ so that the two dimensional algebraic local ring S_1 has regular parameters x_1,y_1 defined by

$$\overline{x}=x_1^m(y_1+\alpha)^{a'},\overline{y}=x_1^q(y_1+\alpha)^{b'}.$$

We have that $R\rightarrow S$ is of type 1 with respect to the regular parameters $\overline{x},\overline{y}$ and u,v. Let $\sigma=gcd(m,pq)$ which is 1 or p.

There exists a unique sequence of quadratic transforms $R\rightarrow R_1$ such that R_1 has regular parameters u_1,v_1 defined by

$$u=u_1^{\overline{m}}(v_1+\beta)^{c'},\overline{v}=u_1^{\overline{q}}(v_1+\beta)^{d'}$$

with $0\ne\beta\in k$ giving a commutative diagram of homomorphisms

$$\begin{array}{ccc} R_1 & \rightarrow & S_1\\ \uparrow & & \uparrow\\ R & \rightarrow & S\end{array}$$

such that $R_1\rightarrow S_1$ is quasi finite. We have that $J(S_1/R_1)=(x_1^{c_1})$ for some positive integer c_1. Further:

0) *If $\frac{q}{m}\ge\frac{\overline{c}}{p-1}$ then $R_1\rightarrow S_1$ is of type 0.*

1) *If $\frac{q}{m}<\frac{\overline{c}}{p-1}$ and $\sigma=1$ then $R_1\rightarrow S_1$ is of type 1 and*

$$\left(\frac{c_1}{p-1}\right)=\left(\frac{\overline{c}}{p-1}\right)m-q.$$

2) *If $\frac{q}{m}<\frac{\overline{c}}{p-1}$ and $\sigma=p$ then $R_1\rightarrow S_1$ is of type 2 and*

$$\left(\frac{c_1}{p-1}\right)=\left(\frac{\overline{c}}{p-1}\right)m-q+1.$$

In cases 1) and 2), $m=\sigma\overline{m}$, $pq=\sigma\overline{q}$ and $\overline{m}c'-\overline{q}d'=1$.

Proof. Define a monomial valuation μ dominating $\hat{S}$ by prescribing that $\mu(\overline{x}) = m$, $\mu(\overline{y}) = q$ and for $0 \neq \sum a_{ij}\overline{x}^i\overline{y}^j \in \hat{S}$, $\nu(\sum a_{ij}\overline{x}^i\overline{y}^j) = \min\{im + jq \mid a_{ij} \neq 0\}$.

Expand

$$\overline{v} = \sum_{i=1}^{d} \gamma_i \overline{x}^{\alpha_i}\overline{y}^{\beta_i} + \sum_{i>d} \gamma_i \overline{x}^{\alpha_i}\overline{y}^{\beta_j} \tag{16}$$

where all $\gamma_i \in k$ are nonzero, $\overline{x}^{\alpha_i}y^{\beta_i}$ have minimal μ value ρ for $1 \leq i \leq d$ and $\overline{x}^{\alpha_i}\overline{y}^{\beta_i}$ have value larger than ρ for $i > d$ and $\beta_1 < \cdots < \beta_d$. By our choice of $\overline{v}$ and (15), we have that $d \leq 2$ and the initial form of $\overline{v}$ with respect to μ has one of three special forms (to be enumerated in the list of three cases in the "finer analysis" later in the proof). Further, the substitution defining $\overline{v}$ ensures that

$$\beta_1 > 0. \tag{17}$$

Substitute

$$\overline{x} = x_1^m(y_1+\alpha)^{a'}, \overline{y} = x_1^q(y_1+\alpha)^{b'}. \tag{18}$$

into u and the expression (16) of $\overline{v}$ to obtain

$$\begin{array}{lll} u & = & x_1^m(y_1+\alpha)^{a'} \\ \overline{v} & = & x_1^{\alpha_1 m+\beta_1 q}(y_1+\alpha)^{a'\alpha_1+b'\beta_1}\Lambda \end{array}$$

where

$$\Lambda = \left(\sum_{i=1}^{d} \gamma_i (y_1+\alpha)^{\frac{\beta_i-\beta_1}{m}} + x_1\Omega\right) \tag{19}$$

with $\frac{\beta_i-\beta_1}{m} \in \mathbb{N}$ for all i. This expression for $\overline{v}$ is shown in [34] and in the proof of [17, Theorem 8.4].

Assume that Λ is a unit. We will show later in this proof that with our choice of variables, if Λ is not a unit then we will reach the case where $R_1 \rightarrow S_1$ is of type 0 in the conclusions of the theorem.

Define

$$\overline{\sigma} = \gcd(m, \alpha_1 m + \beta_1 q) = \gcd(m, \beta_1 q). \tag{20}$$

Let

$$\tau = \text{Det}\begin{pmatrix} m & a' \\ \alpha_1 m + \beta_1 q & \alpha_1 a' + \beta_1 b' \end{pmatrix} = \beta_1(mb' - a'q) = \beta_1 > 0. \tag{21}$$

Let

$$\varphi = \text{Det}\begin{pmatrix} m & 0 \\ \alpha_1 m + \beta_1 q & 1 \end{pmatrix} = m > 0. \tag{22}$$

Let $R \rightarrow R^*$ be defined by $u = u_1^g v_1^{g'}$, $\overline{v} = u_1^h\overline{v}_1^{h'}$ where $g, g', h, h' \in \mathbb{N}$, $gh' - hg' = \pm 1$ and

$$u_1 = x_1^{\overline{\sigma}}(y_1+\alpha)^c\Lambda^e, \overline{v}_1 = x_1^{\overline{\sigma}}(y_1+\alpha)^d\Lambda^f$$

where

$$\text{Det}\begin{pmatrix} \overline{\sigma} & c \\ \overline{\sigma} & d \end{pmatrix} = \overline{\sigma}(d-c) = \tau = \beta_1 \tag{23}$$

and

$$\text{(24)} \qquad \mathrm{Det}\begin{pmatrix} \overline{\sigma} & e \\ \overline{\sigma} & f \end{pmatrix} = \overline{\sigma}(f-e) = \varphi = m.$$

Now perform a single quadratic transform $R^* \to R_1$ so that R_1 is dominated by S_1 and R_1 has regular parameters u_1, v_1 satisfying

$$\text{(25)} \quad u_1 = x_1^{\overline{\sigma}}(y_1+\alpha)^c\Lambda^e, v_1 = \frac{\overline{v}_1}{u_1} - \Lambda(0,0)^{f-e}\alpha^{d-c} = (y_1+\alpha)^{d-c}\Lambda^{f-e} - \Lambda(0,0)^{f-e}\alpha^{d-c}.$$

We have an expression for $R \to R_1$ of the form

$$u = u_1^{\overline{m}}(v_1+\beta)^{\overline{a}_1'}, \overline{v} = u_1^{\overline{q}}(v_1+\beta)^{\overline{b}_1'}$$

where $\overline{m} = \frac{m}{\overline{\sigma}}$, $\overline{q} = \frac{\alpha_1 m + \beta_1 q}{\overline{\sigma}}$ and $\beta = \gamma_1^{\frac{m}{\overline{\sigma}}}\alpha^{\frac{\beta_1}{\overline{\sigma}}}$. We have that $\left(\frac{\overline{v}_1}{u_1}\right)(0,y_1)$ is a polynomial in y_1 since $d-c>0$ and $f-e>0$.

We now make a finer analysis. We have three cases:

1) $\overline{x}^{\overline{c}}\overline{y}$ is the unique minimal value term in the expansion (16).
2) $\overline{x}^{\overline{c}}\overline{y}$ and $\overline{y}^p$ are the two minimal value terms in the expansion (16).
3) $\overline{y}^p$ is the unique minimal value term in the expansion (16).

Suppose that we are in Case 1), so that $\overline{x}^{\overline{c}}\overline{y}$ is the unique minimal value term in the expansion (16), so $\beta_1 = 1$. Further, $\Lambda = \gamma_1 + x_1\Omega$ (with $\gamma_1 \in k \neq 0$). Now $\overline{\sigma}(d-c) = \tau = \beta_1 = 1$, and $\overline{\sigma}(f-e) = m$. Thus $\overline{\sigma} = 1$, $d-c=1$ and $f-e=m$. Thus

$$\begin{aligned} u_1 &= x_1(y_1+\alpha)^c\Lambda^e \\ v_1 &= (y_1+\alpha)(\gamma_1+x_1\Omega)^m - \alpha\gamma_1^m = \gamma_1 y_1 + x_1\Omega'. \end{aligned}$$

Thus $R_1 \to S_1$ is unramifed and we are in Case 0) of the conclusions of the theorem.

Suppose that we are in Case 2), so that $\overline{x}^{\overline{c}}\overline{y}$ and $\overline{y}^p$ are the two minimal value terms in the expansion (16). The expansion of (16) is then

$$\overline{v} = \gamma_1 x^{\overline{c}}\overline{y} + \gamma_2\overline{y}^p + \text{higher value terms.}$$

Since $\mu(\overline{y}^p) = \mu(\overline{x}^{\overline{c}}\overline{y})$ we have that $(p-1)\mu(\overline{y}) = \overline{c}\mu(\overline{x})$. Thus char $k \neq 2$ since $\mu(\overline{y}) \notin \mu(\overline{x})\mathbb{Z}$. Let $\psi = \gcd(p-1,\overline{c})$. Then in the substitution

$$\overline{x} = x_1^m(y_1+\alpha)^{a'}, \overline{y} = x_1^q(y_1+\alpha)^{b'}$$

with $mb' - a'q = 1$ of (18), we have $\psi m = p-1$ and $\psi q = \overline{c}$. Substituting (18) in u and $\overline{v}$, we obtain

$$u = x_1^m(y_1+\alpha)^{a'}$$

and

$$\overline{v} = \gamma_1 x_1^{\overline{c}m+q}(y_1+\alpha)^{a'\overline{c}+b'} + \gamma_2 x_1^{qp}(y_1+\alpha)^{b'p} + \cdots = x_1^{qp}(y_1+\alpha)^{a'\overline{c}+b'}\Lambda$$

where $\Lambda = \gamma_2(y_1+\alpha)^{\psi} + \gamma_1 + x_1\Omega$.

Suppose that Λ is not a unit. Let $d_1 = \mathrm{ord}_{y_1}\left[(y_1+\alpha)^{a'\overline{c}+b'}\Lambda(0,y_1)\right]$. We have that $0 < d_1 < \infty$ since Λ is not a unit. By [14, Proposition 3.1] and since our extension is Galois, we have that the complexity md_1 of $R \to S_1$ divides $p = [L:K]$, which is a contradiction to our assumption that $m>1$ and the fact that m divides $p-1$.

Suppose that Λ is a unit. Following the analysis of the case when Λ is a unit above, we have that $\tau = \beta_1 = 1$ and $\overline{\sigma} = \gcd(m, \overline{c}m+q) = 1$. Thus from (23), we have $d-c=1$ and from (24), we have $f-e=m$. From (25), we obtain

$$u_1 = x_1(y_1+\alpha)^c\Lambda^e$$

and

$$v_1 = (y_1+\alpha)\Lambda^m - \alpha(\gamma_2\alpha^\psi+\gamma_1)^m.$$

We compute

$$v_1(0,y_1) = (y_1+\alpha)(\gamma_2(y_1+\alpha)^\psi+\gamma_1)^m - \alpha(\gamma_2\alpha^\psi+\gamma_1)^m.$$

We have

$$\begin{array}{rcl}\frac{\partial}{\partial y_1}v_1(0,y_1) & = & (\gamma_2(y_1+\alpha)^\psi+\gamma_1)^m + (y_1+\alpha)m(\gamma_2(y_1+\alpha)^\psi+\gamma_1)^{m-1}\psi\gamma_2\psi(y_1+\alpha)^{\psi-1}\\ & = & (\gamma_2(y_1+\alpha)^\psi+\gamma_1)^{m-1}[\gamma_2(y_1+\alpha)^\psi+\gamma_1+m(y_1+\alpha)\psi\gamma_2(y_1+\alpha)^{\psi-1}]\\ & = & (\gamma_2(y_1+\alpha)^\psi+\gamma_1)^{m-1}[\gamma_2(y_1+\alpha)^\psi+\gamma_1+\gamma_2(p-1)(y_1+\alpha)^{\psi-1}]\\ & = & (\gamma_2(y_1+\alpha)^\psi+\gamma_1)^{m-1}\gamma_1\end{array}$$

is a unit so $\mathrm{ord}_{y_1}v_1(0,y_1)=1$. Thus the complexity of $R_1\to S_1$ is $m<p$, so the complexity must be one, so that $R_1\to S_1$ is unramified and we are in Case 0) of the conclusions of the theorem.

Now suppose that we are in Case 3) so that $\overline{y}^p$ is the unique minimal value term of the expansion (16) of $\overline{v}$. Then $\beta_1=p$ in (16) and $\Lambda=\gamma_1+x_1\Omega$ is a unit in (19). In the analysis of the case when Λ is a unit following (19), we have that $d-c=1$ if $\overline{\sigma}=p$ and $d-c=p$ if $\overline{\sigma}=1$, so that

$$\mathrm{ord}_{y_1}v_1(0,y_1) = \mathrm{ord}_{y_1}(y_1+\alpha)^{d-c}\gamma_1^{f-e} - \alpha^{d-c}\gamma_1^{f-e} = \left\{\begin{array}{ll}1 & \text{if } \overline{\sigma}=p\\ p & \text{if } \overline{\sigma}=1.\end{array}\right.$$

Thus from (25), we see that $R_1\to S_1$ is of type 1 if $\overline{\sigma}=p$ and is of type 2 if $\overline{\sigma}=1$.

We now establish that if Case 3) above holds ($\overline{y}^p$ is the unique minimal value term in (16)), then the invariant $\overline{\sigma}=\gcd(m,\alpha_1 m+\beta_1 q)$, defined in (20), is such that $\overline{\sigma}=\gcd(m,pq)$, so that the σ defined in the statement of the theorem is $\overline{\sigma}$. Since $\overline{y}^p$ is the unique minimal value term in (16), then $\alpha_1=0$ and $\beta_1=p$, giving the desired equality.

In our analysis above, we saw that if $\mu(\overline{x}^{\overline{c}}\overline{y})\le\mu(\overline{y}^p)$ then $R_1\to S_1$ is in Case 0) of the conclusions of the theorem and if $\mu(\overline{x}^{\overline{c}}\overline{y})>\mu(\overline{y}^p)$ then $R_1\to S_1$ is in Case 1) or Case 2) of the conclusions of the theorem. Since $\mu(\overline{x}^{\overline{c}}\overline{y})=(m\overline{c}+q)$ and $\mu(\overline{y}^p)=pq$, we have that if $\frac{q}{m}\ge\frac{\overline{c}}{p-1}$ then $R_1\to S_1$ is unramifed and if $\frac{q}{m}<\frac{\overline{c}}{p-1}$ then $R_1\to S_1$ is either of type 1 or type 2.

We now establish the formulas for the Jacobian ideal $J(S_1/R_1)$. We have that

$$J(R_1/R)J(S_1/R_1) = J(S/R)J(S_1/S),$$

where $J(S_1/R_1)=x_1^{c_1}S_1$, $J(S/R)=\overline{x}^{\overline{c}}S$ and $J(S_1/S)=x_1^{m+q-1}S_1$.

We have that $\sigma=\gcd(m,pq)$ which is 1 or p. Thus $m=\sigma\overline{m}$ and $pq=\sigma\overline{q}$. Further, we have shown that $\sigma=1$ implies $R_1\to S_1$ is of type 1 and $\sigma=p$ implies $R_1\to S_1$ is of type 2.

Now $\sigma=1$ implies p divides $\overline{q}$ and $\sigma=p$ implies p does not divide $\overline{q}$, since p divides m implies $q\not|q=\overline{q}$. We have that $J(R_1/R)=u_1^{\overline{m}+\overline{q}-1}R_1$ and thus

$$J(R_1/R)S_1 = \left\{\begin{array}{ll}x_1^{m+pq-1}S_1 & \text{if } \sigma=1\\ x_1^{m+pq-p}S_1 & \text{if } \sigma=p.\end{array}\right.$$

In the case $\sigma=1$ we have $(x_1^{m+pq-1})(x_1^{c_1})S_1=(\overline{x}^{\overline{c}})(x_1^{m+q-1})S_1$ so $(x_1^{c_1})S_1=(x_1^{m\overline{c}-(p-1)q})S_1$ and we obtain the formula of Case 1) of the conclusions of the theorem. In the case $\sigma=p$ we have $(x_1^{m+pq-p})(x_1^{c_1})S_1=(\overline{x}^{\overline{c}})(x_1^{m+q-1})S_1$ so $(x_1^{c_1})S_1=(x_1^{m\overline{c}-(p-1)q+(p-1)})S_1$ and we obtain the formula of Case 2) of the conclusions of the theorem. □

Remark 4.2. *Suppose that ω is a rational rank 1 nondiscrete valuation of L dominating S and $R \rightarrow S$ is of type 1. Let ν be the restriction of ω to K. Let $\overline{x} = u$ and $\overline{y}$ be the difference of y and a nonzero polynomial in $\overline{x}$ so that $\omega(\overline{y}) \notin \omega(\overline{x})\mathbb{Z}$. Let $\overline{v}$ be the change of variables in Theorem 4.1.*

Define m and q to be the unique relatively prime positive integers such that $m\omega(\overline{y}) = q\omega(x)$. We have that $m > 0$. There exist $0 \neq \alpha \in k$ and $a', b' \in \mathbb{N}$ such that $mb' - qa' = 1$ and if $S \rightarrow S_1$ is the sequence of quadratic transforms defined by

$$\overline{x} = x_1^m (y_1 + \alpha)^{a'}, \overline{y} = x_1^q (y_1 + \alpha)^{b'}$$

then ω dominates S_1.

Let ν be the restriction of ω to K. The formulas of Cases 0), 1) and 2) of Theorem 4.1 can then be stated in terms of the valuation ω. They are:

0) *If $\frac{q}{m} \geq \frac{\overline{c}}{p-1}$ then $R_1 \rightarrow S_1$ is unramified.*

1) *If $\frac{q}{m} < \frac{\overline{c}}{p-1}$ and $\sigma = 1$ then $R_1 \rightarrow S_1$ is of type 1 and*

$$\left(\frac{c_1}{p-1}\right)\omega(x_1) = \left(\frac{\overline{c}}{p-1}\right)\omega(x) - \omega(\overline{y}).$$

2) *If $\frac{q}{m} < \frac{\overline{c}}{p-1}$ and $\sigma = p$ then $R_1 \rightarrow S_1$ is of type 2 and*

$$\left(\frac{c_1}{p-1}\right)\omega(x_1) = \left(\frac{\overline{c}}{p-1}\right)\omega(x) - \omega(\overline{y}) + \omega(x_1).$$

In the conclusions of the theorem, suppose that $R_1 \rightarrow S_1$ is of type 1. Then we necessarily have that $\nu(\overline{v}) \notin \nu(u)\mathbb{Z}$ since $\sigma = 1$ and thus $\overline{m} = m > 1$.

Theorem 4.3. *Suppose that $R \rightarrow S$ is of type 2 with respect to regular parameters x, y in S and u, v in R and that $J(S/R) = (x^{\overline{c}})$. Let $g(u) \in k[u]$ be a polynomial with no constant term. Make the change of variables, letting $\overline{v} = v - g(u)$ and $\overline{y} = \overline{v}$, so that $x, \overline{y}$ are regular parameters in S and $u, \overline{v}$ are regular parameters in R.*

Suppose that m, q are positive integers with $gcd(m, q) = 1$. Let α be a nonzero element of k. Consider the sequence of quadratic transforms $S \rightarrow S_1$ so that S_1 has regular parameters x_1, y_1 defined by

$$x = x_1^m (y_1 + \alpha)^{a'}, \overline{y} = x_1^q (y_1 + \alpha)^{b'}$$

where $a', b' \in \mathbb{N}$ are such that $mb' - qa' = 1$.

Let $\sigma = gcd(pm, q)$ which is 1 or p. There exists a unique sequence of quadratic transforms $R \rightarrow R_1$ such that R_1 has regular parameters u_1, v_1 defined by

$$u = u_1^{\overline{m}} (v_1 + \beta)^{c'}, \overline{v} = u_1^{\overline{q}} (v_1 + \beta)^{d'}$$

where $pm = \sigma\overline{m}$, $q = \sigma\overline{q}$, $\overline{m}d' - c'\overline{q} = 1$ and $0 \neq \beta \in k$, giving a commutative diagram of homomorphisms

$$\begin{array}{ccc} R_1 & \rightarrow & S_1 \\ \uparrow & & \uparrow \\ R & \rightarrow & S \end{array}$$

such that $R_1 \rightarrow S_1$ is quasi finite. We have that $J(S_1/R_1) = (x_1^{c_1})$ for some positive integer c_1. Further:

1) *If $\sigma = 1$ then $R_1 \rightarrow S_1$ is of type 1 and*

$$\left(\frac{c_1}{p-1}\right) = \left(\frac{\overline{c}}{p-1}\right) m - m.$$

2) *If $\sigma = p$ then $R_1 \to S_1$ is of type 2 and*

$$\binom{c_1}{p-1} = \binom{\overline{c}}{p-1} m - m + 1.$$

Proof. Substitute

$$x = x_1^m (y_1+\alpha)^{a'}, \overline{y} = x_1^q (y_1+\alpha)^{b'}. \tag{26}$$

into u and v to obtain

$$\begin{array}{rcl} u & = & x_1^{mp}(y_1+\alpha)^{a'p}(\lambda + x_1\Omega) \\ v & = & x_1^q (y_1+\alpha)^{b'} \end{array}$$

where $0 \neq \lambda \in k$. Let

$$\sigma = \gcd(mp, q).$$

Let

$$\tau = \mathrm{Det}\begin{pmatrix} mp & a'p \\ q & b' \end{pmatrix} = p(mb' - a'q) = p. \tag{27}$$

Let $R \to R^*$ be the sequence of quadratic transforms defined by $u = u_1^g \overline{v}_1^{g'}$, $\overline{v} = u_1^h \overline{v}_1^{h'}$ where $g, g', h, h' \in \mathbb{N}$, $gh' - hg' = \pm 1$ and

$$u_1 = x_1^\sigma (y_1+\alpha)^c (\lambda + x_1\Omega)^e, \overline{v}_1 = x_1^\sigma (y_1+\alpha)^d (\lambda + x_1\Omega)^f$$

where

$$\mathrm{Det}\begin{pmatrix} \sigma & c \\ \sigma & d \end{pmatrix} = \sigma(d - c) = \tau = p \tag{28}$$

Now perform a single quadratic transform $R^* \to R_1$ so that R_1 is dominated by S_1 and R_1 has regular parameters u_1, v_1 satisfying

(29)

$$u_1 = x_1^\sigma (y_1+\alpha)^c (\lambda + x_1\Omega)^e, v_1 = \frac{\overline{v}_1}{u_1} - \alpha^{d-c}\lambda^{f-e} = (y_1+\alpha)^{d-c}(\lambda + x_1\Omega)^{f-e} - \alpha^{d-c}\lambda^{f-e}.$$

We have an expression for $R \to R_1$ of the form

$$u = u_1^{\overline{m}}(v_1+\beta)^{\overline{a}'_1}, \overline{v} = u_1^{\overline{q}}(v_1+\beta)^{\overline{b}'_1}$$

where $\overline{m} = \frac{mp}{\sigma}$, $\overline{q} = \frac{q}{\sigma}$ and $\beta = \alpha^{\frac{p}{\sigma}}\gamma(0,0)^{-\frac{q}{\sigma}}$. We have that $v_1(0, y_1) = (y_1+\alpha)^{d-c}\lambda^{f-e} - \alpha^{d-c}\lambda^{f-e}$ with $d - c > 0$. Hence

$$0 < d_1 = \mathrm{ord}_{y_1} v_1(0, y_1) < \infty.$$

If $\sigma = p$ then $d - c = 1$ so that $d_1 = 1$ and the complexity of $R_1 \to S_1 = p$. We then have that $R_1 \to S_1$ is of type 2, so that it is in Case 2 of the conclusions of the theorem. If $\sigma = 1$ then $d - c = p$ and $d_1 = p$ so that the complexity of $R_1 \to S_1$ is p and it is in Case 1 of the conclusions of the theorem.

If $\sigma = 1$ then p divides $\overline{m}$ and $\sigma = p$ implies p does not divide $\overline{m}$, since p then divides q which implies p does not divide $m = \overline{m}$.

We have that $J(R_1/R) = (u_1^{\overline{m}+\overline{q}-1})$ and thus

$$J(R_1/R)S_1 = \begin{cases} x_1^{pm+q-1}S_1 & \text{if } \sigma = 1 \\ x_1^{pm+q-p}S_1 & \text{if } \sigma = p. \end{cases}$$

In the case $\sigma = 1$ we have $(x_1^{pm+q-1})(x_1^{c_1})S_1 = (x^{\overline{c}})(x_1^{m+q-1})S_1$ so $(x_1^{c_1})S_1 = (x_1^{m\overline{c}-m(p-1)})S_1$ and we obtain the formula of Case 1) of the statement of the theorem.

In the case $\sigma = p$ we have

$$(x_1^{pm+q-p})(x_1^{c_1})S_1 = (x^{\overline{c}})(x_1^{m+q-1})S_1$$

so

$$(x_1^{c_1})S_1 = (x_1^{m\overline{c}-(p-1)m+(p-1)})S_1$$

and we obtain the formula of Case 2) of the statement of the theorem. □

Remark 4.4. *Suppose that ω is a nondiscrete rational rank 1 valuation of L dominating S and $R \to S$ is of type 2. Let ν be the restriction of ω to K. Make the change of variables, letting $\overline{v}$ be the difference of v and a polynomial in u so that $\omega(\overline{v}) \notin \omega(u)\mathbb{Z}$ and letting $\overline{y} = \overline{v}$.*

Define m and q to be the unique relatively prime positive integers such that $m\omega(\overline{y}) = q\omega(x)$. There exist $0 \neq \alpha \in k$ and $a', b' \in \mathbb{N}$ such that $mb' - qa' = 1$ and if $S \to S_1$ is the sequence of quadratic transforms defined by

$$x = x_1^m(y_1+\alpha)^{a'}, \overline{y} = x_1^q(y_1+\alpha)^{b'}$$

then ω dominates S_1.

The formulas of Cases 1) and 2) of Theorem 4.3 can then be stated in terms of the valuation ω. They are:

1) *If $\sigma = 1$ then $R_1 \to S_1$ is of type 1 and*

$$\left(\frac{c_1}{p-1}\right)\omega(x_1) = \left(\frac{\overline{c}}{p-1}\right)\omega(x) - \omega(x).$$

2) *If $\sigma = p$ then $R_1 \to S_1$ is of type 2 and*

$$\left(\frac{c_1}{p-1}\right)\omega(x_1) = \left(\frac{\overline{c}}{p-1}\right)\omega(x) - \omega(x) + \omega(x_1).$$

If $R_1 \to S_1$ is of type 2, then we have that $\nu(\overline{y}) \notin \nu(x)\mathbb{Z}$, since $gcd(pm, q) = p$.

We will show that $\nu(\overline{y}) \notin \nu(x)\mathbb{Z}$ if $R_1 \to S_1$ is of type 2. We have that $\gcd(pm, q) = p$. If $\nu(\overline{y}) \in \nu(x)\mathbb{Z}$, then $\nu(\overline{y}) = q\nu(x)$ and since $p \mid q$, we have that $\nu(\overline{v}) = \nu(u^{\overline{q}})$ with $\overline{q} = \frac{q}{p}$, a contradiction to the assumption that $\nu(\overline{v}) \notin \nu(u)\mathbb{Z}$.

5. Switching of types of extensions under blowing up

Suppose that K and L are two dimensional algebraic function fields over an algebraically closed field k of characteristic $p > 0$ and $K \to L$ is an Artin-Schreier extension. Suppose that R is a regular algebraic local ring of K and S is a regular algebraic local ring of S such that S dominates R and $R \to S$ is of type 1 or 2 as defined at the beginning of Section 4. Further assume that the Jacobian ideal $J(S/R)$ satisfies $\sqrt{J(S/R)} = xS$. Let $P = uR$ and $Q = xS$. Then $R \to S$ is well prepared and the regular parameters u, v and x, y are admissible parameters. Such an extension $R \to S$ exists by Remark 3.5 and Proposition 3.9.

Inductively applying Theorems 4.1 and 4.3, and making choices for the construction of $S_i \to S_{i+1}$ consistent with the assumptions of Theorems 4.1 and 4.3, we construct a diagram where the horizontal sequences are birational extensions of regular local rings (sequences of quadratic transforms)

$$\begin{array}{ccccccccc} S & = & S_0 & \to & S_1 & \to & S_2 & \to & \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ R & = & R_0 & \to & R_1 & \to & R_2 & \to & \cdots \end{array} \tag{30}$$

Every $R_i \to S_i$ is well prepared of type 0, 1 or 2. Each R_i has admissible regular parameters (u_i, v_i) and $(u_i, \overline{v}_i)$ and each S_i has admissible regular parameters (x_i, y_i) and $(\overline{x}_i, \overline{y}_i)$. The map $S_i \to S_{i+1}$ is defined by

$$\overline{x}_i = x_{i+1}^{m_{i+1}}(y_{i+1} + \alpha_{i+1})^{a'_{i+1}}, \overline{y}_i = x_{i+1}^{q_{i+1}}(y_{i+1} + \alpha_{i+1})^{b'_{i+1}} \tag{31}$$

and the map $R_i \to R_{i+1}$ is defined by

$$u_i = u_{i+1}^{\overline{m}_{i+1}}(v_{i+1} + \beta_{i+1})^{c'_{i+1}}, \overline{v}_i = u_{i+1}^{\overline{q}_{i+1}}(v_{i+1} + \beta_{i+1})^{d'_{i+1}} \tag{32}$$

where $0 \neq \alpha_{i+1}, 0 \neq \beta_{i+1} \in k$. If $R_i \to S_i$ is of type 1 or of type 2 then $\overline{x}_i$, $\overline{y}_i$ and $\overline{v}_i$ are defined by our changes of variables in Theorem 4.1 or 4.3. If $R_i \to S_i$ is of type 0, then we take $\overline{x}_i = u_i$ and $\overline{y}_i = \overline{v}_i = v_i$. If $R_i \to S_i$ is of type 2, we will impose the extra condition that

$$\overline{m}_{i+1} = \frac{pm_{i+1}}{\gcd(pm_{i+1}, q_{i+1})} > 1. \tag{33}$$

We will say that the sequence (30) switches infinitely often if there are infinitely many i such that $R_i \to S_i$ is of type 1 and there are infinitely may i such that $R_i \to S_i$ is of type 2.

Since $\mathrm{trdeg}_K = 2$, we have that $\cup_{i=1}^{\infty} R_i$ and $\cup_{i=1}^{\infty} S_i$ are valuation rings (by [1, Lemma12]). Further, given $f \in R_i$ (or $f \in S_i$), there exists $j \geq i$ such that there is an expression $f = u_i^{t_i}\gamma_i$ where $t_i \in \mathbb{N}$ and γ_i is a unit in R_j (or $f = x_i^{t_i}\gamma_i$ where $t_i \in \mathbb{N}$ and γ_i is a unit in S_j), as shown for instance in [1].

Let ν and ω be valuations which have these respective valuation rings and such that $\omega|K = \nu$. These valuations are uniquely determined up to equivalence of valuations. We have that ω and ν are nondiscrete rational rank 1 valuations, with value groups

$$\nu K = \cup_{i=1}^{\infty} \frac{1}{\overline{m}_1\overline{m}_2 \cdots \overline{m}_i}\mathbb{Z}\nu(u) \text{ and } \omega L = \cup_{i=1}^{\infty} \frac{1}{m_1 m_2 \cdots m_i}\mathbb{Z}\nu(x).$$

Equation (33) is just the statement that $\nu(\overline{v}_i) \notin \mathbb{Z}\nu(u_i)$. The condition that all $m_{i+1} > 1$ in Theorem 4.1 is just the statement that $\omega(\overline{y}_i) \notin \mathbb{Z}\omega(\overline{x}_i)$.

Suppose that $\overline{\omega}$ is a valuation ring of L which dominates S which is nondiscrete of rational rank 1 and $\overline{\nu} = \overline{\omega}|K$. Then we can inductively construct a sequence (30) so that ω dominates S_i for all i, and so $\mathcal{O}_{\overline{\nu}} = \cup_{i=1}^{\infty} R_i$ and $\mathcal{O}_{\overline{\omega}} = \cup_{i=1}^{\infty} S_i$, so that the valuations ω and ν determined by the sequence are $\overline{\omega}$ and $\overline{\nu}$ respectively (up to equivalence of valuations).

The complexity of the maps $R_i \to S_i$ in the diagram (30) must either be p for all i, or will be p until some i_0 and then the complexity will be 1 for all $j \geq i_0$, so that $R_j \to S_j$ is of type 1 or 2 for $j < i_0$ and $R_j \to S_j$ is unramified (has type 0) for all $j \geq i_0$.

We will say a sequence (30) has stable complexity p if the complexity of $R_i \to S_i$ is p for all $i \geq 0$. With this assumption, each map $R_i \to S_i$ in (30) is either of type 1 or of type 2. We draw the following conclusions from Theorems 4.1 and 4.3.

Assume that the stable complexity of a sequence (30) is p. If $R_r \to S_r$ is of type 1, then $S_r \to S_{r+1}$ is the standard sequence of quadratic transforms along ω. Further, $R_r \to R_{r+1}$ is the standard sequence of quadratic transforms along ν unless $m_{r+1} = p$. In this case, $\overline{m}_{r+1} = 1$, so that the standard sequence of quadratic transforms of R_r along ν dominates R_{r+1}, and $R_{r+1} \to S_{r+1}$ is of type 2.

If $R_r \to S_r$ is of type 2, then $R_r \to R_{r+1}$ is the standard sequence of quadratic transforms along ν. Further, $S_r \to S_{r+1}$ is the standard sequence of quadratic transforms along ω unless $\overline{m}_{r+1} = p$. In this case, $m_{r+1} = 1$, so that the standard sequence of quadratic transforms of S_r along ω dominates S_{r+1}, and $R_{r+1} \to S_{r+1}$ is of type 1.

Proposition 5.1. *Suppose that a sequence (30) has stable complexity p. Then the sequence (30) switches infinitely often if and only if νK is p-divisible.*

Proof. We have that ωL is p-divisible if and only if νK is p-divisible (for instance by (3) of [19, Lemma 7.32]).

Suppose that νK is not p-divisible. Then there exists r_0 such that for $r > r_0$ we have that $p \not| \overline{m}_r$ and $p \not| m_r$. Now if $r > r_0$ and $R_r \rightarrow S_r$ is of type 1 then $R_{r+1} \rightarrow S_{r+1}$ must be of type 1 since $\sigma_{r+1} = p$ in Theorem 4.1 implies p divides m_{r+1}. Further, if $R_r \rightarrow S_r$ is of type 2 then $R_{r+1} \rightarrow S_{r+1}$ must be of type 2 since $\sigma_{r+1} = 1$ in Theorem 4.3 implies p divides $\overline{m}_{r+1}$. Thus for $r > r_0$ there can be no switching.

Suppose that νK is p-divisible. Suppose that (30) doesn't switch infinitely often. Then there exists r_0 such that for $r \geq r_0$ $R_r \rightarrow S_r$ is of the same type as $R_{r_0} \rightarrow S_{r_0}$. Suppose that $R_{r_0} \rightarrow S_{r_0}$ is of type 1. Since ωL is p-divisible, there exists $r \geq r_0$ such that p divides m_{r+1}. But then $R_{r+1} \rightarrow S_{r+1}$ must be of type 2 since $\sigma_{r+1} = p$ in Theorem 4.1. Suppose that $R_{r_0} \rightarrow S_{r_0}$ is of type 2. Since νL is p-divisible, there exists $r \geq r_0$ such that p divides m_{r+1}. Now p divides m_{r+1} implies $p \not| q_{r+1}$ which implies $\sigma_{r+1} = \gcd(pm_{r+1}, q_{r+1}) = 1$ in Theorem 4.3. But then $R_{r+1} \rightarrow S_{r+1}$ must be of type 1. We have arrived at a contradiction. This completes the proof that (30) switches infinitely often. ☐

Remark 5.2. *Suppose that a sequence (30) has stable complexity p. If νK is not p-divisible then $m_r = \overline{m}_r$ for $r \gg 0$ and we have that $m_r > 1$ and $\overline{m}_r > 1$ for $r \gg 0$ in (30).*

As the following Proposition shows, the nicest form that a sequence (30) can take is when $R_r \rightarrow S_r$ is of type 2 for all $r \gg 0$. This is the strongly monomial form (defined in the introduction).

Proposition 5.3. *The following are equivalent for a sequence (30) with stable complexity p, and valuations ν and ω which it determines.*

1) *There exists an r_0 such that $R_r \rightarrow S_r$ is of type 2 in (30) for $r \geq r_0$.*
2) *$[\omega L : \nu K] = p$.*
3) *The valued extension L/K is defectless.*

Proof. Since the stable complexity of the sequence (30) is p, we have that

$$p = [\omega L : \nu K]\delta(\omega/\nu)$$

by Remark 3.8 and Proposition 3.6. Thus statement 2) is equivalent to statement 3). We now prove that statement 1) is equivalent to statement 2). Suppose that there exists r_0 such that $R_r \rightarrow S_r$ is of type 2 in (30) for $r \geq r_0$. Then $u_i = \gamma_i \overline{x}_i^p$ for all $i \geq r_0$ and $m_i = \overline{m}_i$ for $i \geq r_0$ by Theorem 4.3. Thus

$$\nu K = \cup_{i=r_0+1}^{\infty} \frac{1}{\overline{m}_{r_0+1} \cdots \overline{m}_i} \mathbb{Z}\nu(u_{r_0}) = \cup_{i=r_0+1}^{\infty} \frac{1}{m_{r_0+1} \cdots m_i} \mathbb{Z}p\omega(x_{r_0}) = p\omega L.$$

If $\nu K = \omega L$ then $p\omega L = \omega L$ which implies that ωL is p-divisible, a contradiction to Proposition 5.1. Thus $[\omega L : \nu K] = p$.

Suppose that there exists r_0 such that $R_r \rightarrow S_r$ is of type 1 in (30) for $r \geq r_0$. Then $u_i = x_i$ for $i \geq r_0$. Thus

$$\nu K = \cup_{i=r_0}^{\infty} \mathbb{Z}\nu(u_i) = \cup_{i=r_0}^{\infty} \mathbb{Z}\omega(x_i) = \omega L.$$

Finally, suppose that (30) switches infinitely often. Then νK and νL are p-divisible by Proposition 5.1. Since $[L : K] = p$, $\omega L = p\omega L \subset \nu K \subset \omega L$ which implies that $\nu K = \omega L$. ☐

We see that if the sequence (30) switches infinitely often then the extension must be a defect extension. Any configuration of switching is possible. A sequence with prescribed switching can be created by iterating the constructions of Theorems 4.1 and 4.3.

If a sequence stabilizes with $R_r \to S_r$ of type 2 for all $r \geq r_0$, then from iteration of formula 2) of Theorem 4.3, for all $s > 0$ we have that

$$\left(\frac{c_{r_0+s}}{p-1}\right)\frac{1}{m_1\cdots\cdot m_{r_0+s}} = \left(\frac{c_{r_0}}{p-1}\right)\frac{1}{m_1\cdots m_{r_0}} - \frac{1}{m_1\cdots m_{r_0}} + \frac{1}{m_1\cdots + m_{r_0+s}}.$$

By Remarks 4.2 and 4.4, $\omega(J(S_i/R_i)) = c_i\omega(x_i)$ is monotonically decreasing with i. We calculate that for $s \geq 1$,

$$\omega(J(S_{r_0+s}/R_{r_0+s})) = \left[\frac{c_{r_0}}{m_1\cdots m_{r_0}} - \frac{(p-1)}{m_1\cdots m_{r_0}} + \frac{p-1}{m_1\cdots m_{r_0+s}}\right]\omega(x_0).$$

Thus since infinitely many m_i are greater than 1,

$$\inf\{\omega(J(S_i/R_i))\} = \left[\frac{c_{r_0}}{m_1\cdots m_{r_0}} - \frac{(p-1)}{m_1\cdots m_{r_0}}\right]\omega(x_0) \in \omega(L).$$

Thus by Proposition 7.9,

$$\mathrm{dist}(\omega/\nu) = -\frac{1}{p-1}\inf\{\omega(J(S_i/R_i))\} \in \frac{1}{p-1}\omega(L).$$

In contrast, we can get any non-positive real number as the distance $\mathrm{dist}(\omega/\nu)$ on K if we allow a sequence which does not stabilize to type 2, as is shown in the following Theorem.

Theorem 5.4. *Suppose that K is an algebraic function field of transcendence degree 2 over an algebraically closed field k of characteristic $p > 0$, and that A is an algebraic regular local ring of K with regular parameters z and w. Let $\alpha \in \mathbb{R} \geq 0$ and let $\Phi : \mathbb{N} \to \{1, 2\}$ be a function such that $\Phi(n)$ is not identically equal to 2 for $n \gg 0$. Then there exists an Artin-Schreier extension $K \to L$ and a sequence (30) such that $R_0 = A$, $R_r \to S_r$ is of type 1 if $\Phi(r) = 1$ and of type 2 if $\Phi(r) = 2$ and the induced defect extension of valuations satisfies*

$$\mathrm{dist}(\omega/\nu) = -\alpha,$$

where the valuation ν of L is normalized so that $\nu(z) = 1$. We will further have that $m_i > 1$ and $\overline{m}_i > 1$ for all i in the sequence (30).

Proof. First assume that $\Phi(0) = 1$. Let $R_0 = A$, $u_0 = z$ and $v_0 = w$. Let e be a positive integer such that $e > \alpha$. Let $c_0 = (p-1)e$. Let Θ be a root of the Artin-Schreier polynomial $X^p - X - v_0u_0^{-pe}$. Let $L = K(\Theta)$. Set $x_0 = u_0$, $y_0 = u_0^e\Theta$. Let $S_0 = R[y_0]_{(x_0,y_0)}$, which is an algebraic regular local ring of L which dominates R_0. The regular parameters x_0, y_0 in S_0 satisfy $u_0 = x_0, v_0 = y_0^p - x_0^{e(p-1)}y_0$, so that the extension $R \to S$ is of type 1. We have that $J(S_0/R_0) = (x_0^{c_0})$, with $\frac{c_0}{p-1} > \alpha$.

Suppose that we have a sequence

$$\begin{array}{ccc} S_r & \to & S_{r+1} \\ \uparrow & & \uparrow \\ R_r & \to & R_{r+1} \end{array}$$

where $R_r \to S_r$ and $R_{r+1} \to S_{r+1}$ are both of type 1. Then from Theorem 4.1, we have that

(34) $$\left(\frac{c_{r+1}}{p-1}\right)\frac{1}{m_1\cdots m_{r+1}} = \left(\frac{c_r}{p-1}\right)\frac{1}{m_1\cdots m_r} - \frac{q_{r+1}}{m_{r+1}}\left(\frac{1}{m_1\cdots m_r}\right).$$

Suppose that we have a sequence

$$\begin{array}{ccccccccc} S_r & \to & S_{r+1} & \to & \cdots & \to & S_{r+s} & \to & S_{r+s+1} \\ \uparrow & & \uparrow & & & & \uparrow & & \uparrow \\ R_r & \to & R_{r+1} & \to & \cdots & \to & R_{r+s} & \to & R_{r+s+1} \end{array}$$

where $s \geq 1$, $R_i \to S_i$ is of type 1 if $i = r$ or $i = r+s+1$ and $R_i \to S_i$ is of type 2 if $r+1 \leq i \leq r+s$. Then from Theorems 4.1 and 4.3, we have that

$$\left(\frac{c_{r+s+1}}{p-1}\right)\frac{1}{m_1\cdots m_{r+s+1}} = \left(\frac{c_r}{p-1}\right)\frac{1}{m_1\cdots m_r} - \frac{q_{r+1}}{m_{r+1}}\left(\frac{1}{m_1\cdots m_r}\right). \tag{35}$$

Let p' be a prime distinct from p.

We now inductively construct the sequence (30), so that $m_i > 1$ and $\overline{m}_i > 1$ for all i. Suppose that the sequence has been constructed up to $R_r \to S_r$, $\Phi(r) = 1$ and we have that for all $t \leq r$ such that $\Phi(t) = 1$,

$$\alpha < \left(\frac{c_t}{p-1}\right)\frac{1}{m_1\cdots m_t} \text{ and } \left(\frac{c_t}{p-1}\right)\frac{1}{m_1\cdots m_t} < \alpha + \frac{1}{2^t} \text{ if } t > 0. \tag{36}$$

First suppose that $\Phi(r+1) = 1$. There exists $\lambda(r+1) \in \mathbb{Z}_+$ such that there exits $q_{r+1} \in \mathbb{Z}_+$ such that $\gcd(q_{r+1}, p') = 1$ and

$$\frac{c_r}{p-1} - \alpha m_1\cdots m_r > \frac{q_{r+1}}{(p')^{\lambda(r+1)}} > \frac{c_r}{p-1} - \left(\alpha + \frac{1}{2^{r+1}}\right)m_1\cdots m_r. \tag{37}$$

Set $m_{r+1} = (p')^{\lambda(r+1)}$. Then

$$\alpha + \frac{1}{2^{r+1}} > \left(\frac{c_r}{p-1}\right)\frac{1}{m_1\cdots m_r} - \left(\frac{q_{r+1}}{m_{r+1}}\right)\frac{1}{m_1\cdots m_r} > \alpha. \tag{38}$$

Now we have that $\frac{q_{r+1}}{m_{r+1}} < \frac{c_r}{p-1}$ with $\gcd(m_{r+1}, pq_{r+1}) = 1$ so we may define from Theorem 4.1 and the above values of q_{r+1} and m_{r+1} a commutative diagram

$$\begin{array}{ccc} S_r & \to & S_{r+1} \\ \uparrow & & \uparrow \\ R_r & \to & R_{r+1} \end{array}$$

such that $R_{r+1} \to S_{r+1}$ is of type 1. Further, (36) holds for $t = r+1$ by (34) and (38).

Now suppose that $\Phi(r+1) = 2$. Let $s \geq 1$ be the smallest integer such that $\Phi(r+s+1) = 1$. There exists an integer $\lambda(r+1) > 1$ such that there exits $q_{r+1} \in \mathbb{Z}_+$ such that $\gcd(q_{r+1}, p) = 1$ and

$$\frac{c_r}{p-1} - \alpha m_1\cdots m_r > \frac{q_{r+1}}{p^{\lambda(r+1)}} > \frac{c_r}{p-1} - \left(\alpha + \frac{1}{2^{r+1}}\right)m_1\cdots m_r. \tag{39}$$

Set $m_{r+1} = p^{\lambda(r+1)}$. Then

$$\alpha + \frac{1}{2^{r+1}} > \left(\frac{c_r}{p-1}\right)\frac{1}{m_1\cdots m_r} - \left(\frac{q_{r+1}}{m_{r+1}}\right)\frac{1}{m_1\cdots m_r} > \alpha. \tag{40}$$

We have that $\frac{q_{r+1}}{m_{r+1}} < \frac{c_r}{p-1}$ with $\gcd(m_{r+1}, pq_{r+1}) = p$ so we may define from Theorem 4.1 and the above values of q_{r+1} and m_{r+1} a commutative diagram

$$\begin{array}{ccc} S_r & \to & S_{r+1} \\ \uparrow & & \uparrow \\ R_r & \to & R_{r+1} \end{array}$$

such that $R_{r+1} \to S_{r+1}$ is of type 2. We have $\sigma = p$ in Theorem 4.1 and $\overline{m}_{r+1} = \frac{m_{r+1}}{\sigma} > 1$. For $r+1 \le i \le r+s$ define

$$\begin{array}{ccc} S_i & \to & S_{i+1} \\ \uparrow & & \uparrow \\ R_i & \to & R_{i+1} \end{array}$$

from Theorem 4.3 by taking $m_{i+1} = (p')^2$ and $q_{i+1} = p^2$ if $i < s$ and taking $m_{i+1} = p^2$ and $q_{i+1} = (p')^2$ if $i = s$. From Theorem 4.3 we have a commutative diagram

$$\begin{array}{ccccccccc} S_r & \to & S_{r+1} & \to & \cdots & \to & S_{r+s} & \to & S_{r+s+1} \\ \uparrow & & \uparrow & & & & \uparrow & & \uparrow \\ R_r & \to & R_{r+1} & \to & \cdots & \to & R_{r+s} & \to & R_{r+s+1} \end{array}$$

such that $R_i \to S_i$ is of type 1 if $i = r$ or $i = r+s+1$ and $R_i \to S_i$ is of type 2 if $r+1 \le i \le r+s$. Further, (36) is satisfied with $t = r+s+1$ by (35) and (40). We saw above that $m_{r+1} > 1$ and $\overline{m}_{r+1} > 1$. If $r+1 < i \le r+s$, we have $\sigma = p$ in Theorem 4.3 so that $\overline{m}_i = \frac{pm_i}{\sigma} = m_i > 1$. If $i = r+s+1$, then $\sigma = 1$ in Theorem 4.3 and $\overline{m}_{r+s+1} = pm_{r+s+1} > 1$. Thus $m_i > 1$ and $\overline{m}_i > 1$ for $r \le i \le r+s+1$.

Now by Proposition 7.9, we have that

$$-\mathrm{dist}(\omega/\nu) = \frac{1}{p-1} \inf_i \{\omega(J(S_i/R_i))\} = \alpha\omega(x_0) = \alpha.$$

Now suppose that $\Phi(0) = 2$. Using the construction of the above case (when $\Phi(0) = 1$), we can in this case construct an augmented sequence

$$\begin{array}{ccccccccc} B & \to & S_{-1} & \to & S_0 & \to & S_1 & \to & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ A & \to & R_{-1} & \to & R_0 & \to & R_1 & \to & \cdots \end{array} \tag{41}$$

where Φ is extended to the set $\{-1, 0, 1, 2, \ldots\}$ by defining $\Phi(-1) = 1$, and such that the conclusions of the theorem hold for this augmented sequence. We then get the statement of the theorem by forgetting the map $R_{-1} \to S_{-1}$. ☐

Remark 5.5. *In the construction of the sequence (30) in Theorem 5.4, we have $m_i > 1$ and $\overline{m}_i > 1$ for all i, so that $\omega(\overline{y}_i) \notin \omega(\overline{x}_i)\mathbb{Z}$ for all i and $\omega(\overline{v}_i) \notin \omega(u_i)\mathbb{Z}$ for all i. Thus $R_0 \to R_1 \to R_2 \to \cdots$ is the sequence of sequences of standard quadratic transforms along ν and $S_0 \to S_1 \to S_2 \to \cdots$ is the sequence of sequences of standard quadratic transforms along ω.*

6. Calculation of distance in some examples

We give an analysis of the tower of two Artin-Schreier extensions constructed in [19, Theorem 7.38]. The example gives a diagram

$$\begin{array}{ccccc} R_1 & \to & A_1 & \to & S_1 \\ \downarrow & & \downarrow & & \downarrow \\ R_2 & \to & A_2 & \to & S_2 \\ \downarrow & & \downarrow & & \downarrow \\ \vdots & & \vdots & & \vdots \end{array}$$

where the first two columns and the last two columns are diagrams of the type of (30). The union of the S_i is the valuation ring of a rational rank 1 nondiscrete valuation ω. The vertical arrows are all standard sequences of quadratic transforms.

The rows are such that $R_i \to A_i$ is of type 2 if i is odd and of type 1 if i is even. The extension $A_i \to S_i$ is of type 1 if i is odd and of type 2 if i is even. We have that R_i has regular parameters u_i, v_i and S_i has regular parameters x_i, y_i such that

$$u_i = \gamma_i x_i^p, v_i = \tau_i y_i^p + x_i g_i$$

for all i, where γ_i, τ_i are units in S_i and $g_i \in S_i$. A further analysis in [19] shows that strong local monomialization fails for this example.

An example is given in the later paper [16] where the condition of local monomialization itself fails.

The example of Section 7 of [19] is a composite of two defect Artin-Schreier extensions,

$$K = k(u, v) \to K_1 = k(x, v) \to K^* = k(x, y)$$

where

$$u = \frac{x^p}{1 - x^{p-1}}, v = y^p - x^c y \tag{42}$$

with c a positive integer which is divisible by $p-1$. A rational rank 1 valuation ω is given of K^*, which is trivial on k. Let ν_1 be the restriction of ω to K_1 and ν be the restriction of ω to K.

We will determine the distances of these extensions, illustrating an application of Proposition 7.9, showing that the extension $K \to K_1 \to K^*$ is a tower of two dependent defect Artin-Schreier extensions. The first of these extensions was computed by a different method in [23].

We have a sequence of algebraic regular local rings of K, K_1 and K^*,

$$R_1 = k[u, v]_{(u,v)} \to A_1 = k[x, v]_{(x,v)} \to S_1 = k[x, y]_{(x,y)}$$

such that ω dominates S_1. We normalize ω by setting $\omega(x) = 1$. There are sequences of homomorphisms

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow \\
R_{r+1} & \to & A_{r+1} & \to & S_{r+1} \\
\uparrow & & \uparrow & & \uparrow \\
R_r & \to & A_r & \to & S_r \\
\uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow \\
R_2 & \to & A_2 & \to & S_2 \\
\uparrow & & \uparrow & & \uparrow \\
R_1 & \to & A_1 & \to & S_1
\end{array}$$

where the vertical arrows are sequences of quadratic transforms which are dominated by ω and S_r dominates A_r and A_r dominates R_r. These sequences are calculated above R_1 and S_1 in [19] and above A_1 in [14]. The homomorphism $R_k \to A_k$ is of type 1 if k is even, and of type 2 if k is odd. The homomorphism $A_k \to S_k$ is of type 1 if k is odd and of type 1 if k is even.

The local ring A_r has regular parameters (x_{A_k}, v_{A_k}) and $(x_{A_k}, \overline{v}_k)$ such that $A_k \to A_{k+1}$ is defined (equations (70) and (71) of [14]) by

$$x_{A_k} = x_{A_{k+1}}^p (v_{A_{k+1}} + 1), \overline{v}_{A_k} = x_{A_{k+1}} \text{ if } k \text{ is odd} \tag{43}$$

and

$$x_{A_k} = x_{A_{k+1}}^{p^3}(v_{A_{k+1}}+1), \overline{v}_{A_k} = x_{A_{k+1}} \text{ if } k \text{ is even.} \tag{44}$$

The local ring S_r has regular parameters (x_{S_k}, y_{S_k}) and $(x_{S_k}, \overline{y}_{S_k})$ such that $S_k \rightarrow S_{k+1}$ is defined for all $k \geq 1$ (equation (45) of [14]) by

$$x_{S_k} = x_{S_{k+1}}^{p^2}(\overline{y}_{k+1}+1), \overline{y}_k = x_{S_{k+1}}. \tag{45}$$

By [19, Theorem 7.38], the local ring R_r has regular parameters (x_{R_k}, y_{R_k}) such that the homomorphism $R_k \rightarrow S_k$ has a stable form

$$u_{R_k} = \gamma_k x_{S_k}^p, v_{R_k} = \alpha_k y_{S_k}^p + x_{S_k} g_k$$

for all $k \geq 1$, where γ_k and α_k are units in S_k and $g_k \in S_k$. We have that $\omega(x_{A_1}) = 1$ and for $k \geq 2$, we deduce from (43) and (44) that

$$\omega(x_{A_k}) = \begin{cases} \frac{1}{p^{2k-2}} & \text{if } k \text{ is odd,} \\ \frac{1}{p^{2k-3}} & \text{if } k \text{ is even.} \end{cases}$$

Letting $J(A_k/R_k) = (x_{A_k}^{c_k})$, we calculate from 2) of Remark 4.2 that if k is even, then

$$\left(\frac{c_{k+1}}{p-1}\right)\omega(x_{A_{k+1}}) = \left(\frac{c_k}{p-1}\right)\omega(x_{A_k}) - \omega(\overline{v}_{A_k}) + \omega(x_{A_{k+1}}) = \left(\frac{c_k}{p-1}\right)\omega(x_{A_k}) \tag{46}$$

and if k is odd, we calculate from 1) of Remark 4.4 that

$$\left(\frac{c_{k+1}}{p-1}\right)\omega(x_{A_{k+1}}) = \left(\frac{c_k}{p-1}\right)\omega(x_{A_k}) - \omega(x_{A_k}) = \left(\frac{c_k}{p-1}\right)\omega(x_{A_k}) - \frac{1}{p^{2k-2}}. \tag{47}$$

From

$$u_{R_1} = \frac{x_{A_1}^p}{1 - x_{A_1}^{p-1}}, v_{R_1} = v_{A_1}$$

we compute $J(A_1/R_1) = (x_{A_1}^{2p-2})$ so $\omega(J(A_1/R_1) = 2(p-1)$ and $c_1 = 2p-2$. Now we compute from equations (46) and (47) and Proposition 7.9 that

$$-\text{dist}(\nu_1/\nu) = \frac{1}{p-1}\inf_k \omega(J(A_k/R_k)) = 1 - \left(\sum_{i=1}^{\infty}\frac{1}{p^{4i}}\right) = 1 - \left(\frac{1}{p^4-1}\right) = \frac{p^4-2}{p^4-1}.$$

Since $\text{dist}(\nu_1/\nu)$ is less than zero, the extension is dependent. This distance is computed using a different method in [23].

From (45), we compute

$$\omega(x_{S_k}) = \frac{1}{p^{2k-2}} \tag{48}$$

for $k \geq 1$. If k is odd, we compute from 2) of Remark 4.2 that

$$\left(\frac{c_{k+1}}{p-1}\right)\omega(x_{S_{k+1}}) = \left(\frac{c_k}{p-1}\right)\omega(x_{S_k}) - \omega(\overline{y}_{S_k}) + \omega(x_{S_{k+1}}) = \left(\frac{c_k}{p-1}\right)\omega(x_{S_k}). \tag{49}$$

If k is even, we compute from 1) of Remark 4.4 that

$$\left(\frac{c_{k+1}}{p-1}\right)\omega(x_{S_{k+1}}) = \left(\frac{c_k}{p-1}\right)\omega(x_{S_k}) - \omega(x_{S_k}) = \left(\frac{c_k}{p-1}\right)\omega(x_{S_k}) - \frac{1}{p^{2k-2}}. \tag{50}$$

Now we compute from (42) that $J(S_1/A_1) = (x_1^c)$ so $c_1 = c$. Now we compute from equations (49) and (50) and Proposition 7.9 that

$$\begin{aligned} -\mathrm{dist}(\omega/\nu_1) &= \tfrac{1}{p-1}\inf_k \omega(J(S_k/A_k)) \\ &= \tfrac{c}{p-1} - \tfrac{1}{p^2}\left(\textstyle\sum_{i=0}^{\infty} \tfrac{1}{p^{4i}}\right) \\ &= \tfrac{c}{p-1} - \tfrac{1}{p^2}\left(\tfrac{p^4}{p^4-1}\right) = \tfrac{cp^3+(c-1)p^2+cp+c}{p^4-1}. \end{aligned}$$

Now $cp^3+(c-1)p^2+cp+c$ is positive for all positive integers c, so $\mathrm{dist}(\omega/\nu_1)$ is less than zero for all positive integral c and thus the extension is dependent.

7. Appendix

In this appendix we give proofs of the results on defect cuts and ramification cuts of Artin-Schreier extensions from [27]. These results are stated in [27] and their proofs are outlined there. A much more general statement than Theorem 7.7 is proven in [28, Theorem 3.5].

We suppose throughout this section that L is an Artin-Schreier extension of a field K of characteristic p, ω is a rank 1 valuation of L and ν is the restriction of ω to K. We suppose that L is a defect extension of K. We will use the notation of Subsections 2.5 and 2.6.

Let Θ be an Artin-Schreier generator of K. We have that

$$\mathrm{Gal}(L/K) \cong \mathbb{Z}_p = \{\mathrm{id}, \sigma_1, \ldots, \sigma_{p-1}\},$$

where $\sigma_i(\Theta) = \Theta + i$. Since L/K is an immediate extension, the set $\omega(\Theta - K)$ is an initial segment in νK which has no maximal element. Further, $\omega(\Theta) < 0$ by [26, Lemma 2.28]. Let $s = \mathrm{dist}(\omega/\nu) \in \mathbb{R}$, so that

$$\mathrm{dist}(\Theta, K) \uparrow \mathbb{R} = s^- = \mathrm{dist}(\omega/\nu)^- \le 0^-.$$

There exists a sequence $\{c_i\}_{i\in\mathbb{N}}$ in K such that

$$\omega(\Theta - c_i) < \omega(\Theta - c_{i+1})$$

for all i, and

$$\lim_{i\to\infty} \omega(\Theta - c_i) = \mathrm{dist}(\omega/\nu).$$

Let r and t be positive integers with $r < t$. Then

$$\nu(c_t - c_r) = \omega((\Theta - c_r) - (\Theta - c_t)) = \omega(\Theta - c_r).$$

Thus for $r < t < u$,

$$\nu(c_t - c_r) = \omega(\Theta - c_r) < \omega(\Theta - c_t) = \nu(c_u - c_t),$$

so $\{c_i\}$ is a pseudo convergent sequence in K ([24] or Page 39 [32]). By the above, for $r \in \mathbb{N}$, $\nu(c_t = c_r)$ has a common value γ_r for all $t > r$. Further, the above shows that

$$\gamma_r = \omega(\Theta - c_r).$$

The Artin-Schreier generator Θ is a pseudo limit of $\{c_i\}$ ([24] or page 47 [32]).

Lemma 7.1. *The pseudo convergent sequence $\{c_i\}$ does not have a pseudo limit in K.*

Proof. Suppose $c \in K$ is a pseudo limit of $\{c_i\}$ in K. Then

$$\nu(c - c_r) = \gamma_r = \omega(\Theta - c_r) \text{ for all } r.$$

Thus for all r,

$$\omega(\Theta - c) = \omega((\Theta - c_r) + (c_r - c)) \geq \omega(\Theta - c_r)$$

so $\omega(\Theta - c) \geq \operatorname{dist}(\omega/\nu)$, a contradiction. $\square$

Lemma 7.2. *Suppose that $f(x) \in K[x]$ is a polynomial such that $deg(f) < p$. Then there exists $t_0 \in \mathbb{N}$ such that $\nu(f(c_t)) = \nu(f(c_{t_0}))$ for $t \geq t_0$.*

Proof. Since ν has rank 1, by [24, Lemma 10], the smallest degree of a polynomial $g(x) \in K[x]$ such that $\nu(g(c_t))$ does not stabilize for large t is a power of p. Since $\{c_t\}$ does not have a pseudo limit in K, we have that this degree is $\geq p$. $\square$

Proposition 7.3. *Suppose that $f(x) \in K[x]$ is a polynomial of degree $r < p$. Let $\Theta_i = \Theta - c_i$. Then there are polynomials $g_j(Y) \in K[Y]$ of degree $\leq j$ such that*

$$f(\Theta) = g_0(c_i)\Theta_i^r + g_1(c_i)\Theta_i^{r-1} + \cdots + g_{r-1}(c_i)\Theta_i + g_r(c_i) \tag{51}$$

for all i, with $g_0(c_i) = g_0$ a non zero element of K. Further, there exists i_0 and $\lambda_j \in \nu K$ (depending on f) such that

$$\nu(g_j(c_i)) = \lambda_j$$

for $i \geq i_0$ and $0 \leq j \leq r$, and

$$\omega(g_j(c_i)\Theta_i^{r-j}) \neq \omega(g_k(c_i)\Theta_i^{r-k}) \tag{52}$$

for all $0 \leq j < k \leq r$ and $i \geq i_0$.

Proof. We have a factorization $f(x) = f_0 f_1(x) \cdots f_l(x)$ where $f_0 \in K$ and $f_i(x)$ are monic and irreducible for $1 \leq i \leq l$. Let $r_j = \deg(f_j(x))$. Let Ω be an algebraic closure of K containing L. We have factorizations

$$f_j(x) = (x - a_{j1})(x - a_{j2}) \cdots (x - a_{jr_j})$$

with $a_{ji} \in \Omega$ for $1 \leq j \leq r$, giving expressions

$$f_j(x) = x^{r_j} - S_1(a_{j1}, \ldots, a_{jr_j})x^{r_j - 1} + \cdots + (-1)^{r_j} S_{r_j}(a_{j1}, \ldots, a_{jr_j})$$

where S_i is the elementary symmetric function of degree i. Let y be an indeterminate. Then

$$f_j(x+y) = x^{r_j} - S_1(a_{j1} - y, \ldots, a_{jr_j} - y)x^{r_j - 1} + \cdots + (-1)^{r_j} S_{r_j}(a_{j1} - y, \ldots, a_{jr_j} - y).$$

Let $L_j = K(a_{j1}, \ldots, a_{jr_j})$ and set

$$h_i = S_i(a_{j1} - y, \ldots, a_{jr_j} - y) \in L_j[y]$$

for $1 \leq i \leq r_j$. h_i is a polynomial of degree i. h_i is invariant under permutation of the a_{jk} and L_j is Galois over K (since it is a normal extension of K and $r_j < p$). Thus $h_i \in K[y]$ for $i \leq r_j$, and we have an expression

$$f(x+y) = g_0 x^r + g_1(y)x^{r-1} + \cdots + g_{r-1}(y)x + g_r(y) \tag{53}$$

where $g_i(y) \in K[y]$ is a polynomial of degree $\leq i$ ($g_0 = f_0$). For $i \in \mathbb{N}$, we have an expression

$$f(\Theta) = f(\Theta_i + c_i) = g_0\Theta_i^r + g_1(c_i)\Theta_i^{r-1} + \cdots + g_{r-1}(c_i)\Theta_i + g_r(c_i).$$

By Lemma 7.2, there exists i_0 such that $\nu(g_j(c_i))$ is a constant value λ_j for $i \ge i_0$ and $0 \le j \le r$. Now $\omega(\Theta_{i+1}) > \omega(\Theta_i)$ for all i, so for all j, k with $0 \le j < k \le r$, there exists $i(j,k)$ such that

$$\omega(\Theta_i) \ne \frac{\lambda_k - \lambda_j}{t}$$

for t any integer with $1 \le t \le r$, whenever $i > i(j,k)$.

Thus for i such that $i > i_0$ and $i > \max\{i(j,k) \mid 0 \le j < k \le r\}$ and $0 \le j < k \le r$,

$$\omega(g_j(c_i)\Theta_i^{r-j}) \ne \omega(g_k(c_i)\Theta_i^{r-k}).$$

□

Corollary 7.4. *The valuation ring $\mathcal{O}_\omega$ is generated as an $\mathcal{O}_\nu$-module by*

$$\{g\Theta_i^j \mid g \in K, 0 \le j \le p-1, i \in N \text{ and } \omega(g\Theta_i^j) \ge 0\}. \tag{54}$$

Proof. Let M be the $\mathcal{O}_\nu$-module generated by the set (54). The module M is certainly contained in $\mathcal{O}_\omega$. Suppose that $h \in \mathcal{O}_\omega$. Then $\omega(h) \ge 0$ and $h = f(\Theta)$ for some polynomial $f \in K[x]$ of degree $< p$. By Proposition 7.3, we have an expression (51) of h. Taking i sufficiently large (so that (52) holds) we have that

$$0 \le \omega(h) = \min\{\omega(g_j(c_i)\Theta_i^{r-j}) \mid 0 \le j \le r\}. \tag{55}$$

Thus $h \in M$. □

Corollary 7.5. *Suppose that $h \in L$. Then*

$$\omega(\sigma(h) - h) = \omega(\tau(h) - h)$$

for $\sigma, \tau \in Gal(L/K)$ which are both not the identity.

Proof. $h = h(\Theta)$ has an expression of the form (51) of Proposition 7.3 such that (52) holds. We compute using (51) for $0 < j < p$,

$$\begin{aligned}
\sigma_j(h) - h &= h(\Theta + j) - h(\Theta) = h(\Theta_i + j + c_i) - h(\Theta_i + c_i)\\
&= g_0(c_i)(\Theta_i + j)^r + \cdots + g_{r-1}(c_i)(\Theta_i + j) + g_r(c_i)\\
&\quad -[g_0(c_i)\Theta_i^r + \cdots + g_{r-1}(c_i)\Theta_i + g_r(c_i)]\\
&= g_0(c_i)\left(\textstyle\sum_{k=0}^{r-1}\binom{r}{k} j^{r-k}\Theta_i^k\right) + g_1(c_i)\left(\textstyle\sum_{k=0}^{r-2}\binom{r-1}{k} j^{r-1-k}\Theta_i^k\right) + \cdots + g_{r-1}(c_i)j.
\end{aligned}$$

Since $\omega(\Theta_i) < 0$, we have that

$$\omega\left(g_l(c_i)\left(\sum_{k=0}^{r-l-1}\binom{r-l}{k} j^{r-l-k}\Theta_i^k\right)\right) = \omega(g_l(c_i)j\Theta_i^{r-l-1}) = \omega(g_l(c_i)\Theta_i^{r-l-1})$$

for $0 \le l \le r-1$. Thus

$$\omega(\sigma_j(h) - h) = \min\{\omega(g_l(c_i)\Theta_i^{r-l-1}) \mid 0 \le l \le r-1\} \tag{56}$$

for $i \gg 0$ by equation (52). □

For $\alpha \in \omega L$, define the ideal $I_\alpha = \{f \in \mathcal{O}_L \mid \omega(f) \ge \alpha\}$ in $\mathcal{O}_\omega$. Then G_{I_α} (defined in Subsection 2.4) is the subgroup

$$G_{I_\alpha} = \{s \in \mathrm{Gal}(L/K) \mid \omega(s(x) - x) \ge \alpha \text{ for all } x \in \mathcal{O}_\omega\}$$

of $G = \mathrm{Gal}(L/K)$.

Corollary 7.6. *Continuing with our assumption that L/K is a defect Artin-Schreier extension and ω has rank 1, suppose that $\alpha \in \mathbb{R}$. Then*

$$G_{I_\alpha} = \begin{cases} \{1\} & \text{if } \alpha > -dist(\omega/\nu) \\ Gal(L/K) & \text{if } \alpha \le -dist(\omega/\nu). \end{cases}$$

Proof. Suppose that $h \in \mathcal{O}_\omega$. Then h has an expression of the form (51) of Proposition 7.3 such that (52) holds. By the calculation of (55) of the proof of Corollary 7.4 we have that for $i \gg 0$,

$$0 \le \omega(h) = \min\{\omega(g_j(c_i)\Theta_i^{r-j}) \mid 0 \le j \le r\}. \tag{57}$$

By the calculation of (56) of the proof of Corollary 7.5 and (57), we have that for id $\ne \sigma \in \mathrm{Gal}(L/K)$ and $i \gg 0$,

$$\omega(\sigma(h) - h) = \min\{\omega(g_l(c_i)\Theta_i^{r-l-1}) \mid 0 \le l \le r-1\} \ge -\omega(\Theta_i) > -\mathrm{dist}(\omega/\nu).$$

Thus $G_\alpha = \mathrm{Gal}(L/K)$ for $\alpha \le -\mathrm{dist}(\omega/\nu)$.

Given $\varepsilon > 0$ there exists $c \in K$ such that $\omega(\Theta - c) > \mathrm{dist}(\omega/\nu) - \frac{\varepsilon}{2}$. Let $\overline{\Theta} = \Theta - c$. The group ωK is dense in $\mathbb{R}$, so there exists $g \in K$ such that

$$0 \le \omega(g\overline{\Theta}) < \frac{\varepsilon}{2}.$$

We have that $\sigma_1(g\overline{\Theta}) - g\overline{\Theta} = g$, so

$$\omega(\sigma_1(g\overline{\Theta}) - g\overline{\Theta}) = \omega(g) < -\mathrm{dist}(\omega/\nu) + \varepsilon.$$

Thus $G_\alpha = \{1\}$ for $\alpha > -\mathrm{dist}(\omega/\nu)$. □

As a consequence of the above corollary, we obtain the following theorem. A more general version of this theorem is proven in [28].

Theorem 7.7. *(Kuhlmann and Piltant* [27]*) Continuing with our assumption that L/K is a defect Artin-Schreier extension and ω has rank 1, let* $\mathrm{Ram}(\omega/\nu)$ *be the ramification cut of L/K defined in subsection 2.5. Then*

$$\mathrm{dist}(\omega/\nu)^- \cap \nu K = \mathrm{dist}(\Theta, K) \cap \nu K = -\mathrm{Ram}(\omega/\nu) \cap \nu K.$$

Proof. By Corollary 7.6

$$(\mathrm{Ram}(\omega/\nu) \uparrow \mathbb{R})^R = \{\alpha \in \mathbb{R} \mid G_{I_\alpha} = 1\} = \cup\{\alpha \in \mathbb{R} \mid \alpha > -\mathrm{dist}(\omega/\nu)\}.$$

Thus

$$(-\mathrm{Ram}(\omega/\nu) \uparrow \mathbb{R})^L = \{\alpha \in \mathbb{R} \mid \alpha < \mathrm{dist}(\omega/\nu)\}$$

and so $\mathrm{dist}(\Theta, K) \cap \nu K = -\mathrm{Ram}(\omega/\nu) \cap \nu K$. □

Lemma 7.8. *(Kuhlmann and Piltant,* [27]*) Suppose that K and L are two dimensional algebraic function fields over an algebraically closed field k of characteristic $p > 0$ and $K \to L$ is an Artin-Schreier extension. Let ω be a rational rank one nondiscrete valuation of L and let ν be the restriction of ω to K. Suppose that L is a defect extension of K.*

Suppose that R is a regular algebraic local ring of K and S is a regular algebraic local ring of L such that ω dominates S, S dominates R and $R \to S$ is of type 1 or 2. Inductively applying Theorems 4.1 and 4.3, we construct a diagram where the horizontal sequences are birational extensions of regular local rings

$$\begin{array}{ccccccc} S = S_0 & \to & S_1 & \to & S_2 & \to & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ R = R_0 & \to & R_1 & \to & R_2 & \to & \cdots \end{array} \tag{58}$$

with $\cup_{i=1}^{\infty} S_i = \mathcal{O}_\omega$. *Further assume that for each map* $R_i \to S_i$, *there are regular parameters* u, v *in* R_i *and* x, y *in* S_i *such that one of the following forms hold:*

$$u = x, v = f \tag{59}$$

where $\dim_k S_i/(x, f) = p$, *or*

$$u = \delta x^p, v = y \tag{60}$$

where δ *is a unit in* S_i *and in both cases that* $x = 0$ *is a local equation of the critical locus of* $Spec(S_i) \to Spec(R_i)$.

Let

$$J_i = J(S_i/R_i) = \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)$$

be the Jacobian ideal of the map $R_i \to S_i$.

Then there exists $c > 0$ *such that* $J_i = x^c S_i$ *(since the critical locus is supported on* $x = 0$*). Let* σ *be a generator of* $Gal(L/K)$. *Then*

1) $\omega(\sigma(y) - y) = \frac{c}{p-1}\omega(x) = \frac{1}{p-1}\omega(J_i)$ *if (59) holds,*
2) $\omega(\sigma(x) - x) = \frac{c}{p-1}\nu(x) = \frac{1}{p-1}\omega(J_i)$ *if (60) holds.*

Proof. We prove the first statement 1). The proof of the second statement is similar. Suppose that a form (59) holds. Let N be the S_i-ideal $N = \mathrm{Ann}_{S_i}(\Omega^1_{S_i/R_i})$.

Since ω is the unique extension of ν to L and $R_i \to S_i$ is quasi finite with complexity $p = [L : K]$, we have that S_i is the integral closure of R_i in L and is thus a finite R_i-module. There exists a unit $\delta \in S_i$ and $\gamma \in S_i$ such that

$$u = x, v = \delta y^p + x\gamma.$$

Let M be the R_i-module $M = R_i + R_i y + \cdots + R_i y^{p-1}$. We have that

$$y^p = \delta^{-1}v - u\delta^{-1}\gamma \in (u, v)S_i$$

and $x = u \in (u, v)S_i$ so $S_i = M + (u, v)S_i$. Thus $S_i = M$ by Nakayama's lemma. Let $f(t) \in K[t]$ be the minimal polynomial of y over K. The polynomial $f(t)$ has degree p since $[L : K] = p$. Since R_i is normal and y is integral over R_i, by Theorem 4 on page 260 of [38], the coefficients of $f(t)$ are in R_i, and thus $S_i \cong R_i[t]/(f(t))$. We have an isomorphism of S_i-modules

$$\Omega^1_{S_i/R_i} \cong S_i/f'(y)S_i$$

where $f'(t) = \frac{df}{dt}$. Thus $N = (f'(y))$. We compute N in another way, from the right exact sequence

$$\Omega^1_{R_i/k} \otimes_{R_i} S_i \to \Omega^1_{S_i/k} \to \Omega^1_{S_i/R_i} \to 0,$$

showing that we have a presentation

$$S_i^2 \xrightarrow{A} S_i^2 \to \Omega^1_{S_i/R_i} \to 0,$$

where

$$A = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

From the fact that

$$\left\{ \begin{pmatrix} 1 \\ \frac{\partial v}{\partial x} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

is an S_i-basis of S_i^2, we see that $\Omega^1_{S_i/R_i} \cong S_i/\frac{\partial v}{\partial y}S_i$, and so $N = (\frac{\partial v}{\partial y}) = J_i = (x^c)$. Factoring

$$f(t) = \prod_{\tau\in\mathrm{Gal}(L/K)} (t-\tau(y))$$

in $L[t]$, we see that

$$f'(y) = \prod_{id\neq\tau\in\mathrm{Gal}(L/K)} (y-\tau(y)).$$

Thus

$$c\omega(x) = \sum_{id\neq\tau\in\mathrm{Gal}(L/K)} \omega(\tau(y)-y) = (p-1)\omega(\sigma(y)-y)$$

by Corollary 7.5. ☐

Proposition 7.9. *(Kuhlmann and Piltant, [27]) Let assumptions be as in Lemma 7.8. Then the distance dist(ω/ν) is computed by the formula*

$$-\mathrm{dist}(\omega/\nu) = \frac{1}{p-1}\inf_i\{\omega(J(S_i/R_i))\}$$

where the infimum is over the $R_i \to S_i$ in the sequence (58).

Proof. Let σ be a generator of $\mathrm{Gal}(L/K)$. Since $\cup S_i = \mathcal{O}_\omega$, we have that

$$-\mathrm{dist}(\omega/\nu) = \inf\{\omega(\sigma(h)-h) \mid h \in S_i \text{ for some } i\}$$

by Corollaries 7.6 and 7.5. By the proof of Lemma 7.8, $h \in S_i$ implies there exists a polynomial $f(t) \in R_i[t]$ of degree $< p$ such that $h = f(z_i)$, where $z_i = y$ if $R_i \to S_i$ is in case (59), $z_i = x$ if $R_i \to S_i$ is in case (60). Thus we have an expression

$$h = a_0 z_i^{p-1} + a_1 z_i^{p-2} + \cdots + a_{p-1}$$

with $a_0, \ldots, a_{p-1} \in R_i$. For $s \geq 1$, we have a factorization

$$\sigma(z_i)^s - z_i^s = (\sigma(z_i)-z_i)(\sigma(z_i)^{s-1} + z_i\sigma(z_i)^{s-2} + \cdots + z_i^{s-1}).$$

The valuation ω is the unique extension of ν to L, so $\omega(\sigma(z_i)) = \omega(z_i) \geq 0$. Thus $\omega(\sigma(z_i)^s - z_i^s) \geq \omega(\sigma(z_i)-z_i)$ for all $s \geq 1$. We have that

$$\sigma(h) - h = a_0(\sigma(z_i)^{p-1} - z_i^{p-1}) + \cdots + a_{p-2}(\sigma(z_i)-z_i)$$

so $\omega(\sigma(h)-h) \geq \omega(\sigma(z_i)-z_i)$. Thus

$$-\mathrm{dist}(\omega/\nu) = \inf\{\omega(\sigma(z_i)-z_i)\}.$$

The proposition now follows from Lemma 7.8. ☐

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