ESSENTIAL FINITE GENERATION OF VALUATION RINGS IN CHARACTERISTIC ZERO ALGEBRAIC FUNCTION FIELDS

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ABSTRACT. Let K be a characteristic zero algebraic function field with a valuation ν. Let L be a finite extension of K and ω be an extension of ν to L. We establish that the valuation ring V_ω of ω is essentially finitely generated over the valuation ring V_ν of ν if and only if the initial index $\varepsilon(\omega|\nu)$ is equal to the ramification index $e(\omega|\nu)$ of the extension. This gives a positive answer, for characteristic zero algebraic function fields, to a question posed by Hagen Knaf.

1. Introduction

Suppose that K is a field and ν is a valuation of K. Let V_ν be the valuation ring of ν with maximal ideal m_ν and Γ_ν be the value group of ν. Suppose that $K \to L$ is a finite field extension and ω is an extension of ν to L. We have associated ramification and inertia indices of the extension ω over ν

$$e(\omega|\nu) = [\Gamma_\omega : \Gamma_\nu] \text{ and } f(\omega|\nu) = [V_\omega/m_\omega : V_\nu/m_\nu].$$

The defect of the extension of ω over ν is

$$d(\omega|\nu) = \frac{[L^h : K^h]}{e(\omega|\nu)f(\omega|\nu)}$$

where K^h and L^h are henselizations of the valued fields K and L. This is a positive integer (as shown in [12]) which is 1 if V_ν/m_ν has characteristic zero and is a power of p if V_ν/m_ν has positive characteristic p.

Let H be an ordered subgroup of an ordered abelian group G. The initial index $\varepsilon(G|H)$ of H in G is defined ([11, page 138]) as

$$\varepsilon(G|H) = |\{g \in G_{\geq 0} \mid g < H_{>0}\}|,$$

where

$$G_{\geq 0} = \{g \in G \mid g \geq 0\} \text{ and } H_{>0} = \{h \in H \mid h > 0\}.$$

We define the initial index $\varepsilon(\omega|\nu)$ of the extension as $\varepsilon(\Gamma_\omega|\Gamma_\nu)$.

We always have that $\varepsilon(\omega|\nu) \leq e(\omega|\nu)$ ([11, (18.3)]).

If S is a subsemigroup of an abelian semigroup T, we say that T is a finitely generated S-module if there exists a finite number of elements $g_1, \ldots, g_t \in T$ such that

$$T = \cup_{i=1}^t (g_i + S).$$

It is shown in [8, Proposition 3.3] that $\varepsilon(\omega|\nu) = e(\omega|\nu)$ if and only if $(\Gamma_\omega)_{\geq 0}$ is a finitely generated $(\Gamma_\nu)_{\geq 0}$-module. We remark that $(\Gamma_\nu)_{\geq 0}$ is the semigroup of values of elements of the valuation ring V_ν.

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Let $D(\nu, L)$ be the integral closure of V_ν in L. The localizations of $D(\nu, L)$ at its maximal ideals are the valuation rings V_{ω_i} of the extensions ω_i of ν to L. We have the following remarkable theorem.

Theorem 1.1. *([11, Theorem 18.6]) The ring $D(\nu, L)$ is a finite V_ν-module if and only if*

$$d(\omega_i|\nu) = 1 \text{ and } \varepsilon(\omega_i|\nu) = e(\omega_i|\nu)$$

for all extensions ω_i of ν to L.

An equivalent formulation is given in [1, Théorème 2, page 143].

Suppose that A is a subring of a ring B. We will say that B is essentially finitely generated over A (or that B is essentially of finite type over A) if B is a localization of a finitely generated A-algebra.

Hagen Knaf proposed the following interesting question, asking for a local form of the above theorem.

Question 1.2. *(Knaf) Suppose that ω is an extension of ν to L. Is V_ω essentially finitely generated over V_ν if and only if*

$$d(\omega|\nu) = 1 \text{ and } \varepsilon(\omega|\nu) = e(\omega|\nu)?$$

Knaf's question is related to the condition of "normalization finiteness" of algebras and schemes over Non-Noetherian valuation rings, which appears in the paper [19], where inseparable local uniformization is established.

Knaf proved the implies direction of his question; his proof is reproduced in [8, Theorem 4.1].

If $e(\omega|\nu) = 1$, $d(\omega|\nu) = 1$ and V_ω/m_ω is separable over V_ν/m_ν, then the only if direction of the question is true, as is proven in [15]. Also, the only if direction of the question is true if L/K is normal or ω is the unique extension of ν to L by [8, Corollary 2.2].

The only if direction of the question is proven when K is the quotient field of an excellent two-dimensional excellent local domain and ν dominates R in [8, Theorem 1.4]. The only if direction is proven when K is an algebraic function field over a field k, ν is an Abhyankar valuation of K and V_ω/m_ω is separable over k in [8, Theorem 1.5].

The proof of [8, Theorem 1.4] uses the existence of a resolution of excellent surface singularities ([16] or [2]) and local monomialization of defectless extensions of two dimensional excellent local domains ([3, Theorem 3.7] and [9, Theorem 7.3]). The proof of [8, Theorem 1.5] uses the local uniformization theorem for Abhyankar valuations in algebraic function fields of Knaf and Kuhlmann in [14].

In this paper, we give a positive answer to the question for characteristic zero algebraic function fields, as stated in the following theorem.

Theorem 1.3. *Let K be an algebraic function field over a field k of characteristic zero and let ν be a valuation of K/k (ν is trivial on k). Assume that L is a finite extension of K and ω is an extension of ν to L. Then V_ω is essentially finitely generated over V_ν if and only if $e(\omega|\nu) = \varepsilon(\omega|\nu)$.*

Recall that the defect $d(\omega|\nu)$ must be 1 under an extension of equicharacteristic zero valuation rings, as occurs in Theorem 1.3.

The proof of Theorem 1.3 uses an explicit form of embedded local resolution of singularities along a valuation in characteristic zero algebraic function fields, by Zariski [20] for rank 1 valuations and as extended to higher rank valuations by ElHitti in [10]. It also uses the existence of a local monomialization of regular algebraic local rings $R \to S$ of K

and L respectively which are dominated by ω as shown in [5]. Algebraic local rings are defined at the beginning of Section 2. Local monomialization is defined at the beginning of Section 5.

It is shown in the proof of Theorem 1.3, that if $e(\omega|\nu) = \varepsilon(\omega|\nu)$, then there exists a locally monomial extension $R \rightarrow S$ along ω such that if S is a localization of a finitely generated R-algebra $F[z_1, \ldots, z_n]$, then V_ω is a localization of the finitely generated V_ν-algebra $V_\nu[z_1, \ldots, z_r]$.

It is shown in [4] that local monomialization is false in positive characteristic, even in dimension two. However, local monomialization is true for defectless extensions in dimension two ([3, Theorem 3.7] and [9, Theorem 7.3]).

I thank the referee for their careful reading of the paper. I also thank the referee for pointing out that Knaf's question 1.2 and our Theorem 1.3 are related to the condition of "normalization finiteness" of algebras and schemes over Non-Noetherian valuation rings, which appears in the paper [19] establishing inseparable local uniformization.

While this article was in press, Rankeya Datta [18] gave a positive answer to Knaf's question 1.2 for general valued field extensions. His proof uses descent in the Henselization of a valued field extension, and the fact that Knaf's question has a positive answer in a Henselian field extension by Theorem 1.1.

2. Preliminaries and Notation

We will denote the non-negative integers by $\mathbb{N}$ and $\mathbb{Z}_{>0}$ will denote the positive integers. We will denote the maximal ideal of a local ring R by m_R. If R and S are local rings such that R is a subring of S and $m_S \cap R = m_R$ then we say that S dominates R. If A is a domain then $\mathrm{QF}(A)$ will denote the quotient field of A.

Suppose that A is a subring of a ring B. We will say that B is essentially finitely generated over A (or that B is essentially of finite type over A) if B is a localization of a finitely generated A-algebra.

We refer to [21] and [11] for basic facts about valuations.

Suppose that k is a field and K/k is an algebraic function field over k. An algebraic local ring of K is a local domain which is essentially of finite type over k and whose quotient field is K. A birational extension $R \rightarrow R_1$ of an algebraic local ring R of K is an algebraic local ring R_1 of K such that R_1 dominates R.

Suppose that ν is a valuation of K/k (a valuation of K which is trivial on k). Let V_ν be the valuation ring of ν, with maximal ideal m_ν. If A is a subring of V_ν. then we write $A_\nu = A_{m_\nu \cap A}$. If A is a local ring which is a subring of V_ν and $m_\nu \cap A = m_A$ then we say that ν dominates A.

Let $u = \mathrm{rank}\ \nu$ and let

$$0 = P_{\nu,u+1} \subset \cdots \subset P_{\nu,1} = m_\nu$$

be the chain of prime ideals in V_ν. Let Γ_ν be the valuation group of ν with chain of convex subgroups

$$0 = \Gamma_{\nu,0} \subset \Gamma_{\nu,1} \subset \cdots \subset \Gamma_{\nu,u} = \Gamma_\nu.$$

Let s_i be the rational rank of $\Gamma_{\nu,i}/\Gamma_{\nu,i-1}$ for $1 \leq i \leq u$. For $1 \leq i \leq u$, let ν_i be the valuation ring $V_{P_{\nu,i}}$ obtained by specialization of ν. In particular, $\nu_1 = \nu$. The value group of ν_i is $\Gamma_\nu/\Gamma_{\nu,i-1}$.

Suppose that T is an algebraic local ring of K which is dominated by ν. Define prime ideals $P_{T,i} = P_{\nu,i} \cap T$ in T for $1 \leq i \leq u$.

Lemma 2.1. *Suppose that char$(k) = 0$. Then there exists an algebraic regular local ring T of K which is dominated by ν and such that trdeg$_{\mathrm{QF}(T/P_{T,i})}\mathrm{QF}(V_\nu/P_{\nu,i}) = 0$ for $1 \le i \le u$. Further, if $T \to T_1$ is a birational extension along ν then T_1 satisfies this condition.*

Proof. Let A be an algebraic local ring of K which is dominated by ν. Let $z_{ij} \in V_\nu$ be such that $\{z_{ij} + P_{\nu,i}\}_j$ for $1 \le i \le u$ is a transcendence basis of $\mathrm{QF}(V_\nu/P_{\nu,i})$ over $\mathrm{QF}(A/P_{A,i})$. This is a finite set. Let $B = A[z_{ij}]_\nu$. Then B satisfies 1) and if $B \to R$ is a birational extension along ν then R satisfies 1).

□

Suppose that T satisfies the conclusions of Lemma 2.1. Suppose that

$$(1) \qquad x_{1,1}, \dots, x_{1,s_1}, x_{1,s_1+1}, \dots, x_{1,t_1}, x_{2,1}, \dots, x_{2,s_2}, x_{2,s_2+1}, \dots, x_{2,t_2}, x_{3,1}, \dots, x_{u,t_u}$$

are regular parameters in T. The regular parameters (1) are called *good parameters* if $x_{i,1}, \dots, x_{i,t_i} \in P_{T,i} \backslash P_{T,i+1}$ and $\nu(x_{i,1}), \dots, \nu(x_{i,s_i})$ form a rational basis of $(\Gamma_{\nu,i}/\Gamma_{\nu,i-1}) \otimes \mathbb{Q}$ for $1 \le i \le u$. If S is a subset of $\{1, \dots, u\}$ then the regular parameters (1) are called *S-good parameters* if they are good parameters and $P_{T,i} = (x_{i,1}, \dots, x_{i,t_i}, \dots)$ for $i \in S$. We will say that the parameters (1) are *very good* if they are $\{1, 2, \dots, u\}$-good. We remark that good parameters are always $\{1\}$-good.

Suppose that (1) are good parameters and

$$(2) \qquad \overline{x}_{1,1}, \dots, \overline{x}_{1,s_1}, \overline{x}_{1,s_1+1}, \dots, \overline{x}_{1,\overline{t}_1}, \overline{x}_{2,1}, \dots, \overline{x}_{2,s_2}, \overline{x}_{2,s_2+1}, \dots, \overline{x}_{2,\overline{t}_2}, \overline{x}_{3,1}, \dots, \overline{x}_{u,\overline{t}_u}$$

is another system of parameters in T. It is not required that the numbers t_i and $\overline{t}_i$ are the same. The system of regular parameters (2) is called an *S-good change of parameters* if the parameters (2) are S-good and $\overline{x}_{i,j} = x_{i,j}$ for $1 \le i \le u$ and $1 \le j \le s_i$.

3. Perron Transforms

3.1. **Perron transforms of types (1,m), (2,m) and (3,m).** The basic Perron transforms of types (1,1) and (2,1) are defined by Zariski in [20] for rank 1 valuations. They are used in [5] and [6] to prove local monomialization of morphisms. The Perron transforms of types (1,m), (2,m) and (3,m), for use in higher rank, are defined by ElHitti in [10]. The notation (1,m-1), (1,m-1,r) and (2,m-1) used in [10] is a little different from our notation.

We use the notation of Section 2 and assume that k has characteristic zero.

Suppose that T is an algebraic local ring of K which is dominated by ν and that T satisfies the conclusions of Lemma 2.1. Suppose that (1) are S-good parameters in T and $1 \le m \le u$. We define a Perron Transform $T \to T_1$ of type $(1, m)$ along ν. We first define N_j by

$$x_{m,j} = N_1^{a_{j,1}} \cdots N_{s_m}^{a_{j,s_m}} \text{ for } 1 \le j \le s_m$$

where $a_{i,j} \in \mathbb{N}$ are defined by Perron's algorithm, as explained in Sections B I and B II of [20]. We have that $\mathrm{Det}(a_{i,j}) = \pm 1$ and $\nu(N_j) > 0$ for all j.

We define $T_1 = T[N_1, \dots, N_{s_m}]_\nu$, which is a regular local ring. We define regular parameters $\{\overline{x}(1)_{i,j}\}$ in T_1 by

$$\overline{x}(1)_{i,j} = \begin{cases} N_j & \text{if } i = m \text{ and } 1 \le j \le s_m \\ x_{i,j} & \text{otherwise} \end{cases}$$

The regular parameters $\{\overline{x}(1)_{i,j}\}$ are S-good parameters in T_1.

We now define a Perron Transform $T \to T_1$ of type $(2, m)$ along ν. This is a generalization of the Perron transform constructed in Section B III of [20]. Let r be such that

$s_m < r \le t_m$. We first define N_j by

$$x_{m,j} = \begin{cases} N_1^{a_{j,1}} \cdots N_{s_m}^{a_{j,s_m}} N_r^{a_{j,s_m+1}} & \text{if } 1 \le j \le s_m \\ N_1^{a_{s_m+1,1}} \cdots N_{s_m}^{a_{s_m+1,s_m}} N_r^{a_{s_m+1,s_m+1}} & \text{if } j = r \end{cases}$$

where $a_{i,j} \in \mathbb{N}$, $\mathrm{Det}(a_{i,j}) = \pm 1$ and $\nu(N_1), \ldots, \nu(N_{s_m}) > 0$ and $\nu_m(N_r) = 0, \nu(N_r) \ge 0$.

$N_1, \ldots, N_{s_m}, N_r$ satisfying the above conditions always exists, as follows from a small variation in Zariski's algorithm in [20]. We construct the Perron transform of Zariski from $x_{m,1}, \ldots, x_{m,s_m}$ and $x_{m,r}$ for ν_m, as contructed in Section B III of [20]. In this algorithm, the next to last step constructs $M_1, \ldots, M_{s_m}, M_r$ and a $(s_m+1) \times (s_m+1)$-matrix $(b_{i,j})$ such that

$$x_{m,j} = \begin{cases} M_1^{b_{j,1}} \cdots M_{s_m}^{b_{j,s_m}} M_r^{b_{j,s_m+1}} & \text{if } 1 \le j \le s_m \\ M_1^{b_{s_m+1,1}} \cdots M_{s_m}^{b_{s_m+1,s_m}} M_r^{b_{s_m+1,s_m+1}} & \text{if } j = r \end{cases}$$

where $b_{i,j} \in \mathbb{N}$, $\mathrm{Det}(b_{i,j}) = \pm 1$, $\nu_m(M_1), \ldots, \nu_m(M_{s_m}), \nu_m(M_r) > 0$ and $\nu_m(M_r) = \nu_m(M_1) > 0$. We then have that

$$\nu_m\left(\frac{M_r}{M_1}\right) = \nu_m\left(\frac{M_1}{M_r}\right) = 0.$$

If $\nu(\frac{M_r}{M_1}) \ge 0$, define $N_1, \ldots, N_{s_m}, N_r$ by

$$M_i = \begin{cases} N_i & \text{if } i \ne r \\ N_r N_1 & \text{if } i = r \end{cases}$$

If $\nu(\frac{M_1}{M_r}) > 0$, define $N_1, \ldots, N_{s_m}, N_r$ by

$$M_i = \begin{cases} N_1 N_r & \text{if } i = 1 \\ M_i = N_i & \text{if } i \ne 1 \text{ and } i \ne r \\ N_1 & \text{if } i = r \end{cases}$$

We define $T_1 = T[N_1, \ldots, N_{s_m}, N_r]_\nu$, which is a regular local ring. Let $\mathfrak{m} = m_\nu \cap T[N_1, \ldots, N_{s_m}, N_r]$. Choose $y \in T_1$ such that y is the lift to T_1 of a generator of the maximal ideal of

$$\begin{array}{rl} & T_1/(x_{1,1}, \ldots, x_{1,t_1}, \ldots, x_{m-1,t_{m-1}}, N_1, \ldots, N_{s_m}, x_{m,s_m+1}, \ldots, x_{m,r-1}, x_{m,r+1}, \ldots) \\ \cong & (T/m_T)[N_r]_{\mathfrak{m}(T/m_T)[N_r]}. \end{array}$$

Then

$$y, x_{1,1}, \ldots, x_{1,t_1}, \ldots, x_{m-1,t_{m-1}}, N_1, \ldots, N_{s_m}, x_{m,s_m+1}, \ldots, x_{m,r-1}, x_{m,r+1}, \ldots$$

is a system of regular parameters in T_1. There is a smallest natural number λ such that $y \in P_{T_1,\lambda} \setminus P_{T_1,\lambda+1}$. We define regular parameters $\{\overline{x}(1)_{i,j}\}$ in T_1 by

$$\overline{x}(1)_{i,j} = \begin{cases} N_j & \text{if } i = m \text{ and } 1 \le j \le s_m \\ y & \text{if } i = m \text{ and } j = r \\ x_{i,j} & \text{otherwise} \end{cases}$$

if $\lambda = m$, and

$$\overline{x}(1)_{i,j} = \begin{cases} N_j & \text{if } i = m \text{ and } 1 \le j \le s_m \\ y & \text{if } i = \lambda \text{ and } j = t_\lambda + 1 \\ x_{i,j-1} & \text{if } i = m \text{ and } j \ge r+1 \\ x_{i,j} & \text{otherwise} \end{cases}$$

if $\lambda \ne m$.

If $i \in S$ and $i > m$, then $P_{T,i}T_1 = (x_{i,1}, \ldots, x_{i,2}, \ldots)T_1$ is a regular prime of T_1 which has the same height as $P_{T_1,i}$ since T_1 satisfies 1) of Lemma 2.1. Since this prime ideal is contained in $P_{T_1,i}$ we have that $P_{T_1,i} = P_{T,i}T_1$. Thus if $m+1 \in S$, then $\lambda \le m$.

We have that $\{\overline{x}(1)_{i,j}\}$ are S'-good parameters in T_1, where

$$S' = \{j \in S | j > m\}.$$

We now define a Perron transformation of type (3,m). Suppose that $d_1, \ldots, d_{s_m} \in \mathbb{N}$, $k > m$ and $1 \le l \le t_k$. Let

$$N = \frac{x_{k,l}}{x_{m,1}^{d_1} \cdots x_{m,s_m}^{d_{s_m}}}.$$

Then $\nu_k(N) > 0$. Let $T_1 = T[N]_\nu$, which is a regular local ring. Let

$$\overline{x}(1)_{i,j} = \begin{cases} N & \text{if } i = k \text{ and } j = l \\ x_{i,j} & \text{otherwise.} \end{cases}$$

Then $\{\overline{x}(1)_{i,j}\}$ are S-good parameters in T_1.

We will find the following proposition useful.

Proposition 3.1. *Suppose that R is an algebraic regular local ring of K which is dominated by ν and $\{x_{i,j}\}$ are good parameters in R. Suppose that*

$$M_1 = x_{1,1}^{a_{1,1}} \cdots x_{1,s_1}^{a_{1,s_1}} x_{2,1}^{a_{2,1}} \cdots x_{2,s_2}^{a_{2,s_2}} x_{3,1}^{a_{3,1}} \cdots x_{u,s_u}^{a_{u,s_u}}$$

and

$$M_2 = x_{1,1}^{b_{1,1}} \cdots x_{1,s_1}^{b_{1,s_1}} x_{2,1}^{b_{2,1}} \cdots x_{2,s_2}^{b_{2,s_2}} x_{3,1}^{b_{3,1}} \cdots x_{u,s_u}^{b_{u,s_u}}$$

are monomials such that $\nu(M_1) \le \nu(M_2)$. Then there exists a sequence of Perron tranforms of types (1,m) and (3,m) along ν,

$$R \to R_1 \to \cdots \to R_s,$$

such that M_1 divides M_2 in R_s.

If $\nu(M_1) = \nu(M_2)$, then $M_1 = M_2$ since the members of

$$\{\nu(x_{i,j}) | 1 \le i \le u, 1 \le j \le s_i\}$$

are rationally independent.

Proof. Suppose that $\nu(M_1) < \nu(M_2)$. There exists a largest index l such that $\prod_j x_{l,j}^{a_{l,j}} \neq \prod_j x_{l,j}^{b_{l,j}}$. Then $\nu(\prod_j x_{l,j}^{a_{l,j}}) < \nu(\prod_j x_{l,j}^{b_{l,j}})$. By [20, Theorem 2], there exists a sequence of Perron transforms of type (1,l) $R \to R_1$ along ν such that $\prod_j x_{l,j}^{a_{l,j}}$ divides $\prod_j x_{l,j}^{b_{l,j}}$ in R_1. Writing M_1 and M_2 in the regular parameters $\{z(1)_{i,j}\}$ of R_1 as

$$M_1 = \prod x(1)_{i,j}^{a(1)_{i,j}} \text{ and } M_2 = \prod x(1)_{i,j}^{b(1)_{i,j}},$$

where the product is over $1 \le i \le u$ amd $1 \le j \le s_i$, we have that

$$M_2 = \left(\prod_{i<l,j} x(1)_{i,j}^{b(1)_{i,j}}\right)\left(\prod_j x(1)_{l,j}^{b(1)_{l,j}}\right)\left(\prod_{i>l,j} x(1)_{i,j}^{a(1)_{i,j}}\right)$$

with $b(1)_{l,j} - a(1)_{l,j} \ge 0$ for all j and for some j, $b(1)_{l,j} - a(1)_{l,j} > 0$. Without loss of generality, this occurs for $j = 1$. (If $b(1)_{l,j} = a(1)_{l,j}$ for all j, then $a_{l,j} = b_{l,j}$ for all j in contradiction to our choice of l.)

Now perform a sequence of Perron transforms of type (3,n) for $1 \leq n < l$, $R_1 \to R_m$ along ν defined by $x(t)_{l,1} = x(t+1)_{l,1}x(t+1)_{i,j}$ for $i < l$ and j such that $b(t)_{i,j} < a(t)_{i,j}$ where

$$M_1 = \prod x(t)_{i,j}^{a(t)_{i,j}} \text{ and } M_2 = \prod x(t)_{i,j}^{b(t)_{i,j}}$$

to achieve that M_1 divides M_2 in R_m. □

3.2. **Sequences of good monoidal transform sequences.** We now define a *good monoidal transform sequence* along ν, which will be abbreviated as a GMTS. Suppose that T satisfies the conditions of Lemma 2.1 and $\{\overline{x}_{i,j}\}$ are good parameters in T. Let $\{x_{i,j}\}$ be a good change of parameters in T. Let $T \to T_1$ be a Perron transform of one of the types (1,m), (2,m) or (3,m) of the previous subsection, giving good parameters $\{\overline{x}(1)_{i,j}\}$ in T_1. Then we call $T \to T_1$, with the parameters $\{\overline{x}_{i,j}\}$ and good change of parameters $\{x_{i,j}\}$ in T and good parameters $\{\overline{x}(1)_{i,j}\}$ in T_1 a good monoidal transform sequence.

Suppose that $T(0)$ satisfies the conditions of Lemma 2.1 and $\{\overline{x}(0)_{i,j}\}$ are good parameters in T. A sequence of GMTSs is a sequence

$$T(0) \to T(1) \to \cdots \to T(n)$$

of GMTS. The good parameters of $T(i)$ are $\{\overline{x}(i)_{k,l}\}$ as determined by the preceding GMTS $T(i-1) \to T(i)$, and a good change of parameters $\{x(i)_{k,l}\}$.

3.3. **A refinement of the results of** [10]**.** In this paper we will make use of some results from Chapters 4 and 5 of [10]. There is a technical condition in the hypotheses of [10] which is not necessary. We will show in this subsection how this assumption (Condition A of [10, Definition 4.1]) can be eliminated. We first develop some necessary material, and then give the definition of Condition A. We then show how it is not necessary in Chapter 4 and Chapter 5 of [10]. This condition can also be eliminated from the final Chapter 6 of [10], but since we do not require results from Chapter 6, we will not address this.

Suppose that T is a normal algebraic ring. Let $P(\omega)_T$ be a prime ideal of T and ω be a rank one valuation of the quotient field of $T/P(\omega)_T$ which dominates $T/P(\omega)_T$. The valuation ω induces a "pseudo-valuation" of T, where we define $\omega(f) = \omega(\overline{f})$ if the class $\overline{f}$ of f in $T/P(\omega)_T$ is nonzero, and define $\omega(f) = \infty$ if $f \in P(\omega)_T$. We further suppose that V_ω/m_ω is an algebraic field extension of T/m_T. Let

$$Q(T) = \left\{ \begin{array}{l} \text{Cauchy sequences } \{f_n\} \text{ in } T \text{ such that for all } l \in \mathbb{Z}_{>0}, \\ \text{there exists } n_l \in \mathbb{Z}_{>0} \text{ such that } \omega(f_n) \geq l\omega(m_T) \text{ if } n \geq n_l \end{array} \right\}.$$

We have that $Q(T)$ is a prime ideal in $\hat{T}$ and $Q(T) \cap T = P(\omega)_T$. There is a unique extension of ω to a valuation of the quotient field of $\hat{T}/Q(T)$ which dominates $\hat{T}/Q(T)$. It is an immediate extension (there is no extension of the value group or the residue fields of the valuation rings). We define

$$\sigma(T) = \dim \hat{T}/Q(\hat{T}).$$

The objects $Q(T)$ and $\sigma(T)$ are defined in [6], [7] and [10]. Concepts of this type are studied in [13].

The following Lemma is proven in the case that ω is a valuation dominating T (and not just a pseudo valuation) in [6, Lemma 6.3]. The proof is essentially the same here, although a little more notation is required.

Lemma 3.2. *Let notation be as above in this subsection.*

1) *Let $\tilde{T}$ be a normal algebraic local ring such that $T \subset \tilde{T} \subset \hat{T}$ and $\hat{T}$ dominates $\tilde{T}$, so that $\tilde{T}$ is a localization of a finite étale extension of T. Then there exists a unique extension of ω to a valuation dominating $\tilde{T}/Q(T) \cap \tilde{T}$, and thus a unique extension of ω to a pseudo valuation dominating $\tilde{T}$ which has $P(\omega)_{\tilde{T}} = Q(T) \cap \tilde{T}$.*
2) *Let I be a nonzero ideal in $\tilde{T}$ such that $I \not\subset P(\omega)_{\tilde{T}}$. Let $f \in I$ be such that $\omega(f) = \omega(I)$. Let*

$$J = \cup_{j=1}^{\infty} \left(P(\omega)_{\tilde{T}} \tilde{T}[\frac{I}{f}] : I^j \tilde{T}[\frac{I}{f}] \right),$$

which is the strict transform of the ideal $P(\omega)_{\tilde{T}}$ in $\tilde{T}[\frac{I}{f}]$. Then J is a prime ideal in $\tilde{T}[\frac{I}{f}]$, the map $\tilde{T}/P(\omega)_{\tilde{T}} \rightarrow \tilde{T}[\frac{I}{f}]/J$ is birational ($\tilde{T}[\frac{I}{f}]/J$ is of finite type over $\tilde{T}/P(\omega)_{\tilde{T}}$ and both rings have the same quotient field) and there exists a maximal ideal n of $\tilde{T}[\frac{I}{f}]$ containing J such that ω dominates $(\tilde{T}[\frac{I}{f}]/J)_n$ and so ω is a pseudo valuation on $T_1 = \tilde{T}[\frac{I}{f}]_n$ with $P(\omega)_{T_1} = J_n$.
3) *Suppose that T_1 is normal. Then $\sigma(T_1) \leq \sigma(\tilde{T}) = \sigma(T)$.*

Proof. Identify ω with its unique extension to $\hat{T}/Q(T)$ which dominates the maximal ideal. Statement 1) then follows directly from restricting this extension to $\tilde{T}/Q(T) \cap \tilde{T}$.

We now consider Statement 2). Let $\overline{f}$ be the class of f in $\tilde{T}/P(\omega)_{\tilde{T}}$. Since $I \not\subset P(\omega)_{\tilde{T}}$ we have that $\overline{f} \neq 0$ and

$$\tilde{T}\left[\frac{I}{f}\right]/J = (\tilde{T}/P(\omega)_{\tilde{T}})\left[\frac{I(\tilde{T}/P(\omega)_{\tilde{T}})}{\overline{f}}\right]$$

is a birational extension of $\tilde{T}/P(\omega)_{\tilde{T}}$ and all its elements have nonnegative ω-value. Let n be the prime ideal in $\tilde{T}[\frac{I}{f}]$ of elements of positive ω-value. $\tilde{T}/m_{\tilde{T}} \subset \tilde{T}[\frac{I}{f}]/n \subset V_\omega/m_\omega$ and V_ω/m_ω is assumed to be algebraic over $\tilde{T}/m_{\tilde{T}}$. Thus $\tilde{T}[\frac{I}{f}]/n$ is finite over the field $\tilde{T}/m_T$. The domain $\tilde{T}[\frac{I}{f}]/n$ is then a field, so that n is a maximal ideal of $\tilde{T}[\frac{I}{f}]$.

We now establish statement 3). The completion of T_1 at it's maximal ideal is $\hat{T}_1 = \widehat{\hat{T}[\frac{I\hat{T}}{f}]_{\tilde{n}}}$ where $\tilde{n} = m_{\hat{T}_1} \cap \hat{T}[\frac{I\hat{T}}{f}]$. Let

$$\tilde{Q} = \cup_{j=1}^{\infty} \left(Q(T)\hat{T}\left[\frac{I\hat{T}}{f}\right]_{\tilde{n}} : I^j \hat{T}\left[\frac{I\hat{T}}{f}\right]_{\tilde{n}} \right),$$

the strict transform of $Q(T)$ in $\hat{T}[\frac{I\hat{T}}{f}]_{\tilde{n}}$. Since $I \not\subset P(\omega)_{\tilde{T}}$ we have that $\omega(f) = \infty$ if $f \in \tilde{Q}$. Thus $\tilde{Q} \subset Q(T_1)$. Now $\hat{T}/Q(\hat{T}) \rightarrow \hat{T}[\frac{I\hat{T}}{f}]_{\tilde{n}}/\tilde{Q}$ is birational and the residue field extension is finite, so by the dimension formula [17, Theorem 15.6],

$$\sigma(\hat{T}) = \dim \hat{T}[\frac{I\hat{T}}{f}]_{\tilde{n}}/\tilde{Q} = \dim \hat{T}_1/\tilde{Q}T_1$$

since completion is flat. Thus $\sigma(\tilde{T}) \geq \dim \hat{T}_1/Q(T_1) = \sigma(T_1)$. □

In [10, Chapter 4], sequences of étale Perron transforms are constructed. A sequence of étale Perron transforms along ω is a sequence

$$T \rightarrow T_1 \rightarrow \cdots \rightarrow T_n \tag{3}$$

of algebraic regular local rings where each $T_i \to T_{i+1}$ is one of the following three types, generalizing the Perron transforms defined in Subsection 3.1.

1) A transform $T_i \to T_{i+1}$ of the type of a Perron transform of type $(1,1)$.
2) First construct $T_i \to \tilde{T}_i$ as in 1) of Lemma 3.2. Then perform a transformation $\tilde{T}_i \to T_{i+1}$ of the type of a Perron transform of type (2,1).
3) A transform $T_i \to T_{i+1}$ of the type of a Perron transform of type (3,1).

In defining these transforms, we blow up an ideal generated by monomials in a regular system of parameters in $\tilde{T}$ such that not all of these monomials are in $P(\omega)_{\tilde{T}_i}$. By 2) of Lemma 3.2, the pseudo valuation ω dominates T_{i+1} and by 3) of Lemma 3.2, we have that

$$\sigma(T_{i+1}) \leq \sigma(T_i) \tag{4}$$

for all i. In [10, Chapter 4], it is assumed that "Condition A" holds (Definition [10, Definition 4.1]). This condition is assumed in [10, Chapter 4]. The condition A is that $\sigma(T_i)$ is constant in all sequences of étale Perron transforms 3. However, this assumption is not necessary to obtain the conclusions of [10, Chapters 4 and 5].

We now show how to eliminate the Condition A assumption from [10]. We run the algorithms of [10, Chapter 4] without this assumption. If at some point in the construction of a sequence of étale Perron transforms (3) in [10, Chapter 4] we have a change in $\sigma(T_i)$ then we have a decrease in this invariant by (4). We then just start the algorithm again in the T_i where there was a drop. Since $\sigma(T_i)$ is always a positive integer, we will eventually be able to complete the algorithm without a drop in $\sigma(T_j)$ along the way. [10, Theorem 4.3] which constructs a sequence of Perron transforms along ν from a sequence of étale Perron transforms along ω does not require that $\sigma(T_i)$ be constant.

The results from [10, Chapter 5] now do not require the condition A. Thus this condition is not necessary for the results from [10, Chapters 4 and 5] which we will use.

4. Embedded resolution by Perron transforms

We continue to use the notation of Section 2, and to assume that $\mathrm{char}(k) = 0$. We will use our refinement of the results of [10] given in Subsection 3.3 above, where we show that the conclusions of [10, Chapters 4 and 5] hold without assuming "Condition A".

In the following proof we use the fact that the sequences of monoidal transforms constructed in the algorithms of [10] are GMTSs. This fact follows from the proofs in these papers. We have that [10, Theorem 4.3], explaining the construction of a sequence of monodial transforms (with m=1) from a given UTS (a uniformizing transformation sequence) is a special case of the proof of Theorem 4.8 [5]. On line -6 from the bottom of page 79, in Step 3 of [5], it is explicitely stated and shown that the sequence of monoidal transforms is a GMTS.

Theorem 4.1. *Suppose that T satisfies the conditions of Lemma 2.1 and that T has a very good system of parameters $\{\overline{x}_{i,j}\}$. Suppose that $f \in T$. Then there exists a sequence of GMTSs*

$$T = T_0 \to T_1 \to \cdots \to T_m$$

along ν such that

$$f = x_{1,1}(m)^{d_{1,1}} \cdots x(m)_{1,s_1}^{d_{1,s_1}} x(m)_{2,1}^{d_{2,1}} \cdots x(m)_{2,s_2}^{d_{2,s_2}} x(m)_{3,1}^{d_{3,1}} \cdots x(m)_{u,s_u}^{d_{u,s_u}} \gamma$$

where γ is a unit in T_m and $d_{ij} \in \mathbb{N}$ for all i, j.

Proof. Let $k = \min\{i \mid \nu_i(f) < \infty\}$. By [10, Theorem 4.14] (which is applicable when ν has arbitrary rank and $\nu_1(f) < \infty$, using some Perron transforms of type (3,m)) and [10, Theorem 5.6], there exists a sequence of GMTSs along ν_k

$$R_k = T_{P_{T,k}} \rightarrow R_k(1) \rightarrow \cdots R_k(n_{k,1})$$

such that $R_k(n_{k,1})$ has a very good change of parameters

$$x_k(n_{k,1})_{1,1}, \ldots, x_k(n_{k,1})_{1,s_k}, \ldots$$

such that $f = x_k(n_{k,1})_{1,1}^{d_{k,1}} \cdots x_k(n_{k,1})_{1,s_k}^{d_{k,s_k}} \gamma_k$ where $d_{k,1}, \ldots, d_{k,s_k} \in \mathbb{N}$ and $\gamma_k \in R_k(n_{k,1})$ is a unit.

Now by [10, Lemma 5.3] (on line 11 of the statement of the lemma it should be "$(T_1)_{P^1_{T_1}} = R_1$"), [10, Remark 5.4] and [10, Lemma 5.5], there exists a sequence of GMTSs along ν_{k-1},

$$R_{k-1} = T_{P_{T,k-1}} \rightarrow R_{k-1}(1) \rightarrow \cdots \rightarrow R_{k-1}(n_{k-1,1})$$

such that $R_{k-1}(n_{k-1,1})_{P_{R_{k-1}(n_{k-1,1}),k}} = R_k(n_{k,i})$ and $R_{k-1}(n_{k-1,1})$ has a very good change of parameters

$$x_{k-1}(n_{k-1,1})_{1,1}, \ldots, x_{k-1}(n_{k-1,1})_{1,s_{k-1}}, x_{k-1}(n_{k-1,1})_{2,1}, \ldots, x_{k-1}(n_{k-1,1})_{2,s_k}, \ldots$$

such that

$$f = x_{k-1}(n_{k-1,1})_{2,1}^{d_{k,1}} \cdots x_{k-1}(n_{k-1,1})_{2,s_k}^{d_{k,s_k}} \overline{\gamma}_k$$

where $\overline{\gamma}_k \in R_{k-1}(n_{k-1,1})$ satisfies $\nu_k(\overline{\gamma}_k) = 0$.

By descending induction on k, successively applying [10, Theorem 4.14], [10, Theorem 5.6] and then [10, Lemma 5.3], [10, Remark 5.4] and [10, Lemma 5.5], we obtain the conclusions of the theorem.

□

5. Monomial extensions

Suppose that $K \rightarrow L$ is a finite extension of algebraic function fields over a field k of characteristic zero, ν is a valuation of K/k and ω is an extension of ν to L. Let

$$e = e(\omega|\nu) = [\Gamma_\omega : \Gamma_\nu], f = f(\omega|\nu) = [V_\omega/m_\omega : V_\nu/m_\nu].$$

Suppose that $R \rightarrow S$ is an extension of algebraic regular local rings of K and L respectively such that ω dominates S and S dominates R. Suppose that $\{z_i\}$ are regular parameters in R and $\{w_j\}$ are regular parameters in S. We will say that $R \rightarrow S$ is locally monomial (with respect to these systems of parameters) if there exists an $n \times n$ matrix $C = (c_{i,j})$ with coefficients in $\mathbb{N}$, where $n = \dim R = \dim S$ with $\mathrm{Det}(C) \neq 0$, and units $\alpha_i \in S$ such that

$$z_i = \prod_{j=1}^{n} w_j^{c_{i,j}} \alpha_i \text{ for } 1 \leq i \leq n. \tag{5}$$

The existence of a local monomialization in Theorem 5.1 follows from [5, Theorem 1.1]. The fact that the regular parameters can be taken to be very good parameters follows from [9, Theorem 4.8].

Theorem 5.1. *Suppose that $R^* \rightarrow S^*$ is an extension of algebraic regular local rings of K and L respectively such that ω dominates S^* and S^* dominates R^*. There exist sequences of monoidal transforms $R^* \rightarrow R$ and $S^* \rightarrow S$ along ω such that S dominates R and $R \rightarrow S$*

is locally monomial. We can further construct $R \to S$ so that the regular parameters z_i in R and w_j in S giving the monomial form are very good parameters in R and S respectively.

With the notation of the conclusion of Theorem 5.1, we have by [9, Theorem 4.2] that there exists a birational extension $R \to \overline{R}$ such that $\overline{R}$ is normal, S dominates R and S is a localization of the integral closure of $\overline{R}$ in L.

The following proposition is Theorem 6.1 [9].

Proposition 5.2. *Let $g_1, \ldots, g_f$ be a basis of V_ω/m_ω over V_ν/m_ν. Then there exist an algebraic local ring R' of K which is dominated by V_ν such that whenever $R \to S$ is an extension such that R is an algebraic regular local ring of K and S is an algebraic regular local ring of L which is dominated by ω and dominates R such that $R \to S$ is locally monomial with regular parameters satisfying (5) and R dominates R', then $[S/m_S : R/m_R] = f$, $|\mathrm{Det}(C)| = e$, $[\mathrm{QF}(\hat{S}) : \mathrm{QF}(\hat{R})] = ef$ and $g_1, \ldots, g_f$ is a basis of S/m_S over R/m_R.*

Lemma 5.3. *Suppose that R and S satisfy the conclusions of Lemma 2.1 for ν and ω respectively, $R \to S$ is a locally monomial extension and $f \in S$. Then there exists $g \in \overline{R}$ such that f divides g in S.*

Proof. Let F be a Galois closure of L over K and let $\overline{S}$ be the integral closure of $\overline{R}$ in L and $\overline{T}$ be the integral closure of $\overline{R}$ in F. There exists $\gamma \in \overline{S}$ such that γ is a unit in S and $\gamma f \in \overline{S}$. Let $\overline{f} = \gamma f$. Let $G = \mathrm{Gal}(F/K)$ and $g = \prod_{\sigma \in G} \sigma(\overline{f})$. Then $\sigma(g) = g$ for all $\sigma \in G$ so that $g \in K$. Further, $\sigma(\overline{f}) \in \overline{T}$ for all $\sigma \in G$ so g is integral over $\overline{R}$. Thus $g \in \overline{R}$ since $\overline{R}$ is normal. Let $h = \frac{g}{f}$. Then $h \in L$ and $h \in \overline{T}$ so h is integral over $\overline{S}$. Thus $h \in \overline{S}$ since $\overline{S}$ is normal, and so $\gamma h \in \overline{S} \subset S$. Thus f divides g in S. ☐

6. Analysis when $\varepsilon(\omega|\nu) = e(\omega|\nu)$

In this section, let K be an algebraic function field over a field k of characteristic zero and let ν be a valuation of K/k (ν is trivial on k). Assume that L is a finite extension of K and ω is an extension of ν to L. The ramification index $e(\omega|\nu)$ and initial index $\varepsilon(\omega|\nu)$ are defined in Section 1. Let

$$e = e(\omega|\nu) \text{ and } \varepsilon = \varepsilon(\omega|\nu).$$

The following proposition is Proposition 7.1 [8]. It holds very generally for finite extensions of valued fields $(K, \nu) \to (L, \omega)$.

Proposition 6.1. *Suppose that K is a field, ν is a valuation of K, L is a finite extension field of K and ω is an extension of ν to L such that*

$$1 < \varepsilon(\omega|\nu) = e(\omega|\nu).$$

Let $\Gamma_{\nu,1}$ be the first convex subgroup of Γ_ν and $\Gamma_{\omega,1}$ be the first convex subgroup of Γ_ω. Then $\Gamma_{\omega,1} \cong \mathbb{Z}$ and in the short exact sequence of groups

$$0 \to \Gamma_{\omega,1}/\Gamma_{\nu,1} \to \Gamma_\omega/\Gamma_\nu \to (\Gamma_\omega/\Gamma_{\omega,1})/(\Gamma_\nu/\Gamma_{\nu,1}) \to 0 \tag{6}$$

we have that

$$(\Gamma_\omega/\Gamma_{\omega,1})/(\Gamma_\nu/\Gamma_{\nu,1}) = 0$$

and

$$\Gamma_\omega/\Gamma_\nu \cong \Gamma_{\omega,1}/\Gamma_{\nu,1} \cong \mathbb{Z}_e.$$

The following proposition generalizes Proposition 7.4 of [8] from Abhyankar valuations on algebraic function fields to arbitrary valuations on characteristic zero algebraic function fields.

Proposition 6.2. *Suppose that $e(\omega|\nu)=\varepsilon(\omega|\nu)$. Then there exist algebraic regular local rings R of K and S of L which are dominated by ω and ν respectively such that S dominates R, R dominates the ring R' of Proposition 5.2 and R has good regular parameters $\{x_{i,j}\}$ and S has good regular parameters $\{y_{i,j}\}$ such that there is an expression*

$$x_{1,1}=\gamma y_{1,1}^e \text{ and } x_{i,j}=y_{i,j} \text{ if } i>1 \text{ or } j\ge 2$$

where γ is a unit in S. Further, if $e>1$, then $\nu(x_{1,1})$ is a generator of $\Gamma_{\nu,1}$ and $\omega(y_{1,1})$ is a generator of $\Gamma_{\omega,1}$. If $e=1$, then $\gamma=1$.

Remark 6.3. *We can assume that the parameters $\{x_{ij}\}$ and $\{y_{ij}\}$ are very good parameters in the conclusions of Proposition 6.2.*

Proof. By Theorem 5.1 and Proposition 5.2 there exist algebraic regular local rings R_0 of K and S_0 of L such that ω dominates S_0, S_0 dominates R_0, R_0 has very good parameters $\{x_{i,j}\}$ and S_0 has very good parameters $\{y_{i,j}\}$ satisfying the conclusions of Proposition 5.2.

We reindex the very good parameters $\{x_{i,j}\}$ and $\{y_{i,j}\}$ by

$$x_j=\begin{cases} x_{1,j} & \text{if } j\le t_1\\ x_{l,i} & \text{if } j=t_1+\cdots+t_l+i \text{ with } 1\le i\le t_{l+1}\end{cases}$$

and

$$y_j=\begin{cases} y_{1,j} & \text{if } j\le t_1\\ y_{l,i} & \text{if } j=t_1+\cdots+t_l+i \text{ with } 1\le i\le t_{l+1}.\end{cases}$$

We point out that the above parameters are a reindexing of the parameters of [9, Theorem 4.8]. The parameters of [9, Theorem 4.8] have values in monotonically decreasing convex (isolated) subgroups of Γ_ω and Γ_ν while the parameters defined above have values in monotonically increasing convex subgroups of Γ_ω and Γ_ν.

These parameters have a monomial form

$$x_j=\gamma_j y_1^{c_{j,1}}\cdots y_n^{c_{j,n}} \text{ for } 1\le j\le n \tag{7}$$

where $C=(c_{i,j})$ is an $n\times n$ matrix with $\mathrm{Det}(C)\ne 0$ and γ_j are units in S_0. By the proof of [9, Theorem 4.10] (the following statements are not effected by reindexing the regular systems of parameters $\{x_i\}$ and $\{y_j\}$), we can assume that

$$\begin{aligned}\Gamma_\omega/\Gamma_\nu &\cong (\textstyle\sum\omega(y_i)\mathbb{Z})/(\sum\nu(x_i)\mathbb{Z})\\ &\cong \mathbb{Z}^n/C^t\mathbb{Z}^n\end{aligned} \tag{8}$$

so that

$$|\mathrm{Det}(C)|=e. \tag{9}$$

First suppose that $\varepsilon(\omega|\nu)=e(\omega|\nu)>1$. Then $\Gamma_{\omega,1}\cong\mathbb{Z}$ and

$$\Gamma_{\omega,1}/\Gamma_{\nu,1}\cong\mathbb{Z}_e \tag{10}$$

by Proposition 6.1. In particular, $s_1=1$.

Recall that our choice of regular parameters is a reindexing of the one of [9, Theorem 4.8]. In equation (13) of [9], we have that M_r is the $t_1 \times t_1$ matrix

$$\begin{pmatrix} c_{1,1} & 0 \\ 0 & I_{t_1-1} \end{pmatrix}$$

so that by [9, Theorem 4.8], we may assume that

$$x_1 = \gamma y_1^{c_{1,1}}, x_2 = y_2, \ldots, x_{t_1} = y_{t_1}$$

where γ is a unit in S_0. Then from (9) and (10), we have that $c_{1,1} = e$ and $|\mathrm{Det}(\overline{C})| = 1$ where

$$\overline{C} = \begin{pmatrix} c_{t_1+1,t_1+1} & \cdots & c_{t_1+1,n} \\ \vdots & & \\ c_{n,t_1+1} & \cdots & c_{n,n} \end{pmatrix}.$$

We define a birational extension along ν, $R_0 \to R_1 = R_0[x(1)_1, \ldots, x(1)_n]_\nu$ by

$$x_j = \begin{cases} x(1)_j & \text{for } 1 \le j \le t_1 \\ x(1)_j x(1)_2^{c_{j,2}} \cdots x(1)_{t_1}^{c_{j,t_1}} & \text{for } t_1 < j \le n \end{cases}$$

to get that S_0 dominates R_1 and $R_1 \to S_0$ is locally monomial with

$$x(1)_j = \begin{cases} \gamma y_1^e & \text{if } j = 1 \\ y_j & \text{if } 1 < j \le t_1 \\ y_1^{c_{j,1}} y_{t_1+1}^{c_{j,t_1+1}} \cdots y_n^{c_{j,n}} \alpha_j & \text{if } t_1 + 1 \le j \le n \end{cases}$$

where $\alpha_j \in S_0$ are units.

Since $\mathrm{Det}(\overline{C}) = \pm 1$, there exist $r_{t_1+1}, \ldots, r_n \in \mathbb{Z}$ such that

$$\overline{C} \begin{pmatrix} r_{t_1+1} \\ \vdots \\ r_n \end{pmatrix} = - \begin{pmatrix} c_{t_1+1,1} \\ \vdots \\ c_{n,1} \end{pmatrix}.$$

Define $d_{t_1+1}, \ldots, d_n \in \mathbb{N}$ by

$$\begin{pmatrix} d_{t_1+1} \\ \vdots \\ d_n \end{pmatrix} = \overline{C} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

There exists $v \in \mathbb{Z}_{>0}$ such that $r_i + ve > 0$ for all i. Perform the sequence of GMTSs of type (3,1) $S_0 \to S_1$ along ω where S_1 has good parameters $\{y(1)_i\}$ defined by

$$y_i = \begin{cases} y(1)_i & \text{if } 1 \le i \le t_1 \\ y(1)_i y(1)_1^{r_i + ve} & \text{if } t_1 < i \le n. \end{cases}$$

We have that $S_1 = S_0[y_1(1), \ldots, y_n(1)]_\omega$ dominates R_1 and $R_1 \to S_1$ is locally monomial. There exist units $\gamma_i' \in S_1$ such that

$$(11) \qquad x(1)_i = \begin{cases} \gamma y(1)_1^e & \text{if } i = 1 \\ y(1)_i & \text{if } 1 < i \le t_1 \\ \gamma_i' y(1)_1^{evd_i} y(1)_{t_1+1}^{c_{i,t_1+1}} \cdots y(1)_n^{c_{i,n}} & \text{if } t_1 < i \le n. \end{cases}$$

Now perform the sequence of GMTSs $R_1 \to R_2$ of type (3,1) along ν defined by

$$x(1)_i = \begin{cases} x(2)_1 & \text{if } 1 \le i \le t_1 \\ x(2)_1^{vd_i} x(2)_i & \text{if } t_1 < i \le n. \end{cases}$$

Then S_1 dominates R_2 and there exist units $\gamma(1)_i \in S_1$ such that

$$x(2)_i = \begin{cases} \gamma y(1)_1^e & \text{if } i=1 \\ y(1)_i & \text{if } 1 < i \le t_1 \\ \gamma(1)_i y(1)_{t_1+1}^{c_{i,t_1+1}} \cdots y(1)_n^{c_{i,n}} & \text{if } t_1 < i \le n. \end{cases} \tag{12}$$

Let $B = \overline{C}^{-1}$. Write

$$B = \begin{pmatrix} b_{t_1+1,t_1+1} & \cdots & b_{t_1+1,n} \\ & \vdots & \\ b_{n,t_1+1} & \cdots & b_{n,n} \end{pmatrix}$$

with $b_{i,j} \in \mathbb{Z}$. We now replace the $y(1)_i$ with the product of the unit $\gamma(1)_{t_1+1}^{-b_{i,t_1+1}} \cdots \gamma(1)_n^{-b_{i,n}}$ and $y(1)_i$ for $t_1+1 \le i \le n$ to get $\gamma_i(1) = 1$ for $t_1+1 \le i \le n$ in (12).

Now define a birational transformation $R_2 \to R_3$ along ν by $R_3 = R_2[x(3)_{t_1+1}, \ldots, x(3)_n]_\nu$ where R_3 has regular parameters $\{x(3)_i\}$ defined by

$$x(2)_i = \begin{cases} x(3)_i & \text{if } 1 \le i \le t_1 \\ x(3)_{t_1+1}^{c_{i,t_1+1}} \cdots x(3)_n^{c_{i,n}} & \text{for } t_1+1 \le i \le n. \end{cases}$$

The ring R_3 is a regular local ring with regular parameters $x(3)_1, \ldots, x(3)_n$. We have that S_1 dominates R_3 and

$$x(3)_i = \begin{cases} \gamma y(1)_1^e & \text{if } i = 1 \\ y(1)_i & \text{if } 2 \le i \le n \end{cases}$$

where γ is a unit in S_1. Going back to (8), we see that $\omega(y(1)_1)$ is a generator of $\Gamma_{\omega,1}$ and $\nu(x(3)_1)$ is a generator of $\Gamma_{\nu,1}$. We thus have the conclusions of the proposition.

Now suppose that $e = 1$. This case is much simpler. In (9) we then have that $\mathrm{Det}(C) = \pm 1$. Taking $B = C^{-1} = (b_{i,j})$, we can then make the change of variables in S_0 replacing the y_i with the product of the unit $\gamma_1^{-b_{i,1}} \cdots \gamma_n^{-b_{i,n}}$ times y_i for $1 \le i \le n$ to get $\gamma_i = 1$ for $1 \le i \le n$ in (7).

Now define a birational transformation $R_0 \to R_1$ along ν by $R_1 = R_0[x(1)_1, \ldots, x(1)_n]_\nu$ where

$$x_i = x(1)_1^{c_{i,1}} \cdots x(1)_n^{c_{i,n}} \text{ for } 1 \le i \le n.$$

The ring R_1 is a regular local ring with regular parameters $x(1)_1, \ldots, x(1)_n$. We have that R_1 is dominated by S, and

$$x(1)_i = y(1)_i \text{ for } 1 \le i \le n,$$

giving the conclusions of the proposition. □

Proposition 6.4. *Suppose that $e(\omega|\nu) = \varepsilon(\omega|\nu)$ and $R_0 \to S_0$ has the form of the conclusions of Proposition 6.2 for good parameters $\{\overline{x}(0)_{i,j}\}$ in R_0 and good parameters $\{\overline{y}(0)_{i,j}\}$ in S_0. Then there exist $z_1, \ldots, z_m \in V_\omega$ such that $S_0 = R_0[z_1, \ldots, z_m]_\omega$.*

Let $R_0 \to R_1$ be a GMTS along ν, constructed from a good change of parameters $\{x(0)_{i,j}\}$ in R_0, and giving good parameters $\{\overline{x}(1)_{i,j}\}$ in R_1.

We then have a good change of parameters $\{y(0)_{i,j}\}$ in S_0 defined by

$$y(0)_{i,j} = \begin{cases} \overline{y}(0)_{i,j} & \text{if } i = 1 \text{ and } j = 1 \\ x(0)_{i,j} & \text{otherwise.} \end{cases}$$

The good parameters $\{x(0)_{i,j}\}$ and $\{y(0)_{i,j}\}$ continue to have the form of the conclusions of Proposition 6.2.

There exists a GMTS $S_0 \to S_1$ along ω, constructed from the above good parameters $\{y(0)_{i,j}\}$ in S_0, and giving good parameters $\{\overline{y}(1)_{i,j}\}$ in S_1, such that there is a good change *of parameters $\{y'_{i,j}\}$ in S_1 such that $\{x(1)_{i,j}\}$ and $\{y'_{i,j}\}$ are related by an expression of the form of the conclusions of Proposition 6.2 and we have that $S_1 = R_1[z_1,\ldots,z_m]_\omega$.*

Proof. The expression $S_0 = R_0[z_1,\ldots,z_m]_\omega$ follows since S_0 is essentially of finite type over k.

Suppose that the GMTS $R_0 \to R_1$ is of type (1,m). Then $R_1 = R[N_1,\ldots,N_{s_m}]_\nu$ where

$$x(0)_{m,j} = N_1^{a_{j,1}} \cdots N_{s_m}^{a_{j,s_m}} \text{ for } 1 \le j \le s_m$$

and the good regular parameters $\overline{x}(1)_{i,j}$ in R_1 are defined by

$$\overline{x}(1)_{i,j} = \begin{cases} N_j & \text{if } i = m \text{ and } 1 \le j \le s_m \\ x(0)_{i,j} & \text{otherwise} \end{cases}$$

If $e > 1$ then $\Gamma_{\nu,1}$ has rational rank 1 ($s_1 = 1$), and we then cannot perform a GMTS of type (1,1). In particular, we have $e = 1$ if $m = 1$.

We define the Perron transform $S_0 \to S_1[N_1,\ldots,N_{s_m}]_\omega$ of type (1,m), giving good regular parameters $\overline{y}(1)_{i,j}$ such that

$$\overline{y}(1)_{i,j} = \begin{cases} N_j = \overline{x}(1)_{i,j} & \text{if } i = m \text{ and } 1 \le j \le s_m \\ y(1)_{i,j} & \text{otherwise.} \end{cases}$$

We then have that the good parameters $\{\overline{x}(1)_{i,j}\}$ and $\{\overline{y}(1)_{i,j}\}$ are related by an expression of the form of Proposition 6.2 and we have that $S_1 = R_1[z_1,\ldots,z_m]_\omega$.

Suppose that the GMTS $R_0 \to R_1$ is of type (2,m). Then $R_1 = R_0[N_1,\ldots N_{s_m},N_r]_\nu$ where $s_m < r \le t_m$,

$$x(0)_{m,j} = \begin{cases} N_1^{a_{j,1}} \cdots N_{s_m}^{a_{j,s_m}} N_r^{a_{j,s_m+1}} & \text{if } 1 \le j \le s_m \\ N_1^{a_{s_m+1,1}} \cdots N_{s_m}^{a_{s_m+1,s_m}} N_r^{a_{s_m+1,s_m+1}} & \text{if } j = r \end{cases}$$

where $a_{i,j} \in \mathbb{N}$, $\mathrm{Det}(a_{i,j}) = \pm 1$ and $\nu(N_1),\ldots,\nu(N_{s_m}) > 0$ and $\nu_m(N_r) = 0, \nu(N_r) \ge 0$.

Let $\mathfrak{m} = m_\nu \cap R_0[N_1,\ldots,N_{s_m},N_r]$. Choose $y \in R_1$ such that y is the lift to R_1 of a generator of the maximal ideal of

(13)
$$\begin{array}{cl} & R_1/(x(0)_{1,1},\ldots,x(0)_{m-1,t_{m-1}},N_1,\ldots,N_{s_m},x(0)_{m,s_m+1},\ldots,x(0)_{m,r-1},x(0)_{m,r+1},\ldots) \\ \cong & (R_0/m_{R_0})[N_r]_{\mathfrak{m}(R_0/m_{R_0})[N_r]}. \end{array}$$

Let λ be the smallest natural number such that $y \in P_{R_1,\lambda} \setminus P_{R_1,\lambda+1}$. Then the regular parameters $\{\overline{x}(1)_{i,j}\}$ in R_1 are defined by

$$\overline{x}(1)_{i,j} = \begin{cases} N_j & \text{if } i = m \text{ and } 1 \le j \le s_m \\ y & \text{if } i = m \text{ and } j = r \\ x(0)_{i,j} & \text{otherwise} \end{cases}$$

if $\lambda = m$, and

$$\overline{x}(1)_{i,j} = \begin{cases} N_j & \text{if } i = m \text{ and } 1 \le j \le s_m \\ y & \text{if } i = \lambda \text{ and } j = t_\lambda + 1 \\ x(0)_{i,j-1} & \text{if } i = m \text{ and } j \ge r+1 \\ x(0)_{i,j} & \text{otherwise} \end{cases}$$

if $\lambda \ne m$.

First suppose that $m > 1$ or $m = e = 1$.

We define the Perron transform $S_0 \to S_1 = S_0[N_1, \ldots, N_{s_m}, N_r]_\omega$ of type (2,m). Let $\mathfrak{n} = m_\omega \cap S_0[N_1, \ldots, N_{s_m}, N_r]$. We have

(14)

$$\begin{array}{ll} & S_1/(y(0)_{1,1}, \ldots, y(0)_{m-1,t_{m-1}}, N_1, \ldots, N_{s_m}, y(0)_{m,s_m+1}, \ldots, y(0)_{m,r-1}, y(0)_{m,r+1}, \ldots) \\ \cong & (S_0/m_{S_0})[N_r]_{\mathfrak{n}(S_0/m_{S_0})[N_r]}. \end{array}$$

By (13) and (14), the dominant homomorphism $R_1 \to S_1$, induces a dominant homomorphism

$$(R_0/m_{R_0})[N_r]_{\mathfrak{m}(R_0/m_{R_0})[N_r]} \to (S_0/m_{S_0})[N_r]_{\mathfrak{n}(S_0/m_{S_0})[N_r]}.$$

Suppose that y is the lift of $\overline{y} \in (R_0/m_{R_0})[N_r]_{\mathfrak{m}(R_0/m_{R_0})[N_r]}$. We can assume that $\overline{y} \in (R_0/m_{R_0})[N_r]$ is irreducible. Then $\overline{y}$ is a separable polynomial in the polynomial ring $(R_0/m_{R_0})[N_r]$ since R_0/m_{R_0} has characteristic zero. Thus $\overline{y} \in \mathfrak{n}(S_0/m_{S_0})[N_r] \subset (S_0/m_{S_0})[N_r]$ is separable and hence $\overline{y}$ is a generator of the maximal ideal of

$$(S_0/m_{S_0})[N_r]_{\mathfrak{n}(S_0/m_{S_0})[N_r]}.$$

We may thus define our good parameters $\overline{y}(1)_{i,j}$ in S_1 by

$$\overline{y}(1)_{i,j} = \begin{cases} N_j & \text{if } i = m \text{ and } 1 \le j \le s_m \\ y & \text{if } i = m \text{ and } j = r \\ y(0)_{i,j} & \text{otherwise} \end{cases}$$

if $\lambda = m$, and

$$\overline{y}(1)_{i,j} = \begin{cases} N_j & \text{if } i = m \text{ and } 1 \le j \le s_m \\ y & \text{if } i = \lambda \text{ and } j = t_\lambda + 1 \\ y(0)_{i,j-1} & \text{if } i = m \text{ and } j \ge r+1 \\ y(0)_{i,j} & \text{otherwise} \end{cases}$$

if $\lambda \ne m$.

We have that the good parameters $\{\overline{x}(1)_{i,j}\}$ in R_1 and $\{\overline{y}(1)_{i,j}\}$ in S_1 are related by an expression of the form of Proposition 6.2 and we have that $S_1 = R_1[z_1, \ldots, z_m]_\omega$.

Now suppose that $m = 1$ and $e > 1$. Then $x(0)_{1,1} = \gamma y(0)_{1,1}^e$ and $x(0)_{1,r} = y(0)_{1,1}$. We then have that $s_1 = 1$, $\nu(x(0)_{1,1}$ is a generator of $\Gamma_{\nu,1} \cong \mathbb{Z}$ and $\nu(y(0)_{1,r})$ is a generator of $\Gamma_{\omega,1} \cong \mathbb{Z}$. Thus there exists $a \in \mathbb{Z}_+$ such that $\nu(x(0)_{1,r}) = a\nu(x(0)_{1,1})$ and so the equations defining $R_0 \to R_1$ are $x(0)_{1,1} = N_1$ and $x(0)_{1,r} = N_1^a N_r$. We have that $\omega(y(0)_{1,r}) = ea\omega(y(0)_{1,1})$. Define a GMTS along ω, $S_0 \to S_1$, by $S_1 = S_0[M_1, M_r]_\omega$ where $y(0)_{1,1} = M_1, y(0)_{1,r} = M_1^{ea} M_r$. We have that $M_r = \gamma^a N_r$ so $S_1 = R_1[z_1, \ldots, z_m]_\omega$. As in the case $m > 1$ or $m = e = 1$, we may define our good parameters $\{\overline{y}(1)_{i,j}\}$ in S_1 so that the good parameters $\{\overline{x}(1)_{i,j}\}$ in R_1 and the good parameters $\{\overline{y}(1)_{i,j}\}$ in S_1 are related by an expression of the form of Proposition 6.2.

Suppose that the MTS $R_0 \to R_1$ is of the type (3,m). Then $R_1 = R[N]_\nu$ where

$$N = \frac{x(0)_{k,l}}{x(0)_{m,1}^{d_1} \cdots x(0)_{m,s_1}^{d_{s_m}}}.$$

for some $d_1, \ldots, d_{s_m} \in \mathbb{N}$, $k > m$ and $1 \le l \le t_k$. The good parameters $\{\overline{x}(1)_{1,1}\}$ in R_1 are defined by

$$\overline{x}(1)_{i,j} = \begin{cases} N & \text{if } i = k \text{ and } j = l \\ x(0)_{i,j} & \text{otherwise.} \end{cases}$$

If $m>1$ or $e=m=1$, then

$$N=\frac{y(0)_{k,l}}{y(0)_{m,1}^{d_1}\cdots y(0)_{m,s_m}^{d_{s_m}}}$$

and if $m=1$ and $e>1$, then

$$N=\gamma^{-d_1}\frac{y(0)_{k,l}}{y(0)_{1,1}^{ed_1}}.$$

We may thus define a GMTS along ω, $S_0\to S_1$, of type (3,m) by $S_1=S_0[N]_\omega$.

The good parameters $\{\overline{y}(1)_{i,j}\}$ in S_1 defined by the GMTS are such that after making a good change of parameters, replacing $\overline{y}(1)_{k,l}$ with $x(1)_{k,l}$, the good parameters $\{\overline{x}(1)_{i,j}\}$ in R_1 and the good parameters $\{\overline{y}(1)_{ij}\}$ in S_1 are related by an expression of the form of Proposition 6.2. We have that $S_1=R_1[z_1,\ldots,z_m]_\omega$. ☐

We now prove Theorem 1.3 from Section 1, which we restate in Theorem 6.5.

Theorem 6.5. *Let K be an algebraic function field over a field k of characteristic zero and let ν be a valuation of K/k (ν is trivial on k). Assume that L is a finite extension of K and ω is an extension of ν to L. Then V_ω is essentially finitely generated over V_ν if and only if $e(\omega|\nu)=\varepsilon(\omega|\nu)$.*

Proof. If V_ω is essentially finitely generated over V_ν then $e(\omega|\nu)=\varepsilon(\omega|\nu)$ by Theorem 4.1 [8].

Suppose that $e(\omega|\nu)=\varepsilon(\omega|\nu)$. We will show that V_ω is essentially finitely generated over V_ν. Let $R_0\to S_0$ be such that R_0 satisfies the conclusions of Lemma 2.1 and the conclusions of Proposition 6.2 with respect to very good parameters in R_0 and very good parameters in S_0. Write $S_0=R_0[z_1,\ldots,z_m]_\omega$. We will show that $V_\omega=V_\nu[z_1,\ldots,z_n]_\omega$.

Suppose that $f\in V_\omega$. Write $f=\frac{g}{h}$ with $g,h\in S_0$. By Lemma 5.3, and since with the conclusions of Proposition 6.2 S_0 is a localization of the integral closure of R_0 in L, there exists $c\in R_0=\overline{R_0}$ such that gh divides c in S_0. By Theorem 4.1, there exists a sequence of GMTSs $R_0\to R_1\to\cdots\to R_m$ such that

$$c=x(m)_{1,1}^{d_{1,1}}\cdots x(m)_{1,s_1}^{d_{1,s_1}}x(m)_{2,1}^{d_{2,1}}\cdots x(m)_{2,s_2}^{d_{2,s_2}}x(m)_{3,1}^{d_{3,1}}\cdots x(m)_{u,s_u}^{d_{u,s_u}}\gamma$$

where γ is a unit in R_m and $d_{ij}\in\mathbb{N}$ for all i,j.

By Proposition 6.4, there exists a sequence of GMTSs $S_0\to S_1\to\cdots\to S_m$ such that $S_m=R_m[z_1,\ldots,z_n]_\omega$ and there are good parameters $\{y_{i,j}(m)\}$ in S_m such that $x(m)_{i,j}=y(m)_{i,j}$ if $1<i\le u$ and $1\le j\le s_i$ or if $i=1$ and $j>1$. We further have that

$$x(m)_{1,1}=\begin{cases}\tau y(m)_{1,1}^{e}\text{ where }\tau\in S_m\text{ is a unit} & \text{if } e>1\\ y(m)_{1,1} & \text{if } e=1.\end{cases}$$

Thus in S_m, c has an expression

$$c=y(m)_{1,1}^{ed_{1,1}}y(m)_{1,2}^{d_{1,2}}\cdots y(m)_{1,s_1}^{d_{1,s_1}}y(m)_{2,1}^{d_{2,1}}\cdots y(m)_{2,s_2}^{d_{2,s_2}}x(m)_{3,1}^{d_{3,1}}\cdots y(m)_{u,s_u}^{d_{u,s_u}}\gamma'$$

where γ' is a unit in S_m ($\gamma'=\gamma$ if $e=1$).

Since gh divides c in S_m, we have expressions

$$g=y(m)_{1,1}^{a_{1,1}}\cdots y(m)_{1,s_1}^{a_{1,s_1}}y(m)_{2,1}^{a_{2,1}}\cdots y(m)_{2,s_2}^{a_{2,s_2}}x(m)_{3,1}^{a_{3,1}}\cdots y(m)_{u,s_u}^{a_{u,s_u}}\alpha$$

where α is a unit in S_m and $a_{ij}\in\mathbb{N}$ for all i,j, and

$$h=y(m)_{1,1}^{b_{1,1}}\cdots y(m)_{1,s_1}^{b_{1,s_1}}y(m)_{2,1}^{b_{2,1}}\cdots y(m)_{2,s_2}^{b_{2,s_2}}x(m)_{3,1}^{b_{3,1}}\cdots y(m)_{u,s_u}^{b_{u,s_u}}\beta$$

where β is a unit in S_m and $b_{ij} \in \mathbb{N}$ for all i,j. Let

$$W_1 = x(m)_{1,1}^{a_{1,1}} x(m)_{1,2}^{ea_{1,2}} \cdots x(m)_{1,s_1}^{ea_{1,s_1}} x(m)_{2,1}^{ea_{2,1}} \cdots x(m)_{2,s_2}^{ea_{2,s_2}} x(m)_{3,1}^{ea_{3,1}} \cdots x(m)_{u,s_u}^{ea_{u,s_u}}$$

and

$$W_2 = x(m)_{1,1}^{b_{1,1}} x(m)_{1,2}^{eb_{1,2}} \cdots x(m)_{1,s_1}^{eb_{1,s_1}} x(m)_{2,1}^{eb_{2,1}} \cdots x(m)_{2,s_2}^{eb_{2,s_2}} x(m)_{3,1}^{eb_{3,1}} \cdots x(m)_{u,s_u}^{eb_{u,s_u}}.$$

We have that $\nu(W_1) = e\omega(g) \geq e\omega(h) = \nu(W_2)$.

By Proposition 3.1, there exists a sequence of GMTSs

$$R_m \to R_{m+1} \to \cdots \to R_v$$

of types (1,m) and (3,m) such that W_2 divides W_1 in R_v.

By Proposition 6.4, there exists a sequence of GMTSs $S_m \to S_{m+1} \to \cdots \to S_v$ such that $S_v = R_v[z_1,\ldots,z_n]_\omega$ and there exists a good change of parameters $\{y(v)_{i,j}\}$ in S_v such that the good parameters $\{x(v)_{i,j}\}$ of R_v have the good form of the conclusions of Proposition 6.4. Thus W_2 divides W_1 in S_v. Now $W_1 = g^e\alpha^{-e}\tau^{a_{1,1}}$ and $W_2 = h^e\beta^{-e}\tau^{b_{1,1}}$. Thus h^e divides g^e in S_v. Now g and h are monomials in the good parameters of S_v times units. Thus h divides g in S_v and so $f = \frac{g}{h} \in S_v$. Thus $f \in V_\nu[z_1,\ldots,z_m]_\omega$. Since this is true for all $f \in V_\omega$, we have that $V_\omega = V_\nu[z_1,\ldots,z_m]_\omega$ is essentially finitely generated over V_ν.

□

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