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Resolvent and Proximal Compositions

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Abstract
We introduce the resolvent composition, a monotonicity-preserving operation between a linear operator and a set-valued operator, as well as the proximal composition, a convexity-preserving operation between a linear operator and a function. The two operations are linked by the fact that, under mild assumptions, the subdifferential of the proximal composition of a convex function is the resolvent composition of its subdifferential. The resolvent and proximal compositions are shown to encapsulate known concepts, such as the resolvent and proximal averages, as well as new operations pertinent to the analysis of equilibrium problems. A large core of properties of these compositions is established and several instantiations are discussed. Applications to the relaxation of monotone inclusion and convex optimization problems are presented.

Keywords Monotone operator · Proximal average · Proximal composition · Proximal point algorithm · Relaxed monotone inclusion · Resolvent average · Resolvent composition · Resolvent mixture.

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1 Introduction

Throughout, $\mathcal{H}$ and $\mathcal{G}$ are real Hilbert spaces, $2^{\mathcal{H}}$ is the power set of $\mathcal{H}$, $\mathrm{Id}_{\mathcal{H}}$ is the identity operator of $\mathcal{H}$, and $\mathcal{B}(\mathcal{H},\mathcal{G})$ is the space of bounded linear operators from $\mathcal{H}$ to $\mathcal{G}$. Let $B\colon\mathcal{G}\to 2^{\mathcal{G}}$ be a set-valued operator and denote by J_B its resolvent, that is,

$$J_B=(B+\mathrm{Id}_{\mathcal{G}})^{-1}. \tag{1.1}$$

The resolvent operator is a central tool in nonlinear analysis [2, 9, 14, 28, 51], largely owing to the fact that its set of fixed points $\{y\in\mathcal{G}\,|\,y\in J_By\}$ coincides with the set of zeros $\{y\in\mathcal{G}\,|\,0\in By\}$ of B, which models equilibria in many fields; see for instance

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[3, 15, 18, 21, 29, 30, 33, 34, 40, 52, 57]. A standard operation between B and a linear operator $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ that induces an operator from $\mathcal{H}$ to $2^{\mathcal{H}}$ is the composition

$$L^* \circ B \circ L. \tag{1.2}$$

Early manifestations of this construct can be found in [17, 50]. A somewhat dual operation is the parallel composition $L^* \triangleright B \colon \mathcal{H} \to 2^{\mathcal{H}}$ defined by [9, 13] (see [16, 28] for further applications)

$$L^* \triangleright B = \big(L^* \circ B^{-1} \circ L\big)^{-1}. \tag{1.3}$$

The objective of the present article is to investigate alternative compositions, which we call the *resolvent composition* and the *resolvent cocomposition*.

Definition 1.1 *Let $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ and $B \colon \mathcal{G} \to 2^{\mathcal{G}}$. The resolvent composition of B with L is the operator $L \diamond B \colon \mathcal{H} \to 2^{\mathcal{H}}$ given by*

$$L \diamond B = L^* \triangleright (B + \mathrm{Id}_{\mathcal{G}}) - \mathrm{Id}_{\mathcal{H}} \tag{1.4}$$

and the resolvent cocomposition of B with L is $L \blacklozenge B = (L \diamond B^{-1})^{-1}$.

The terminology in Definition 1.1 stems from the following composition rule, which results from (1.1), (1.4), and (1.3).

Proposition 1.2 *Let $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ and $B \colon \mathcal{G} \to 2^{\mathcal{G}}$. Then $J_{L \diamond B} = L^* \circ J_B \circ L$.*

The resolvent composition will be shown to encapsulate known concepts as well as new operations pertinent to the analysis of equilibrium problems. As an illustration, we recover below the resolvent average.

Example 1.3 (**resolvent average**) Let $0 \neq p \in \mathbb{N}$ and, for every $k \in \{1, \ldots, p\}$, let $B_k \colon \mathcal{H} \to 2^{\mathcal{H}}$ and $\omega_k \in \,]0, +\infty[$. Additionally, let $\mathcal{G}$ be the standard product vector space $\mathcal{H}^p$, with generic element $\boldsymbol{y} = (y_k)_{1 \leqslant k \leqslant p}$, equipped with the scalar product $(\boldsymbol{y}, \boldsymbol{y}') \mapsto \sum_{k=1}^{p} \omega_k \langle y_k \mid y_k' \rangle$, and set $L \colon \mathcal{H} \to \mathcal{G} \colon x \mapsto (x, \ldots, x)$ and $B \colon \mathcal{G} \to 2^{\mathcal{G}} \colon \boldsymbol{y} \mapsto B_1 y_1 \times \cdots \times B_p y_p$. Then $L^* \colon \mathcal{G} \to \mathcal{H} \colon \boldsymbol{y} \mapsto \sum_{k=1}^{p} \omega_k y_k$ and we derive from (1.4) that

$$L \diamond B = \left(\sum_{k=1}^{p} \omega_k \big(B_k + \mathrm{Id}_{\mathcal{H}}\big)^{-1}\right)^{-1} - \mathrm{Id}_{\mathcal{H}} = \left(\sum_{k=1}^{p} \omega_k J_{B_k}\right)^{-1} - \mathrm{Id}_{\mathcal{H}}. \tag{1.5}$$

In particular, if $\sum_{k=1}^{p} \omega_k = 1$, then (1.5) is the *resolvent average* of the operators $(B_k)_{1 \leqslant k \leqslant p}$. This operation is studied in [4, 12], while $\sum_{k=1}^{p} \omega_k J_{B_k} = J_{L \diamond B}$ shows up in common zero problems [24, 37].

Given $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ and a proper convex function $g \colon \mathcal{G} \to]-\infty, +\infty]$ with subdifferential ∂g, a question we shall address is whether the resolvent composition $L \diamond \partial g$ is itself a subdifferential operator and, if so, of which function. Answering this question will lead to the introduction of the following operations, where $\square$ denotes infimal convolution and where $\mathscr{Q}_{\mathcal{H}} = \|\cdot\|_{\mathcal{H}}^2/2$ and $\mathscr{Q}_{\mathcal{G}} = \|\cdot\|_{\mathcal{G}}^2/2$ are the canonical quadratic forms of $\mathcal{H}$ and $\mathcal{G}$, respectively (see Section 2 for notation).

Definition 1.4 *Let $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ and let $g \colon \mathcal{G} \to]-\infty, +\infty]$ be proper. The proximal composition of g with L is the function $L \diamond g \colon \mathcal{H} \to]-\infty, +\infty]$ given by*

$$L \diamond g = \big((g^* \,\square\, \mathscr{Q}_{\mathcal{G}}) \circ L\big)^* - \mathscr{Q}_{\mathcal{H}} \tag{1.6}$$

and the proximal cocomposition of g with L is $L \blacklozenge g = (L \diamond g^*)^*$.

In connection with the above question, if $\|L\| \leqslant 1$ and if g is lower semicontinuous and convex in Definition 1.4, the proximal composition will be shown to be linked to the resolvent composition through the subdifferential identity

$$\partial(L \diamond g) = L \diamond \partial g, \tag{1.7}$$

and its proximity operator to be decomposable as $\mathrm{prox}_{L \diamond g} = L^* \circ \mathrm{prox}_g \circ L$, which explains the terminology in Definition 1.4. Furthermore, we shall see that the proximal composition captures notions such as the proximal average of convex functions.

We provide notation and preliminary results in Section 2. Examples of resolvent compositions are presented in Section 3. In Section 4, various properties of the resolvent composition are investigated. Section 5 is devoted to the proximal composition and its properties. Applications to monotone inclusion and variational problems are discussed in Section 6.

2 Notation and Preliminary Results

We refer to [9] for a detailed account of the following elements of convex and nonlinear analysis. In addition to the notation introduced in Section 1, we designate the direct Hilbert sum of $\mathcal{H}$ and $\mathcal{G}$ by $\mathcal{H} \oplus \mathcal{G}$. The scalar product of a Hilbert space is denoted by $\langle \cdot \mid \cdot \rangle$ and the associated norm by $\| \cdot \|$.

Let $A : \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued operator. We denote by $\mathrm{gra}\, A = \{(x, x^*) \in \mathcal{H} \times \mathcal{H} \mid x^* \in Ax\}$ the graph of A, by $\mathrm{dom}\, A = \{x \in \mathcal{H} \mid Ax \neq \varnothing\}$ the domain of A, by $\mathrm{ran}\, A = \{x^* \in \mathcal{H} \mid (\exists x \in \mathcal{H})\ x^* \in Ax\}$ the range of A, by $\mathrm{zer}\, A = \{x \in \mathcal{H} \mid 0 \in Ax\}$ the set of zeros of A, by $\mathrm{Fix}\, A = \{x \in \mathcal{H} \mid x \in Ax\}$ the set of fixed points of A, and by A^{-1} the inverse of A, which is the set-valued operator with graph $\{(x^*, x) \in \mathcal{H} \times \mathcal{H} \mid x^* \in Ax\}$. The parallel sum of A and $B \colon \mathcal{H} \to 2^{\mathcal{H}}$ is

$$A \,\Box\, B = \big(A^{-1} + B^{-1}\big)^{-1}. \tag{2.1}$$

The resolvent of A is $J_A = (A + \mathrm{Id}_{\mathcal{H}})^{-1} = A^{-1} \,\Box\, \mathrm{Id}_{\mathcal{H}}$ and the Yosida approximation of A of index $\gamma \in \left]0, +\infty\right[$ is ${}^{\gamma}\!A = \gamma^{-1}(\mathrm{Id}_{\mathcal{H}} - J_{\gamma A})$. Furthermore, A is injective if

$$(\forall x_1 \in \mathcal{H})(\forall x_2 \in \mathcal{H}) \quad Ax_1 \cap Ax_2 \neq \varnothing \quad \Rightarrow \quad x_1 = x_2, \tag{2.2}$$

monotone if

$$\big(\forall (x_1, x_1^*) \in \mathrm{gra}\, A\big)\big(\forall (x_2, x_2^*) \in \mathrm{gra}\, A\big) \quad \langle x_1 - x_2 \mid x_1^* - x_2^* \rangle \geqslant 0, \tag{2.3}$$

α-strongly monotone for some $\alpha \in \left]0, +\infty\right[$ if $A - \alpha\, \mathrm{Id}_{\mathcal{H}}$ is monotone, and maximally monotone if

$$(\forall x_1 \in \mathcal{H})(\forall x_1^* \in \mathcal{H}) \quad \big[(x_1, x_1^*) \in \mathrm{gra}\, A \ \Leftrightarrow\ (\forall (x_2, x_2^*) \in \mathrm{gra}\, A)\ \langle x_1 - x_2 | x_1^* - x_2^* \rangle \geqslant 0\big]. \tag{2.4}$$

Let D be a nonempty subset of $\mathcal{H}$ and let $T \colon D \to \mathcal{H}$. Then T is nonexpansive if it is 1-Lipschitzian, firmly nonexpansive if

$$(\forall x_1 \in D)(\forall x_2 \in D) \quad \|Tx_1 - Tx_2\|^2 + \|(\mathrm{Id}_{\mathcal{H}} - T)x_1 - (\mathrm{Id}_{\mathcal{H}} - T)x_2\|^2 \leqslant \|x_1 - x_2\|^2, \tag{2.5}$$

and strictly nonexpansive if

$$(\forall x_1 \in D)(\forall x_2 \in D) \quad x_1 \neq x_2 \quad \Rightarrow \quad \|Tx_1 - Tx_2\| < \|x_1 - x_2\|. \tag{2.6}$$

Let $\beta \in \left]0, +\infty\right[$. Then T is β-cocoercive if βT is firmly nonexpansive.

A function $f\colon \mathcal{H} \to]-\infty,+\infty]$ is proper if $\operatorname{dom} f = \{x \in \mathcal{H} \mid f(x) < +\infty\} \neq \varnothing$, in which case the set of global minimizers of f is denoted by Argmin f; if Argmin f is a singleton, its unique element is denoted by $\operatorname{argmin}_{x\in\mathcal{H}} f(x)$. The conjugate of $f\colon \mathcal{H} \to [-\infty,+\infty]$ is the function

$$f^*\colon \mathcal{H} \to [-\infty,+\infty]\colon x^* \mapsto \sup_{x\in\mathcal{H}} \big(\langle x \mid x^*\rangle - f(x)\big). \tag{2.7}$$

The infimal convolution of $f\colon \mathcal{H} \to]-\infty,+\infty]$ and $g\colon \mathcal{H} \to]-\infty,+\infty]$ is

$$f \,\Box\, g\colon \mathcal{H} \to [-\infty,+\infty]\colon x \mapsto \inf_{z\in\mathcal{H}} \big(f(z) + g(x-z)\big) \tag{2.8}$$

and the Moreau envelope of f of index $\gamma \in]0,+\infty[$ is

$$^{\gamma}f = f \,\Box\, \big(\gamma^{-1}\mathscr{Q}_{\mathcal{H}}\big). \tag{2.9}$$

The infimal postcomposition of $f\colon \mathcal{H} \to [-\infty,+\infty]$ by $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ is

$$L \triangleright f\colon \mathcal{G} \to [-\infty,+\infty]\colon y \mapsto \inf f\big(L^{-1}\{y\}\big) = \inf_{\substack{x\in\mathcal{H}\\ Lx=y}} f(x), \tag{2.10}$$

and it is denoted by $L \blacktriangleright f$ if, for every $y \in L(\operatorname{dom} f)$, there exists $x \in \mathcal{H}$ such that $Lx = y$ and $(L \triangleright f)(y) = f(x) \in]-\infty,+\infty]$. We denote by $\Gamma_0(\mathcal{H})$ the class of proper lower semicontinuous convex functions $f\colon \mathcal{H} \to]-\infty,+\infty]$. Now let $f \in \Gamma_0(\mathcal{H})$. The subdifferential of f is

$$\partial f\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto \{x^* \in \mathcal{H} \mid (\forall z \in \mathcal{H})\ \ \langle z - x \mid x^*\rangle + f(x) \leqslant f(z)\} \tag{2.11}$$

and its inverse is

$$(\partial f)^{-1} = \partial f^*. \tag{2.12}$$

Fermat's rule states that

$$\operatorname{Argmin} f = \operatorname{zer} \partial f. \tag{2.13}$$

The proximity operator of f is

$$\operatorname{prox}_f = J_{\partial f}\colon \mathcal{H} \to \mathcal{H}\colon x \mapsto \underset{z\in\mathcal{H}}{\operatorname{argmin}} \left(f(z) + \frac{1}{2}\|x - z\|^2\right), \tag{2.14}$$

and we have

$$\operatorname{Argmin} f = \operatorname{Fix} \operatorname{prox}_f. \tag{2.15}$$

We say that f is α-strongly convex for some $\alpha \in]0,+\infty[$ if $f - \alpha\mathscr{Q}_{\mathcal{H}}$ is convex.

Let C be a subset of $\mathcal{H}$. The interior of C is denoted by int C, the indicator function of C by ι_C, and the distance function to C by d_C. If C is nonempty, closed, and convex, the projection operator onto C is denoted by proj_C, i.e., $\operatorname{proj}_C = \operatorname{prox}_{\iota_C} = J_{N_C}$, and the normal cone operator of C is $N_C = \partial\iota_C$.

Next, we state a few technical facts that will assist us in our analysis.

Lemma 2.1 *Let $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$, let $\beta \in]0,+\infty[$, let D be a nonempty subset of $\mathcal{G}$, and let $T\colon D \to \mathcal{G}$ be β-cocoercive. Then the following hold:*

(i) *Suppose that $L \neq 0$. Then $L^* \circ T \circ L$ is $\beta\|L\|^{-2}$-cocoercive.*
(ii) *Suppose that T is firmly nonexpansive and that $\|L\| \leqslant 1$. Then $L^* \circ T \circ L$ is firmly nonexpansive.*
(iii) *Suppose that $D = \mathcal{H}$, T is firmly nonexpansive, and $\|L\| \leqslant 1$. Then $L^* \circ T \circ L$ is maximally monotone.*

Proof (i): Set $R = L^* \circ T \circ L$ and take x_1 and x_2 in $\operatorname{dom} R = L^{-1}(D)$. Then

$$\begin{aligned}\langle x_1 - x_2 \mid Rx_1 - Rx_2\rangle &= \langle Lx_1 - Lx_2 \mid T(Lx_1) - T(Lx_2)\rangle \\ &\geqslant \beta\|T(Lx_1) - T(Lx_2)\|^2 \\ &\geqslant \beta\|L\|^{-2}\|Rx_1 - Rx_2\|^2. \end{aligned} \tag{2.16}$$

(ii): The firm nonexpansiveness is clear when $L = 0$, and it otherwise follows from (i) with $\beta = 1$.

(iii): This follows from (ii) and [9, Example 20.30]. □

The following result, essentially due to Minty [41], illuminates the interplay between nonexpansiveness and monotonicity.

Lemma 2.2 ([9, Proposition 23.8]) *Let D be a nonempty subset of $\mathcal{H}$, let $T\colon D \to \mathcal{H}$, and set $A = T^{-1} - \mathrm{Id}_{\mathcal{H}}$. Then the following hold:*

(i) $T = J_A$.
(ii) *T is firmly nonexpansive if and only if A is monotone.*
(iii) *T is firmly nonexpansive and $D = \mathcal{H}$ if and only if A is maximally monotone.*

Lemma 2.3 *Let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$. Then the following hold:*

(i) *Let $\gamma \in\,]0, +\infty[$. Then* ${}^{\gamma}\!A = \big(\gamma\mathrm{Id}_{\mathcal{H}} + A^{-1}\big)^{-1} = \big(J_{\gamma^{-1}A^{-1}}\big) \circ \gamma^{-1}\mathrm{Id}_{\mathcal{H}}$.
(ii) $\mathrm{Id}_{\mathcal{H}} \,\square\, A = J_{A^{-1}} = \mathrm{Id}_{\mathcal{H}} - J_A$.
(iii) $\operatorname{zer} A = \operatorname{Fix} J_A$.
(iv) $(A - \mathrm{Id}_{\mathcal{H}})^{-1} = (\mathrm{Id}_{\mathcal{H}} - A^{-1})^{-1} - \mathrm{Id}_{\mathcal{H}}$.
(v) *Suppose that A is monotone and let $\alpha \in\,]0, +\infty[$. Then A is α-strongly monotone if and only if J_A is $(\alpha + 1)$-cocoercive.*

Proof (i): See [9, Proposition 23.7(ii)].

(ii): Apply (i) with $\gamma = 1$.

(iii): $\operatorname{zer} A = \{x \in \mathcal{H} \mid x \in (A + \mathrm{Id}_{\mathcal{H}})x\} = \{x \in \mathcal{H} \mid x \in (A + \mathrm{Id}_{\mathcal{H}})^{-1}x\}$.

(iv): By (ii), $A^{-1} = J_{A-\mathrm{Id}_{\mathcal{H}}} = \mathrm{Id}_{\mathcal{H}} - J_{(A-\mathrm{Id}_{\mathcal{H}})^{-1}} = \mathrm{Id}_{\mathcal{H}} - (\mathrm{Id}_{\mathcal{H}} + (A - \mathrm{Id}_{\mathcal{H}})^{-1})^{-1}$. So $(\mathrm{Id}_{\mathcal{H}} - A^{-1})^{-1} = \mathrm{Id}_{\mathcal{H}} + (A - \mathrm{Id}_{\mathcal{H}})^{-1}$ as claimed.

(v): See [9, Proposition 23.13]. □

Lemma 2.4 ([1, Theorem 2.1]) *Let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ be a maximally monotone operator and let $B\colon \mathcal{H} \to 2^{\mathcal{H}}$ be a monotone operator such that* $\operatorname{dom} B = \mathcal{H}$ *and $A - B$ is monotone. Then $A - B$ is maximally monotone.*

Lemma 2.5 ([45, Theorem 5]) *Let $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ and let $B\colon \mathcal{G} \to 2^{\mathcal{G}}$ be 3^* monotone, that is,*

$$\big(\forall (y_1, y_1^*) \in \operatorname{dom} B \times \operatorname{ran} B\big) \;\; \sup\{\langle y_1 - y_2 | y_2^* - y_1^*\rangle | (y_2, y_2^*) \in \operatorname{gra} B\} < +\infty. \tag{2.17}$$

Suppose that $L^* \circ B \circ L$ is maximally monotone. Then the following hold:

(i) $\operatorname{int} L^*(\operatorname{ran} B) \subset \operatorname{ran}(L^* \circ B \circ L)$.
(ii) $L^*(\operatorname{ran} B) \subset \overline{\operatorname{ran}}\,(L^* \circ B \circ L)$.

Lemma 2.6 *Let $f \in \Gamma_0(\mathcal{H})$. Then the following hold:*

(i) [9, Theorem 9.20] *f admits a continuous affine minorant.*

(ii) [9, Corollary 13.38] $f^* \in \Gamma_0(\mathcal{H})$ *and* $f^{**} = f$.

Lemma 2.7 [9, Theorem 18.15] *Let* $f\colon \mathcal{H} \to \mathbb{R}$ *be continuous and convex, and let* $\beta \in\,]0, +\infty[$. *Then the following are equivalent:*

(i) f *is Fréchet differentiable on* $\mathcal{H}$ *and* ∇f *is* β*-Lipschitz continuous.*
(ii) f^* *is* β^{-1}*-strongly convex.*

Lemma 2.8 (Moreau [44]) *Let* $T : \mathcal{H} \to \mathcal{H}$ *be nonexpansive. Then* T *is a proximity operator if and only if there exists a differentiable convex function* $h\colon \mathcal{H} \to \mathbb{R}$ *such that* $T = \nabla h$. *In this case,* $T = \operatorname{prox}_f$, *where* $f = h^* - \mathscr{Q}_{\mathcal{H}}$.

Lemma 2.9 (Moreau [44]) *Let* $f \in \Gamma_0(\mathcal{H})$. *Then the following hold:*

(i) ∂f *is maximally monotone.*
(ii) $f \,\square\, \mathscr{Q}_{\mathcal{H}}\colon \mathcal{H} \to \mathbb{R}$ *is convex and Fréchet differentiable.*
(iii) $(f \,\square\, \mathscr{Q}_{\mathcal{H}})^* = f^* + \mathscr{Q}_{\mathcal{H}}$ *and* $(f + \mathscr{Q}_{\mathcal{H}})^* = f^* \,\square\, \mathscr{Q}_{\mathcal{H}}$.
(iv) $\operatorname{prox}_f = \nabla(f^* \,\square\, \mathscr{Q}_{\mathcal{H}})$.
(v) prox_f *is firmly nonexpansive.*
(vi) $f \,\square\, \mathscr{Q}_{\mathcal{H}} + f^* \,\square\, \mathscr{Q}_{\mathcal{H}} = \mathscr{Q}_{\mathcal{H}}$.
(vii) $\operatorname{prox}_f + \operatorname{prox}_{f^*} = \mathrm{Id}_{\mathcal{H}}$.
(viii) $\partial(f + \mathscr{Q}_{\mathcal{H}}) = \partial f + \mathrm{Id}_{\mathcal{H}}$.

3 Examples of Resolvent Compositions

We provide a few examples that expose various facets of the resolvent composition. The first one describes a scenario in which the compositions (1.2), (1.3), and (1.4) happen to coincide.

Example 3.1 Suppose that $L \in \mathcal{B}(\mathcal{H}, \mathscr{Q})$ is a surjective isometry and let $B\colon \mathcal{G} \to 2^{\mathcal{G}}$. Then $L \diamond B = L^* \circ B \circ L = L^* \triangleright B$.

Proof Since $L^{-1} = L^*$, (1.4) yields $L \diamond B = L^* \triangleright (B + \mathrm{Id}_{\mathcal{G}}) - \mathrm{Id}_{\mathcal{H}} = (L^{-1} \circ (B + \mathrm{Id}_{\mathcal{G}})^{-1} \circ L)^{-1} - \mathrm{Id}_{\mathcal{H}} = L^{-1} \circ (B + \mathrm{Id}_{\mathcal{G}}) \circ L - \mathrm{Id}_{\mathcal{G}} = L^{-1} \circ B \circ L = (L^{-1} \circ B^{-1} \circ L)^{-1} = L^{-1} \triangleright B = L^* \triangleright B$. ☐

Example 3.2 Let $\alpha \in \mathbb{R} \smallsetminus \{0\}$, let $B\colon \mathcal{H} \to 2^{\mathcal{H}}$, and set $L = \alpha^{-1}\mathrm{Id}_{\mathcal{H}}$. Then $L \diamond B = (\alpha^2 - 1)\mathrm{Id}_{\mathcal{H}} + \alpha B \circ (\alpha \mathrm{Id}_{\mathcal{H}})$.

The broad potential of Definition 1.1 is illustrated below by deploying it in product spaces.

Example 3.3 (multivariate resolvent mixture) Let $0 \neq m \in \mathbb{N}$ and $0 \neq p \in \mathbb{N}$. For every $i \in \{1, \ldots, m\}$ and every $k \in \{1, \ldots, p\}$, let $\mathcal{H}_i$ and $\mathcal{G}_k$ be real Hilbert spaces, let $L_{ki} \in \mathcal{B}(\mathcal{H}_i, \mathcal{G}_k)$, let $\omega_k \in]0, +\infty[$, and let $B_k\colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$. Let $\mathcal{H}$ be the standard product vector space $\mathcal{H}_1 \times \cdots \times \mathcal{H}_m$, with generic element $\boldsymbol{x} = (x_i)_{1 \leqslant i \leqslant m}$, and equipped with the scalar product $(\boldsymbol{x}, \boldsymbol{x}') \mapsto \sum_{i=1}^m \langle x_i | x_i' \rangle$. Let $\mathcal{G}$ be the standard product vector space $\mathcal{G}_1 \times \cdots \times \mathcal{G}_p$, with generic element $\boldsymbol{y} = (y_k)_{1 \leqslant k \leqslant p}$, and equipped with the scalar product $(\boldsymbol{y}, \boldsymbol{y}') \mapsto \sum_{k=1}^p \omega_k \langle y_k \mid y_k' \rangle$. Set

$$L\colon \mathcal{H} \to \mathcal{G}\colon \boldsymbol{x} \mapsto \left(\sum_{i=1}^m L_{1i} x_i, \ldots, \sum_{i=1}^m L_{pi} x_i \right) \tag{3.1}$$

and

$$B\colon \mathcal{G}\to 2^{\mathcal{G}}\colon \boldsymbol{y}\mapsto B_1y_1\times\cdots\times B_py_p. \tag{3.2}$$

Then Proposition 1.2 yields

$$J_{L\diamond B}\colon \mathcal{H}\to 2^{\mathcal{H}}\colon \boldsymbol{x}\mapsto \left(\sum_{k=1}^{p}\omega_k L_{k1}^*\left(J_{B_k}\left(\sum_{i=1}^{m}L_{ki}x_i\right)\right),\ldots,\sum_{k=1}^{p}\omega_k L_{km}^*\left(J_{B_k}\left(\sum_{i=1}^{m}L_{ki}x_i\right)\right)\right) \tag{3.3}$$

and we call $L\diamond B=(J_{L\diamond B})^{-1}-\mathrm{Id}_{\mathcal{H}}$ a *multivariate resolvent mixture*.

When $m=1$ in Example 3.3, we obtain the following construction.

Example 3.4 (resolvent mixture) Let $0\neq p\in\mathbb{N}$ and, for every $k\in\{1,\ldots,p\}$, let $\mathcal{G}_k$ be a real Hilbert space, let $L_k\in\mathcal{B}(\mathcal{H},\mathcal{G}_k)$, let $\omega_k\in\left]0,+\infty\right[$, and let $B_k\colon\mathcal{G}_k\to 2^{\mathcal{G}_k}$. Define $\mathcal{G}$ and B as in Example 3.3, and set $L\colon\mathcal{H}\to\mathcal{G}\colon x\mapsto(L_1x,\ldots,L_px)$. Then we obtain the *resolvent mixture*

$$L\diamond B=\left(\sum_{k=1}^{p}\omega_k L_k^*\circ J_{B_k}\circ L_k\right)^{-1}-\mathrm{Id}_{\mathcal{H}}=\left(\sum_{k=1}^{p}\omega_k J_{L_k\diamond B_k}\right)^{-1}-\mathrm{Id}_{\mathcal{H}} \tag{3.4}$$

and $J_{L\diamond B}=\sum_{k=1}^{p}\omega_k L_k^*\circ J_{B_k}\circ L_k$. In particular, if, for every $k\in\{1,\ldots,p\}$, $\mathcal{G}_k=\mathcal{H}$ and $L_k=\mathrm{Id}_{\mathcal{H}}$, then (3.4) reduces to Example 1.3, which itself encompasses the resolvent average.

Example 3.5 (linear projector) Let V be a closed vector subspace of $\mathcal{H}$ and let $B\colon\mathcal{H}\to 2^{\mathcal{H}}$. Then $\mathrm{proj}_V\diamond B=(\mathrm{proj}_V\circ J_B\circ\mathrm{proj}_V)^{-1}-\mathrm{Id}_{\mathcal{H}}$. Here are noteworthy special cases of this construction:

(i) Let C be a nonempty closed convex subset of $\mathcal{H}$ and suppose that $B=N_C$. Then $\mathrm{proj}_V\diamond B=(\mathrm{proj}_V\circ\mathrm{proj}_C\circ\mathrm{proj}_V)^{-1}-\mathrm{Id}_{\mathcal{H}}$. This operator was employed in [19] to construct an instance of weak – but not strong – convergence of the Douglas-Rachford algorithm.

(ii) Define $\mathcal{G}$, $(B_k)_{1\leqslant k\leqslant p}$, and B as in Example 1.3, with $\sum_{k=1}^{p}\omega_k=1$. In addition, define $V=\{\boldsymbol{y}\in\mathcal{G}\mid y_1=\cdots=y_p\}$, let $A=(\sum_{k=1}^{p}\omega_k J_{B_k})^{-1}-\mathrm{Id}_{\mathcal{H}}$ be the resolvent average of $(B_k)_{1\leqslant k\leqslant p}$ (see (1.5)), let $\boldsymbol{y}\in\mathcal{G}$, and set $\overline{y}=\sum_{k=1}^{p}\omega_k y_k$. Then we derive from Proposition 1.2 and [9, Propositions 23.18 and 29.16] that $J_{\mathrm{proj}_V\diamond B}\,\boldsymbol{y}=(\sum_{k=1}^{p}\omega_k J_{B_k}\overline{y},\ldots,\sum_{k=1}^{p}\omega_k J_{B_k}\overline{y})=(J_A\overline{y},\ldots,J_A\overline{y})$. In the case of convex feasibility problems, where each B_k is the normal cone to a nonempty closed convex set, this type of construction was first proposed in [47, 48].

The next example places the subdifferential identity (1.7) in a rigorous framework.

Example 3.6 (subdifferential) Suppose that $L\in\mathcal{B}(\mathcal{H},\mathcal{G})$ satisfies $0<\|L\|\leqslant 1$ and let $g\colon\mathcal{G}\to\left]-\infty,+\infty\right]$ be a proper function that admits a continuous affine minorant. Then the following hold:

(i) $g^*\in\Gamma_0(\mathcal{G})$.
(ii) $L\diamond g\in\Gamma_0(\mathcal{H})$.
(iii) $L\diamond\partial g^{**}=\partial(L\diamond g)$.

(iv) $\operatorname{prox}_{L \diamond g} = L^* \circ \operatorname{prox}_{g^{**}} \circ L$.
(v) Suppose that $g \in \Gamma_0(\mathcal{G})$. Then $L \diamond \partial g = \partial(L \diamond g)$.
(vi) Suppose that $g \in \Gamma_0(\mathcal{G})$. Then $\operatorname{prox}_{L \diamond g} = L^* \circ \operatorname{prox}_g \circ L$.

Proof Set $h = ((g^* \,\square\, \mathscr{Q}_{\mathcal{G}}) \circ L)^* - \|L\|^{-2}\mathscr{Q}_{\mathcal{H}}$. On the one hand, by [9, Proposition 13.13], g^* is lower semicontinuous and convex. On the other hand, by [9, Propositions 13.10(ii) and 13.12(ii)], g^* is proper. Thus,

$$g^* \in \Gamma_0(\mathcal{G}) \tag{3.5}$$

and it follows from Lemma 2.9 that $g^* \,\square\, \mathscr{Q}_{\mathcal{G}} \colon \mathcal{G} \to \mathbb{R}$ is Fréchet differentiable on $\mathcal{G}$ with nonexpansive gradient $\mathrm{Id}_{\mathcal{G}} - \operatorname{prox}_{g^*}$. In turn,

$$\nabla\big((g^* \,\square\, \mathscr{Q}_{\mathcal{G}}) \circ L\big) = L^* \circ (\mathrm{Id}_{\mathcal{G}} - \operatorname{prox}_{g^*}) \circ L \tag{3.6}$$

has Lipschitz constant $\|L\|^2$ and we derive from Lemma 2.7 that

$$\big((g^* \,\square\, \mathscr{Q}_{\mathcal{G}}) \circ L\big)^* \;\text{ is }\; \|L\|^{-2}\text{-strongly convex,} \tag{3.7}$$

We also record the fact that (3.5) and Lemma 2.6(ii) imply that $g^{**} \in \Gamma_0(\mathcal{G})$.

(i): See (3.5).

(ii): We infer from (3.7) that $h \in \Gamma_0(\mathcal{H})$. Hence, since $\|L\|^{-2} > 1$, we conclude that

$$L \diamond g = h + \big(\|L\|^{-2} - 1\big)\mathscr{Q}_{\mathcal{H}} \in \Gamma_0(\mathcal{H}). \tag{3.8}$$

(iii): Note that, on account of Lemma 2.6(i), g^{**} admits a continuous affine minorant. Using (1.4), (1.3), (2.12), Lemma 2.9(viii), Lemma 2.9(iii), [9, Proposition 13.16(iii) and Corollary 16.53(i)], we get

$$\begin{aligned} L \diamond \partial g^{**} + \mathrm{Id}_{\mathcal{H}} &= L^* \triangleright (\partial g^{**} + \mathrm{Id}_{\mathcal{G}}) \\ &= \big(L^* \circ \big(\partial g^{**} + \mathrm{Id}_{\mathcal{G}}\big)^{-1} \circ L\big)^{-1} \\ &= \big(L^* \circ \big(\partial (g^{**} + \mathscr{Q}_{\mathcal{G}})\big)^{-1} \circ L\big)^{-1} \\ &= \big(L^* \circ \partial (g^{**} + \mathscr{Q}_{\mathcal{G}})^* \circ L\big)^{-1} \\ &= \big(L^* \circ \partial (g^{***} \,\square\, \mathscr{Q}_{\mathcal{G}}) \circ L\big)^{-1} \\ &= \big(L^* \circ \partial (g^{*} \,\square\, \mathscr{Q}_{\mathcal{G}}) \circ L\big)^{-1} \\ &= \partial\big((g^* \,\square\, \mathscr{Q}_{\mathcal{G}}) \circ L\big)^*. \end{aligned} \tag{3.9}$$

Since $0 < \|L\| \leqslant 1$, we deduce from (3.7) that $((g^* \,\square\, \mathscr{Q}_{\mathcal{G}}) \circ L)^* - \mathscr{Q}_{\mathcal{H}} \in \Gamma_0(\mathcal{H})$. Hence, appealing to Lemma 2.9(viii) and (1.6), we obtain

$$\begin{aligned} \partial\big((g^* \,\square\, \mathscr{Q}_{\mathcal{G}}) \circ L\big)^* &= \partial\Big(\big((g^* \,\square\, \mathscr{Q}_{\mathcal{G}}) \circ L\big)^* - \mathscr{Q}_{\mathcal{H}} + \mathscr{Q}_{\mathcal{H}}\Big) \\ &= \partial\Big(\big((g^* \,\square\, \mathscr{Q}_{\mathcal{G}}) \circ L\big)^* - \mathscr{Q}_{\mathcal{H}}\Big) + \partial \mathscr{Q}_{\mathcal{H}} \\ &= \partial(L \diamond g) + \mathrm{Id}_{\mathcal{H}}. \end{aligned} \tag{3.10}$$

The sought identity follows by combining (3.9) and (3.10).

(iv): In view of (ii), $\operatorname{prox}_{L \diamond g}$ is well defined and, combining (iii) and Proposition 1.2, we obtain $\operatorname{prox}_{L \diamond g} = J_{\partial(L \diamond g)} = J_{L \diamond \partial g^{**}} = L^* \circ J_{\partial g^{**}} \circ L = L^* \circ \operatorname{prox}_{g^{**}} \circ L$.

(v)–(vi): These identities follow from Lemma 2.6(ii), (iii), and (iv). □

Example 3.7 (proximity operator) Suppose that $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ satisfies $0 < \|L\| \leqslant 1$ and let $g \in \Gamma_0(\mathcal{G})$. Then we derive from Lemma 2.9(iv) and Example 3.6(v) that

$$L \diamond \operatorname{prox}_g = \partial\big(L \diamond (g^* \,\square\, \mathscr{Q}_{\mathcal{G}})\big). \tag{3.11}$$

Example 3.8 (projection operator) Suppose that $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ satisfies $0 < \|L\| \leqslant 1$, let C be a nonempty closed convex subset of $\mathcal{G}$, and set $g = \iota_C$. Then $g \,\square\, \mathscr{Q}_{\mathcal{G}} = d_C^2/2$ and Lemma 2.9(vi) yields $g^* \,\square\, \mathscr{Q}_{\mathcal{G}} = \mathscr{Q}_{\mathcal{G}} - d_C^2/2$. Altogether, we derive from Example 3.7 and Lemma 2.9(vii) that

$$L \diamond \operatorname{proj}_C = \partial\big(L \diamond (\mathscr{Q}_{\mathcal{G}} - d_C^2/2)\big) \quad \text{and} \quad L \diamond (\operatorname{Id}_{\mathcal{G}} - \operatorname{proj}_C) = \partial\big(L \diamond (d_C^2/2)\big). \tag{3.12}$$

Example 3.9 (frames) Suppose that $(e_k)_{k\in\mathbb{N}}$ is a frame in $\mathcal{H}$ [22], i.e., there exist $\alpha \in \left]0, +\infty\right[$ and $\beta \in \left]0, +\infty\right[$ such that

$$(\forall x \in \mathcal{H}) \quad \alpha\|x\|^2 \leqslant \sum_{k\in\mathbb{N}} |\langle x \mid e_k\rangle|^2 \leqslant \beta\|x\|^2. \tag{3.13}$$

We set $\mathcal{G} = \ell^2(\mathbb{N})$, denote by $L\colon \mathcal{H} \to \mathcal{G}\colon x \mapsto (\langle x | e_k\rangle)_{k\in\mathbb{N}}$ the frame analysis operator, and let $(\phi_k)_{k\in\mathbb{N}}$ be functions in $\Gamma_0(\mathbb{R})$ such that $(\forall k \in \mathbb{N})$ $\phi_k \geqslant \phi_k(0) = 0$. Further, we set $B\colon \mathcal{G} \to 2^{\mathcal{G}}\colon (\eta_k)_{k\in\mathbb{N}} \mapsto \{(\nu_k)_{k\in\mathbb{N}} \in \mathcal{G} \mid (\forall k \in \mathbb{N})\ \nu_k \in \partial\phi_k(\eta_k)\}$. Then

$$L \diamond B = \left(\sum_{k\in\mathbb{N}} \big(\operatorname{prox}_{\phi_k}\langle \cdot \mid e_k\rangle\big) e_k\right)^{-1} - \operatorname{Id}_{\mathcal{H}}. \tag{3.14}$$

Proof Set $\varphi\colon \mathcal{G} \to \left]-\infty, +\infty\right]\colon (\eta_k)_{k\in\mathbb{N}} \mapsto \sum_{k\in\mathbb{N}} \phi_k(\eta_k)$ and note that $L^*\colon \mathcal{G} \to \mathcal{H}\colon (\eta_k)_{k\in\mathbb{N}} \mapsto \sum_{k\in\mathbb{N}} \eta_k e_k$. As shown in [31], $\varphi \in \Gamma_0(\mathcal{G})$, $B = \partial\varphi$, and $J_B\colon (\eta_k)_{k\in\mathbb{N}} \mapsto (\operatorname{prox}_{\phi_k}\eta_k)_{k\in\mathbb{N}}$. Thus, $(L^* \triangleright (B + \operatorname{Id}_{\mathcal{G}}))^{-1} = L^* \circ J_B \circ L = \sum_{k\in\mathbb{N}} \big(\operatorname{prox}_{\phi_k}\langle\cdot \mid e_k\rangle\big)e_k$. $\square$

Our last example parallels Example 3.6 in the case of the proximal cocomposition of Definition 1.4.

Example 3.10 (subdifferential) Suppose that $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ satisfies $0 < \|L\| \leqslant 1$ and let $g\colon \mathcal{G} \to \left]-\infty, +\infty\right]$ be a proper function that admits a continuous affine minorant. Then the following hold:

(i) $L \blacklozenge g \in \Gamma_0(\mathcal{H})$.
(ii) $L \blacklozenge \partial g^{**} = \partial(L \blacklozenge g)$.
(iii) $\operatorname{prox}_{L \blacklozenge g} = \operatorname{Id}_{\mathcal{H}} - L^* \circ L + L^* \circ \operatorname{prox}_{g^{**}} \circ L$.
(iv) Suppose that $g \in \Gamma_0(\mathcal{G})$. Then $L \blacklozenge \partial g = \partial(L \blacklozenge g)$.
(v) Suppose that $g \in \Gamma_0(\mathcal{G})$. Then $\operatorname{prox}_{L \blacklozenge g} = \operatorname{Id}_{\mathcal{H}} - L^* \circ L + L^* \circ \operatorname{prox}_g \circ L$.

Proof By virtue of Example 3.6(i) and Lemma 2.6(i), g^* is in $\Gamma_0(\mathcal{G})$ and it admits a continuous affine minorant. As a consequence of Example 3.6(ii), we record the fact that

$$L \diamond g^* \in \Gamma_0(\mathcal{H}). \tag{3.15}$$

(i): We invoke (3.15) and Lemma 2.6(ii) to deduce that $L \blacklozenge g = (L \diamond g^*)^* \in \Gamma_0(\mathcal{H})$.

(ii): It follows from Lemma 2.6(ii) that $g^{**} \in \Gamma_0(\mathcal{G})$. Hence, using Definition 1.1, (3.15), (2.12), and Definition 1.4, we obtain

$$L \blacklozenge \partial g^{**} = \big(L \diamond (\partial g^{**})^{-1}\big)^{-1} = \big(L \diamond \partial g^{***}\big)^{-1} = \big(\partial(L \diamond g^*)\big)^{-1} = \partial(L \diamond g^*)^* = \partial(L \blacklozenge g). \tag{3.16}$$

(iii): Property (i) ensures that $\mathrm{prox}_{L\blacklozenge g}$ is well defined. Further, we deduce from (3.15), Lemma 2.9(vii), and Example 3.6(iv) that

$$\mathrm{prox}_{L\blacklozenge g} = \mathrm{Id}_{\mathcal{H}} - \mathrm{prox}_{L\diamond g^*} = \mathrm{Id}_{\mathcal{H}} - L^*\circ \mathrm{prox}_{g^{***}}\circ L = \mathrm{Id}_{\mathcal{H}} - L^*\circ (\mathrm{Id}_{\mathcal{G}} - \mathrm{prox}_{g^{**}})\circ L. \tag{3.17}$$

(iv)–(v): Since $g = g^{**}$ by Lemma 2.6(ii), these follow from (ii) and (iii). ☐

4 Properties of the Resolvent Composition

We start with basic facts.

Proposition 4.1 *Let $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ and let $B\colon \mathcal{G} \to 2^{\mathcal{G}}$. Then the following hold:*

(i) $L \diamond B = (L^*\circ J_B\circ L)^{-1} - \mathrm{Id}_{\mathcal{H}}$.
(ii) $L\blacklozenge B = (\mathrm{Id}_{\mathcal{H}} - L^*\circ L + L^*\circ J_B\circ L)^{-1} - \mathrm{Id}_{\mathcal{H}}$.
(iii) *Suppose that L is an isometry. Then $L\diamond B = L\blacklozenge B$.*
(iv) $(L\diamond B)^{-1} = L\blacklozenge B^{-1} = (\mathrm{Id}_{\mathcal{H}} - L^*\circ J_B\circ L)^{-1} - \mathrm{Id}_{\mathcal{H}}$.
(v) $J_{L\blacklozenge B} = \mathrm{Id}_{\mathcal{H}} - L^*\circ L + L^*\circ J_B\circ L$.
(vi) $\mathrm{gra}\,(L\diamond B) = \big\{(x, x^*) \in \mathcal{H}\times\mathcal{H} \;\big|\; (x + x^*, x) \in \mathrm{gra}\,(L^*\circ J_B\circ L)\big\}$.
(vii) $\mathrm{gra}\,(L\blacklozenge B) = \big\{(x, x^*) \in \mathcal{H}\times\mathcal{H} \;\big|\; (x + x^*, (L^*\circ L)(x + x^*) - x^*) \in \mathrm{gra}\,(L^*\circ J_B\circ L)\big\}$.
(viii) $\mathrm{dom}\,(L\diamond B) \subset L^*(\mathrm{dom}\,B)$.
(ix) $\mathrm{ran}\,(L\diamond B) \subset \mathrm{ran}\,(\mathrm{Id}_{\mathcal{H}} - L^*\circ L) + L^*(\mathrm{ran}\,B)$.
(x) $\mathrm{dom}\,(L\blacklozenge B) \subset \mathrm{ran}\,(\mathrm{Id}_{\mathcal{H}} - L^*\circ L) + L^*(\mathrm{dom}\,B)$.
(xi) $\mathrm{ran}\,(L\blacklozenge B) \subset L^*(\mathrm{ran}\,B)$.
(xii) $\mathrm{zer}\,(L\diamond B) = \mathrm{Fix}(L^*\circ J_B\circ L)$.
(xiii) $L^{-1}(\mathrm{zer}\,B) \subset \mathrm{zer}\,(L\blacklozenge B)$.
(xiv) $(L\diamond B)\,\square\,\mathrm{Id}_{\mathcal{H}} + L^*\circ (B^{-1}\,\square\,\mathrm{Id}_{\mathcal{G}})\circ L = \mathrm{Id}_{\mathcal{H}}$.
(xv) $(L\blacklozenge B)\,\square\,\mathrm{Id}_{\mathcal{H}} = L^*\circ (B\,\square\,\mathrm{Id}_{\mathcal{G}})\circ L$.

Proof (i): A consequence of (1.1) and Proposition 1.2.

(ii): In view of (i), Lemma 2.3(iv), and Lemma 2.3(ii), $L\blacklozenge B = (L\diamond B^{-1})^{-1} = ((L^*\circ J_{B^{-1}}\circ L)^{-1} - \mathrm{Id}_{\mathcal{H}})^{-1} = (\mathrm{Id}_{\mathcal{H}} - L^*\circ J_{B^{-1}}\circ L)^{-1} - \mathrm{Id}_{\mathcal{H}} = (\mathrm{Id}_{\mathcal{H}} - L^*\circ(\mathrm{Id}_{\mathcal{G}} - J_B)\circ L)^{-1} - \mathrm{Id}_{\mathcal{H}}$.

(iii): Since $L^*\circ L = \mathrm{Id}_{\mathcal{H}}$, this follows from (i) and (ii).

(iv): The first identity is clear by inspecting Definition 1.1. To establish the second, note that (i) and Lemma 2.3(iv) yield

$$(L\diamond B)^{-1} = \big((L^*\circ J_B\circ L)^{-1} - \mathrm{Id}_{\mathcal{H}}\big)^{-1} = (\mathrm{Id}_{\mathcal{H}} - L^*\circ J_B\circ L)^{-1} - \mathrm{Id}_{\mathcal{H}}. \tag{4.1}$$

(v): A consequence of (ii).

(vi): Let $(x, x^*) \in \mathcal{H}\times\mathcal{H}$. Then (i) yields $(x, x^*) \in \mathrm{gra}(L\diamond B) \Leftrightarrow x^* \in (L^*\circ J_B\circ L)^{-1}x - x \Leftrightarrow x \in (L^*\circ J_B\circ L)(x + x^*)$.

(vii): Let $(x, x^*) \in \mathcal{H}\times\mathcal{H}$. By (vi) and Lemma 2.3(ii), $(x, x^*) \in \mathrm{gra}\,(L\blacklozenge B) \Leftrightarrow (x^*, x) \in \mathrm{gra}\,(L\diamond B^{-1}) \Leftrightarrow x + x^* \in (L^*\circ J_{B^{-1}}\circ L)^{-1}x^* \Leftrightarrow x^* \in (L^*\circ J_{B^{-1}}\circ L)(x + x^*) = (L^*\circ L)(x + x^*) - (L^*\circ J_B\circ L)(x + x^*) \Leftrightarrow (L^*\circ L)(x + x^*) - x^* \in (L^*\circ J_B\circ L)(x + x^*)$.

(viii): In view of (i) and Proposition 1.2,

$$\mathrm{dom}\,(L\diamond B) = \mathrm{dom}\,(L^*\circ J_B\circ L)^{-1} = \mathrm{ran}\,(L^*\circ J_B\circ L) \subset L^*(\mathrm{ran}\,J_B) = L^*(\mathrm{dom}\,B). \tag{4.2}$$

(ix): We invoke (iv) and Lemma 2.3(ii) to get

$$\begin{aligned}
\operatorname{ran}(L \diamond B) &= \operatorname{dom}(L \diamond B)^{-1} \\
&= \operatorname{dom}\big(\mathrm{Id}_{\mathcal{H}} - L^* \circ J_B \circ L\big)^{-1} \\
&= \operatorname{ran}\big(\mathrm{Id}_{\mathcal{H}} - L^* \circ J_B \circ L\big) \\
&= \operatorname{ran}\big(\mathrm{Id}_{\mathcal{H}} - L^* \circ L + L^* \circ (\mathrm{Id}_{\mathcal{G}} - J_B) \circ L\big) \\
&= \operatorname{ran}\big(\mathrm{Id}_{\mathcal{H}} - L^* \circ L + L^* \circ J_{B^{-1}} \circ L\big) \qquad (4.3)\\
&\subset \operatorname{ran}(\mathrm{Id}_{\mathcal{H}} - L^* \circ L) + \operatorname{ran}(L^* \circ J_{B^{-1}} \circ L) \\
&\subset \operatorname{ran}(\mathrm{Id}_{\mathcal{H}} - L^* \circ L) + L^*(\operatorname{ran} J_{B^{-1}}) \\
&= \operatorname{ran}(\mathrm{Id}_{\mathcal{H}} - L^* \circ L) + L^*(\operatorname{dom} B^{-1}) \\
&= \operatorname{ran}(\mathrm{Id}_{\mathcal{H}} - L^* \circ L) + L^*(\operatorname{ran} B), \qquad (4.4)
\end{aligned}$$

which furnishes the desired inclusion.

(x): In view of (ix), $\operatorname{dom}(L \blacklozenge B) = \operatorname{ran}(L \diamond B^{-1}) \subset \operatorname{ran}(\mathrm{Id}_{\mathcal{H}} - L^* \circ L) + L^*(\operatorname{ran} B^{-1}) = \operatorname{ran}(\mathrm{Id}_{\mathcal{H}} - L^* \circ L) + L^*(\operatorname{dom} B)$.

(xi): In view of (viii), $\operatorname{ran}(L \blacklozenge B) = \operatorname{dom}(L \diamond B^{-1}) \subset L^*(\operatorname{dom} B^{-1}) = L^*(\operatorname{ran} B)$.

(xii): Combine Lemma 2.3(iii) and Proposition 1.2.

(xiii): Let $x \in \mathcal{H}$. With the help of Lemma 2.3(ii)–(iii) and Proposition 1.2, we derive that

$$\begin{aligned}
x \in L^{-1}(\operatorname{zer} B) &\Leftrightarrow 0 \in Lx - J_B(Lx) \\
&\Rightarrow 0 \in L^*\big((\mathrm{Id}_{\mathcal{G}} - J_B)Lx\big) \\
&\Leftrightarrow 0 \in L^*\big(J_{B^{-1}} Lx\big) \\
&\Leftrightarrow 0 \in J_{L \diamond B^{-1}} x \\
&\Leftrightarrow x \in \big(\mathrm{Id}_{\mathcal{G}} - J_{L \diamond B^{-1}}\big)x \\
&\Leftrightarrow x \in J_{(L \diamond B^{-1})^{-1}} x \\
&\Leftrightarrow x \in \operatorname{zer}\big(L \blacklozenge B\big). \qquad (4.5)
\end{aligned}$$

(xiv): It follows from Lemma 2.3(ii) that $(L \diamond B) \,\Box\, \mathrm{Id}_{\mathcal{H}} + (L \diamond B)^{-1} \,\Box\, \mathrm{Id}_{\mathcal{H}} = \mathrm{Id}_{\mathcal{H}}$. On the other hand, Proposition 1.2 yields $(L \diamond B)^{-1} \,\Box\, \mathrm{Id}_{\mathcal{H}} = J_{L \diamond B} = L^* \circ (B^{-1} \,\Box\, \mathrm{Id}_{\mathcal{H}}) \circ L$.

(xv): It follows from (1.1), (iv), and Proposition 1.2 that $(L \blacklozenge B) \,\Box\, \mathrm{Id}_{\mathcal{H}} = J_{(L \blacklozenge B)^{-1}} = J_{L \diamond B^{-1}} = L^* \circ J_{B^{-1}} \circ L = L^* \circ (B \,\Box\, \mathrm{Id}_{\mathcal{G}}) \circ L$. ◻

Remark 4.2 (isometry) In connection with Proposition 4.1(iii), here are some important settings in which L is an isometry:

(i) Example 3.4 under the assumption that $\sum_{k=1}^{p} \omega_k L_k^* \circ L_k = \mathrm{Id}_{\mathcal{H}}$.
(ii) The resolvent average of Example 1.3, as a realization of (i).
(iii) Example 3.9 under the assumption that $(e_k)_{k \in \mathbb{N}}$ is a *Parseval frame*, i.e., $\alpha = \beta = 1$ in (3.13).

Proposition 4.3 *Let $\mathcal{K}$ be a real Hilbert space, let $Q \in \mathcal{B}(\mathcal{H}, \mathcal{G})$, let $L \in \mathcal{B}(\mathcal{G}, \mathcal{K})$, and let $B\colon \mathcal{K} \to 2^{\mathcal{K}}$. Then $Q \diamond (L \diamond B) = (L \circ Q) \diamond B$.*

Proof It follows from Proposition 4.1(i) and Proposition 1.2 that $Q \diamond (L \diamond B) = (Q^* \circ J_{L \diamond B} \circ Q)^{-1} - \mathrm{Id}_{\mathcal{H}} = (Q^* \circ L^* \circ J_B \circ L \circ Q)^{-1} - \mathrm{Id}_{\mathcal{H}} = ((L \circ Q)^* \circ J_B \circ (L \circ Q))^{-1} - \mathrm{Id}_{\mathcal{H}} = (L \circ Q) \diamond B$. ◻

The next results bring into play monotonicity. A key fact is that, if L is nonexpansive, then the resolvent composition preserves monotonicity and maximal monotonicity.

Proposition 4.4 *Let $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ and let $B\colon \mathcal{G} \to 2^{\mathcal{G}}$ be monotone. Then the following hold:*

(i) *Suppose that $\|L\| \leqslant 1$. Then $L \diamond B$ is monotone.*
(ii) *Suppose that $\|L\| \leqslant 1$. Then $L \blacklozenge B$ is monotone.*
(iii) *Suppose that $L \neq 0$, let $\alpha \in [0, +\infty[$ be such that $B - \alpha\,\mathrm{Id}_{\mathcal{G}}$ is monotone, set $\beta = (\alpha + 1)\|L\|^{-2} - 1$, and suppose that one of the following is satisfied:*

  (a) $\|L\| < \sqrt{\alpha + 1}$.
  (b) $\|L\| \leqslant 1$ *and* $\alpha > 0$, *i.e., B is α-strongly monotone.*
  (c) $\|L\| < 1$.

  Then $L \diamond B$ is β-strongly monotone.

Proof (i): We set $R = L^* \circ J_B \circ L$ and note that R is single-valued on its domain since Lemma 2.2(i)-(ii) states that J_B is. Now take $(x_1, x_1^*) \in \operatorname{gra}(L \diamond B)$ and $(x_2, x_2^*) \in \operatorname{gra}(L \diamond B)$. By Proposition 4.1(iv), $(x_1 + x_1^*, x_1) \in \operatorname{gra} R$ and $(x_2 + x_2^*, x_2) \in \operatorname{gra} R$, i.e., $x_1 = R(x_1 + x_1^*)$ and $x_2 = R(x_2 + x_2^*)$. However, since R is firmly nonexpansive by Lemma 2.1(ii), we get

$$
\begin{aligned}
\langle x_1 - x_2 \mid x_1^* - x_2^* \rangle &= \langle R(x_1 + x_1^*) - R(x_2 + x_2^*) \mid (x_1 + x_1^*) - (x_2 + x_2^*) \rangle - \|x_1 - x_2\|^2 \\
&\geqslant \|R(x_1 + x_1^*) - R(x_2 + x_2^*)\|^2 - \|x_1 - x_2\|^2 \\
&= 0, \qquad (4.6)
\end{aligned}
$$

which establishes (2.3).

(ii): Since monotonicity is preserved under inversion, B^{-1} is monotone, and so is $L \diamond B^{-1}$ by (i). In turn, if $L \blacklozenge B = (L \diamond B^{-1})^{-1}$ is monotone as well.

(iii): We consider only property (iii)(a), which implies that $\beta > 0$, since (iii)(b) and (iii)(c) are special cases of it. In view of Lemma 2.2(ii) (for $\alpha = 0$) and Lemma 2.3(v) (for $\alpha > 0$), J_B is $(\alpha + 1)$-cocoercive and $L^* \circ J_B \circ L$ is therefore $(\alpha + 1)\|L\|^{-2}$-cocoercive on account of Lemma 2.1(i). This shows that $(L^* \circ J_B \circ L)^{-1}$ is $(\alpha + 1)\|L\|^{-2}$-strongly monotone. Appealing to Proposition 4.1(i), we conclude that $L \diamond B = (L^* \circ J_B \circ L)^{-1} - \mathrm{Id}_{\mathcal{H}}$ is β-strongly monotone. □

The theorem below significantly improves Proposition 4.4(i)-(ii) and Proposition 4.1(viii)–(xi) in the case of maximally monotone operators.

Theorem 4.5 *Let $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ be such that $\|L\| \leqslant 1$ and let $B\colon \mathcal{G} \to 2^{\mathcal{G}}$ be maximally monotone. Then the following hold:*

(i) *$L \diamond B$ is maximally monotone.*
(ii) *$L \blacklozenge B$ is maximally monotone.*
(iii) *Suppose that L is injective and that B is at most single-valued. Then $L \diamond B$ is at most single-valued.*
(iv) *Suppose that L and B are injective. Then $L \diamond B$ is injective.*
(v) $\operatorname{int}\operatorname{dom}(L \diamond B) = \operatorname{int} L^*(\operatorname{dom} B)$.
(vi) $\overline{\operatorname{dom}}\,(L \diamond B) = \overline{L^*(\operatorname{dom} B)}$.
(vii) $\operatorname{int}\operatorname{ran}\,(L \diamond B) = \operatorname{int}(\operatorname{ran}(\mathrm{Id}_{\mathcal{H}} - L^* \circ L) + L^*(\operatorname{ran} B))$.
(viii) $\overline{\operatorname{ran}}\,(L \diamond B) = \overline{\operatorname{ran}\,(\mathrm{Id}_{\mathcal{H}} - L^* \circ L) + L^*(\operatorname{ran} B)}$.
(ix) $\operatorname{int}\operatorname{dom}(L \blacklozenge B) = \operatorname{int}(\operatorname{ran}(\mathrm{Id}_{\mathcal{H}} - L^* \circ L) + L^*(\operatorname{dom} B))$.

(x) $\overline{\text{dom}}\,(L\blacklozenge B) = \overline{\text{ran}\,(\text{Id}_{\mathcal{H}} - L^* \circ L) + L^*(\text{dom}\,B)}$.
(xi) $\text{int}\,\text{ran}\,(L\blacklozenge B) = \text{int}\,L^*(\text{ran}\,B)$.
(xii) $\overline{\text{ran}}\,(L\blacklozenge B) = \overline{L^*(\text{ran}\,B)}$.

Proof It follows from Lemma 2.2(iii) that $J_B : \mathcal{G} \to \mathcal{G}$ is firmly nonexpansive. Hence, we derive from Lemma 2.1(iii) that

$$L^* \circ J_B \circ L \text{ is maximally monotone.} \tag{4.7}$$

(i): It follows from (4.7) that $(L^* \circ J_B \circ L)^{-1}$ is maximally monotone. In view of Proposition 4.1(i), Proposition 4.4(i), and Lemma 2.4, we conclude that $L \diamond B$ is maximally monotone.

(ii): Since maximal monotonicity is preserved under inversion, B^{-1} is maximally monotone. In view of (i), this renders $L \diamond B^{-1}$ maximally monotone. We then infer that $L\blacklozenge B = (L \diamond B^{-1})^{-1}$ is maximally monotone.

(iii): Let us first recall that a maximally monotone operator is at most single-valued if and only if its resolvent is injective [11, Theorem 2.1(iv)]. Hence, J_B is injective and, appealing to (i) and Proposition 1.2, it is enough to show that $L^* \circ J_B \circ L$ is injective. Let $x_1 \in \mathcal{H}$ and $x_2 \in \mathcal{H}$ be such that $(L^* \circ J_B \circ L)x_1 = (L^* \circ J_B \circ L)x_2$. Then, since Lemma 2.2(iii) asserts that J_B is firmly nonexpansive,

$$\begin{aligned} 0 &= \langle (L^* \circ J_B \circ L)x_1 - (L^* \circ J_B \circ L)x_2 \mid x_1 - x_2 \rangle \\ &= \langle J_B(Lx_1) - J_B(Lx_2) \mid Lx_1 - Lx_2 \rangle \\ &\geqslant \| J_B(Lx_1) - J_B(Lx_2) \|^2. \end{aligned} \tag{4.8}$$

Therefore $J_B(Lx_1) = J_B(Lx_2)$ and, since J_B is injective, $Lx_1 = Lx_2$. Finally, the injectivity of L yields $x_1 = x_2$.

(iv): Using the fact that a maximally monotone operator is injective if and only if its resolvent is strictly nonexpansive [11, Theorem 2.1(ix)], we obtain the strict nonexpansiveness of J_B. Furthermore, according to (i) and Proposition 1.2, it is enough to show that $L^* \circ J_B \circ L$ is strictly nonexpansive. To this end, we let $x_1 \in \mathcal{H}$ and $x_2 \in \mathcal{H}$ be such that

$$\| (L^* \circ J_B \circ L)x_1 - (L^* \circ J_B \circ L)x_2 \| = \| x_1 - x_2 \|. \tag{4.9}$$

Then, since $\|L^*\| = \|L\| \leqslant 1$,

$$\begin{aligned} \|x_1 - x_2\| &= \| (L^* \circ J_B \circ L)x_1 - (L^* \circ J_B \circ L)x_2 \| \\ &\leqslant \| J_B(Lx_1) - J_B(Lx_2) \| \\ &\leqslant \| Lx_1 - Lx_2 \| \\ &\leqslant \| x_1 - x_2 \|. \end{aligned} \tag{4.10}$$

Thus, $\|J_B(Lx_1) - J_B(Lx_2)\| = \|Lx_1 - Lx_2\|$ and, since J_B is strictly nonexpansive, we obtain $Lx_1 = Lx_2$. In view of the injectivity of L, this means that $x_1 = x_2$. As Lemma 2.2(iii) and Lemma 2.1(ii) imply that $L^* \circ J_B \circ L$ is nonexpansive, we conclude that it is strictly nonexpansive.

(v)–(vi): Arguing as in (4.2), we observe that

$$\text{ran}\,(L^* \circ J_B \circ L) = \text{dom}\,(L \diamond B) \subset L^*(\text{dom}\,B). \tag{4.11}$$

On the other hand, [9, Example 25.20(ii)] asserts that J_B is 3^* monotone. Therefore, we derive from (4.7) and Lemma 2.5(i) that

$$\operatorname{int} L^*(\operatorname{dom} B) = \operatorname{int} L^*(\operatorname{ran} J_B) \subset \operatorname{ran}\,(L^* \circ J_B \circ L) = \operatorname{dom}\,(L \diamond B) \subset L^*(\operatorname{dom} B), \tag{4.12}$$

which yields (v). Let us turn to (vi). Proceeding as above and invoking Lemma 2.5(ii), (4.11) yields

$$L^*(\operatorname{dom} B) = L^*(\operatorname{ran} J_B) \subset \overline{\operatorname{ran}}\,(L^* \circ J_B \circ L) = \overline{\operatorname{dom}}\,(L \diamond B) \subset \overline{L^*(\operatorname{dom} B)} \tag{4.13}$$

and, therefore, $\overline{\operatorname{dom}}\,(L \diamond B) = \overline{L^*(\operatorname{dom} B)}$.

(vii)–(viii): Set

$$\begin{cases} A = \operatorname{Id}_{\mathcal{H}} - L^* \circ L \\ \boldsymbol{L} \colon \mathcal{H} \to \mathcal{H} \oplus \mathcal{G} \colon x \mapsto (x, Lx) \\ \boldsymbol{B} \colon \mathcal{H} \oplus \mathcal{G} \to 2^{\mathcal{H}} \times 2^{\mathcal{G}} \colon (x, y) \mapsto Ax \times \{J_{B^{-1}} y\}. \end{cases} \tag{4.14}$$

Since $\boldsymbol{L}^* \colon \mathcal{H} \oplus \mathcal{G} \to \mathcal{H} \colon (x^*, y^*) \mapsto x^* + L^* y^*$, we deduce from (4.3) and (4.4) that

$$\operatorname{ran}\,(\boldsymbol{L}^* \circ \boldsymbol{B} \circ \boldsymbol{L}) = \operatorname{ran}\,(A + L^* \circ J_{B^{-1}} \circ L) = \operatorname{ran}\,(L \diamond B) \subset \operatorname{ran} A + L^*(\operatorname{ran} B) = \boldsymbol{L}^*(\operatorname{ran} \boldsymbol{B}). \tag{4.15}$$

In addition, since

$$(\forall x \in \mathcal{H}) \quad \langle x \,|\, L^*(Lx)\rangle = \|Lx\|^2 \geqslant \|L\|^2 \, \|Lx\|^2 \geqslant \|L^*(Lx)\|^2, \tag{4.16}$$

the operator $L^* \circ L$ is firmly nonexpansive and so is therefore $A = \operatorname{Id}_{\mathcal{H}} - L^* \circ L$, which is thus maximally monotone by virtue of [9, Example 20.30]. In view of [9, Proposition 25.16], this means that A is 3^* monotone. On the other hand, since B^{-1} is maximally monotone, we derive from [9, Example 25.20(iii)] that $J_{B^{-1}}$ is 3^* monotone. Thus, $\boldsymbol{B}$ is 3^* monotone. Moreover, since [9, Proposition 20.23] implies that $\boldsymbol{B}$ is maximally monotone and since $\operatorname{dom} \boldsymbol{B} = \mathcal{H} \oplus \mathcal{G}$, it follows from [9, Corollary 25.6] that $\boldsymbol{L}^* \circ \boldsymbol{B} \circ \boldsymbol{L}$ is maximally monotone. We can therefore invoke Lemma 2.5(i) to obtain

$$\operatorname{int} \boldsymbol{L}^*(\operatorname{ran} \boldsymbol{B}) \subset \operatorname{ran}\,(\boldsymbol{L}^* \circ \boldsymbol{B} \circ \boldsymbol{L}). \tag{4.17}$$

In view of (4.15), this proves (vii). Similarly, Lemma 2.5(ii) guarantees that

$$\boldsymbol{L}^*(\operatorname{ran} \boldsymbol{B}) \subset \overline{\operatorname{ran}}\,(\boldsymbol{L}^* \circ \boldsymbol{B} \circ \boldsymbol{L}) \tag{4.18}$$

and, using (4.15), we arrive at (viii).

(ix): Using (vii), we obtain

$$\begin{aligned} \operatorname{int} \operatorname{dom} L \blacklozenge B &= \operatorname{int} \operatorname{ran} L \diamond B^{-1} \\ &= \operatorname{int} \big(\operatorname{ran}\,(\operatorname{Id}_{\mathcal{H}} - L^* \circ L) + L^*(\operatorname{ran} B^{-1})\big) \\ &= \operatorname{int} \big(\operatorname{ran}\,(\operatorname{Id}_{\mathcal{H}} - L^* \circ L) + L^*(\operatorname{dom} B)\big). \end{aligned} \tag{4.19}$$

(x): Using (viii), we obtain

$$\begin{aligned} \overline{\operatorname{dom}}\, L \blacklozenge B &= \overline{\operatorname{ran}}\, L \diamond B^{-1} \\ &= \overline{\operatorname{ran}\,(\operatorname{Id}_{\mathcal{H}} - L^* \circ L) + L^*(\operatorname{ran} B^{-1})} \\ &= \overline{\operatorname{ran}\,(\operatorname{Id}_{\mathcal{H}} - L^* \circ L) + L^*(\operatorname{dom} B)}. \end{aligned} \tag{4.20}$$

(xi): Using (v), we obtain $\operatorname{int} \operatorname{ran} L \blacklozenge B = \operatorname{int} \operatorname{dom} L \diamond B^{-1} = \operatorname{int} L^*(\operatorname{dom} B^{-1}) = \operatorname{int} L^*(\operatorname{ran} B)$.

(xii): Using (vi), we obtain $\overline{\operatorname{ran}}\, L \blacklozenge B = \overline{\operatorname{dom}}\, L \diamond B^{-1} = \overline{L^*(\operatorname{dom} B^{-1})} = \overline{L^*(\operatorname{ran} B)}$. □

Corollary 4.6 *Suppose that $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ satisfies $\|L\| \leqslant 1$ and let $B : \mathcal{G} \to 2^{\mathcal{G}}$ be maximally monotone. Then the following hold:*

(i) *Suppose that $L^*(\text{dom } B) = \mathcal{H}$. Then $\text{dom } (L \diamond B) = \mathcal{H}$.*
(ii) *Suppose that $\text{ran } (\text{Id}_{\mathcal{H}} - L^* \circ L) + L^*(\text{ran } B) = \mathcal{H}$. Then $L \diamond B$ is surjective.*
(iii) *Suppose that $\text{ran } (\text{Id}_{\mathcal{H}} - L^* \circ L) + L^*(\text{dom } B) = \mathcal{H}$. Then $\text{dom } (L \blacklozenge B) = \mathcal{H}$.*
(iv) *Suppose that $L^*(\text{ran } B) = \mathcal{H}$. Then $L \blacklozenge B$ is surjective.*

Proof We deduce (i) from Theorem 4.5(v), (ii) from Theorem 4.5(vii), (iii) from Theorem 4.5(ix), and (iv) from Theorem 4.5(xi). □

Example 4.7 Going back to Example 3.5, let $B : \mathcal{H} \to 2^{\mathcal{H}}$ be maximally monotone and suppose that $V \neq \{0\}$ is a closed vector subspace of $\mathcal{H}$ such that $(\forall v \in V)\ (v + V^{\perp}) \cap \text{ran } B \neq \varnothing$. Then $\text{proj}_V \diamond B$ is surjective.

Proof Set $L = \text{proj}_V$. Then $\|L\| = 1$ and $\text{ran } (\text{Id}_{\mathcal{H}} - L^* \circ L) = \text{ran } (\text{Id}_{\mathcal{H}} - \text{proj}_V) = V^{\perp}$. On the other hand, $(\forall v \in V)(\exists x^* \in \text{ran } B)\, x^* \in v + V^{\perp} = \text{proj}_V^{-1} v$. Therefore $L^*(\text{ran } B) = \text{proj}_V(\text{ran } B) = V$. Thus, $\text{ran } (\text{Id}_{\mathcal{H}} - L^* \circ L) + L^*(\text{ran } B) = V + V^{\perp} = \mathcal{H}$ and the result follows from Corollary 4.6(ii). □

Proposition 4.8 *Suppose that $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$, let $\beta \in \left]0 + \infty\right[$, let D be a nonempty subset of $\mathcal{G}$, let $B : D \to \mathcal{G}$ be $\beta-$cocoercive, suppose that $0 < \|L\| < \sqrt{\beta + 1}$, and set $\alpha = (\beta + 1)\|L\|^{-2} - 1$. Then $L \blacklozenge B$ is $\alpha-$cocoercive.*

Proof Since B^{-1} is β-strongly monotone, Lemma 2.3(v) entails that $J_{B^{-1}}$ is $(\beta + 1)$-cocoercive. In turn, by Lemma 2.1(i), $L^* \circ J_{B^{-1}} \circ L$ is $(\beta + 1)\|L\|^{-2}$-cocoercive, which makes $(L^* \circ J_{B^{-1}} \circ L)^{-1}$ a $(\beta + 1)\|L\|^{-2}$-strongly monotone operator. In view of Proposition 4.1(iv) and Proposition 4.1(i), we conclude that

$$(L \blacklozenge B)^{-1} = L \diamond B^{-1} = \big(L^* \circ J_{B^{-1}} \circ L\big)^{-1} - \text{Id}_{\mathcal{H}} \tag{4.21}$$

is $\alpha-$strongly monotone and hence that $L \blacklozenge B$ is $\alpha-$cocoercive. □

Proposition 4.9 *Let $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ be such that $\|L\| \leqslant 1$, let D be a nonempty subset of $\mathcal{G}$, and let $B : D \to \mathcal{G}$ be monotone and nonexpansive. Then $L \blacklozenge B$ is monotone and nonexpansive.*

Proof The monotonicity of $L \blacklozenge B$ is established in Proposition 4.4(ii). Let us show its nonexpansiveness. Since B is nonexpansive, it follows from [9, Proposition 4.4] and Lemma 2.2(ii) that there exists a monotone operator $E : \mathcal{G} \to 2^{\mathcal{G}}$ such that $B = 2J_E - \text{Id}_{\mathcal{G}}$. Now set $M = \text{Id}_{\mathcal{H}} - L^* \circ L + L^* \circ E \circ L$. Since $\|L\| \leqslant 1$, $\text{Id}_{\mathcal{H}} - L^* \circ L$ is monotone, while $L^* \circ E \circ L$ is monotone by [9, Proposition 20.10]. The sum M of these two operators is therefore monotone, which renders J_M firmly nonexpansive by Lemma 2.2(ii), and hence $2J_M - \text{Id}_{\mathcal{H}}$ nonexpansive. On the other hand, Proposition 4.1(v) yields

$$\begin{aligned} J_{L \blacklozenge B} &= \text{Id}_{\mathcal{H}} - L^* \circ L + L^* \circ (B + \text{Id}_{\mathcal{G}})^{-1} \circ L \\ &= \text{Id}_{\mathcal{H}} - L^* \circ L + L^* \circ (2J_E)^{-1} \circ L \\ &= \big(2\,\text{Id}_{\mathcal{H}} - 2L^* \circ L + L^* \circ (E + \text{Id}_{\mathcal{G}}) \circ L\big) \circ (\text{Id}_{\mathcal{H}}/2) \\ &= (\text{Id}_{\mathcal{H}} + M) \circ (\text{Id}_{\mathcal{H}}/2) \\ &= (2J_M)^{-1}. \end{aligned} \tag{4.22}$$

We have thus verified that $L \blacklozenge B = 2J_M - \text{Id}_{\mathcal{H}}$ is nonexpansive. □

Remark 4.10 (resolvent average) Consider the setting of Example 1.3, where $\sum_{k=1}^{p} \omega_k = 1$, and let A be the resolvent average of the operators $(B_k)_{1\leqslant k\leqslant p}$ defined in (1.5). Then, as discussed in Example 1.3, Remark 4.2(ii), and Proposition 4.1(iii), $A = L \diamond B = L \blacklozenge B$, where $L\colon x \mapsto (x, \ldots, x)$ is an isometry with adjoint $L^*\colon (y_k)_{1\leqslant k\leqslant p} \mapsto \sum_{k=1}^{p} \omega_k y_k$ and $B\colon (y_k)_{1\leqslant k\leqslant p} \mapsto B_1 y_1 \times \cdots \times B_p y_p$. We can therefore establish at once from the above results various properties of the resolvent average, such as the following:

(i) Proposition 4.1(iv) yields $A^{-1} = L \diamond B^{-1} = (\sum_{k=1}^{p} \omega_k (B_k^{-1} + \mathrm{Id}_{\mathcal{H}})^{-1})^{-1} - \mathrm{Id}_{\mathcal{H}}$ (see [4, Theorem 2.2]).

(ii) Suppose that the operators $(B_k)_{1\leqslant k\leqslant p}$ are monotone. Then Theorem 4.5(i) asserts that A is maximally monotone if the operators $(B_k)_{1\leqslant k\leqslant p}$ are. In addition, Proposition 4.1(viii) asserts that $\operatorname{dom} A \subset \sum_{k=1}^{p} \omega_k \operatorname{dom} B_k$ and Proposition 4.1(xi) that $\operatorname{ran} A \subset \sum_{k=1}^{p} \omega_k \operatorname{ran} B_k$ (see [4, Proposition 2.7] and note that maximality is not required in the last two properties).

(iii) Suppose that the operators $(B_k)_{1\leqslant k\leqslant p}$ are maximally monotone. Then Theorem 4.5(v)–(vi) yields $\operatorname{int} \operatorname{dom} A = \operatorname{int} \sum_{k=1}^{p} \omega_k \operatorname{dom} B_k$ and $\overline{\operatorname{dom}} A = \overline{\sum_{k=1}^{p} \omega_k \operatorname{dom} B_k}$, while Theorem 4.5(xi)–(xii) yields $\operatorname{int} \operatorname{ran} A = \operatorname{int} \sum_{k=1}^{p} \omega_k \operatorname{ran} B_k$, and $\overline{\operatorname{ran}} A = \overline{\sum_{k=1}^{p} \omega_k \operatorname{ran} B_k}$ (see [4, Theorem 2.11]).

(iv) Suppose that the operators $(B_k)_{1\leqslant k\leqslant p}$ are maximally monotone and strongly monotone. Then it follows from Proposition 4.4(iii)(b) that A is strongly monotone (see [4, Theorem 3.20], where the strong monotonicity of A is established under the more general assumption that only one of the operators $(B_k)_{1\leqslant k\leqslant p}$ is strongly monotone).

(v) Suppose that, for every $k \in \{1, \ldots, p\}$, $B_k\colon \mathcal{G}_k \to \mathcal{G}_k$ is monotone and nonexpansive. Then it follows from Proposition 4.9 that A is monotone and nonexpansive (see [4, Theorem 4.16]).

Remark 4.11 (parametrization) A parameter $\gamma \in \left]0, +\infty\right[$ can be introduced in Definition 1.1 by putting

$$L \overset{\gamma}{\diamond} B = L^* \triangleright (B + \gamma^{-1} \mathrm{Id}_{\mathcal{G}}) - \gamma^{-1} \mathrm{Id}_{\mathcal{H}}. \tag{4.23}$$

In the special case of the resolvent average discussed in Example 1.3, (4.23) leads to the parametrized version of (1.5) considered in [4], namely $L \overset{\gamma}{\diamond} B = (\sum_{k=1}^{p} \omega_k (B_k + \gamma^{-1} \mathrm{Id}_{\mathcal{H}})^{-1})^{-1} - \gamma^{-1} \mathrm{Id}_{\mathcal{H}}$. In general, with the assistance of Lemma 2.3(i) and Proposition 1.2, we obtain

$$J_{\gamma(L \overset{\gamma}{\diamond} B)} = L^* \circ J_{\gamma B} \circ L = J_{L \diamond (\gamma B)}. \tag{4.24}$$

This shows that the parametrized version (4.23) is closely related to the original one (1.4) since $\gamma(L \overset{\gamma}{\diamond} B) = L \diamond (\gamma B)$. The proximal composition of Definition 1.4 can be parametrized similarly by putting $L \overset{\gamma}{\diamond} g = ((g^* \,\square\, (\gamma \mathscr{Q}_{\mathcal{G}})) \circ L)^* - \gamma^{-1} \mathscr{Q}_{\mathcal{H}}$.

Remark 4.12 (warping) An extension of Definition 1.1 can be devised using the theory of warped resolvents [20]. Let $\mathcal{X}$ and $\mathcal{Y}$ be reflexive real Banach spaces, let $K_{\mathcal{Y}}\colon \mathcal{Y} \supset D_{\mathcal{Y}} \to \mathcal{Y}^*$, let $L \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, and let $B : \mathcal{Y} \to 2^{\mathcal{Y}^*}$. Then, under suitable conditions, the warped resolvent of B with kernel $K_{\mathcal{Y}}$ is $J_B^{K_{\mathcal{Y}}} = (B + K_{\mathcal{Y}})^{-1} \circ K_{\mathcal{Y}}$ (for instance, if $h\colon \mathcal{Y} \to \left]-\infty, +\infty\right]$ is a Legendre function such that $\operatorname{dom} B \subset \operatorname{int} \operatorname{dom} h$ and $K_{\mathcal{Y}} = \nabla h$, then $J_B^{K_{\mathcal{Y}}}$ is the D-resolvent of B [6]). For a suitable kernel $K_{\mathcal{X}}\colon \mathcal{X} \supset D_{\mathcal{X}} \to \mathcal{X}^*$, we then define the *warped resolvent composition* $L \diamond B = K_{\mathcal{X}} \circ (L^* \triangleright (K_{\mathcal{Y}}^{-1} \circ (B + K_{\mathcal{Y}}))) - K_{\mathcal{X}}$, which yields $J_{L \diamond B}^{K_{\mathcal{X}}} = L^* \circ J_B^{K_{\mathcal{Y}}} \circ L$.

5 The Proximal Composition

This section is dedicated to the study of some aspects of the proximal composition operations introduced in Definition 1.4 and further discussed in Examples 3.6 and 3.10.

Remark 5.1 The proximal composition was linked to the resolvent composition in Example 3.6(v). We can also motivate this construction via Moreau's theory of proximity operators and envelopes [42–44]. Indeed, let $g \in \Gamma_0(\mathcal{G})$, suppose that $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ satisfies $0 < \|L\| \leqslant 1$, and set $T = L^* \circ \operatorname{prox}_g \circ L$. Then T is nonexpansive since prox_g and L are. On the other hand, we infer from Lemma 2.9(iv) that $T = L^* \circ \nabla(g^* \,\square\, \mathscr{Q}_\mathcal{G}) \circ L = \nabla((g^* \,\square\, \mathscr{Q}_\mathcal{G}) \circ L)$. Altogether, Lemma 2.8 implies that $T = \operatorname{prox}_f$, where $f = ((g^* \,\square\, \mathscr{Q}_\mathcal{G}) \circ L)^* - \mathscr{Q}_\mathcal{H}$. The function f is precisely the proximal composition $L \diamond g$. Thus, up to an additive constant, $L \diamond g$ is the function the proximity operator of which is $L^* \circ \operatorname{prox}_g \circ L$.

Let us now establish some properties of proximal compositions.

Proposition 5.2 *Let $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$, let $g\colon \mathcal{G} \to \left]-\infty, +\infty\right]$ and $h\colon \mathcal{G} \to \left]-\infty, +\infty\right]$ be proper functions such that $h \leqslant g$, and let $\breve{g}$ be the largest lower semicontinuous convex function majorized by g. Then the following hold:*

(i) $L \diamond h \leqslant L \diamond g$.
(ii) *Suppose that $h \geqslant \breve{g}$ and that g admits a continuous affine minorant. Then $L \diamond g = L \diamond h$.*
(iii) *Suppose that g admits a continuous affine minorant. Then $L \diamond g = L \diamond g^{**}$.*

Proof (i): In view of (2.7) and (2.8), $h^* \geqslant g^*$ and hence $h^* \,\square\, \mathscr{Q}_\mathcal{G} \geqslant g^* \,\square\, \mathscr{Q}_\mathcal{G}$. Thus, $(h^* \,\square\, \mathscr{Q}_\mathcal{G}) \circ L \geqslant (g^* \,\square\, \mathscr{Q}_\mathcal{G}) \circ L$ and therefore $((h^* \,\square\, \mathscr{Q}_\mathcal{G}) \circ L)^* \leqslant ((g^* \,\square\, \mathscr{Q}_\mathcal{G}) \circ L)^*$. Appealing to (1.6), we conclude that $L \diamond h \leqslant L \diamond g$.

(ii): Let a be a continuous affine minorant of g. Then $-\infty < a = \breve{a} \leqslant \breve{g} \leqslant g \not\equiv +\infty$ and $\breve{g}$ is therefore proper. In addition, $\breve{g} \leqslant h \leqslant g$. Hence, [9, Proposition 13.16] yields $h^* = g^*$ and the conclusion follows from (1.6).

(iii): Since $\breve{g} = g^{**}$ [9, Proposition 13.45], the assertion follows from (ii). □

Proposition 5.3 *Suppose that $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ satisfies $0 < \|L\| \leqslant 1$ and let $g\colon \mathcal{G} \to \left]-\infty, +\infty\right]$ be a proper function that admits a continuous affine minorant. Then the following hold:*

(i) $L \diamond g = L^* \triangleright (g^{**} + \mathscr{Q}_\mathcal{G}) - \mathscr{Q}_\mathcal{H}$.
(ii) $\operatorname{dom}(L \diamond g) = L^*(\operatorname{dom} g^{**})$.
(iii) $(L \diamond g)^* = (\mathscr{Q}_\mathcal{H} - (g^* \,\square\, \mathscr{Q}_\mathcal{G}) \circ L)^* - \mathscr{Q}_\mathcal{H}$.
(iv) $(L \diamond g)^* = L \blacklozenge g^*$.
(v) $(L \blacklozenge g)^* = L \diamond g^*$.
(vi) $(L \diamond g) \,\square\, \mathscr{Q}_\mathcal{H} + (L \blacklozenge g^*) \,\square\, \mathscr{Q}_\mathcal{H} = \mathscr{Q}_\mathcal{H}$.
(vii) *Suppose that L is an isometry. Then $L \diamond g = L \blacklozenge g$.*

Proof By Example 3.6(i) and Lemma 2.6(i), g^* is in $\Gamma_0(\mathcal{G})$ and it admits a continuous affine minorant. In turn, we deduce from Lemma 2.9(ii) that $g^* \,\square\, \mathscr{Q}_\mathcal{G} \in \Gamma_0(\mathcal{G})$ and hence that $(g^* \,\square\, \mathscr{Q}_\mathcal{G}) \circ L \in \Gamma_0(\mathcal{H})$. We then deduce from Lemma 2.6(ii) that $((g^* \,\square\, \mathscr{Q}_\mathcal{G}) \circ L)^* \in \Gamma_0(\mathcal{H})$.

(i): Since $\operatorname{dom}(g^* \,\square\, \mathscr{Q}_\mathcal{G}) = \mathcal{G}$ and $g^* \,\square\, \mathscr{Q}_\mathcal{G} \in \Gamma_0(\mathcal{G})$, it follows from [9, Corollary 15.28(i)] and Lemma 2.9(iii) that $L \diamond g + \mathscr{Q}_\mathcal{H} = ((g^* \,\square\, \mathscr{Q}_\mathcal{G}) \circ L)^* = L^* \triangleright (g^* \,\square\, \mathscr{Q}_\mathcal{G})^* = L^* \triangleright (g^{**} + \mathscr{Q}_\mathcal{G})$.

(ii): We invoke (i) and [9, Proposition 12.36(i)] to get $\operatorname{dom}(L \diamond g) = \operatorname{dom}(L^* \triangleright (g^{**} + \mathscr{Q}_\mathcal{G})) = L^*(\operatorname{dom}(g^{**} + \mathscr{Q}_\mathcal{G})) = L^*(\operatorname{dom} g^{**})$.

(iii): Since $((g^* \,\square\, \mathscr{Q}_{\mathcal{G}}) \circ L)^* \in \Gamma_0(\mathcal{H})$, it follows from Definition 1.4 and [9, Proposition 13.29] that

$$(L \diamond g)^* = \Big(\big((g^* \,\square\, \mathscr{Q}_{\mathcal{G}}) \circ L\big)^* - \mathscr{Q}_{\mathcal{H}}\Big)^*$$
$$= \Big(\mathscr{Q}_{\mathcal{H}} - \big((g^* \,\square\, \mathscr{Q}_{\mathcal{G}}) \circ L\big)^{**}\Big)^* - \mathscr{Q}_{\mathcal{H}}$$
$$= \big(\mathscr{Q}_{\mathcal{H}} - (g^* \,\square\, \mathscr{Q}_{\mathcal{G}}) \circ L\big)^* - \mathscr{Q}_{\mathcal{H}}. \tag{5.1}$$

(iv): Proposition 5.2(iii) yields $L \blacklozenge g^* = (L \diamond g^{**})^* = (L \diamond g)^*$.

(v): Example 3.6(i)–(ii) implies that $L \diamond g^* \in \Gamma_0(\mathcal{G})$. In turn, Lemma 2.6(ii) yields $(L \blacklozenge g)^* = (L \diamond g^*)^{**} = L \diamond g^*$.

(vi): Combine Example 3.6(ii), Lemma 2.9(vi), and (iv).

(vii): Since $\mathscr{Q}_{\mathcal{H}} = \mathscr{Q}_{\mathcal{G}} \circ L$, we derive from Lemma 2.9(vi) and (iii) that

$$L \diamond g = \big((g^* \,\square\, \mathscr{Q}_{\mathcal{G}}) \circ L\big)^* - \mathscr{Q}_{\mathcal{H}},$$
$$= \big((\mathscr{Q}_{\mathcal{G}} - g^{**} \,\square\, \mathscr{Q}_{\mathcal{G}}) \circ L\big)^* - \mathscr{Q}_{\mathcal{H}}$$
$$= \big(\mathscr{Q}_{\mathcal{H}} - (g^{**} \,\square\, \mathscr{Q}_{\mathcal{G}}) \circ L\big)^* - \mathscr{Q}_{\mathcal{H}}$$
$$= (L \diamond g^*)^*$$
$$= L \blacklozenge g, \tag{5.2}$$

as claimed. □

The next result concerns the case when L is an isometry.

Proposition 5.4 *Suppose that $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ is an isometry and let $g : \mathcal{G} \to \left]-\infty, +\infty\right]$ be a proper function that admits a continuous affine minorant. Then $(g^* \circ L)^* \leqslant L \diamond g \leqslant g \circ L$.*

Proof We recall from Example 3.6(ii) that $L \diamond g \in \Gamma_0(\mathcal{H})$. Fix $x \in \mathcal{H}$ and recall that $g^{**} \leqslant g$ [9, Proposition 13.16(i)]. By Proposition 5.3(i),

$$(L \diamond g)(x) = \inf_{\substack{y \in \mathcal{G} \\ L^* y = x}} \big(g^{**}(y) + \mathscr{Q}_{\mathcal{G}}(y)\big) - \mathscr{Q}_{\mathcal{H}}(x) \leqslant \inf_{\substack{y \in \mathcal{G} \\ L^* y = x}} \big(g(y) + \mathscr{Q}_{\mathcal{G}}(y)\big) - \mathscr{Q}_{\mathcal{H}}(x). \tag{5.3}$$

Now set $y = Lx$. Then $L^* y = L^*(Lx) = x$ and $\mathscr{Q}_{\mathcal{G}}(Lx) = \mathscr{Q}_{\mathcal{H}}(x)$. Therefore, (5.3) yields

$$(L \diamond g)(x) \leqslant g(Lx) + \mathscr{Q}_{\mathcal{G}}(Lx) - \mathscr{Q}_{\mathcal{H}}(x) = (g \circ L)(x), \tag{5.4}$$

which provides the second inequality. To prove the first one, we recall from Example 3.6(i) that $g^* \in \Gamma_0(\mathcal{G})$. Therefore, g^* admits a continuous affine minorant by Lemma 2.6(i). In turn, (5.4) yields $L \diamond g^* \leqslant g^* \circ L$ and hence $(L \diamond g^*)^* \geqslant (g^* \circ L)^*$. We then invoke successively Proposition 5.2(i), Proposition 5.3(iv), and Proposition 5.3(vii) to obtain

$$L \diamond g \geqslant L \diamond g^{**} = \big(L \diamond g^*\big)^* \geqslant \big(g^* \circ L\big)^*, \tag{5.5}$$

as announced. □

Let us take a closer look at the proximal composition for functions in $\Gamma_0(\mathcal{G})$.

Theorem 5.5 *Suppose that $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ satisfies $0 < \|L\| \leqslant 1$ and let $g \in \Gamma_0(\mathcal{G})$. Then the following hold:*

(i) $L \diamond g = L^* \triangleright (g + \mathscr{Q}_{\mathcal{G}}) - \mathscr{Q}_{\mathcal{H}}$.
(ii) $\operatorname{dom} (L \diamond g) = L^*(\operatorname{dom} g)$.

(iii) $\operatorname{Argmin}(L \diamond g) = \operatorname{Fix}(L^* \circ \operatorname{prox}_g \circ L)$.
(iv) $(L \diamond g) \,\square\, \mathscr{Q}_{\mathcal{H}} = \mathscr{Q}_{\mathcal{H}} - (g^* \,\square\, \mathscr{Q}_{\mathcal{G}}) \circ L$.
(v) $(L \blacklozenge g) \,\square\, \mathscr{Q}_{\mathcal{H}} = (g \,\square\, \mathscr{Q}_{\mathcal{G}}) \circ L$.
(vi) $L^{-1}(\operatorname{Argmin} g) \subset \operatorname{Argmin}(L \blacklozenge g) = \operatorname{Argmin}((g \,\square\, \mathscr{Q}_{\mathcal{G}}) \circ L)$.

Proof We recall from Lemma 2.6(i) that g admits a continuous affine minorant and from Example 3.6(ii) that $L \diamond g \in \Gamma_0(\mathcal{H})$.

(i)–(ii): These follow from Proposition 5.3(i)–(ii) and Lemma 2.6(ii).

(iii): Example 3.6(vi) and (2.15) yield $\operatorname{Argmin}(L \diamond g) = \operatorname{Fix} \operatorname{prox}_{L \diamond g} = \operatorname{Fix}(L^* \circ \operatorname{prox}_g \circ L)$.

(iv): It follows from (i) that $(L \diamond g) + \mathscr{Q}_{\mathcal{H}} = L^* \rhd (g + \mathscr{Q}_{\mathcal{G}})$. Therefore, using Example 3.6(ii), Lemma 2.9(iii), and [9, Proposition 13.24(iv)], we derive that

$$(L \diamond g)^* \,\square\, \mathscr{Q}_{\mathcal{H}} = \big((L \diamond g) + \mathscr{Q}_{\mathcal{H}}\big)^* = \big(L^* \rhd (g + \mathscr{Q}_{\mathcal{G}})\big)^* = (g + \mathscr{Q}_{\mathcal{G}})^* \circ L = (g^* \,\square\, \mathscr{Q}_{\mathcal{G}}) \circ L. \tag{5.6}$$

Hence, it follows from Lemma 2.9(vi) that

$$(L \diamond g) \,\square\, \mathscr{Q}_{\mathcal{H}} + (g^* \,\square\, \mathscr{Q}_{\mathcal{G}}) \circ L = (L \diamond g) \,\square\, \mathscr{Q}_{\mathcal{H}} + (L \diamond g)^* \,\square\, \mathscr{Q}_{\mathcal{H}} = \mathscr{Q}_{\mathcal{H}}. \tag{5.7}$$

(v): We use Example 3.10(i), Lemma 2.9(vi), Proposition 5.3(v), (iv), and Lemma 2.6(ii) to obtain

$$\begin{aligned}(L \blacklozenge g) \,\square\, \mathscr{Q}_{\mathcal{H}} &= \mathscr{Q}_{\mathcal{H}} - (L \blacklozenge g)^* \,\square\, \mathscr{Q}_{\mathcal{H}} \\ &= \mathscr{Q}_{\mathcal{H}} - (L \diamond g^*) \,\square\, \mathscr{Q}_{\mathcal{H}} \\ &= \mathscr{Q}_{\mathcal{H}} - \big(\mathscr{Q}_{\mathcal{H}} - (g^{**} \,\square\, \mathscr{Q}_{\mathcal{G}}) \circ L\big) \\ &= (g \,\square\, \mathscr{Q}_{\mathcal{G}}) \circ L. \end{aligned} \tag{5.8}$$

(vi): We derive from (2.13), Proposition 4.1(xiii) with $B = \partial g$, and Example 3.10(iv) that

$$L^{-1}(\operatorname{Argmin} g) = L^{-1}(\operatorname{zer} \partial g) \subset \operatorname{zer}(L \blacklozenge \partial g) = \operatorname{zer} \partial (L \blacklozenge g) = \operatorname{Argmin}(L \blacklozenge g). \tag{5.9}$$

Next, since $L \blacklozenge g \in \Gamma_0(\mathcal{H})$ by Example 3.10(i), [9, Proposition 17.5] and (v) yield $\operatorname{Argmin}(L \blacklozenge g) = \operatorname{Argmin}((L \blacklozenge g) \,\square\, \mathscr{Q}_{\mathcal{H}}) = \operatorname{Argmin}((g \,\square\, \mathscr{Q}_{\mathcal{G}}) \circ L)$. ◻

Proposition 5.6 *Suppose that $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$, satisfies $0 < \|L\| \leqslant 1$, let $\alpha \in [0, +\infty[$, let $g \in \Gamma_0(\mathcal{G})$ be such that $g - \alpha \mathscr{Q}_{\mathcal{G}}$ is convex, and set $\beta = (\alpha + 1)\|L\|^{-2} - 1$. Suppose that one of the following is satisfied:*

(i) *$\alpha > 0$, i.e., g is α-strongly convex.*
(ii) $\|L\| < 1$.

Then $L \diamond g$ is β-strongly convex.

Proof By assumption, $g - \alpha \mathscr{Q}_{\mathcal{G}} \in \Gamma_0(\mathcal{G})$ and hence, by Lemma 2.9(i), $\partial(g - \alpha \mathscr{Q}_{\mathcal{G}})$ is maximally monotone. However, by Lemma 2.9(viii),

$$\partial g = \partial\big((g - \alpha \mathscr{Q}_{\mathcal{G}}) + \alpha \mathscr{Q}_{\mathcal{G}}\big) = \partial(g - \alpha \mathscr{Q}_{\mathcal{G}}) + \alpha \operatorname{Id}_{\mathcal{G}} \tag{5.10}$$

and therefore $\partial g - \alpha \operatorname{Id}_{\mathcal{G}} = \partial(g - \alpha \mathscr{Q}_{\mathcal{G}})$ is monotone. Moreover, by [55, Remark 3.5.3], ∂g is α-strongly monotone in (i). Altogether, it follows from Example 3.6(v) and Proposition 4.4(iii) that $\partial(L \diamond g) = L \diamond \partial g$ is β-strongly monotone. Appealing to [55, Remark 3.5.3] again, we conclude that $L \diamond g$ is β-strongly convex. ◻

Proposition 5.7 *Suppose that $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ satisfies $0 < \|L\| \leqslant 1$, let $\alpha \in]0,+\infty[$, let $g\colon \mathcal{G} \to \mathbb{R}$ be convex and differentiable, with a α^{-1}-Lipschitzian gradient, and set $\beta = (\alpha+1)\|L\|^{-2} - 1$. Then $L \blacklozenge g$ is differentiable on $\mathcal{H}$ and its gradient is β^{-1}-Lipschitzian.*

Proof We derive from Lemma 2.7 that g^* is α-strongly convex. In turn, Proposition 5.3(v) and Proposition 5.6(ii) imply that $(L \blacklozenge g)^* = L \diamond g^*$ is β-strongly convex. Invoking Lemma 2.7 once more, we obtain the assertion. □

The remainder of this section is devoted to examples of proximal compositions.

Example 5.8 (linear projection) Let V be a closed vector subspace of $\mathcal{H}$ and let $g\colon \mathcal{H} \to]-\infty,+\infty]$ be a proper function that admits a continuous affine minorant. Then $\mathrm{proj}_V \diamond g = \iota_V + \big(g^* + d_V^2/2\big)^*$.

Proof Let $x \in \mathcal{H}$. By Proposition 5.3(ii), $\mathrm{dom}\,(\mathrm{proj}_V \diamond g) = \mathrm{proj}_V(\mathrm{dom}\,g^{**}) \subset V$. Therefore, if $x \notin V$, then $(\mathrm{proj}_V \diamond g)(x) = +\infty$. Now suppose that $x \in V$ and note that, by Pythagoras' identity, $(\forall v \in V^\perp)\ \mathscr{Q}_{\mathcal{H}}(x-v) = \mathscr{Q}_{\mathcal{H}}(x) + \mathscr{Q}_{\mathcal{H}}(v)$. Hence, using Proposition 5.3(i) and basic conjugation calculus [9, Chapter 13], we get

$$
\begin{aligned}
\big(\mathrm{proj}_V \diamond g\big)(x) &= \min_{\substack{y\in\mathcal{H}\\ \mathrm{proj}_V y = x}} \; g^{**}(y) + \mathscr{Q}_{\mathcal{H}}(y) - \mathscr{Q}_{\mathcal{H}}(x)\\
&= \min_{y \in x+V^\perp} g^{**}(y) + \mathscr{Q}_{\mathcal{H}}(y) - \mathscr{Q}_{\mathcal{H}}(x)\\
&= \min_{v\in V^\perp} g^{**}(x-v) + \mathscr{Q}_{\mathcal{H}}(x-v) - \mathscr{Q}_{\mathcal{H}}(x)\\
&= \min_{v\in\mathcal{H}} g^{**}(x-v) + \iota_{V^\perp}(v) + \mathscr{Q}_{\mathcal{H}}(v)\\
&= \big(g^{**} \,\square\, (\iota_{V^\perp} + \mathscr{Q}_{\mathcal{H}})\big)(x)\\
&= \big((g^*)^* \,\square\, (d_V^2/2)^*\big)(x)\\
&= \big(g^* + d_V^2/2\big)^*(x),
\end{aligned}
\tag{5.11}
$$

which establishes the identity. □

Example 5.9 (proximal mixture) Let $0 \neq p \in \mathbb{N}$ and, for every $k \in \{1,\dots,p\}$, let $\mathcal{G}_k$ be a real Hilbert space, let $L_k \in \mathcal{B}(\mathcal{H},\mathcal{G}_k)$, let $\omega_k \in]0,+\infty[$, and let $g_k \in \Gamma_0(\mathcal{G}_k)$. Suppose that $0 < \sum_{k=1}^p \omega_k\|L_k\|^2 \leqslant 1$ and let $\mathcal{G}$ be the standard product vector space $\mathcal{G}_1 \times \cdots \times \mathcal{G}_p$, with generic element $\boldsymbol{y} = (y_k)_{1\leqslant k\leqslant p}$, and equipped with the scalar product $(\boldsymbol{y},\boldsymbol{y}') \mapsto \sum_{k=1}^p \omega_k\langle y_k \mid y_k'\rangle$. Set $L\colon \mathcal{H} \to \mathcal{G}\colon x \mapsto (L_k x)_{1\leqslant k\leqslant p}$ and $g\colon \mathcal{G} \to]-\infty,+\infty]\colon \boldsymbol{y} \mapsto \sum_{k=1}^p \omega_k g_k(y_k)$. Then $\mathscr{Q}_{\mathcal{G}}\colon \mathcal{G} \to \mathbb{R}\colon \boldsymbol{y} \mapsto \sum_{k=1}^p \omega_k \mathscr{Q}_{\mathcal{G}_k}(y_k)$, $L^*\colon \mathcal{G} \to \mathcal{H}\colon \boldsymbol{y} \mapsto \sum_{k=1}^p \omega_k L_k^* y_k$, $\mathrm{prox}_g\colon \mathcal{G} \to \mathcal{G}\colon \boldsymbol{y} \mapsto (\mathrm{prox}_{g_k} y_k)_{1\leqslant k\leqslant p}$, and $g^*\colon \mathcal{G} \to]-\infty,+\infty]\colon \boldsymbol{y}^* \mapsto \sum_{k=1}^p \omega_k g_k^*(y_k^*)$. Thus, $g \in \Gamma_0(\mathcal{G})$, $0 < \|L\| \leqslant 1$, (1.6) produces the *proximal mixture*

$$
L \diamond g = \left(\sum_{k=1}^p \omega_k \big(g_k^* \,\square\, \mathscr{Q}_{\mathcal{G}_k}\big) \circ L_k\right)^* - \mathscr{Q}_{\mathcal{H}},
\tag{5.12}
$$

and Example 3.6 yields

$$
L \diamond g \in \Gamma_0(\mathcal{H}) \quad \text{and} \quad \mathrm{prox}_{L\diamond g} = \sum_{k=1}^m \omega_k L_k^* \circ \mathrm{prox}_{g_k} \circ L_k.
\tag{5.13}
$$

In particular if, for every $k \in \{1, \ldots, p\}$, $\mathcal{G}_k = \mathcal{H}$ and $L_k = \mathrm{Id}_{\mathcal{H}}$, then (5.12) is the *proximal average*

$$L \diamond g = \left(\sum_{k=1}^{p} \omega_k \big(g_k^* \,\square\, \mathscr{Q}_{\mathcal{H}} \big) \right)^* - \mathscr{Q}_{\mathcal{H}}, \tag{5.14}$$

which has been studied in [10] (see also [39] for illustrations and numerical aspects). The fact that $\sum_{k=1}^{m} \omega_k \mathrm{prox}_{g_k}$ is a proximity operator was first observed by Moreau [43, 44] as a consequence of Lemma 2.8.

Remark 5.10 (proximal sum) In Example 5.9, if $\sum_{k=1}^{p} \omega_k \|L_k\|^2 > 1$, the proximal mixture (5.12) may not be a function in $\Gamma_0(\mathcal{H})$. In the case of (5.14) with $p = 2$ and $\omega_1 = \omega_2 = 1$, conditions under which the *proximal sum* $L \diamond g = (g_1^* \,\square\, \mathscr{Q}_{\mathcal{H}} + g_2^* \,\square\, \mathscr{Q}_{\mathcal{H}})^* - \mathscr{Q}_{\mathcal{H}}$ is in $\Gamma_0(\mathcal{H})$ are provided in [7, 28, 56].

Remark 5.11 (proximal average) As in Remark 4.10, we can specialize the above results to establish in a straightforward fashion various properties of the proximal average (5.14). In this context, we define $\mathcal{G}$ and g as in Example 5.9 with $\mathcal{G}_1 = \cdots = \mathcal{G}_p = \mathcal{H}$ and $\sum_{k=1}^{p} \omega_k = 1$, and set $L\colon \mathcal{H} \to \mathcal{G}\colon x \mapsto (x, \ldots, x)$. Then L is an isometry and the resulting proximal average $f = L \diamond g = L \blacklozenge g$ of (5.14) (see Proposition 5.3(vii)) possesses in particular the following properties:

(i) Example 3.6(ii) yields $f \in \Gamma_0(\mathcal{H})$ (see [10, Corollary 5.2]).
(ii) Example 3.6(vi) yields $\mathrm{prox}_f = \sum_{k=1}^{p} \omega_k \,\mathrm{prox}_{g_k}$ (see [10, Theorem 6.7]).
(iii) Proposition 5.3(v) yields $f^* = (\sum_{k=1}^{p} \omega_k (g_k \,\square\, \mathscr{Q}_{\mathcal{H}}))^* - \mathscr{Q}_{\mathcal{H}}$ (see [10, Theorem 5.1]).
(iv) Proposition 5.4 yields $(\sum_{k=1}^{p} \omega_k g_k^*)^* \leqslant f \leqslant \sum_{k=1}^{p} \omega_k g_k$ (see [10, Theorem 5.4]).
(v) Theorem 5.5(ii) yields $\mathrm{dom}\, f = \sum_{k=1}^{p} \omega_k \mathrm{dom}\, g_k$ (see [10, Theorem 4.6]).
(vi) Theorem 5.5(v) yields $f \,\square\, \mathscr{Q}_{\mathcal{H}} = \sum_{k=1}^{p} \omega_k (g_k \,\square\, \mathscr{Q}_{\mathcal{H}})$ (see [10, Theorem 6.2(i)]).
(vii) Theorem 5.5(vi) yields $\mathrm{Argmin}(f \,\square\, \mathscr{Q}_{\mathcal{H}}) = \mathrm{Argmin} \sum_{k=1}^{p} \omega_k (g_k \,\square\, \mathscr{Q}_{\mathcal{H}})$ (see [10, Corollary 6.4]).
(viii) Suppose that the functions $(g_k)_{1 \leqslant k \leqslant p}$ are strongly convex. Then it follows from Proposition 5.6(i) that f is strongly convex (see [4, Corollary 3.23], where the strong convexity of f is shown to hold more generally under the assumption that one of the functions $(g_k)_{1 \leqslant k \leqslant p}$ is strongly convex).

6 Application to Monotone Inclusion Models

On the numerical side, in monotone inclusion problems, the advantage of the resolvent composition over compositions such as (1.2) or (1.3) is that its resolvent is readily available through Proposition 1.2. Hence, processing it efficiently in an algorithm does not require advanced splitting techniques. In particular, in minimization problems, one deals with monotone operators which are subdifferentials and handling a proximal composition $L \diamond g$ is more straightforward than the compositions $g \circ L$ or $L^* \triangleright g$ thanks to Example 3.6(vi). On the modeling side, while these compositions are not interchangeable in general, replacing the standard composition (1.2) by a resolvent composition, may also be of interest. For instance, in the special case of the basic proximal average (5.14), replacing $g \circ L = \sum_{k=1}^{p} \omega_k g_k$ by $L \diamond g = (\sum_{k=1}^{p} \omega_k (g_k^* \,\square\, \mathscr{Q}_{\mathcal{H}}))^* - \mathscr{Q}_{\mathcal{H}}$ in variational problems has been advocated in [38, 54]. More generally, the computational and modeling benefits of employing resolvent compositions in place of classical ones in concrete applications is a natural topic of investigation, and it will be pursued elsewhere.

The focus of this section is on the use of resolvent and proximal compositions in the context of the following constrained inclusion problem.

Problem 6.1 Suppose that $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ satisfies $0 < \|L\| \leqslant 1$, let $B: \mathcal{G} \to 2^{\mathcal{G}}$ be maximally monotone, and let $V \neq \{0\}$ be a closed vector subspace of $\mathcal{H}$. The task is to

$$\text{find } x \in V \text{ such that } 0 \in B(Lx). \tag{6.1}$$

As will be illustrated in the examples below, (6.1) models a broad spectrum of problems in applied analysis. Of special interest to us are situations in which, due to modeling errors, $L(V) \cap \operatorname{zer} B = \varnothing$, which means that Problem 6.1 has no solution. As a surrogate to it with adequate approximate solutions in such instances, we propose the following formulation. It is based on the resolvent composition and will be seen to be solvable by a simple implementation of the proximal point algorithm.

Problem 6.2 Suppose that $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ satisfies $0 < \|L\| \leqslant 1$, let $B: \mathcal{G} \to 2^{\mathcal{G}}$ be maximally monotone, let $V \neq \{0\}$ be a closed vector subspace of $\mathcal{H}$, let $\gamma \in]0, +\infty[$, and set $A = L \blacklozenge (\gamma B)$. The task is to

$$\text{find } x \in \mathcal{H} \text{ such that } 0 \in \big(\operatorname{proj}_V \diamond A\big)x. \tag{6.2}$$

A justification of the fact that Problem 6.2 is an adequate relaxation of Problem 6.1 is given in item (v) below.

Theorem 6.3 *Consider the settings of Problems 6.1 and 6.2, and let S_1 and S_2 be their respective sets of solutions. Then the following hold:*

(i) $\operatorname{proj}_V \diamond A$ *is maximally monotone.*
(ii) $J_{\operatorname{proj}_V \diamond A} = \operatorname{proj}_V \circ (\mathrm{Id}_{\mathcal{H}} - L^* \circ L + L^* \circ J_{\gamma B} \circ L) \circ \operatorname{proj}_V$.
(iii) S_1 *and* S_2 *are closed convex sets.*
(iv) $S_2 = \operatorname{Fix}(\operatorname{proj}_V \circ (\mathrm{Id}_{\mathcal{H}} - L^* \circ L + L^* \circ J_{\gamma B} \circ L))$.
(v) *Problem 6.2 is an exact relaxation of Problem 6.1 in the sense that* $S_1 \neq \varnothing \Rightarrow S_2 = S_1$.
(vi) $S_2 = \operatorname{zer}(N_V + L^* \circ ({}^{\gamma}B) \circ L)$.

Proof (i): Theorem 4.5(ii) asserts that A is maximally monotone. In view of Theorem 4.5(i), this makes $\operatorname{proj}_V \diamond A$ maximally monotone.

(ii): It follows from Proposition 1.2 and Proposition 4.1(v) that

$$\begin{aligned} J_{\operatorname{proj}_V \diamond A} &= \operatorname{proj}_V \circ J_{L \blacklozenge (\gamma B)} \circ \operatorname{proj}_V \\ &= \operatorname{proj}_V \circ \big(\mathrm{Id}_{\mathcal{H}} - L^* \circ L + L^* \circ J_{\gamma B} \circ L\big) \circ \operatorname{proj}_V. \end{aligned} \tag{6.3}$$

(iii): The maximal monotonicity of B implies that $\operatorname{zer} B$ is closed and convex [9, Proposition 23.39]. Hence, since L is continuous and linear, $L^{-1}(\operatorname{zer} B)$ is closed and convex, and so is therefore $S_1 = V \cap L^{-1}(\operatorname{zer} B)$. Likewise, it follows from (i) that $S_2 = \operatorname{zer}(\operatorname{proj}_V \diamond A)$ is closed and convex.

(iv): It results from Lemma 2.3(iii) and (ii) that

$$S_2 = \operatorname{zer}\big(\operatorname{proj}_V \diamond A\big) = \operatorname{Fix} J_{\operatorname{proj}_V \diamond A} = \operatorname{Fix}\Big(\operatorname{proj}_V \circ \big(\mathrm{Id}_{\mathcal{H}} - L^* \circ L + L^* \circ J_{\gamma B} \circ L\big)\Big). \tag{6.4}$$

(v): Suppose that $\overline{x} \in S_1$ and $x \in S_2$. Then $\overline{x} = \operatorname{proj}_V \overline{x}$ and $0 \in B(L\overline{x})$, i.e., by Lemma 2.3(iii), $L\overline{x} = J_{\gamma B}(L\overline{x})$ and therefore $\overline{x} = (\mathrm{Id}_{\mathcal{H}} - L^* \circ L)\overline{x} + L^*(L\overline{x}) = (\mathrm{Id}_{\mathcal{H}} - L^* \circ L)\overline{x} + L^*(J_{\gamma B}(L\overline{x}))$. Altogether, bringing into play (iv), we get

$$\overline{x} = \operatorname{proj}_V \overline{x} = \operatorname{proj}_V\big((\mathrm{Id}_{\mathcal{H}} - L^* \circ L)\overline{x} + (L^* \circ J_{\gamma B} \circ L)\overline{x}\big) \in S_2. \tag{6.5}$$

It remains to show that $x \in S_1$, i.e., as (iv) yields $x \in V$, that $0 \in B(Lx)$. Since $L\overline{x} \in \operatorname{zer} B$, Lemma 2.3(iii) entails that ${}^{\gamma}B(L\overline{x}) = 0$. Hence,

$$(\forall v \in V) \quad \big\langle v \mid L^*\big({}^{\gamma}B(L\overline{x})\big)\big\rangle = 0. \tag{6.6}$$

On the other hand, we derive from (iv) that

$$x = \operatorname{proj}_V\Big(x - L^*\big((\operatorname{Id}_{\mathcal{H}} - J_{\gamma B})(Lx)\big)\Big) = \big(N_V + \operatorname{Id}_{\mathcal{H}}\big)^{-1}\Big(x - \gamma L^*\big({}^{\gamma}B(Lx)\big)\Big). \tag{6.7}$$

Thus, $-L^*({}^{\gamma}B(Lx)) \in N_V x = V^{\perp}$, i.e.,

$$(\forall v \in V) \quad \big\langle v | L^*\big({}^{\gamma}B(Lx)\big)\big\rangle = 0. \tag{6.8}$$

Since $x - \overline{x} \in V$, we deduce from (6.6) and (6.8) that

$$\big\langle x - \overline{x} \mid L^*\big({}^{\gamma}B(Lx) - {}^{\gamma}B(L\overline{x})\big)\big\rangle = 0. \tag{6.9}$$

Thus,

$$\langle Lx - L\overline{x} \,|\, {}^{\gamma}B(Lx) - {}^{\gamma}B(L\overline{x})\rangle = 0 \tag{6.10}$$

and, since ${}^{\gamma}B$ is γ-cocoercive [9, Corollary 23.11(iii)], we obtain

$$\gamma \|{}^{\gamma}B(Lx)\|^2 = \gamma \|{}^{\gamma}B(Lx) - {}^{\gamma}B(L\overline{x})\|^2 \leqslant \langle Lx - L\overline{x} \,|\, {}^{\gamma}B(Lx) - {}^{\gamma}B(L\overline{x}) = 0. \tag{6.11}$$

We conclude that ${}^{\gamma}B(Lx) = 0$ and hence that $Lx \in \operatorname{zer} {}^{\gamma}B = \operatorname{Fix} J_{\gamma B} = \operatorname{zer} B$.

(vi): Let $x \in \mathcal{H}$. Then, arguing as in (6.7),

$$\begin{aligned} x \in S_2 &\Leftrightarrow x - L^*\big(Lx - J_{\gamma B}(Lx)\big) \in (N_V + \operatorname{Id}_{\mathcal{H}})x \\ &\Leftrightarrow 0 \in N_V x + L^*\big((\operatorname{Id}_{\mathcal{G}} - J_{\gamma B})(Lx)\big) \\ &\Leftrightarrow x \in \operatorname{zer}\big(N_V + L^* \circ ({}^{\gamma}B) \circ L\big), \end{aligned} \tag{6.12}$$

which provides the desired identity. □

Remark 6.4 (isometry) Suppose that L is an isometry in Theorem 6.3 (see Remark 4.2). In view of Proposition 4.1(iii) and Proposition 4.3, the relaxed problem (6.2) is then to find a zero of

$$\operatorname{proj}_V \diamond A = \operatorname{proj}_V \diamond \big(L \diamond (\gamma B)\big) = (L \circ \operatorname{proj}_V) \diamond (\gamma B), \tag{6.13}$$

and it follows from Theorem 6.3(iv) that its set of solutions is $S_2 = \operatorname{Fix}(\operatorname{proj}_V \circ L^* \circ J_{\gamma B} \circ L)$.

Next, we propose an algorithm for solving Problem 6.2 which is based on the most elementary method for solving monotone inclusions, namely the proximal point algorithm [51].

Proposition 6.5 *Suppose that Problem 6.2 has a solution, let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $]0,2[$ such that $\sum_{n\in\mathbb{N}} \lambda_n(2-\lambda_n) = +\infty$, and let $x_0 \in V$. Iterate*

$$\begin{array}{l} \textit{for } n = 0, 1, \ldots \\ \left\lfloor \begin{array}{l} y_n = Lx_n \\ q_n = J_{\gamma B} y_n - y_n \\ z_n = L^* q_n \\ x_{n+1} = x_n + \lambda_n \operatorname{proj}_V z_n. \end{array} \right. \end{array} \tag{6.14}$$

Then $(x_n)_{n\in\mathbb{N}}$ converges weakly to a solution to Problem 6.2.

Proof Set $M = \mathrm{proj}_V \diamond (L\blacklozenge(\gamma B))$. Since $(x_n)_{n\in\mathbb{N}}$ lies in V, it follows from Theorem 6.3 (i)–(ii) that $(x_n)_{n\in\mathbb{N}}$ is generated by the proximal point algorithm, to wit,

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = x_n + \lambda_n(J_M x_n - x_n). \tag{6.15}$$

Therefore, we derive from [26, Lemma 2.2(vi)] that $(x_n)_{n\in\mathbb{N}}$ converges weakly to a point in zer M, i.e., a solution to (6.2). □

Remark 6.6 (weak convergence) The weak convergence of $(x_n)_{n\in\mathbb{N}}$ in Proposition 6.5 cannot be improved to strong convergence in general. Indeed, suppose that, in Problem 6.2, $\mathcal{G} = \mathcal{H}$, $L = \mathrm{Id}_{\mathcal{H}}$, and $B = N_C$, where C is a nonempty closed convex subset of $\mathcal{H}$. Then, if we take the parameters $(\lambda_n)_{n\in\mathbb{N}}$ to be 1, the proximal point algorithm (6.14) reduces to the alternating projection method $(\forall n \in \mathbb{N})\ x_{n+1} = \mathrm{proj}_V(\mathrm{proj}_C x_n)$. In [35], a hyperplane V and a cone C are constructed for which $(x_n)_{n\in\mathbb{N}}$ fails to converge strongly. Note, however, that using the strongly convergent modifications of (6.15) discussed in [8, 53], it is straightforward to obtain strongly convergent methods to solve Problem 6.2. Let us add that, as shown in [26, Lemma 2.2(vi)], the weak convergence result in Proposition 6.5 remains valid if q_n is defined as $q_n = J_{\gamma B} y_n + c_n - y_n$ in (6.14), where $(c_n)_{n\in\mathbb{N}}$ is a sequence modeling approximate implementations of $J_{\gamma B}$ and satisfies $\sum_{n\in\mathbb{N}} \lambda_n \|c_n\| < +\infty$.

Henceforth, we specialize Problems 6.1 and 6.2 to scenarios of interest.

Example 6.7 (feasibility problem) Let $0 \neq m \in \mathbb{N}$ and let $(\mathsf{C}_i)_{1\leqslant i\leqslant m}$ be nonempty closed convex subsets of a real Hilbert space H. Set $\mathcal{H} = \bigoplus_{i=1}^m \mathsf{H}$, $V = \{(\mathsf{x}, \ldots, \mathsf{x}) \in \mathcal{H} \mid \mathsf{x} \in \mathsf{H}\}$, and $C = \mathsf{C}_1 \times \cdots \times \mathsf{C}_m$. Since V is isomorphic to H, Problem 6.1 with $\mathcal{G} = \mathcal{H}$, $L = \mathrm{Id}_{\mathcal{H}}$, and $B = N_C = A$ amounts to finding a point in $V \cap C$, i.e., a point in $\bigcap_{i=1}^m \mathsf{C}_i$, while Theorem 6.3(iv) asserts that the relaxation given in Problem 6.2 amounts to finding a fixed point of $\mathrm{proj}_V \circ \mathrm{proj}_C$, i.e., of $(1/m)\sum_{i=1}^m \mathrm{proj}_{\mathsf{C}_i}$ or, equivalently, a minimizer of $\sum_{i=1}^m d_{\mathsf{C}_i}^2$. This product space framework for relaxing inconsistent feasibility problems was proposed in [46, Section II.2] and re-examined in [5, 23].

Example 6.8 (resolvent mixtures) Let $0 \neq p \in \mathbb{N}$, let $\gamma \in \left]0, +\infty\right[$, and let $V \neq \{0\}$ be a closed vector subspace of $\mathcal{H}$. For every $k \in \{1, \ldots, p\}$, let $\mathcal{G}_k$ be a real Hilbert space, let $L_k \in \mathcal{B}(\mathcal{H}, \mathcal{G}_k)$, let $\omega_k \in \left]0, +\infty\right[$, and let $B_k\colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$ be maximally monotone. Suppose that $0 < \sum_{k=1}^p \omega_k \|L_k\|^2 \leqslant 1$ and define $\mathcal{G}$, L, and B as in Example 3.4. Then the objective of Problem 6.1 is to

$$\text{find } x \in V \text{ such that } (\forall k \in \{1, \ldots, p\}) \quad 0 \in B_k(L_k x). \tag{6.16}$$

Now let M be the resolvent mixture of the operators $((\gamma B_k)^{-1})_{1\leqslant k\leqslant p}$ (see Example 3.4). Then the relaxed Problem 6.2 is to

$$\text{find } x \in \mathcal{H} \text{ such that } 0 \in \big(\mathrm{proj}_V \diamond M^{-1}\big)x \tag{6.17}$$

or, equivalently, upon invoking Theorem 6.3(vi), to

$$\text{find } x \in \mathcal{H} \text{ such that } 0 \in N_V x + \sum_{k=1}^p \omega_k L_k^*\big({}^{\gamma}B_k(L_k x)\big). \tag{6.18}$$

In addition, it follows from Proposition 6.5 that, given $x_0 \in V$ and a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $]0, 2[$ such that $\sum_{n\in\mathbb{N}} \lambda_n(2-\lambda_n) = +\infty$, the sequence $(x_n)_{n\in\mathbb{N}}$ constructed by the algorithm

$$
\begin{array}{l}
\text{for } n = 0, 1, \ldots \\
\left\lfloor
\begin{array}{l}
\text{for } k = 1, \ldots, p \\
\left\lfloor
\begin{array}{l}
y_{k,n} = L_k x_n \\
q_{k,n} = J_{\gamma B_k} y_{k,n} - y_{k,n}
\end{array}
\right. \\
z_n = \sum_{k=1}^{p} \omega_k L_k^* q_{k,n} \\
x_{n+1} = x_n + \lambda_n \operatorname{proj}_V z_n
\end{array}
\right.
\end{array}
\tag{6.19}
$$

converges weakly to a solution to the relaxed problem if one exists.

Example 6.9 (common zero problem) Suppose that, in Example 6.8, we have $(\forall k \in \{1, \ldots, p\})$ $\mathcal{G}_k = \mathcal{H}$ and $L_k = \mathrm{Id}_{\mathcal{H}}$. Then (6.16) consists of finding $x \in V \cap \bigcap_{k=1}^{p} \operatorname{zer} B_k$ and its relaxation (6.17)/(6.18) consists of finding a zero of $N_V + \sum_{k=1}^{p} \omega_k \, {}^{\gamma}B_k$. This relaxation was proposed in [25] and it originates in Legendre's method of least-squares [36] to relax inconsistent systems of linear equations (see [27, Example 4.3]).

Example 6.10 (Wiener systems) In Example 6.8, suppose that, for every $k \in \{1, \ldots, p\}$, $B_k = (\mathrm{Id}_{\mathcal{G}_k} - F_k + p_k)^{-1} - \mathrm{Id}_{\mathcal{G}_k}$, where $F_k : \mathcal{G}_k \to \mathcal{G}_k$ is firmly nonexpansive and $p_k \in \mathcal{G}_k$. Then we recover the Wiener system setting investigated in [32]. Specifically, (6.16) reduces to the nonlinear reconstruction problem [32, Problem 1.1]

$$
\text{find } x \in V \text{ such that } (\forall k \in \{1, \ldots, p\}) \quad F_k(L_k x) = p_k \tag{6.20}
$$

and (6.17) yields the relaxed problem [32, Problem 1.3]

$$
\text{find } x \in V \text{ such that } (\forall y \in V) \ \sum_{k=1}^{p} \omega_k \langle L_k y - L_k x \mid F_k(L_k x) - p_k \rangle = 0. \tag{6.21}
$$

In addition, given $x_0 \in V$ and a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $]0, 2[$ such that $\sum_{n\in\mathbb{N}} \lambda_n(2-\lambda_n) = +\infty$, the sequence $(x_n)_{n\in\mathbb{N}}$ constructed by the algorithm

$$
\begin{array}{l}
\text{for } n = 0, 1, \ldots \\
\left\lfloor
\begin{array}{l}
\text{for } k = 1, \ldots, p \\
\left\lfloor
\begin{array}{l}
y_{k,n} = L_k x_n \\
q_{k,n} = p_k - F_k y_{k,n}
\end{array}
\right. \\
z_n = \sum_{k=1}^{p} \omega_k L_k^* q_{k,n} \\
x_{n+1} = x_n + \lambda_n \operatorname{proj}_V z_n
\end{array}
\right.
\end{array}
\tag{6.22}
$$

converges weakly to a solution to the relaxed problem if one exists (see [32, Proposition 4.3] for existence conditions).

Proof For every $k \in \{1, \ldots, p\}$, it follows from (2.5) that $\mathrm{Id}_{\mathcal{G}_k} - F_k + p_k : \mathcal{G}_k \to \mathcal{G}_k$ is firmly nonexpansive and therefore from Lemma 2.2 that B_k is maximally monotone, with $J_{B_k} = \mathrm{Id}_{\mathcal{G}_k} - F_k + p_k$ and ${}^{1}B_k = F_k - p_k$. In addition, we observe that this choice of the operators $(B_k)_{1\leqslant k\leqslant p}$ makes (6.20) a realization of (6.16), and (6.22) a realization of (6.19). At the same time, (6.17)/(6.18) with $\gamma = 1$ becomes

$$
\text{find } x \in \mathcal{H} \text{ such that } 0 \in N_V x + \sum_{k=1}^{p} \omega_k L_k^* \big(F_k(L_k x) - p_k \big), \tag{6.23}
$$

which is precisely (6.21). $\square$

Example 6.11 (proximal composition) In Problem 6.1, suppose that $B = \partial g$, where $g \in \Gamma_0(\mathcal{G})$. Then (6.1) becomes

$$\text{find}\;\; x \in V \;\;\text{such that}\;\; Lx \in \operatorname{Argmin} g. \tag{6.24}$$

Now set $f = L^{\blacklozenge}(\gamma g)$. Then the relaxation (6.2) becomes

$$\underset{x\in\mathcal{H}}{\text{minimize}}\; \big(\text{proj}_V \diamond f\big)(x) \tag{6.25}$$

or, equivalently,

$$\underset{x\in V}{\text{minimize}}\; \big({}^{\gamma}\!g\big)(Lx). \tag{6.26}$$

In addition, given $x_0 \in V$ and a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $]0,2[$ such that $\sum_{n\in\mathbb{N}} \lambda_n(2-\lambda_n) = +\infty$, the algorithm

$$\begin{array}{l}\text{for } n = 0, 1, \ldots\\ \left\lfloor \begin{array}{l} y_n = Lx_n\\ q_n = \text{prox}_{\gamma g} y_n - y_n\\ z_n = L^* q_n\\ x_{n+1} = x_n + \lambda_n \,\text{proj}_V z_n \end{array}\right. \end{array} \tag{6.27}$$

produces a sequence $(x_n)_{n\in\mathbb{N}}$ that converges weakly to a solution to the relaxed problem if one exists.

Proof The fact that (6.1) yields (6.24) is a consequence of Fermat's rule (2.13). Next, we derive from Example 3.10(iv) that, in Problem 6.2,

$$A = L^{\blacklozenge}(\gamma B) = L^{\blacklozenge}\partial(\gamma g) = \partial\big(L^{\blacklozenge}(\gamma g)\big) = \partial f. \tag{6.28}$$

Thus, by Example 3.6(v),

$$\text{proj}_V \diamond A = \text{proj}_V \diamond \partial f = \partial\big(\text{proj}_V \diamond f\big). \tag{6.29}$$

Therefore, by Fermat's rule (2.13), the solution set of Problem 6.2 is

$$\text{zer}\,\big(\text{proj}_V \diamond A\big) = \operatorname{Argmin}\,\big(\text{proj}_V \diamond f\big). \tag{6.30}$$

On the other hand, since $\text{dom}\,{}^{\gamma}\!g = \mathcal{G}$, [9, Example 23.3 and Theorem 16.47(i)] yield

$$N_V + L^* \circ \big({}^{\gamma}(\partial g)\big) \circ L = \partial\iota_V + L^* \circ \big(\nabla{}^{\gamma}\!g\big) \circ L = \partial\big(\iota_V + ({}^{\gamma}\!g) \circ L\big). \tag{6.31}$$

Thus, we deduce from Theorem 6.3(vi) and (2.13) that

$$\text{zer}\,\big(\text{proj}_V \diamond A\big) = \text{zer}\,\big(N_V + L^* \circ {}^{\gamma}(\partial g) \circ L\big) = \operatorname{Argmin}\big(\iota_V + ({}^{\gamma}\!g) \circ L\big). \tag{6.32}$$

In view of (6.29), this confirms the equivalence between (6.25) and (6.26). The last claim is an application of Proposition 6.5 using (2.14). □

Example 6.12 (proximal mixture) In the context of Example 6.11, choose $\mathcal{G}$, L, and g as in Example 5.9. Then the initial problem (6.24) is to

$$\text{find}\;\; x \in V \;\;\text{such that}\;\; (\forall k \in \{1,\ldots,p\})\;\; L_k x \in \operatorname{Argmin} g_k. \tag{6.33}$$

Now let m be the proximal mixture of the functions $((\gamma g_k)^*)_{1\leqslant k\leqslant p}$ (see Example 5.9). Then the relaxation of (6.33) given by (6.25) is to

$$\underset{x\in\mathcal{H}}{\text{minimize}}\; \big(\text{proj}_V \diamond m^*\big)(x) \tag{6.34}$$

or, equivalently, via (6.26), to

$$\underset{x \in V}{\text{minimize}} \sum_{k=1}^{p} \omega_k \big({}^{\gamma}g_k\big)(L_k x). \tag{6.35}$$

This problem can be solved via (6.19), where $J_{\gamma B_k}$ is replaced by $\text{prox}_{\gamma g_k}$.

Remark 6.13 (proximal average) In Example 6.11, suppose that $V = \mathcal{H}$ and that L is an isometry. Then it follows from Proposition 5.3(vii) that the relaxed problem (6.25) consists of minimizing the proximal composition $f = L \diamond (\gamma g)$. In particular, if f is the proximal average of the functions $(g_k)_{1 \leqslant k \leqslant p}$ (see (5.14)), it follows from Example 6.12 that minimizing it is an exact relaxation of the problem of finding a common minimizer of the functions $(g_k)_{1 \leqslant k \leqslant p}$. This provides a principled interpretation for methodologies adopted in [38, 54].

Example 6.14 (split feasibility) Suppose that, in Example 6.12, for every $k \in \{1, \ldots, p\}$, $g_k = \iota_{D_k}$, where D_k is a nonempty closed convex subset of $\mathcal{G}_k$. Then (6.33) is the split feasibility problem [49]

$$\text{find } x \in V \text{ such that } (\forall k \in \{1, \ldots, p\}) \quad L_k x \in D_k, \tag{6.36}$$

while the relaxation (6.34)/(6.35) is to

$$\underset{x \in V}{\text{minimize}} \sum_{k=1}^{p} \omega_k d_{D_k}^2 (L_k x). \tag{6.37}$$

This problem can be solved via (6.19), where $J_{\gamma B_k}$ is replaced by proj_{D_k}.

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