MORE ON LINES IN EUCLIDEAN RAMSEY THEORY

DAVID CONLON AND YU-HAN WU

Abstract. Let ℓ_m be a sequence of m points on a line with consecutive points at distance one. Answering a question raised by Fox and the first author and independently by Arman and Tsaturian, we show that there is a natural number m and a red/blue-colouring of $\mathbb{E}^n$ for every n that contains no red copy of ℓ_3 and no blue copy of ℓ_m.

1. Introduction

Let $\mathbb{E}^n$ denote n-dimensional Euclidean space, that is, $\mathbb{R}^n$ equipped with the Euclidean metric. Given two sets $X_1, X_2 \subset \mathbb{E}^n$, we write $\mathbb{E}^n \to (X_1, X_2)$ if every red/blue-coloring of $\mathbb{E}^n$ contains either a red copy of X_1 or a blue copy of X_2, where a copy for us will always mean an isometric copy. Conversely, $\mathbb{E}^n \not\to (X_1, X_2)$ means that there is some red/blue-coloring of $\mathbb{E}^n$ which contains neither a red copy of X_1 nor a blue copy of X_2.

The study of which sets $X_1, X_2 \subset \mathbb{E}^n$ satisfy $\mathbb{E}^n \to (X_1, X_2)$ is a particular case of the Euclidean Ramsey problem, which has a long history going back to a series of seminal papers [6, 7, 8] of Erdős, Graham, Montgomery, Rothschild, Spencer and Straus in the 1970s. Despite the vintage of the problem, surprisingly little progress has been made since these foundational papers (though see [9, 12] for some important positive results). For instance, it is an open problem, going back to the papers of Erdős et al. [7], as to whether, for every n, there is m such that $\mathbb{E}^n \not\to (X, X)$ for every $X \subset \mathbb{E}^n$ with $|X| = m$.

Write ℓ_m for the set consisting of m points on a line with consecutive points at distance one. Perhaps because it is a little more accessible than the general problem, the question of determining which n and X satisfy the relation $\mathbb{E}^n \to (\ell_2, X)$ has received considerable attention. For instance, it is known [11, 14] that $\mathbb{E}^2 \to (\ell_2, X)$ for every four-point set $X \subset \mathbb{E}^2$ and that $\mathbb{E}^2 \to (\ell_2, \ell_5)$. On the other hand [5], there is a set X of 8 points in the plane, namely, a regular heptagon with its center, such that $\mathbb{E}^2 \not\to (\ell_2, X)$.

In higher dimensions, by combining results of Szlam [13] and Frankl and Wilson [10], it was observed by Fox and the first author [4] that $\mathbb{E}^n \to (\ell_2, \ell_m)$ provided $m \leq 2^{cn}$ for some positive constant c (see also [1, 2] for some better bounds in low dimensions). Our concern here will be with a question raised independently by Fox and the first author [4] and also by Arman and Tsaturian [2], namely, as to whether an analogous result holds with ℓ_2 replaced by ℓ_3. That is, for every natural number m, is there a natural number n such that $\mathbb{E}^n \to (\ell_3, \ell_m)$? We answer this question in the negative.

Theorem 1.1. *There exists a natural number m such that $\mathbb{E}^n \not\to (\ell_3, \ell_m)$ for all n.*

Before our work, the best result that was known in this direction was a 50-year-old result of Erdős et al. [6], who showed that $\mathbb{E}^n \not\to (\ell_6, \ell_6)$ for all n. Their proof uses a spherical colouring, where all points at the same distance from the origin receive the same colour. We will also use a spherical colouring, though, unlike the colouring in [6], which is entirely explicit, our colouring will be partly random.

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2. Preliminaries

In this short section, we note two key lemmas that will be needed in our proof. The first says that certain real-valued quadratic polynomials are reasonably well-distributed modulo a prime q.

Lemma 2.1. *Let $p(x) = x^2 + \alpha x + \beta$, where α and β are real numbers, and let q be a prime number. Then, for $m = q^3$, the set $\{p(i)\}_{i=1}^m$ overlaps with at least $q/6$ of the intervals $[j, j+1)$ with $0 \le j \le q-1$ when considered mod q.*

Proof. By a standard argument using the pigeonhole principle, there exists some $k \le q^2$ such that $|k\alpha| \le 1/q$ mod q. We split into two cases, depending on whether k is a multiple of q or not.

Suppose first that $k \not\equiv 0$ mod q and consider the set of values $\{p(ki)\}_{i=1}^q$. Note first that $\{i^2\}_{i=1}^q$ is a set of $(q+1)/2$ distinct integers mod q, so, since k is not a multiple of q, the same is also true of the set $\{k^2 i^2\}_{i=1}^q$. Hence, letting $p_1(x) = x^2 + \beta$, we see that the set $\{p_1(ki)\}_{i=1}^q$ overlaps with at least $q/2$ of the intervals $[j, j+1)$ with $0 \le j \le q-1$ when considered mod q. But $|ki\alpha| \le 1$ mod q for all $1 \le i \le q$, so that $|p(ki) - p_1(ki)| \le 1$ for all $1 \le i \le q$. Therefore, since exactly three different intervals are within distance one of any particular interval, the set $\{p(ki)\}_{i=1}^q$ overlaps with at least $q/6$ of the intervals $[j, j+1)$ mod q.

Suppose now that $k = sq$ for some $s \le q$. Then $sq\alpha = rq + \epsilon$ for some $|\epsilon| \le 1/q$, which implies that $\alpha = \frac{r}{s} + \epsilon'$, where $|\epsilon'| \le 1/q^2$. Without loss of generality, we may assume that r and s have no common factors. Consider now the polynomial $p_2(x) = x^2 + \frac{r}{s}x$ and the set $\{p_2(si)\}_{i=1}^q$. Since $p_2(si) = s^2 i^2 + ri$, it is easy to check that $p_2(si) \equiv p_2(sj)$ mod q if and only if $s^2(i+j) + r \equiv 0$ mod q. Since r and s are coprime, this implies that the set $\{p_2(si)\}_{i=1}^q$ takes at least $q/2$ values mod q. Hence, letting $p_3(x) = x^2 + \frac{r}{s}x + \beta$, we see that the set $\{p_3(si)\}_{i=1}^q$ overlaps with at least $q/2$ of the intervals $[j, j+1)$ with $0 \le j \le q-1$ when considered mod q. But, since $|\alpha - r/s| \le 1/q^2$, we have that $|p(si) - p_3(si)| = |\alpha - \frac{r}{s}| si \le 1$, so that, as above, the set $\{p(si)\}_{i=1}^q$ overlaps with at least $q/6$ of the intervals $[j, j+1)$ mod q. □

Given M real polynomials $p_1, \dots, p_M$ in N variables, a vector $\sigma \in \{-1, 0, 1\}^M$ is called a sign pattern of $p_1, \dots, p_M$ if there exists some $x \in \mathbb{R}^N$ such that the sign of $p_i(x)$ is σ_i for all $1 \le i \le M$. The second result we need is the Oleinik–Petrovsky–Thom–Milnor theorem (see, for example, [3]), which, for N fixed, gives a polynomial bound for the number of sign patterns.

Lemma 2.2. *For $M \ge N \ge 2$, the number of sign patterns of M real polynomials in N variables, each of degree at most D, is at most $\left(\frac{50DM}{N}\right)^N$.*

3. Proof of Theorem 1.1

Suppose that $a_1, a_2, a_3 \in \mathbb{R}^n$ form a copy of ℓ_3 with $|a_1 - a_2| = |a_2 - a_3| = 1$. If the points are at distances x_1, x_2 and x_3, respectively, from the origin o and the angle $a_1 a_2 o$ is θ, then we have

$$x_1^2 = x_2^2 + 1 - 2x_2 \cos\theta$$

and

$$x_3^2 = x_2^2 + 1 + 2x_2 \cos\theta.$$

Adding the two gives

$$x_1^2 + x_3^2 = 2x_2^2 + 2.$$

Similarly, if $a_1, a_2, \dots, a_m \in \mathbb{R}^n$ form a copy of ℓ_m with $|a_i - a_{i+1}| = 1$ for all $i = 1, 2, \dots, m-1$, then, again writing x_i for the distance of a_i from the origin, we have

$$x_{i-1}^2 + x_{i+1}^2 = 2x_i^2 + 2$$

for all $i = 2, \dots, m-1$. Given these observations, our aim will be to colour $\mathbb{R}_{\ge 0}$ so that there is no red solution to $y_1 + y_3 = 2y_2 + 2$ and no blue solution to the system $y_{i-1} + y_{i+1} = 2y_i + 2$ with

$i = 2, \dots, m-1$. Assuming that we have such a colouring χ, we can simply colour a point $a \in \mathbb{R}^n$ by $\chi(|a|^2)$ and it is easy to check that there is no red copy of ℓ_3 and no blue copy of ℓ_m.

We have therefore moved our problem to one of finding a natural number m and a colouring χ of $\mathbb{R}_{\geq 0}$ with no red solution to $y_1 + y_3 = 2y_2 + 2$ and no blue solution to the system $y_{i-1} + y_{i+1} = 2y_i + 2$ with $i = 2, \dots, m-1$. Let q be a prime number. We will take $m = q^3$ and define χ by choosing an appropriate colouring χ' of $\mathbb{Z}_q$ and then setting $\chi(y) = \chi'(\lfloor y \rfloor \bmod q)$ for all $y \in \mathbb{R}_{\geq 0}$. Our aim now is to show that there is a suitable choice for χ'. For this, we consider a random red/blue-colouring χ' of $\mathbb{Z}_q$ and show that, for q sufficiently large, the probability that χ contains either of the banned configurations is small.

Concretely, suppose that $\mathbb{Z}_q$ is coloured randomly in red and blue with each element of $\mathbb{Z}_q$ coloured red with probability $p = q^{-3/4}$ and blue with probability $1-p$. With this choice, the expected number of solutions in red to any of the equations $y_1 + y_3 = 2y_2 + c$ with $c \in \{1, 2, 3\}$ is at most

$$3p^3q^2 + 9p^2q < 12q^{-1/4} < \frac{1}{2},$$

where we used that there are at most $3q$ solutions to any of our 3 equations with two of the variables $\{y_1, y_2, y_3\}$ being equal and that q is sufficiently large. Note that if there are indeed no red solutions to these three equations over $\mathbb{Z}_q$, then there is no red solution to $y_1 + y_3 = 2y_2 + 2$ in the colouring χ of $\mathbb{R}$. Indeed, if $y_i = n_i + \epsilon_i$ with $0 \leq \epsilon_i < 1$, then n_i is coloured red in χ' and

$$n_1 + n_3 = 2n_2 + 2 + 2\epsilon_2 - \epsilon_1 - \epsilon_3.$$

But $|2\epsilon_2 - \epsilon_1 - \epsilon_3| < 2$, so we must have

$$n_1 + n_3 = 2n_2 + c$$

for $c \in \{1, 2, 3\}$. However, we know that there are no red solutions to any of these equations in the colouring χ', so there is no red solution to $y_1 + y_3 = 2y_2 + 2$ in the colouring χ.

For the blue configurations, we first observe that if the y_i satisfy the equations $y_{i-1} + y_{i+1} = 2y_i + 2$ with $i = 2, \dots, m-1$ with $y_1 = a$ and $y_2 = a + d$, then $y_i = a + (i-1)d + (i^2 - 3i + 2)$. In particular, by Lemma 2.1, at least $q/6$ elements of the sequence $y_1, \dots, y_m$ lie in different intervals $[j, j+1)$ with $0 \leq j \leq q-1$ when considered mod q.

Our aim now is to apply Lemma 2.2 to count the number of different ways in which a set of solutions $(y_1, y_2, \dots, y_m)$ to our system of equations can overlap the collection of intervals $[j, j+1)$ mod q. Without loss of generality, we may assume that $0 \leq a, d < q$. Since, under this assumption, any set of solutions over $\mathbb{R}$ to our system of equations is contained in the interval $[0, 2m^2)$, it will suffice to count the number of feasible overlaps with the intervals $[j, j+1)$ with $0 \leq j \leq 2m^2 - 1$. Since we need to check at most two linear inequalities in the two variables a and d to check whether each of the m points are placed in each of the $2m^2$ intervals, we can apply Lemma 2.2 with $N = 2$, $D = 1$ and $M = 2 \cdot m \cdot 2m^2 = 4m^3$ to conclude that the points $y_1, \dots, y_m$ overlap the intervals $[j, j+1)$ with $0 \leq j \leq 2m^2 - 1$ in at most $(100m^3)^2 = 10^4 m^6$ different ways. But now, since at least $q/6$ of the y_i must always be in distinct intervals, a union bound implies that the probability we have a blue solution to our system of equations is at most

$$10^4 m^6 (1 - q^{-3/4})^{q/6} < \frac{1}{2}$$

for m sufficiently large. Combined with our earlier estimate for the probability of a red solution to $y_1 + y_3 = 2y_2 + 2$, we see that for m sufficiently large ($m = 10^{50}$ will suffice) there exists a colouring with no red ℓ_3 and no blue ℓ_m, as required.

4. Concluding remarks

We say that a set $X \subset \mathbb{E}^d$ is *Ramsey* if for every natural number r there exists n such that every r-colouring of $\mathbb{E}^n$ contains a monochromatic copy of X. In [4], it was shown that a set X is

Ramsey if and only if for every natural number m and every fixed $K \subset \mathbb{E}^m$ there exists n such that $\mathbb{E}^n \to (X, K)$. We suspect that there may be an even simpler characterisation.

Conjecture 4.1. *A set X is Ramsey if and only if for every natural number m there exists n such that $\mathbb{E}^n \to (X, \ell_m)$.*

Of course, by the result mentioned above, we already know that if X is Ramsey, then $\mathbb{E}^n \to (X, \ell_m)$ for n sufficiently large. It therefore remains to show that if X is not Ramsey, then there exists m such that $\mathbb{E}^n \nrightarrow (X, \ell_m)$ for all n. To prove this in full generality might be difficult. However, an important result of Erdős et al. [6] says that if X is Ramsey, then it must be spherical, in the sense that it must be contained in the surface of a sphere of some dimension. Thus, a first step towards Conjecture 4.1 might be to prove the following.

Conjecture 4.2. *For every non-spherical set X, there exists a natural number m such that $\mathbb{E}^n \nrightarrow (X, \ell_m)$ for all n.*

The simplest example of a non-spherical set is the line ℓ_3, so our main result may be seen as a verification of Conjecture 4.2 in this particular case. The next case of interest seems to be when X consists of three points a_1, a_2, a_3 on a line, but now with $|a_1 - a_2| = 1$ and $|a_2 - a_3| = \alpha$ for some irrational α.

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Department of Mathematics, California Institute of Technology, Pasadena, CA 91125, USA
Email address: dconlon@caltech.edu

École Normale Supérieure - PSL, Paris, France
Email address: yu-han.wu@ens.psl.eu