

MONOCHROMATIC COMPONENTS WITH MANY EDGES

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Abstract. Given an r -edge-coloring of the complete graph K_n , what is the largest number of edges in a monochromatic connected component? This natural question has only recently received the attention it deserves, with work by two disjoint subsets of the authors resolving it for the first two special cases, when $r = 2$ or 3 . Here we introduce a general framework for studying this problem and apply it to fully resolve the $r = 4$ case, showing that any 4-edge-coloring of K_n contains a monochromatic component with at least $\frac{1}{12}n^2$ edges, where the constant $\frac{1}{12}$ is optimal only when the coloring matches a certain construction of Gyárfás.

1. Introduction

Given an r -coloring of the edges of the complete graph K_n , how large is the largest monochromatic connected component? A partial answer to this question was provided in 1977 by Gyárfás [6], who showed that any such r -coloring always contains a monochromatic connected component with at least $n/(r-1)$ vertices. Moreover, this estimate is best possible whenever $r-1$ is a prime power and n is a multiple of $(r-1)^2$. An alternative proof of this result, as a simple corollary of his fractional version of Ryser's conjecture, was later found by Füredi [4, 5], who also showed that if there is no plane of order $r-1$, then the bound can be improved to $(r-1)n/(r^2-2r)$.

There are many variants of this question. For instance, what happens when the complete graph K_n is replaced by another graph, say a subgraph of the complete graph [8] or the complete bipartite graph [3] of high minimum degree? Or what happens when we insist that our component has small diameter [10]? Here we will be concerned with another variant, a rather basic one which has received surprisingly little attention in the literature, namely, given an r -coloring of K_n , what is the largest number of edges in a monochromatic connected component?

This question was first raised by Conlon and Tyomkyn [1] because of its close relation with another problem, that of determining the Ramsey number for trails and circuits. However, the components problem is arguably the more fundamental question. If we write $M(n; r)$ for the largest natural number such that every r -coloring of K_n contains a monochromatic connected component with at least $M(n; r)$ edges, then the main result of [1] may be interpreted as saying that $M(n; 2) = \frac{2}{9}n^2 + o(n^2)$. In fact, a more careful analysis of their argument implies that $M(n; 2) = \frac{1}{9}(2n^2 - n - 1)$, with, where divisibility allows it, the example consisting of two disjoint red cliques of orders $\frac{2n+1}{3}$ and $\frac{n-1}{3}$ with all blue edges between showing that this is best possible.

To say something about the general case, we first look at Gyárfás' construction of r -colorings where each monochromatic component has at most $n/(r-1)$ vertices. As noted earlier, his construction, which relies on the existence of the plane of order $r-1$, works when $r-1$ is a prime power and n is a multiple of $(r-1)^2$. Concretely, the plane of order $r-1$ corresponds to a copy of $K_{(r-1)^2}$ together with r different decompositions of this graph into $r-1$ vertex-disjoint copies of K_{r-1} (that is, r different K_{r-1} -factors) with the property that any edge is contained in exactly one of the $r(r-1)$ copies of K_{r-1} . By giving the edges in the i th K_{r-1} -factor color i , we

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obtain an r -coloring where every monochromatic component has at most $r - 1$ vertices. Moreover, when n is a multiple of $(r - 1)^2$, we can simply blow up this coloring to obtain an r -coloring where every monochromatic component has at most $n/(r - 1)$ vertices.

As noted by Conlon and Tyomkyn [1], essentially the same construction works in the edge case to show that, when $r - 1$ is a prime power, there are r -colorings where each monochromatic component has at most $(\frac{1}{r(r-1)} + o(1)) \frac{n}{2}$ edges (the only caveat is that we should use each color roughly the same number of times within each of the blown-up vertices). They also showed that this bound is correct up to a constant and conjectured that, for $r = 3$, it is asymptotically tight. That this is the case was verified by Luo [9], who proved that $M(n; 3) = d \frac{1}{6} \frac{n}{2}$ for n sufficiently large. Moreover, by giving a tight lower bound for the largest number of edges in a connected component in a graph of given density, he was able to show that $M(n; r) \sim \frac{1}{r(r-1)} \frac{n}{2}$ in the general case, a result which was later strengthened to $M(n; r) \sim \frac{1}{r(r-1)} \frac{n}{2}$ in a revised version of the paper. Our concern here will be with the following conjectural improvement to this bound.

Conjecture 1. For any natural numbers n and r with $r \geq 3$,

$$M(n; r) \leq \frac{1}{r(r-1)} \frac{n}{2} :$$

Moreover, when there is no plane of order $r - 1$, there exists a constant $\epsilon_r > 0$ such that

$$M(n; r) \leq \frac{1}{r(r-1)} \frac{n}{2} + \epsilon_r \frac{n}{2} :$$

The result of [9] proves this conjecture when $r = 3$, while the result of [1] shows that the conjectured bound does not extend to the case $r = 2$. Our main result is a proof of the next open case, when $r = 4$. Note that, in this case, Gyárfás' construction corresponds to a 4-coloring of K_9 where each color class is the union of three vertex-disjoint triangles. In the statement below, by saying that a coloring matches Gyárfás' construction, we mean that the set of components and the intersection pattern of their vertex sets match those in this construction.

Theorem 2. In every 4-coloring of the edges of K_n , there is a monochromatic component with at least $\frac{1}{12} \frac{n}{2}$ edges. That is, $M(n; 4) \geq \frac{1}{12} \frac{n}{2}$. Moreover, unless the coloring matches Gyárfás' construction, there is a monochromatic component with at least $\frac{1}{12} + \epsilon \frac{n}{2}$ edges, where $\epsilon = \frac{2}{14+96} - \frac{1}{12} > 0.0007$.

Our proof of Theorem 2 consists of first showing that any 4-coloring of K_n has one of a bounded number of component structures and then that each such component structure contains a component with enough edges. For instance, one of the possible component structures is that each color has precisely three components. But then one of these 12 components clearly contains at least $\frac{1}{12}$ of the edges, as required. For the other possible component structures, our arguments are not usually so simple, relying instead on a key observation, Proposition 3 below. This says that if a certain union of components is large in the vertex sense, but none of these components is large in the edge sense, then some one of the remaining components will be large in the edge sense. In fact, even this is not quite enough and, inspired by Erdős' approach to the vertex case, we must allow for weighted or fractional unions of components. We will describe our general framework and how it may be applied in more detail in the next section.

2. A general framework

Suppose $r \geq 2$ and x an r -coloring of the edges of the complete graph $G = (V; E) \subseteq K_n$. For $1 \leq i \leq r$, let G_i be the graph of color i , i.e., the subgraph of K_n formed by the edges with color

i , where we include vertices that are isolated in that color in the graph. In particular, $|G_i| = n$ for all i . Let C_i be the set of connected components in G_i and let C be the set of all monochromatic components, i.e., $C = \bigcup_{i=1}^r C_i$. Since each vertex of G is in exactly one component of each color, we have

$$\sum_{C \in C} |V(C)| = r|V(G)| = rn.$$

Similarly, since each edge of G is in exactly one component of exactly one color, we have

$$\sum_{C \in C} |E(C)| = |E(G)| = \frac{n^2}{2}.$$

The following observation will be central to our approach.

Proposition 3. Let $X \subseteq C$ be a set of connected monochromatic components in an r -coloring of K_n and let $x = |X|$. Suppose $\alpha \in [0; r]$ and $z \in \mathbb{R}^+$ are constants such that $\sum_{C \in X} |V(C)| \leq \alpha n$ and $\max_{C \in C} |E(C)| \leq \frac{nz}{2}$. Then

$$z(r - \alpha)^2 \leq \max(1 - xz; 0)^2. \quad (2.1)$$

Proof. If $\alpha = r$, then $\sum_{C \in X} |V(C)| = rn = \sum_{C \in C} |V(C)|$, so $X = C$. Then

$$xz = \frac{\sum_{C \in C} |V(C)|}{z} \cdot \frac{\sum_{C \in C} |E(C)|}{\frac{n^2}{2}} = 1;$$

so both sides of (2.1) are zero. Thus, we can assume $\alpha < r$.

For $1 \leq i \leq r$, let H_i be the induced subgraph of G_i on $V(G) \setminus \bigcup_{C \in X} V(C)$. Then

$$\sum_{i=1}^r |E(H_i)| = |E(K_n)| - \sum_{C \in X} |E(C)| \leq \frac{n^2}{2} - \max(1 - xz) \frac{n^2}{2};$$

while

$$\sum_{i=1}^r |V(H_i)| = rn - \sum_{C \in X} |V(C)| \leq (r - \alpha)n.$$

Note that $|V(H_i)| = 0$ implies $|E(H_i)| = 0$. Thus, by a standard averaging argument, there is some j with $|V(H_j)| = 0$ and

$$\frac{|E(H_j)|}{|V(H_j)|} \leq \frac{\sum_{i=1}^r |E(H_i)|}{\sum_{i=1}^r |V(H_i)|} \leq \frac{1}{2} \max\left(r - \frac{xz}{r}, 0\right).$$

Let C_j^0 be the set of components of H_j , so, by the same averaging argument, there is some $C^0 \in C_j^0$ such that

$$\frac{|E(C^0)|}{|V(C^0)|} \leq \frac{\sum_{C \in C_j^0} |E(C)|}{\sum_{C \in C_j^0} |V(C)|} = \frac{|E(H_j)|}{|V(H_j)|} \leq \frac{1}{2} \max\left(r - \frac{xz}{r}, 0\right). \quad (2.2)$$

Since $|E(C^0)| \leq \frac{|V(C^0)|^2}{2}$, we have

$$\frac{n}{2} - \frac{1}{2} \leq \frac{|V(C^0)|}{2} - \frac{1}{2} \leq \frac{|E(C^0)|}{|V(C^0)|},$$

which, combined with (2.2), implies $\frac{1-xz}{r} \geq 1$. We therefore have

$$\begin{aligned} z &\geq \frac{n}{2} \frac{jE(C^0)j}{jV(C^0)j} = \frac{n}{2} \frac{jE(C^0)j}{jV(C^0)j} \frac{jV(C^0)j}{jV(C^0)j} = \frac{n}{2} \frac{jE(C^0)j}{jV(C^0)j} \frac{jV(C^0)j}{jV(C^0)j} + 1 \\ &= \frac{n}{2} \frac{jE(C^0)j}{jV(C^0)j} + 1 = \frac{n}{2} \max\left(\frac{1-xz}{r}, 0\right) + 1 \\ &= \frac{\max(1-xz; 0)^2 n}{(r-xz)^2} \geq \frac{n}{2} \end{aligned}$$

which rearranges to the desired bound.

Note that taking $X = \emptyset$, we have $x = 0$, so that $z \geq \frac{1}{2}$, which immediately yields the simple lower bound $M(n; r) \geq \frac{1}{2}n$, as proved in Corollary 4 of [9]. For $1 \leq xz \leq r$, $x > 0$, and $0 \leq z \leq 1$, (2.1) is equivalent to

$$z \geq \frac{(r-xz)^2 + 2xz}{2x^2} = \frac{r^2 - 2rxz + x^2z^2 + 2xz}{2x^2} = \frac{r^2 - 2rxz + x^2z^2 + 2xz}{2x^2} \quad (2.3)$$

In order to go beyond the bound $M(n; r) \geq \frac{1}{2}n$ from Theorem 1 of [9], we must investigate the possible component structures in our r -coloring. For instance, in the case $r = 3$, the proof of Theorem 2 in [9] shows that there are three possible structures for the components in our coloring. Either:

- (a) Some color has exactly one component ($x = 1; x = 1$ above),
- (b) Each color has exactly two components ($x = 3; x = 6$), or
- (c) There is a component of each color such that every vertex is covered by at least two of these three components ($x = 2; x = 3$).

Applying Proposition 3 to cases (a) and (c) yields, in each case, a lower bound on z higher than the tight bound of $\frac{1}{2}$, while case (b), with $r = 0$, implies $1 \leq xz \leq 0$, so $z \geq \frac{1}{2}$, which is tight.

For general r , in order to prove a lower bound of the form $M(n; r) \geq \frac{1}{2}n$, it suffices to find, in any given r -coloring, a set $X \subseteq C$ yielding values of $x \in \mathbb{Z}^+$ and $z \in [0; r]$ such that $x \geq 1$ and (2.1) does not hold, that is, $z(r-xz)^2 \geq (1-xz)^2$. This rearranges to

$$x \geq \frac{1-r}{z} \quad (2.4)$$

We can state our conclusions concisely as follows.

Corollary 4. Let $z \in \mathbb{R}^+$. If there exists a set of components $X \subseteq C$ such that, for some $x \in \mathbb{Z}^+$ and $z \in [0; r]$, $|X| = x$, $\sum_{C \in X} |V(C)| = n$, and (2.4) holds, then there is a component $C \in C$ with at least $\frac{1}{2}n$ edges.

In fact, a simple probabilistic argument allows us to strengthen Corollary 4 to a fractional form allowing for non-integer values of x .

Proposition 5. Let $z \in \mathbb{R}^+$. If there exists a function $w : C \rightarrow [0; 1]$ such that, for some $x \in \mathbb{R}^+$ and $z \in [0; r]$, $\sum_{C \in C} w(C) = x$, $\sum_{C \in C} w(C)|V(C)| = n$, and (2.4) holds, then there is a component $C \in C$ with at least $\frac{1}{2}n$ edges.

Proof. Our proof takes advantage of the fact that (2.4) is linear in x and z . Given a function w with the required properties, we construct a random subset $X \subseteq C$ by taking each component $C \in C$

with probability $w(C)$. Abusing notation by letting x and $\bar{p}$ be the random variables with $x = |X|$ and $\bar{p} = \frac{1}{n} \sum_{C \in \mathcal{C}} |V(C)|$, we have

$$E \left[x + \frac{r \bar{p}}{z} \right] = \sum_{C \in \mathcal{C}} w(C) + \frac{r}{z} \cdot \frac{1}{n} \sum_{C \in \mathcal{C}} w(C) |V(C)|$$

$$x + \frac{r \bar{p}}{z} \geq \frac{1}{z} \sum_{C \in \mathcal{C}} w(C) |V(C)|$$

so there is some choice of X that satisfies the conditions of Corollary 4, yielding a component with the desired size.

In practice, we will apply Proposition 5 in the following form, allowing us to work with the convenient bound (2.3).

Corollary 6. Let $w : \mathcal{C} \rightarrow [0; 1]$, $x \in \mathbb{R}^+$, and $z \in [0; r]$ and suppose $\sum_{C \in \mathcal{C}} w(C) = x$ and $\sum_{C \in \mathcal{C}} w(C) |V(C)| \leq n$. Then (2.3) holds for $z = \frac{1}{n} \sum_{C \in \mathcal{C}} w(C) |V(C)|$.

Note that the condition $\sum_{C \in \mathcal{C}} w(C) |V(C)| \leq n$ is satisfied if, for every vertex $v \in V(G)$, we have

$$\sum_{C \ni v} w(C) \leq 1$$

i.e., the function w gives a fractional cover of the vertices by components that cover each vertex with weight at least $\frac{1}{r}$. This allows us to convert our problem into a linear program, akin to Furedi's approach to the vertex case, namely, we wish to minimize $x = \sum_{C \in \mathcal{C}} w(C)$ subject to the constraints that $0 \leq w(C) \leq 1$ for all C and $\sum_{C \ni v} w(C) \leq 1$ for all $v \in V(G)$. In particular, to show that $z = \frac{r}{r+1}$ works, it would suffice to show that $x \geq \frac{r}{r+1}$.

3. The case $r = 4$

In the case $r = 4$, we can investigate the set of possible component structures as in the case $r = 3$, though the analysis is now considerably more intricate. Recall that our aim is to show there exists a component with at least $\frac{1}{12} n$ edges; moreover, we would like to show that this bound is only asymptotically tight when the components match the extremal configuration described by Gyrfas in [7].

We will make use of the following fact shown in the course of handling the $r = 3$ case in [9]. An equivalent result also appears as Lemma 4.19 in [2].

Lemma 7. In any 2-coloring of the complete bipartite graph between two vertex sets A_1 and A_2 , one of the following holds:

- (a) Some color has exactly one component,
- (b) Each color has exactly two components, each of which intersects both A_1 and A_2 , or
- (c) There is one component of each color such that the intersection of their vertex sets contains one of A_1 and A_2 and their union contains both.

In each of these cases, one may assign weights of $\frac{1}{2}$ or 1 to the components involved to get weights summing to at most 2 that cover every vertex involved to weight at least 1.

It will sometimes be convenient to specify further subcases within the cases of Lemma 7. We say that a pair of vertex sets $(A_1; A_2)$ "satisfies case (a) for color c " if c is a color with exactly one component covering $A_1 \cup A_2$. We also say that $(A_1; A_2)$ "satisfies case (c) directed toward A_1 " (or, equivalently, directed away from A_2) if A_1 is contained within the intersection of the two components.

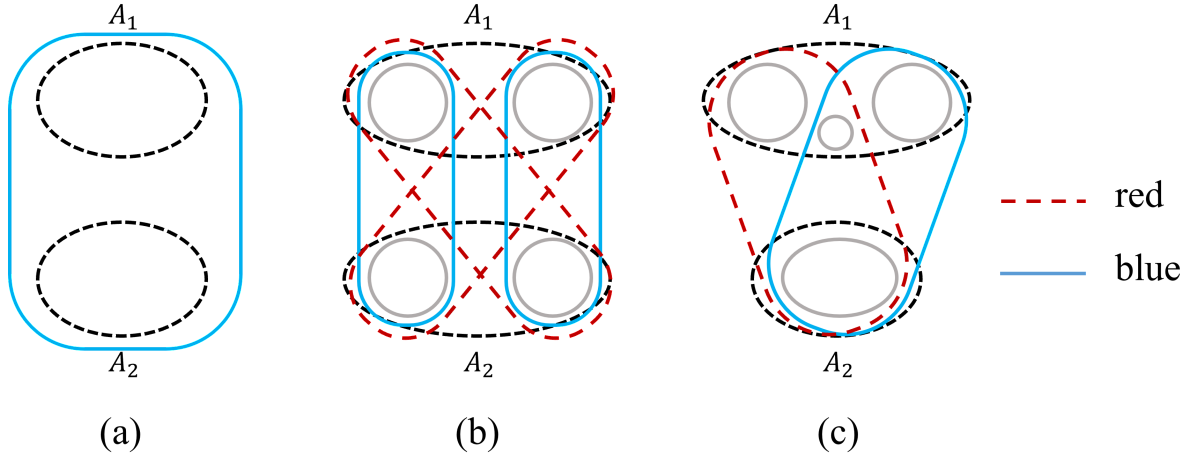


Figure 1. The cases of Lemma 7

Denote the G_i and C_i as before and, for convenience, name the colors red, orange, yellow, and blue. We begin by establishing a lower bound on $\max_{C_2 \subseteq C} |E(C)|$ in the following "degenerate" case where some three components with distinct colors cover the entire vertex set.

Lemma 8. If, in a 4-coloring of the edges of K_n , there are three components $C_1; C_2; C_3 \subseteq C$ of distinct colors such that $C_1 \cup C_2 \cup C_3 = V$, then

$$\frac{\max_{C_2 \subseteq C} |E(C)|}{\frac{n}{2}} \geq \frac{2}{14 + \frac{p}{96}} > \frac{1}{12}.$$

Proof. If there is one component C_1 with $V(C_1) = V$, then applying Proposition 3 with $X = \{C_1\}$ and $(x) = (1; 1)$ yields, via (2.3), the bound

$$z \geq \frac{2}{11 + \frac{p}{117}} \geq \frac{2}{22} \geq \frac{2}{14 + \frac{p}{96}};$$

as desired.

Next, suppose that there are two components $C_1; C_2$ with $C_1 \cup C_2 = V$, but with both $V_1 := C_1 \cap C_2$ and $V_2 := C_2 \cap C_1$ non-empty. Since we have not yet distinguished the colors in any way, we can assume without loss of generality that C_1 is red and C_2 is orange. Then all edges between V_1 and V_2 are either yellow or blue. Applying Lemma 7 to the pair of vertex sets $(V_1; V_2)$ yields a way to choose weights on the yellow and blue components summing to at most 2 such that every vertex in $V_1 \cup V_2$ is covered by components with weights summing to at least 1. Starting from this choice of weights and adding a weight of 1 to each of C_1 and C_2 then allows us to apply Corollary 6 with $(x) = (2; 4)$, which yields the bound

$$z \geq \frac{2}{12 + \frac{p}{80}} \geq \frac{2}{21} \geq \frac{2}{14 + \frac{p}{96}};$$

which again succeeds.

The remaining case to consider is where there are three components $C_1; C_2; C_3$ with $C_1 \cup C_2 \cup C_3 = V$, but all of $V_1 := C_1 \cap (C_2 \cup C_3)$, $V_2 := C_2 \cap (C_3 \cup C_1)$, and $V_3 := C_3 \cap (C_1 \cup C_2)$ are non-empty. Without loss of generality, we can assume that C_1 is red, C_2 is orange, and C_3 is yellow. For each pair of the V_i , only two colors are possible on the edges between them, one of which is blue. Our aim is to apply Lemma 7 to each of these pairs and then combine the results into an appropriate choice of weights on the components of the overall coloring of K_n .

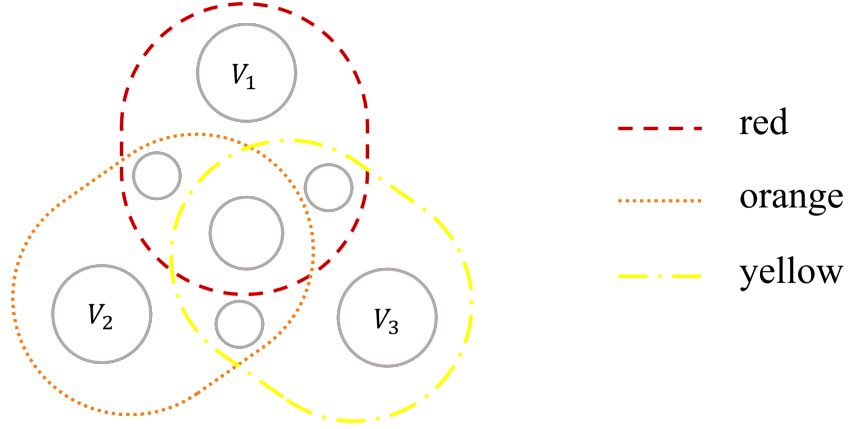


Figure 2. The vertex partition in Lemma 8

For each of the pairs $(V_i; V_j)$, we apply Lemma 7 to the complete bipartite graph between V_i and V_j , yielding one of the cases (a), (b), or (c) described therein. We further split case (a) into (a1) and (a2), depending on whether the corresponding component is blue (case (a1)) or not (some cases are not mutually exclusive, but this will not be an issue).

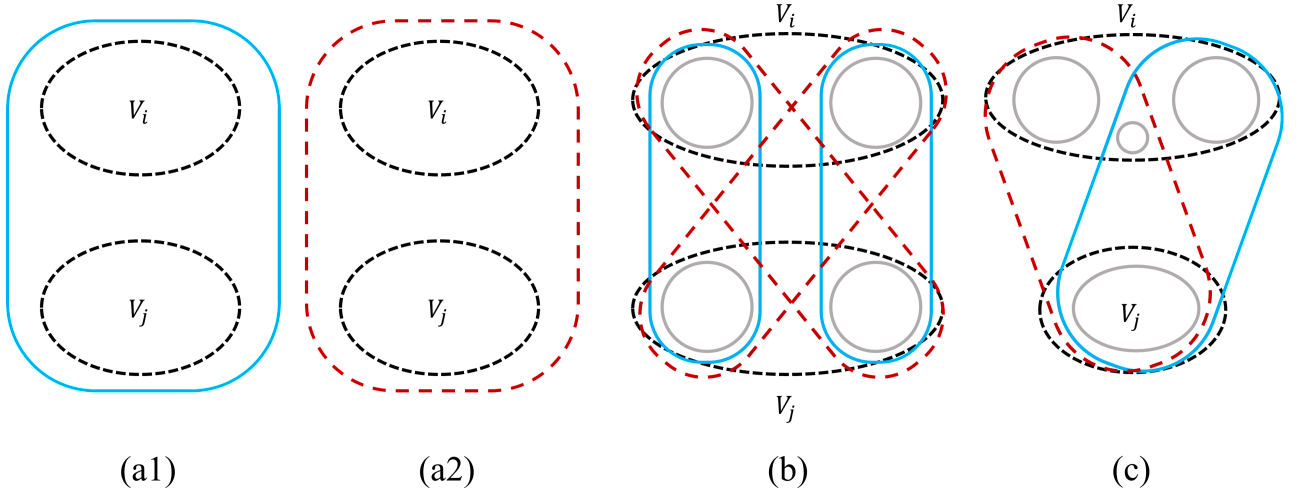


Figure 3. The cases between pairs in Lemma 8

For any case constellation between the pairs $(V_i; V_j)$, we aim to exhibit at most two components other than $C_1; C_2; C_3$ covering $V_1 \sqcup V_2 \sqcup V_3$. Together with C_1, C_2 , and C_3 , this will yield a 2-cover of the entire graph K_n by at most 5 components (since all vertices outside $V_1 \sqcup V_2 \sqcup V_3$ are in at least two of the C_i and so are already 2-covered). Applying Corollary 6 with $(; x) = (2; 5)$ would then give the claimed bound of

$$z \leq \frac{2}{14 + \frac{2}{96}} :$$

If case (a) (that is, either (a1) or (a2)) occurs for at least two pairs $(V_i; V_j)$, then the relevant components readily give a cover of $V_1 \sqcup V_2 \sqcup V_3$. Hence, from now on we can assume that this is not the case.

Suppose case (a1) occurs on one of the pairs, say $(V_1; V_2)$. If $(V_1; V_3)$ satisfies case (b), then $V_1 [V_2 [V_3$ is covered by a single blue component. If $(V_1; V_3)$ satisfies case (c) (in either direction), then $V_1 [V_2$ is covered by a blue component B and $V_3 \cap B$ is covered by a non-blue component. Thus, we can assume from now on that case (a1) never happens.

If case (b) occurs, say on $(V_1; V_2)$, then one of the two remaining pairs, say $(V_1; V_3)$, must satisfy case (b) or (c). If $(V_1; V_3)$ satisfies case (b), then $V_1 [V_2 [V_3$ is covered by at most two blue components. The same is true if $(V_1; V_3)$ satisfies case (c) directed away from V_1 . Lastly, if $(V_1; V_3)$ satisfies (c) directed toward A_1 , then $V_1 [V_2$ is covered by a blue component B^0 and $V_3 \cap B^0$ is covered by a non-blue component. Hence, we may also assume from now on that case (b) never happens. In other words, each pair satisfies either case (a2) or (c) and the former occurs at most once.

If all three pairs $(V_i; V_j)$ satisfy case (c), then the respective ‘case (c) directions’ result in an auxiliary 3-vertex tournament, which is either cyclic or transitive. In the cyclic case we again have that $V_1 [V_2 [V_3$ is contained in a single blue component. In the transitive case, without loss of generality, let each $(V_i; V_j)$ for $i < j$ be directed toward V_i . Then again $V_1 [V_2$ lies in a single blue component B^{00} and $V_3 \cap B^{00}$ can be covered by a non-blue component.

Hence, we may assume that one pair, say $(V_1; V_2)$, is of type (a2), with D being the non-blue component between them, and the other two pairs are of type (c). If either of these type (c) pairs is directed toward V_3 , then $V_1 [V_2 [V_3$ is covered by D and the corresponding blue component. Thus, the last case to consider is when both $(V_1; V_3)$ and $(V_2; V_3)$ are directed away from V_3 .

In that case, let B_1 and D_1 be the blue and non-blue components between $(V_1; V_3)$, respectively, and define B_2 and D_2 similarly with respect to $(V_2; V_3)$. If $B_1 \setminus B_2 \setminus V_3 = \emptyset$, then B_1 and B_2 coalesce into a single component B in K_n and $V_3 \cap B$ can be covered by a non-blue component. On the other hand, if $B_1 \setminus B_2 \setminus V_3 \neq \emptyset$, then

$$D_1 [D_2 (V_3 \cap B_1) [(V_3 \cap B_2) = V_3$$

and, since D_1 and D_2 cover V_1 and V_2 , respectively, we again obtain that two components, namely D_1 and D_2 , cover $V_1 [V_2 [V_3$. Thus, in every case, we have a component with at least $\frac{2}{14+96}n^2$ edges, as desired.

The lower bound of $\frac{2}{14+96}n^2$ in the conclusion of Lemma 8 can be improved slightly through a more careful analysis. However, we chose to omit this more involved proof, since the improved estimate is not needed for the tight case in Theorem 2.

For any vertex v and any component C with $v \in C$, we can assume there is a vertex $w \in C$ not in any other component containing v ; otherwise, we are done by Lemma 8. In particular, we can assume that no component has its vertex set contained entirely within the vertex set of another component.

For the sake of clarity, we say that two vertices v_1 and v_2 are joined in a given color if the edge between them is of that color, while a set of vertices is connected in a given color if they share a component of that color. Call two vertices v_1 and v_2 equivalent if they are contained in a common component, that is, they are connected, in each color. Clearly, this is an equivalence relation, so we may speak of equivalence classes with respect to it. Note that if v_1 and v_2 share exactly three components, then, considering a vertex w outside of these three components (if there is no such w , we are again done by Lemma 8), both v_1 and v_2 must be joined to w via the fourth color, meaning that they share all four components, a contradiction. Hence, we can assume that any two vertices are either equivalent or share at most two components. If v_1 and v_2 share precisely two components, let us call them biconnected.

The next two lemmas cover a pair of general cases where we again get a lower bound of at least $\frac{2}{14+96} \frac{n}{2}$ edges.

Lemma 9. If there is a pair of components of different colors that do not intersect, then there is a monochromatic component with at least $\frac{2}{14+96} \frac{n}{2}$ edges.

Proof. By symmetry, we can assume that the two disjoint components R and B are red and blue, respectively. All edges between R and B are orange or yellow, so we can apply Lemma 7 to the complete bipartite graph $G_{R;B}$ between them. If either R or B is contained entirely in a single orange or yellow component in $G_{R;B}$ and thus in G , we are done by Lemma 8, so assume otherwise. Then only case (b) of Lemma 7 can apply, where there are exactly two orange components and two yellow components in $G_{R;B}$ (none of which can coalesce together in G). It is easy to see that in this case any two vertices in R that share an orange component also share a yellow component and are therefore equivalent. Hence, there are exactly two equivalence classes of vertices in R and, likewise, exactly two equivalence classes in B . Taking representatives $v_1, v_2 \in R$ and $w_1, w_2 \in B$ of these equivalence classes, let

$$X = \{C \in \mathcal{C} : C \setminus \{v_1, v_2, w_1, w_2\} = \emptyset \text{ or } R; B\}$$

Define $w(C) = 1$ when $C \in X$ and $w(C) = 0$ otherwise. We claim that this choice of w satisfies the conditions of Corollary 6 with $(\gamma; x) = (3; 8)$. If $v_1, v_2 \in C$ for any component $C = R$, then $R \in X$, a contradiction, so v_1 and v_2 do not share any non-red components; likewise, w_1 and w_2 do not share any non-blue components. Thus, X contains exactly two components of each color, so $\sum_{C \in X} w(C) = 3$. Moreover, by construction, for $u \in R \setminus B$, we have

Now consider a vertex $u \notin R \setminus B$. Since the two equivalence classes in R do not share any non-red components, there are at least two colors among the edges between u and R and, similarly, at least two colors among the edges between u and B . If there are exactly two colors among the edges between u and $R \setminus B$, then these colors must be orange and yellow, a contradiction because $R \setminus B$ cannot be covered by the union of an orange component and a yellow one. So there are at least three colors among the edges between u and $R \setminus B$, which means $\sum_{C \ni u} w(C) \geq 3$. Thus, we can apply Corollary 6 with $(\gamma; x) = (3; 8)$, yielding a monochromatic component with at least $\frac{2}{17+33} \frac{n}{2} > \frac{2}{14+96} \frac{n}{2}$ edges, as desired.

Lemma 10. If there is a pair of biconnected vertices, then there is a monochromatic component with at least $\frac{2}{14+96} \frac{n}{2}$ edges.

Proof. Suppose that v_1 and v_2 are biconnected. By symmetry, we can assume that the two components they share are red and orange and that there are at least as many orange components as red components in our coloring. We can also assume that there are at least two red components; otherwise, we are done by Lemma 8. If there are exactly two orange components, then applying Proposition 3 to the set of red and orange components with $(\gamma; x) = (2; 4)$ yields $z \geq \frac{2}{12+80}$ and we are again done, so we can assume that there are at least three orange components. Let R and O be the joint red and orange components, respectively, of v_1 and v_2 and, for $i \in \{1, 2\}$, let the yellow and blue components containing v_i be Y_i and B_i , respectively. Let $T = V \setminus R$.

Since $\{R; O; Y_1; B_1\}$ are the four components of v_1 , their union is the whole of V . Hence, $D := B_1 \cap (R \setminus O) = \emptyset$, as otherwise $\{R; O; Y_1\}$ would satisfy the assumptions of Lemma 8 and, analogously, $E := Y_1 \cap (R \setminus O) = \emptyset$. Observe that v_2 must be joined in yellow to all vertices in D and in blue to all vertices in E . This means that $D \setminus E = \emptyset$ and, consequently, v_1 is joined to all of D in blue and to all of E in yellow. Furthermore, $E = B_2 \cap (R \setminus O)$ and $D = Y_2 \cap (R \setminus O)$.

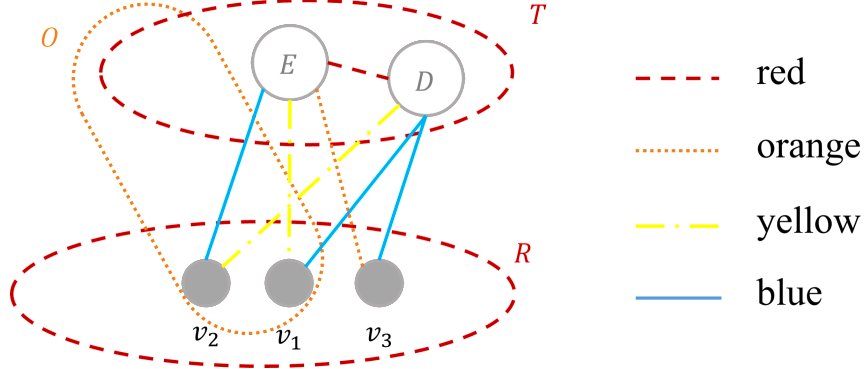


Figure 4. The situation in Lemma 10

If there is a vertex $w \in Y_1 \setminus B_1 \setminus R \cap O$, then w shares exactly three components with v_1 , a contradiction. Hence, we may conclude that

$$Y_1 \setminus B_1 \cap O = Y_1 \setminus B_1 \cap (R \cup O) = D \setminus E = ; \quad (3.1)$$

and, similarly, $Y_2 \setminus B_2 \cap O = ;$.

Now consider a vertex $v_3 \in R \cap O$. If v_3 is joined in orange to all of $T \cap O = D \cup E$, then either there are only two orange components (O and the component of v_3) or there is an orange component whose vertex set is contained within R ; either case is a contradiction. Therefore, v_3 is joined in blue or yellow to some vertex in $T \cap O$, which means it is in one of $B_1; B_2; Y_1; Y_2$. By symmetry, we can assume it is in B_1 , so by (3.1) we have $v_3 \notin Y_1$. We conclude that v_3 must be joined to all of E in orange (see Figure 4). Since v_3 was chosen from $R \cap O$ arbitrarily, every vertex in $R \cap O$ is joined either to all of D or to all of E in orange. If any orange component does not intersect R , we are done by Lemma 9. Otherwise, we have at most three orange components in total, so by our earlier assumption it must be exactly three. In particular, there will be a vertex in $R \cap O$ connected in orange to all of D and none of E . It follows that every edge between D and E must be red, so all of $D \cup E = V \cap (R \cup O)$ is contained in a single red component. If there is a third red component, it must lie entirely inside O , so we are done by Lemma 8. Therefore, we can assume there are exactly two red components.

Next, note that v_1 and v_3 share components in red and blue but not in orange, which means v_1 and v_3 are biconnected. Therefore, by the same argument as above (with orange and blue swapped), the total number of blue components is also three. Let B_3 be the third blue component and note that $B_3 \cap R \cup O$. Consequently, we must have $B_3 \setminus R \cap O = ;$, as otherwise B_3 would be contained in O . Let $v_4 \in B_3 \setminus R \cap O$ be an arbitrary vertex. By using similar reasoning as for v_3 , we have that v_4 is in one of $B_1; B_2; Y_1; Y_2$; since $v_4 \in B_3$, it must be in one of Y_1 and Y_2 . We conclude, as before for v_3 and blue, that there are exactly three yellow components.

Thus, we have exactly two red components and exactly three components of every other color, for a total of exactly 11 monochromatic components. By the pigeonhole principle, one of these components must have at least $\frac{1}{11} \cdot \frac{n}{2} > \frac{2}{14+96} \cdot \frac{n}{2}$ edges, as needed.

We are now ready to complete the proof of our main theorem.

Proof of Theorem 2. If the hypotheses of any of Lemmas 8, 9, or 10 hold, we have a component with at least $\frac{2}{14+96} \cdot \frac{n}{2}$ edges, as needed, so we can assume otherwise. Thus, every pair of components of different colors intersects and any pair of vertices sharing at least two components are equivalent.

We claim that in this case there are at most three components of each color. Without loss of generality, let red be a color with the fewest components. Let R be a red component and x a vertex $v \notin R$. Let O , Y , and B be the orange, yellow, and blue components of v , respectively. By Lemma 10, we can assume that all vertices in $R \setminus O$ are equivalent and similarly for $R \setminus Y$ and $R \setminus B$. However, $O \cap B \cap Y \cap R$, as every vertex in R is adjacent to v via a non-red edge. So the vertices of R can be partitioned into at most three equivalence classes. Thus, there are at most three components of each color that intersect $V(R)$ and since, by Lemma 9, we can assume that R intersects every non-red component, there are at most three components of each color in total, as claimed.

If there is a color with fewer than three components, then $jC_j \leq 11$ and there is some component with at least $\frac{1}{11} \frac{n}{2}$ edges. Otherwise, we will show that our coloring matches the extremal construction claimed. Indeed, by assumption, the equivalence class of a vertex is determined by its red and orange components. Let $R = R_1; R_2; R_3$ be the red components and $O_1; O_2; O_3$ the orange components and let $V_{ij} = R_i \setminus O_j$ for $1 \leq i, j \leq 3$. The V_{ij} form a partition of $V(G)$ and each is an equivalence class of vertices. Since we can assume that every pair of components of different colors intersects, the V_{ij} are all non-empty.

Every component contains exactly three equivalence classes: one for each orange component if it is red and one for each red component if it is not red. Each pair of the three vertex sets $V_{11}; V_{22}; V_{33}$ must share at least one component, each of which must be yellow or blue; one of the two colors occurs at least twice, say yellow. Then $V_{11} \cup V_{22} \cup V_{33}$ is connected in yellow and in fact must form a yellow component, since each yellow component contains exactly three equivalence classes. Then neither V_{13} nor V_{31} shares a red, orange, or yellow component with V_{22} , so $V_{13} \cup V_{22} \cup V_{31}$ forms a blue component and, similarly, so do $V_{12} \cup V_{21} \cup V_{33}$ and $V_{11} \cup V_{23} \cup V_{32}$. Repeating this argument shows that $V_{12} \cup V_{23} \cup V_{31}$ forms another yellow component, as does $V_{13} \cup V_{21} \cup V_{32}$. Hence, we are in exactly the extremal configuration claimed.

Thus, only the claimed extremal configuration can have fewer than $\frac{2}{14+\frac{1}{96}} \frac{n}{2}$ edges in every component and in this extremal configuration, since there are 12 components, we instead get a lower bound of $\frac{1}{12} \frac{n}{2}$ edges, as desired.

As in [1] and [9], it is possible to amend our argument to show that every 4-coloring of K_n contains a monochromatic trail or circuit of length at least $(\frac{1}{24} + o(1))n^2$. We omit the details, but briefly note the main idea, which is to delete a sparse subgraph in order to guarantee that each component is Eulerian and then work around these omitted edges.

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