On the size-Ramsey number of grids

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Abstract

We show that the size-Ramsey number of the $\sqrt{n} \times \sqrt{n}$ grid graph is $O(n^{5/4})$, improving a previous bound of $n^{3/2+o(1)}$ by Clemens, Miralaei, Reding, Schacht, and Taraz.

1 Introduction

For graphs G and H, we say that G is Ramsey for H, and write $G \to H$, if every 2-colouring of the edges of G contains a monochromatic copy of H. In 1978, Erdős, Faudree, Rousseau, and Schelp [9] pioneered the study of the size-Ramsey number $\hat{r}(H)$, defined as the smallest integer m for which there exists a graph G with m edges such that $G \to H$. The existence of the usual Ramsey number $r(H)$ shows that this notion is sensible, since, for any H, it is easy to see that $\hat{r}(H) \leq \binom{r(H)}{2}$. When H is a complete graph, this inequality is an equality, a simple fact first observed by Chvátal.

An early example showing that size-Ramsey numbers can exhibit interesting behaviour was found by Beck [1], who showed that P_n, the path with n vertices, satisfies $\hat{r}(P_n) = O(n)$, which is significantly smaller than the $O(n^2)$ bound that follows from applying the inequality above and the corresponding bound $r(P_n) = O(n)$ for the usual Ramsey number of P_n. In a follow-up paper, Beck [2] asked whether a similar phenomenon occurs for all bounded-degree graphs, that is, whether, for any integer $\Delta \geq 3$, there exists a constant c such that any graph H with n vertices and maximum degree Δ has size-Ramsey number at most cn. Although Rödl and Szemerédi [19] showed that this question has a negative answer already for $\Delta = 3$, much work has gone into extending Beck's result to other natural families of graphs, including: cycles [14], bounded-degree trees [10], powers of paths and bounded-degree trees [3, 5, 13], and more besides.

Most of the known families with linear size-Ramsey numbers have a bounded structural parameter, such as bandwidth [5] or, more generally, treewidth [15] (though see the recent papers [8, 18] for examples with a somewhat different flavour). However, a fairly simple family of graphs which does not fall into any of these categories, but may still have linear size-Ramsey numbers, is the family of two-dimensional grid graphs. For $s \in \mathbb{N}$, the $s \times s$ grid is the graph with vertex set $[s] \times [s]$ where two pairs are adjacent if and only if they differ by one in exactly one coordinate. Obviously, the maximum degree of the $s \times s$ grid is four, but its bandwidth and treewidth are both exactly s (see, e.g., [4]), so the problem of estimating the size-Ramsey number of this graph,

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and usually we will take $s = \sqrt{n}$ so that the graph has n vertices, provides an interesting test case for exploring new ideas and techniques.

Regarding upper bounds for the size-Ramsey number of the $\sqrt{n} \times \sqrt{n}$ grid, an important result of Kohayakawa, Rödl, Schacht, and Szemerédi [17], which says that every graph H with n vertices and maximum degree Δ satisfies $\hat{r}(H) \leq n^{2-1/\Delta+o(1)}$, immediately yields the bound $n^{7/4+o(1)}$. This was recently improved by Clemens, Miralaei, Reding, Schacht, and Taraz [6] to $n^{3/2+o(1)}$ (and an alternative proof of this bound was also noted in our recent paper [7]). The goal of this short note is to provide an elementary proof of an improved upper bound.

Theorem 1.1. *There exists a constant $C > 0$ such that the size-Ramsey number of the $\sqrt{n} \times \sqrt{n}$ grid graph is at most $Cn^{5/4}$.*

Like much of the work on size-Ramsey numbers, the previous bounds for grids were obtained by applying the sparse regularity method to show that every 2-colouring of the edges of the Erdős–Rényi random graph $G_{n,p}$, for some appropriate density p, contains a monochromatic copy of the grid. However, it is a simple exercise in the first moment method to show that for $p \ll n^{-1/2}$ the random graph $G_{n,p}$ with high probability does not contain the $s \times s$ grid graph as a subgraph if $s = \Theta(\sqrt{n})$, so the bound $O(n^{3/2})$ is the best that one can hope to achieve using this procedure.

To see how it is that we gain on this bound, suppose that $s = \sqrt{n}$. It is known [14] that there are $K, \Delta > 0$ and a graph H with Ks vertices and maximum degree at most Δ which is Ramsey for C_s, the cycle of length s. Consider now a 'blow-up' Γ of H obtained by replacing every $x \in V(H)$ by an independent set V_x of order $\Theta(s)$ and every $xy \in H$ by a bipartite graph $\Gamma(V_x, V_y)$ in which every edge exists independently with probability $p = \Theta(s^{-1/2})$. With high probability, such a blow-up contains $\Theta(s^{5/2}) = \Theta(n^{5/4})$ edges. That is, instead of revealing a random graph $G_{n,p}$ on all $n = \Theta(s^2)$ vertices, we only reveal edges that lie within $\Theta(s)$ bipartite subgraphs, each with parts of order $\Theta(s)$. This salvages a significant number of edges which would otherwise go to waste.

Consider now a 2-colouring of Γ and recall that H was chosen so that $H \to C_s$. A key lemma, Lemma 2.3 below, then allows us to conclude that there are sets $V_1, \ldots, V_s$ in Γ and a collection $U_i \subseteq V_i$ of large subsets such that all (U_i, U_{i+1}) with $i \in [s]$, where addition is taken modulo s, are 'regular' in the same colour. We may then sequentially embed the vertices of the grid so that the first row is embedded into $U_1, \ldots, U_s$, the second into $U_2, \ldots, U_s, U_1$, and so on.

2 Definitions and key lemmas

In this section, we recall several standard definitions and note two key lemmas that will be needed in the proof of Theorem 1.1. Most of these revolve around the concept of sparse regularity (for a thorough overview of which we refer the reader to the survey by Gerke and Steger [12]).

For $\varepsilon > 0$ and $p \in (0, 1]$, a pair of sets (V_1, V_2) is said to be (ε, p)-lower-regular in a graph G if, for all $U_i \subseteq V_i$, $i \in \{1, 2\}$, with $|U_i| \geq \varepsilon |V_i|$, the density $d_G(U_1, U_2) = e_G(U_1, U_2)/(|U_1||U_2|)$ of edges between U_1 and U_2 satisfies

$$d_G(U_1, U_2) \geq (1 - \varepsilon)p.$$

Immediately from this definition, we get that in every (ε, p)-lower-regular pair (V_1, V_2), for each $i \in \{1, 2\}$, all but at most $\varepsilon |V_i|$ vertices in V_i have degree at least $(1 - \varepsilon)p|V_{3-i}|$ into V_{3-i} — a fact we will make use of in the proof of Theorem 1.1. Another useful and well-known property is that lower-regularity is inherited on large sets.

Lemma 2.1. Let 0 " , p P p0; 1s, and let pV1; V2q be an p"; pq-lower-regular pair. Then any pair of subsets V_i^1 „ V_i, i P t1; 2u, with $|V_i^1|$ ¥ $|V_i|$ form an p"{; pq-lower-regular pair.

For ¡ 0 and p P p0; 1s, a graph G is said to be p; pq-uniform if, for all disjoint X; Y „ V pGq with |X|; |Y | ¥ |V pGq|, the density of edges between X and Y satises d_GpX; Y q p1 qp. If only the upper bound holds, the graph is said to be upper-uniform.[1] For example, it is easy to see that the random graph $G_{n;p}$ is with high probability pop1q; pq-uniform whenever p " 1{n. If G pV1; V2; Eq is bipartite, we say that G is p; pq-uniform or upper-uniform if the same conditions hold for all X „ V1 and Y „ V2 with |X| ¥ |V1| and |Y | ¥ |V2|. In order to prove our main technical lemma, we rely on the following result, a simple corollary of [16, Lemma 6], whose proof follows a density increment argument. The same conclusion can also be obtained by an application of the sparse regularity lemma.

Lemma 2.2. For all 0 " 1{2 and P p0; 1q, there exists ¡ 0 such that the following holds for every p P p0; 1s. Let G pV1; V2; Eq be a p; pq-upper-uniform bipartite graph with |V1| |V2| and |E| ¥ |V1||V2|p. Then there exist U_i „ V_i, i P t1; 2u, with $|U_i|$ $|V_i|$ such that pU1; U2q is p"; pq-lower-regular in G.

The next lemma is the crux of our argument. Here and elsewhere, we say that pX; Y ; Eq is lower-regular if pX; Y q is lower-regular with respect to the set of edges E.

Lemma 2.3. For every r; ¥ 2 and " ¡ 0, there exists ¡ 0 such that the following holds for every p P p0; 1s. Let H be a graph on at least two vertices with pHq ⁄ and let be obtained by replacing every x P V pHq with an independent set V_x of suciently large order n and every xy P H by a p; pq-uniform bipartite graph between V_x and V_y. Then, for every r-colouring of the edges of , there exists an r-colouring ’ of the edges of H and, for every x P V pHq, a subset U_x „ V_x of order $|U_x|$ n such that pU_x; U_y; $E_{’pxyq}$q is p"; p{p2rqq-lower-regular for each xy P H, where $E_{’pxyq}$ „ Ep q stands for the edges in colour ’pxyq.

Proof. Given ", r, and , we let 1{p2rq, " 1 : ", 1 2:2p" 1; q, and, for every i ; : : : ; 1, sequentially take "i "i 1i 1 and i 2:2p"i; q. Lastly, let iPr 1s i.

Fix any r-colouring of (the edges of) and, for every c P rrs, let c stand for the subgraph (in terms of edges) in colour c. Note that H has edge-chromatic number at most 1. In other words, there exists a partition of the edges of H into H_1; : : : ; H 1 such that each H_i is a matching. We nd the required collection tU_xu$_{xPV pHq}$ by maintaining the following condition for every i P r 1s: for every x P V pHq, there exists a chain V_x U_x^0 …U_x^1 …U_x^2 … …U_x^i such that

(i) $|U_x^j|$ j $|U_x^{j1}|$ for all j P ris and

(ii) for every xy P j⁄i H_j, pU_x^i; U_y^iq is p"i; pq-lower-regular in c for some c P rrs.

Consequently, for i 1, we obtain sets U_x „ V_x, for every x P V pHq, of order $|U_x|$ p iPr 1s iqn n such that pU_x; U_yq is p" 1; pq-lower-regular and, thus, p"; pq-lower-regular for every xy P H. It remains to show that we can indeed do this.

[1]For consistency with the existing literature and for historical reasons, we use both ‘regular’ and ‘uniform’ as terms, even though they are basically the same concept.

Consider first $i = 1$. For each $xy \in H_1$, let $c \in [r]$ be the majority colour in $\Gamma[V_x, V_y]$. As $e_{\Gamma_c}(V_x, V_y) \geq (1-\gamma)n^2p/r$, we may apply Lemma 2.2 with ε_1 (as ε) and $\Gamma_c[V_x, V_y]$ (as G) to obtain sets U_x^1, U_y^1 with the desired properties. For every $x \in V(H)$ which is isolated in H_1, we simply take an arbitrary subset $U_x^1 \subseteq V_x$ of order $\alpha_1|U_x|$. Thus, the required condition holds for $i = 1$.

Suppose now that the condition holds for some $i \geq 1$ and let us show that it also holds for $i+1$. As above, for every $xy \in H_{i+1}$, let $c \in [r]$ be the majority colour in $\Gamma[V_x, V_y]$. Since $\Gamma[V_x, V_y]$ is (p, γ)-uniform and, by (i), $|U_x^i|, |U_y^i| \geq \alpha_i n$, we have $e_\Gamma(U_x^i, U_y^i) \geq (1-\gamma)|U_x^i||U_y^i|p$ and, hence,

$$(1-\gamma)|U_x^i||U_y^i|p/r \leq e_{\Gamma_c}(U_x^i, U_y^i) \leq (1+\gamma)|U_x^i||U_y^i|p.$$

Lemma 2.2 applied to $\Gamma_c[U_x^i, U_y^i]$ with ε_{i+1} (as ε) gives sets $U_x^{i+1} \subseteq U_x^i$ and $U_y^{i+1} \subseteq U_y^i$ of order

$$|U_x^{i+1}| = \alpha_{i+1}|U_x^i| \quad \text{and} \quad |U_y^{i+1}| = \alpha_{i+1}|U_y^i|$$

for which (U_x^{i+1}, U_y^{i+1}) is (ε_{i+1}, p)-lower-regular in Γ_c. For every $x \in V(H)$ which is isolated in H_{i+1}, we again take an arbitrary subset $U_x^{i+1} \subseteq U_x^i$ of order $\alpha_{i+1}|U_x^i|$. Observe also that, for every $xz \in \bigcup_{j \leq i} H_j$, since (U_x^i, U_z^i) was (ε_i, p)-lower-regular in $\Gamma_{c'}$ for some $c' \in [r]$ and $|U_x^{i+1}| = \alpha_{i+1}|U_x^i|$, Lemma 2.1 and the fact that $\varepsilon_i/\alpha_{i+1} \leq \varepsilon_{i+1}$ imply that (U_x^{i+1}, U_z^{i+1}) is (ε_{i+1}, p)-lower-regular in $\Gamma_{c'}$, as desired. This completes the proof. □

We also need a variant of a result from our previous paper [7, Lemma 3.5] about regularity inheritance. While that result was stated for the usual (full) notion of regularity, we only need lower-regularity here, allowing us to save a factor of $(\log n)^{1/2}$.

Lemma 2.4. For all $\varepsilon, \beta, \gamma > 0$, there exist positive constants $\varepsilon'(\varepsilon, \beta)$ and $C(\varepsilon, \beta, \gamma)$ such that for $p \geq Cn^{-1/2}$, with probability at least $1 - o(n^{-5})$, the random graph $\Gamma \sim G_{n,p}$ has the following property.

Suppose $G \subseteq \Gamma$ and $V_1, V_2 \subseteq V(\Gamma)$ are disjoint subsets of order $\tilde{n} \geq \gamma n$ such that (V_1, V_2) is (ε', p)-lower-regular in G. Then there exists $B \subseteq V(\Gamma)$ of order $|B| \leq \beta\tilde{n}$ such that, for each $v, w \in V(\Gamma) \setminus (V_1 \cup V_2 \cup B)$ (not necessarily distinct), the following holds: for any two subsets $N_v \subseteq N_\Gamma(v, V_1)$ and $N_w \subseteq N_\Gamma(w, V_2)$ of order $\tilde{n}p/4$, both (N_v, V_2) and (N_v, N_w) are (ε, p)-lower-regular in G.

Sketch of the proof. The proof proceeds along the same lines as the proof of [7, Lemma 3.5]. The only difference is that there we made use of an inheritance lemma for full regularity (namely, Corollary 3.5 in [20]), which requires the sets on which regularity is inherited to be of order at least $C \log n/p$, resulting in the requirement that $p \geq C(\log n/n)^{1/2}$. However, for lower-regularity, one can instead use the inheritance lemma of Gerke, Kohayakawa, Rödl, and Steger [11, Corollary 3.8], which only requires the sets to be of order at least C/p, resulting in $p \geq Cn^{-1/2}$. The rest of the proof remains exactly the same. □

3 Proof of Theorem 1.1

Since it requires no additional work, we will actually prove the r-colour analogue of Theorem 1.1. More precisely, we will show that for every integer $r \geq 2$ there exists a graph of order n with $O(n^{5/4})$ edges for which every r-colouring of the edges contains a monochromatic copy of the $\gamma\sqrt{n} \times \gamma\sqrt{n}$ grid for some $\gamma > 0$.

By a result of Haxell, Kohayakawa, and Łuczak [14, Theorem 10], there exist constants $K, \Delta > 0$, both depending only on r, such that, for every sufficiently large $s \in \mathbb{N}$, there is a graph H on Ks vertices with maximum degree at most Δ which has the property that every r-colouring of its edges contains a monochromatic copy of C_ℓ, the cycle of length ℓ, for every $\log s \leq \ell \leq s$. Let

$$\gamma = 1/(2r), \quad \varepsilon = \gamma/256, \quad \varepsilon_1 = \varepsilon_{2.4}(\varepsilon/9, \gamma), \quad \delta = \delta_{2.3}(r, \gamma, \varepsilon_1), \quad \text{and} \quad \beta = \min\{1/(4K), \varepsilon/4\}.$$

We show that the size-Ramsey number of the $s \times s$ grid is $O(s^{5/2})$, which, for $s = \sqrt{n}$, implies the desired statement.

Let Γ be a graph obtained by replacing every vertex $x \in V(H)$ by an independent set V_x of order s and every edge $xy \in H$ by a bipartite graph between V_x and V_y in which each edge exists independently with probability $p = Cs^{-1/2}$ for some sufficiently large constant $C > 0$. With high probability, Γ has the following property:

(A1) $e(V_x', V_y') = (1 \pm \delta)|V_x'||V_y'|p$ for every $xy \in H$ and $V_x' \subseteq V_x$ and $V_y' \subseteq V_y$ with $|V_x'||V_y'|p \geq \frac{1}{100}s^{3/2}$.

This is a standard feature of random graphs and follows from the Chernoff bound together with an application of the union bound. In particular, it establishes that with high probability $\Gamma[V_x, V_y]$ is (δ, p)-uniform for every $xy \in H$ and, therefore, Γ has at most

$$Ks\Delta/2 \cdot (1 + \delta)s^2 p = O(s^{5/2})$$

edges. Additionally, with high probability, Γ is such that every $\Gamma[V_x \cup V_y \cup V_z]$ has the property of Lemma 2.4 (applied with $\varepsilon/9$ as ε, $\beta/3$ as β, and $3s$ as n) for every path xyz of length two in H.[2] This again follows from the union bound, as there are $O(s)$ such paths in total and the conclusion of Lemma 2.4 holds with probability $1 - o(s^{-5})$ for every fixed path. We now fix an outcome of Γ which satisfies all of these properties.

Consider some r-colouring of the edges of Γ and let φ be the colouring of the edges of H given by Lemma 2.3 (applied with ε_1 as ε). By the choice of H, this colouring contains a monochromatic copy of C_s, which, without loss of generality, we may assume has vertices $1, \dots, s$. Therefore, there is a colour $c \in [r]$ and sets U_i of order $\tilde{s} \geq \gamma s$ in V_i such that, for every $i \in [s]$, the pair (U_i, U_{i+1}) is (ε_1, p)-lower-regular in the subgraph of Γ induced by colour c, where we identify $s + i$ with i. Let G be the graph induced by these sets whose edges are the edges of Γ of colour c. We will show that G contains the $s \times s$ grid as a subgraph.

For every $i \in [s]$, let $B \subseteq U_i \cup U_{i+1} \cup U_{i+2}$ be the set given by Lemma 2.4 (which was applied with $\varepsilon/9$ as ε, $\beta/3$ as β, and $3s$ as n) on $\Gamma[U_i \cup U_{i+1} \cup U_{i+2}]$, which is a set of 'bad vertices' for the pair (U_{i+1}, U_{i+2}). As each U_i is a part of three such applications, by the chosen properties of Γ, for every $i \in [s]$ there exists a set $B_i \subseteq U_i$ of order $|B_i| \leq \varepsilon\tilde{s}$ such that:

(B1) $(N_v, U_{i+2} \setminus B_{i+2})$ is (ε, p)-lower-regular[3] in G for every $v \in U_i \setminus B_i$ and $N_v \subseteq N_G(v, U_{i+1})$ of order $\tilde{s}p/4$ and

(B2) (N_v, N_u) is (ε, p)-lower-regular in G for every $v \in U_i \setminus B_i$, $u \in U_{i+1} \setminus B_{i+1}$ and $N_v \subseteq N_G(v, U_{i+1})$, $N_u \subseteq N_G(u, U_{i+2})$, each of order $\tilde{s}p/4$.

Our plan is to embed the vertex (i, j) of the $s \times s$ grid into U_{i+j-1}. The next claim helps us achieve this.

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[2] Technically, to apply the lemma, we must also temporarily reveal the edges between V_x and V_z and within each V_x, V_y, V_z, but, unless xz is itself an edge of H, these are all then removed from Γ.

[3] The conclusion of Lemma 2.4 states that (N_v, U_{i+2}) is $(\varepsilon/9, p)$-lower-regular, but, as B_{i+2} is small, Lemma 2.1 implies that $(N_v, U_{i+2} \setminus B_{i+2})$ is (ε, p)-lower-regular.

Claim 3.1. Let $i \in [s]$. Suppose that sets $S_{i+j-1} \subseteq U_{i+j-1} \setminus B_{i+j-1}$ of order $\tilde{n}p/4$ are given for each $j \in [s]$ and that (S_{i+j-1}, S_{i+j}) and $(S_{i+j-1}, U_{i+j} \setminus B_{i+j})$ are (ε, p)-lower-regular. Then, for every $Q_{i+j-1} \subseteq U_{i+j-1}$, $j \in [s]$, of order $|Q_{i+j-1}| \le 2\varepsilon\tilde{n}$, there exists a path $v_1, \dots, v_s$ with each $v_j \in S_{i+j-1}$ such that $|N_G(v_j, U_{i+j} \setminus Q_{i+j})| \ge \tilde{n}p/4$.

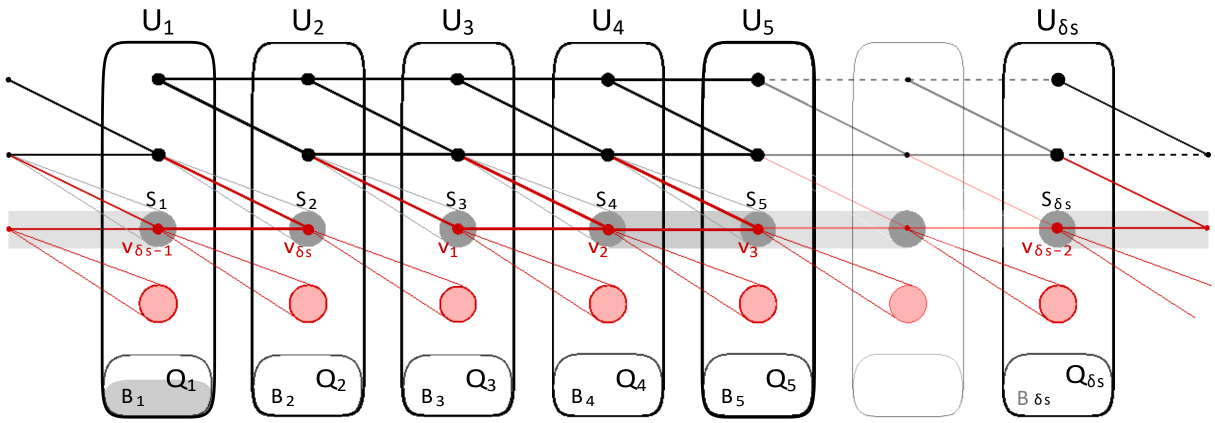

Figure 1: A picture showing the first two rows of the grid already embedded (the thick black lines), the candidate sets for the third row (the grey blobs $S_3, S_4, \dots, S_s, S_1, S_2$), and (in red) the path $v_1, v_2, \dots, v_s$ given by Claim 3.1, together with the corresponding neighbourhoods $N_G(v_j, U_{i+j} \setminus Q_{i+j})$ (the red blobs).

Before proving the claim, we show how to complete the embedding of the grid assuming that it holds. We start by embedding the first row. Let $v_1 \in U_1 \setminus B_1$ be a vertex for which there is $S_2 \subseteq N_G(v_1, U_2 \setminus B_2)$ of order $\tilde{n}p/4$ such that $(S_2, U_3 \setminus B_3)$ is (ε, p)-lower-regular. As $(U_1 \setminus B_1, U_2 \setminus B_2)$ is $(2\varepsilon^{1}, p)$-lower-regular, there are at least $(1-2\varepsilon^{1})(1-\varepsilon)\tilde{n}$ vertices $v \in U_1 \setminus B_1$ that satisfy

$$\deg_G(v, U_2 \setminus B_2) \ge (1-2\varepsilon^{1})|U_2 \setminus B_2|p \ge \tilde{n}p/4,$$

by our choice of constants. Thus, by property (B1) almost any choice of $v_1 \in U_1 \setminus B_1$ will do. Sequentially, for every $i \ge 2$, let $v_i \in S_i$ be a vertex for which there is $S_{i+1} \subseteq N_G(v_i, U_{i+1} \setminus B_{i+1})$ of order $\tilde{n}p/4$ and both $(S_{i+1}, U_{i+2} \setminus B_{i+2})$ and (S_i, S_{i+1}) are (ε, p)-lower-regular. This is possible as $(S_i, U_{i+1} \setminus B_{i+1})$ is (ε, p)-lower-regular and properties (B1) and (B2) hold. We continue until we have embedded the first row of the grid as $v_1, \dots, v_s$, with $v_i \in U_i$ for every $i \in [s]$.

Consider now sets $S_2, \dots, S_s, S_1$ which we previously chose, where we note that S_1 was defined when we embedded v_s. In particular, $S_{1+j} \subseteq U_{1+j} \setminus B_{1+j}$ and (S_{1+j}, S_{2+j}) and $(S_{1+j}, U_{2+j} \setminus B_{2+j})$ are both (ε, p)-regular for every $j \in [s]$. Then, by setting $Q_{1+j} := B_{1+j} \cup \{v_{1+j}\}$ and invoking Claim 3.1 with $i = 2$, we can embed the second row of the grid as $u_1, \dots, u_s$, with $u_j \in S_{1+j}$ for every $j \in [s]$. By the conclusion of Claim 3.1 and a slight abuse of notation, there is a collection of sets $S_{2+j} \subseteq N_G(u_j, U_{2+j} \setminus Q_{2+j})$ for every $j \in [s]$, each of order $\tilde{n}p/4$, which, by (B1) and (B2), as $u_j \in U_{1+j} \setminus B_{1+j}$ and $u_{j+1} \in U_{1+j+1} \setminus B_{1+j+1}$, are such that (S_{2+j}, S_{2+j+1}) and $(S_{2+j}, U_{2+j+1} \setminus B_{2+j+1})$ are (ε, p)-lower-regular.

The same process can now be repeated for any $i \ge 3$ by setting the sets $Q_{i+j-1} \subseteq U_{i+j-1}$ for every $j \in [s]$ to be the union of B_{i+j-1} and the vertices of the grid that were previously embedded into U_{i+j-1}, that is, the images of the vertices $(1, i+j-1), (2, i+j-2), \dots, (i-1, j+1)$. Since $|B_{i+j-1}| \le \varepsilon\tilde{n}$, ε, and the lower-regularity conditions hold by (B1) and (B2), we may apply Claim 3.1 to embed the ith row. It only remains to prove this claim.

Proof of Claim 3.1. Without loss of generality, we may assume that all the $Q_{i\ j1}$ are of order 2"s̃, as we can take arbitrary supersets if this is not the case. Let $S_{i\ j1}$ „ $S_{i}^{1}{}_{j1}$ be the set of all v P $S_{i\ j1}$ with at least s̃p{4 neighbours in $U_{i\ j}$ r $Q_{i\ j}$. On the one hand, as p$S_{i\ j1}$; $U_{i\ j}$ r $B_{i\ j}$q is p"; pq-lower-regular and, thus, there are fewer than "|$S_{i\ j1}$| vertices in $S_{i\ j1}$ with degree less than s̃p{2 in $U_{i\ j}$ r $B_{i\ j}$, we have

$$e_G pS_{i\ j1}\ r\ S_{i\ j1};Q_{i\ j}q\ ¥\ \ |S_{i\ j1}\ r\ S_{i\ j1}|_1"|S_{i\ j1}|s̃p\{4:$$

On the other hand, assuming $S_{i\ j1}$ r $S_{i}^{1}{}_{j1}$ is of order at least s̃p{16 and, hence,

$$|S_{i\ j1}\ r\ S_{i\ j1}^{1}||Q_{i\ j}|p\ ¥\ s̃p\{16\ \ 2"s̃p\ ¥\ \ 100s\{^2$$

for C ¡ 0 suciently large, property (A1) implies that

$$e_G pS_{i\ j1}\ r\ S_{i\ j1}^{1};Q_{i\ j}q\ ⁄\ \ p1 \qquad q2"s̃|S_{i\ j1}\ r\ S_{i\ j1}^{1}|p:$$

Since " {128, this is a contradiction. Therefore, there are sets $S_{i}^{1}{}_{j1}$ „ $S_{i\ j1}$ of order at least |$S_{i\ j1}$| s̃p{16 for each j P rss such that every v P $S_{i\ j1}$ satises |$N_G$pv; $U_{i\ j}$ r $Q_{i\ j}$q| ¥ s̃p{4.

We will now nd a collection of sets $S_{i}^{2}{}_{j1}$ „ $S_{i\ j1}^{1}$ of order at least |$S_{i\ j1}$| s̃p{8 such that, for every 2 ⁄ j ⁄ s, every v P $S_{i\ j2}$ has a non-empty $N_G$pv; $S_{i\ j1}$q. First, choose $S_{i\ s1}$ „ $S_{i\ s1}$ of order |$S_{i\ s1}^{1}$| s̃p{8 arbitrarily, noting that such a set exists by the bound on |$S_{i\ s1}$|. Having chosen $S_{i\ j1}$ for some 2 ⁄ j ⁄ s, we choose $S_{i\ j2}$ as follows. Recall that p$S_{i\ j2}$; $S_{i\ j1}$q is p"; pq-lower-regular and, thus, by Lemma 2.1 and the bounds on the orders of $S_{i\ j2}$ and $S_{i\ j1}$, p$S_{i\ j2}$; $S_{i\ j1}$q is p2"; pq-lower-regular. It follows that there are at least p1 2"q|$S_{i\ j2}$| ¥ |$S_{i\ j2}$| s̃p{8 vertices v P $S_{i\ j2}^{1}$ which satisfy

$$\deg_G pv;S_{i\ j1}^{2}q\ ¥\ \ p1\ \ 2"q|S_{i\ j1}^{2}|p\ ¥\ \ ^2s̃p^2\{16\ ¡\ \ 0:$$

We declare the set of such vertices to be $S_{i\ j2}^{2}$ and continue on to the next index j.

Starting with an arbitrary v_1 P S_i^2 and sequentially choosing v_j P N_Gpv_{j1}; $S_{i\ j1}^{2}$q now completes the proof. □

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