Long-time L^2 stability for an IMEX discretization of the 1D Fujita Equation

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Abstract

We study an efficient time-stepping scheme for the 1D Fujita equation that is implicit for the linear terms but explicit for the nonlinear terms. We analyze the long-time stability of the scheme for varying parameter values, which reveal parameter value regimes in which the method is stable. We provide numerical results that illustrate the theory and show the analytically derived stability conditions are sufficient to achieve long-time stability results.

1 Introduction

We consider the Fujita semilinear heat equation which is given in 1D by [3]:

$$\begin{aligned} u_t - \Delta u + \beta\left(u - u_g\right) &= \alpha |u|^{\gamma} \quad \text{on} \quad (a,b), \\ u(a,t) = u(b,t) &= u_g, \\ u(x,0) &= u_g, \end{aligned} \tag{1}$$

where $u = u(x,t)$ is the unknown temperature, $\alpha > 0$, $\beta > 0$, and $\gamma > 0$ are problem dependent constants, and u_g is the (assumed constant) temperature of the surrounding medium. This system models various chemical and physical phenomenon such as diffusion and reaction, combustion, quantum mechanics, and fluid mechanics [7]. Our particular interest is the modeling of viscoelastic materials under cyclic deformation due to vibrational loading accompanied by self-heating [13]. In some cases, this self-heating can cause catastrophic heat explosion which can change the mechanical properties of the material, and the material itself will ultimately fail [5].

Our goal is to study a numerical discretization for (1), and in particular its long-time stability. As there are many different notions of stability [8, 9, 10, 12], uniform boundedness of the solution in the L^2 norm will be understood to be our definition of stability throughout the paper. Understanding the stability properties of discretizations often reveals information about its well-posedness, robustness, and accuracy. Moreover, the stability of our discretization is directly related

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Gamma	Sufficient Conditions
$0 < \gamma < 1$	$\beta > 2\alpha\left(\lvert\Omega\rvert + 1\right)$
$\gamma = 1$	$\beta > \frac{3\alpha}{2}$
$1 < \gamma < 3$	$\Delta t \le min\{1, \frac{2}{\beta}\}$, $\alpha \le min\left\{\frac{1}{4}, \frac{1}{2\left(\lvert\Omega\rvert 2^\gamma \left(2\int_\Omega \left(\frac{2}{\beta}\left(\frac{2}{3}\right)^2\right) + \frac{2}{\beta}\Vert u_g^\gamma\Vert^2\right)\frac{4}{\beta} + \lvert\Omega\rvert 2^\gamma \left(\left(2\int_\Omega\left(\frac{2}{\beta}\left(\frac{2}{3}\right)^2\right)+\frac{2}{\beta}\Vert u_g^\gamma\Vert^2\right)\frac{4}{\beta}\right)^2\right)}\right\}$
$\gamma = 3$	$\Delta t \le min\{1, \frac{1}{\beta}\}$, $\alpha \le \frac{1}{2\left(\lvert\Omega\rvert 2^\gamma\left(\frac{1}{2\beta}(2^\gamma)^2\alpha^2\Vert u_g^\gamma\Vert^2\right)\frac{2}{\beta} + \lvert\Omega\rvert 2^\gamma\left(\frac{1}{2\beta}(2^\gamma)^2\alpha^2\Vert u_g^\gamma\Vert^2\right)^2\frac{4}{\beta^2}\right)}$
$\gamma > 3$	$\Delta t \le \frac{2}{\beta}, \alpha \le \frac{\rho}{16\lvert\Omega\rvert\left(2\left(\frac{2}{\beta}\left(\frac{1}{2\beta}(2^\gamma)^2\alpha^2\Vert u_g^\gamma\Vert^2\right)\right)^6 + \left(\frac{2}{\beta}\left(\frac{1}{2\beta}(2^\gamma)^2\alpha^2\Vert u_g^\gamma\Vert^2\right)\right)^3\right)}$

Table 1: Shown above are the parameter restrictions for the different ranges of γ.

to the stability of (1), and thus can give insight into when blowup phenomenon/material failure may occur.

A great deal of literature exists on this Fujita equation, including conditions for its well-posedness, blowup criteria, and other mathematical properties. In much of this literature, the Cauchy problem (no boundary conditions) is considered [3, 4, 11], and Newtonian relaxation is neglected ($\beta = 0$). However, there is much less in the literature regarding the Fujita equation with physical boundary conditions and even less on numerical methods, and to our knowledge none for long-time stability of numerical methods.

Before discretizing, we first rewrite the system (1) in terms of a new variable $\hat{u}$ which is defined by $\hat{u} = u - u_g$. This transforms (1) into

$$\begin{aligned} \hat{u}_t - \Delta\hat{u} + \beta\hat{u} &= \alpha|\hat{u} + u_g|^\gamma \quad \text{on} \quad (a, b), \\ \hat{u}(a, t) = \hat{u}(b, t) &= 0, \\ \hat{u}(x, 0) &= 0. \end{aligned}$$

We consider a backward Euler temporal discretization together with a first order explicit approximation of the nonlinear term to get, on a domain $\Omega = (a, b)$,

$$\begin{aligned} \frac{\hat{u}^{n+1} - \hat{u}^n}{\Delta t} - \Delta\hat{u}^{n+1} + \beta\hat{u}^{n+1} &= \alpha|\hat{u}^n + u_g|^\gamma, \\ \hat{u}^0 &= 0, \end{aligned} \tag{2}$$

where $\Delta t > 0$ denotes the time step size. The purpose of this paper is to prove long-time stability results for (2), and we show herein that the stability properties depend on γ, α, and β values as well as (mild) restrictions on Δt. A summary of our results is shown in Table 1. From the table, observe that there are different p arameter r estrictions i n t he d ifferent ra nges of γ. In fa ct, the stability proof structures change in each of these cases as well, as each seems to require a different proof technique. Section 3 of this paper proves these results. In Section 4, we perform numerical tests that illustrate our theory that picking parameters in the ranges yields long-time stability; on the other hand, picking them outside of these ranges can lead to blow up.

2 Preliminaries

We assume the domain $\Omega \subset \mathbb{R}$ is a bounded open interval. The space $L^2(\Omega)$ is defined as

$$L^2(\Omega) := \{v : \Omega \to \mathbb{R}, \int_\Omega |v|^2 dx < \infty\},$$

and the $L^2(\Omega)$ inner product and norm will be defined as

$$(v, w) := \int_\Omega v \cdot w \, dx,$$
$$||v|| := (v, v)^{\frac{1}{2}}.$$

We utilize Young's inequality: for $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

for any $a, b \in \mathbb{R}$.

We define the space X to be

$$X := H_0^1(\Omega).$$

Poincare's inequality also holds in X: for all vectors, there is a constant C_p dependent only on the size of the domain such that

$$||v|| \leq C_p ||\nabla v||.$$

We also utilize a polarization identity that states for $a, b \in \mathbb{R}$,

$$(a - b)a = \frac{1}{2}(a^2 - b^2 + (a - b)^2).$$

Our analysis uses the following inequality taken from [2], which is valid in 1D:

$$||f||_{L^p} \leq C_{GN}(p) ||f'||_{L^2}^{\theta} ||f||_{L^2}^{1-\theta} \quad \text{if} \quad p \in (2, \infty), \tag{3}$$

with $\theta = \frac{p-2}{2p}$ and C_{GN} only dependent upon p.

We also use the 1D Agmon inequality [1]: For any $u \in H_0^1(\Omega)$ with $\Omega \subset \mathbb{R}$, there exists a constant C such that

$$||u||_{L^\infty(\Omega)} \leq C ||u||^{\frac{1}{2}} ||u||_{H^1(\Omega)}^{\frac{1}{2}}.$$

Lemma 1 *Suppose that a and b are nonnegative real numbers and $\gamma > 0$. Then,*

$$(a + b)^\gamma \leq \max\left(1, 2^{\gamma-1}\right)\left(a^\gamma + b^\gamma\right). \tag{4}$$

Proof. For $\gamma = 1$, (4) is trivially true. Therefore, let us assume that $\gamma \in (0, 1) \cup (1, \infty)$. If $a = b \geq 0$, the inequality trivially holds. Without loss of generality, let $0 \leq a < b$. Then, we define x as $x := \frac{a}{b} \in [0, \infty)$. Dividing both sides of (4) by b^γ, we get the following:

$$(x+1)^\gamma \le \max\left(1, 2^{\gamma-1}\right)(x^\gamma + 1) \quad \forall x \in [0,\infty). \tag{5}$$

We will now prove that (5) holds for all $\gamma \in (0,1) \cup (1,\infty)$.

Case I When $\gamma \in (0,1)$, the function $f : x \in [0,1] \mapsto (x^\gamma + 1)^{\frac{1}{\gamma}}$ is concave. Therefore, the secant connecting the points $(0, f(0))$ and $(1, f(1))$ lies below the graph of f, i.e.,

$$\tfrac{f(1)-f(0)}{1-0}(x-0) + f(0) \le f(x) \quad \forall x \in [0,1].$$

Because $f(0) = 1,\ f(1) - f(0) = 2^{\frac{1}{\gamma}} - 1 \ge 1$ and $1 = \max(1, 2^{\gamma-1})$, we have that

$$(x+1)^\gamma \le \max\left(1, 2^{\gamma-1}\right)(x^\gamma + 1) \quad \forall x \in [0,1]. \tag{6}$$

If $x \in [1,\infty)$, then $\frac{1}{x} \in (0,1]$, and therefore, upon replacing x by $\frac{1}{x}$ in inequality (6) and multiplying both sides of the resulting inequality by x^γ, we deduce that (4) also holds for all $x \in [1,\infty)$.

Case II When $\gamma \in (1,\infty)$, the function $g : x \in [0,\infty) \mapsto x^\gamma$ is convex. Therefore, for all $x, y \in [0,\infty)$ and all $\theta \in [0,1]$,

$$g(\theta x + (1-\theta)y)) \le \theta g(x) + (1-\theta)g(y).$$

Thus, in particular, for $y = 1$ and $\theta = \frac{1}{2}$ we have that

$$\left(\tfrac{x+1}{2}\right)^\gamma \le \tfrac{1}{2}x^\gamma + \tfrac{1}{2}1^\gamma \quad \forall x \in [0,\infty).$$

If we multiply this inequality by 2^γ and observe that $2^{\gamma-1} = \max(1, 2^{\gamma-1})$, this inequality reduces to (5).

Thus, (5) holds for all $\gamma \in (0,1)\cup(1,\infty)$, and therefore, (4) is true for all positive γ and nonnegative a and b. □

Lemma 2 [Lemma 2.5 from [6]] *Suppose constants r and B satisfy $r > 1$, $B \ge 0$. Then if the sequence of real numbers a_n satisfies*

$$ra_{n+1} \le a_n + B,$$

we have that

$$a_{n+1} \le a_0\left(\tfrac{1}{r}\right)^{n+1} + \tfrac{B}{r-1}.$$

3 Numerical Scheme and Stability Analysis

For our numerical scheme below, we use a finite difference method which is equivalent to a spatial finite element method with a conforming finite element space, $V_h \subset H_0^1((a,b))$. We note that if we use continuous piecewise linears on a uniform subdivision of V_h, the analysis that follows can be repeated verbatim.

The time stepping algorithm we study reads as follows: Given $\hat{u}^n$, form $\hat{u}^{n+1}$ satisfying

$$\frac{\hat{u}^{n+1}-\hat{u}^n}{\Delta t} - \Delta\hat{u}^{n+1} + \beta\hat{u}^{n+1} = \alpha|\hat{u}^n + u_g|^\gamma, \qquad \hat{u}^0 = 0. \tag{7}$$

In this section, we will analyze (7) for long-time stability for varying γ. There appears to be distinct regimes for these analyses, $\gamma < 1, \gamma = 1, 1 < \gamma < 3, \gamma = 3, \gamma > 3$. We begin with $\gamma = 1$.

Remark. The theorems below prove long-time stability under certain conditions on parameters α, β, γ. These conditions are only sufficient for the results, and a different analysis could potentially provide slightly different results.

Theorem 1 *Suppose that $\gamma = 1$ in algorithm (7) and $\beta > \frac{3\alpha}{2}$. Then the solution is long-time stable: for any given $n \in \mathbb{N}$,*

$$||\hat{u}^n|| \le \frac{\frac{\alpha}{2}||u_g||^2}{C_p^{-2}+\beta-\frac{3\alpha}{2}}.$$

Proof. Multiplying (7) by $\hat{u}^{n+1}$ and taking the integral over Ω gives us

$$\left(\tfrac{\hat{u}^{n+1}-\hat{u}^n}{\Delta t}, \hat{u}^{n+1}\right) - (\Delta\hat{u}^{n+1}, \hat{u}^{n+1}) + \beta||\hat{u}^{n+1}||^2 = \alpha\left(|\hat{u}^n + u_g|, \hat{u}^{n+1}\right).$$

We can rewrite our first term using the polarization identity via

$$\left(\tfrac{\hat{u}^{n+1}-\hat{u}^n}{\Delta t}, \hat{u}^{n+1}\right) = \tfrac{1}{2\Delta t}\left(||\hat{u}^{n+1}||^2 - ||\hat{u}^n||^2 + ||\hat{u}^{n+1} - \hat{u}^n||^2\right).$$

For the second term, we use Green's Theorem:

$$-\left(\Delta\hat{u}^{n+1}, \hat{u}^{n+1}\right) = ||\nabla\hat{u}^{n+1}||^2 - \int_{\delta\Omega}\left(\nabla\hat{u}^{n+1}\cdot u\right)\hat{u}^{n+1}du = ||\nabla\hat{u}^{n+1}||^2.$$

Combining the terms on our left-hand side, we now have

$$\frac{1}{2\Delta t}\left(||\hat{u}^{n+1}||^2 - ||\hat{u}^n||^2 + ||\hat{u}^{n+1} - \hat{u}^n||^2\right) + ||\nabla\hat{u}^{n+1}||^2 + \beta||\hat{u}^{n+1}||^2 = \alpha\left(|\hat{u}^n + u_g|, \hat{u}^{n+1}\right). \tag{8}$$

Decomposing the right-hand side term, we obtain

$$\alpha\left(|\hat{u}^n + u_g|, \hat{u}^{n+1}\right) \le \alpha\left(|\hat{u}^n|, \hat{u}^{n+1}\right) + \alpha\left(|u_g|, \hat{u}^{n+1}\right),$$

and by the Cauchy-Schwarz inequality, we can bound these terms by

$$\alpha\left(|\hat{u}^n|, \hat{u}^{n+1}\right) + \alpha\left(|u_g|, \hat{u}^{n+1}\right) \le \alpha||\hat{u}^n||||\hat{u}^{n+1}|| + \alpha||u_g||||\hat{u}^{n+1}||.$$

By Young's inequality, we can further upper bound this inequality and obtain

$$\alpha||\hat{u}^n||||\hat{u}^{n+1}|| + \alpha||u_g||||\hat{u}^{n+1}|| \le \alpha\left(\frac{1}{2}||\hat{u}^n||^2 + \frac{1}{2}||\hat{u}^{n+1}||^2\right) + \alpha\left(\frac{1}{2}||u_g||^2 + \frac{1}{2}||\hat{u}^{n+1}||^2\right). \tag{9}$$

Combining (8) with (9) and rearranging terms, we get that

$$\frac{1}{2\Delta t}||\hat{u}^{n+1}||^2+\frac{1}{2\Delta t}||\hat{u}^{n+1}-\hat{u}^n||^2+||\nabla\hat{u}^{n+1}||^2+\beta||\hat{u}^{n+1}||^2-\alpha||\hat{u}^{n+1}||^2 \le \frac{\alpha}{2}||\hat{u}^n||^2+\frac{1}{2\Delta t}||\hat{u}^n||^2+\frac{\alpha}{2}||u_g||^2$$

Next, we drop the nonnegative term $\frac{1}{2\Delta t}||\hat{u}^{n+1}-\hat{u}^n||^2$ from the left-hand side of our bound. Thus, we now have that

$$\frac{1}{2\Delta t}||\hat{u}^{n+1}||^2+||\nabla\hat{u}^{n+1}||^2+\beta||\hat{u}^{n+1}||^2-\alpha||\hat{u}^{n+1}||^2 \le \frac{\alpha}{2}||\hat{u}^n||^2+\frac{1}{2\Delta t}||\hat{u}^n||^2+\frac{\alpha}{2}||u_g||^2.$$

Using Poincare's inequality, we can lower bound our left-hand side even further via

$$\left(\frac{1}{2\Delta t}+C_p^{-2}+\beta-\alpha\right)||\hat{u}^{n+1}||^2 \le \left(\frac{\alpha}{2}+\frac{1}{2\Delta t}\right)||\hat{u}^n||^2+\frac{\alpha}{2}||u_g||^2.$$

Dividing both sides by $\frac{\alpha}{2}+\frac{1}{2\Delta t}$, we find that

$$\left(\frac{\frac{1}{2\Delta t}+C_p^{-2}+\beta-\alpha}{\frac{\alpha}{2}+\frac{1}{2\Delta t}}\right)||\hat{u}^{n+1}||^2 \le ||\hat{u}^n||^2+\left(\frac{1}{\frac{\alpha}{2}+\frac{1}{2\Delta t}}\right)\frac{\alpha}{2}||u_g||^2.$$

Let $r := \frac{\frac{1}{2\Delta t}+C_p^{-2}+\beta-\alpha}{\frac{\alpha}{2}+\frac{1}{2\Delta t}}$ and note that since $\beta > \frac{3\alpha}{2}$, we can now use Lemma 2 to get

$$||\hat{u}^{n+1}||^2 \le ||\hat{u}^0||^2\left(\frac{1}{r}\right)^{n+1}+\frac{\left(\frac{1}{\frac{\alpha}{2}+\frac{1}{2\Delta t}}\right)\frac{\alpha}{2}||u_g||^2}{r-1} = 0\left(\frac{1}{r}\right)^{n+1}+\frac{\left(\frac{1}{\frac{\alpha}{2}+\frac{1}{2\Delta t}}\right)\frac{\alpha}{2}||u_g||^2}{\frac{C_p^{-2}+\beta-\frac{3\alpha}{2}}{\frac{1}{2\Delta t}+\frac{\alpha}{2}}} = \frac{\frac{\alpha}{2}||u_g||^2}{C_p^{-2}+\beta-\frac{3\alpha}{2}},$$

which finishes the proof. □

For the case of $0<\gamma<1$, we use the $\gamma=1$ result in the proof.

Theorem 2 *Suppose that $0<\gamma<1$ in algorithm (7) and $\beta > 2\alpha(|\Omega|+1)$. Then the solution is long-time stable: for any given $n\in\mathbb{N}$,*

$$||\hat{u}^n|| \le \frac{\alpha||u_g||^2}{C_p^{-2}+\beta-2\alpha(|\Omega|+1)}.$$

Proof. Following as in the proof of Theorem 1 up to (8), we get

$$\frac{1}{2\Delta t}\left(||\hat{u}^{n+1}||^2-||\hat{u}^n||^2+||\hat{u}^{n+1}-\hat{u}^n||^2\right)+||\nabla\hat{u}^{n+1}||^2+\beta||\hat{u}^{n+1}||^2=\alpha\left(|\hat{u}^n+u_g|^\gamma,\hat{u}^{n+1}\right). \quad (10)$$

By Lemma 1, we have that

$$\begin{aligned}\alpha\left(|\hat{u}^n+u_g|^\gamma,\hat{u}^{n+1}\right) &\le \alpha\int_\Omega |\hat{u}^n+u_g|^\gamma|\hat{u}^{n+1}| \\ &\le \alpha\int_\Omega 2^\gamma\left(|\hat{u}^n|^\gamma+|u_g|^\gamma\right)|\hat{u}^{n+1}| \\ &\le 2^\gamma\alpha\int_\Omega|\hat{u}^n|^\gamma|\hat{u}^{n+1}|+2^\gamma\alpha\int_\Omega|u_g^\gamma||\hat{u}^{n+1}|. \end{aligned} \quad (11)$$

We upper bound the first term of (11) by utilizing Young's inequality and obtain

$$2^\gamma\alpha\int_\Omega|\hat{u}^n|^\gamma|\hat{u}^{n+1}| \le 2^\gamma\alpha\int_\Omega\left(\frac{\gamma}{2}|\hat{u}^n|^2+\frac{2-\gamma}{2}|\hat{u}^{n+1}|^{\frac{2}{2-\gamma}}\right)=2^\gamma\alpha\int_\Omega\frac{\gamma}{2}|\hat{u}^n|^2+2^\gamma\alpha\int_\Omega\frac{2-\gamma}{2}|\hat{u}^{n+1}|^{\frac{2}{2-\gamma}}.$$

Rewriting these terms as their equivalent L^p norms, we get

$$2^\gamma \alpha \int_\Omega \tfrac{\gamma}{2} |\hat{u}^n|^2 + 2^\gamma \alpha \int_\Omega \tfrac{2-\gamma}{2} |\hat{u}^{n+1}|^{\frac{2}{2-\gamma}} = 2^\gamma \alpha \tfrac{\gamma}{2} ||\hat{u}^n||^2 + 2^\gamma \alpha \tfrac{2-\gamma}{2} ||\hat{u}^{n+1}||_{L^{\frac{2}{2-\gamma}}}^{\frac{2}{2-\gamma}}.$$

By our assumption that $0 < \gamma < 1$, we can further upper bound these terms via

$$2^\gamma \alpha \frac{\gamma}{2} ||\hat{u}^n||^2 + 2^\gamma \alpha \frac{2-\gamma}{2} ||\hat{u}^{n+1}||_{L^{\frac{2}{2-\gamma}}}^{\frac{2}{2-\gamma}} \leq \alpha ||\hat{u}^n||^2 + 2\alpha |\Omega| ||\hat{u}^{n+1}||^2. \tag{12}$$

For the second term of (11), we utilize Cauchy-Schwarz and Young inequalities to obtain

$$2^\gamma \alpha \int_\Omega |u_g^\gamma| |\hat{u}^{n+1}| \leq 2^\gamma \alpha ||u_g^\gamma|| ||\hat{u}^{n+1}|| \leq 2^\gamma \alpha \left(\frac{1}{2} ||u_g^\gamma||^2 + \frac{1}{2} ||\hat{u}^{n+1}||^2 \right) \leq \alpha ||u_g^\gamma||^2 + \alpha ||\hat{u}^{n+1}||^2. \tag{13}$$

Combining (10) with (12) and (13) and dropping the nonnegative term $\frac{1}{2\Delta t} ||\hat{u}^{n+1} - \hat{u}^n||^2$ from the left-hand side, we get that

$$\tfrac{1}{2\Delta t} ||\hat{u}^{n+1}||^2 + ||\nabla \hat{u}^{n+1}||^2 + \beta ||\hat{u}^{n+1}||^2 - (2\alpha|\Omega| + \alpha) ||\hat{u}^{n+1}||^2 \leq \left(\tfrac{1}{2\Delta t} + \alpha \right) ||\hat{u}^n||^2 + \alpha ||u_g||^2.$$

Now using Poincare's inequality, we lower bound our left-hand side even further via

$$\left(\tfrac{1}{2\Delta t} + C_p^{-2} + \beta - 2\alpha|\Omega| - \alpha \right) ||\hat{u}^{n+1}||^2 \leq \left(\tfrac{1}{2\Delta t} + \alpha \right) ||\hat{u}^n||^2 + \alpha ||u_g||^2.$$

Dividing both sides by $\frac{1}{2\Delta t} + \alpha$, we find that

$$\left(\frac{\frac{1}{2\Delta t} + C_p^{-2} + \beta - 2\alpha|\Omega| - \alpha}{\frac{1}{2\Delta t} + \alpha} \right) ||\hat{u}^{n+1}||^2 \leq ||\hat{u}^n||^2 + \left(\frac{\alpha}{\frac{1}{2\Delta t} + \alpha} \right) ||u_g||^2.$$

Let $r := \frac{\frac{1}{2\Delta t} + C_p^{-2} + \beta - 2\alpha|\Omega| - \alpha}{\frac{1}{2\Delta t} + \alpha}$ and note that since $\beta > 2\alpha(1 + |\Omega|)$, we can now use Lemma 2 to get

$$||\hat{u}^{n+1}||^2 \leq ||\hat{u}^0||^2 \left(\tfrac{1}{r}\right)^{n+1} + \frac{\left(\frac{\alpha}{\frac{1}{2\Delta t}+\alpha}\right) ||u_g||^2}{r-1} = 0 \left(\tfrac{1}{r}\right)^{n+1} + \frac{\left(\frac{\alpha}{\frac{1}{2\Delta t}+\alpha}\right) ||u_g||^2}{\frac{C_p^{-2} + \beta - 2\alpha|\Omega| - 2\alpha}{\frac{1}{2\Delta t}+\alpha}} = \frac{\alpha ||u_g||^2}{C_p^{-2} + \beta - 2\alpha(|\Omega| + 1)},$$

which completes the proof. □

Next we consider $\gamma = 3$. We note that this is the critical exponent for blowup of the Cauchy problem for the Fujita equation [4] when $\beta = 0$, and also seems to be an inflection point for our stability analysis approach.

Theorem 3 *Suppose that $\gamma = 3$ in the scheme (7). Assume that $\Delta t \leq \min\{1, \frac{1}{\beta}\}$, and*

$$\alpha \leq \frac{1}{2\left(|\Omega| 2^\gamma \left(\frac{1}{2\beta} (2^\gamma)^2 \alpha^2 ||u_g^\gamma||^2 \right) \frac{2}{\beta} + |\Omega| 2^\gamma \left(\frac{1}{2\beta} (2^\gamma)^2 \alpha^2 ||u_g^\gamma||^2 \right)^2 \frac{4}{\beta^2} \right)}.$$

Then the solution is long-time stable: for any given $n \in \mathbb{N}$,

$$||\hat{u}^n|| \leq \sqrt{\tfrac{2}{\beta} \left(\tfrac{1}{2\beta} (2^\gamma)^2 \alpha^2 ||u_g^\gamma||^2 \right)}.$$

Proof. Following as in the proof of Theorem 1 up to (11), we get

$$\alpha \left(|\hat{u}^n + u_g|^\gamma, \hat{u}^{n+1}\right) \leq 2^\gamma \alpha \int_\Omega |\hat{u}^n|^\gamma |\hat{u}^{n+1}| + 2^\gamma \alpha \int_\Omega |u_g^\gamma||\hat{u}^{n+1}|. \tag{14}$$

We add and subtract $\hat{u}^n$ to $\hat{u}^{n+1}$ in the first right-hand side term. From the triangle inequality,

$$2^\gamma \alpha \int_\Omega |\hat{u}^n|^\gamma |\hat{u}^{n+1}| \leq 2^\gamma \alpha \int_\Omega |\hat{u}^n|^\gamma |\hat{u}^{n+1} - \hat{u}^n| + 2^\gamma \alpha \int_\Omega |\hat{u}^n|^{\gamma+1} \tag{15}$$

$$= 2^\gamma \alpha \int_\Omega |\hat{u}^n|^\gamma |\hat{u}^{n+1} - \hat{u}^n| + 2^\gamma \alpha ||\hat{u}^n||_{L^{\gamma+1}}^{\gamma+1}. \tag{16}$$

We utilize Cauchy-Schwarz and Young on the first term of (16) and multiply by $\frac{2\Delta t}{2\Delta t}$ to obtain

$$2^\gamma \alpha \int_\Omega |\hat{u}^n|^\gamma |\hat{u}^{n+1} - \hat{u}^n| \leq 2^\gamma \alpha \left(\tfrac{1}{4\Delta t}||\hat{u}^{n+1} - \hat{u}^n||^2 + \Delta t||\left(\hat{u}^n\right)^\gamma||^2\right).$$

Rewriting the last term in its equivalent L^p norm, we find that

$$2^\gamma \alpha \left(\frac{1}{4\Delta t}||\hat{u}^{n+1} - \hat{u}^n||^2 + \Delta t||\left(\hat{u}^n\right)^\gamma||^2\right) = 2^\gamma \alpha \left(\frac{1}{4\Delta t}||\hat{u}^{n+1} - \hat{u}^n||^2 + \Delta t||\hat{u}^n||_{L^{2\gamma}}^{2\gamma}\right). \tag{17}$$

Combining the last term of (16) with (17) gives us

$$2^\gamma \alpha \int_\Omega |\hat{u}^n|^\gamma |\hat{u}^{n+1} - \hat{u}^n| + 2^\gamma \alpha \int_\Omega |\hat{u}|^{\gamma+1} \leq 2^\gamma \alpha \left(\frac{1}{4\Delta t}||\hat{u}^{n+1} - \hat{u}^n||^2 + \Delta t||\hat{u}^n||_{L^{2\gamma}}^{2\gamma}\right) + 2^\gamma \alpha ||\hat{u}^n||_{L^{\gamma+1}}^{\gamma+1}. \tag{18}$$

We also utilize Cauchy-Schwarz and Young inequalities on the second term of (14) and multiply it by $\frac{\beta}{\beta}$ to obtain

$$2^\gamma \alpha \int_\Omega |u_g^\gamma||\hat{u}^{n+1}| \leq \frac{1}{2\beta}\left(2^\gamma\right)^2 \alpha^2 ||u_g^\gamma||^2 + \frac{\beta}{2}||\hat{u}^{n+1}||^2. \tag{19}$$

Combining (10) with (18) and (19) and dropping the nonnegative term $||\hat{u}^{n+1} - \hat{u}^n||^2$ from the left-hand side, we get that

$$\frac{1}{2\Delta t}||\hat{u}^{n+1}||^2 + ||\nabla \hat{u}^{n+1}||^2 + \frac{\beta}{2}||\hat{u}^{n+1}||^2$$
$$\leq 2^\gamma \alpha \Delta t ||\hat{u}^n||_{L^{2\gamma}}^{2\gamma} + 2^\gamma \alpha ||\hat{u}^n||_{L^{\gamma+1}}^{\gamma+1} + \frac{1}{2\Delta t}||\hat{u}^n||^2 + \frac{1}{2\beta}\left(2^\gamma\right)^2 \alpha^2 ||u_g^\gamma||^2. \tag{20}$$

By our assumption that $\gamma = 3$ and from (3) with $\theta = \frac{1}{3}$, we can upper bound the first term on the right-hand side of (20) to find that

$$2^\gamma \alpha \Delta t ||\hat{u}^n||_{L^{2\gamma}}^{2\gamma} \leq C 2^\gamma \alpha \Delta t ||\hat{u}^n||^4 ||\nabla \hat{u}^n||^2, \tag{21}$$

with C depending only on Ω.

Similarly, we upper bound the second-term on the right-hand side of (19) with the L^∞ norm and Agmon's inequality to get

$$2^\gamma \alpha ||\hat{u}^n||_{L^{\gamma+1}}^{\gamma+1} \leq C 2^\gamma \alpha ||\hat{u}^n||^2 ||\nabla \hat{u}^n||^2, \tag{22}$$

with C again depending only on Ω.

Combining (20) with (21) and (22), we have that

$$\left(\frac{1}{2\Delta t} + \frac{\beta}{2}\right) ||\hat{u}^{n+1}||^2 + ||\nabla \hat{u}^{n+1}||^2$$
$$\leq \left(|\Omega| 2^\gamma \alpha \Delta t ||\hat{u}^n||^4 + |\Omega| 2^\gamma \alpha ||\hat{u}^n||^2\right) ||\nabla \hat{u}^n||^2 + \frac{1}{2\Delta t} ||\hat{u}^n||^2 + \frac{1}{2\beta} \left(2^\gamma\right)^2 \alpha^2 ||u_g^\gamma||^2. \tag{23}$$

From here, we utilize mathematical induction to prove that $\forall n$, $||\hat{u}^n||^2$ is bounded above by problem data.

Base Case. For $n = 0$, by (23), we have that

$$\left(\frac{1}{2\Delta t} + \frac{\beta}{2}\right) ||\hat{u}^1||^2 + ||\nabla \hat{u}^1||^2 \leq \frac{1}{2\beta} \left(2^\gamma\right)^2 \alpha^2 ||u_g^\gamma||^2.$$

Dropping the nonnegative term $||\nabla \hat{u}^1||^2$ and dividing by $\frac{1}{2\Delta t} + \frac{\beta}{2}$, we find that

$$||\hat{u}^1||^2 \leq \frac{\frac{1}{2\beta} \left(2^\gamma\right)^2 \alpha^2 ||u_g^\gamma||^2}{\frac{1}{2\Delta t} + \frac{\beta}{2}}.$$

We define λ as $\lambda := \frac{\frac{1}{2\Delta t}}{\frac{1}{2\Delta t} + \frac{\beta}{2}} = \frac{1}{1+\beta\Delta t} < 1$. Therefore, we have that

$$||\hat{u}^1||^2 \leq \frac{\frac{1}{2\beta} \left(2^\gamma\right)^2 \alpha^2 ||u_g^\gamma||^2}{\frac{1}{2\Delta t} + \frac{\beta}{2}} = 2\Delta t \lambda \left(\frac{1}{2\beta} \left(2^\gamma\right)^2 \alpha^2 ||u_g^\gamma||^2\right).$$

By our assumptions on β and Δt, we can immediately infer that $\frac{1}{2} \leq \lambda < 1$. With this,

$$||\hat{u}^1||^2 \leq 2\Delta t \lambda \left(\frac{1}{2\beta} \left(2^\gamma\right)^2 \alpha^2 ||u_g^\gamma||^2\right)$$
$$\leq \frac{2}{\beta} \left(\frac{1}{2\beta} \left(2^\gamma\right)^2 \alpha^2 ||u_g^\gamma||^2\right).$$

Induction Step. By mathematical induction, for the *kth* step, our induction hypothesis is that

$$||\hat{u}^k||^2 + 2\lambda\Delta t ||\nabla \hat{u}^k||^2 \leq \frac{2}{\beta} \left(\frac{1}{2\beta} \left(2^\gamma\right)^2 \alpha^2 ||u_g^\gamma||^2\right).$$

Therefore, the $k+1$ solution satisfies, by (23),

$$\left(\frac{1}{2\Delta t}+\frac{\beta}{2}\right)||\hat{u}^{k+1}||^2+||\nabla\hat{u}^{k+1}||^2$$
$$\leq\left(|\Omega|2^{\gamma}\alpha\Delta t||\hat{u}^k||^4+|\Omega|2^{\gamma}\alpha||\hat{u}^k||^2\right)||\nabla\hat{u}^k||^2+\frac{1}{2\Delta t}||\hat{u}^k||^2+\frac{1}{2\beta}\left(2^{\gamma}\right)^2\alpha^2||u_g^{\gamma}||^2. \quad (24)$$

By our assumptions on α and bounds on λ, we have that

$$\alpha\leq\frac{1}{2\left(|\Omega|2^{\gamma}\left(\frac{1}{2\beta}\left(2^{\gamma}\right)^2\alpha^2||u_g^{\gamma}||^2\right)\frac{2}{\beta}+|\Omega|2^{\gamma}\left(\frac{1}{2\beta}\left(2^{\gamma}\right)^2\alpha^2||u_g^{\gamma}||^2\right)^2\frac{4}{\beta^2}\right)}$$
$$\leq\frac{\lambda}{\left(|\Omega|2^{\gamma}\left(\frac{1}{2\beta}\left(2^{\gamma}\right)^2\alpha^2||u_g^{\gamma}||^2\right)\frac{2}{\beta}+\Delta t|\Omega|2^{\gamma}\left(\frac{1}{2\beta}\left(2^{\gamma}\right)^2\alpha^2||u_g^{\gamma}||^2\right)^2\frac{4}{\beta^2}\right)}.$$

Hence by our induction hypothesis, we obtain

$$\alpha\leq\frac{\lambda}{|\Omega|2^{\gamma}||\hat{u}^k||^2+\Delta t|\Omega|2^{\gamma}||\hat{u}^k||^4}. \quad (25)$$

Combining (24) with (25) yields

$$\left(\frac{1}{2\Delta t}+\frac{\beta}{2}\right)||\hat{u}^{k+1}||^2+||\nabla\hat{u}^{k+1}||^2\leq\frac{1}{2\Delta t}||\hat{u}^k||^2+\lambda||\nabla\hat{u}^k||^2+\frac{1}{2\beta}\left(2^{\gamma}\right)^2\alpha^2||u_g^{\gamma}||^2.$$

Utilizing our induction hypothesis once again, we get that

$$\left(\frac{1}{2\Delta t}+\frac{\beta}{2}\right)||\hat{u}^{k+1}||^2+||\nabla\hat{u}^{k+1}||^2\leq\left(\frac{\beta\Delta t+1}{\beta\Delta t}\right)\frac{1}{2\beta}\left(2^{\gamma}\right)^2\alpha^2||u_g^{\gamma}||^2.$$

Dropping the nonnegative term $||\nabla\hat{u}^{k+1}||^2$ and dividing both sides by $\frac{1}{2\Delta t}+\frac{\beta}{2}$, we obtain

$$||\hat{u}^{k+1}||^2\leq\left(\frac{1}{\frac{1}{2\Delta t}+\frac{\beta}{2}}\right)\left(\frac{\beta\Delta t+1}{\beta\Delta t}\right)\frac{1}{2\beta}\left(2^{\gamma}\right)^2\alpha^2||u_g^{\gamma}||^2$$
$$=\frac{2}{\beta}\left(\frac{1}{2\beta}\left(2^{\gamma}\right)^2\alpha^2||u_g^{\gamma}||^2\right),$$

which is precisely the induction hypothesis for step $k+1$.

Thus, by mathematical induction, $||\hat{u}^n||\leq\sqrt{\frac{2}{\beta}\left(\frac{1}{2\beta}\left(2^{\gamma}\right)^2\alpha^2||u_g^{\gamma}||^2\right)}$ holds for every natural number n. □

To prove stability for $1<\gamma<3$, the proof technique follows a similar strategy as the $\gamma=3$ case.

Theorem 4 *Suppose that $1<\gamma<3$ in the scheme (7). Assume that $\Delta t\leq\min\{1,\frac{2}{\beta}\}$, $\alpha<\frac{1}{4}$, and*

$$\alpha \leq \frac{1}{2\left(|\Omega|2^{\gamma}\left(2\int_{\Omega}\left(\frac{2}{\beta}\left(\frac{2}{3}\right)^2\right)+\frac{2}{\beta}||u_g^{\gamma}||^2\right)\frac{4}{\beta}+|\Omega|2^{\gamma}\left(\left(2\int_{\Omega}\left(\frac{2}{\beta}\left(\frac{2}{3}\right)^2\right)+\frac{2}{\beta}||u_g^{\gamma}||^2\right)\frac{4}{\beta}\right)^2\right)}.$$

Then the soultion is long-time stable: for any given $n \in \mathbb{N}$,

$$||\hat{u}^n|| \leq \sqrt{\frac{4}{\beta}\left(2\int_{\Omega}\left(\frac{2}{\beta}\left(\frac{2}{3}\right)^2\right)+\frac{2}{\beta}||u_g^{\gamma}||^2\right)}, \forall n \in \mathbb{N}.$$

Proof. Following as in the proof of Theorem 1 up to (11), we get

$$\alpha\left(|\hat{u}^n+u_g|^{\gamma},\hat{u}^{n+1}\right) \leq 2^{\gamma}\alpha\int_{\Omega}|\hat{u}^n|^{\gamma}|\hat{u}^{n+1}|+2^{\gamma}\alpha\int_{\Omega}|u_g^{\gamma}||\hat{u}^{n+1}|. \tag{26}$$

Rewriting the first term of the right-hand side and using Young's inequality, we get that

$$2^{\gamma}\alpha\int_{\Omega}\left(|\hat{u}^n|^{\gamma}\cdot 1\right)|\hat{u}^{n+1}| \leq 2^{\gamma}\alpha\int_{\Omega}\frac{\gamma}{3}|\hat{u}^n|^3|\hat{u}^{n+1}|+2^{\gamma}\alpha\int_{\Omega}\frac{3-\gamma}{3}|\hat{u}^{n+1}|.$$

Since $1<\gamma<3$, we further bound this term and obtain

$$2^{\gamma}\alpha\int_{\Omega}\frac{\gamma}{3}|\hat{u}^n|^3|\hat{u}^{n+1}|+2^{\gamma}\alpha\int_{\Omega}\frac{3-\gamma}{3}|\hat{u}^{n+1}| \leq 2^{\gamma}\alpha\int_{\Omega}|\hat{u}^n|^3|\hat{u}^{n+1}|+2^{\gamma}\alpha\int_{\Omega}\frac{2}{3}|\hat{u}^{n+1}|. \tag{27}$$

Combining (26) and (27), we see that

$$\alpha\left(|\hat{u}^n+u_g|^{\gamma},\hat{u}^{n+1}\right) \leq 2^{\gamma}\alpha\int_{\Omega}|\hat{u}^n|^3|\hat{u}^{n+1}|+2^{\gamma}\alpha\int_{\Omega}\frac{2}{3}|\hat{u}^{n+1}|+2^{\gamma}\alpha\int_{\Omega}|u_g^{\gamma}||\hat{u}^{n+1}|. \tag{28}$$

We add and subtract $\hat{u}^n$ to $\hat{u}^{n+1}$ in the first right-hand side term. From the triangle inequality,

$$2^{\gamma}\alpha\int_{\Omega}|\hat{u}^n|^3|\hat{u}^{n+1}-\hat{u}^n+\hat{u}^n| \leq 2^{\gamma}\alpha\int_{\Omega}|\hat{u}^n|^3|\hat{u}^{n+1}-\hat{u}^n|+2^{\gamma}\alpha\int_{\Omega}|\hat{u}^n|^4. \tag{29}$$

We utilize Cauchy-Schwarz and Young inequalities on the first term of (29) and multiply by $\frac{2\Delta t}{2\Delta t}$ to obtain

$$2^{\gamma}\alpha\int_{\Omega}|\hat{u}^n|^3|\hat{u}^{n+1}-\hat{u}^n| \leq 2^{\gamma}\alpha\left(\frac{1}{4\Delta t}||\hat{u}^{n+1}-\hat{u}^n||^2+\Delta t||\left(\hat{u}^n\right)^3||^2\right).$$

Rewriting the last term as an L^6 norm and combining with (29) gives us

$$2^{\gamma}\alpha\int_{\Omega}|\hat{u}^n|^3|\hat{u}^{n+1}-\hat{u}^n|+2^{\gamma}\alpha\int_{\Omega}|\hat{u}^n|^4 \leq 2^{\gamma}\alpha\left(\frac{1}{4\Delta t}||\hat{u}^{n+1}-\hat{u}^n||^2+\Delta t||\hat{u}^n||_{L^6}^6\right)+2^{\gamma}\alpha\int_{\Omega}|\hat{u}^n|^4.$$

Rewriting the last term as an L^4 norm and combining with (28) gives us

$$\alpha\left(|\hat{u}^n+u_g|^{\gamma},\hat{u}^{n+1}\right) \leq 2^{\gamma}\alpha\left(\frac{1}{4\Delta t}||\hat{u}^{n+1}-\hat{u}^n||^2+\Delta t||\hat{u}^n||_{L^6}^6\right) + 2^{\gamma}\alpha||\hat{u}^n||_{L^4}^4+2^{\gamma}\alpha\int_{\Omega}\frac{2}{3}|\hat{u}^{n+1}|+2^{\gamma}\alpha\int_{\Omega}|u_g^{\gamma}||\hat{u}^{n+1}|. \tag{30}$$

By Lemma 2, we see that

$$\alpha\left(|\hat{u}^n+u_g|^\gamma,\hat{u}^{n+1}\right)\le\frac{2^\gamma\alpha}{4\Delta t}||\hat{u}^{n+1}-\hat{u}^n||^2+\left(C\alpha 2^\gamma||\hat{u}^n||^4+C\alpha 2^\gamma||\hat{u}^n||^2\right)||\nabla\hat{u}^n||^2 + 2^\gamma\alpha\int_\Omega\frac{2}{3}|\hat{u}^{n+1}|+2^\gamma\alpha\int_\Omega|u_g^\gamma||\hat{u}^{n+1}|,$$

with C depending only on Ω.

Utilizing Young on the third term of (30) and multiplying by $\frac{4\beta}{4\beta}$, we get that

$$\alpha\left(|\hat{u}^n+u_g|^\gamma,\hat{u}^{n+1}\right)\le\frac{2^\gamma\alpha}{4\Delta t}||\hat{u}^{n+1}-\hat{u}^n||^2+\left(|\Omega|\alpha 2^\gamma||\hat{u}^n||^4+|\Omega|\alpha 2^\gamma||\hat{u}^n||^2\right)||\nabla\hat{u}^n||^2 + 2^\gamma\alpha\int_\Omega\left(\frac{2}{\beta}\left(\frac{2}{3}\right)^2\right)+\frac{2^\gamma\alpha\beta}{8}||\hat{u}^{n+1}||^2+2^\gamma\alpha\int_\Omega|u_g^\gamma||\hat{u}^{n+1}|.$$

We use Cauchy-Schwarz and Young inequalities on the last term of (30) and after multiplying by $\frac{\beta}{\beta}$, we obtain

$$\alpha\left(|\hat{u}^n+u_g|^\gamma,\hat{u}^{n+1}\right)\le\frac{2^\gamma\alpha}{4\Delta t}||\hat{u}^{n+1}-\hat{u}^n||^2+\left(|\Omega|\alpha 2^\gamma||\hat{u}^n||^4+|\Omega|\alpha 2^\gamma||\hat{u}^n||^2\right)||\nabla\hat{u}^n||^2 + 2^\gamma\alpha\int_\Omega\left(\frac{2}{\beta}\left(\frac{2}{3}\right)^2\right)+\frac{2^\gamma\alpha\beta}{8}||\hat{u}^{n+1}||^2+\frac{1}{2\beta}\left(2^\gamma\right)^2\alpha^2||u_g^\gamma||^2+\frac{\beta}{2}||\hat{u}^{n+1}||^2. \quad (31)$$

Taking advantage of our assumptions that $\gamma<3$ and $\alpha<\frac{1}{4}$,we further upper bound the right-hand side in order to get that

$$\alpha\left(|\hat{u}^n+u_g|^\gamma,\hat{u}^{n+1}\right)\le\frac{1}{2\Delta t}||\hat{u}^{n+1}-\hat{u}^n||^2+\left(|\Omega|\alpha 2^\gamma||\hat{u}^n||^4+|\Omega|\alpha 2^\gamma||\hat{u}^n||^2\right)||\nabla\hat{u}^n||^2+\frac{\beta}{4}||\hat{u}^{n+1}||^2 + \frac{\beta}{2}||\hat{u}^{n+1}||^2+2\int_\Omega\left(\frac{2}{\beta}\left(\frac{2}{3}\right)^2\right)+\frac{2}{\beta}||u_g^\gamma||^2. \quad (32)$$

Combining (10) with (32) and combining like terms, we get

$$\left(\frac{1}{2\Delta t}+\frac{\beta}{4}\right)||\hat{u}^{n+1}||^2+||\nabla\hat{u}^{n+1}||^2\le\frac{1}{2\Delta t}||\hat{u}^n||^2+\left(|\Omega|\alpha 2^\gamma||\hat{u}^n||^4+|\Omega|\alpha 2^\gamma||\hat{u}^n||^2\right)||\nabla\hat{u}^n||^2 + 2\int_\Omega\left(\frac{2}{\beta}\left(\frac{2}{3}\right)^2\right)+\frac{2}{\beta}||u_g^\gamma||^2. \quad (33)$$

From here, we utilize mathematical induction to prove that $\forall n$, $||\hat{u}^n||^2$ is bounded above by problem data.

Base Case. For $n = 0$, by (33), we have that

$$\left(\frac{1}{2\Delta t} + \frac{\beta}{4}\right) ||\hat{u}^1||^2 + ||\nabla \hat{u}^1||^2 \le 2\int_\Omega \left(\frac{2}{\beta}\left(\frac{2}{3}\right)^2\right) + \frac{2}{\beta}||u_g^\gamma||^2.$$

Dropping the nonnegative term $||\nabla \hat{u}^1||^2$ and dividing by $\frac{1}{2\Delta t} + \frac{\beta}{4}$ gives us

$$||\hat{u}^1||^2 \le \frac{2\int_\Omega \left(\frac{2}{\beta}\left(\frac{2}{3}\right)^2\right) + \frac{2}{\beta}||u_g^\gamma||^2}{\frac{1}{2\Delta t} + \frac{\beta}{4}}.$$

We define $\epsilon := \frac{\frac{1}{2\Delta t}}{\frac{1}{2\Delta t} + \frac{\beta}{4}} = \frac{1}{1 + \frac{\beta \Delta t}{2}} < 1$. Therefore, we have that

$$||\hat{u}^1||^2 \le \frac{2\int_\Omega \left(\frac{2}{\beta}\left(\frac{2}{3}\right)^2\right) + \frac{2}{\beta}||u_g^\gamma||^2}{\frac{1}{2\Delta t} + \frac{\beta}{4}} = 2\Delta t \epsilon \left(2\int_\Omega \left(\frac{2}{\beta}\left(\frac{2}{3}\right)^2\right) + \frac{2}{\beta}||u_g^\gamma||^2\right)$$

By our assumptions on β and Δt, we can immediately infer that $\frac{1}{2} \le \epsilon < 1$. With this,

$$\begin{aligned} ||\hat{u}^1||^2 &\le 2\Delta t \epsilon \left(2\int_\Omega \left(\frac{2}{\beta}\left(\frac{2}{3}\right)^2\right) + \frac{2}{\beta}||u_g^\gamma||^2\right) \\ &\le \frac{4}{\beta}\left(2\int_\Omega \left(\frac{2}{\beta}\left(\frac{2}{3}\right)^2\right) + \frac{2}{\beta}||u_g^\gamma||^2\right) \end{aligned}$$

Induction Step. By mathematical induction, for the kth step, our induction hypothesis is that

$$||\hat{u}^k||^2 + 2\Delta t \epsilon ||\nabla \hat{u}^k||^2 \le \frac{4}{\beta}\left(2\int_\Omega \left(\frac{2}{\beta}\left(\frac{2}{3}\right)^2\right) + \frac{2}{\beta}||u_g^\gamma||^2\right)$$

Therefore, the $k + 1$ solution satisfies, by (33),

$$\begin{aligned} \left(\frac{1}{2\Delta t} + \frac{\beta}{4}\right) ||\hat{u}^{k+1}||^2 + ||\nabla \hat{u}^{k+1}||^2 \le \frac{1}{2\Delta t}||\hat{u}^k||^2 + \left(|\Omega|\alpha 2^\gamma ||\hat{u}^k||^4 + |\Omega|\alpha 2^\gamma ||\hat{u}^k||^2\right) ||\nabla \hat{u}^k||^2 \\ + 2\int_\Omega \left(\frac{2}{\beta}\left(\frac{2}{3}\right)^2\right) + \frac{2}{\beta}||u_g^\gamma||^2. \quad (34) \end{aligned}$$

By our assumptions on α and our bounds on ϵ, we have that

$$\begin{aligned} \alpha &\le \frac{1}{2\left(|\Omega| 2^\gamma \left(2\int_\Omega \left(\frac{2}{\beta}\left(\frac{2}{3}\right)^2\right) + \frac{2}{\beta}||u_g^\gamma||^2\right)\frac{4}{\beta} + |\Omega| 2^\gamma \left(\left(2\int_\Omega \left(\frac{2}{\beta}\left(\frac{2}{3}\right)^2\right) + \frac{2}{\beta}||u_g^\gamma||^2\right)\frac{4}{\beta}\right)^2\right)} \\ &\le \frac{\epsilon}{\left(|\Omega| 2^\gamma \left(2\int_\Omega \left(\frac{2}{\beta}\left(\frac{2}{3}\right)^2\right) + \frac{2}{\beta}||u_g^\gamma||^2\right)\frac{4}{\beta} + |\Omega| 2^\gamma \left(\left(2\int_\Omega \left(\frac{2}{\beta}\left(\frac{2}{3}\right)^2\right) + \frac{2}{\beta}||u_g^\gamma||^2\right)\frac{4}{\beta}\right)^2\right)}. \end{aligned}$$

Hence by our induction hypothesis, we obtain

$$\alpha \leq \frac{\epsilon}{|\Omega|2^\gamma||\hat{u}^k||^2 + |\Omega|2^\gamma||\hat{u}^k||^4}. \tag{35}$$

Combining (34) with (35) yields

$$\left(\frac{1}{2\Delta t} + \frac{\beta}{4}\right)||\hat{u}^{k+1}||^2 + ||\nabla\hat{u}^{k+1}||^2 \leq \frac{1}{2\Delta t}||\hat{u}^k||^2 + \epsilon||\nabla\hat{u}^k||^2 + 2\int_\Omega \left(\frac{2}{\beta}\left(\frac{2}{3}\right)^2\right) + \frac{2}{\beta}||u_g^\gamma||^2.$$

Utilizing our induction hypothesis once again, we get that

$$\left(\frac{1}{2\Delta t} + \frac{\beta}{4}\right)||\hat{u}^{k+1}||^2 + ||\nabla\hat{u}^{k+1}||^2 \leq \left(\frac{2+\beta\Delta t}{\beta\Delta t}\right)\left(2\int_\Omega \left(\frac{2}{\beta}\left(\frac{2}{3}\right)^2\right) + \frac{2}{\beta}||u_g^\gamma||^2\right).$$

Dropping the nonnegative term $||\nabla\hat{u}^{k+1}||^2$ and dividing both sides by $\frac{1}{2\Delta t} + \frac{\beta}{4}$, we obtain

$$\begin{aligned}||\hat{u}^{k+1}||^2 &\leq \left(\frac{1}{\frac{1}{2\Delta t} + \frac{\beta}{4}}\right)\left(\frac{2+\beta\Delta t}{\beta\Delta t}\right)\left(2\int_\Omega \left(\frac{2}{\beta}\left(\frac{2}{3}\right)^2\right) + \frac{2}{\beta}||u_g^\gamma||^2\right)\\ &= \frac{4}{\beta}\left(2\int_\Omega \left(\frac{2}{\beta}\left(\frac{2}{3}\right)^2\right) + \frac{2}{\beta}||u_g^\gamma||^2\right),\end{aligned}$$

which is precisely the induction hypothesis for step $k+1$.

Thus, by mathematical induction, $||\hat{u}^n|| \leq \sqrt{\frac{4}{\beta}\left(2\int_\Omega \left(\frac{2}{\beta}\left(\frac{2}{3}\right)^2\right) + \frac{2}{\beta}||u_g^\gamma||^2\right)}$ holds $\forall n \in \mathbb{N}$. □

From here, we prove that sufficient conditions exist for any $\gamma > 3$. We pick $\gamma = 4$ for simplicity as the powers of θ from (3) become increasingly complex with higher values for γ.

Theorem 5 *Suppose that $\gamma = 4$ in the scheme (7). Assume that $\Delta t \leq \frac{2}{\beta}$, and*

$$\alpha \leq \frac{\rho}{16|\Omega|\left(2\left(\frac{2}{\beta}\left(\frac{1}{2\beta}\left(2^\gamma\right)^2\alpha^2||u_g^\gamma||^2\right)\right)^6 + \left(\frac{2}{\beta}\left(\frac{1}{2\beta}\left(2^\gamma\right)^2\alpha^2||u_g^\gamma||^2\right)\right)^3\right)},$$

with $0 < \rho < 1$ arbitrary but fixed. Then the solution is long-time stable: for any given $n \in \mathbb{N}$,

$$||\hat{u}^n|| \leq \sqrt{\frac{2}{\beta}\left(\frac{1}{2\beta}\left(2^\gamma\right)^2\alpha^2||u_g^\gamma||^2\right)}.$$

Proof. Following as in the proof of Theorem 3 up to (20), we have that

$$\frac{1}{2\Delta t}||\hat{u}^{n+1}||^2 + ||\nabla\hat{u}^{n+1}||^2 + \frac{\beta}{2}||\hat{u}^{n+1}||^2 \leq 2^\gamma\alpha\Delta t||\hat{u}^n||^{2\gamma}_{L^{2\gamma}} + 2^\gamma\alpha||\hat{u}^n||^{\gamma+1}_{L^{\gamma+1}} + \frac{1}{2\Delta t}||\hat{u}^n||^2 + \frac{1}{2\beta}\left(2^\gamma\right)^2\alpha^2||u_g^\gamma||^2. \tag{36}$$

Using our assumption that $\gamma = 4$ as well as (3) with $p = 8$ on our first right-hand side term, we obtain

$$2^{\gamma}\alpha\Delta t||\hat{u}^n||_{L^{2\gamma}}^{2\gamma} \leq 16C\alpha\Delta t||\nabla\hat{u}^n||^3||\hat{u}||^5, \tag{37}$$

where C is only dependent upon Ω.

Bounding the second right-hand side term by the L^{∞} norm as well as the 1D Agmon inequality, we get that

$$2^{\gamma}\alpha||\hat{u}^n||_{L^{\gamma+1}}^{\gamma+1} \leq 16C\alpha||\nabla\hat{u}^n||^{\frac{5}{2}}||\hat{u}^n||^{\frac{5}{2}}, \tag{38}$$

with C only dependent upon Ω.

Combining (36) with (37) and (38), we see that

$$\frac{1}{2\Delta t}||\hat{u}^{n+1}||^2+||\nabla\hat{u}^{n+1}||^2+\frac{\beta}{2}||\hat{u}^{n+1}||^2 \leq \left(16|\Omega|\alpha\Delta t||\nabla\hat{u}^n||||\hat{u}^n||^5 + 16|\Omega|\alpha||\nabla\hat{u}^n||^{\frac{1}{2}}||\hat{u}^n||^{\frac{5}{2}}\right)||\nabla\hat{u}^n||^2 + \frac{1}{2\Delta t}||\hat{u}^n||^2 + \frac{1}{2\beta}\left(2^{\gamma}\right)^2\alpha^2||u_g^{\gamma}||^2. \tag{39}$$

From here, we utilize mathematical induction to prove that $\forall n, ||\hat{u}^n||^2$ is bounded above by problem data.

Base Case. For $n = 0$, by (39), we have that

$$\left(\frac{1}{2\Delta t}+\frac{\beta}{2}\right)||\hat{u}^1||^2 + ||\nabla\hat{u}^1||^2 \leq \frac{1}{2\beta}\left(2^{\gamma}\right)^2\alpha^2||u_g^{\gamma}||^2.$$

Dropping the nonnegative term $||\nabla\hat{u}^1||^2$ and dividing by $\frac{1}{2\Delta t}+\frac{\beta}{2}$, we find that

$$||\hat{u}^1||^2 \leq \frac{\frac{1}{2\beta}\left(2^{\gamma}\right)^2\alpha^2||u_g^{\gamma}||^2}{\frac{1}{2\Delta t}+\frac{\beta}{2}}.$$

Utilizing our definition of λ from Theorem 3, we see that

$$||\hat{u}^1||^2 \leq \frac{\frac{1}{2\beta}\left(2^{\gamma}\right)^2\alpha^2||u_g^{\gamma}||^2}{\frac{1}{2\Delta t}+\frac{\beta}{2}} = 2\Delta t\lambda\left(\frac{1}{2\beta}\left(2^{\gamma}\right)^2\alpha^2||u_g^{\gamma}||^2\right).$$

By our assumptions on β and Δt, we can immediately infer that $\frac{1}{2} \leq \lambda \leq 1$. With this,

$$\begin{aligned}||\hat{u}^1||^2 &\leq 2\Delta t\lambda\left(\frac{1}{2\beta}\left(2^{\gamma}\right)^2\alpha^2||u_g^{\gamma}||^2\right)\\ &\leq \frac{2}{\beta}\left(\frac{1}{2\beta}\left(2^{\gamma}\right)^2\alpha^2||u_g^{\gamma}||^2\right).\end{aligned}$$

Induction Step. Our induction hypothesis is that

$$||\hat{u}^k||^2 + 2\rho\Delta t||\nabla\hat{u}^k||^2 \leq \frac{2}{\beta}\left(\frac{1}{2\beta}\left(2^\gamma\right)^2\alpha^2||u_g^\gamma||^2\right).$$

Therefore, the $k+1$ solution satisfies, by (39),

$$\frac{1}{2\Delta t}||\hat{u}^{k+1}||^2 + ||\nabla\hat{u}^{k+1}||^2 + \frac{\beta}{2}||\hat{u}^{k+1}||^2 \leq \left(16|\Omega|\alpha\Delta t||\nabla\hat{u}^k||||\hat{u}^k||^5 + 16|\Omega|\alpha||\nabla\hat{u}^k||^{\frac{1}{2}}||\hat{u}^k||^{\frac{5}{2}}\right)||\nabla\hat{u}^k||^2 + \frac{1}{2\Delta t}||\hat{u}^k||^2 + \frac{1}{2\beta}\left(2^\gamma\right)^2\alpha^2||u_g^\gamma||^2. \quad (40)$$

Applying our induction hypothesis to our assumption on α and Δt, we obtain

$$\alpha \leq \frac{\rho}{16|\Omega|2||\nabla\hat{u}^k||||\hat{u}^k||^5 + 16|\Omega|||\nabla\hat{u}^k||^{\frac{1}{2}}||\hat{u}^k||^{\frac{5}{2}}}. \quad (41)$$

Combining (40) and (41) yields

$$\left(\frac{1}{2\Delta t} + \frac{\beta}{2}\right)||\hat{u}^{k+1}||^2 + ||\nabla\hat{u}^{k+1}||^2 \leq \rho||\nabla\hat{u}^k||^2 + \frac{1}{2\Delta t}||\hat{u}^k||^2 + \frac{1}{2\beta}\left(2^\gamma\right)^2\alpha^2||u_g^\gamma||^2.$$

Utilizing our induction hypothesis once again, we get that

$$\left(\frac{1}{2\Delta t} + \frac{\beta}{2}\right)||\hat{u}^{k+1}||^2 + ||\nabla\hat{u}^{k+1}||^2 \leq \left(\frac{\beta\Delta t + 1}{\beta\Delta t}\right)\frac{1}{2\beta}\left(2^\gamma\right)^2\alpha^2||u_g^\gamma||^2.$$

Dropping the nonnegative term $||\nabla\hat{u}^{k+1}||^2$ and dividing both sides by $\frac{1}{2\Delta t} + \frac{\beta}{2}$, we obtain

$$\begin{aligned}||\hat{u}^{k+1}||^2 &\leq \left(\frac{1}{\frac{1}{2\Delta t} + \frac{\beta}{2}}\right)\left(\frac{\beta\Delta t + 1}{\beta\Delta t}\right)\frac{1}{2\beta}\left(2^\gamma\right)^2\alpha^2||u_g^\gamma||^2 \\ &= \frac{2}{\beta}\left(\frac{1}{2\beta}\left(2^\gamma\right)^2\alpha^2||u_g^\gamma||^2\right),\end{aligned}$$

which is precisely the induction hypothesis for step $k+1$.

Thus, by mathematical induction, $||\hat{u}^n|| \leq \sqrt{\frac{2}{\beta}\left(\frac{1}{2\beta}\left(2^\gamma\right)^2\alpha^2||u_g^\gamma||^2\right)}$ holds for every natural number n. □

4 Numerical Results

We now show the results of our method applied to a test problem. The problem is to solve (1) on $[0, 12.7]$ with $T = 1980$, $u_g = 21.45$, and $u_0 = u_h +$ small random noise that comes from the measurement error. Thus setup is an effort t o m imic a r ecent l aboratory e xperiment performed here at Clemson. The initial condition is plotted in figure 1.

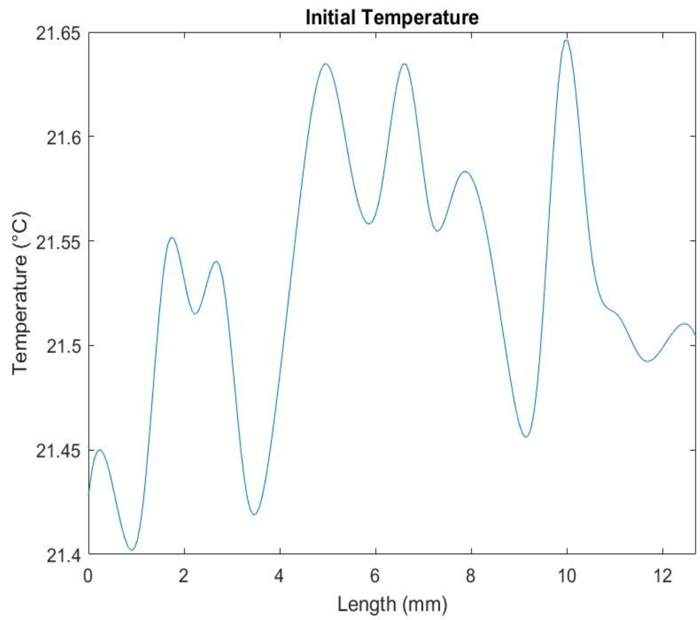

Figure 1: Shown above is the initial temperature profile for the numerical tests.

We use our method combined with a second order finite difference approximation and equally spaced points to approximate varying values of α, β, and γ, using time step size $\Delta t = .1563$ and spatial step size $\Delta x = .0638$. We believe that these discretization parameters were sufficiently small to resolve solutions.

The analysis of our scheme suggests different regimes based on the γ parameter. Hence, we experiment with $\gamma = \frac{1}{2}, 1, 2, 3, 4, 6$. For each γ, we test various values of α and β, and ask the question: is the method stable or not, which is determined by whether the final time step's solution has L^2 norm less that 10^6 (when unstable, the solution norm is always at least 10^{30}, and when stable, it is less than 10^4). Plots of stability regions are shown in figure 2, and we observe a linear relationship between the α and β parameters, as our analysis predicts. We observe that as γ grows, β must also grow in order to maintain stability for the same values of α; in particular, β must grow at a rate 10 times greater than γ. These observations are consistent with the parameter assumptions from our analysis that guarantee that solutions will lie within the stability regions for each value of γ.

5 Conclusions and Future Directions

We have shown conditions on the data that provide long-time stability of our scheme for any positive γ. Numerical tests illustrate the theory. For future work, we plan to consider extending to 2D, which will be challenging due to weaker Sobolev and Agmon inequalities.

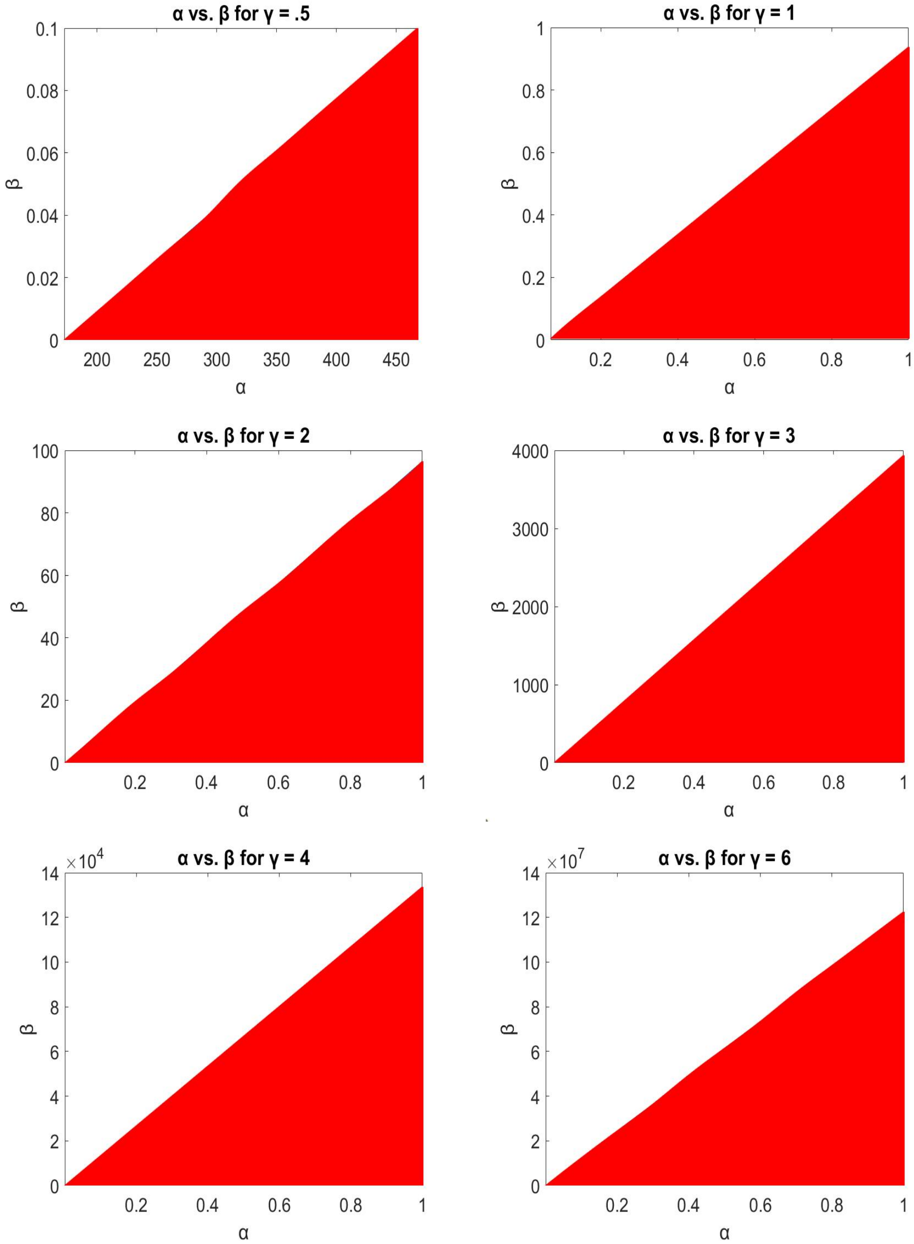

Figure 2: Shown above are the stability regions and instability regions (shaded in red) for the numerical tests with varying parameters.

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