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Unitary Representations of Locally Compact Groups as Metric Structures

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Abstract For a locally compact group G, we show that it is possible to present the class of continuous unitary representations of G as an elementary class of metric structures, in the sense of continuous logic. More precisely, we show how nondegenerate $*$-representations of a general $*$-algebra A (with some mild assumptions) can be viewed as an elementary class, in a many-sorted language, and use the correspondence between continuous unitary representations of G and nondegenerate $*$-representations of $L^1(G)$. We relate the notion of ultraproduct of logical structures, under this presentation, with other notions of ultraproduct of representations appearing in the literature, and we characterize property (T) for G in terms of the definability of the sets of fixed points of L^1 functions on G.

Introduction

When suggesting a model-theoretic treatment of a mathematical object or of a class of such objects, one must first present the said object(s) as logical structures. In other words, to each of the objects in question we associate a logical structure, from which the original object can be recovered.

In some cases, such as fields or groups, this step is so straightforward that it is hardly noticed. In others, such as valued fields, there is some (very small) degree of freedom, and one would say they can be viewed as structures in this language or in that. When dealing with metric structures in the formalism of continuous logic, a new difficulty arises, namely that one might wish to consider an unbounded metric space, such as a Banach space, in a logic that can only consider bounded metric spaces.

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We know of two potential remedies for this. First, one can sometimes extend the logic to one that does allow unbounded structures, with some price to pay at the level of technical complexity. Second, one can sometimes argue that the unbounded structure can in fact be represented by a bounded structure (possibly many-sorted). The second solution applies quite frequently in the case of Banach space structures. Indeed, one can recover the entire Banach space from its unit ball equipped with the structure of a convex space. This practice of restricting the domain of quantification to the unit ball is quite standard in other contexts—for example, when defining the operator norm.

Whether such a presentation is "correct" is very context dependent, of course, but usually there are two or three good indicators for that, especially when considering a class of objects that arises naturally in some area of mathematics (below we consider two such examples: Banach spaces and continuous unitary representations of a topological group G). Either of the second or third conditions implies the first, and all three are "usually" satisfied (or not) simultaneously.

(i) The class of (presentations of) objects should be closed under logical ultraproducts.
(ii) The class of (presentations of) objects should be elementary.
(iii) When there already exists a useful intrinsic notion of ultraproduct in the class, it should agree with the logical one.

The unit ball of a Banach space is a good example of the positive case, where the intrinsic ultraproduct of Banach spaces (see Dacunha-Castelle and Krivine [5]) coincides with the logical ultraproduct of their unit balls.

A failure of a logical presentation to satisfy these criteria often comes in one of two flavors, which we refer to as *missing points* and *extraneous points*, respectively. As an example, let us consider two natural yet misguided attempts to represent the class of continuous unitary representations of a fixed nondiscrete topological group G.

1. A first naive approach would be to consider the structure consisting of the unit ball of a Hilbert space, together with a function symbol for the action of each $g \in G$. While the map $g \mapsto g\xi$ is continuous (at $1 \in G$) for each ξ in the structure, the family of such maps is not necessarily equicontinuous, and a logical ultrapower may well contain ξ such that $g \mapsto g\xi$ is not continuous at all. This is an "extraneous point" (one kind of nonlogical ultraproduct of continuous representations is defined exactly by excluding such points from the Banach space ultraproduct).
2. A next attempt might be to consider, for each modulus of continuity at 1, the sort of all ξ in the unit ball such that $g \mapsto g\xi$ satisfies that modulus of continuity. In such a structure, associated to a continuous representation, G acts continuously, and even equicontinuously, on each sort, and this is preserved by ultraproducts.

   However, such a structure must also satisfy a subtler condition. Say A and B are two moduli of continuity, with B stronger than A. Then the associated sorts must satisfy an inclusion relation $S_B \subseteq S_A$. Moreover, any point of S_A that happens to satisfy the modulus B must belong to S_B—and this property need not be preserved under ultraproducts. If this fails, then we may say that the sort S_B is *missing* some points.

In the case of a locally compact G we propose a solution, by splitting the representation into sorts not by moduli of continuity, but as images of the action of $L^1(G)$. The resulting structure may be presented most elegantly as a nondegenerate representation of the $*$-algebra $L^1(G)$. We therefore begin with a general discussion of the presentation of nondegenerate representations of $*$-algebras as logical structures in Section 1. How this specializes to representations of a locally compact G is discussed in Section 2 (this is fairly standard and mostly included for the sake of completeness). In Section 3 we put the notion of ultraproduct associated with our logical structures in the context of notions of ultraproduct of unitary representations existing in the literature. Finally, in Section 4 we give a model-theoretic characterization of Kazhdan's property (T) in G in terms of definability of the set(s) of fixed points in the associated structure.

1 Nondegenerate $*$-Representations

Throughout, by an *algebra* we mean a complex algebra, that is, a ring A, not necessarily commutative or unital, that is also a complex vector space, satisfying $\alpha(ab) = (\alpha a)b = a(\alpha b)$ for all $a, b \in A$ and $\alpha \in \mathbf{C}$. An algebra equipped with a conjugate-linear involution $*$ satisfying $(ab)^* = b^*a^*$ is a *$*$-algebra*. An algebra equipped with a norm satisfying $\|ab\| \le \|a\|\|b\|$ is a *normed algebra*, and a *Banach algebra* if it is complete. If it is both a normed (Banach) algebra and a $*$-algebra, and $\|a^*\| = \|a\|$, then it is a *normed (Banach) $*$-algebra*. A *C^*-algebra* is a Banach $*$-algebra in which $\|aa^*\| = \|a\|^2$.

A *morphism* of normed algebras is a bounded linear map that respects multiplication, and a *$*$-morphism* of normed $*$-algebras is a normed algebra morphism that respects the involution. A *$*$-representation* of a $*$-algebra A in a Hilbert space E is a $*$-morphism $\pi: A \to B(E)$.

Fact 1.1 *Let A be a Banach $*$-algebra, and let B be a C^*-algebra. Then any $*$-morphism $\varphi: A \to B$ is contractive. In particular, any $*$-representation of a Banach $*$-algebra is contractive.*

Proof In the case where B is a *unital* C^*-algebra, this is proved in Folland [6, Proposition 1.24(b)]. In order to reduce from the general case to the unital case, one may always add a unit (see Folland [6, Proposition 1.27]). □

Given a Banach $*$-algebra A, a $*$-representation of A can be naturally viewed as a single-sorted metric structure. Indeed, all we need to do is take the unit ball of a Hilbert space E, and for each $a \in A$ of norm at most 1, name the operator $\pi(a)$ in the language. The class of all such structures is elementary, defined by universal axioms (modulo the axioms for a unit ball of a Banach space), and if A is separable, then by choosing a dense subfamily of A the language can be made countable.

This is a little too easy and falls short of what we want to achieve: in view of Fact 2.3, we want to consider $*$-representations that are also nondegenerate. Nondegeneracy may be thought of as an analogue of the condition $\pi(1) = \mathrm{id}$ for nonunital $*$-algebras (see, e.g., Cherix, Cowling, and Straub [4, Section 4]).

Definition 1.2 Let A be a Banach $*$-algebra, and let $\pi: A \to B(E)$ be a $*$-representation. The *nondegenerate part* of the representation, let us call it E^π, is the closed subspace generated by all $\pi(a)\xi$ for $a \in A$ and $\xi \in E$. If $E^\pi = E$, then π is *nondegenerate*.

Fact 1.3 *Let $\pi\colon A \to B(E)$ be a $*$-representation of a Banach $*$-algebra A, and let $\xi \in E$.*

(i) *We have $\xi \perp E^\pi$ if and only if $\pi(a)\xi = 0$ for every $a \in A$. In particular, the $*$-representation restricts to a nondegenerate one on E^π.*

(ii) *Assume that (e_α) is a left approximate unit for A. Then $\xi \in E^\pi$ if and only if $\pi(e_\alpha)\xi \to \xi$. Equivalently (given the first item), for any $\xi \in E$, the sequence $\pi(e_\alpha)\xi$ converges to the orthogonal projection of ξ to E^π.*

Proof The first item follows from the identity $\langle \xi, \pi(a)\zeta\rangle = \langle \pi(a^*)\xi, \zeta\rangle$. For the second, if $\pi(e_\alpha)\xi \to \xi$, then $\xi \in E^\pi$ by definition. For the converse, we may assume that ξ is of the form $\pi(a)\zeta$. Then, by Fact 1.1:

$$\left\|\pi(e_\alpha)\xi - \xi\right\| = \left\|\pi(e_\alpha a)\zeta - \pi(a)\zeta\right\| \leq \|e_\alpha a - a\|\|\zeta\| \to 0. \qquad \square$$

If A does not have a unit, then the class of nondegenerate $*$-representations of A, presented naively as above, need not be elementary, so something better needs to be done. The problem is that we cannot express an infinite disjunction such as "there exists $a \in A$ such that ξ is close to the image of $\pi(a)$." We solve this by using a many-sorted language: for each $a \in A$ we shall have a sort S_a, consisting of the closure of the image of the unit ball under $\pi(a)$, and all that is left is to express the interactions between these sorts.

In what follows, by a *symmetric convex space*, we mean a closed convex subset of a Banach space closed under opposite. Following [2], a (bounded) symmetric convex space will be considered as a metric structure in the language $\{0, -, \frac{x+y}{2}\}$, where $-$ is the unary opposite operation and $\frac{x+y}{2}$ is the binary average operation. If E is a real Banach space and $C \subseteq E$ is a symmetric convex set that generates a dense subset of E, then E can be recovered from C. Linear maps can be recovered using the following easy result.

Lemma 1.4 *Let E and F be real normed spaces, and let $C \subseteq E$ be a symmetric convex generating subset. Assume that $f\colon C \to F$ is bounded in the sense that $\|f(x)\| < \alpha\|x\|$ for some $\alpha \in \mathbf{R}$. Then the following are equivalent.*

(i) *The map f respects the convex structure: $f(0) = 0$ and $f(\frac{x+y}{2}) = \frac{f(x)+f(y)}{2}$.*

(ii) *The map f is additive: $f(x+y) = f(x) + f(y)$ whenever $x, y, x+y \in C$.*

(iii) *The map f extends to a linear bounded map $E \to F$.*

Proof (i) $\Rightarrow$ (ii). For $x \in C$, we have $f(x/2) = f(\frac{x+0}{2}) = \frac{f(x)+0}{2} = f(x)/2$. Therefore, if $x, y, x+y \in C$, then

$$f(x+y) = 2f\left(\frac{x+y}{2}\right) = 2\frac{f(x)+f(y)}{2} = f(x) + f(y).$$

(ii) $\Rightarrow$ (iii). If f is additive on C, then it extends to an additive map $E \to F$. Such a map is necessarily $\mathbf{Q}$-linear and bounded with the same constant α, so it is $\mathbf{R}$-linear.

(iii) $\Rightarrow$ (i). This is immediate. $\square$

Definition 1.5 Let A be a Banach $*$-algebra, and let $\pi\colon A \to B(E)$ be a nondegenerate $*$-representation. We associate to it a multisorted structure $M = M(E, \pi)$ constructed as follows.

- For each $a \in A$, M admits a sort $S_a = \overline{\pi(a)E_{\leq 1}}$, where $E_{\leq 1}$ denotes the unit ball of E.
- Each sort is equipped with the structure of a symmetric convex space, as well as with a symbol for multiplication by i.
- For any $a, b \in A$, we name the restriction to $S_a \times S_b$ of the real part of the inner product: $[\xi, \zeta] = \Re\langle \xi, \zeta \rangle$. Since one such predicate exists for each pair of sorts, we may sometimes write $[\cdot, \cdot]_{a,b}$.
- For any $a, b \in A$, the map $\pi(a): S_b \to S_{ab}$ is named by a function symbol π_a.

All the symbols are bounded and uniformly continuous in a manner that does not depend on the choice of (E, π), so these can all be viewed as structures in a common language, call it $\mathcal{L}^A$. Of course, the same construction applies even if (E, π) has a degenerate part, but this degenerate part will not be reflected in any way in the structure $M(E, \pi)$.

We define T^A to be the theory consisting of the following axiom schemes that we explain shortly. All the axioms are either stated in continuous logic or can easily be. Axioms Definition Conv to Definition Complex are universal, with implicit universal quantifiers.

$$\text{Each } S_a \text{ is a symmetric convex space of radius at most } \|a\|, \tag{Conv}$$

$$[\xi, \zeta] = [\zeta, \xi], \tag{Sym}$$

$$\left[\xi, \frac{\zeta + \zeta'}{2}\right] = \frac{[\xi, \zeta] + [\xi, \zeta']}{2}, \tag{Lin1}$$

$$[\xi, 0] = 0, \tag{Lin2}$$

$$[\xi, \xi] = d(\xi, 0)^2, \tag{Norm}$$

$$\sum_{i,j=1}^{n} [\xi_i, \xi_j] \geq 0. \tag{Pos}$$

From here on, let $\| \sum \xi_i \|$ be short for $\sqrt{\sum [\xi_i, \xi_j]}$.

$$\left\| \sum \pi_a \xi_i \right\| \leq \|a\| \left\| \sum \xi_i \right\|, \tag{Pi1}$$

$$a \mapsto \pi_a \quad \text{is a } *\text{-morphism}, \tag{Pi2}$$

$$i: S_a \to S_a \quad \text{respects all other symbols, } i^2\xi = -\xi, \tag{Complex}$$

$$\sup_{\xi_i \in S_{b_i}} \inf_{\zeta \in S_a} \left\| \sum \pi_a \xi_i - \zeta \right\| \leq \left| \left\| \sum \xi_i \right\| - 1 \right| \|a\|, \tag{BallImg}$$

$$\sup_{\xi \in S_{ab}} \inf_{\zeta \in S_b} \|\pi_a \zeta - \xi\| = 0, \tag{DenseImg}$$

$$\sup_{\xi \in S_a} \inf_{\zeta \in S_b} \|\zeta - \xi\| \leq \|a - b\|. \tag{HausDist}$$

(According to our notational convention, in the last axiom, $\|\zeta - \xi\|^2 = [\xi, \xi] - 2[\xi, \zeta] + [\zeta, \zeta]$, while $\|a - b\|$ is the norm in A, and similarly for the other axioms.)

Clearly, every structure of the form $M(E, \pi)$ is a model of T^A. Now let M be any model of T^A, or, at a first time, merely of Definition Conv to Definition Complex.

(i) Axiom Definition Conv requires that each sort S_a be a symmetric convex space of radius at most $\|a\|$. This is indeed expressible in continuous logic and the generated real normed space can be recovered, call it E_a (see, e.g., [2]).

(ii) For each pair $a, b \in A$, axioms Definition Sym to Definition Lin2 require $[\cdot,\cdot]$ on $S_a \times S_b$ to be symmetric and $\mathbf{R}$-bilinear—that is to say that it extends (uniquely) to an $\mathbf{R}$-bilinear form on $E_a \times E_b$, as per Lemma 1.4. By axiom Definition Norm, it defines the norm on E_a.

(iii) Let $F^1 = \bigoplus_{a\in A} E_a$. If $\xi, \zeta \in F^1$, say $\xi = \sum \xi_i$ and $\zeta = \sum \zeta_i$ where $\xi_i, \zeta_i \in E_{a_i}$, let $[\xi,\zeta] = \sum_{i,j}[\xi_i, \zeta_j]$. This defines a symmetric $\mathbf{R}$-bilinear form on F^1, and axiom Definition Pos requires it to be positive semidefinite. It follows that $\|\sum \xi_i\| = \sqrt{\sum[\xi_i,\xi_j]}$ is a seminorm. Let $F^0 \subseteq F^1$ be its kernel. Then $F = \overline{F^1/F^0}$ with the induced norm is a real Hilbert space.

(iv) Axiom Definition Pi1 implies, first of all, that $\pi_a\colon S_b \to S_{ab} \subseteq E_{ab} \subseteq F$ is bounded:

$$\|\pi_a\xi\| \leq \|a\|\|\xi\|. \tag{1}$$

The axiom also implies that $\pi_a\colon S_b \to F$ is additive, in the sense of Lemma 1.4. Indeed, if $\xi \in S_b$, then $\|\xi + (-\xi)\| = 0$, and therefore $\|\pi_a\xi + \pi_a(-\xi)\| = 0$, so $\pi_a(-\xi) = -\pi_a\xi$. If ξ, ζ, and $\chi = \xi + \zeta$ all belong to S_b, then $\|\xi + \zeta - \chi\| = 0$, and therefore $\|\pi_a\xi + \pi_a\zeta - \pi_a\chi\| = 0$ as well. Therefore, π_a extends uniquely to an $\mathbf{R}$-linear map $\pi_a\colon E_b \to F$. These maps combine to a single $\mathbf{R}$-linear map $\pi_a\colon F^1 \to F$, and axiom Definition Pi1 implies that this combined map also satisfies (1). Therefore, it induces an operator $\sigma(a) \in B(F)$ of norm at most $\|a\|$.

(v) Now that we have a map $\sigma\colon A \to B(F)$, we require it to be a $*$-morphism (with respect to the real inner product $[\cdot,\cdot]$). This consists of a long list of identities, which we chose to omit (axiom Definition Pi2).

(vi) Axiom Definition Complex means that multiplication by i is isometric and linear (Lemma 1.4 again), putting a complex structure on F. Since i commutes with each π_a, each $\sigma(a)$ is $\mathbf{C}$-linear. Following the convention that a sesquilinear form is linear in the first argument and semilinear in the second, we may recover a complex inner product by $\langle\xi,\zeta\rangle = [\xi,\zeta] + i[\xi, i\zeta]$.

If (E,π) is a nondegenerate $*$-representation and $M = M(E,\pi)$, then the isometric embeddings $S_a = \overline{\pi(a)E_{\leq 1}} \hookrightarrow F^1 \to F$ glue (by nondegeneracy) to a canonical isometric linear bijection $E \to F$, which is the desired isomorphism of $*$-representations $(E,\pi) \cong (F,\sigma)$ (if (E,π) is degenerate, then we recover its nondegenerate part).

If M is an arbitrary model of Definition Conv to Definition Complex, then we recover a $*$-representation (F,σ). However, $M(F,\sigma)$ need not be isomorphic to M, since we still need to say that $S_a \subseteq F$ is exactly $\overline{\sigma(a)F_{\leq 1}}$. This will follow from the three last axioms, provided that we make one additional hypothesis regarding A: that for every $a \in A$ there exists $b \in A$ of norm 1, such that ab is arbitrarily close to a. This holds, in particular, if A admits a right approximate identity (e_α) of norm 1 (in which case (e_α^*) is a left approximate identity, and $(e_\alpha + e_\alpha^* - e_\alpha e_\alpha^*)$ is a two-sided approximate identity, albeit not necessarily of norm 1).

(i) The inclusion $S_a \supseteq \overline{\sigma(a)F_{\leq 1}}$ is exactly axiom Definition BallImg.
(ii) For the opposite inclusion, let $a \in A$ and $\varepsilon > 0$, and choose $b \in A$ of norm 1 such that $\|a - ab\| < \varepsilon$. By axioms Definition DenseImg and Definition HausDist, for any $\xi \in S_a$ there exists $\zeta \in S_b$ such that $\|\pi_a \zeta - \xi\| < \varepsilon$, and $\|\zeta\| \leq \|b\| = 1$.

Putting this all together, we have proved the following.

Theorem 1.6 *Let A be a Banach $*$-algebra. Assume that A admits an approximate one-sided identity of norm 1, or merely that every $a \in A$ belongs to $\overline{aA_{\leq 1}}$. Then the class of nondegenerate $*$-representations of A can be identified with the class of models of T^A and therefore may be considered to be an elementary class.*

In particular, if A satisfies the hypotheses of Theorem 1.6, then the class of nondegenerate $*$-representations of A is closed under the ultraproduct/ultrapower construction applied to $\mathcal{L}^A$-structures. This coincides with the *nondegenerate ultraproduct/ultrapower*, as proposed in [4, Section 4], obtained by taking the Banach space ultraproduct/ultrapower of E (see Section 3), which is naturally a $*$-representation, and taking its nondegenerate part.

2 Continuous Unitary Representations of Locally Compact Groups

Let G be a topological group, and let $M(G)$ denote the space of regular complex Borel measures on G. For every $\mu \in M(G)$, there exists a unique finite positive measure $|\mu|$ such that $d\mu = \alpha d|\mu|$, where $\alpha: G \to \mathbf{C}$ is a Borel function into the unit circle. We set the *total variation* of μ to be $\|\mu\| = |\mu|(G)$. For $\mu, \nu \in M(G)$, we may define an *involution* and a *convolution* by

$$\mu^* = \overline{i_* \mu}, \qquad \mu * \nu = m_*(\mu \otimes \nu),$$

where $m: G^2 \to G$ is the group law and $i: G \to G$ is inversion. It is easy to check that μ^* and $\mu * \nu$ belong to $M(G)$, and that equipped with these two operations and with the norm $\|\mu\|$, $M(G)$ is a Banach $*$-algebra.

For every $g \in G$ we have a Dirac measure $\delta_g \in M(G)$, and δ_e is the unit of $M(G)$. The map $g \mapsto \delta_g$ is injective and respects the algebraic structure: $\delta_g^* = \delta_{g^{-1}}$, $\delta_g * \delta_h = \delta_{gh}$. (However, the norm topology on $M(G)$ induces the discrete topology on G under this identification.) In particular, G acts isometrically on $M(G)$ on either side by convolution with the Dirac measure:

$$f\mu h = \delta_f * \mu * \delta_h, \qquad \|f\mu h\| = \|\mu\|.$$

A *continuous (always unitary) representation* of G consists of a Hilbert space E equipped with a continuous unitary action $G \curvearrowright E$, or equivalently, with a continuous morphism $\pi: G \to U(E)$, where $U(E)$ is equipped with the strong (equivalently, weak) operator topology. This means in particular that we may write $g\xi$ and $\pi(g)\xi$ interchangeably when $g \in G$ and $\xi \in E$. For a unitary action $G \curvearrowright E$ to be (jointly) continuous it suffices that for each $\xi \in E$ (separately), the map $g \mapsto g\xi$ is continuous at the identity.

Any continuous representation of G in E can be extended naturally to a $*$-representation of $M(G)$ by

$$\pi(\mu) = \int_G \pi(g)\, d\mu(g).$$

By this we mean that $\pi(\mu)\xi \in E$ is the unique vector such that

$$\langle \pi(\mu)\xi, \zeta \rangle = \int_G \langle g\xi, \zeta \rangle \, d\mu(g). \tag{2}$$

We have $\|\pi(\mu)\| \leq \|\mu\|$ by Cauchy–Schwarz, giving rise to a map $\pi\colon M(G) \to B(E)$. It is easy to check that $\pi(\mu^*) = \pi(\mu)^*$ and $\pi(\mu * \nu) = \pi(\mu)\pi(\nu)$, so π is indeed a $*$-representation. In addition, $\pi(\delta_g) = \pi(g)$, so this representation extends the original one via our identification $G \subseteq M(G)$.

Lemma 2.1 *Let $\xi \in E$, and let $\mu \in M(G)$ be a probability measure concentrated on the set $\{g \in G : \|g\xi - \xi\| \leq r\}$. Then $\|\pi(\mu)\xi - \xi\| \leq r$.*

Proof For all $\zeta \in E$, we have $|\langle \pi(\mu)\xi - \xi, \zeta \rangle| \leq r\|\zeta\|$ by a direct application of (2) and the hypotheses. □

Let us add the hypothesis that G is locally compact, and choose a left Haar measure H. Then $H(fAh) = \Delta(h)H(A)$, where $\Delta\colon G \to (\mathbf{R}^{>0}, \cdot)$ is the modular function on G, a group morphism that depends only on G. We shall write dg for $dH(g)$.

We may identify $\varphi \in L^1(G)$ (with respect to H) with $\mu_\varphi \in M(G)$ defined by $d\mu_\varphi = \varphi dH$. The map $\varphi \mapsto \mu_\varphi$ is a linear isometry, so $L^1(G) \subseteq M(G)$ is a Banach subspace. If $\varphi, \psi \in L^1(G)$, then (under the identification of φ with μ_φ)

$$\varphi^*(g) = \Delta(g^{-1})\overline{\varphi}(g^{-1})$$

and

$$(\psi * \varphi)(g) = \int_G \psi(h)\varphi(h^{-1}g)\, dh = \int_G \Delta(h^{-1})\psi(gh^{-1})\varphi(h)\, dh.$$

More generally, for any $\varphi \in L^1(G)$ and $\mu \in M(G)$,

$$(\mu * \varphi)(g) = \int_G \varphi(h^{-1}g)\, d\mu(h),$$
$$(\varphi * \mu)(g) = \int_G \Delta(h^{-1})\varphi(gh^{-1})\, d\mu(h),$$

so

$$(\delta_f * \varphi * \delta_h)(g) = \Delta(h^{-1})\varphi(f^{-1}gh^{-1}).$$

In particular, $L^1(G) \subseteq M(G)$ is a $*$-subalgebra, so to every continuous representation (E, π) of G corresponds a canonical $*$-representation of $L^1(G)$ through the restriction of the $*$-representation of $M(G)$:

$$\pi(\varphi) = \int_G \varphi(g)\pi(g)\, dg.$$

The usefulness of this representation is due to the following classical fact.

Fact 2.2 *Let $\varphi \in L^1(G)$. Then $\delta_f * \varphi * \delta_h \to \varphi$ in $L^1(G)$ as $f, h \to e$.*

Proof When φ is bounded, this follows from dominated convergence, and the general case follows by a density argument. □

Let (U_α) be a basis of compact neighborhoods of 1, and let $\alpha \leq \beta$ when $U_\alpha \supseteq U_\beta$. Let $e_\alpha \in C_c(G) \subseteq L^1(G)$ be continuous, positive, of norm 1, supported in U_α. Then as a net, (e_α) is an approximate identity of $L^1(G)$, that is to say that $\|e_\alpha * \varphi - \varphi\| \to 0$ and $\|\varphi * e_\alpha - \varphi\| \to 0$ for every $\varphi \in L^1(G)$. If (E, π) is a continuous representation of G, then $\pi(e_\alpha)\xi \to \xi$ for every $\xi \in E$ (e.g., by Lemma 2.1). In particular, $E^\pi = E$ and $\pi\colon L^1(G) \to B(E)$ is nondegenerate.

Conversely, let $\pi\colon L^1(G) \to B(E)$ be any nondegenerate $*$-representation. For any $g \in G$, the map $\varphi \mapsto \delta_g * \varphi$ is isometric. If $\xi \in E$, then $\pi(e_\alpha)\xi \to \xi$ by nondegeneracy, so $\pi(\delta_g * e_\alpha)\xi$ must converge as well; call its limit $\pi(g)\xi$ or $g\xi$. This defines a group morphism $\pi\colon G \to U(E)$. A combination of Fact 2.2 with the rate of convergence of $\pi(e_\alpha)\xi$ to ξ yields that $g \mapsto g\xi$ is continuous at e for any fixed $\xi \in E$, so $\pi\colon G \to U(E)$ is a continuous representation.

Fact 2.3 *These operations are one the inverse of the other, yielding a bijective correspondence between continuous representations of G on E and nondegenerate $*$-representations of $L^1(G)$ on E.*

Proof See Folland [6, Theorem 3.11]. □

Since $L^1(G)$ admits an approximate identity of norm 1, we have the following corollary to Theorem 1.6.

Corollary 2.4 *Let us identify a continuous representation $\pi\colon G \to U(E)$ with the corresponding nondegenerate $*$-representation $\pi\colon L^1(G) \to B(E)$, as per Fact 2.3. As in the previous section, let us identify the latter with the $\mathcal{L}^{L^1(G)}$-structure $M(E, \pi)$.*

With these identifications, the class of continuous representations of G is elementary, axiomatized by the theory $T^{L^1(G)}$.

If G is discrete, a unitary representation of G can also be considered as a single-sorted structure, consisting of the unit ball of a Hilbert space with each unitary operator $\pi(g)$ (restricted to the unit ball) named in the language (see, e.g., Berenstein [3]). In this case, the Haar measure is (a multiple of) the counting measure, and $\Delta \equiv 1$. Identifying $g \in G$ with δ_g, we have $G \subseteq L^1(G)$. Viewing a representation of G as an $\mathcal{L}^{L^1(G)}$-structure, the sort associated to δ_g is the entire unit ball for any $g \in G$, and $\pi(\delta_h)$ acts on it as $\pi(h)$ for all $h \in G$. In other words, we may recover the single-sorted structure alluded to above as a reduct of the $\mathcal{L}^{L^1(G)}$-structure. Conversely, we can interpret the multisorted $\mathcal{L}^{L^1(G)}$-structure in the single-sorted one. The full details would involve more definitions than interesting results, so we shall omit them.

3 Ultraproduct Constructions

Let us discuss possible ultraproduct constructions for unitary actions. Throughout this discussion, I is a set and $\mathcal{U}$ is an ultrafilter on I. Let $\prod^{\mathrm{B}}{}_{\mathcal{U}} E_i$ denote the Banach space ultraproduct of a family of Banach spaces, possibly with additional structure.

In particular, if $E = \prod^{\mathrm{B}}{}_{\mathcal{U}} E_i$ and $C = \prod^{\mathrm{B}}{}_{\mathcal{U}} B(E_i)$, then C is again a C^*-algebra, and there is a natural isometric embedding of C^*-algebras:

$$\prod_{\mathcal{U}}^{\mathrm{B}} B(E_i) \subseteq B(E),$$

where $[T_i] \in \prod^{\mathrm{B}}{}_{\mathcal{U}} B(E_i)$ acts on E by $[T_i][\xi_i] = [T_i \xi_i]$.

If each (E_i, π) is a unitary representation (not necessarily continuous) of G, then $(\prod^{\mathrm{B}}{}_{\mathcal{U}} E_i, \pi_{\mathcal{U}})$ is again such a representation, where $\pi_{\mathcal{U}}(g) = [\pi_i(g)] \in \prod^{\mathrm{B}}{}_{\mathcal{U}} B(E_i) \subseteq B(E)$. We call this the *naive* ultraproduct. Even if each E_i is a continuous representation, the naive ultraproduct need not be so. In the literature one finds two main ideas for remedying this deficiency.

First, any unitary representation admits a continuous part.

Definition 3.1 Let E be a Hilbert space, and let $G \to U(E)$ be a unitary representation, not necessarily continuous. We define E^{c} to consist of all $\xi \in E$ for which the map $g \mapsto g\xi$ is continuous. We call it the *continuous part* of the unitary representation E.

In particular, if (E_i) are unitary representations (say, continuous, but this is not required for this definition), we define the *continuous ultraproduct* as

$$\prod_{\mathcal{U}}^{\mathrm{c}} E_i = \Big(\prod_{\mathcal{U}}^{\mathrm{B}} E_i\Big)^{\mathrm{c}}.$$

The following is easy.

Fact 3.2 *With the hypotheses of Definition 3.1, E^{c} is a Hilbert subspace of E. It is moreover G-invariant, and the restricted representation $G \to U(E^c)$ is continuous.*

In particular, the continuous ultraproduct of (continuous) unitary representations of G is a continuous unitary representation.

For a slightly different, a priori stronger, approach, recall that a *seminorm* on G is a function $\rho: G \to \mathbf{R}^+$ satisfying $\rho(1) = 0$ and $\rho(g^{-1} f) \leq \rho(g) + \rho(f)$. It is a *norm* if $\rho(g) = 0$ implies $g = 1$. Let us write $\{\rho < \varepsilon\}$ for $\{g \in G : \rho(g) < \varepsilon\}$. A seminorm is continuous if and only if $\{\rho < \varepsilon\}$ is a neighborhood of 1 for all $\varepsilon > 0$. In particular, if $\rho' \leq \rho$ are seminorms and ρ is continuous, then so is ρ'.

Whenever a group G acts on a metric space X by isometry, every $x \in X$ gives rise to a seminorm $\rho_x(g) = d(x, gx)$. We encounter seminorms of this form regularly. For example, we can restate Fact 2.2 as the following.

Fact 3.3 *Let $\varphi \in L^1(G)$. Then $\rho_\varphi(g) = \|g\varphi - \varphi\|_1$ is a continuous seminorm.*

Similarly, if E is a unitary representation and $\xi \in E$, then $\xi \in E^{\mathrm{c}}$ if and only if $\rho_\xi(g) = \|g\xi - \xi\|$ is continuous.

Definition 3.4 Let $(E_i : i \in I)$ be continuous unitary representations of G. We define their *equicontinuous ultraproduct*, denoted $\prod^{\mathrm{ec}}{}_{\mathcal{U}} E_i$, to consist of all $\xi = [\xi_i] \in \prod^{\mathrm{B}}{}_{\mathcal{U}} E_i$ such that for some continuous seminorm ρ, we have $\rho_{\xi_i} \leq \rho$ for all (or equivalently, $\mathcal{U}$-many) i, as well as the limits of such:

$$\prod_{\mathcal{U}}^{\mathrm{ec}} E_i = \overline{\Big\{[\xi_i] \in \prod_{\mathcal{U}}^{\mathrm{B}} E_i : \rho_{\xi_i} \leq \rho \text{ for some common continuous seminorm } \rho\Big\}}.$$

It is clear that $\prod^{\mathrm{ec}}{}_{\mathcal{U}} E_i$ is a G-invariant Hilbert space, and if $\rho_{\xi_i} \leq \rho$ for some continuous ρ and all i, then $\rho_\xi \leq \rho$ as well. Therefore,

$$\prod_{\mathcal{U}}^{\mathrm{ec}} E_i \subseteq \prod_{\mathcal{U}}^{\mathrm{c}} E_i \subseteq \prod_{\mathcal{U}}^{\mathrm{B}} E_i.$$

Proposition 3.5 *Assume that G is locally compact. Let (E_i, π_i) be continuous representations for $i \in I$, and let $E = \prod^{\mathrm{B}}{}_{\mathcal{U}} E_i$. Each π_i extends to a $*$-morphism $\pi_i\colon M(G) \to B(E_i)$, giving rise to $\pi\colon M(G) \to \prod^{\mathrm{B}}{}_{\mathcal{U}} B(E_i) \subseteq B(E)$. In addition, since $\prod^{\mathrm{c}}{}_{\mathcal{U}} E_i = E^c$ is a continuous representation, it gives rise to a $*$-morphism $\sigma\colon M(G) \to B(E^c)$. Then the following are equivalent for $\xi \in E$:*

(i) *$\xi \in E^{\mathrm{c}}$ and $\pi(\varphi)\xi = \sigma(\varphi)\xi$ for all $\varphi \in L^1(G)$;*
(ii) *ξ belongs to the nondegenerate part of the $*$-representation $\pi\colon L^1(G) \to B(E)$ (i.e., to the nondegenerate ultraproduct in the sense of [4]);*
(iii) *$\xi \in \prod^{\mathrm{ec}}{}_{\mathcal{U}} E_i$.*

Proof (i) $\Rightarrow$ (ii). Let (e_α) be any approximate identity of norm 1 in $L^1(G)$. Since $\sigma\colon G \to U(E^{\mathrm{c}})$ is continuous, the $*$-representation $\sigma\colon L^1(G) \to B(E^{\mathrm{c}})$ is nondegenerate. Therefore $\pi(e_\alpha)\xi = \sigma(e_\alpha)\xi \to \xi$, so ξ is in the nondegenerate part of $\pi\colon L^1(G) \to B(E)$.

(ii) $\Rightarrow$ (iii). Assume that $\xi = \pi(\varphi)\zeta$ for $\varphi \in L^1(G)$ and $\zeta = [\zeta_i]$ in E. We may assume that $\|\zeta_i\| \leq \|\zeta\|$ for all i, and that $\xi = [\xi_i]$, where $\xi_i = \pi_i(\varphi)\zeta_i$. Then $\rho_{\xi_i} < \|\zeta_i\|\rho_\varphi \leq \|\zeta\|\rho_\varphi$, and $\xi \in \prod^{\mathrm{ec}}{}_{\mathcal{U}} E_i$. Since $\prod^{\mathrm{ec}}{}_{\mathcal{U}} E_i$ is complete, this is enough.

(iii) $\Rightarrow$ (i). Finally, let $\xi \in \prod^{\mathrm{ec}}{}_{\mathcal{U}} E_i \subseteq \prod^{\mathrm{c}}{}_{\mathcal{U}} E_i$, and let $\varphi \in L^1(G)$. We need to show that $\pi(\varphi)\xi = \sigma(\varphi)\xi$. The latter is a closed condition in both ξ and φ. We may therefore assume that $\xi = [\xi_i]$, ρ is a continuous seminorm, $\rho_{\xi_i} \leq \rho$ for all i (and therefore $\rho_\xi \leq \rho$), and φ has compact support. Let $\varepsilon > 0$. By a partition of unity argument, we can express φ as a finite sum $\sum_{m<n} \varphi_m$, where the φ_m are supported on disjoint sets, each contained in a single translate $g_m\{\rho < \varepsilon\}$. Then, by Lemma 2.1, we have

$$\begin{aligned}\Big\|\sigma(\varphi)\xi - \sum_{m<n} \|\varphi_m\| g_m\xi\Big\| &\leq \sum_{m<n} \big\|\sigma(\varphi_m)\xi - \|\varphi_m\| g_m\xi\big\| \\ &\leq \sum_{m<n} \varepsilon\|\varphi_m\| = \varepsilon\|\varphi\|.\end{aligned}$$

By the same argument, for all i we have

$$\Big\|\pi_i(\varphi)\xi_i - \sum_{m<n} \|\varphi_m\| g_m\xi_i\Big\| \leq \varepsilon\|\varphi\|,$$

and therefore, in the ultraproduct,

$$\Big\|\pi(\varphi)\xi - \sum_{m<n} \|\varphi_m\| g_m\xi\Big\| \leq \varepsilon\|\varphi\|.$$

It follows that $\|\sigma(\varphi)\xi - \pi(\varphi)\xi\| \leq 2\varepsilon\|\varphi\|$, and since ε was arbitrary, $\sigma(\varphi)\xi = \pi(\varphi)\xi$. $\square$

This means in particular that the nondegenerate ultraproduct of $*$-representations of $L^1(G)$ and the equicontinuous ultraproduct of representations of G are, in essence, the same construction.

Question 3.6 Find an example where $\prod^{\mathrm{ec}}{}_{\mathcal{U}} E_i \subsetneq \prod^{\mathrm{c}}{}_{\mathcal{U}} E_i$ (or show that this can never happen).

4 Property (T)

Suppose that G is a topological group, $Q \subseteq G$, and $\varepsilon > 0$. Let E be a unitary representation of G.

A vector $\xi \in E$ is (Q,ε)-*invariant* if $\sup_{g\in Q} \|g\xi - \xi\| < \varepsilon\|\xi\|$. Note in particular that a (Q,ε)-invariant vector must be nonzero. We say that $\xi \in E$ is G-*invariant* if $g\xi = \xi$ for all $g \in G$. We say that (Q,ε) is a *Kazhdan pair* for G if, whenever E is a unitary representation of G with a (Q,ε)-invariant vector, then E has a nonzero G-invariant vector. If there is $\varepsilon > 0$ such that (Q,ε) is a Kazhdan pair for G, then we say that Q is a *Kazhdan set* for G. Finally, G is said to have *property (T)* if it has a compact Kazhdan set.

Fact 4.1 *A locally compact group G with property (T) is compactly generated. Moreover, if $Q \subseteq G$ has nonempty interior, then it is generating if and only if it is a Kazhdan set.*

Proof See Bekka, de la Harpe, and Valette [1, Theorem 1.3.1, Proposition 1.3.2]. □

For $\varphi \in L^1(G)$ and $m \in \mathbf{N}$, define an open neighborhood of the identity by

$$U_{\varphi,m} = \left\{g \in G : \|\varphi - g\varphi\|_1 < 2^{-m}\right\}.$$

For $K \subseteq G$ compact, choose $K_{\varphi,m} \subseteq K$ finite so that $K \subseteq K_{\varphi,m}U_{\varphi,m}$.

Let us consider a continuous unitary representation of G as a model of $T^{L^1(G)}$, in the language $\mathcal{L}^{L^1(G)}$, as described in Section 1. Let $\varphi \in L^1(G)$, and recall that $S_\varphi \subseteq E$ is the closed image of the unit ball under π_φ. Define

$$\mathrm{Fix}_\varphi = \{\xi \in S_\varphi : g\xi = \xi \text{ for all } g \in G\}.$$

For $\xi \in S_\varphi$, define

$$\alpha_{K,\varphi,m}(\xi) = \max_{g\in K_{\varphi,m}} \|\xi - g\xi\|,$$
$$\Phi_{K,\varphi}(\xi) = \sup_m 2^{-m}\alpha_{K,\varphi,m}(\xi).$$

Since $K_{\varphi,m}$ is finite, $\alpha_{K,\varphi,m}$ is definable on S_φ by a formula. Consequently, $\Phi_{K,\varphi}$ is a definable predicate on S_φ, as a uniform limit of such.

If E is such a representation and $\xi \in \mathrm{Fix}_\varphi$, then $\Phi_{K,\varphi}(\xi) = 0$. Conversely, if $\xi \in S_\varphi$ and $\Phi_{K,\varphi}(\xi) = 0$, then $g\xi = \xi$ for all $g \in K$. In particular, if K is a compact generating set, then

$$\xi \in \mathrm{Fix}_\varphi \iff \Phi_{K,\varphi}(\xi) = 0.$$

Theorem 4.2 *Suppose that G is a compactly generated locally compact group. Then G has property (T) if and only if, for each $\varphi \in L^1(G)$, we have that* Fix_φ *is a definable subset of S_φ in the sense of the theory $T^{L^1(G)}$.*

Proof First, suppose that G has property (T). Let K be a compact generating set for G, and let $\varphi \in L^1(G)$ be arbitrary. Since the result is trivial for $\varphi = 0$, we assume that $\varphi \neq 0$. By Fact 4.1, K is a Kazhdan set for G.

Let $\varepsilon > 0$ be such that (K, ε) is a Kazhdan pair for G. We already know that Fix_φ is the zero-set of $\Phi_{K,\varphi}$, and we need to show that if $\Phi_{K,\varphi}(\xi)$ is small, then ξ is close to Fix_φ. Indeed, assume that $\Phi_{K,\varphi}(\xi) < 2^{-2m-2}$. Then $\alpha_{K,\varphi,m+1}(\xi) < 2^{-m-1}$. Let $f \in K$. Then $f = gh$, where $g \in K_{\varphi,m+1}$ and $h \in U_{\varphi,m+1}$. The former implies that $\|\xi - g\xi\| < 2^{-m-1}$, and the latter that $\|\xi - h\xi\| < 2^{-m-1}$. Since the action of g is isometric,

$$\|\xi - f\xi\| \leq \|\xi - g\xi\| + \|g\xi - gh\xi\| \leq \|\xi - g\xi\| + \|\xi - h\xi\| < 2^{-m}.$$

In other words, ξ is $(K, 2^{-m})$-invariant. By [1, Proposition 1.1.9], there exists a G-invariant vector $\zeta \in E$ such that $\|\xi - \zeta\| \leq \frac{\|\xi\|}{2^m \varepsilon}$ and $\|\zeta\| \leq \|\xi\| \leq \|\varphi\|_1$. The latter, together with the fact that ζ is fixed, implies that $\zeta \in S_\varphi$, so $\zeta \in \mathrm{Fix}_\varphi$. Thus indeed, if $\Phi_{K,\varphi}(\xi)$ is sufficiently small, then ξ is as close as desired to Fix_φ.

For the converse implication, let us fix a compact generating K for G, with nonempty interior. Set $\varphi = \frac{1_K}{H(K)} \in L^1(G)$. We shall prove that (K, ε) is a Kazhdan pair for G for $\varepsilon > 0$ small enough. By hypothesis, Fix_φ is definable in S_φ in all continuous unitary representations of G, viewed as models of $T^{L^1(G)}$. Since Fix_φ is the zero-set of $\Phi_{K,\varphi}$, there is $\delta > 0$ such that, if $\Phi_{K,\varphi}(\xi) < \delta$, then $d(\xi, \mathrm{Fix}_\varphi) < \frac{1}{2}$: this holds uniformly in every model of $T^{L^1(G)}$. We may now choose $\varepsilon > 0$ such that in any such representation, if $\xi \in S_\varphi$ is $(K, 3\varepsilon)$-invariant, then $\Phi_{K,\varphi}(\xi) < \delta$.

Fix a continuous unitary representation of G, and suppose that ξ is a (K, ε)-invariant unit vector. Let $\hat{\xi} = \pi(\varphi)\xi \in S_\varphi$. By Lemma 2.1, applied to $d\mu = \varphi dH$, we have $\|\hat{\xi} - \xi\| \leq \varepsilon$, so $\hat{\xi}$ is a $(K, 3\varepsilon)$-invariant vector. Since $\hat{\xi} \in S_\varphi$, it follows that $\Phi_{K,\varphi}(\hat{\xi}) < \delta$, whence $d(\hat{\xi}, \mathrm{Fix}_\varphi) < \frac{1}{2}$. If $\zeta \in \mathrm{Fix}_\varphi$ is such that $d(\hat{\xi}, \zeta) < \frac{1}{2}$, then $d(\xi, \zeta) < \varepsilon + \frac{1}{2}$. We may assume that $\varepsilon < \frac{1}{2}$, so $\zeta \neq 0$. We have thus found a nonzero G-invariant vector, which shows that (K, ε) is a Kazhdan pair for G. □

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