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Letter Editors' Suggestion

Exactly solvable lattice models for interacting electronic insulators in two dimensions

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In the past decade, tremendous efforts have been made towards understanding fermionic symmetry-protected topological (FSPT) phases in interacting systems. Nevertheless, for systems with continuum symmetry, e.g., electronic insulators, it is still unclear how to construct an exactly solvable model with a finite-dimensional Hilbert space in general. In this Letter, we give a lattice model construction and classification for two-dimensional (2D) interacting electronic insulators. Based on the physical picture of $\mathrm{U}(1)_f$ charge decorations, we illustrate the key idea by considering the well-known 2D interacting topological insulator. Then we generalize our construction to an arbitrary 2D interacting electronic insulator with symmetry $G_f = \mathrm{U}(1)_f \rtimes_{\rho_1,\omega_2} G$, where $\mathrm{U}(1)_f$ is the charge conservation symmetry and ρ_1, ω_2 are additional data which fully characterize the group structure of G_f. Finally, we study more examples, including the full interacting classification of 2D crystalline topological insulators.

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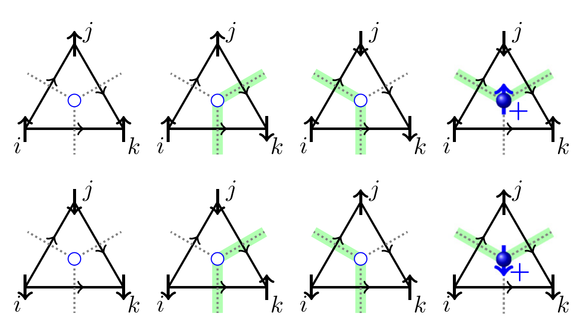
Introduction. In recent years, remarkable progress has been made in the theoretical understanding of gapped phases in quantum many-body systems, in particular for fermionic symmetry-protected topological (FSPT) phases [1–33], which include topological band insulators as the most familiar example [34,35]. Exactly solvable lattice Hamiltonians, whose ground states are fixed-point wave functions, have played a vital role in these developments, which often serve as proof-of-principle models for the existence of interacting topological phases and facilitate the extraction of universal physical properties to characterize the topological order. They can often be turned into exact tensor network states, offering a convenient starting point for the study of more realistic systems. However, known constructions of SPT phases typically feature local Hilbert space isomorphic to the protecting symmetry group, which becomes problematic if the symmetry is continuous. To date, no systematic exactly solvable constructions are available for generic electronic insulators, except for a couple of isolated examples. In this Letter, we generalize the decorated domain wall construction of interacting FSPT with finite total symmetry group G_f into interacting electronic insulators involving $\mathrm{U}(1)_f$ charge conservation symmetry. As a simple application, we will derive the full interacting classification of two-dimensional (2D) crystalline topological insulators [36,37]. Our method can also be applied to systems with other continuum symmetries such as SU(2) spin rotational symmetry.

2D interacting topological insulator from $\mathrm{U}(1)_f$ *charge decorations.* We begin with a concrete example of a 2D FSPT state protected by $G_f = [\mathrm{U}(1)_f \rtimes \mathbb{Z}_4^T]/\mathbb{Z}_2$. It is the well-known topological insulator with $\mathrm{U}(1)_f$ charge conservation and time-reversal symmetries. Let us consider a triangular lattice shown in Fig. 1. On each vertex i, there is a bosonic Ising spin $\sigma_i = \uparrow / \downarrow = \pm 1$. At the center of each triangle $\langle ijk \rangle$, there are spin-1/2 fermionic degrees of freedom c_{ijk}^{σ} ($\sigma = \uparrow / \downarrow$). While the bosonic spin σ_i does not carry $\mathrm{U}(1)_f$ charge, the $\mathrm{U}(1)_f$ charge of the fermion c_{ijk}^{σ} is chosen to be $+1$ (-1) if $\langle ijk \rangle$ is an up-pointing triangle $\vartriangle$ (a down-pointing triangle $\triangledown$). On the other hand, the time-reversal symmetry flips the bosonic spin σ_i between $\uparrow$ and $\downarrow$, and transforms the spin-1/2 fermion as $c_{ijk}^{\uparrow} \to c_{ijk}^{\downarrow}$ and $c_{ijk}^{\downarrow} \to -c_{ijk}^{\uparrow}$. In summary, the total Hilbert space of our model is the tensor product of $\mathrm{U}(1)_f$ chargeless spin-1/2 bosons on vertices and $\mathrm{U}(1)_f$ charge-± 1 spin-1/2 fermions at the center of triangles.

The fixed-point wave function is obtained by decorating fermionic $\mathrm{U}(1)_f$ charges to the symmetry domain walls of $\{\sigma_i\}$. A simpler bosonic U(1) charge decoration can be found in Refs. [38,39]. To be more specific, let us consider the domain wall configurations of a single triangle $\langle ijk \rangle$. There are in total $2^3 = 8$ different spin (black arrow) configurations or four domain wall (green line) configurations, for example, in an up-pointing triangle:



(1)

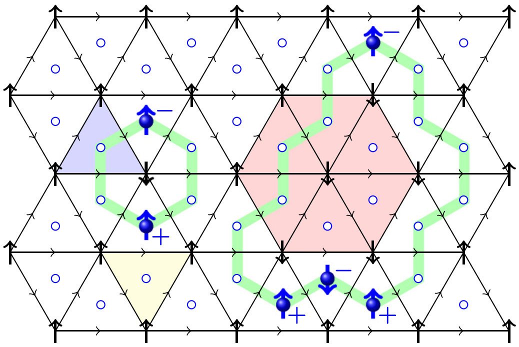
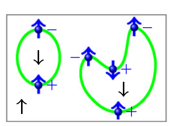
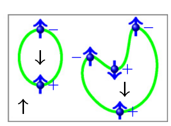


FIG. 1. Fermionic $\mathrm{U}(1)_f$ charge decoration. Fermions with $\mathrm{U}(1)_f$ charge $+1$ and -1 (blue dots) are decorated at the minimum and maximum points of the domain walls (green lines), respectively. The spin of the fermion (blue arrow) depends on the bosonic spin (black arrow) at the left vertex σ_i of the corresponding triangle. The terms $P_\triangle$, $P_\triangledown$, and A_s of the Hamiltonian are associated with triangles illustrated by blue, yellow, and red colors.

If the configuration satisfies $\sigma_i = -\sigma_j = \sigma_k$ (see the two rightmost figures above), a fermion $c_{ijk}^{\sigma_i}$ with spin σ_i and $\mathrm{U}(1)_f$ charge $+1$ (-1) will be decorated at the center when the triangle $\langle ijk\rangle$ is up pointing (down pointing). An explicit example of the decorations can be found in Fig. 1. The fixed-point wave function is a superposition of all possible bosonic spin configurations decorated with fermionic $\mathrm{U}(1)_f$ charges using the rules above:

$$|\Psi\rangle = \sum_{\text{all conf.}} \Psi \Bigg($$  $$\Bigg) \Bigg|$$  $$\Bigg\rangle. \tag{2}$$

By solving the consistency conditions (symmetry condition and twisted supercocycle equation), we will show later that the coefficient $\Psi(c)$ for each configuration c is always ± 1 depending on the order of the decorated fermions. The above $\mathrm{U}(1)_f$ charge decoration is compatible with the symmetry G_f. For each domain wall loop, the numbers of minimum and maximum points are the same. Therefore, the total $\mathrm{U}(1)_f$ charge of the decorated configuration is always zero. On the other hand, the time-reversal symmetry flips all the bosonic and fermionic spins in the configuration. By choosing the coefficient $\Psi(c)$ appropriately, one can make the ground state $|\Psi\rangle$ time-reversal invariant.

Commuting-projector Hamiltonian and edge state. As a fixed-point wave function, the $\mathrm{U}(1)_f$ charge decorated state Eq. (2) is the ground state of an exactly solvable commuting-projector Hamiltonian with finite-dimensional local Hilbert spaces (see Fig. 1):

$$H = -\sum_{\triangle} P_\triangle - \sum_{\triangledown} P_\triangledown - \sum_{\text{site } s} \frac{1 + A_s}{2} \prod_{\triangle} P_\triangle \prod_{\triangledown} P_\triangledown. \tag{3}$$

The triangle terms $P_\triangle$ and $P_\triangledown$ are projections enforcing the decoration rules such as Eq. (1) for each triangle. The operator A_s in the last term flips the bosonic spin at site s, and changes the fermionic $\mathrm{U}(1)_f$ charge decorations accordingly for the six surrounding triangles. We present more details of the Hamiltonian in the Supplemental Material [40]. In the literature, there are other constructions for the interacting topological insulator. Compared to the sophisticated method of decorating multiple Majorana chains [36,37], the state Eq. (2) we constructed is much simpler and can be systematically generalized to other symmetry group G_f, which we will describe later.

The state Eq. (2) is the interacting counterpart of the free-fermion topological insulator with charge conservation and time-reversal symmetries. They share the same nontrivial gapped, symmetry-breaking edge state and belong to the same phase. In fact, we can consider a position-dependent Zeeman field on the boundary, such that there are two edge spin domain walls, whose local profiles are related to each other via time-reversal symmetry. Due to the $\mathrm{U}(1)_f$ charge conservation of the domain wall loop, these two edge domain walls should have total $\mathrm{U}(1)_f$ charge ± 1. If the edge is particle-hole symmetric, each domain wall will have half $\mathrm{U}(1)_f$ charge (see Supplemental Material [40] for the formal derivation).

Symmetries of interacting electronic insulators. Before generalizing the above constructions to other systems, we first need to introduce some notations and definitions about the symmetry group G_f . For insulators, there is a $\mathrm{U}(1)_f$ charge conservation symmetry. The element of this group is $U_\theta = e^{i\theta Q}$, where Q is the $\mathrm{U}(1)_f$ charge operator. As the fermion parity operator is the order-2 element U_π in this group, we will denote the charge conservation symmetry by $\mathrm{U}(1)_f$ with a subscript f. The action of U_θ on a bosonic/fermionic annihilation operator with $\mathrm{U}(1)_f$ charge q is $U_\theta c_j^{\sigma,q} U_\theta^\dagger = e^{-iq\theta} c_j^{\sigma,q}$, where j is the lattice site and σ is the combination of other indices such as orbital and spin, etc. As $\mathrm{U}(1)_f$ charge symmetry is always a normal subgroup of the total symmetry G_f for electronic insulators, we have the following short exact sequence,

$$1 \to \mathrm{U}(1)_f \to G_f \to G \to 1, \tag{4}$$

where $G := G_f/\mathrm{U}(1)_f$ is the quotient group. In this Letter, we assume that G is a finite group.

Conversely, given $\mathrm{U}(1)_f$ and G, we can recover the group $G_f = \mathrm{U}(1)_f \rtimes_{\rho_1,\omega_2} G$ by using two ingredients ρ_1 and ω_2. The 1-cocycle $\rho_1 \in H^1(G, \mathbb{Z}_2)$ is a homomorphism from G to $\mathrm{Aut}[\mathrm{U}(1)_f] = \mathbb{Z}_2$. It implements the charge conjugation action of G on $U_\theta = e^{i\theta Q} \in \mathrm{U}(1)_f$ as

$$g \times U_\theta \times g^{-1} = (U_\theta)^{(-1)^{\rho_1(g)}} = U_{(-1)^{\rho_1(g)}\theta}. \tag{5}$$

The second ingredient ω_2 is related to the extension of G. As a set, G_f is the same as $\mathrm{U}(1)_f \times G$, so the elements of G_f can be parametrized as (U_θ, g). But the multiplication in G_f reads

$$(1, g) \times (1, h) = (U_{2\pi\omega_2(g,h)}, gh) \in G_f, \tag{6}$$

where $\omega_2(g, h) \in \mathbb{R}/\mathbb{Z} \simeq \mathrm{U}(1)_f$ is a phase associated with $g, h \in G$. The associativity condition of G_f implies that ω_2 is a 2-cocycle in $H^2_{\rho_1}[G, \mathrm{U}(1)_f]$ [41], where the subscript ρ_1 indicates the G action on the coefficient $\mathrm{U}(1)_f$.

The two cocycles ρ_1 and ω_2 fully characterize the group structure of $G_f = \mathrm{U}(1)_f \rtimes_{\rho_1,\omega_2} G$, but the action of the group G or G_f on the wave functions is still not full determined yet. When there is an antiunitary symmetry in G, we should also introduce a third ingredient s_1 to specify its action on the wave functions with $i \to -i$:

$$s_1(g) = \begin{cases} 0, & \text{if } g \text{ is unitary,} \\ 1, & \text{if } g \text{ is antiunitary.} \end{cases} \tag{7}$$

Apparently, s_1 is also a 1-cocycle in $H^1(G, \mathbb{Z}_2)$.

In general, the 1-cocycles s_1 and ρ_1 are not the same. Combining Eqs. (5) and (7), the G action on the charge operator Q in $U_\theta = e^{i\theta Q} \in \mathrm{U}(1)_f$ should be

$$g \times Q \times g^{-1} = (-1)^{\rho_1(g)+s_1(g)} Q. \tag{8}$$

So the $\mathrm{U}(1)_f$ charges change sign under the g action if and only if $\rho_1(g)$ and $s_1(g)$ are different.

Generalization to symmetry $G_f = \mathrm{U}(1)_f \rtimes_{\rho_1,\omega_2} G$. Now we want to generalize the construction of $\mathrm{U}(1)_f$ charge decoration to arbitrary 2D interacting electronic insulators protected by $G_f = \mathrm{U}(1)_f \rtimes_{\rho_1,\omega_2} G$. The degrees of freedom (DOF) of our lattice model are as follows. We first triangulate the 2D spatial manifold with a branching structure. On each vertex i, we put a $|G|$-level spin Hilbert space spanned by $|g_i\rangle$ ($g_i \in G$). At the center of each triangle $\langle ijk\rangle$, we put a Hilbert space spanned by bosons/fermions $c_{ijk}^{\sigma,q}$ ($\sigma \in G$, $q \in \mathbb{Z}$, $|q| \leqslant \Lambda$). Here, q is the $\mathrm{U}(1)_f$ charge of the boson/fermion, and Λ is a finite positive integer depending on G [42]. We choose the DOF $c_{ijk}^{\sigma,q}$ to be a fermion (boson) if q is odd (even) [43]. So the (anti)commutation relation reads

$$c_{ijk}^{\sigma,q}\big(c_{i'j'k'}^{\sigma',q'}\big)^\dagger - (-1)^{qq'}\big(c_{i'j'k'}^{\sigma',q'}\big)^\dagger c_{ijk}^{\sigma,q} = \delta_{ijk,i'j'k'}\delta_{\sigma\sigma'}\delta_{qq'}. \tag{9}$$

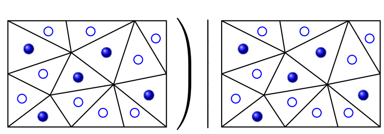
Under the symmetries $U_\theta \in \mathrm{U}(1)_f$ and $g \in G$, these DOF transform as

$$\begin{aligned} U_\theta|g_i\rangle &= |g_i\rangle, \quad U(g)|g_i\rangle = |gg_i\rangle, \\ U_\theta c_{ijk}^{\sigma,q} U_\theta^\dagger &= e^{-iq\theta} c_{ijk}^{\sigma,q}, \\ U(g) c_{ijk}^{\sigma,q} U(g)^\dagger &= e^{-2\pi i \omega_2(g,\sigma)(-1)^{\rho_1(g)+s_1(g)}q} c_{ijk}^{g\sigma,(-1)^{\rho_1(g)+s_1(g)}q}. \end{aligned} \tag{10}$$

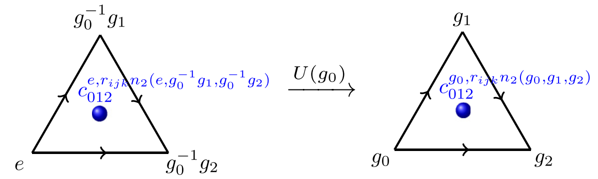
In this way, both the bosonic and fermionic DOF support linear representations of the total symmetry group G_f (see Supplemental Material [40] for a proof).

To obtain a 2D G_f-FSPT state, we can decorate $\mathrm{U}(1)_f$ charges to the domain wall junctions of G. After proliferating G domain walls, we will obtain a symmetric gapped FSPT state protected by symmetry G_f. Schematically, the wave function would have the form

$$|\Psi\rangle = \sum_{\text{all conf.}} \Psi \Bigg(\cdots \Bigg) \Bigg| \cdots \Bigg\rangle,$$



where the blue dots are the decorated $\mathrm{U}(1)_f$ charges similar to Eq. (2). Now we try to decorate the $\mathrm{U}(1)_f$ charges $c_{ijk}^{\sigma,q}$ to the domain wall junctions (triangle centers) of G. The decoration is specified by an integral charge function $n_2(g_i, g_j, g_k) \in \mathbb{Z}$. For a triangle $\langle ijk\rangle$ with orientation $r_{ijk} = \pm 1$ and vertex spin labels $e, g_0^{-1}g_1, g_0^{-1}g_2 \in G$, we decorate the $\mathrm{U}(1)_f$ charge $c_{ijk}^{e,r_{ijk}n_2(e,g_0^{-1}g_1,g_0^{-1}g_2)}$ at the center. All other charges $c_{ijk}^{\sigma,q}$ of this triangle with $\sigma \neq e$ or $q \neq r_{ijk}n_2(e, g_0^{-1}g_1, g_0^{-1}g_2)$ remain empty or in the vacuum state. From this standard triangle decoration, we can obtain the decoration for arbitrary triangle under the action of $U(g_0)$:



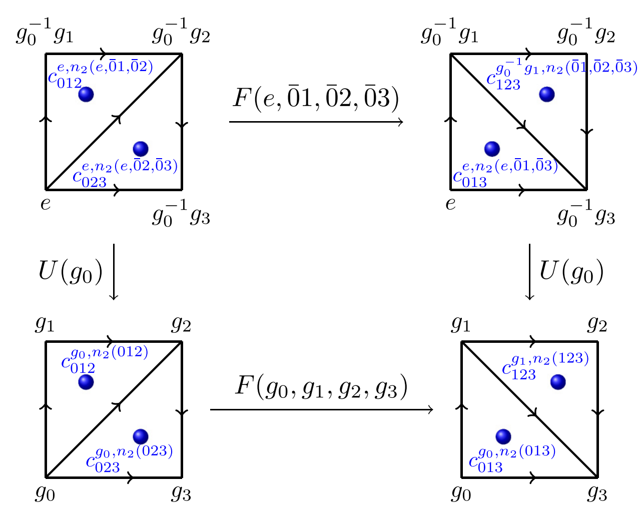
(11)

To be consistent with the symmetry transformation Eq. (10), the function n_2 should satisfy

$$n_2(g_0, g_1, g_2) = (-1)^{\rho_1(g_0)+s_1(g_0)} n_2\big(e, g_0^{-1}g_1, g_0^{-1}g_2\big). \tag{12}$$

So n_2 is a 2-cochain in $C^2_{\rho_1+s_1}(G, \mathbb{Z})$ with a G action on the integral charges indicated by the subscript $\rho_1 + s_1$. This nontrivial action can be traced back to Eq. (8).

$\mathrm{U}(1)_f$*-symmetric fermionic* F *moves.* To make the wave function Eq. (11) well defined, we have to check several consistency conditions. The easiest way is to consider wave functions on different triangulations of the spatial manifold. They are related to each other by elementary local changes called Pachner moves (F moves). Since we want the wave function to be G_f symmetric, the F moves should respect the symmetry. So we have the following commuting square:



(13)

Given the standard F move with the first vertex labeled by $e \in G$, we can use the above commuting diagram to derive the nonstandard one with generic $g_0 \in G$. They have the following explicit expressions,

$$\begin{aligned} F(e,\bar{0}1,\bar{0}2,\bar{0}3) := \nu_3(e,\bar{0}1,\bar{0}2,\bar{0}3)\big(c_{012}^{e,n_2(e,\bar{0}1,\bar{0}2)}\big)^\dagger \\ \times \big(c_{023}^{e,n_2(e,\bar{0}2,\bar{0}3)}\big)^\dagger c_{013}^{e,n_2(e,\bar{0}1,\bar{0}3)} c_{123}^{g_0^{-1}g_1,n_2(\bar{0}1,\bar{0}2,\bar{0}3)}, \end{aligned} \tag{14}$$

$$
\begin{aligned}
F(g_0,g_1,g_2,g_3) &= U(g_0)F(e,g_0^{-1}g_1,g_0^{-1}g_2,g_0^{-1}g_3)U(g_0)^{-1}\\
&:= \nu_3(g_0,g_1,g_2,g_3)\big(c_{012}^{g_0,n_2(012)}\big)^\dagger\\
&\quad\times\big(c_{023}^{g_0,n_2(023)}\big)^\dagger c_{013}^{g_0,n_2(013)}c_{123}^{g_1,n_2(123)}, \quad (15)
\end{aligned}
$$

where we use abbreviations $\bar{i}j$ for $g_i^{-1}g_j$ and $n_2(ijk)$ for $n_2(g_i,g_j,g_k)$. We also set $r_{ijk}=1$ for all the triangles shown above. From the $U(g_0)$ action on the complex numbers and bosonic/fermionic $\mathrm{U}(1)_f$ charges in Eq. (10), the F move coefficient $\nu_3\in C^3_{s_1}[G,\mathrm{U}(1)]$ has the symmetry condition

$$
\begin{aligned}
\nu_3(g_0,g_1,g_2,g_3) &= \big[\nu_3\big(e,g_0^{-1}g_1,g_0^{-1}g_2,g_0^{-1}g_3\big)\big]^{1-2s_1(g_0)}\\
&\quad\times e^{-2\pi i\omega_2(g_0,g_0^{-1}g_1)n_2(g_1,g_2,g_3)}. \quad (16)
\end{aligned}
$$

Here, we use the normalization condition $\omega_2(g_0,e)=0$.

Besides the G symmetry, the F should also preserve the $\mathrm{U}(1)_f$ charges. By counting the $\mathrm{U}(1)_f$ charges on the two sides of the F move Eq. (14), we have the integer equation,

$$
\begin{aligned}
&(d_{\rho_1+s_1}n_2)(g_1,g_2,g_3)\\
&= (-1)^{\rho_1(g_1)+s_1(g_1)}n_2(g_2,g_3)-n_2(g_1g_2,g_3)\\
&\quad+n_2(g_1,g_2g_3)-n_2(g_1,g_2)=0, \quad (17)
\end{aligned}
$$

where we define the inhomogeneous cochain $n_2(g_1,g_2):=n_2(e,g_1,g_1g_2)$ to be the homogeneous one with the first argument being $e\in G$. One can also show that adding coboundaries to n_2 can be gauged away by symmetric local unitaries. Therefore, n_2 is in fact a 2-cocycle in $H^2_{\rho_1+s_1}(G,\mathbb{Z})$. Here, we use the subscript ρ_1+s_1 to indicate the possibly nontrivial G action on the $\mathrm{U}(1)_f$ charge appearing in the first term of the second line of Eq. (17). This action originates from Eqs. (8) and (12).

Twisted supercocycle equation. Given two triangulations of the spatial manifold, there are possibly many different sequences of F moves connecting them. Since the initial and the final states are fixed, we should have the same result from different sequences. The smallest loop among these sequences is the twisted version of the supercocycle equation [4].

Let us choose the label of the first vertex to be $e\in G$. In this way, the standard supercocycle equation reads

$$
\begin{aligned}
&F(e,\bar{0}1,\bar{0}2,\bar{0}3)\cdot F(e,\bar{0}1,\bar{0}3,\bar{0}4)\cdot F(\bar{0}1,\bar{0}2,\bar{0}3,\bar{0}4)\\
&\quad= F(e,\bar{0}2,\bar{0}3,\bar{0}4)\cdot F(e,\bar{0}1,\bar{0}2,\bar{0}4). \quad (18)
\end{aligned}
$$

The nonstandard ones are automatically satisfied by simply a symmetry action $U(g_0)$. Using the symmetry condition $F(\bar{0}1,\bar{0}2,\bar{0}3,\bar{0}4)=U(\bar{0}1)F(e,\bar{1}2,\bar{1}3,\bar{1}4)U(\bar{0}1)^\dagger$ from Eq. (15), we can convert the above equation to a formula that only involves the standard F moves Eq. (14). After eliminating all the $c_{ijk}^{\sigma,q}$ operators, the final result is a twisted cocycle equation for the ν_3 as

$$
d_{s_1}\nu_3 = e^{2\pi i\left(\omega_2\smile n_2+\frac{1}{2}n_2\smile n_2\right)}. \quad (19)
$$

Here, the differential d_{s_1} of the inhomogeneous cochain $\nu_3(g_1,g_2,g_3):=\nu_3(e,g_1,g_1g_2,g_1g_2g_3)$ is defined as

$$
\begin{aligned}
&(d_{s_1}\nu_3)(g_1,g_2,g_3,g_4)\\
&\quad= \frac{\nu_3(g_2,g_3,g_4)^{1-2s_1(g_1)}\nu_3(g_1,g_2g_3,g_4)\nu_3(g_1,g_2,g_3)}{\nu_3(g_1g_2,g_3,g_4)\nu_3(g_1,g_2,g_3g_4)}, \quad (20)
\end{aligned}
$$

and the first cup product on the right-hand side of Eq. (19) reads

$$
\begin{aligned}
&(\omega_2\smile n_2)(g_1,g_2,g_3,g_4)\\
&\quad= \omega_2(g_1,g_2)(-1)^{\rho_1(g_1g_2)+s_1(g_1g_2)}n_2(g_3,g_4). \quad (21)
\end{aligned}
$$

It has a simpler expression $(\omega_2\smile n_2)(e,g_1,g_2,g_3,g_4)=\omega_2(e,g_1,g_2)n_2(g_2,g_3,g_4)$ in the homogeneous notation, where the G-action sign $(-1)^{\rho_1+s_1}$ is absorbed in $n_2(g_2,g_3,g_4)$. The second cup product $(-1)^{n_2\smile n_2}$ has a similar expression with sign $(-1)^{\rho_1+s_1}$ and comes from the reordering of the $c_{ijk}^{\sigma,q}$ operators when they are fermions.

Using the solutions (n_2,ν_3) of the obstruction Eqs. (17) and (19), we can construct a G_f-symmetric wave function Eq. (11) by decorating $\mathrm{U}(1)_f$ charges. It can be shown that the decoration data (n_2,ν_3) of the same cohomology class would give us equivalent wave functions related by fermionic symmetric local unitary transformations. Moreover, as discussed in the Supplemental Material [40], ν_3 and $\nu_3e^{2\pi i\omega_2\smile n_1}$ with $n_1\in H^1_{\rho_1+s_1}(G,\mathbb{Z})$ are also equivalent. Therefore, the final classification data of interacting electronic insulators are n_2 and ν_3, which are elements in $H^2_{\rho_1+s_1}(G,\mathbb{Z})$ and $C^3_{s_1}[G,\mathrm{U}(1)]/B^3_{s_1}[G,\mathrm{U}(1)]/\Gamma^3$, where Γ^3 is the trivialization subgroup due to the 1D anomalous SPT states [44].

More examples. Let us consider some simple examples of G_f-FSPT with charge conservation symmetry.

(1) $G_f=\mathrm{U}(1)_f\times\mathbb{Z}_2$. In this case, we have $G=\mathbb{Z}_2$ and $\rho_1=s_1=\omega_2=0$. It can be shown easily that the nontrivial fermion decoration $n_2\in H^2(\mathbb{Z}_2,\mathbb{Z})$ is obstruction free. After gauging G and considering only the $\mathbb{Z}_2^f$ subgroup of $\mathrm{U}(1)_f$, the state is identical to the fermionic toric code [45]. With a nontrivial bosonic SPT (BSPT) protected by G only, the full classification of G_f-FSPT is $\mathbb{Z}_4$. In fact, the root state of this $\mathbb{Z}_4$ is the $\nu=2$ state of the $\mathbb{Z}_8$ classification of $G_f=\mathbb{Z}_2^f\times\mathbb{Z}_2$ FSPT [5].

(2) $G_f=\mathrm{U}(1)_f\rtimes\mathbb{Z}_2^T$. Now $\rho_1=s_1$ is nontrivial and ω_2 is trivial. One can show that the $\mathrm{U}(1)_f$ charge decoration n_2 is obstructed. There is also no BSPT state. So there is only a trivial G_f-FSPT state.

(3) By applying the fermionic crystalline equivalence [46–51] where a mirror reflection symmetry action should be mapped onto a time-reversal symmetry action, and that spinless (spin-1/2) fermionic systems should be mapped into spin-1/2 (spinless) fermionic systems, we can also derive the complete interacting classification of 2D crystalline topological insulators. In Supplemental Material [40], we list the classification results for all 17 wallpaper groups.

Discussion and conclusion. In this Letter, we construct and classify interacting electronic insulators in two spatial dimensions with arbitrary symmetry group $G_f=\mathrm{U}(1)_f\rtimes_{\rho_1,\omega_2}G$. The construction is obtained by decorating $\mathrm{U}(1)_f$ charges to the G symmetry domain wall junctions. This decoration is specified by a 2-cocycle $n_2\in H^2_{\rho_1+s_1}(G,\mathbb{Z})$. The second piece of classification data $\nu_3\in C^3_{s_1}[G,\mathrm{U}(1)]/B^3_{s_1}[G,\mathrm{U}(1)]/\Gamma^3$ is the wave-function coefficient satisfying the supercocycle equation (19). As an explicit example, we construct the fixed-point wave function and commuting-projector Hamiltonian of a topological insulator with charge conservation and time-reversal symmetries. By applying the crystalline equivalence principle, we also derive the complete interacting classification of 2D crystalline topological insulators. Apparently, our classification data can also classify interacting electronic insulators with both internal and space group symmetry.

Finally, we stress that our constructions and classification scheme can be easily generalized to other continuous groups by decorating the corresponding continuous-symmetry-protected states to discrete-symmetry domain walls. It can be also generalized from two dimensions to higher dimensions, though the corresponding obstruction functions could become more complicated.

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[41] Since cohomologous ω_2's will give isomorphic G_f, we also have to mod out the 2-coboundaries.
[42] Λ is the biggest number of $|n_2(g_i, g_j, g_k)|$ for all possible $g_i, g_j, g_k \in G$ and 2-cocycles $n_2 \in H^2_{\rho_1+s_1}(G, \mathbb{Z})$.
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