

A FAMILY OF IMMERSED FINITE ELEMENT SPACES AND APPLICATIONS TO THREE-DIMENSIONAL $\mathbf{H}(\text{curl})$ INTERFACE PROBLEMS*

LONG CHEN[†], RUCHI GUO[‡], AND JUN ZOU[§]

Abstract. Efficient and accurate computation of $\mathbf{H}(\text{curl})$ interface problems is of great importance in many electromagnetic applications. Unfitted mesh methods are especially attractive in three-dimensional (3D) computation as they can circumvent generating complex 3D interface-fitted meshes. However, many unfitted mesh methods rely on nonconforming approximation spaces, which may cause a loss of accuracy for solving Maxwell-type equations, and the widely used penalty techniques in the literature may not help in recovering the optimal convergence. In this article, we provide a remedy by developing Nédélec-type immersed finite element (IFE) spaces with a Petrov–Galerkin scheme that is able to produce optimal-convergent solutions. To establish a systematic framework, we analyze all the H^1 , $\mathbf{H}(\text{curl})$, and $\mathbf{H}(\text{div})$ IFE spaces and form a discrete de Rham complex. Based on these fundamental results, we further develop a fast solver using a modified Hiptmair–Xu preconditioner which works for both the generalized minimal residual (GMRES) and conjugate gradient (CG) methods for solving the nonsymmetric linear algebraic system. The approximation capabilities of the proposed IFE spaces will be also established.

Key words. interface problems, Maxwell equations, immersed finite element methods, Petrov–Galerkin formulation, de Rham complex, Nédélec elements, Raviart–Thomas elements, preconditioner, $\mathbf{H}(\text{curl}; \Omega)$ -elliptic equations

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See reproducibility of computational results at end of the article.

1. Introduction. Efficient and accurate computation of electromagnetic interface problems is of great importance since many applications involve multiple media with different electromagnetic properties, such as electromagnetic motors and actuators for electric drive vehicles [12, 35], and some nondestructive detection techniques based on electromagnetic fields [1, 24].

Let $\mathbf{u}$ be an electric or magnetic field in a domain Ω in $\mathbb{R}^3$ that covers multiple subdomains representing media with different electromagnetic properties. For simplicity, we assume that there are only two subdomains of Ω denoted as $\Omega^\pm$ separated by an interface Γ . Then, the mathematical model for the interface conditions of $\mathbf{u}$ at Γ can be generally written as

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[†]Department of Mathematics, University of California at Irvine, Irvine, CA 92697 USA (chenlong@uci.edu).

[‡]Department of Mathematics, University of California at Irvine, Irvine, CA 92697 USA, The Chinese University of Hong Kong, Hong Kong (ruchig@uci.edu).

[§]Department of Mathematics, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong (zou@math.cuhk.edu.hk).

$$(1.1a) \quad [\mathbf{u} \times \mathbf{n}]_{\Gamma} := \mathbf{u}^+ \times \mathbf{n} - \mathbf{u}^- \times \mathbf{n} = \mathbf{0},$$

$$(1.1b) \quad [(\alpha \operatorname{curl} \mathbf{u}) \times \mathbf{n}]_{\Gamma} := \alpha^+ \operatorname{curl} \mathbf{u}^+ \times \mathbf{n} - \alpha^- \operatorname{curl} \mathbf{u}^- \times \mathbf{n} = \mathbf{0},$$

$$(1.1c) \quad [\beta \mathbf{u} \cdot \mathbf{n}]_{\Gamma} := \beta^+ \mathbf{u}^+ \cdot \mathbf{n} - \beta^- \mathbf{u}^- \cdot \mathbf{n} = 0,$$

where $\mathbf{n}$ denotes the normal vector to Γ from Ω^- to Ω^+ , and $\alpha = \alpha^{\pm}$ and $\beta = \beta^{\pm}$ in $\Omega^{\pm}$ are assumed to be positive piecewise constant functions with different electromagnetic meanings in various situations. In this work, we shall develop immersed finite element (IFE) spaces to capture those conditions.

The interface conditions in (1.1) appear in many formulations of Maxwell equations. For example, the following time-dependent curl-curl equation can be used to model an electric field $\mathbf{u} = \mathbf{E}$:

$$(1.2) \quad \epsilon \mathbf{u}_{tt} + \sigma \mathbf{u}_t + \operatorname{curl}(\mu^{-1} \operatorname{curl} \mathbf{u}) = \mathbf{J}_t$$

subject to some boundary and initial conditions, where ϵ , σ , and μ are the electric permeability, conductivity, and magnetic permeability of the media, respectively, and $\mathbf{J}$ is the applied current density. Then, solving (1.2) at each time step by an implicit method yields the $\mathbf{H}(\operatorname{curl})$ interface system

$$(1.3) \quad \operatorname{curl}(\alpha \operatorname{curl} \mathbf{u}) + \beta \mathbf{u} = \mathbf{f}.$$

In this model, the jump conditions in (1.1) admit $\alpha = \mu^{-1}$ and $\beta = \epsilon \Delta t^{-2} + \sigma \Delta t^{-1}$. We refer readers to [17, 27, 30, 47] for more details of this formulation.

It is widely known that traditional finite element (FE) methods can be applied on geometrically fitted meshes to solve interface problems, but generating a high-quality fitted mesh is nontrivial, especially for the considered three-dimensional (3D) space. For complex geometry, a global adjustment is needed to generate high-quality meshes, which can be computationally expensive. Locally modifying the mesh near the interface is more efficient [26, 28] but cannot guarantee shape regularity or even the maximal angle condition [14]. For arbitrarily shaped interface surfaces, the so-called slivers having edges nearly coplanar are difficult to completely remove from tetrahedral meshes [36], which may deteriorate the accuracy. For all these approaches, small edges of anisotropic meshes may make the conditioning of the resulting linear algebraic system even worse and pose a severer restriction on the time step size for time-dependent problems.

In contrast, some special numerical methods are designed for solving interface problems on unfitted meshes, which completely avoids the mesh generation issue. Typical examples include penalty-type methods [15, 16, 61], nonmatching mesh methods [21, 22, 27, 54], and IFE methods to be discussed in this article. The methodology of IFE methods is to encode the jump conditions in the construction of approximation spaces, i.e., the IFE spaces, which then have optimal approximation capabilities for functions satisfying the related jump conditions. We also refer readers to FDTD methods [77] based on a finite difference formulation for Maxwell interface equations. All these methods have been successfully applied to various interface problems arising from fluid, elasticity, wave propagation, and so on; see [6, 37, 39, 42, 45] and the references therein. However, due to the reason explained below, currently, there seems no satisfactory methodology for the Maxwell-type interface problems.

For almost all the aforementioned unfitted-mesh FE methods, one trade-off is the loss of the conformity of the approximation spaces. The proposed IFE space is also $\mathbf{H}(\operatorname{curl})$ -nonconforming, i.e., not tangential continuous across interface faces;

see Figure 3.1 for an illustration. For many situations, the discontinuity can be well handled by suitable penalties such as Nitsche's penalty [15, 60]. The essential idea is to treat those spaces through a discontinuous Galerkin (DG) formulation. But it becomes one of the main obstacles for Maxwell-type equations due to the underlying $\mathbf{H}(\text{curl})$ space. Here let us briefly explain the issue. Solutions of (1.3) generally admit low regularity, especially near the interface [31, 32, 33]. Suppose that $\mathbf{u} \in \mathbf{H}^s(\text{curl}; \Omega)$, $s > 0$, and an approximation space is nonconforming on a face F ; then the penalty $h^{-1} \int_F [\mathbf{u}_h \times \mathbf{n}] \cdot [\mathbf{v}_h \times \mathbf{n}] ds$ is generally needed to ensure the stabilization. As a result, in the error estimation, one needs the inequality

$$(1.4) \quad h_K^{-1/2} \|\mathbf{u} - \pi_F \mathbf{u}\|_{L^2(F)} \lesssim h_K^{s-1} \|\mathbf{u}\|_{\mathbf{H}^s(\text{curl}; K)},$$

where π_F is a projection to some polynomial space on F . The inequality in (1.4) will not cause any issue if the solution regularity is high enough; see the recent work [61]. In [61], the authors developed a framework of penalty-type methods for all the H^1 , $\mathbf{H}(\text{curl})$, and $\mathbf{H}(\text{div})$ interface problems that can uniformly handle arbitrarily high order spaces. The penalty idea was also employed for unfitted mesh methods based on a patch-reconstruction technique in [59]. However, note that even for a moderate regularity $s = 1$, (1.4) immediately leads to a failure of convergence. This issue has been studied in a series of articles in the literature [9, 21, 22, 69], and it seems to be essential rather than caused by the limitation of analysis techniques as it can be verified numerically in the aforementioned works.

Some approaches in the literature can overcome this issue. One is to search for a conforming subspace of the underlying nonconforming space, including standard DG methods [51, 52] and the mortar FE method [54] relying on a “nested mesh” assumption. As the unfitted mesh methods modify their spaces near the interface, the conforming subspaces may not exist. Without resorting to conformity, some approaches can use the appropriate scaling in the stabilization [7, 13, 19] to overcome this issue. But for many unfitted mesh methods, such a scaling factor may not yield a stable scheme.

Therefore, it would be highly desirable if the penalty could be completely avoided. With this motivation, here we employ a Petrov–Galerkin (PG) formulation that uses IFE trial spaces but keeps the standard conforming FE test spaces, which addresses the nonconformity issue by avoiding estimate (1.4). For the resulting matrix being square, the trial and test spaces should have the same number of degrees of freedom (DoFs). Thus, we shall design the IFE spaces to be isomorphic to the standard FE spaces through the usual DoFs, which is shown by the mapping between the middle and bottom sequences in Figure 1.1; see the definition of the spaces and operators in section 3.3. The original idea of PG formulation can be found in the fundamental work [3] of Babuška, Caloz, and Osborn and is then used in [50] for the multiscale FE method and in [49] for IFE methods on H^1 interface problems.

There are multiple novelties in this work. We construct both the 3D $\mathbf{H}(\text{curl})$ and $\mathbf{H}(\text{div})$ IFE spaces according to their corresponding jump conditions and show that they have the same edge and face DoFs as the standard Nédélec and Raviart–Thomas elements, respectively. Although the local IFE spaces have been introduced in [20], we establish a few fundamental properties including unisolvence, approximation capabilities, and the de Rham complex, as shown by the middle sequence in Figure 1.1 that is identical to the FE counterparts. The resulting PG-IFE method has the optimal convergence, and its condition number is independent of small-cutting elements.

There have been extensive works on developing efficient preconditioners for unfitted mesh methods for various interface problems; see [38, 34, 62, 73]. However, to

$$\begin{array}{ccccccccc}
\mathbb{R} & \hookrightarrow & H^2(\beta; \Omega) & \xrightarrow{\text{grad}} & \mathbf{H}^1(\text{curl}, \alpha, \beta; \Omega) & \xrightarrow{\text{curl}} & \mathbf{H}^1(\text{div}, \alpha; \Omega) & \xrightarrow{\text{div}} & H^1(\Omega) \rightarrow 0 \\
& & \downarrow I_h^n & & \downarrow I_h^e & & \downarrow I_h^f & & \downarrow \Pi^0 \\
\mathbb{R} & \hookrightarrow & S_h^n & \xrightarrow{\text{grad}} & \mathbf{S}_h^e & \xrightarrow{\text{curl}} & \mathbf{S}_h^f & \xrightarrow{\text{div}} & Q_h \rightarrow 0 \\
& & \uparrow I_h^n \downarrow (I_h^n)^{-1} & & \uparrow I_h^e \downarrow (I_h^e)^{-1} & & \uparrow I_h^f \downarrow (I_h^f)^{-1} & & \uparrow I \\
\mathbb{R} & \hookrightarrow & \tilde{S}_h^n & \xrightarrow{\text{grad}} & \tilde{\mathbf{S}}_h^e & \xrightarrow{\text{curl}} & \tilde{\mathbf{S}}_h^f & \xrightarrow{\text{div}} & Q_h \rightarrow 0
\end{array}$$

FIG. 1.1. Diagrams of IFE spaces. $S_h^n, \mathbf{S}_h^e$, and $\mathbf{S}_h^f$ are IFE spaces. $\tilde{S}_h^n, \tilde{\mathbf{S}}_h^e$, and $\tilde{\mathbf{S}}_h^f$ are standard FE spaces. They are isomorphisms through the canonical interpolation operators defined by DoFs.

the best of our knowledge, there seems to be no study on preconditioners for those methods on the $\mathbf{H}(\text{curl})$ system. In fact, it is well known that solving the linear algebraic system from the $\mathbf{H}(\text{curl})$ system is challenging, as the kernel space of the curl operator is quite large. An exact sequence is demanded to describe the kernel space precisely. In this paper, based on the de Rham complex, we develop an IFE version of the HX preconditioner [48]. We design a special smoother by computing the exact inverse of the non-SPD submatrix near the interface. The idea is closely related to domain decomposition methods [76]. We demonstrate by numerical experiments that the resulting fast solver works well for generalized minimal residual (GMRES) methods and even conjugate gradient (CG) methods which typically do not converge for non-SPD systems. It is worthwhile to mention that CG-like methods have been developed for more general semisymmetric matrices [63]. We also refer readers to multigrid methods for H^1 -interface problems in [75, 53].

The article contains six additional sections. In the next section, we describe the geometry and set up some notation. In section 3, we establish the IFE spaces and their de Rham complex. In section 4, we describe the PG-IFE formulation, the fast solver, and a numerical test for the *inf-sup* condition. In section 5, we show the optimal approximation capabilities. In section 6, we present some numerical experiments. In section 7, we offer some concluding remarks.

2. Preliminary. In this section, we describe an unfitted background mesh and introduce some spaces and notation which will be frequently used in this paper.

2.1. Meshes. An electromagnetic wave generally propagates over the whole space, and a widely used approach is to cover the modeling media by a box as a modeling and computational domain. So in this paper, we shall assume our domain is cubic and generate a background Cartesian mesh in which each cubic element is then partitioned into several tetrahedra. For example, we show two types of triangulations in the left two plots of Figure 2.1, where the triangulation in Type I does not introduce extra nodes, but Type II introduces an extra node on each face and one inside the element. (We show only 4 tetrahedra in the second plot of Figure 2.1 for better illustration). All these tetrahedra form an interface-unfitted shape-regular mesh denoted by $\mathcal{T}_h$ which will be used for discretization. Let $\mathcal{N}_h, \mathcal{E}_h$, and $\mathcal{F}_h$ be the collection of nodes, edges, and faces of $\mathcal{T}_h$, respectively. For each element K , we let $\mathcal{N}_K, \mathcal{E}_K$, and $\mathcal{F}_K$ be the collection of nodes, edges, and faces of K with $|\mathcal{N}_K| = 4$, $|\mathcal{E}_K| = 6$, and $|\mathcal{F}_K| = 4$.

Given the signed-distance level-set function of the interface denoted by φ^Γ , we can compute its linear interpolation φ_h^Γ on the mesh $\mathcal{T}_h$; namely, φ_h^Γ interpolates φ^Γ at

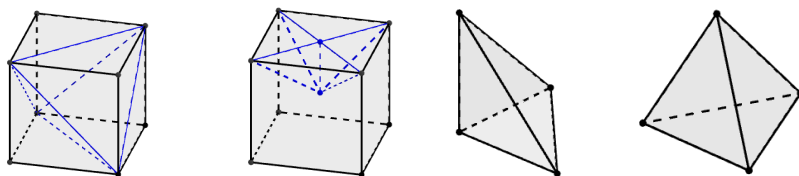


FIG. 2.1. Triangulation of a cube. The left two plots show two triangulation approaches resulting in 5 tetrahedra (Type I) and 24 tetrahedra (Type II). The right two plots show two types of tetrahedra from the triangulation: a trirectangular tetrahedron, which is a tetrahedron that has three isometric orthogonal edges meeting at one vertex, and a regular tetrahedron, whose edges have equal length.

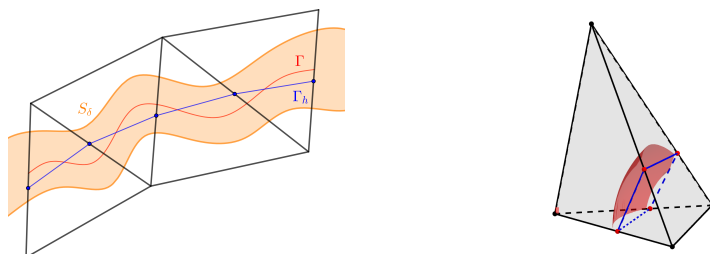


FIG. 2.2. An illustration of the mismatching region of an element and the δ strip. The intersection points of Γ_h^K are always coplanar, while those of Γ may not be.

the mesh nodes and is piecewise linear on each element. Then, the numerical interface is defined as

$$(2.1) \quad \Gamma_h = \{\mathbf{x} : \varphi_h^\Gamma(\mathbf{x}) = 0\}.$$

We note that as φ_h^Γ is piecewise linear, Γ_h is also piecewise linear and forms a polyhedron as a linear approximation to the exact interface. See Figure 2.2 for illustration. Linear interpolation of signed-distance functions has been widely used for discretizing interface numerically [71, 67]. Note that Γ_h may not be the linear interpolation of Γ . For a smooth interface, the geometric error is still in the order of $\mathcal{O}(h^2)$ (see Lemma 5.3), which is sufficient for a linear method [58]. Then, we let Γ_h cut Ω into $\Omega_h^\pm$. An element K is called an interface element if

$$(2.2) \quad \min_{\mathbf{x} \in \mathcal{N}_K} \varphi_h^\Gamma(\mathbf{x}) \max_{\mathbf{x} \in \mathcal{N}_K} \varphi_h^\Gamma(\mathbf{x}) < 0.$$

As φ_h^Γ interpolates φ^Γ at the mesh nodes, (2.2) also implies that K intersects the exact interface. Let $\mathcal{T}_h^i$ be the collection of interface elements. In addition, given a $K \in \mathcal{T}_h^i$, define the interface patch $\Gamma_h^K = \Gamma_h \cap K$. Let $K_h^\pm$ be the two subelements cut by the interface patch Γ_h^K .

2.2. Sobolev Spaces. Given a subdomain $D \subseteq \Omega$, for a nonnegative number s , we let $H^s(D)$ be the standard Sobolev space. In addition, we introduce the following vector function spaces:

$$(2.3a) \quad \mathbf{H}^s(\text{curl}; D) = \{\mathbf{u} \in \mathbf{H}^s(D) : \text{curl } \mathbf{u} \in \mathbf{H}^s(D)\},$$

$$(2.3b) \quad \mathbf{H}^s(\text{div}; D) = \{\mathbf{u} \in \mathbf{H}^s(D) : \text{div } \mathbf{u} \in H^s(D)\}.$$

Given a subdomain $D \subseteq \Omega$, if $D \cap \Gamma \neq \emptyset$, we let $D^\pm = \Omega^\pm \cap D$ and further denote $H^s(\cup D^\pm)$, $\mathbf{H}^s(\text{curl}; \cup D^\pm)$, and $\mathbf{H}^s(\text{div}; \cup D^\pm)$ consisting of functions that belong to the corresponding Sobolev spaces on each $D^\pm$. Given a face F in the mesh, we let rot_F be the two-dimensional (2D) rotatory operator on F and define the space

$$(2.4) \quad \mathbf{H}^s(\text{rot}; F) = \{\mathbf{u} \in \mathbf{H}^s(F) : \text{rot}_F \mathbf{u} \in H^s(F)\}.$$

The H^1 -interface problem reads as

$$(2.5) \quad -\nabla \cdot (\beta \nabla u) = f \quad \text{in } \Omega^- \cup \Omega^+,$$

with $f \in L^2(\Omega)$, subject to $u = 0$ on $\partial\Omega$ and the jump conditions

$$(2.6a) \quad [u]_\Gamma := u^+ - u^- = 0,$$

$$(2.6b) \quad [\beta \nabla u \cdot \mathbf{n}]_\Gamma := \beta^+ \nabla u^+ \cdot \mathbf{n} - \beta^- \nabla u^- \cdot \mathbf{n} = 0,$$

where the parameter β is the same as the one in (1.1c) that has a similar physical meaning related to conductivity. The $\mathbf{H}(\text{div})$ -interface problem is given by

$$(2.7) \quad -\nabla \text{div}(\mathbf{u}) + \alpha \mathbf{u} = \mathbf{f},$$

with $\mathbf{f} \in \mathbf{H}(\text{curl}; \Omega)$, subject to $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$ and the jump conditions

$$(2.8a) \quad [\mathbf{u} \cdot \mathbf{n}]_\Gamma := \mathbf{u}^+ \cdot \mathbf{n} - \mathbf{u}^- \cdot \mathbf{n} = 0,$$

$$(2.8b) \quad [\alpha \mathbf{u} \times \mathbf{n}]_\Gamma := \alpha^+ \mathbf{u}^+ \times \mathbf{n} - \alpha^- \mathbf{u}^- \times \mathbf{n} = \mathbf{0},$$

$$(2.8c) \quad [\text{div}(\mathbf{u})]_\Gamma := \text{div}(\mathbf{u}^+) - \text{div}(\mathbf{u}^-) = 0.$$

This system comes from the first-order least-squares formulation of the elliptic interface problem [18] or preconditioning for the mixed FE using a gradient formulation [2]. The related FE approximation for interface problems can be found in [46].

Now we encode the jump conditions for the H^1 case (2.6), the $\mathbf{H}(\text{curl})$ case (1.1), and the $\mathbf{H}(\text{div})$ case (2.8) into the definition of the following special Sobolev spaces:

$$(2.9a) \quad H^2(\beta; D) := \{u \in H^2(\cup D^\pm) \cap H^1(D) : \beta \nabla u \in \mathbf{H}(\text{div}; D)\},$$

$$(2.9b) \quad \mathbf{H}^1(\alpha, \beta, \text{curl}; D) := \{\mathbf{u} \in \mathbf{H}^1(\text{curl}; \cup D^\pm) \cap \mathbf{H}(\text{curl}; D) : \\ \beta \mathbf{u} \in \mathbf{H}(\text{div}; D), \alpha \text{curl} \mathbf{u} \in \mathbf{H}(\text{curl}; D)\},$$

$$(2.9c) \quad \mathbf{H}^1(\alpha, \text{div}; D) := \{\mathbf{u} \in \mathbf{H}^1(\text{div}; \cup D^\pm) \cap \mathbf{H}(\text{div}; D) : \\ \alpha \mathbf{u} \in \mathbf{H}(\text{curl}; D), \text{div} \mathbf{u} \in H^1(D)\}.$$

An alternative interpretation of these interface conditions is to regard α and β as Hodge star operators mapping between fields in different Sobolev spaces, which is illustrated by the following diagram (2.10). The construction of IFE spaces is just to mimic the diagram locally on each interface element.

$$(2.10) \quad \begin{array}{ccccccc} H^2(\beta; \Omega) & \xrightarrow{\text{grad}} & \mathbf{H}^1(\text{curl}, \alpha, \beta; \Omega) & \xrightarrow{\text{curl}} & \mathbf{H}^1(\text{div}, \alpha; \Omega) & \xrightarrow{\text{div}} & H^1(\Omega) \\ \downarrow I & & \downarrow \beta & & \downarrow \alpha & & \downarrow I \\ L^2(\Omega) & \xleftarrow{\text{div}} & \mathbf{H}(\text{div}; \Omega) & \xleftarrow{\text{curl}} & \mathbf{H}(\text{curl}; \Omega) & \xleftarrow{\text{grad}} & H^1(\Omega) \end{array}$$

3. Immersed finite element spaces. In this section, we develop Nédélec-type and Raviart–Thomas-type IFE spaces that admit the edge and face DoFs, respectively. We shall use α_h and β_h for the discontinuous coefficients partitioned by Γ_h instead of Γ . In the following discussion, we let $\mathbb{P}_k(D)$ be the k th degree polynomial space on a domain D . For the sake of the readers, we recall the lowest-order local Nédélec [65, 66] and Raviart–Thomas spaces [68]:

$$\begin{aligned}\mathcal{ND}_0(K) &:= \{\mathbf{a} \times \mathbf{x} + \mathbf{b} : \mathbf{a} \in \mathbb{R}^3, \mathbf{b} \in \mathbb{R}^3\}, \\ \mathcal{RT}_0(K) &:= \{c\mathbf{x} + \mathbf{a} : c \in \mathbb{R}, \mathbf{a} \in \mathbb{R}^3\}.\end{aligned}$$

The local 3D H^1 IFE space has been constructed in [57], and the $\mathbf{H}(\text{div})$ and $\mathbf{H}(\text{curl})$ can be found in a recent work [20], which will be reviewed in section 3.1. However, this work does not show any DoFs of these local spaces and the related unisolvence. A typical continuous-type Galerkin scheme requires the usual nodal, edge, and face DoFs to glue those local spaces together. This is also crucial for the proposed PG formulation. In section 3.2, we analyze those DoFs with unisolvence and point out that this is highly nontrivial due to the jump conditions. In fact, unlike the usual FE spaces, the unisolvence demands certain geometrical conditions of the elements. Finally, we establish the de Rham complex for these spaces.

3.1. The local H^1 , $\mathbf{H}(\text{curl})$, and $\mathbf{H}(\text{div})$ IFE spaces. Consider the two piecewise constant vector spaces

$$(3.1a) \quad \mathbf{A}_h(K) = \{\mathbf{c} : \mathbf{c}^\pm = \mathbf{c}|_{K_h^\pm} \in [\mathbb{P}_0(K_h^\pm)]^3, \mathbf{c} \in \mathbf{H}(\text{div}; K), \alpha_h \mathbf{c} \in \mathbf{H}(\text{curl}; K)\},$$

$$(3.1b) \quad \mathbf{B}_h(K) = \{\mathbf{c} : \mathbf{c}^\pm = \mathbf{c}|_{K_h^\pm} \in [\mathbb{P}_0(K_h^\pm)]^3, \mathbf{c} \in \mathbf{H}(\text{curl}; K), \beta_h \mathbf{c} \in \mathbf{H}(\text{div}; K)\}.$$

We can derive the specific format for functions in (3.1). Given each interface element with the approximate plane Γ_h^K , we let $\bar{\mathbf{n}}_K$, $\bar{\mathbf{t}}_{K,1}$, and $\bar{\mathbf{t}}_{K,2}$ be its unit normal vector and two orthogonal unit tangential vectors. Furthermore, we denote $T_K = [\bar{\mathbf{n}}_K, \bar{\mathbf{t}}_{K,1}, \bar{\mathbf{t}}_{K,2}]$ and define the transformation matrices

$$(3.2) \quad A_K = T_K \begin{bmatrix} 1 & 0 & 0 \\ 0 & \tilde{\alpha} & 0 \\ 0 & 0 & \tilde{\alpha} \end{bmatrix} T_K^\top \quad \text{and} \quad B_K = T_K \begin{bmatrix} \tilde{\beta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} T_K^\top,$$

where $\tilde{\alpha} = \alpha^+/\alpha^-$ and $\tilde{\beta} = \beta^+/\beta^-$. Then, the spaces $\mathbf{A}_h(K)$ and $\mathbf{B}_h(K)$ can be rewritten as

$$\begin{aligned}\mathbf{A}_h(K) &= \{\mathbf{c} : \mathbf{c}^\pm \in [\mathbb{P}_0(K_h^\pm)]^3, \mathbf{c}^- = A_K \mathbf{c}^+\}, \\ \mathbf{B}_h(K) &= \{\mathbf{c} : \mathbf{c}^\pm \in [\mathbb{P}_0(K_h^\pm)]^3, \mathbf{c}^- = B_K \mathbf{c}^+\}.\end{aligned}$$

With these two piecewise constant vector spaces, we are able to construct the H^1 , $\mathbf{H}(\text{curl})$, and $\mathbf{H}(\text{div})$ IFE spaces, denoted by $S_h^n(K)$, $\mathbf{S}_h^e(K)$, and $\mathbf{S}_h^f(K)$, which are reported in Table 3.1 including their corresponding jump conditions and general function format. One can easily check that the IFE spaces above belong to the corresponding Sobolev spaces. The jump conditions for functions in $\mathbf{S}_h^e(K)$ or $\mathbf{S}_h^f(K)$ are satisfied only at one point $\mathbf{x}_K$. Moreover, the formats are identical to the standard Lagrange, Nédélec, and Raviart–Thomas elements in which the only difference is the vectors $\mathbf{a}_h$ and $\mathbf{b}_h$ replacing the simple constant vectors in $[\mathbb{P}_0(K)]^3$. Then, it is easy to see the following exact sequence:

TABLE 3.1

The IFE spaces, the Sobolev spaces to which the IFE spaces belong, the IEF spaces' function format, and the jump conditions. Here, $\mathbf{x}_K$ is chosen as the centroid of the interface patch Γ_h^K , but other choices are also acceptable; see Remark 3.1.

IFE spaces	$S_h^n(K)$	$S_h^e(K)$	$S_h^f(K)$
Sobolev spaces	$H^1(K)$	$\mathbf{H}(\text{curl}; K)$	$\mathbf{H}(\text{div}; K)$
Function format	$\mathbf{b}_h \cdot (\mathbf{x} - \mathbf{x}_K) + c$ $\mathbf{b}_h \in \mathbf{B}_h(K)$, $c \in \mathbb{P}_0(K)$	$\mathbf{a}_h \times (\mathbf{x} - \mathbf{x}_K) + \mathbf{b}_h$ $\mathbf{a}_h \in \mathbf{A}_h(K)$, $\mathbf{b}_h \in \mathbf{B}_h(K)$	$c(\mathbf{x} - \mathbf{x}_K) + \mathbf{a}_h$ $c \in \mathbb{P}_0(K)$, $\mathbf{a}_h \in \mathbf{A}_h(K)$
Jump conditions	$[v_h]_{\Gamma_h^K} = 0$ $[\beta_h \nabla v_h \cdot \bar{\mathbf{n}}]_{\Gamma_h^K} = 0$	$[\mathbf{v}_h \times \bar{\mathbf{n}}]_{\Gamma_h^K} = \mathbf{0}$ $[\beta_h \mathbf{v}_h \cdot \bar{\mathbf{n}}]_{\mathbf{x}_K} = 0$ $[\alpha_h \text{curl} \mathbf{v}_h \times \bar{\mathbf{n}}]_{\Gamma_h^K} = \mathbf{0}$	$[\mathbf{v}_h \cdot \bar{\mathbf{n}}]_{\Gamma_h^K} = 0$ $[\alpha_h \mathbf{v}_h \times \bar{\mathbf{n}}]_{\mathbf{x}_K} = \mathbf{0}$ $[\text{div} \mathbf{v}_h]_{\Gamma_h^K} = 0$

$$(3.3) \quad \mathbb{R} \xrightarrow{\hookrightarrow} S_h^n(K) \xrightarrow{\text{grad}} S_h^e(K) \xrightarrow{\text{curl}} S_h^f(K) \xrightarrow{\text{div}} \mathbb{P}_0(K) \rightarrow 0.$$

Remark 3.1. Our numerical experience and the analysis in section 5 show that $\mathbf{x}_K$ can be quite arbitrary as long as it is located within the $\mathcal{O}(h_K)$ distance to K . Indeed, different choices of $\mathbf{x}_K$ generally lead to different IFE functions. In computation, we chose $\mathbf{x}_K$ as the centroid of Γ_h^K . However, as an interesting observation, for the nodal IFE space, various $\mathbf{x}_K$ in fact result in the same space, although the shape functions might be different. To see this, we take $\mathbf{x}'_K$ as another point on Γ_h^K , and let $\bar{\mathbf{t}} = \mathbf{x}'_K - \mathbf{x}_K$. Let us take the nodal space $S_h^n(K)$ as an example. For any $\mathbf{b}_h \in \mathbf{B}_h(K)$, we have $\mathbf{b}_h^- \cdot \bar{\mathbf{t}} = (\mathbf{b}_h^+)^T B_K \bar{\mathbf{t}} = \mathbf{b}_h^+ \cdot \bar{\mathbf{t}}$. Thus, a function $v_h \in S_h^n(K)$ can be expressed as

$$(3.4) \quad v_h = \mathbf{b}_h \cdot (\mathbf{x} - \mathbf{x}'_K) + \mathbf{b}_h \cdot \bar{\mathbf{t}} + c = \mathbf{b}_h \cdot (\mathbf{x} - \mathbf{x}_K) + c',$$

showing that the functions defined with $\mathbf{x}'_K$ also belong to $S_h^n(K)$.

3.2. Degrees of freedom. For the H^1 case, it has been proved in [57] that the functions in $S_h^n(K)$ admit nodal DoFs for the two types of tetrahedra shown in the right two plots of Figure 2.1, i.e., a trirectangular tetrahedron that has three isometric orthogonal edges meeting at one vertex and a regular tetrahedron that has six isometric edges. Namely, there exist $\phi_i^n \in S_h^n(K)$, $i = 1, 2, 3, 4$, such that $\phi_i^n(\mathbf{z}_j) = \delta_{ij}$, where $\mathbf{z}_j$, $j = 1, 2, 3, 4$, are the nodes of K . Then, the global H^1 IFE space is defined as

$$(3.5) \quad S_h^n = \{v_h \in L^2(\Omega) : v_h|_K \in S_h^n(K) \text{ if } K \in \mathcal{T}_h^i, \text{ and } v_h|_K \in \mathbb{P}_1(K) \text{ if } K \in \mathcal{T}_h^n, \\ v_h \text{ is continuous at } \mathbf{x} \in \mathcal{N}_h\}.$$

We then consider the unsolvence of $S_h^f(K)$ and begin by deriving a formula for computing $\mathbf{H}(\text{div})$ IFE shape functions. Given an interface element K with 4 faces F_i , $i = 1, 2, \dots, 4$, we introduce $v_i = \int_{F_i} \mathbf{v}_h \cdot \mathbf{n}_{F_i} ds$, with $\mathbf{n}_{F_i}$ being the unit normal vector to F_i , for any function $\mathbf{v}_h = c(\mathbf{x} - \mathbf{x}_K) + \mathbf{a}_h \in S_h^f(K)$ based on Table 3.1, and we let $v_0 = v_1 + v_2 + v_3 + v_4$. Determining c is straightforward:

$$(3.6) \quad c = \frac{\text{div}(\mathbf{v}_h)}{3} = \frac{1}{3|K|} \int_K \text{div}(\mathbf{v}_h) d\mathbf{x} = \frac{1}{3|K|} \sum_{i=1}^4 \int_{F_i} \mathbf{v}_h \cdot \mathbf{n}_{F_i} ds = \frac{v_0}{3|K|}.$$

Then, by (3.1) $\mathbf{a}_h$ should satisfy the following equations: for $i = 1, \dots, 4$,

$$(3.7) \quad |F_i^+| \mathbf{n}_{F_i}^\top \mathbf{a}_h^+ + |F_i^-| \mathbf{n}_{F_i}^\top A_K \mathbf{a}_h^+ = v_i - \frac{v_0}{3|K|} \int_{F_i} (\mathbf{x} - \mathbf{x}_K) \cdot \mathbf{n}_{F_i} \, ds.$$

Imposing (3.7) on any three faces F_1, F_2 , and F_3 , we obtain a linear system for $\mathbf{a}_h^+$:

$$(3.8) \quad M_K \mathbf{a}_h^+ = \bar{\mathbf{v}}, \quad \text{with } M_K = \Phi^+ N_K + \Phi^- N_K A_K,$$

where $\Phi^\pm = \text{diag}(|F_1^\pm|, |F_2^\pm|, |F_3^\pm|)$, $N_K = [\mathbf{n}_{F_1}, \mathbf{n}_{F_2}, \mathbf{n}_{F_3}]^\top$, and $\bar{\mathbf{v}}$ is given by the right-hand side of (3.7). The solvability of $\mathbf{a}_h^+$ is equivalent to the invertibility of M_K .

LEMMA 3.2. *Given a trirectangular tetrahedral or regular tetrahedral interface element K , the space $\mathbf{S}_h^f(K)$ has the DoFs $\int_{F_i} \mathbf{v}_h \cdot \mathbf{n}_{F_i} \, ds$, $i = 1, 2, 3, 4$.*

Proof. It remains to show the invertibility of M_K . We treat the two types of tetrahedra separately.

If K is a trirectangular tetrahedron, we choose F_i , $i = 1, 2, 3$, as the three orthogonal faces. Then, N_K is an orthonormal matrix, and we can rewrite

$$(3.9) \quad M_K N_K^\top = \Phi^+ + \Phi^- N_K A_K N_K^\top = \Phi (I + \Phi^{-1} \Phi^- N_K (A_K - I) N_K^\top),$$

where $\Phi = \text{diag}(|F_1|, |F_2|, |F_3|) = \Phi^- + \Phi^+$. Without loss of generality, we can assume $\tilde{\alpha} = \alpha^+/\alpha^- \leq 1$; otherwise we simply consider the matrix $M_K A_K^{-1} N_K^\top = \Phi^+ N_K A_K N_K^\top + \Phi^-$. Let $\lambda_{\max}(\cdot)$ be the largest magnitude of eigenvalues of a matrix. Note that $\lambda_{\max}(\Phi^{-1} \Phi^-) \leq 1$ and $\lambda_{\max}(N_K (A_K - I) N_K^\top) = \lambda_{\max}(A_K - I) = 1 - \tilde{\alpha}$. Since $\Phi^{-1} \Phi^-$ is a diagonal matrix and $N_K (A_K - I) N_K^\top$ is symmetric, it is well known that $\Phi^{-1} \Phi^- N_K (A_K - I) N_K^\top$ has real eigenvalues, and

$$(3.10) \quad \lambda_{\max}(\Phi^{-1} \Phi^- N_K (A_K - I) N_K^\top) \leq \lambda_{\max}(\Phi^{-1} \Phi^-) \lambda_{\max}(N_K (A_K - I) N_K^\top) \leq 1 - \tilde{\alpha}.$$

Therefore, the eigenvalues of $I + \Phi^{-1} \Phi^- N_K (A_K - I) N_K^\top$ must be bounded below by $1 - (1 - \tilde{\alpha}) = \tilde{\alpha}$, and thus, from (3.9), the matrix M_K is nonsingular.

If K is a regular tetrahedron, such a simple proof is not available. Instead, we employ a computer-aided argument to verify that M_K is nonsingular. As it is elementary and technical, we refer readers to the supplementary materials (120943_1_supp_515743_rq18d5_sc.pdf [local/web 406KB]) for details. $\square$

Lemma 3.2 guarantees the existence of the shape functions $\phi_i^f \in \mathbf{S}_h^f(K)$, $i = 1, 2, 3, 4$, satisfying

$$(3.11) \quad \int_{F_j} \phi_i^f \cdot \mathbf{n} \, ds = \delta_{ij}, \quad j = 1, 2, 3, 4,$$

regardless of interface location. Then, the global $\mathbf{H}(\text{div})$ IFE space is defined as

$$(3.12) \quad \mathbf{S}_h^f = \left\{ \mathbf{v}_h \in L^2(\Omega) : \mathbf{v}_h|_K \in \mathbf{S}_h^f(K) \text{ if } K \in \mathcal{T}_h^i, \text{ and } \mathbf{v}_h|_K \in \mathcal{RT}_0(K) \text{ if } K \in \mathcal{T}_h^n, \right. \\ \left. \int_F [\mathbf{v}_h \cdot \mathbf{n}] \, ds = 0 \, \forall F \in \mathcal{F}_h \right\}.$$

With the nodal and face DoFs, we are ready to analyze the DoFs of $\mathbf{S}_h^e(K)$.

LEMMA 3.3. *Given a trirectangular tetrahedral or regular tetrahedral interface element K , the space $\mathbf{S}_h^e(K)$ has the DoFs $\int_{e_i} \mathbf{v}_h \cdot \mathbf{t}_i \, ds$, $i = 1, 2, \dots, 6$.*

Proof. Given an interface element K with six edges e_i , $i = 1, \dots, 6$, we let $v_i = \int_{e_i} \mathbf{v}_h \cdot \mathbf{t}_i \, ds$ for a function $\mathbf{v}_h \in \mathbf{S}_h^e(K)$ where $\mathbf{t}_i$'s have some assigned orientation. We need to show that $\mathbf{v}_h$ uniquely exists for arbitrary v_i . As the dimensions match, we only need to prove the existence by explicitly constructing

$$(3.13) \quad \phi^e = \phi^f \times (\mathbf{x} - \mathbf{x}_K)/2 + \nabla \phi^n,$$

satisfying the DoFs. For each face F_j of K , we let

$$(3.14) \quad w_j = \sum_{e_i \subset \partial F_j} c_{j,e_i} v_i, \quad j = 1, \dots, 4, \quad \text{and} \quad \phi^f = \sum_{j=1}^4 w_j \phi_j^f,$$

where $c_{j,e_i} = 1$ or -1 is a sign to correct the orientation such that the edges associated with each face have the counterclockwise orientation. So, trivially $\sum_{j=1}^4 w_j = 0$ and thus ϕ^f is div-free, i.e., $\phi^f \in \text{curl} \mathbf{S}_h^e(K)$ by the local exact sequence in (3.3). We then proceed to construct the component in $\mathbf{S}_h^n(K)$. Let the four nodes of K be $\mathbf{z}_l$, $l = 1, \dots, 4$. Picking any node, say $\mathbf{z}_4$, without loss of generality, we let the three neighbor edges be e_i pointing to $\mathbf{z}_i$, $i = 1, 2, 3$. Define the orientation coefficients $d_i = 1$ if $\mathbf{t}_i$ points from $\mathbf{z}_4$ to another ending point of e_i ; otherwise $d_i = -1$. We then construct

$$\phi^n = \sum_{i=1,2,3} d_i \left(v_i - \int_{e_i} (\phi^f \times (\mathbf{x} - \mathbf{x}_K)) \cdot \mathbf{t}_i / 2 \, ds \right) \phi_i^n.$$

It is straightforward to see that ϕ^e constructed in (3.13) matches the DoFs on e_i , $i = 1, 2, 3$. The verification of the rest of the edges can be shown by the DoFs of ϕ^f and ϕ^n with the formula $\int_F \text{curl} \phi^e \cdot \mathbf{n} \, ds = \int_F \text{rot}_F \phi^e \, ds = \int_{\partial F} \phi^e \cdot \mathbf{t} \, ds$ for each face F . $\square$

Thanks to the same DoFs, the whole construction procedure exactly mimics the standard FE spaces. Lemma 3.3 guarantees the existence of Nédélec IFE shape functions $\phi_i^e \in \mathbf{S}_h^e(K)$, $i = 1, 2, \dots, 6$, satisfying

$$(3.15) \quad \int_{e_j} \phi_i^e \cdot \mathbf{t} \, ds = \delta_{ij}, \quad j = 1, 2, \dots, 6.$$

Then, the global $\mathbf{H}(\text{curl})$ IFE space can be constructed as

$$(3.16) \quad \mathbf{S}_h^e = \left\{ \mathbf{v}_h \in L^2(\Omega) : \mathbf{v}_h|_K \in \mathbf{S}_h^e(K) \text{ if } K \in \mathcal{T}_h^i, \text{ and } \mathbf{v}_h|_K \in \mathcal{ND}_0(K) \text{ if } K \in \mathcal{T}_h^n, \right. \\ \left. \int_e (\mathbf{v}_h|_{K_1} - \mathbf{v}_h|_{K_2}) \cdot \mathbf{t} \, ds = 0 \, \forall e \in \mathcal{E}_h, \forall K_1, K_2 \text{ sharing } e \right\}.$$

Remark 3.4. As $\mathbf{v}_h \cdot \mathbf{t}$ is piecewise constant on $e_h^\pm$, $\int_e (\mathbf{v}_h|_{K_1} - \mathbf{v}_h|_{K_2}) \cdot \mathbf{t} \, ds = 0$ cannot imply $(\mathbf{v}_h|_{K_1} - \mathbf{v}_h|_{K_2}) \cdot \mathbf{t} = 0$. So the IFE spaces are nonconforming. In fact, even the weak continuity $\int_F (\mathbf{v}_h|_{K_1} - \mathbf{v}_h|_{K_2}) \times \mathbf{n}_F \, ds = \mathbf{0}$ does not hold. We illustrate this kind of discontinuity in Figure 3.1 for an edge IFE function $\mathbf{v}_h$ whose tangential traces $\mathbf{v}_h|_{K_i} \times \mathbf{n}$ on F from the two neighboring elements, K_i , $i = 1, 2$, do not match.

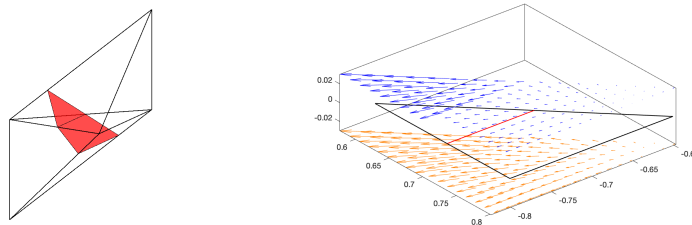


FIG. 3.1. Left: two neighbor elements. Right: the tangential traces of an IFE function on the sharing face from the two neighbor elements where the blue field is from the top and the orange field is from the bottom. (Color figure available online.)

Finally, we define the following interpolation operators:

$$\begin{aligned} I_h^n : H^2(\beta; \Omega) &\rightarrow S_h^n, & \text{satisfying } I_h^n u(\mathbf{x}) &= u(\mathbf{x}) \quad \forall \mathbf{x} \in \mathcal{N}_h, \\ I_h^e : \mathbf{H}^1(\alpha, \beta, \text{curl}; \Omega) &\rightarrow \mathbf{S}_h^e, & \text{satisfying } \int_e I_h^e \mathbf{u} \cdot \mathbf{t} \, ds &= \int_e \mathbf{u} \cdot \mathbf{t} \, ds \quad \forall e \in \mathcal{E}_h, \\ I_h^f : \mathbf{H}^1(\alpha, \text{div}; \Omega) &\rightarrow \mathbf{S}_h^f, & \text{satisfying } \int_F I_h^f \mathbf{u} \cdot \mathbf{n} \, ds &= \int_F \mathbf{u} \cdot \mathbf{n} \, ds \quad \forall F \in \mathcal{F}_h. \end{aligned}$$

Note that $\int_e \mathbf{u} \cdot \mathbf{t} \, ds$ is well defined since $\mathbf{u}$ has piecewise $\mathbf{H}^1(\text{curl})$ regularity.

3.3. The de Rham complex. Now we recover the well-known and important de Rham complex in Figure 1.1, where Q_h denotes the piecewise constant space on the mesh $\mathcal{T}_h$, and Π^0 denotes the L^2 projection to Q_h .

THEOREM 3.5. *The sequence in the middle of Figure 1.1 is a complex.*

Proof. First, it is clear that $\mathbb{R} \subset S_h^n$. In addition, with the local result in (3.3), we know that the functions mapped by the operators ∇ , curl , and div must satisfy the corresponding jump conditions on each element. Thus, we only need to verify the corresponding integral conditions on edges and faces in (3.16) and (3.12), respectively. For each edge $e \in \mathcal{E}_h$ with the nodes $\mathbf{z}_1$ and $\mathbf{z}_2$, we let K_1 and K_2 be any two elements sharing e . Then, given any $v_h \in S_h^n$, the continuity implies

$$\int_e \nabla v_h|_{K_1} \cdot \mathbf{t} \, ds = v_h|_{K_1}(\mathbf{z}_2) - v_h|_{K_1}(\mathbf{z}_1) = v_h|_{K_2}(\mathbf{z}_2) - v_h|_{K_2}(\mathbf{z}_1) = \int_e \nabla v_h|_{K_2} \cdot \mathbf{t} \, ds,$$

where $\mathbf{t}$ orients from $\mathbf{z}_1$ to $\mathbf{z}_2$. So, we conclude $\nabla v_h \in \mathbf{S}_h^e$. Next, for each face $F \in \mathcal{F}_h$, we let K_1 and K_2 be the two neighbor elements. Given any $\mathbf{v}_h \in \mathbf{S}_h^e$, we have

$$\int_F \text{curl } \mathbf{v}_h|_{K_1} \cdot \mathbf{n}_F \, ds = \int_F \text{rot}_F \mathbf{v}_h|_{K_1} \, ds = \int_F \text{rot}_F \mathbf{v}_h|_{K_2} \, ds = \int_F \text{curl } \mathbf{v}_h|_{K_2} \cdot \mathbf{n}_F \, ds.$$

Therefore, we have $\text{curl } \mathbf{v}_h \in \mathbf{S}_h^f$. Finally, it is trivial that $\text{div}(\mathbf{v}_h) \in Q_h$ for any $\mathbf{v}_h \in \mathbf{S}_h^f$. $\square$

THEOREM 3.6. *The diagram formed by the sequences in the middle and top of Figure 1.1 is commutative.*

Proof. Let us only consider $I_h^f \circ \text{curl} = \text{curl} \circ I_h^e$, and the proof for other cases is similar. Given $\mathbf{u} \in \mathbf{H}^1(\alpha, \beta, \text{curl}; \Omega)$, there holds

$$\int_F I_h^f \text{curl } \mathbf{u} \cdot \mathbf{n} \, ds = \int_F \text{curl } \mathbf{u} \cdot \mathbf{n} \, ds = \int_{\partial F} \mathbf{u} \cdot \mathbf{t} \, ds = \int_{\partial F} I_h^e \mathbf{u} \cdot \mathbf{t} \, ds = \int_F \text{curl } I_h^e \mathbf{u} \cdot \mathbf{n} \, ds$$

which finishes the proof. $\square$

To show the exactness, let us first describe the isomorphism between the IFE spaces and standard FE spaces. Let $\tilde{S}_h^n$, $\tilde{S}_h^e$, and $\tilde{S}_h^f$ be the standard Lagrange, Nédélec, and Raviart–Thomas spaces defined on the same mesh $\mathcal{T}_h$. Note that the interpolations I_h^n , I_h^e , and I_h^f are all well defined on these standard spaces, which lead to isomorphism; namely their inversions from the IFE spaces to FE spaces are also well defined. We shall keep the same notation I_h^n , I_h^e , and I_h^f as the isomorphisms and illustrate this structure by the diagram formed from the middle and bottom sequences in 1.1, where I is the identity operator.

LEMMA 3.7. *The diagram formed by the middle and bottom sequences in Figure 1.1 is commutative.*

Proof. The argument is the same as for Theorem 3.6 by verifying the DoFs. $\square$

LEMMA 3.8. *The operators I_h^n , I_h^e , and I_h^f and their inversions preserve the kernels of ∇ , curl, and div:*

$$(3.17) \quad \ker(d) \cap \tilde{S}_h^s \xrightleftharpoons[(I_h^s)^{-1}]{I_h^s} \ker(d) \cap \mathbf{S}_h^s, \quad (d, s) = (\nabla, n), (\text{curl}, e), (\text{div}, f).$$

Proof. By the same argument of Theorem 3.6, we can show that the sequences in the middle and top of Figure 1.1 are commutative for both I_h^s and $(I_h^s)^{-1}$, which then implies the desired result. $\square$

THEOREM 3.9. *The sequence in the middle of Figure 1.1 is exact.*

Proof. The result directly follows from Lemma 3.8 and the exactness for the standard FE spaces. $\square$

4. The Petrov–Galerkin IFE method for the $\mathbf{H}(\text{curl})$ interface problem.

In this section, we shall present the PG-IFE scheme by using the $\mathbf{H}(\text{curl})$ IFE space as the trial space and the standard Nédélec FE space as the test space to the $\mathbf{H}(\text{curl})$ interface problem. We then adapt the HX preconditioner for solving the resulting nonsymmetric linear algebraic system.

4.1. Schemes. For simplicity of presentation, here we consider only the perfect electric conductor (PEC) or correspondingly the perfect magnetic conductor (PMC) boundary condition, i.e., $\mathbf{u} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$. For the quasi-static equation (1.3), the proposed PG-IFE scheme is to find $\mathbf{u}_h \in \mathbf{S}_{h,0}^e$, i.e., the subspace of $\mathbf{S}_h^e$ with the zero trace, such that

$$(4.1) \quad \begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) &= (\mathbf{f}, \mathbf{v})_\Omega \quad \forall \mathbf{v}_h \in \tilde{\mathbf{S}}_{h,0}^e \\ \text{with } a_h(\mathbf{u}_h, \mathbf{v}_h) &:= (\alpha_h \text{curl } \mathbf{u}_h, \text{curl } \mathbf{v}_h)_\Omega + (\beta_h \mathbf{u}_h, \mathbf{v}_h)_\Omega, \end{aligned}$$

where $(\cdot, \cdot)_\Omega$ denotes the standard L^2 inner product. Note that (4.1) follows from the integration by parts of (1.3) which always holds as long as the test function space is $\mathbf{H}(\text{curl})$ -conforming. Thanks to the isomorphism between the IFE and FE spaces, the resulting linear system is square. So the PG formulation completely avoids the nonconformity issue mentioned in the introduction.

The PG scheme plays an important role in achieving optimal convergence. Let us briefly explain why the schemes with the $H(\text{curl})$ -nonconforming IFE space as the test function space with or without penalties all result in merely suboptimal convergence. Suppose $\mathbf{u} \in \mathbf{H}^s(\alpha, \beta, \text{curl}; \Omega)$ is the exact solution. Testing (1.3) by $\mathbf{v}_h \in \mathbf{S}_h^e$ and applying integration by parts yields

$$(4.2) \quad (\alpha \operatorname{curl} \mathbf{u}, \operatorname{curl} \mathbf{v}_h)_\Omega + (\beta \mathbf{u}, \mathbf{v}_h)_\Omega + \sum_{F \in \mathcal{F}_h^i} \int_F (\alpha \operatorname{curl} \mathbf{u}) \cdot [\mathbf{v}_h \times \mathbf{n}]_F \, ds = (\mathbf{f}, \mathbf{v}_h)_\Omega,$$

where $\mathcal{F}_h^i$ denotes the collection of the faces intersecting with the interface, and $[\cdot]_F$ denotes the jump of a quantity on F . The term of integration on faces in (4.2) appears as $\mathbf{v}_h$ is generally not tangential continuous on F , which typically causes the loss of convergence. This is a widely observed issue for not just IFE methods [40, 43, 60] but also other unfitted mesh methods [15, 16, 21, 22], as almost all of them are built on nonconforming approximation spaces. A typical solution is to apply suitable penalties to handle the nonconformity. However, as the $\mathbf{u}$ merely has the $\mathbf{H}^s(\operatorname{curl})$ regularity, the penalty terms, particularly the stabilization term will cause the suboptimal convergence shown in (1.4). We refer readers to [20] for a numerical example.

4.2. A fast solver: HX preconditioner. The resulting linear system of (4.1) is denoted by

$$(4.3) \quad \mathbf{B}^{PG} \bar{\mathbf{u}} = \bar{\mathbf{f}},$$

where $\mathbf{B}^{PG}$ is not symmetric positive definite (SPD). However, the majority portion of $\mathbf{B}^{PG}$ is indeed SPD, and the “problematic” portion is only around the interface. So, naturally, it may help if we employ the exact solver on the subsystem around the interface and use an iterative solver on the rest of the system. We shall apply the solver developed in our recent work [20] and briefly recall it below.

Define the collections of elements and edges around the interface:

$$(4.4) \quad \begin{aligned} \mathcal{T}_h^l &= \{K \in \mathcal{T}_h : \bar{K} \cap \bar{K}' \neq \emptyset \text{ for some } K' \in \mathcal{T}_h^{l-1}\}, \\ \mathcal{E}_h^l &= \{e \in \mathcal{E}_h : e \in \mathcal{E}_K \text{ for some } K \in \mathcal{T}_h^l\}, \end{aligned}$$

where $\mathcal{T}_h^0 = \mathcal{T}_h^i$ is simply the set of interface elements. It is not hard to see that $\mathcal{T}_h^l$ and $\mathcal{E}_h^l$ can expand to the whole domain from the interface when increasing l . But with relatively small l , the set $\mathcal{E}_h^l$ is sufficient to cover the “problematic” DoFs. Assuming that the DoFs associated with $\mathcal{E}_h^l$ are indexed at the end, we can write

$$(4.5) \quad \mathbf{B}^{PG} = \begin{bmatrix} \mathbf{B}_l^{(1,1)} & \mathbf{B}_l^{(2,1)} \\ \mathbf{B}_l^{(1,2)} & \mathbf{B}_l^{(2,2)} \end{bmatrix},$$

where $\mathbf{B}_l^{(1,1)}$ is SPD but $\mathbf{B}_l^{(2,2)}$ is non-SPD and also $\mathbf{B}_l^{(1,2)} \neq (\mathbf{B}_l^{(2,1)})^\top$. Based on (4.5), we then construct a special smoother:

$$(4.6) \quad \mathbf{R}_l^{PG} = \begin{bmatrix} \operatorname{diag}(\mathbf{B}_l^{(1,1)}) & 0 \\ 0 & \mathbf{B}_l^{(2,2)} \end{bmatrix},$$

In the following discussion, $\mathbf{R}_l^{PG}$ and l shall be frequently referred to as a “block diagonal smoother” and an “expanding width,” respectively.

We write $\mathbf{B}^\nabla$ as the matrix associated with the IFE discretization of

$$(4.7) \quad a_{h,1}^\nabla(\mathbf{u}_h, \mathbf{v}_h) := (\alpha_h \nabla \mathbf{u}_h, \nabla \mathbf{v}_h)_\Omega + (\beta_h \mathbf{u}_h, \mathbf{v}_h)_\Omega \quad \forall \mathbf{u}_h, \mathbf{v}_h \in [S_{h,0}^n]^3,$$

and write B^∇ as the matrix of the bilinear form

$$(4.8) \quad a_{h,2}^\nabla(u_h, v_h) := (\alpha_h \nabla u_h, \nabla v_h)_\Omega \quad \forall u_h, v_h \in S_{h,0}^n.$$

Then, the IFE version of the HX preconditioner [48] is

$$(4.9) \quad \mathbf{B}^{\text{curl}} = [\mathbf{R}_l^{PG}]^{-1} + P_{\text{curl}}(\mathbf{B}^\nabla)^{-1}P_{\text{curl}}^\top + G(B^\nabla)^{-1}G^\top.$$

Here G is the matrix representation of the operator $\nabla : \tilde{\mathbf{S}}_h^n \rightarrow \tilde{\mathbf{S}}_h^e$, which is indeed the incidence matrix between edges and vertices: there are two nonzero entries, ± 1 , on each row, and P_{curl} is the matrix associated with the interpolation operator $\Pi_h^{\text{curl}} : [\tilde{\mathbf{S}}_h^n]^3 \rightarrow \tilde{\mathbf{S}}_h^e$ such that $\Pi_h^{\text{curl}} \mathbf{w}|_e \cdot \mathbf{t} = 0.5(\mathbf{w}_1 + \mathbf{w}_2) \cdot \mathbf{t} \ \forall e \in \mathcal{E}_h$ with $\mathbf{w}_1$ and $\mathbf{w}_2$ being the nodal values of $\mathbf{w}$ at the nodes of e . We remark that these two matrices can be identically defined on the IFE spaces as the IFE spaces share exactly the same DoFs as the FE spaces. Hence, forming such matrices highly relies on the isomorphism shown in Figure 1.1. The matrices $\mathbf{B}^\nabla$ and B^∇ are SPD, and their inverses are approximated by a few algebraic multigrid (AMG) cycles. The $[\mathbf{R}_l^{PG}]^{-1}$, particularly the block $(\mathbf{B}_l^{(2,2)})^{-1}$, is computed by the direct solver (the backslash in MATLAB) as it is of small size, and we shall see that this direct solver only has a very subtle effect on the total computational time but is the key to handling the “problematic” DoFs near the interface. We adopt the HX preconditioner implemented in *iFEM* [25].

4.3. A numerical test of the *inf-sup* condition. Define the energy norm

$$\|\mathbf{w}_h\|_{h,D}^2 = \int_D \alpha_h \operatorname{curl} \mathbf{w}_h \cdot \operatorname{curl} \mathbf{w}_h \, dx + \int_D \beta_h \mathbf{w}_h \cdot \mathbf{w}_h \, dx \quad \forall \mathbf{w}_h \in \mathbf{S}_h^e \text{ or } \tilde{\mathbf{S}}_h^e.$$

Clearly, there holds the identity $\|\mathbf{v}_h\|_{h,\Omega}^2 = a_h(\mathbf{v}_h, \mathbf{v}_h)$ and the norm equivalence $\|\cdot\|_{\mathbf{H}(\operatorname{curl};\Omega)} \simeq \|\cdot\|_{h,\Omega}$. With the optimal approximation capabilities of the $\mathbf{H}(\operatorname{curl})$ IFE spaces discussed later, one can immediately show the optimal convergence for the PG-IFE scheme as long as the following *inf-sup* condition holds:

$$(4.10) \quad \eta_s := \sup_{\mathbf{v}_h \in \tilde{\mathbf{S}}_{h,0}^e} \frac{a_h(\mathbf{u}_h, \mathbf{v}_h)}{\|\mathbf{v}_h\|_{h,\Omega} \|\mathbf{u}_h\|_{h,\Omega}} \geq C_s \quad \forall \mathbf{u}_h \in \mathbf{S}_{h,0}^e.$$

However, it is generally quite challenging to show the *inf-sup* condition. The approach of Babuška, Caloz, and Osborn in [3] assumes a certain structure of the elliptic coefficient. The proof of the 2D PG-IFE method [41] highly relies on the symmetry of the high-order curl-curl term, which does not hold anymore for the 3D case. In this work, instead of pursuing a rigorous theoretical analysis, we provide a numerical test for the *inf-sup* condition (4.10).

Denote $\mathbf{B}^{FE}$ and $\mathbf{B}^{IF}$ as the matrices corresponding to the FE and IFE schemes on the same mesh, i.e., the matrices associated with the bilinear forms $a_h(\mathbf{w}_h, \mathbf{v}_h)$, $\mathbf{w}_h, \mathbf{v}_h \in \tilde{\mathbf{S}}_{h,0}^e$ and $a_h(\mathbf{w}_h, \mathbf{v}_h)$, $\mathbf{w}_h, \mathbf{v}_h \in \mathbf{S}_{h,0}^e$, respectively. Furthermore, we let N be the total number of edge DoFs. Then, (4.10) is equivalent to

$$(4.11) \quad \sup_{\bar{\mathbf{v}} \in \mathbb{R}^N} \frac{\bar{\mathbf{v}}^\top \mathbf{B}^{PG} \bar{\mathbf{u}}}{\sqrt{\bar{\mathbf{u}}^\top \mathbf{B}^{IF} \bar{\mathbf{u}}} \sqrt{\bar{\mathbf{v}}^\top \mathbf{B}^{FE} \bar{\mathbf{v}}}} \geq C_s \quad \forall \bar{\mathbf{u}} \in \mathbb{R}^N.$$

In general, one may verify (4.11) by constructing a suitable test function $\bar{\mathbf{v}} = \mathbf{P} \bar{\mathbf{u}}$, where $\mathbf{P}$ is a nonsingular matrix, such that

$$(4.12) \quad \frac{\bar{\mathbf{u}}^\top \mathbf{P}^\top \mathbf{B}^{PG} \bar{\mathbf{u}}}{\sqrt{\bar{\mathbf{u}}^\top \mathbf{B}^{IF} \bar{\mathbf{u}}} \sqrt{\bar{\mathbf{u}}^\top \mathbf{P}^\top \mathbf{B}^{FE} \mathbf{P} \bar{\mathbf{u}}}} \geq C_s \quad \forall \bar{\mathbf{u}} \in \mathbb{R}^N.$$

It is not hard to see that (4.12) implies (4.14) as it takes sup over $\bar{\mathbf{v}} \in \mathbb{R}^N$. We shall focus on (4.12).

Note that one natural choice of the test function $\mathbf{v}_h = (I_h^e)^{-1} \mathbf{u}_h$. Consequently, in (4.12) $\bar{\mathbf{u}} = \bar{\mathbf{v}}$, i.e., $\mathbf{P}$ is an identity matrix, and thus an equivalent condition for (4.12) to hold is $\mathbf{B}^{PG} + (\mathbf{B}^{PG})^\top$ being SPD. Unfortunately, numerical results show that $\mathbf{B}^{PG} + (\mathbf{B}^{PG})^\top$ has negative eigenvalues. Our experience suggests that such a local construction approach may not be appropriate. Hence, we consider a global construction approach. Given each IFE function $\mathbf{u}_h \in \mathbf{S}_{h,0}^e$, we define its projection to the FE space denoted by $\pi_h \mathbf{u}_h \in \tilde{\mathbf{S}}_{h,0}^e$ through $a_h(\cdot, \cdot)$, namely

$$(4.13) \quad a_h(\mathbf{u}_h, \mathbf{v}_h) = a_h(\pi_h \mathbf{u}_h, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \tilde{\mathbf{S}}_{h,0}^e.$$

Note that (4.13) is always well defined since a_h is SPD on the FE space. The corresponding vector solution of $\pi_h \mathbf{u}_h$ is $\bar{\mathbf{v}} = (\mathbf{B}^{FE})^{-1} \mathbf{B}^{PG} \bar{\mathbf{u}}$, i.e., $\mathbf{P} = (\mathbf{B}^{FE})^{-1} \mathbf{B}^{PG}$. Notice that

$$(4.14) \quad \bar{\mathbf{u}}^\top \mathbf{P}^\top \mathbf{B}^{PG} \bar{\mathbf{u}} = \bar{\mathbf{u}}^\top (\mathbf{B}^{PG})^\top (\mathbf{B}^{FE})^{-1} \mathbf{B}^{PG} \bar{\mathbf{u}} = \bar{\mathbf{u}}^\top \mathbf{P}^\top \mathbf{B}^{FE} \mathbf{P} \bar{\mathbf{u}}.$$

Thus, (4.12) becomes the following minimization problem:

$$(4.15) \quad \eta_s = \min_{\bar{\mathbf{u}} \in \mathbb{R}^N} \frac{\sqrt{\bar{\mathbf{u}}^\top (\mathbf{B}^{PG})^\top (\mathbf{B}^{FE})^{-1} \mathbf{B}^{PG} \bar{\mathbf{u}}}}{\sqrt{\bar{\mathbf{u}}^\top \mathbf{B}^{IF} \bar{\mathbf{u}}}}.$$

Similar to [5, 23], the solution to the minimization problem in (4.15) is the smallest generalized eigenvalue:

$$(4.16) \quad (\mathbf{B}^{PG})^\top (\mathbf{B}^{FE})^{-1} \mathbf{B}^{PG} \bar{\mathbf{w}} = \lambda \mathbf{B}^{IF} \bar{\mathbf{w}}.$$

Here, λ must be nonnegative as the matrix on the left is semi-SPD and the right one is SPD. Thus, we have $\eta_s = \sqrt{\lambda_{\min}}$ which can be used as a numerical estimate of the *inf-sup* constant. We refer readers to section 6 for the numerical results.

5. Approximation capabilities. In this section, we proceed to show the optimal approximation capabilities of the IFE spaces. Due to the interface conditions, it will be very different from the standard FE spaces in [30, 64]. Note that each subelement of an interface element can be highly irregular, and the desired error bounds should be independent of the irregular subelements, i.e., the generic constants are uniformly bounded regardless of how the interface cuts an element. This is one key difficulty in our analysis. In the following discussion, we shall employ the notation $x \lesssim y$ to represent the relation $x \leq Cy$, where C is a generic constant that is independent of the irregular subelements. As the nodal interpolation of the H^1 IFE space has been analyzed in [57], we shall concentrate on the $\mathbf{H}(\text{curl})$ and $\mathbf{H}(\text{div})$ cases here.

5.1. Some fundamental estimates. Note that there is a mismatching region sandwiched by Γ_h and Γ . We denote it as $K_{\text{int}} := (K^+ \cap K_h^-) \cup (K^- \cap K_h^+)$ and then define $\Omega_{\text{int}} := \cup_{K \in \mathcal{T}_h^i} K_{\text{int}}$. We introduce a δ -strip $S_\delta = \{\mathbf{x} : \text{dist}(\mathbf{x}, \Gamma) \leq \delta\}$ and refer readers to Figure 2.2 for illustration. Note that Lemma 5.3 below immediately implies that

$$(5.1) \quad \Omega_{\text{int}} \subset S_\delta \quad \text{for some } \delta \in (0, c_0 h^2].$$

We can control the L^2 -norm on the δ -strip by its width.

LEMMA 5.1 (see [58]). *It holds for any $z \in H_0^1(\Omega^\pm)$ that*

$$(5.2) \quad \|z\|_{L^2(S_\delta)} \leq C \sqrt{\delta} \|z\|_{H^1(\Omega)}.$$

Next, we recall the $\mathbf{H}^1(\text{curl}; \Omega)$ - and $\mathbf{H}^1(\text{div}; \Omega)$ -extension operators; see Theorem 3.4 and Corollary 3.5 in [47] and Theorem 3.2 in [46]. To simplify the discussion, we shall adopt a general differential operator $\mathbf{d} = \text{curl}$ or div .

THEOREM 5.2 (see [47, 46]). For $d = \text{curl}$ or div , there exist two bounded linear operators

$$(5.3) \quad \mathbf{E}_d^\pm : \mathbf{H}^1(d; \Omega^\pm) \rightarrow \mathbf{H}^1(d; \Omega),$$

such that for each $\mathbf{u} \in \mathbf{H}^1(d; \Omega^\pm)$,

1. $\mathbf{E}_d^\pm \mathbf{u} = \mathbf{u}$ a.e. in $\Omega^\pm$;
2. $\|\mathbf{E}_d^\pm \mathbf{u}\|_{\mathbf{H}^1(d; \Omega)} \leq C_E \|\mathbf{u}\|_{\mathbf{H}^1(d; \Omega^\pm)}$ with the constant C_E only depending on Ω .

With this theorem, we define $\mathbf{u}_{E,d}^\pm = \mathbf{E}_d^\pm \mathbf{u}^\pm$ which are the keys in the analysis. Furthermore, for any subdomain $D \subset \Omega$, we employ the following notation for simplicity:

$$(5.4) \quad \|\mathbf{u}\|_{E,d,s,D} = \|\mathbf{u}_{E,d}^+\|_{\mathbf{H}^s(d,D)} + \|\mathbf{u}_{E,d}^-\|_{\mathbf{H}^s(d,D)},$$

and $\|\mathbf{u}\|_{E,s,D}$ simply denotes the usual Sobolev norm without the curl or div terms.

Next, we introduce a patch ω_K of each interface element K :

$$(5.5) \quad \omega_K = \cup \{K' : \partial K' \cap \partial K \neq \emptyset\}.$$

Then, for each ω_K , define $\Gamma^{\omega_K} := \Gamma \cap \omega_K$ and $\Gamma_h^{\omega_K} := \hat{\Gamma}_h^K \cap \omega_K$, where $\hat{\Gamma}_h^K$ is the entire plane containing Γ_h^K . Namely, $\Gamma_h^{\omega_K}$ is the extension of Γ_h^K to the patch. The following geometric estimates regard the closeness between Γ^{ω_K} and $\Gamma_h^{\omega_K}$.

LEMMA 5.3. Suppose Γ is smooth enough such that φ^Γ is also smooth near the interface, and let the mesh size be sufficiently small. Denote $\mathbf{n}$ and $\bar{\mathbf{n}}$ as the normal vectors to Γ^{ω_K} and $\Gamma_h^{\omega_K}$ which have the same orientation. For each $\mathbf{x} \in \Gamma_h^{\omega_K}$, let $\mathbf{x}_\perp$ be the projection of $\mathbf{x}$ onto $\Gamma_h^{\omega_K}$. Then, for every interface element K and its patch ω_K , there holds that

$$(5.6) \quad \|\mathbf{x} - \mathbf{x}_\perp\| \leq c_{\Gamma,1} h_K^2 \quad \text{and} \quad \|\mathbf{n} - \bar{\mathbf{n}}\| \leq c_{\Gamma,2} h_K.$$

Proof. The proof simply follows from the similar argument in [56] by using interpolation on ω_K . $\square$

5.2. Special quasi-interpolation operators. To develop quasi-interpolation for handling the discontinuity of the IFE functions across the interface, by Table 3.1 and (3.1), let us rewrite the functions in $\mathbf{S}_h^e(K)$ in the following format:

$$(5.7) \quad \mathbf{v}_h = \begin{cases} \mathbf{v}_h^+ = \mathbf{a}_h^+ \times (\mathbf{x} - \mathbf{x}_K) + \mathbf{b}_h^+ & \text{in } K_h^+, \\ \mathbf{v}_h^- = \mathcal{C}_K^{\text{curl}}(\mathbf{v}_h^+) := (A_K \mathbf{a}_h^+) \times (\mathbf{x} - \mathbf{x}_K) + (B_K \mathbf{b}_h^+) & \text{in } K_h^-. \end{cases}$$

Similarly, the functions in $\mathbf{S}_h^f(K)$ can be represented as

$$(5.8) \quad \mathbf{v}_h = \begin{cases} \mathbf{v}_h^+ = c_0(\mathbf{x} - \mathbf{x}_K) + \mathbf{a}_h^+ & \text{in } K_h^+, \\ \mathbf{v}_h^- = \mathcal{C}_K^{\text{div}}(\mathbf{v}_h^+) := c_0(\mathbf{x} - \mathbf{x}_K) + (A_K \mathbf{a}_h^+) & \text{in } K_h^-. \end{cases}$$

The operators $\mathcal{C}_K^{\text{curl}}$ and $\mathcal{C}_K^{\text{div}}$ can be considered as discrete extensions mapping the polynomial component on the “+” side to the “−” side through the jump conditions. The “−” piece of the IFE functions can be completely determined by the “+” piece.

In addition, we also need an interpolation: $\mathcal{P}_{\omega_K}^d : \mathbf{H}(d; \omega_K) \rightarrow \tilde{\mathbf{S}}_h(K)$, where $\tilde{\mathbf{S}}_h(K) = \mathcal{ND}_0(K)$ if $d = \text{curl}$ or $\tilde{\mathbf{S}}_h(K) = \mathcal{RT}_0(K)$ if $d = \text{div}$ such that

$$(5.9) \quad \begin{aligned} \mathcal{P}_{\omega_K}^{\text{curl}} \mathbf{v} &:= (\Pi_{\omega_K}^0 \text{curl } \mathbf{v}) \times (\mathbf{x} - \mathbf{x}_K)/2 + \Pi_{\omega_K}^0 \mathbf{v}, \\ \mathcal{P}_{\omega_K}^{\text{div}} \mathbf{v} &:= (\Pi_{\omega_K}^0 \text{div } \mathbf{v})(\mathbf{x} - \mathbf{x}_K)/3 + \Pi_{\omega_K}^0 \mathbf{v}, \end{aligned}$$

where $\Pi_{\omega_K}^0$ is simply the projection onto $[\mathbb{P}_0(\omega_K)]^3$ or $[\mathbb{P}_0(\omega_K)]$ depending on the dimension. Note that $\mathcal{P}_{\omega_K}^d$ will be applied only to each Sobolev extension $\mathbf{u}_{E,d}^\pm$ which are smooth functions involving no discontinuity. The quasi-interpolation has the commutative diagram property:

$$(5.10) \quad d \circ \mathcal{P}_{\omega_K}^d = \Pi_{\omega_K}^0 \circ d,$$

which is illustrated in the diagram in Figure 5.1.

Now, with $\mathcal{C}_K^d$ and $\mathcal{P}_{\omega_K}^d$, we define the quasi-interpolation $J_K^d : \mathbf{H}(d; K) \rightarrow \mathbf{S}_h(K)$ where $\mathbf{S}_h(K) = \mathbf{S}_h^e(K)$ if $d = \text{curl}$ or $\mathbf{S}_h(K) = \mathbf{S}_h^f(K)$ if $d = \text{div}$, such that

$$(5.11) \quad J_K^d \mathbf{u} = \begin{cases} J_K^{d,+} \mathbf{u} = \mathcal{P}_{\omega_K}^d \mathbf{u}_{E,d}^+ & \text{in } K_h^+, \\ J_K^{d,-} \mathbf{u} = \mathcal{C}_K^d(\mathcal{P}_{\omega_K}^d \mathbf{u}_{E,d}^+) & \text{in } K_h^-. \end{cases}$$

Remark 5.4. Let us briefly explain the motivation for the quasi-interpolation in (5.11) by establishing a diagram in Figure 5.2. Suppose $\mathcal{P}_{\omega_K}^d(\mathbf{u}_{E,d}^+)$ has sufficient approximation to $\mathbf{u}_{E,d}^+$. Then, its discrete extension $\mathcal{C}_K^d(\mathcal{P}_{\omega_K}^d \mathbf{u}_{E,d}^+)$ is also expected to be a good approximation to $\mathbf{u}_{E,d}^-$. So the quasi-interpolation J_K^d behaves close to a Hermitian-type interpolation. The quasi-interpolation plays an important role in our analysis. In fact, roughly speaking, the robustness of the approximation capabilities of the IFE spaces with respect to irregular subelements is attributed to the fact that $\mathcal{C}_K^d$ can stably extend the polynomials from one subelement to another in the sense of $\|A_K\|_2 \lesssim 1$ and $\|B_K\|_2 \lesssim 1$.

Remark 5.4 suggests the following decomposition:

$$(5.12) \quad \mathbf{u}^- - J_K^{d,-} \mathbf{u} = (\mathbf{u}^- - \mathcal{P}_{\omega_K}^d \mathbf{u}_{E,d}^-) + (\mathcal{P}_{\omega_K}^d \mathbf{u}_{E,d}^- - J_K^{d,-} \mathbf{u}) := \boldsymbol{\xi}_{\mathbf{u}} + \boldsymbol{\eta}_{\mathbf{u}}.$$

Now we estimate each term above. Let us begin with $\boldsymbol{\xi}_{\mathbf{u}}$ in the following lemma.

LEMMA 5.5. For $\mathbf{w} \in \mathbf{H}^1(d; \omega_K)$, there holds

$$(5.13) \quad \|\mathbf{w} - \mathcal{P}_{\omega_K}^d \mathbf{w}\|_{\mathbf{H}(d; \omega_K)} \lesssim h_K \|\mathbf{w}\|_{\mathbf{H}^1(d; \omega_K)}.$$

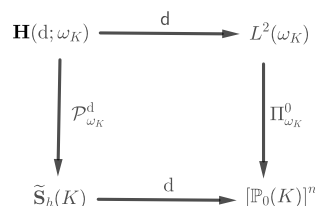


FIG. 5.1. The commutative diagram for $\mathcal{P}_{\omega_K}^d$.

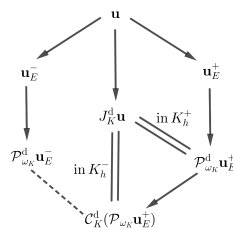


FIG. 5.2. The diagram for interpolation.

Proof. The estimate for the L^2 -norm directly follows from the approximation result of $\Pi_{\omega_K}^0$ and $\|\mathbf{x} - \mathbf{x}_K\| \lesssim h_K$ for $\mathbf{x} \in \omega_K$. The estimate for the $\mathbf{d}$ -seminorm additionally follows from (5.10). $\square$

The estimate of $\boldsymbol{\eta}_{\mathbf{u}}$ is included in the following lemma.

LEMMA 5.6. For $\mathbf{u} \in \mathbf{H}^1(\alpha, \beta, \text{curl}; \Omega)$ with $\mathbf{d} = \text{curl}$ and $\mathbf{u} \in \mathbf{H}^1(\alpha, \text{div}; \Omega)$ with $\mathbf{d} = \text{div}$,

$$(5.14) \quad \|\mathcal{P}_{\omega_K}^{\mathbf{d}} \mathbf{u}_{E, \mathbf{d}}^- - \mathcal{C}_K^{\mathbf{d}}(\mathcal{P}_{\omega_K}^{\mathbf{d}} \mathbf{u}_{E, \mathbf{d}}^+)\|_{\mathbf{H}(\mathbf{d}; \omega_K)} \lesssim h_K \|\mathbf{u}\|_{E, \mathbf{d}, 1, \omega_K}.$$

Proof. As the argument is lengthy and similar to the 2D case [41], for the sake of completeness, we put it in section SM3 of the supplementary materials. $\square$

Now, we are ready to estimate $J_K^{\mathbf{d}}$.

LEMMA 5.7. For $\mathbf{u} \in \mathbf{H}^1(\alpha, \beta, \text{curl}; \Omega)$ or $\mathbf{u} \in \mathbf{H}^1(\alpha, \text{div}; \Omega)$, there holds

$$(5.15) \quad \|\mathbf{u}_{E, \mathbf{d}}^{\pm} - J_K^{\mathbf{d}, \pm} \mathbf{u}\|_{\mathbf{H}(\mathbf{d}; \omega_K)} \lesssim h_K \|\mathbf{u}\|_{E, \mathbf{d}, 1, \omega_K}.$$

Proof. The result immediately follows from the definition (5.11) and applying the triangular inequality to (5.12) together with Lemmas 5.5 and 5.6. $\square$

5.3. Face and edge interpolation estimates. Lemma 5.7 only gives the local approximation capabilities. We then estimate the global spaces through the edge and face interpolations. Let us first present the following estimates for the bounds of the face and edge shape functions for which the proof immediately follows from Lemmas 3.2 and 3.3, respectively.

LEMMA 5.8. Given a trirectangular tetrahedral or regular tetrahedral interface element K , it holds that

$$(5.16) \quad \|\phi_i^f\|_{L^\infty(K)} \lesssim h_K^{-2} \quad \text{and} \quad \|\text{div} \phi_i^f\|_{L^\infty(K)} \lesssim h_K^{-3}, \quad i = 1, 2, \dots, 4,$$

$$(5.17) \quad \|\phi_i^e\|_{L^\infty(K)} \lesssim h_K^{-1} \quad \text{and} \quad \|\text{curl} \phi_i^e\|_{L^\infty(K)} \lesssim h_K^{-2}, \quad i = 1, 2, \dots, 6.$$

Let us briefly explain the idea of interpolation estimation. The key is to employ $J_K^{\mathbf{d}}$ as the bridge. Let I_h be I_h^e if $\mathbf{d} = \text{curl}$ or I_h^f if $\mathbf{d} = \text{div}$. Then, the following triangular inequality holds:

$$(5.18) \quad \|\mathbf{u} - I_h \mathbf{u}\|_{L^2(K)} \leq \|\mathbf{u} - J_K^{\mathbf{d}} \mathbf{u}\|_{L^2(K)} + \|J_K^{\mathbf{d}} \mathbf{u} - I_h \mathbf{u}\|_{L^2(K)}.$$

For the first term in the right-hand side of (5.18), Lemma 5.7 almost gives the desired result, but special attention must be paid to the different partitions of $\mathbf{u}$ and $J_K \mathbf{u}$ which are defined with Γ and Γ_h^K , respectively.

LEMMA 5.9. For $\mathbf{u} \in \mathbf{H}^1(\alpha, \beta, \text{curl}; \Omega)$ or $\mathbf{u} \in \mathbf{H}^1(\alpha, \text{div}; \Omega)$, there holds

$$(5.19) \quad \sum_{K \in \mathcal{T}_h^i} \|\mathbf{u} - J_K^{\mathbf{d}} \mathbf{u}\|_{L^2(K)} \lesssim h(\|\mathbf{u}\|_{\mathbf{H}^1(\mathbf{d}; \Omega^-)} + \|\mathbf{u}\|_{\mathbf{H}^1(\mathbf{d}; \Omega^+)}).$$

Proof. On each interface element K , we have

$$(5.20) \quad \|\mathbf{u} - J_K^{\mathbf{d}} \mathbf{u}\|_{L^2(K)} \leq \sum_{s=\pm} \|\mathbf{u}_{E, \mathbf{d}}^s - J_K^{\mathbf{d}, s} \mathbf{u}\|_{L^2(K_h^s)} + \|\mathbf{u}_{E, \mathbf{d}}^+ - \mathbf{u}_{E, \mathbf{d}}^-\|_{L^2(K_{\text{int}})},$$

where the estimate of the first term is readily given by Lemma 5.7. Summing (5.20) over all the interface elements and applying (5.1), Lemma 5.1, as well as Theorem 5.2, we have (5.19). $\square$

Next, we turn to the term $\|J_K^d \mathbf{u} - I_h \mathbf{u}\|_{L^2(K)}$ in (5.18). We first prepare the following estimate.

LEMMA 5.10. For $\mathbf{u} \in \mathbf{H}^1(\alpha, \beta, \text{curl}; \Omega)$ or $\mathbf{u} \in \mathbf{H}^1(\alpha, \text{div}; \Omega)$, given a δ -strip with $\delta \lesssim h^2$, there holds

$$(5.21) \quad \sum_{K \in \mathcal{T}_h^i} \|J_K^{d,\pm} \mathbf{u}\|_{L^2(S_\delta \cap \omega_K)} \lesssim h \|\mathbf{u}\|_{\mathbf{H}^1(d; \Omega^+)}.$$

Proof. For the “+” component, with $|S_\delta \cap \omega_K| \lesssim h_K^4$ and $\|\mathbf{x} - \mathbf{x}_K\|_{L^2(S_\delta \cap \omega_K)} \lesssim h_K^3$, we have

$$(5.22) \quad \begin{aligned} \|J_K^{d,+} \mathbf{u}\|_{L^2(S_\delta \cap \omega_K)} &\lesssim h_K^{3/2} \|\Pi_{\omega_K}^0 d\mathbf{u}_{E,d}^+\|_{L^2(\omega_K)} + h_K^{1/2} \|\Pi_{\omega_K}^0 \mathbf{u}_{E,d}^+\|_{L^2(\omega_K)} \\ &\lesssim h_K^{3/2} \|d\mathbf{u}_{E,d}^+\|_{L^2(\omega_K)} + h_K^{1/2} \|\mathbf{u}_{E,d}^+\|_{L^2(\omega_K)}, \end{aligned}$$

where we have used the boundedness property of the projection $\Pi_{\omega_K}^0$ in the last inequality. Then, the desired estimate follows from summing (5.22) over all the interface elements, using the finite overlapping of ω_K , and applying the δ -strip argument from Lemma 5.1 with $\delta \simeq h_K$ as well as Theorem 5.2. For the “−” side, by (3.2), we have the boundedness $\|A_K\| \lesssim 1$ and $\|B_K\| \lesssim 1$. Therefore, it also holds that

$$(5.23) \quad \|J_K^{d,-} \mathbf{u}\|_{L^2(S_\delta \cap \omega_K)} \lesssim h_K^3 \|\Pi_{\omega_K}^0 d\mathbf{u}_{E,d}^+\| + h_K^2 \|\Pi_{\omega_K}^0 \mathbf{u}_{E,d}^+\|$$

for which the estimate is the same as (5.22). $\square$

Now, with the estimate in Lemma 5.9, we proceed to show the second term in (5.18) and split the discussion for the edge and face elements.

5.3.1. Face interpolation. Let us first consider the face interpolation and fix $d = \text{div}$. We need an assumption to handle the mismatching portion of faces:

(A1) For each interface element $K \in \mathcal{T}_h^i$ and each face of F , define $\tilde{F} = F \cap K_{\text{int}}$.

Assume that there exists a δ -strip S_{δ_1} with $\delta_1 \simeq O(h^2)$ such that for every $K \in \mathcal{T}_h^i$, there is a pyramid $P_{\tilde{F}} \subset \omega_K \cap S_{\delta_1}$ with $\tilde{F}$ as its base satisfying that the height $l_{\tilde{F}}$ of $P_{\tilde{F}}$ supporting $\tilde{F}$ is $O(h_K)$.

We refer readers to Figure SM4.2 in the supplementary materials for an illustration of (A2).

THEOREM 5.11. For $\mathbf{u} \in \mathbf{H}^1(\alpha, \text{div}; \Omega)$, under assumption (A1), there holds

$$(5.24) \quad \|\mathbf{u} - I_h^f \mathbf{u}\|_{L^2(\Omega)} \lesssim h(\|\mathbf{u}\|_{\mathbf{H}^1(\text{div}; \Omega^-)} + \|\mathbf{u}\|_{\mathbf{H}^1(\text{div}; \Omega^+)}).$$

Proof. We only consider interface elements. Given $K \in \mathcal{T}_h^i$, by (5.18) and Lemma 5.9, it remains to estimate $\|J_K^d \mathbf{u} - I_h^f \mathbf{u}\|_{L^2(K)}$. We need an auxiliary function $\mathbf{w}$ such that $\mathbf{w}^\pm = J_K^{d,\pm} \mathbf{u} - \mathbf{u}_{E,d}^\pm$ in $K^\pm$, i.e., it is partitioned with Γ . Note that $\mathbf{w}$ is slightly different from $J_K^d \mathbf{u} - \mathbf{u}$ as $J_K^d \mathbf{u}$ is defined with Γ_h^K , and thus this difference is only on K_{int} . To control the mismatch, by assumption (A1) and the trace inequality for polynomials [74], we have

$$(5.25) \quad \|J_K^{d,+} \mathbf{u} - J_K^{d,-} \mathbf{u}\|_{L^2(\tilde{F})} \lesssim h_K^{-1/2} (\|J_K^{d,+} \mathbf{u}\|_{L^2(S_{\delta_1} \cap \omega_K)} + \|J_K^{d,-} \mathbf{u}\|_{L^2(S_{\delta_1} \cap \omega_K)}).$$

Note that each component of $\mathbf{w}^\pm$ can be naturally extended to the whole element and whole patch. As $\mathbf{w}^\pm \in \mathbf{H}^{1/2}(F)$ for each face of F , by the trace inequality and Lemma 5.7 we have

$$(5.26) \quad \begin{aligned} \|\mathbf{w}^\pm\|_{L^2(F^\pm)} &\lesssim \|\mathbf{w}^\pm\|_{L^2(F)} \\ &\lesssim h_K^{-1/2} \|\mathbf{w}\|_{L^2(K)} + h_K^{1/2} \|\mathbf{w}\|_{H^1(K)} \lesssim h_K^{1/2} \|\mathbf{u}\|_{E, \text{div}, 1, \omega_K}. \end{aligned}$$

Now, by the definition of the face interpolation, we obtain

$$\begin{aligned} \|J_K^d \mathbf{u} - I_h^f \mathbf{u}\|_{L^2(K)} &= \|I_h^f(J_K^d \mathbf{u} - \mathbf{u})\|_{L^2(K)} \\ &\lesssim \|\phi_i^f\|_{L^2(K)} \left[\sum_{i=1}^4 |F_i|^{1/2} (\|\mathbf{w}^+ \cdot \mathbf{n}\|_{L^2(F_i)} + \|\mathbf{w}^- \cdot \mathbf{n}\|_{L^2(F_i)}) \right. \\ &\quad \left. + |\tilde{F}_i|^{1/2} \|J_K^{d,+} \mathbf{u} - J_K^{d,-} \mathbf{u}\|_{L^2(\tilde{F}_i)} \right] \\ &\lesssim h_K \|\mathbf{u}\|_{E, \text{div}, 1, \omega_K} + (\|J_K^{d,+} \mathbf{u}\|_{L^2(S_{\delta_1} \cap \omega_K)} + \|J_K^{d,-} \mathbf{u}\|_{L^2(S_{\delta_1} \cap \omega_K)}), \end{aligned}$$

where we have used (5.25), (5.26), and Lemma 5.8. Summing the estimates above over all the elements, using the finite overlapping property of ω_K , and applying Lemma 5.9 and Theorem 5.2 yields the desired result. $\square$

The estimate of $\text{div}(\mathbf{u} - I_h^f \mathbf{u})$ is trivial as $\text{div} I_h^f \mathbf{u} = \Pi_0 \text{div} \mathbf{u}$ by Lemma 3.7. At last, we present the following estimate.

THEOREM 5.12. *For $\mathbf{u} \in \mathbf{H}^1(\alpha, \text{div}; \Omega)$, there holds*

$$(5.27) \quad \|\text{div}(\mathbf{u} - I_h^f \mathbf{u})\|_{L^2(\Omega)} \lesssim h(\|\mathbf{u}\|_{\mathbf{H}^1(\text{div}; \Omega^-)} + \|\mathbf{u}\|_{\mathbf{H}^1(\text{div}; \Omega^+)}).$$

5.3.2. Edge interpolation. Next, we proceed to estimate the approximation capabilities of $\mathbf{S}_h^e$ through I_h^e . We need a similar assumption to (A1).

(A2) For each interface element $K \in \mathcal{T}_h^i$ and each edge e , define $\tilde{e} = e \cap K_{\text{int}}$, i.e., the portion of the edge contained in the mismatching region K_{int} . Assume that there exists a δ -strip S_{δ_2} with $\delta_2 \simeq O(h^2)$ such that for every $K \in \mathcal{T}_h^i$, there is a tetrahedron $T_{\tilde{e}} \subset \omega_K \cap S_{\delta_2}$ with $\tilde{e}$ as one of its edges satisfying $|T_{\tilde{e}}| \gtrsim |\tilde{e}|^2 h_K$.

Also see Figure SM4.2 in the supplementary materials for illustration. In the following discussion, we fix $d = \text{curl}$.

THEOREM 5.13. *For $\mathbf{u} \in \mathbf{H}^1(\alpha, \beta, \text{curl}; \Omega)$, under assumption (A3), there holds*

$$(5.28) \quad \|\mathbf{u} - I_h^e \mathbf{u}\|_{L^2(\Omega)} \lesssim h(\|\mathbf{u}\|_{\mathbf{H}^1(\text{curl}; \Omega^-)} + \|\mathbf{u}\|_{\mathbf{H}^1(\text{curl}; \Omega^+)}).$$

Proof. We still only consider interface elements. Similar to the argument of Theorem 5.11, we only need to estimate $\|J_K^d \mathbf{u} - I_h^e \mathbf{u}\|_{L^2(K)}$. Define the same auxiliary function $\mathbf{w}$ partitioned with Γ : $\mathbf{w}^\pm = J_K^{d,\pm} \mathbf{u} - \mathbf{u}_{E,d}^\pm$ in $K^\pm$. Let us estimate $J_K^{d,+} \mathbf{u} - J_K^{d,-} \mathbf{u}$ on the small subedge $\tilde{e}$. By assumption (A3) and the trace inequality for polynomials [74], we obtain

$$(5.29) \quad \|J_K^{d,+} \mathbf{u} - J_K^{d,-} \mathbf{u}\|_{L^2(\tilde{e})} \lesssim h_K^{-1/2} |\tilde{e}|^{-1/2} (\|J_K^{d,+} \mathbf{u}\|_{L^2(S_{\delta_2} \cap \omega_K)} + \|J_K^{d,-} \mathbf{u}\|_{L^2(S_{\delta_2} \cap \omega_K)}).$$

In addition, as $\mathbf{u}_{E,d}^\pm \in \mathbf{H}^1(\text{curl}; \Omega)$, the first moment $\int_e \mathbf{u}_{E,d}^\pm \cdot \mathbf{t} ds$ is well defined on each edge e . Furthermore, we have $\mathbf{u}_{E,d}^\pm \in \mathbf{H}^{1/2}(F) \cap \mathbf{H}(\text{rot}; F)$ on each face F . Thus, by Lemma 4.4 in [8] and the trace inequality, we obtain for each face F of K and any edge e that

$$\begin{aligned} (5.30) \quad \|\mathbf{w}^\pm \cdot \mathbf{t}\|_{L^1(e^\pm)} &\lesssim \|\mathbf{w}^\pm \cdot \mathbf{t}\|_{L^1(e)} \\ &\lesssim \|\mathbf{w}^\pm\|_{L^2(F)} + h_F^{1/2} \|\mathbf{w}^\pm\|_{H^{1/2}(F)} + h_F \|\text{rot}_F \mathbf{w}^\pm\|_{L^2(F)} \\ &\lesssim h_K^{-1/2} \|\mathbf{w}^\pm\|_{L^2(K)} + h_K^{1/2} \|\mathbf{w}^\pm\|_{H^1(K)} \\ &\quad + h_K^{1/2} \|\text{curl} \mathbf{w}^\pm\|_{L^2(K)} + h_K^{3/2} \|\text{curl} \mathbf{w}^\pm\|_{H^1(K)} \lesssim h_K^{1/2} \|\mathbf{u}\|_{E, \text{curl}, 1, \omega_K}, \end{aligned}$$

where we have used Lemma 5.7 in the fourth inequality with the following estimates:

$$\begin{aligned}\|\mathbf{w}^\pm\|_{H^1(K)} &\lesssim \|\mathbf{u}_{E,d}^\pm\|_{H^1(K)} + \|\operatorname{curl} J_K^{\mathbf{d},\pm} \mathbf{u}\|_{L^2(K)} \lesssim \|\mathbf{u}\|_{E,\operatorname{curl},1,\omega_K}, \\ \|\operatorname{curl} \mathbf{w}^\pm\|_{H^1(K)} &\lesssim \|\operatorname{curl} \mathbf{u}_{E,d}^\pm\|_{H^1(K)} \lesssim \|\mathbf{u}\|_{E,\operatorname{curl},1,\omega_K}.\end{aligned}$$

Then, it follows from the definition of $J_K^{\mathbf{d}}$ together with (5.29), (5.30), and Lemma 5.8 that

$$\begin{aligned}(5.31) \quad \|J_K^{\mathbf{d}} \mathbf{u} - I_h^e \mathbf{u}\|_{L^2(K)} &= \|I_h^e(J_K^{\mathbf{d}} \mathbf{u} - \mathbf{u})\|_{L^2(K)} \\ &\leq \left(\sum_{i=1}^6 \int_{e_i} |\mathbf{w} \cdot \mathbf{t}| \, ds + |\tilde{e}_i|^{1/2} \|J_K^{\mathbf{d},+} \mathbf{u} - J_K^{\mathbf{d},-} \mathbf{u}\|_{L^2(\tilde{e}_i)} \right) \|\phi_i^e\|_{L^2(K)} \\ &\lesssim h_K \|\mathbf{u}\|_{E,\operatorname{curl},1,\omega_K} + (\|J_K^{\mathbf{d},+} \mathbf{u}\|_{L^2(S_{\delta_2 \cap \omega_K})} + \|J_K^{\mathbf{d},-} \mathbf{u}\|_{L^2(S_{\delta_2 \cap \omega_K})}).\end{aligned}$$

Summing (5.31) over all the interface elements, using the finite overlapping property of ω_K , and applying Lemma 5.9 and Theorem 5.2 leads to the desired estimate. $\square$

Remark 5.14. One key in the proof is to use assumption (A3) with the trace inequality for polynomials to control $J_K^{\mathbf{d},\pm} \mathbf{u}$ on edges. If this is done for $\mathbf{u}_{E,d}^\pm$, one must be careful about bounding $\|\mathbf{u}_{E,d}\|_{L^2(\tilde{e})}$ since it may not be well-defined for $\mathbf{u} \in \mathbf{H}^1(\omega_K)$.

Finally, as $\operatorname{curl} I_h^e \mathbf{u} = I_h^f \operatorname{curl} \mathbf{u}$, Theorem 5.11 immediately yields the estimate for $\operatorname{curl}(\mathbf{u} - I_h^e \mathbf{u})$, $\mathbf{u} \in \mathbf{H}^1(\alpha, \beta, \operatorname{curl}; \Omega)$.

THEOREM 5.15. *Under assumption (A3), for $\mathbf{u} \in \mathbf{H}^1(\alpha, \beta, \operatorname{curl}; \Omega)$, there holds*

$$(5.32) \quad \|\operatorname{curl}(\mathbf{u} - I_h^e \mathbf{u})\|_{L^2(\Omega)} \lesssim h(\|\mathbf{u}\|_{\mathbf{H}^1(\operatorname{curl}; \Omega^-)} + \|\mathbf{u}\|_{\mathbf{H}^1(\operatorname{curl}; \Omega^+)}).$$

6. Numerical experiments. In this section, we present a group of numerical experiments to demonstrate the performance of the proposed method. We fix the cubic domain to be $\Omega = (-1, 1)^3$ but with varying interface shapes. The background Cartesian mesh is generated by N^3 uniform cubes, each of which is partitioned into tetrahedra. In [41] for the 2D case, numerical experiments were provided to show that the standard Galerkin methods with or without penalties do not converge. Similar behavior can also be observed in the 3D case, and here we omit their comparison.

6.1. Test 1 (stability). In the first test, we study the numerical stability of the method with respect to the mesh size and the interface location. We first provide a group of numerical estimates for the quantity η_s in (4.10) indicating a lower bound of the *inf-sup* constant. We consider the spherical interface shape shown in the middle plot of Figure 6.1 for the three groups of parameters:

$$(6.1) \quad \left(\frac{\max(\beta^+, \beta^-)}{\min(\beta^+, \beta^-)}, \frac{\max(\alpha^+, \alpha^-)}{\min(\alpha^+, \alpha^-)} \right) = (200, 100), (200, 1000), (2000, 100),$$

where the minimal values are always set as 1; see the detailed parameter illustration in the right corner of plots in Figure 6.2. Then, we consider a sequence of meshes with $N = 10, 20, 30, 40, 50, 60, 70$. We plot the values of η_s in Figure 6.2. Here, the computational results for the first two groups of parameters are presented in the first two plots, but for the third group, we separate the parameter $(\beta^+, \alpha^-) = (2000, 100)$ in a single plot to show more details. Overall, we can see that η_s is bounded below. More specifically, we can make the following numerical observations:

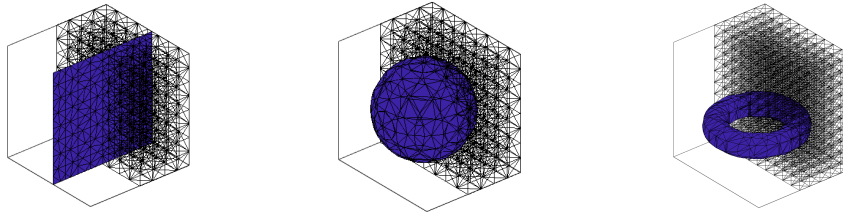
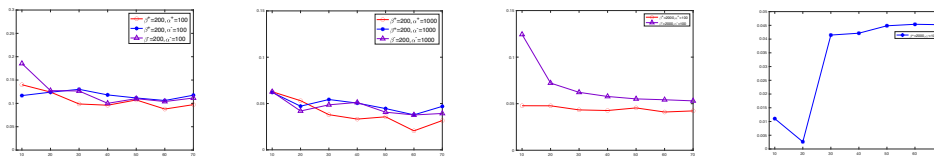


FIG. 6.1. A flat interface (left), a spherical interface (middle), and a torus interface (right).

FIG. 6.2. Estimates of η_s for various α , β and mesh sizes.TABLE 6.1
Condition numbers for the flat interface with various ϵ .

ϵ	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
$\text{cond}(\mathbf{B}^{PG})$	88267	83052	78834	84570	78253	81324

- η_s slightly decreases at the beginning, and then the curve proceeds to be flat as the mesh becomes finer.
- η_s highly depends on the contrast of the parameters, and the larger contrast leads to smaller η_s . Particularly, the values decrease from the first to the third group of parameters.
- For the third group where the parameters have the ratio (2000, 100), when $(\alpha^-, \beta^+) = (100, 2000)$, $\eta_s \approx 2.6 \times 10^{-3}$ for the mesh $N = 20$, which is indeed close to zero. But it turns back to around 0.05 for finer meshes.

These observations are consistent with the analysis in two dimensions (cf. [41]) that the *inf-sup* condition only holds for sufficiently small mesh size.

Next, we test the condition number of $\mathbf{B}^{PG}$ with respect to various interface locations. Let us fix mesh size $N = 10$ and choose $(\alpha^-, \alpha^+) = (1, 100)$, $(\beta^-, \beta^+) = (1, 200)$. Consider a flat interface $(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{n} = 0$ passing through $\mathbf{x}_0 = [\epsilon, 0, 0]$ with $\epsilon = 10^{-j}$, $j = 1, 2, \dots, 6$ and the normal vector $\mathbf{n} = [1, 0, 0]^\top$ such that the interface gradually moves to one face/edge of interface elements. Thus, small subelements exist for every interface element. The condition numbers, estimated through the `condtest` command in MATLAB, are shown in Table 6.1 for various ϵ . In fact, as the interface element shrinks to a noninterface element, the IFE functions will converge to standard FE functions, which is “good” for computation.

6.2. Test 2 (optimal convergence). In the second test, we investigate the optimal convergence rates with respect to the mesh size and speed of the fast solver. Consider a spherical interface with a radius r_1 shown in the left plot of Figure 6.1. We still employ (1.3) as the model equation and use the benchmark example from [41] in which the analytical solution is given by

$$(6.2) \quad \mathbf{u} = \begin{cases} \frac{1}{\beta^-} \mathbf{x} + \frac{1}{\alpha^-} n_1 R_1(\mathbf{x}) [(x_2 - x_3), (x_3 - x_1), (x_1 - x_2)]^\top & \text{in } \Omega^-, \\ \frac{1}{\beta^+} \mathbf{x} + \frac{1}{\alpha^+} n_2 R_1(\mathbf{x}) R_2(\mathbf{x}) [(x_2 - x_3), (x_3 - x_1), (x_1 - x_2)]^\top & \text{in } \Omega^+, \end{cases}$$

where $\mathbf{x} = [x_1, x_2, x_3]^\top$, $R_1(\mathbf{x}) = r_1^2 - \|\mathbf{x}\|^2$, $R_2(\mathbf{x}) = r_2^2 - \|\mathbf{x}\|^2$, and $n_1 = n_2(r_2^2 - r_1^2)$. We let $(\alpha^-, \alpha^+) = (\beta^-, \beta^+) = (1, 100)$ and $(1, 1000)$, and denote $\rho = \alpha^+ / \alpha^- = \beta^+ / \beta^-$. We present the numerical results for $N = 10, \dots, 70$ in Figure 6.3 which clearly show the optimal convergence rates.

To solve the linear system, we employ the HX preconditioner described in section 4.2 with both GMRES and CG iteration strategies. It involves a direct solver on a smaller matrix for which the size depends on the expanding width l , i.e., the size of $\mathbf{B}_l^{(2,2)}$ in (4.5). But the block diagonal smoother in (4.6) can help in greatly reducing the iteration steps for convergence, and consequently significantly reduce the computational time. In fact, as l is small, in practice we store the LU factorization of $\mathbf{B}_l^{(2,2)}$ to avoid repeatedly computing the inverse. In Table 6.2, we present both the number of iterations and computational time for GMRES. It is noteworthy that the preconditioning strategy even works for a CG method which in general does not converge for non-SPD matrices. The results are presented in Table 6.3, and we can observe that the CG method must need $l = 1$ at least for convergence. But once the block diagonal smoother in (4.6) is used, the CG method has better performance than GMRES. We may refer to our detailed discussions in section 4.2 and point out again the important fact that the majority portion of $\mathbf{B}^{PG}$ is symmetric, and the problematic portion is only around the interface which can be well handled by the direct inverse $(\mathbf{B}_l^{(2,2)})^{-1}$. Overall, for both methods, we observe that the expanding width l depends on the contrast ρ . For a larger contrast, there needs to be a relatively larger l to ensure fast convergence. With an appropriately chosen expanding

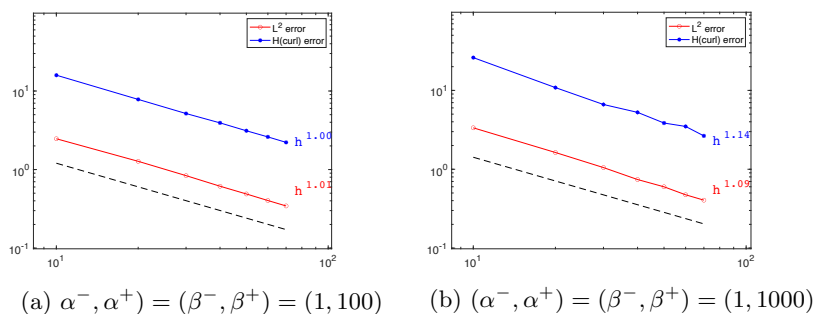


FIG. 6.3. Errors for Test 2 where the dashed line is the reference line indicating $\mathcal{O}(h)$. Left: $(\alpha^-, \alpha^+) = (\beta^-, \beta^+) = (1, 100)$. Right: $(\alpha^-, \alpha^+) = (\beta^-, \beta^+) = (1, 1000)$.

TABLE 6.2
Iteration numbers and computational time for the GMRES solver.

# DoFs	7930	59660	197190	462520	897650	1544580	2445310
$\rho = 100, l = 0$	63(2.6s)	66(14s)	66(29s)	67(51s)	68(108s)	68(191s)	68(283s)
$\rho = 100, l = 1$	35(2.1s)	40(7.9s)	39(14s)	40(28s)	40(56s)	38(100s)	41(147s)
$\rho = 1000, l = 0$	212(8.9s)	244(60s)	263(117s)	223(185s)	246(404s)	272(817s)	234(1017s)
$\rho = 1000, l = 1$	55(8.4s)	63(13.7s)	76(33s)	48(35s)	46(70s)	52(143s)	54(243s)
$\rho = 1000, l = 2$	42(1.9s)	47(11.2s)	46(21s)	52(39s)	46(83s)	51(144s)	51(209s)
$\rho = 1000, l = 3$	41(2.2s)	39(11s)	42(20s)	39(40s)	42(75s)	42(135s)	42(191s)

TABLE 6.3
Iteration numbers and computational time for the CG solver.

# DoFs	7930	59660	197190	462520	897650	1544580	2445310
$\rho = 100, l = 0$	—	84(18s)	78(29s)	—	—	—	—
$\rho = 100, l = 1$	37(1.6s)	39(9.8s)	40(18s)	41(30s)	42(56s)	42(99s)	43(140s)
$\rho = 100, l = 2$	36(1.3s)	39(10s)	41(17s)	42(32s)	42(66s)	42(110s)	42(170s)
$\rho = 1000, l = 0$	—	—	—	—	—	—	—
$\rho = 1000, l = 1$	40(1.4s)	42(17s)	42(30s)	42(30s)	43(59s)	45(100s)	43(140s)
$\rho = 1000, l = 2$	37(1.4s)	39(8.4s)	40(18s)	41(29s)	42(61s)	44(110s)	42(150s)

width, the algorithm converges in a small number of iterations, which indeed takes significantly less computational time.

6.3. Test 3 (application to a time-domain Maxwell equation). In this test, we consider the scattering problem modeled by (1.2) in a time domain $[0, T]$. We partition it into M subintervals: $t_0 = 0 < t_1 < t_2 < \cdots < t_M = T$ with equal length $\tau = T/M$. We then apply the implicit time-discretization method introduced in [30] which reads as

$$(6.3) \quad (\epsilon \partial_\tau^2 \mathbf{u}_h^n, \mathbf{v}_h)_\Omega + (\sigma \partial_\tau \mathbf{u}_h^n, \mathbf{v}_h)_\Omega + (\mu^{-1} \operatorname{curl} \mathbf{u}_h^n, \operatorname{curl} \mathbf{v}_h)_\Omega = (\mathbf{J}_t, \mathbf{v}_h)_\Omega \quad \forall \mathbf{v}_h \in \mathbf{S}_h^e,$$

where, for $n \geq 2$, $\partial_\tau \mathbf{u}_h^n = (\mathbf{u}_h^n - \mathbf{u}_h^{n-1})/\tau$ and $\partial_\tau^2 \mathbf{u}_h^n = (\partial_\tau \mathbf{u}_h^n - \partial_\tau \mathbf{u}_h^{n-1})/\tau$. Then, at each time step it can be discretized by the PG-IFE scheme with the parameters $\alpha = \mu^{-1}$ and $\beta = \epsilon\tau^{-2} + \sigma\tau^{-1}$ and solved with the proposed fast solver. Here, we consider a simulation problem of electromagnetic waves propagating through a torus where the analytical solution is difficult, if not impossible, to derive, but instead we compare the results of the IFE method and the standard FE method on fitted meshes.

The torus interface shown in the right plot of Figure 6.1 is described by a level-set signed-distance function:

$$\gamma(\mathbf{x}) = (\sqrt{(x_1 - z_1)^2 + (x_2 - z_2)^2} - r_2)^2 + (x_3 - z_3)^2 - r_1^2,$$

where $z_1 = z_2 = 0$, $z_3 = -0.3$ and $r_1 = 0.2$, $r_2 = \pi/5$. We further set the medium parameters as $\epsilon^+ = 0.05$, $\epsilon^- = 2\epsilon^+$, $\sigma^+ = 0.1$, $\sigma^- = 1$, and $\mu^+ = 4\pi$, $\mu^- = 3\mu^+$. Initially, we impose a Gaussian pulse

$$(6.4) \quad \mathbf{u} = \exp(-b(a(x_1 - z_0) - \omega t)^2)[0, 1, 0]^\top,$$

where $a = \omega\sqrt{\epsilon^+\mu^+}$ modeling the speed of the electromagnetic wave. In addition, the boundary condition is set to match (6.4) except on the faces at $\mathbf{x} = -1, 1$ where it is set as $\mathbf{0}$. Set the source as $\mathbf{J} = \mathbf{0}$, let the time domain be $[0, 1.5]$, and let the step size be $\tau = 1.5/128$. The IFE method is computed on an unfitted mesh containing 1,872,064 DoFs, while the FE method is computed on a fitted mesh containing 2,002,947 DoFs.

To compare the simulation results, we first plot the evolution of the x_2 component at some slices in Figure 6.4 at $t = 12\tau$, $t = 24\tau$, $t = 36\tau$, and $t = 48\tau$. The results of the IFE and FE methods are almost identical, demonstrating the effectiveness of the proposed method. In addition, we can clearly observe the delayed electromagnetic wave inside the torus agreeing with the physics. Next, we plot the electromagnetic fields inside the torus in Figure 6.5 such that we can also examine the direction. The plots still clearly show that the two methods have very similar simulation results.

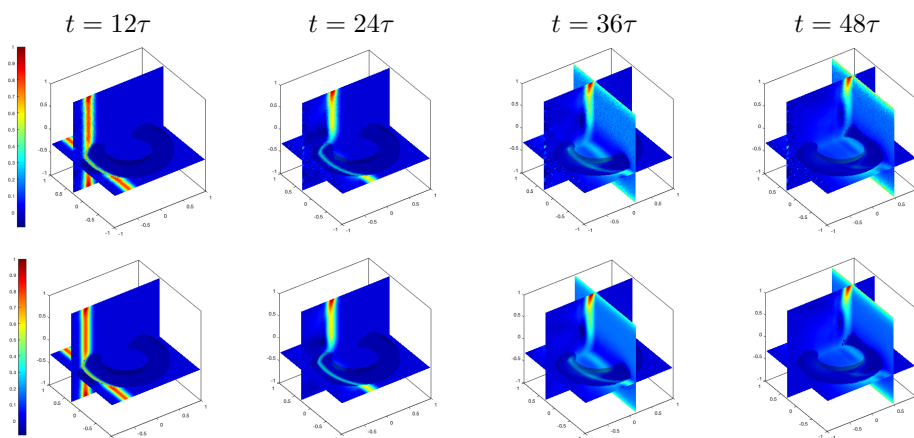


FIG. 6.4. Comparison of FE (top) and IFE (bottom) solutions at slices $x_1 = 5/32$, $x_2 = 5/16$, $x_3 = -5/16$.

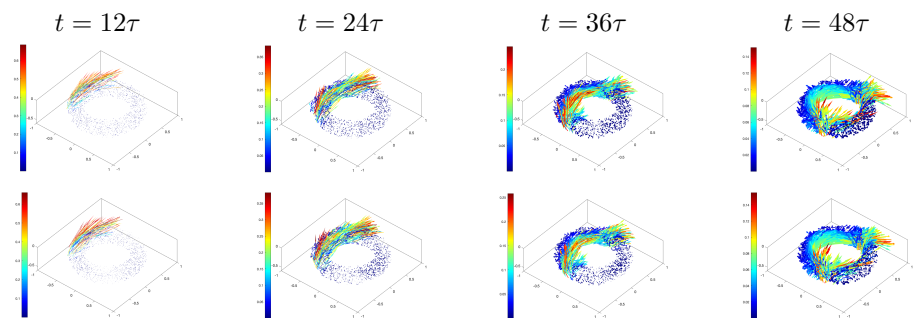


FIG. 6.5. Comparison of FE (top) and IFE (bottom) solutions of vector fields.

7. Concluding remarks. In this work, we have proposed an IFE method in a Petrov–Galerkin scheme for solving a $\mathbf{H}(\text{curl})$ interface system. This type of interface problem is much more challenging than the usual H^1 interface problems due to the low regularity. The proposed method can overcome the issue of suboptimal convergence caused by the nonconforming approximation spaces and the widely used penalty techniques. Nevertheless, it is also worthwhile to mention that, for the H^1 elliptic interface problems, some suitable penalties may help in achieving numerical solutions robust to the high contrast of the discontinuous coefficients; see the discussion in [16, 29, 44, 55]. However, to the best of our knowledge, we are not aware of any work that deals with high-contrast Maxwell-type interface problems. It is indeed interesting and critical to study those problems as they widely appear in practice. Note that in some situations β may vanish on one medium but may not vanish on another medium.

Another contribution of this work is to develop an efficient preconditioner for the considered lowest-order case. We point out that it can be more challenging to develop preconditioners for high-order FE methods. Existing works can be found for solving Maxwell equations with standard FE methods, say [70, 72, 11, 4]. However, there seems no result in the literature for unfitted mesh methods.

Developing such an efficient solver is of great importance, especially for the high-frequency case [10].

Reproducibility of computational results. This paper has been awarded the “SIAM Reproducibility Badge: code and data available”, as a recognition that the authors have followed reproducibility principles valued by SISC and the scientific computing community. Code and data that allow readers to reproduce the results in this paper are available in <https://github.com/lyc102/ifem> OR in the Supplementary Materials.

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