Family of Quantum Codes with Exotic Transversal Gates

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Recently, an algorithm has been constructed that shows that the binary icosahedral group 2I together with a T-like gate forms the most efficient single-qubit universal gate set. To carry out the algorithm fault tolerantly requires a code that implements 2I transversally. However, no such code has ever been demonstrated in the literature. We fill this void by constructing a family of distance $d = 3$ codes that all implement 2I transversally. A surprising feature of this family is that the codes can be deduced entirely from symmetry considerations that only 2I affords.

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Introduction.—Let $((n, K, d))$ denote an n-qubit quantum error-correcting code with a code space of dimension K and distance d. The Eastin-Knill theorem [1] shows that when a code is nontrivial ($d \geq 2$), the logical operations in $\mathrm{SU}(K)$ that can be implemented transversally are always a *finite* subgroup $\mathsf{G} \subset \mathrm{SU}(K)$. A logical gate g is called transversal if g can be implemented as $U_1 \otimes \cdots \otimes U_n$, where each $U_i \in \mathrm{U}(2)$. Transversal gates are considered naturally fault tolerant because they do not propagate errors between physical qubits.

Our focus will be on encoding a single logical qubit into n physical qubits ($K = 2$). In this case, the Eastin-Knill theorem shows that the transversal gates must be a finite subgroup of SU(2). The finite subgroups of SU(2) are the cyclic groups, the dicyclic groups, and three exceptional groups. We are primarily interested in the three exceptional groups: the binary tetrahedral group 2T, the binary octahedral group 2O, and the binary icosahedral group 2I. These three groups correspond to the lift through the double cover $\mathrm{SU}(2) \to \mathrm{SO}(3)$ of the symmetry groups of the tetra-, octa-, and icosahedron, respectively (see Fig. 1). For more information on the finite subgroups of SU(2), see the Supplemental Material [2].

The group 2O is better known as the single-qubit Clifford group C. Many codes implement 2O transversally, for example, the $[[7, 1, 3]]$ Steane code and the $[[2^{2r-1} - 1, 1, 2^r - 1]]$ quantum punctured Reed-Muller codes. More generally, all doubly even self-dual CSS codes implement 2O transversally. The group 2T is a subgroup of the Clifford group and there are also many codes with transversal gate group 2T, the most famous example being the $[[5, 1, 3]]$ code.

In stark contrast, no code has ever been explicitly demonstrated to implement 2I transversally. This omission is particularly glaring given the role 2I plays in the "optimal absolute super golden gate set" proposed in [32] as the best single-qubit universal gate set.

Super golden gates: A single-qubit universal gate set is a finite collection of gates that generates a dense subset of SU(2). The Solovay-Kitaev theorem [33] says that a universal gate set can approximate any gate in SU(2) up to some ϵ precision using at most $\mathcal{O}[\log^c(1/\epsilon)]$ gates for some constant c (see [34–36] for bounds on c). Roughly, given a universal gate set, we can approximate any single-qubit gate using a relatively small number of gates.

In the context of fault tolerance, we usually think of a universal gate set as $\mathsf{G} + \tau$, where G is a finite group of gates considered "cheap" to implement and τ is a single gate outside the group, which is considered "expensive." This abstractly models how magic state distillation [3] works in practice; G is a set of transversal gates for some code (and so naturally fault tolerant) and τ is a gate that must be "simulated" using magic states, distillation, and teleportation and is usually quite costly to implement (cf. [37,38]).

A "super golden gate set" [32] is a universal gate set $\mathsf{G} + \tau$ that possesses optimal navigation properties and minimizes the number of expensive τ gates that are used (see Secs. 2.2 and 2.3 of [39] for a precise definition). We already know from the Solovay-Kitaev theorem that the total number of gates in any approximation will be small, but a super golden gate set in addition guarantees there will not be too many expensive τ gates. There are only finitely many super golden gate sets, including one for each

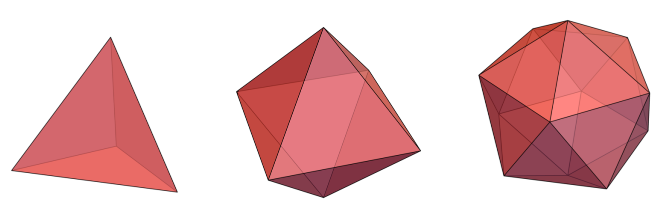

FIG. 1. From left to right, the following Platonic solids: tetrahedron, octahedron, icosahedron.

of the symmetry groups of the platonic solids shown in Fig. 1.

The most familiar example of a super golden gate set is Clifford $+ T$ or, equivalently, $2O + T$. Here T is the square root of the phase gate (also known as the $\pi/8$ gate). For this gate set, Clifford operations are indeed cheap since there are many codes that can implement them transversally, e.g., the $[[7, 1, 3]]$ Steane code. Implementing the T gate fault tolerantly is standard in the magic state literature. As of writing, the best navigation algorithm for $2O + T$ can efficiently factor any gate in SU(2) to within ϵ precision using at most $\frac{7}{3}\log_2(1/\epsilon^3)$ expensive T gates (see Theorem 1 of [40]).

Another example of a super golden gate set is $2I + \tau_{60}$ defined in [32]. Here, the cheap gates form the group 2I, while the expensive gate is called τ_{60}. It is defined as

$$\tau_{60} \coloneqq \frac{1}{\sqrt{5\varphi + 7}} \begin{pmatrix} i(2+\varphi) & 1+i \\ -1+i & -i(2+\varphi) \end{pmatrix}, \quad (1)$$

where $\varphi = (1 + \sqrt{5})/2$ denotes the golden ratio. The best navigation algorithm for $2I + \tau_{60}$ can efficiently factor any gate in SU(2) to within ϵ precision using at most $\frac{7}{3}\log_{59}(1/\epsilon^3)$ expensive τ_{60} gates (see Theorem 1 of [39]).

Notice that the only difference between the number of τ gates in these two cases is in the base of the logarithm (which is related to the structure of the super golden gate set). Since $\log_2(x) = \log_2(59)\log_{59}(x)$, using the universal gate set $2I + \tau_{60}$, instead of $2O + T$, gives a $\log_2(59) \approx 5.9$ times reduction in the number of expensive τ gates (in the worst case).

For example, if we want to approximate any gate in SU(2) up to a precision of $\epsilon = 10^{-10}$, then $2O + T$ would need at most 233 T gates, whereas $2I + \tau_{60}$ would only need at most 40 τ_{60} gates. Out of all of the super golden gate sets, $2I + \tau_{60}$ has the largest logarithm base and so it is optimal.

Summary of results: A practical implementation of the $2I + \tau_{60}$ super golden gate set requires a cheap way to implement gates from 2I. The most natural solution is to proceed as in the case of the Clifford $+ T$ super golden gate set and find quantum error-correcting codes that implement 2I transversally. As already mentioned, no such code has even been demonstrated.

In what follows, we fill this void. We first show that any code that supports 2I transversally must be a nonadditive code (Theorem 1). Then we construct a 7-qubit code that can correct an arbitrary error (i.e., $d = 3$) and we show that it is the smallest code that can implement 2I transversally (Theorem 2). We then prove a correspondence between spin codes and multiqubit codes (Lemma 2) which we use to construct $d = 3$ codes that implement 2I transversally for all odd n except 1,3,5,9,11,15,21 (Theorem 3). This result implies that the fast navigation algorithm for $2I + \tau_{60}$ can be performed for nearly all odd numbers of qubits. This abundance of codes is due to a symmetry phenomenon that is unique to 2I among all finite subgroups of SU(2) (Theorem 4).

Preliminaries.—Single-qubit quantum gates are usually presented as elements of the unitary group U(2). However, $U(2) = e^{i\theta}SU(2)$, so it is sufficient to consider quantum gates from the special unitary group SU(2). We will denote matrices from SU(2) by sans serif font. See Table I for our chosen correspondence. Most of the gates are standard with the notable exception of the "facet gate" F (see the Supplemental Material [2] for discussion).

The single-qubit Clifford group C is generated as $\mathsf{C} = \langle \mathsf{X}, \mathsf{Z}, \mathsf{F}, \mathsf{H}, \mathsf{S} \rangle$ and has 48 elements. This group is isomorphic to 2O. Although there are infinitely many (conjugate) realizations of 2O in SU(2), $\langle \mathsf{X}, \mathsf{Z}, \mathsf{F}, \mathsf{H}, \mathsf{S} \rangle$ is the only version that contains the Pauli group $\mathsf{P} = \langle \mathsf{X}, \mathsf{Z} \rangle$ and so it is the canonical choice. A subgroup of 2O is the binary tetrahedral group 2T with 24 elements. Again, 2T

TABLE I. Correspondence between traditional gates in U(2) and gates in SU(2).

	U(2)	SU(2)
Pauli X	$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\mathsf{X} = -iX$
Pauli Y	$Y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$	$\mathsf{Y} = -iY$
Pauli Z	$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\mathsf{Z} = -iZ$
Hadamard	$H = (1/\sqrt{2})\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$\mathsf{H} = -iH$
Phase	$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$	$\mathsf{S} = e^{-i\pi/4}S$
Facet	$F = HS^\dagger$	$\mathsf{F} = (e^{-i\pi/4}/\sqrt{2})\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} = \mathsf{H}\mathsf{S}^\dagger$
$\pi/8$ gate	$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$	$\mathsf{T} = \begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{pmatrix}$
$(2\pi/2^r)$ phase	$Ph(2\pi/2^r) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i2\pi/2^r} \end{pmatrix}$	$\mathsf{Ph}(2\pi/2^r) = \begin{pmatrix} e^{-i\pi/2^r} & 0 \\ 0 & e^{i\pi/2^r} \end{pmatrix}$

has infinitely many realizations, but the canonical choice is $\langle \mathsf{X}, \mathsf{Z}, \mathsf{F} \rangle$, showcasing the role of the F gate.

The binary icosahedral group 2I of order 120 also has infinitely many realizations. Unlike 2O and 2T there is not a single canonical choice of 2I subgroup, but rather two, related by Clifford conjugation. One version of 2I is $\langle \mathsf{X}, \mathsf{Z}, \mathsf{F}, \Phi \rangle$ where

$$\Phi = \frac{1}{2}\begin{pmatrix} \varphi + i\varphi^{-1} & 1 \\ -1 & \varphi - i\varphi^{-1} \end{pmatrix}. \tag{2}$$

The other version of 2I is $\langle \mathsf{X}, \mathsf{Z}, \mathsf{F}, \Phi^\star \rangle$ where we obtain $\Phi^\star$ from Φ by making the replacement $\sqrt{5} \to -\sqrt{5}$ and then taking the complex conjugate.

The $r+1$ level of the (special) 1-qubit Clifford hierarchy is defined recursively as

$$\mathsf{C}_{r+1} \coloneqq \{\mathsf{U} \in \mathrm{SU}(2) : \mathsf{U}\mathsf{P}\mathsf{U}^\dagger \subset \mathsf{C}_r\}, \tag{3}$$

where $\mathsf{C}_1 \coloneqq \mathsf{P}$ is the 1-qubit Pauli group [4]. Clearly C_2 is the 1-qubit Clifford group C. The T gate is in C_3, and in general, $\mathsf{Ph}(2\pi/2^r)$ is in C_r. In fact, every gate in Table I is in some level of the Clifford hierarchy. On the contrary, we have the following.

Lemma 1.—The Φ gate is not in the Clifford hierarchy.

The essence of the proof—worked out in the Supplemental Material [2]—is that the golden ratio φ cannot be expressed in terms of iterated square roots of 2. Not being in the Clifford hierarchy is the sense in which we call the Φ gate *exotic*. In fact, the only gates from 2I that are in the Clifford hierarchy are the gates forming the subgroup $2\mathrm{T} = \langle \mathsf{X}, \mathsf{Z}, \mathsf{F} \rangle$. The 96 other gates in 2I are exotic (see the Supplemental Material [2]).

On the other hand, it is known that the transversal gate group of a "stabilizer" code must lie in a finite level of the Clifford hierarchy [41–43]. In other words, exotic gates cannot be in the transversal gate group of a stabilizer code. This along with Lemma 1 implies our first claim.

Theorem 1.—Any code that implements 2I transversally must be nonadditive.

A quantum code is called "nonadditive" if it is not equivalent via nonentangling gates to any stabilizer code. For more background about nonadditive codes, see the Supplemental Material [2].

Let $g \in \mathrm{U}(2)$ be a logical gate for an $((n, 2, d))$ code. We say that g is "exactly transversal" if the physical gate $g^{\otimes n}$ implements logical g on the code space. We say g is "h-strongly transversal" if there exists some $h \in \mathrm{U}(2)$, not necessarily equal to g, such that the physical gate $h^{\otimes n}$ implements logical g on the code space.

An $((n, 2, d))$ code that implements the group G strongly transversally must transform in a two-dimensional faithful irreducible representation (irrep) of G. For 2I, there are only two such irreps, the fundamental representation π_2 and the closely related representation $\overline{\pi_2}$, which is just permuted by an outer automorphism (the character table for 2I can be found in the Supplemental Material [2]).

The smallest 2I code.—Using a computerized search over 2I invariant subspaces, we found a ((7,2,3)) code that implements 2I transversally. A normalized basis for the code space is

$$\begin{aligned} |\bar{0}\rangle &= \frac{\sqrt{15}}{8}|D_0^7\rangle + \frac{\sqrt{7}}{8}|D_2^7\rangle + \frac{\sqrt{21}}{8}|D_4^7\rangle - \frac{\sqrt{21}}{8}|D_6^7\rangle, \\ |\bar{1}\rangle &= -\frac{\sqrt{21}}{8}|D_1^7\rangle + \frac{\sqrt{21}}{8}|D_3^7\rangle + \frac{\sqrt{7}}{8}|D_5^7\rangle + \frac{\sqrt{15}}{8}|D_7^7\rangle. \end{aligned} \tag{4}$$

Here $|D_w^n\rangle$ is a Dicke state [44–48] defined as the (normalized) uniform superposition over all $\binom{n}{w}$ of the n-qubit states with Hamming weight w. For example,

$$|D_2^3\rangle = \frac{1}{\sqrt{3}}(|011\rangle + |110\rangle + |101\rangle). \tag{5}$$

The weight enumerator coefficients [49] of the ((7,2,3)) 2I code are

$$A = (1, 0, 7, 0, 7, 0, 49, 0), \tag{6a}$$

$$B = (1, 0, 7, 42, 7, 84, 49, 66). \tag{6b}$$

We immediately observe that the code distance is $d = 3$ since $A_i = B_i$ for each $i = 0, 1, 2$.

Both X and Z are exactly transversal since $X^{\otimes n}$ sends $|D_w^n\rangle$ to $|D_{n-w}^n\rangle$ and $Z^{\otimes n}$ sends $|D_w^n\rangle$ to $(-1)^w|D_w^n\rangle$. Thus, $\mathsf{X}^{\otimes 7}$ implements logical $-\mathsf{X}$ and $\mathsf{Z}^{\otimes 7}$ implements logical $-\mathsf{Z}$. Logical F is strongly F^* transversal and logical F is strongly F^* transversal, where $*$ denotes complex conjugation. The $[[7, 1, 3]]$ Steane code also implements logical F and F in this way, in contrast with the $[[5, 1, 3]]$ code, where both F and F are exactly transversal. Finally, logical $\Phi^\star$ is strongly Φ transversal. It follows that the code in Eq. (4) implements 2I transversally. This is the first code to have a transversal implementation of a gate outside of the Clifford hierarchy. Note that since logical $\Phi^\star$ is being implemented, rather than logical Φ, this code lives in the $\overline{\pi_2}$ irrep (as opposed to the π_2 irrep).

Theorem 2.—The smallest nontrivial code ($d \geq 2$) with 2I strongly transversal is the ((7,2,3)) code in Eq. (4). It is the unique 2I transversal code in 7-qubits.

The proof—given in the Supplemental Material [2]—is a basic application of branching rules. The minimal error-correcting codes that implement 2T, 2O, and 2I transversally are the $[[5, 1, 3]]$ code, the Steane $[[7, 1, 3]]$ code, and the ((7,2,3)) code, respectively. It is with this observation that the ((7,2,3)) code should be regarded as fundamental.

A family of 2I codes.—In [5], the author considers the problem of encoding a qubit into a single large spin. Spin j corresponds to the unique $2j+1$-dimensional irrep of

SU(2), which is spanned by the eigenvectors of J_z (the z component of angular momentum) denoted $|j,m\rangle$ for $|m| \leq j$. Each $g \in \mathrm{SU}(2)$ has a natural action on the space via the Wigner D rotation operators $D^j(g)$.

Encoding a qubit into this space means choosing a two-dimensional subspace. If $D^j(g)$ preserves the code space, then it will implement a logical gate. The collection of logical gates forms a finite group G. We will be interested in the cases for which the logical gate is implemented as $\lambda(g)$, where λ is an irrep of G.

We measure a spin code's performance based on how well it can correct small order isotropic errors. This is equivalent to correcting products of angular momenta. Analogous to multiqubit codes, we say a spin code has "distance" d if for all code words $|\bar{u}\rangle$ and $|\bar{v}\rangle$ we have

$$\langle \bar{u}|J_{\alpha_1} \cdots J_{\alpha_p}|\bar{v}\rangle = C\langle \bar{u}|\bar{v}\rangle \quad \text{for } 0 \leq p < d. \tag{7}$$

Here J_{α_i} is either J_z or a ladder operator $J_\pm$ [6] and the constant C is allowed to depend on $\alpha_1, \ldots, \alpha_p$, but not on the code words. These are the Knill-Laflamme conditions (KL) for spin codes [5,7,50].

A spin j system is isomorphic to the permutationally invariant subspace of the tensor product of $n = 2j$ many spin $1/2$ systems [6]. An explicit isomorphism is the Dicke state mapping

$$|j,m\rangle \overset{\mathcal{D}}{\mapsto} |D^{2j}_{j-m}\rangle. \tag{8}$$

The Dicke state mapping $\mathcal{D}$ behaves as an intertwiner between the natural action of SU(2) on a spin j irrep and the natural action of SU(2) on an $n = 2j$-qubit system via the tensor product,

$$\mathcal{D}[D^j(g)|j,m\rangle] = g^{\otimes n}\mathcal{D}|j,m\rangle. \tag{9}$$

The main implication of this property is that $\mathcal{D}$ converts logical gates of a spin code into logical gates of the corresponding multiqubit code. On the other hand, the Dicke state mapping $\mathcal{D}$ also behaves well with respect to error-correcting properties.

Lemma 2.—A spin j code with distance $d = 3$ that implements logical gates from G corresponds under $\mathcal{D}$ to a permutationally invariant G transversal $n = 2j$ multiqubit code with distance $d = 3$.

A proof is given in the Supplemental Material [2]. This lemma means we can focus on constructing spin codes with good distance that transform in the group 2I. To guarantee a $d = 3$ spin code, we need to satisfy the KL conditions [Eq. (7)] for p-fold products of angular momentum where $p = 0$, 1, 2 (henceforth called the "rank").

If the code words are orthonormal, then the rank-0 conditions are automatically satisfied. Thus we need to find code words such that the KL conditions hold for the rank-1 errors J_α and for the rank-2 errors $J_\alpha J_\beta$.

Very surprisingly, the rank-1 conditions are *always* satisfied when the logical group is 2I and the irrep is $\overline{\pi_2}$.

Lemma 3 (rank 1).—A 2I spin code transforming in the $\overline{\pi_2}$ irrep satisfies all rank-1 KL conditions *automatically.*

On the other hand, rank-2 errors are satisfied quite generically. Similar ideas can be found in [5,7,51].

Lemma 4 (rank 2).—Suppose a spin code implements logical X and logical Z using the physical gates $D^j(\mathsf{X})$ and $D^j(\mathsf{Z})$, respectively. If the code words are real and the rank-1 KL conditions are satisfied, then the rank-2 KL conditions are also satisfied.

The proof of each of these lemmas is given in the Supplemental Material [2]. In particular, a 2I spin code with real code words that transforms in the $\overline{\pi_2}$ irrep will satisfy both of these lemmas.

To be sure, this means that real $(2\mathrm{I}, \overline{\pi_2})$ spin codes have distance $d = 3$ automatically. In other words, these spin codes are deduced *entirely* from symmetry. We can now use Lemma 2 to immediately get a distance $d = 3$ multiqubit code family.

Theorem 3 (family of error-correcting 2I codes).—There is an $((n, 2, 3))$ multiqubit code that implements 2I transversally for all odd n except 1,3,5,9,11,15,21.

The exceptions are easy to understand. Only odd tensor powers of qubits branch to faithful irreps of 2I and $\overline{\pi_2}$ does not appear in the permutationally invariant subspace for any of the seven odd values of n listed. We give a concrete construction of the code words in the Supplemental Material [2].

It is worth emphasizing that rank-2 errors are satisfied fairly generically. So long as you implement the logical Pauli group and choose your code words to be real, Lemma 4 says that you can bootstrap an error *detecting* distance $d = 2$ spin code to an error *correcting* distance $d = 3$ spin code for free. In contrast, the rank-1 error condition from Lemma 3 was particular to the $\overline{\pi_2}$ irrep of 2I. One might wonder if this restriction was unnecessary, e.g., are there any other pairs (G, λ) for which this automatic rank-1 condition is true? The answer is no. Only the binary icosahedral group 2I affords enough symmetry.

Theorem 4.—The automatic rank-1 protection property from Lemma 3 is *unique* to the pair $(2\mathrm{I}, \overline{\pi_2})$ among all finite subgroups of SU(2).

Conclusion.—In this Letter, we have constructed codes that implement the binary icosahedral group 2I transversally, thereby completing the first half of a practical implementation of the fast icosahedral navigation algorithm.

One notable consequence of our search for a 2I code was the discovery that 2I codes satisfying certain transversality properties are automatically guaranteed to be error correcting (permutationally invariant real codes transforming in the $\overline{\pi_2}$ irrep). This is the first time that the error-correcting properties of a code have been deduced purely from transversality considerations. This suggests a deep connection between transversality and error correction that requires further investigation.

Most of the demonstrated advantage of nonadditive codes over stabilizer codes has been confined to marginal improvements in the parameter K relative to fixed n and d. However, our 2I code family motivates the study of nonadditive codes from a different, and much stronger, perspective. Namely, we show that nonadditive codes can achieve transversality properties that are forbidden for any stabilizer code.

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