Time-Energy Uncertainty Relation for Noisy Quantum Metrology

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Detection of very weak forces and precise measurement of time are two of the many applications of quantum metrology to science and technology. To sense an unknown physical parameter, one prepares an initial state of a probe system, allows the probe to evolve as governed by a Hamiltonian H for some time t, and then measures the probe. If H is known, we can estimate t by this method; if t is known, we can estimate classical parameters on which H depends. The accuracy of a quantum sensor can be limited by either intrinsic quantum noise or by noise arising from the interactions of the probe with its environment. In this work, we introduce and study a fundamental trade-off, which relates the amount by which noise reduces the accuracy of a quantum clock to the amount of information about the energy of the clock that leaks to the environment. Specifically, we consider an idealized scenario in which a party Alice prepares an initial pure state of the clock, allows the clock to evolve for a time that is not precisely known, and then transmits the clock through a noisy channel to a party Bob. Meanwhile, the environment (Eve) receives any information about the clock that is lost during transmission. We prove that Bob's loss of quantum Fisher information about the elapsed time is equal to Eve's gain of quantum Fisher information about a complementary energy parameter. We also prove a similar, but more general, trade-off that applies when Bob and Eve wish to estimate the values of parameters associated with two noncommuting observables. We derive the necessary and sufficient conditions for the accuracy of the clock to be unaffected by the noise, which form a subset of the Knill-Laflamme error-correction conditions. A state and its local time-evolution direction, if they satisfy these conditions, are said to form a metrological code. We provide a scheme to construct metrological codes in the stabilizer formalism. We show that there are metrological codes that cannot be written as a quantum error-correcting code with similar distance in which the Hamiltonian acts as a logical operator, potentially offering new schemes for constructing states that do not lose any sensitivity upon application of a noisy channel. We discuss applications of the trade-off relation to sensing using a quantum many-body probe subject to erasure or amplitude-damping noise.

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I. INTRODUCTION

Quantum mechanics places fundamental limits on how well we can measure a physical quantity when using a quantum system as a probe [1]. *Quantum metrology* is an active research area addressing how physical quantities can be estimated based on observations of a probe system [2–4]. As methods for accurately controlling quantum systems steadily advance, increasingly sophisticated measurement strategies are becoming feasible [5,6], leading for example to more sensitive gravitational wave detectors [7], improved frequency standards [8], and ultra-precise quantum clocks [9]. These technological developments accentuate the need for a precise theoretical understanding of the potential of quantum

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metrology and of the ultimate limits on measurement accuracy.

Fundamental accuracy limits in quantum metrology can often be phrased in terms of uncertainty relations, wherein the accuracy of one physical quantity trades off against the accuracy of a complementary quantity. For example, a particle with a definite position has a highly uncertain momentum, and vice versa. Such trade-offs may be captured conveniently by entropic uncertainty relations [10,11]. One may envision a two-party scenario, where the entropic uncertainty relation connects the first party's ignorance about a quantity A with the second party's lack of knowledge about a complementary quantity B. Typically these quantities are values of non-commuting observables.

In this work, we focus on a related but fundamentally different type of uncertainty relation. Rather than a trade-off between the values of two observables, we consider an information-theoretic trade-off between time and energy. Specifically, we envision preparing a probe state ρ_{init}, which then evolves for a time t as determined by some Hamiltonian H. By measuring the probe $\rho(t)$ at time t, we attempt to infer the value of t [12]. The time-energy uncertainty relation relates the accuracy of our estimate of t to the energy fluctuations of the probe state $\rho(t)$ [13,14]; a state with larger energy fluctuations evolves more rapidly, allowing the elapsed time to be estimated more precisely. Here, too, it is helpful to envision two parties, one attempting to measure time, the other attempting to measure energy. Indeed, such entropic time-energy uncertainty relations have recently been established [15,16].

For our purposes, a *clock* is a quantum system used to measure a time interval. The clock is initialized at some initial time and is measured at a later time, with the aim of the measurement being to reveal the difference in time between the initialization and the measurement. We are particularly interested in how a noise channel affects the accuracy of a clock. For that purpose we consider the following idealized scenario, involving three parties referred to as Alice, Bob, and Eve, which is amenable to precise mathematical analysis (see Fig. 1). Alice prepares a noiseless clock in the pure state vector $|\psi_{\text{init}}\rangle$, then allows that clock to evolve until some (*a priori* unknown) time t. Rather than measuring the clock herself for the purpose of estimating t, Alice stops the evolution of the clock and sends it to Bob through a noisy quantum channel $\mathcal{N}_{A\to B}$. As with any noisy channel, we can represent $\mathcal{N}_{A\to B}$ as an isometric map from Alice's system A to BE, where B is Bob's system and E is the channel's environment, after which E is discarded. In our scenario, Bob receives B and Eve receives E. We wish to study the trade-off between what Bob can learn about the elapsed time by measuring B and what Eve can learn about the energy of the clock by measuring E. Intuitively, such a trade-off is expected, because leakage to the environment of information about

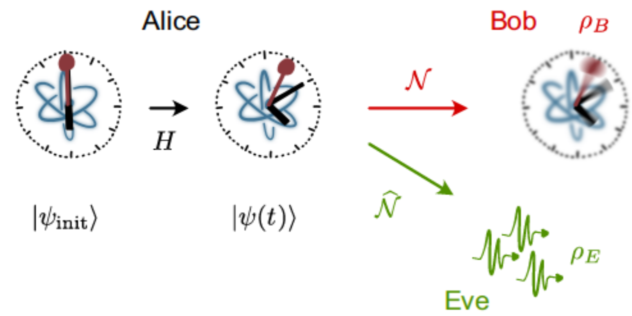

FIG. 1. A noiseless clock is initialized by Alice in $|\psi_{\text{init}}\rangle$ and evolves for a time t under the Hamiltonian H. Then Alice sends the clock through an instantaneous noisy channel $\mathcal{N}_{A\to B}$ to Bob, who receives the state ρ_B, measures it, and estimates t. The complementary channel $\widehat{\mathcal{N}}_{A\to E}$ describes the quantum information that leaks to the environment. Eve receives the state $\rho_E = \widehat{\mathcal{N}}(\psi(t))$, measures, and estimates the energy parameter of $|\psi(t)\rangle$. Our main result describes the trade-off between Bob's ability to estimate the time and Eve's ability to estimate the energy.

the clock's energy causes the clock to dephase in the energy-eigenstate basis, obscuring its evolution.

We consider the setting of *local parameter estimation*. This means that the value of a parameter is already approximately known, and we wish to determine it to greater accuracy. In this setting, the optimal estimate of the parameter is determined by the *quantum Fisher information* (QFI). For example, if $F_{\text{Alice},t}$ denotes the QFI of Alice's state with respect to the parameter t, then by performing the optimal measurement on her state, Alice can estimate the value of t with a mean-square error of $1/F_{\text{Alice},t}$. For the purpose of locally estimating $t = t_0 + \Delta t$ to first order in Δt, it suffices to know the quantum state $\rho(t_0)$ and its first time derivative, and indeed the QFI is determined by just these quantities.

Bob's noisy clock, degraded by transmission through the noisy channel $\mathcal{N}_{A\to B}$, has a reduced QFI compared to Alice's clock, and correspondingly Bob's optimal measurement yields a less accurate estimate of the time t than Alice's. On the other hand, Eve receives the state of Alice's clock after transmission through the *complementary* noisy channel $\widehat{\mathcal{N}}_{A\to E}$, the channel obtained if B is discarded after A is isometrically mapped to BE. We imagine that Eve wishes to learn about the *energy* of Alice's clock, rather than about the elapsed time. More precisely, Eve's goal is to determine an "energy parameter" denoted η and defined in Sec. II B, which is complementary to the time t. Because Eve, like Bob, receives a state of the clock degraded by noise, the QFI of her state with respect to η is in general less than Alice's. Our main result is an equality relating Bob's QFI about t to Eve's QFI about η given by

$$\frac{F_{\text{Bob},t}}{F_{\text{Alice},t}} + \frac{F_{\text{Eve},\eta}}{F_{\text{Alice},\eta}} = 1. \tag{1}$$

This time-energy uncertainty relation, derived in Sec. III and Appendix E using semidefinite programming duality, substantially differs from previous results [12,17,18] in that it characterizes the trade-off between Bob's and Eve's QFI, rather than the trade-off between the inherent energy variance and time uncertainty of the noiseless clock.

Figure 2 illustrates the setting of Eq. (1) in a concrete example. Alice initializes a single qubit in the pure state vector $|+\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$, which evolves under the Hamiltonian $H = \omega Z/2$. Here and in the following, X, Y, Z denote the qubit Pauli-X, Y, Z operators, respectively. The qubit basis states are denoted by $|\uparrow\rangle, |\downarrow\rangle$ for consistency with which state is excited with respect to the Hamiltonian H, with $Z|\uparrow\rangle = |\uparrow\rangle$ and $Z|\downarrow\rangle = -|\downarrow\rangle$. Later in this work, we also use the alternative notation $|0\rangle \equiv |\uparrow\rangle$ and $|1\rangle \equiv |\downarrow\rangle$ whenever necessary to facilitate the representation of states of multiple qubits using bit strings or for consistency with the literature on quantum error-correcting codes. At time $t = t_0 + \Delta t$, the partially dephasing channel $\mathcal{N}_p = (1-p)\mathrm{id} + p\mathcal{D}_Z$ is applied to Alice's qubit, where $\mathcal{D}_Z(\cdot) = \langle\uparrow|(\cdot)|\uparrow\rangle\, |\uparrow\rangle\langle\uparrow| + \langle\downarrow|(\cdot)|\downarrow\rangle\, |\downarrow\rangle\langle\downarrow|$. We may describe this channel by saying that the environment (Eve) measures the qubit with probability p in the energy-eigenstate basis (i.e., along the Z axis of the Bloch sphere). The partial dephasing attenuates the t dependence of Bob's state $\rho_B(t)$ by the factor $1-p$, hindering his ability to estimate the time. Equation (1) captures the trade-off between Bob's information about the time (proportional to $1-p$) and Eve's information gain about the energy (proportional to p).

The trade-off relation Eq. (1) can be a useful tool for deriving upper bounds on QFI. The QFI for a mixed state can be tricky to characterize in cases where a diagonal representation of the state is not easily obtained. Along these lines, it is useful to note that QFI obeys a data processing inequality, which ensures that, for any state ρ and any quantum channel $\mathcal{N}$, the QFI of $\mathcal{N}(\rho)$ is no larger than the QFI of ρ [19]. We can imagine that Eve applies a channel to her state ρ_E, obtaining the state ρ'_E, which she then measures for the purpose of estimating η. Using the data processing inequality, we conclude that

$$\frac{F_{\mathrm{Bob},t}}{F_{\mathrm{Alice},t}} + \frac{F'_{\mathrm{Eve},\eta}}{F_{\mathrm{Alice},\eta}} \leqslant 1, \tag{2}$$

where now $F'_{\mathrm{Eve},\eta}$ denotes the QFI of ρ'_E with respect to η. Even if the QFI of ρ_E is difficult to compute, the QFI of ρ'_E may be easy to compute if the channel taking ρ_E to ρ'_E is artfully chosen; then Eq. (2) provides a computable upper bound on $F_{\mathrm{Bob},t}$. For example, in the case where $\mathcal{N}_{A\to B}$ is an amplitude damping noise channel, a useful upper bound on Bob's QFI can be derived by applying a completely dephasing channel to Eve's state ρ_E. We apply this idea to an Ising spin chain in Sec. VIII.

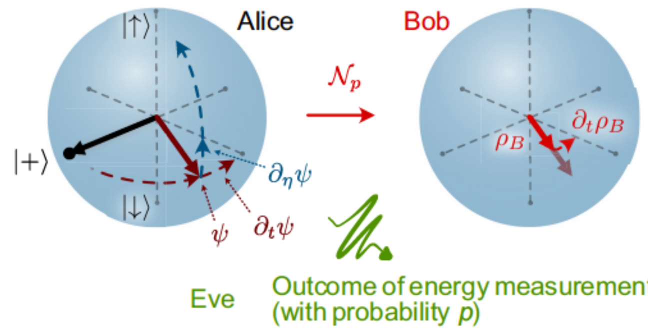

FIG. 2. Illustration of Eq. (1) for a single qubit subjected to partial dephasing. Alice's clock state is initialized as $|+\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ and evolves according to the Hamiltonian $H = \omega Z/2$, where Z denotes the qubit Pauli-Z operator. At time t, the channel $\mathcal{N}_p(\cdot) = (1-p)(\cdot) + p\,|\uparrow\rangle\langle\uparrow|(\cdot)|\uparrow\rangle\langle\uparrow| + p\,|\downarrow\rangle\langle\downarrow|(\cdot)|\downarrow\rangle\langle\downarrow|$ is instantaneously applied to Alice's clock state. In effect, Eve measures the energy observable Z with probability p, and Bob receives the partially dephased clock. Equation (1) relates Bob's reduced information about the elapsed time to Eve's information gain about the clock's energy. Unitary evolution in Eve's complementary energy variable η, generated by an optimal local time-sensing observable, rotates the state into a direction that is orthogonal to the direction of the original evolution in time t (see Sec. II B).

One consequence of Eq. (1) is a necessary and sufficient condition for the clock's sensitivity to be unaffected by transmission through the noisy channel $\mathcal{N}_{A\to B}$: $F_{\mathrm{Bob},t} = F_{\mathrm{Alice},t}$ if and only $F_{\mathrm{Eve},\eta} = 0$. This condition can be usefully restated in terms of the Kraus operators $\{E_k\}$ of the channel $\mathcal{N}_{A\to B}$. Recall that we aim to estimate the time $t = t_0 + \Delta t$ in the setting of local parameter estimation, i.e., to linear order in Δt. Suppose that after evolution for time t_0, the state of Alice's clock is $|\psi\rangle$, and that $|\xi\rangle = (H - \langle H\rangle_\psi)|\psi\rangle = P_\psi^\perp H|\psi\rangle$ with $P_\psi^\perp = \mathbb{1} - |\psi\rangle\langle\psi|$. Then the condition $F_{\mathrm{Eve},\eta} = 0$ is equivalent to

$$\langle\xi|E_k^\dagger E_j|\psi\rangle + \langle\psi|E_k^\dagger E_j|\xi\rangle = 0 \quad \text{for all } k, j. \tag{3}$$

Intuitively, Eq. (3) means that the action of the channel on the clock cannot be confused with genuine time evolution.

Equation (3) may be recognized as a weakened version of the Knill-Laflamme condition for quantum error correction, the necessary and sufficient condition for the action of a noisy channel on an encoded subspace to be reversible by a suitable recovery channel [20]. This condition may be stated as $\Pi_L E_k^\dagger E_j \Pi_L \propto \Pi_L$ for all k and j, where Π_L is the projector onto the encoded subspace. To write Eq. (3) in a similar form, consider the two-dimensional subspace spanned by the mutually orthogonal state vectors $|\psi\rangle$ and $|\xi\rangle$; we call this two-dimensional space a "virtual qubit." Using the notation $|+\rangle_L := |\psi\rangle$, $|-\rangle_L := \||\xi\rangle\|^{-1}|\xi\rangle$, the orthogonal projector onto the virtual qubit is $\Pi_L = |+\rangle\langle+|_L + |-\rangle\langle-|_L$, and $Z_L = |+\rangle\langle-|_L + |-\rangle\langle+|_L$ is the logical Z Pauli operator acting on the virtual qubit. In this language, Eq. (3) becomes

$$\mathrm{tr}\big(\Pi_L E_k^\dagger E_j \Pi_L Z_L\big) = 0 \quad \text{for all } k, j\,. \tag{4}$$

The condition Eq. (4) is reminiscent of a recently formulated condition for quantum coding to improve how measurement sensitivity scales with increasing sensing time [21,22]. In Sec. VII, we explain how time-covariant quantum error-correcting codes automatically fulfill Eq. (3), providing some simple examples. In particular, we consider spins on a graph with Ising or Heisenberg interactions and construct a state vector $|\psi\rangle$ that fulfills Eq. (3), where the noise model inflicts a single located erasure.

We have derived the trade-off relation Eq. (1) in a highly idealized setting, in which noiseless evolution of Alice's clock is followed by transmission to Bob through the noisy channel $\mathcal{N}_{A\to B}$. For an actual clock, the noise acts continuously as the clock evolves, rather than after the time evolution is complete. By focusing on the idealized setting, we have been able to perform a particularly elegant analysis of the time-energy trade-off. But in Sec. VI we connect our results to the more realistic case of continuous Markovian noise described by a master equation in Lindblad form, noting that the two settings are actually equivalent, or nearly equivalent, under certain conditions. One can decompose the Lindbladian into a Hamiltonian part and a noise part that contains all the jump operators; if, for example, these two parts define commuting channels, then the Markovian evolution for time t is equivalent to Hamiltonian evolution for time t followed by a noise channel $\mathcal{N}_t$. Other cases where the Lindblad evolution is compatible with a trade-off relation of the form Eq. (1) (at least to a good approximation) are identified in Sec. VI.

Although the time-energy trade-off provided the primary motivation for this work, we find that a trade-off relation similar to Eq. (1) can be derived in a more general setting. Suppose that A and B are Hermitian operators, and that $\psi = |\psi\rangle\langle\psi|$ is a pure quantum state. We may consider the "flow" in Hilbert space generated by A or by B. That is, we consider a one-parameter family of pure states close to ψ, generated by A and parameterized by a, and a one-parameter family generated by B and parametrized by b, such that

$$\partial_a \psi = -i[A, \psi], \quad \partial_b \psi = -i[B, \psi]. \tag{5}$$

In the setting of local parameter estimation, we suppose that Bob wishes to estimate the parameter a and Eve wants to estimate the parameter b, where a and b are both small. Alice's QFI about a is $F_{\mathrm{Alice},a}$, but Bob receives the state via the noisy channel $\mathcal{N}_{A\to B}$, so his QFI about a ($F_{\mathrm{Bob},a}$) is in general smaller than Alice's. Alice's QFI about b is $F_{\mathrm{Alice},b}$, but Eve receives the state via the complementary channel $\widehat{\mathcal{N}}_{A\to E}$, so her QFI about b ($F_{\mathrm{Eve},b}$) is in general smaller than Alice's. In Sec. III we derive the trade-off relation

$$\frac{F_{\mathrm{Bob},a}}{F_{\mathrm{Alice},a}} + \frac{F_{\mathrm{Eve},b}}{F_{\mathrm{Alice},b}} \leqslant 1 + 2\sqrt{1 - \frac{\langle i[A,B]\rangle_\psi^2}{4\sigma_A^2\sigma_B^2}}, \tag{6}$$

where $\sigma_M := [\langle M^2\rangle_\psi - \langle M\rangle_\psi^2]^{1/2}$ denotes the standard deviation of the observable M. Note that, in contrast to Eq. (1), this relation is an inequality rather than an equality. It is reminiscent of the Robertson uncertainty relation, with the commutator quantifying the incompatibility of the observables A and B.

Figure 3 summarizes the structure of this work and provides an overview of our results. In Sec. II, we introduce the setting of local parameter estimation, recall some useful properties of the QFI, define the energy parameter η, and review the concept of a complementary quantum channel. We sketch the proof of the trade-off relation Eq. (1) and its generalization Eq. (6) in Sec. III (more details can be found in Appendix E), and discuss some examples in Sec. IV. We use the trade-off relation to derive upper bounds on the QFI in Sec. V. In Sec. VI we discuss how the setting in Fig. 1 is connected with the more realistic setting of continuous Markovian noise. In Sec. VII we derive the necessary and sufficient condition Eq. (3) for the clock's sensitivity to be undiminished by transport through the noisy channel $\mathcal{N}_{A\to B}$, and discuss some of the implications of this condition. Numerical results for our upper bound on QFI in many-body systems are reported in Sec. VIII. We summarize and comment on our results in Sec. IX. Many further details are presented in the appendices.

II. SETTING

We review the standard setting in quantum metrology of single-parameter estimation. We then introduce our noise model and the quantities that are relevant to formulate our uncertainty relation.

A. Quantum parameter estimation

Consider a quantum state $\rho(t)$ that depends on a single parameter t. The task we study is how well the parameter t can be estimated by performing suitable measurements (Fig. 4). In the context of this work, the parameter t is identified with physical time, although the results hold for any general real parameter that the quantum state might depend on.

We consider the setting of local sensitivity, where the goal of the quantum measurement is to refine the precision to which we determine the parameter if the value of the parameter is already known to be close to a given value t_0. More precisely, we seek a measurement operator T with minimal variance such that the expectation value of

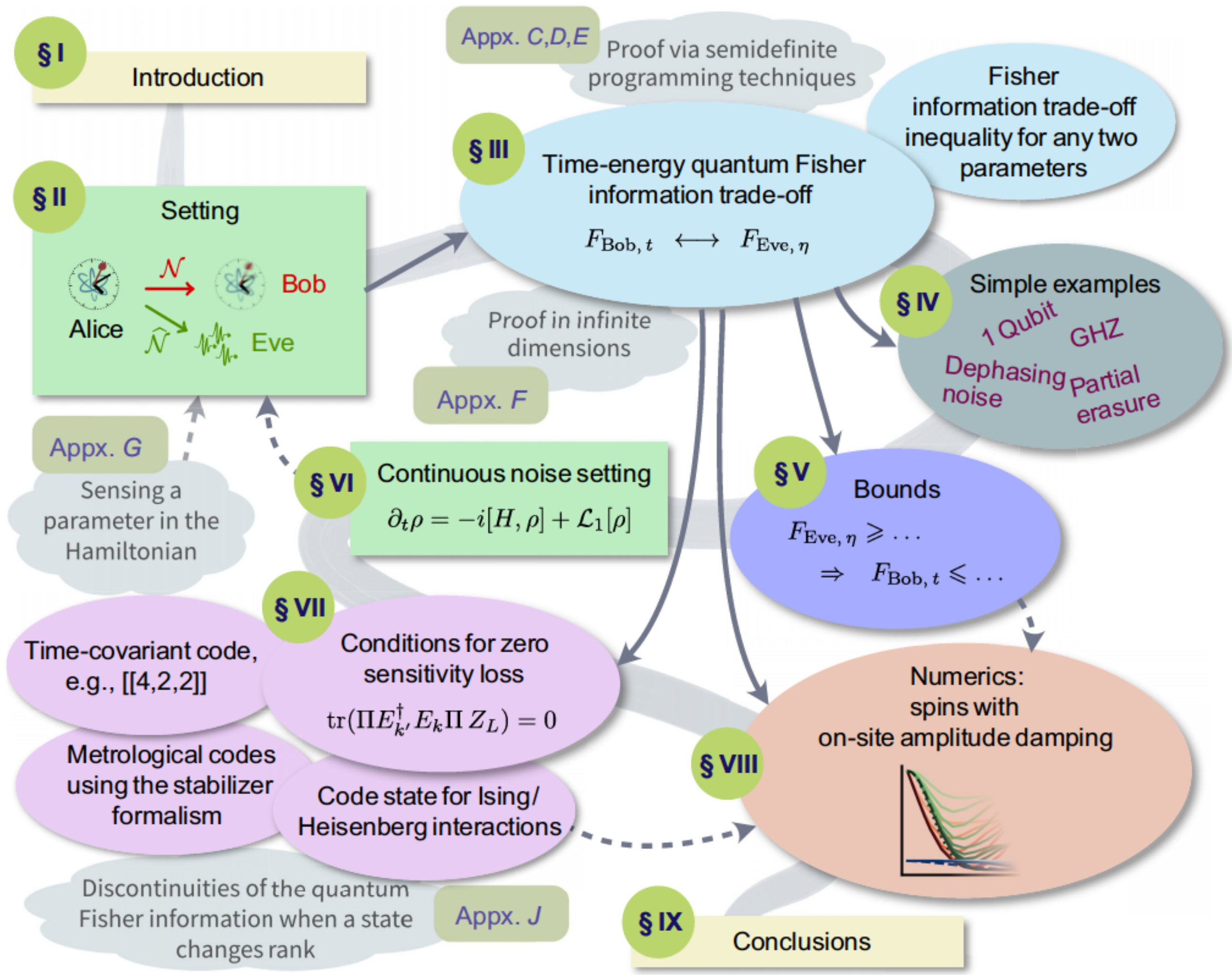

FIG. 3. Overview of our main results and structure of this work.

T reveals the value of the parameter locally around t_0 to first order in dt, i.e.,

$$\langle T\rangle_{\rho(t_0+dt)} = t_0 + dt + O(dt^2). \tag{7}$$

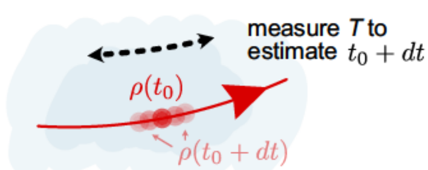

FIG. 4. In the setting of quantum parameter estimation, the task is to infer a parameter t in a one-parameter family of states $t \mapsto \rho(t)$ through suitable measurements. For *local* parameter estimation, we assume the value of the parameter is already known to lie in the neighborhood of a given value t_0. The measurement is required to refine the parameter estimation by optimally distinguishing $\rho(t_0)$ from $\rho(t_0 + dt)$ to first order in dt. This setting is standard in the field of quantum metrology, and the optimal sensitivity is quantified by a quantity known as the Fisher information.

Identifying the orders in dt we see that Eq. (7) is equivalent to

$$\langle T\rangle_{\rho(t_0)} = t_0 \quad \text{and} \quad \mathrm{tr}\big(T\,\partial_t\rho\,(t_0)\big) = 1, \tag{8}$$

using the notation $\partial_t\rho = \partial\rho/\partial t$. (We write a partial derivative instead of a total derivative in anticipation of other variables, which will be introduced later.) In the literature, it is common to reuse the symbol t for both the parameter on which ρ depends as well as the reference value of the parameter t_0. We keep the distinction for clarity.

Here, we restricted the measurement to be projective, as described by the Hermitian observable T. A more general positive operator-valued measure (POVM) does not offer any more sensitivity in sensing the parameter [1,12].

A central result in quantum metrology is the quantum Cramér-Rao bound, which states that the optimal sensitivity to which one can determine the parameter t locally around t_0 is determined by a quantity called the quantum

Fisher information [1,23,24]. The quantum Fisher information of the state $\rho(t_0)$ with respect to a direction $\partial_t\rho\,(t_0)$ is defined as

$$F\big(\rho\,;\,\partial_t\rho\big) = \mathrm{tr}\big(\rho\, R^2\big), \tag{9}$$

where R is any Hermitian operator that solves the equation $1/2\{\rho, R\} = 1/2\big(\rho R + R\rho\big) = \partial_t\rho$, and where the quantities ρ and $\partial_t\rho$ are evaluated at t_0. The Cramér-Rao bound can be formulated for our purposes as follows: for any observable T that satisfies Eq. (8), we must have

$$\big\langle (T-t_0)^2\big\rangle_{\rho(t_0)} \geqslant \frac{1}{F\big(\rho(t_0)\,;\,\partial_t\rho\,(t_0)\big)}, \tag{10}$$

and furthermore, the equality in Eq. (10) can always be achieved by a suitable choice of T. We refer to a choice of T, which is optimal in Eq. (10) as an *optimal local sensing observable for t*.

The operator R in Eq. (9) is called a *symmetric logarithmic derivative*. When ρ and $\partial_t\rho$ commute, we can choose $R = \rho^{-1}\partial_t\rho = (\partial/\partial t)\ln\rho$. A general construction of R in terms of an eigendecomposition of ρ is given as follows [25]. Consider an eigenbasis $\{|k\rangle\}$ of ρ that spans the full Hilbert space, such that $\rho = \sum_k \lambda_k |k\rangle\langle k|$ and $k = 1, 2, \ldots, \dim(\mathscr{H})$, then

$$R = \sum_{\substack{k,k':\\ \lambda_k+\lambda_{k'}\neq 0}} \frac{2}{\lambda_k + \lambda_{k'}} \big\langle k\big|\partial_t\rho\big|k'\big\rangle\, |k\rangle\langle k'|, \tag{11}$$

where the sum ranges over all pairs of indices k, k' except those for which both $\lambda_k = 0$ and $\lambda_{k'} = 0$. The expression for the Fisher information becomes $F\big(\rho\,;\,\partial_t\rho\big)$, where

$$F\big(\rho\,;\,\partial_t\rho\big) = \sum_{\substack{k,k':\\ \lambda_k+\lambda_{k'}\neq 0}} \frac{2}{\lambda_k + \lambda_{k'}} \big|\big\langle k\big|\partial_t\rho\big|k'\big\rangle\big|^2. \tag{12}$$

The solution to the anticommutator equation $1/2\{\rho, R\} = \partial_t\rho$ is unique up to transformations of the form $R \mapsto R + P_\rho^\perp M P_\rho^\perp$, where M is an arbitrary Hermitian operator, where $P_\rho^\perp = \mathbb{1} - P_\rho$, and where P_ρ denotes the projector onto the support of ρ. In the event that $P_\rho^\perp \frac{d\rho}{dt} P_\rho^\perp \neq 0$, there is no solution for R. In such a situation, the optimal estimation variance (10) is zero and the Fisher information is not defined; such cases do not arise in the setting we consider in this work.

We review the solutions to the anticommutator equation $1/2\{\rho, R\} = \partial_t\rho$ in Appendix B. In Appendix C, the definition and elementary properties of the Fisher information are reviewed using simple techniques based on semidefinite programming. In Appendix D, we review a derivation of the Cramér-Rao bound using these methods.

Observables T that estimate the time parameter t with an accuracy that achieves the Cramér-Rao bound (10), i.e., the optimal local sensing observables, turn out to be the projective measurements with outcomes associated with the eigenspaces of a symmetric logarithmic derivative [1,23]. Specifically, any optimal local sensing observable for t is of the form

$$T = t_0 + \frac{1}{F\big(\rho\,;\,\frac{d\rho}{dt}\big)} R, \tag{13}$$

where R is as above any solution to the anticommutator equation $1/2\{\rho, R\} = d\rho/dt$ (see Appendix D for a review of the proof). Due to the freedom in the choice of R, all optimal local sensing observables for t differ by a term of the form $P_\rho^\perp M P_\rho^\perp$ where M is any Hermitian operator.

In the remaining part of this section, we review a few properties of the Fisher information for later use (see Appendix C for details). First is a scaling property: if $0 < \alpha \leqslant 1$ and $\beta \in \mathbb{R}$, we have

$$F\big(\alpha\rho\,;\,\beta\,\partial_t\rho\big) = \frac{\beta^2}{\alpha} F\big(\rho\,;\,\partial_t\rho\big), \tag{14}$$

where the definition (9) is formally extended to positive semidefinite operators ρ that satisfy $\mathrm{tr}\,\rho \leqslant 1$. Second, in case the state ρ and derivative $\partial_t\rho$ commute, the Fisher information takes the simple form

$$[\rho, \partial_t\rho] = 0 \quad\Rightarrow\quad F\big(\rho\,;\,\partial_t\rho\big) = \mathrm{tr}\big[\rho^{-1}\,(\partial_t\rho)^2\big]. \tag{15}$$

Finally, for general $\rho, \partial_t\rho$, we can express the Fisher information in terms of a pair of convex optimization problems [26–28] as

$$\frac{1}{4}F\big(\rho\,;\,\partial_t\rho\big)$$

$$= \max_{S=S^\dagger}\Big\{\mathrm{tr}\big[(\partial_t\rho)\,S\big] - \mathrm{tr}\big[\rho\,S^2\big]\Big\} \tag{16a}$$

$$= \min_{L\ \text{arb.}}\Big\{\mathrm{tr}(L^\dagger L)\ :\ \rho^{1/2}L + L^\dagger\rho^{1/2} = \partial_t\rho\Big\}. \tag{16b}$$

These two optimizations can be cast as semidefinite problems that are dual to each other. These optimizations are convenient to derive bounds on the Fisher information, as it suffices to exhibit suitable candidates in Eqs. (16a) or (16b).

B. Time and energy parameters of the noiseless clock

Now we turn to the setup depicted in Fig. 1, in which Alice possesses a noiseless quantum clock, which she sends to Bob through a given noisy channel. In this subsection, we study Alice's noiseless quantum clock, and in the following subsection we study the effect of the noise.

1. The noiseless clock

Suppose that Alice prepares a quantum clock in a pure state living in a finite-dimensional Hilbert space $\mathscr{H}_A$. She lets it evolve according to a Hamiltonian $H(t)$, generating a one-parameter family of state vectors $t \mapsto |\psi(t)\rangle$. The time evolution of $\psi(t) = |\psi(t)\rangle\langle\psi(t)|$ is governed by the standard Schrödinger time evolution

$$\partial_t\psi := \frac{\partial\psi}{\partial t} = -i\,[H,\psi]. \tag{17}$$

We now compute the Fisher information associated with Alice's clock locally around a time of interest t_0, following the definition (9). For any t_0, we can choose $R = 2\,\partial_t\psi = -2i[H,\psi]$, because $\{\partial_t\psi,\psi\} = \partial_t(\psi^2) = \partial_t\psi$. Alice's Fisher information $F_{\mathrm{Alice},t}$ for the evolution $|\psi(t)\rangle$ at the time of interest t_0 is therefore given by

$$F_{\mathrm{Alice},t} := F\Big(\psi\,;-i[H,\psi]\Big) = 4\sigma_H^2, \tag{18}$$

where ψ and H are evaluated at time t_0, and where again, we denote by $\sigma_M = [\langle M^2\rangle_\psi - \langle M\rangle_\psi^2]^{1/2}$ the standard deviation of an observable M. Alternative expressions of the standard deviation are given by

$$\sigma_M^2 = \big\langle (M-\langle M\rangle)^2\big\rangle = -\big\langle [M,\psi]^2\big\rangle, \tag{19}$$

writing $\langle M\rangle := \langle M\rangle_\psi$ for brevity.

Around the point t_0, any optimal local sensing observable for t takes the form given by Eq. (13), which we can rewrite in this context as

$$T = t_0 - \frac{i[H,\psi]}{2\sigma_H^2} + P_\psi^\perp M P_\psi^\perp, \tag{20}$$

where M is any Hermitian operator. In the case where H is time independent, then $F_{\mathrm{Alice},t}$ does not depend on the time of interest t_0, but the optimal sensing observable T depends on t_0 not only directly but also indirectly through ψ and $\partial_t\psi$. In the following, we fix t_0 and we consider only the evolution $|\psi(t)\rangle$ locally at t_0. Furthermore, we use the shorthand $|\psi\rangle := |\psi(t_0)\rangle$.

2. The energy parameter

The optimal local time-sensing observable T in Eq. (20), being a Hermitian operator, can be used to generate a different evolution in an alternative direction in the space of quantum states. In our setup, we define $\eta_0 = \langle H\rangle_\psi$ and we consider any family of state vectors $\eta \mapsto |\psi(\eta)\rangle$ such that $|\psi(\eta=\eta_0)\rangle = |\psi\rangle = |\psi(t=t_0)\rangle$ and such that at the point $|\psi(\eta=\eta_0)\rangle$ we have

$$\partial_\eta\psi = i[T,\psi]. \tag{21}$$

This evolution can be interpreted as a Schrödinger equation with the effective Hamiltonian $-T$. An example

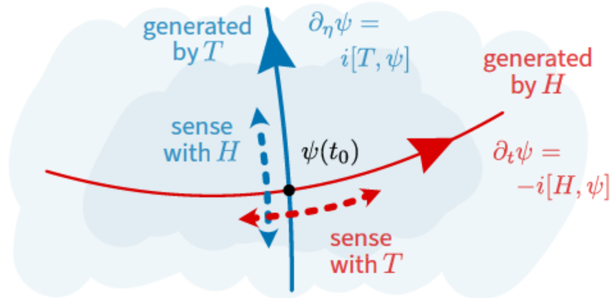

FIG. 5. We define a parameter η that is complementary to time evolution and that represents the energy of the state. Consider a quantum clock modeled as a pure state ψ evolving according to the Schrödinger equation $\partial_t\psi = -i[H,\psi]$, where H is the Hamiltonian. Locally around t_0, the observable T that optimally distinguishes the neighboring states $\psi(t_0)$ and $\psi(t_0+dt)$ defines an *optimal local time-sensing observable*. T is the relevant measurement to carry out to optimally read out the information about time stored in the quantum clock. We now consider locally around $\psi(t_0)$ the direction in state space defined by $\partial_\eta\psi = i[T,\psi]$, i.e., a Schrödinger-type evolution with $-T$ playing the role of an effective Hamiltonian. It turns out that the optimal estimation procedure for the parameter η is to measure H itself. Therefore, the parameter η represents the energy of $\psi(\eta)$. The parameters t and η are, therefore, complementary to each other in the sense that the generator associated with one parameter optimally distinguishes neighboring values of the other parameter and *vice versa*.

of such an evolution is

$$|\psi(\eta)\rangle = e^{iT(\eta-\eta_0)}\,|\psi\rangle. \tag{22}$$

Interestingly, the evolution generated in this way locally around $|\psi\rangle$ turns out to be complementary to time evolution in the sense that we can derive a meaningful uncertainty relation and that the parameter η can be identified with the average energy of the state vector $|\psi(\eta)\rangle$ (see Fig. 5).

More formally and to clarify the dependencies of $|\psi\rangle$ on t and η, we consider a two-parameter family of state vectors $(t,\eta) \mapsto |\psi(t,\eta)\rangle$ with $|\psi(t_0,\eta_0)\rangle = |\psi\rangle$ and such that at the point (t_0,η_0) we have

$$\partial_t\psi\;(t_0,\eta_0) = -i[H,\psi], \quad \partial_\eta\psi\;(t_0,\eta_0) = i[T,\psi], \tag{23}$$

where T is given by Eq. (20). For example, we could choose

$$|\psi(t,\eta)\rangle = \exp\big\{-i[(t-t_0)H - (\eta-\eta_0)T]\big\}\,|\psi\rangle. \tag{24}$$

Unless indicated otherwise, the state vector $|\psi\rangle$ and the corresponding derivatives $\partial_t\psi, \partial_\eta\psi$ are henceforth implicitly evaluated at (t_0,η_0). We use the shorthands $|\psi(t)\rangle := |\psi(t,\eta_0)\rangle$ and $|\psi(\eta)\rangle := |\psi(t_0,\eta)\rangle$ to denote the respective evolutions according to t and η in which the other parameter is fixed to η_0 or t_0, respectively; the name of the argument (t or η) determines which evolution is meant.

Let us re-express the derivative $\partial_\eta\psi$ of $\psi = |\psi\rangle\langle\psi|$ in terms of the Hamiltonian. Using Eq. (20), we have

$$\partial_\eta\psi = i\big[T,\psi\big] = \frac{1}{2\sigma_H^2}\,\big[[H,\psi],\psi\big]. \tag{25}$$

A brief computation reveals that $\big[[H,\psi],\psi\big] = H\psi + \psi H - 2\langle H\rangle\,\psi = \{H - \langle H\rangle, \psi\}$ and therefore

$$\partial_\eta\psi = \frac{1}{2\sigma_H^2}\,\big\{H - \langle H\rangle, \psi\big\}. \tag{26}$$

Alice's Fisher information with respect to the parameter η is given by the same expression as Eq. (18), but with H and t replaced by $-T$ and η, to get

$$F_{\text{Alice},\eta} := F\Big(\psi\,;\, i[T,\psi]\Big) = 4\sigma_T^2 = \frac{1}{\sigma_H^2}, \tag{27}$$

where the last equality follows from

$$\sigma_T^2 = \big\langle (T-t_0)^2\big\rangle = \frac{-\langle [H,\psi]^2\rangle}{4\sigma_H^4} = \frac{1}{4\sigma_H^2}. \tag{28}$$

To justify that the parameter η in the evolution (21) can be associated with the energy of the state vector locally around $|\psi\rangle$, we compute the optimal sensing observable for η and show that it is the Hamiltonian H itself (up to terms lying outside of the support of ψ). The optimal local-sensing observable that distinguishes $\psi(\eta)$ from $\psi(\eta + d\eta)$ is given by Eq. (13), but with the parameter t replaced by the parameter η. Using Eq. (26), observe that the operator $R' = (H - \langle H\rangle)/\sigma_H^2$ solves the equation $\{\psi, R'\}/2 = \partial_\eta\psi$. From Eq. (13) and substituting t by η, we see that the optimal local-sensing observable for η is simply $\eta_0 + H - \langle H\rangle = H$. That is, the optimal measurement distinguishing $|\psi(\eta_0)\rangle$ from $|\psi(\eta_0 + d\eta)\rangle$ is the Hamiltonian H itself, up to a term $P_\psi^\perp M P_\psi^\perp$ for any Hermitian M. [Alternatively, the same conclusion would have been reached had we started from Eq. (20) with t, H replaced by $\eta, -T$. A more detailed computation is provided in Appendix D.] Therefore, the parameter η describes an evolution along which, locally around $|\psi\rangle$, we have $\eta_0 + d\eta = \langle H\rangle_{\psi(\eta_0+d\eta)}$. In this sense, η represents the energy of the probe $|\psi(\eta)\rangle$ locally around η_0.

To summarize, the evolution of $\psi(t) = |\psi(t)\rangle\langle\psi(t)|$ is generated by the Hamiltonian H; nearby states $\psi(t_0)$ and $\psi(t_0 + dt)$ are optimally distinguished by a local time-sensing observable T. The complementary evolution $\psi(\eta)$ is one that inverts the roles of H and T: the evolution $\psi(\eta)$ is generated by T, and H is the operator that optimally distinguishes neighboring states $\psi(\eta_0)$ and $\psi(\eta_0 + d\eta)$.

3. Single-qubit example

Consider a qubit initialized in the state vector $|\psi_{\text{init}}\rangle = |+\rangle$, where $|\pm\rangle = [|\uparrow\rangle \pm |\downarrow\rangle]/\sqrt{2}$, and let the qubit evolve according to the Hamiltonian $H = \omega Z/2$ (i.e., Alice's system in Fig. 2). The time evolution of the clock is given by $|\psi(t)\rangle = U_t|+\rangle$, where $U_t = e^{-iHt}$; we see that

$$\begin{aligned}|\psi(t)\rangle &= \frac{1}{\sqrt{2}}\big[e^{-\frac{i\omega t}{2}}|\uparrow\rangle + e^{\frac{i\omega t}{2}}|\downarrow\rangle\big]\\ &= \cos\Big(\frac{\omega t}{2}\Big)\,|+\rangle - i\sin\Big(\frac{\omega t}{2}\Big)\,|-\rangle.\end{aligned} \tag{29}$$

It is also convenient to note that

$$\psi(t) = U_t|+\rangle\langle+|U_t^\dagger = \frac{\mathbb{1}}{2} + \frac{1}{2}U_t X U_t^\dagger \tag{30a}$$

$$= \frac{\mathbb{1}}{2} + \frac{1}{2}\big[\cos(\omega t)\,X + \sin(\omega t)\,Y\big], \tag{30b}$$

using the identity $|+\rangle\langle+| = [\mathbb{1} + X]/2$ along with $e^{-iaZ}Xe^{iaZ} = \cos(2a)\,X + \sin(2a)\,Y$. The time derivative of the state is

$$\begin{aligned}\partial_t\psi(t) &= -i[H,\psi(t)] = -\frac{i\omega}{2}\,\big[Z\,,\,U_t|+\rangle\langle+|U_t^\dagger\big]\\ &= -\frac{i\omega}{2}\,U_t\Big[Z\,,\,\frac{\mathbb{1}+X}{2}\Big]U_t^\dagger\\ &= \frac{\omega}{2}\,U_t Y U_t^\dagger\end{aligned} \tag{31a}$$

$$= \frac{\omega}{2}\big[\cos(\omega t)\,Y - \sin(\omega t)\,X\big], \tag{31b}$$

using $e^{-iaZ}Ye^{iaZ} = \cos(2a)\,Y - \sin(2a)\,X$. The expressions (30a) and (31a) manifest the fact that the state and the derivative evolve in time by rotation around the Z axis of the Bloch sphere, whereas we can read out from the expressions (30b) and (31b) the information about the time evolution of the components of the Bloch vector. The average energy $\langle H\rangle_{\psi(t)}$ is

$$\langle H\rangle_{\psi(t)} = \operatorname{tr}\Big[U_t|+\rangle\langle+|U_t^\dagger\,\frac{\omega Z}{2}\Big] = 0 \tag{32}$$

for all t, noting that U_t commutes with Z and that $\langle+|Z|+\rangle = 0$. The energy's standard deviation at time t is then

$$\sigma_H^2 = \big\langle H^2\big\rangle_{\psi(t)} = \frac{\omega^2}{4}, \tag{33}$$

noting that $Z^2 = \mathbb{1}$.

We now compute the time sensitivity and the optimal time-sensing observable locally around a given time t_0. We write $|\psi\rangle = |\psi(t_0)\rangle$ for short. The optimal time-sensing observable is given by Eq. (20), which we can compute as (ignoring the degree of freedom $P_\psi^\perp M P_\psi^\perp$),

$$T - t_0 = \frac{1}{2\sigma_H^2}\,\partial_t\psi\,(t_0) = \frac{1}{\omega}\,U_{t_0} Y U_{t_0}^\dagger = \frac{1}{\omega}\big[\cos(\omega t_0)\,Y - \sin(\omega t_0)\,X\big]. \quad (34)$$

The optimal sensing observable T is therefore aligned with the direction on the Bloch sphere that is tangent to the state's evolution.

We now determine the parameter η. It is generated by T as per Eq. (21), and we can compute the associated derivative using Eq. (26) as

$$\{H - \langle H\rangle, \psi\} = \frac{\omega}{2}\Big\{Z, \frac{\mathbb{1}+X}{2}\Big\} = \frac{\omega}{2}Z, \quad (35)$$

$$\partial_\eta\psi = \frac{1}{2\sigma_H^2}\frac{\omega}{2}Z = \frac{Z}{\omega}, \quad (36)$$

recalling that the Pauli matrices along different directions anticommute. The direction associated with the η parameter is aligned with the Z axis of the Bloch sphere (Fig. 2), which is the direction in which the Hamiltonian is oriented.

We can now compute the sensitivities with respect to t and η using Eqs. (18) and (27) as

$$F_{\mathrm{Alice},t} = 4\sigma_H^2 = \omega^2, \quad F_{\mathrm{Alice},\eta} = \frac{1}{\sigma_H^2} = \frac{4}{\omega^2}. \quad (37)$$

Finally, we can check that H is an optimal local-sensing observable for η. First observe with $\eta_0 = \langle H\rangle_\psi = 0$ that

$$\langle H\rangle_{\psi(t_0,\eta_0+d\eta)} = d\eta\,\mathrm{tr}\big(H\,\partial_\eta\psi\big) = d\eta, \quad (38)$$

using Eq. (36) along with $Z^2 = \mathbb{1}$. Hence, H satisfies the condition (7) for the parameter η. The variance of this observable was computed above as

$$\big\langle (H - \langle H\rangle)^2\big\rangle = \sigma_H^2 = \frac{\omega^2}{4} = \frac{1}{F_{\mathrm{Alice},\eta}}, \quad (39)$$

and therefore H also saturates the Cramér-Rao bound. It is an optimal local-sensing observable.

C. The noisy channel and the environment

1. The noisy clock

Suppose that Alice sends the clock from its noiseless environment to a receiver Bob through a noisy channel $\mathcal{N}_{A\to B}$ (Fig. 1). Bob has access to the noisy clock state

$$\rho_B(t) = \mathcal{N}_{A\to B}(\psi(t)). \quad (40)$$

We consider the sensitivity of Bob's clock locally around t_0, i.e., we ask how well Bob can distinguish $\rho_B(t_0)$ from $\rho_B(t_0 + dt)$. We assume that the noisy channel $\mathcal{N}_{A\to B}$ does not depend on t. This setting is nonstandard in the context of quantum metrology. Usually, one considers a quantum clock that is exposed to continuous noise as it evolves in time instead of the noise being applied separately and instantaneously after the system has evolved unitarily for a given amount of time. This alternative setting represents the situation where Alice would like to send a quantum reference frame to Bob over a noisy channel [29]. We defer the discussion of the connections between these two settings to Sec. VI.

Locally around t_0, Bob's optimal sensitivity is given via the Cramér-Rao bound (10) by Bob's Fisher information with respect to time,

$$F_{\mathrm{Bob},t} := F\big(\rho_B(t_0)\,;\,\partial_t\rho_B\,(t_0)\big). \quad (41)$$

We may furthermore express ρ_B and $\partial_t\rho_B$ as

$$\begin{aligned}\rho_B &= \mathcal{N}_{A\to B}(\psi),\\ \partial_t\rho_B &= \mathcal{N}_{A\to B}\big(\partial_t\psi\big) = \mathcal{N}_{A\to B}\big(-i[H,\psi]\big).\end{aligned} \quad (42)$$

Determining $F_{\mathrm{Bob},t}$ in principle requires the usage of a general expression of the Fisher information for mixed states such as Eq. (9) or Eq. (12), which can be significantly more cumbersome to manipulate as opposed to computing the variance of the Hamiltonian in the case of a pure-state evolution.

2. The environment

Any quantum channel $\mathcal{N}_{A\to B}$ can be expressed as a unitary evolution over a larger system, where the environment is initialized in a pure state. This construction is known as a Stinespring dilation. The initial pure state of the environment can be contracted with the global unitary to give a more concise description of the Stinespring dilation in terms of an isometry $A \to BE$. More precisely, any quantum channel $\mathcal{N}_{A\to B}$ can be written as

$$\mathcal{N}_{A\to B}(\cdot) = \mathrm{tr}_E\big(V_{A\to BE}\,(\cdot)\,V^\dagger\big), \quad (43)$$

where E is a suitable environment system, and where $V_{A\to BE}$ is an isometry mapping states of A into $B\otimes E$.

The system E, which we call Eve, represents the quantum information that is discarded by the channel $\mathcal{N}_{A\to B}$. Instead, we can consider a quantum channel that describes what Eve gets if Bob's system B is discarded. By tracing out B instead of E in Eq. (43) we obtain the *complementary channel*,

$$\widehat{\mathcal{N}}_{A\to E}(\cdot) = \mathrm{tr}_B\big(V_{A\to BE}\,(\cdot)\,V^\dagger\big). \quad (44)$$

If we write the noisy channel in an operator-sum representation with Kraus operators $\{E_k\}$ as

$$\mathcal{N}(\cdot) = \sum_k E_k(\cdot)E_k^\dagger, \tag{45}$$

we may write a corresponding complementary channel as

$$\widehat{\mathcal{N}}(\cdot) = \sum_{k,k'} \mathrm{tr}\big(E_{k'}^\dagger E_k(\cdot)\big)\, |k\rangle\langle k'|_E, \tag{46}$$

for some orthonormal basis $\{|k\rangle\}$ on the environment system E. The complementary channel is unique up to a partial isometry on the environment system.

Our main result involves Eve's sensitivity to the η parameter of the state that she obtains if Alice's quantum clock is sent to E via the complementary channel. Namely, we define

$$\rho_E(\eta) = \widehat{\mathcal{N}}_{A\to E}(\psi(\eta)). \tag{47}$$

Recalling Eq. (26), we have

$$\partial_\eta \rho_E = \frac{1}{2\sigma_H^2}\widehat{\mathcal{N}}_{A\to E}(\{H - \langle H\rangle, \psi\}). \tag{48}$$

As for ψ, $\partial_t\psi$, and $\partial_\eta\psi$, the states ρ_B, ρ_E and the derivatives $\partial_t\rho_B$, $\partial_\eta\rho_E$ are implicitly evaluated at (t_0, η_0) unless specified otherwise. We also abbreviate $\mathcal{N}_{A\to B}$ and $\widehat{\mathcal{N}}_{A\to B}$ by $\mathcal{N}$ and $\widehat{\mathcal{N}}$ for convenience and whenever it is unambiguous to do so.

III. BIPARTITE UNCERTAINTY RELATION FOR THE FISHER INFORMATION

A. Equality Fisher information trade-off for time and energy and expression for sensitivity loss

Sending Alice's clock to Bob through the noisy channel $\mathcal{N}_{A\to B}$ reduces the clock's sensitivity to the time parameter t. On the other hand, sending the clock to Eve through the complementary channel $\widehat{\mathcal{N}}_{A\to E}$ enables Eve to gain sensitivity with respect to the energy parameter η. Our main result characterizes how these two effects are related.

Theorem 1 (Bipartite time-energy uncertainty relation).—Suppose Alice prepares a probe in a quantum state vector $|\psi\rangle$ and consider the local parameters t, η defining directions in state space generated by H and T and centered at $|\psi\rangle = |\psi(t_0, \eta_0)\rangle$ as in Eq. (23). Alice sends her probe to Bob through a channel $\mathcal{N}_{A\to B}$; let Eve represent the output of the corresponding complementary channel $\widehat{\mathcal{N}}_{A\to E}$ (see Fig. 1). Then

$$\frac{F_{\mathrm{Bob},t}}{F_{\mathrm{Alice},t}} + \frac{F_{\mathrm{Eve},\eta}}{F_{\mathrm{Alice},\eta}} = 1, \tag{49}$$

provided the rank of $\mathcal{N}(\psi(t))$ does not change at t_0.

Recalling Eqs. (18) and (27), our uncertainty relation is equivalently stated as

$$\frac{F_{\mathrm{Bob},t}}{4\sigma_H^2} + \sigma_H^2 F_{\mathrm{Eve},\eta} = 1. \tag{50}$$

Using the Cramér-Rao bound (10) we can relate the optimal sensing accuracies $\langle \delta t^2_{\mathrm{Bob,est}}\rangle$, $\langle \delta\eta^2_{\mathrm{Eve,est}}\rangle$ associated with the parameters t, η on Bob and Eve's systems,

$$\frac{1}{4\sigma_H^2}\frac{1}{\langle \delta t^2_{\mathrm{Bob,est}}\rangle} + \frac{1}{4\sigma_T^2}\frac{1}{\langle \delta\eta^2_{\mathrm{Eve,est}}\rangle} \leqslant 1, \tag{51}$$

noting that equality can be achieved with sensing strategies that saturate the Cramér-Rao bound provided the rank of $\mathcal{N}(\psi(t))$ does not change at $t = t_0$.

A proof of Theorem 1 proceeds by writing the Fisher information on Bob's end, i.e., after the application of the noise channel, in terms of the Bures metric. The environment Eve is introduced as the purifying space over which the fidelity is computed via Uhlmann's theorem. The resulting expression is expressed as a semidefinite program as in Refs. [30,31]; suitably manipulating the corresponding dual problem yields the relation (49). The full proof is provided in Appendix E. We also provide an alternative proof using a semidefinite characterization of the Fisher information.

The condition the rank of $\mathcal{N}(\psi(t))$ does not change locally at the time t_0 ensures that we avoid edge cases where the correspondence between the Fisher information and the Bures metric is incomplete [32–34]. In edge cases where this condition is violated, the uncertainty relation (49) can be shown to hold as an inequality instead of an equality (see below and Appendix E). The no rank change condition is typically associated with situations where the quantum Fisher information is discontinuous. In such cases its operational relevance can be questioned; we further discuss these points below in the context of independent and identically distributed (IID) noise as well as in Appendix F.

The condition that the rank of $\mathcal{N}(\psi(t))$ does not change at $t = t_0$ is formalized by requiring that for any eigenvalue $p_k(t)$ of $\mathcal{N}(\psi(t))$ such that $p_k(t_0) = 0$ we also have $\partial_t^2 p_k(t_0) = 0$. This more precise formulation is the form of the assumption that is used in the proof. Observe that any eigenvalue $p_k(t)$ of $\mathcal{N}(\psi(t))$ that satisfies $p_k(t_0) = 0$ necessarily also satisfies $\partial_t p_k(t_0) = 0$, since the value zero is necessarily a minimum for $p_k(t)$.

Another equivalent form of our uncertainty relation (49) is one that quantifies directly the difference between the sensitivity of the noiseless clock and the resulting sensitivity on Bob's end. Let us define

$$\Delta F_{\mathrm{Bob},t} = F_{\mathrm{Alice},t} - F_{\mathrm{Bob},t} = 4\sigma_H^2 - F_{\mathrm{Bob},t}. \tag{52}$$

A few simple algebraic manipulations of Eq. (49) lead to

$$F_{\mathrm{Alice},t} - F_{\mathrm{Bob},t} = \frac{F_{\mathrm{Alice},t}}{F_{\mathrm{Alice},\eta}} F_{\mathrm{Eve},\eta}, \tag{53}$$

which gives us an expression for $\Delta F_{\mathrm{Bob},t}$. We can further spell out this expression using Eqs. (18) and (27) along with simple scaling properties that follow from the definition of the Fisher information to find

$$\Delta F_{\mathrm{Bob},t} = (2\sigma_H^2)^2 F_{\mathrm{Eve},\eta} = F\big(\rho_E\,;\,2\sigma_H^2\partial_\eta\rho_E\big) = F\big(\rho_E\,;\,\widehat{\mathcal{N}}(\{\bar{H},\psi\})\big), \tag{54}$$

where we have used Eq. (48) in the last equality.

Summarizing the above argument, we obtain an alternative form of Theorem 1 as an expression for the sensitivity loss $\Delta F_{\mathrm{Bob},t}$ in terms of the Fisher information that Eve obtains with respect to a direction associated with the anticommutator of H and ψ.

Corollary 1 (Expression for Bob's sensitivity loss via Eve).—consider the setting of Theorem 1 and assume that the rank of $\mathcal{N}(\psi(t))$ does not change locally at t_0. Then

$$\Delta F_{\mathrm{Bob},t} = F\big(\widehat{\mathcal{N}}(\psi)\,;\,\widehat{\mathcal{N}}(\{\bar{H},\psi\})\big), \tag{55}$$

where $\Delta F_{\mathrm{Bob},t} = F_{\mathrm{Alice},t} - F_{\mathrm{Bob},t}$ and where we recall the shorthand $\bar{H} = H - \langle H\rangle$. As a consequence,

$$F_{\mathrm{Bob},t} = 4\sigma_H^2 - F\big(\widehat{\mathcal{N}}(\psi)\,;\,\widehat{\mathcal{N}}(\{\bar{H},\psi\})\big). \tag{56}$$

Two extreme cases can readily be identified. One is where there is no noise and $\mathcal{N} = \mathrm{id}$ is the identity process; in this case, the complementary channel is a channel that outputs a constant state regardless of the input, $\widehat{\mathcal{N}}(\cdot) = \mathrm{tr}(\cdot)\,\tau_E$ for some state τ_E. In this case Eve obtains no information about the probe's energy, which can be seen in our formalism by the fact that $\widehat{\mathcal{N}}(\{\bar{H},\psi\}) = 0$ and therefore $\Delta F_{\mathrm{Bob},t} = 0$. In the opposite extreme case, the noise destroys its input entirely and sends it to the environment, with correspondingly $\widehat{\mathcal{N}} = \mathrm{id}$. In this case, Eve has maximal sensitivity to the η parameter, $F_{\mathrm{Eve},\eta} = F_{\mathrm{Alice},\eta}$, and therefore $F_{\mathrm{Bob},t} = 0$ and $\Delta F_{\mathrm{Bob},t} = 4\sigma_H^2$.

B. Trade-off relation in terms of a virtual qubit

In this section we simplify the setting required to produce the relation in Theorem 1, in an effort to identify the fundamental concepts required for our uncertainty relation to hold. It turns out that Theorem 1 can be rephrased as an uncertainty relation between Bob and Eve distinguishing states, respectively, along the Y and Z Pauli directions of a virtual qubit space, which in the setting of Theorem 1 is defined by the clock state vector $|\psi\rangle$ and its image $H|\psi\rangle$ under application of the Hamiltonian.

Consider the subspace of Alice's Hilbert space spanned by the probe state $|\psi\rangle$ and its time derivative $\propto H|\psi\rangle$. This subspace defines a virtual qubit. We choose to identify the probe state with the $+1$ Pauli-X eigenvector. It turns out that our uncertainty relation admits a restatement as a relation between the sensitivity that Bob and Eve can achieve with respect to Pauli-Y and logical Pauli-Z directions of the virtual qubit. More precisely, we first define

$$|\xi\rangle = P_\psi^\perp H|\psi\rangle = \big(H - \langle H\rangle\big)\,|\psi\rangle = \bar{H}|\psi\rangle, \tag{57}$$

recalling $P_\psi^\perp = \mathbb{1} - \psi$. The norm of $|\xi\rangle$ satisfies

$$\big\||\xi\rangle\big\|^2 = \langle\xi|\xi\rangle = \sigma_H^2. \tag{58}$$

Here, we assume that $|\xi\rangle \neq 0$, otherwise the probe does not evolve in time and all the terms in our uncertainty relation are trivially zero. We can write the following derivatives in terms of $|\xi\rangle$:

$$\partial_t\psi = -i[\bar{H},\psi] = -i\big(|\xi\rangle\langle\psi| - |\psi\rangle\langle\xi|\big), \tag{59a}$$

$$2\sigma_H^2\,\partial_\eta\psi = \{\bar{H},\psi\} = |\xi\rangle\langle\psi| + |\psi\rangle\langle\xi|. \tag{59b}$$

An orthonormal basis of the virtual qubit can be chosen as

$$|+\rangle_L = |\psi\rangle, \quad |-\rangle_L = \frac{1}{\sigma_H}\,|\xi\rangle. \tag{60}$$

As the logical computational basis of the virtual qubit, we choose

$$|0\rangle_L = \frac{1}{\sqrt{2}}\big[|+\rangle_L + |-\rangle_L\big], \quad |1\rangle_L = \frac{1}{\sqrt{2}}\big[|+\rangle_L - |-\rangle_L\big]. \tag{61}$$

This choice of basis is motivated to match the qubit operators of a single spin-1/2 particle prepared in the $+X$ eigenstate and evolving according to a magnetic field pointing along the Z axis.

Consider the logical Pauli-X, Y and Z operators defined as usual with respect to the basis (61). They are expressed in the $|\pm\rangle_L$ basis as

$$X_L = |+\rangle\langle+|_L - |-\rangle\langle-|_L, \tag{62a}$$

$$Y_L = -i|-\rangle\langle+|_L + i|+\rangle\,\langle-|_L, \tag{62b}$$

$$Z_L = |-\rangle\langle+|_L + |+\rangle\langle-|_L, \tag{62c}$$

with furthermore

$$Y_L = \frac{-i}{\sigma_H}[\bar{H},\psi], \quad Z_L = \frac{1}{\sigma_H}\{\bar{H},\psi\}. \tag{63}$$

We see that the logical Pauli-Y and Pauli-Z operators are parallel to the evolution, respectively, along t and along η locally at $|\psi\rangle = |\psi(t_0,\eta_0)\rangle$, as we recall Eq. (59). Our uncertainty relation can be stated in terms of a metrological logical qubit as follows.

Theorem 2 (Uncertainty relation for the metrological logical qubit).—let A, B, and E be finite-dimensional quantum systems. Let $\mathcal{N}_{A\to B}$ be a completely positive, trace nonincreasing map. Let $V_{A\to BE}$ be such that $\mathcal{N}_{A\to B}(\cdot) = \mathrm{tr}_E\big(V(\cdot)V^\dagger\big)$ and $V^\dagger V \leqslant \mathbb{1}$, i.e., V is a Stinespring dilation of $\mathcal{N}$. Consider the complementary channel $\widehat{\mathcal{N}}_{A\to E}(\cdot) = \mathrm{tr}_B\big(V(\cdot)V^\dagger\big)$. Let $|\pm\rangle_L$ be any two orthogonal and normalized state vectors on system A, and let X_L, Y_L, Z_L be defined via Eq. (62). If $(P^\perp_{\rho_B} \otimes P^\perp_{\rho_E})V|-\rangle_L = 0$, then

$$F\big(\mathcal{N}(\psi)\,;\mathcal{N}(Y_L)\big) + F\big(\widehat{\mathcal{N}}(\psi);\,\widehat{\mathcal{N}}(Z_L)\big) = 4\langle -|\mathcal{N}^\dagger(\mathbb{1})|-\rangle_L. \quad (64)$$

If $(P^\perp_{\rho_B} \otimes P^\perp_{\rho_E})V|-\rangle_L \neq 0$, then we have the inequality

$$F\big(\mathcal{N}(\psi)\,;\mathcal{N}(Y_L)\big) + F\big(\widehat{\mathcal{N}}(\psi)\,;\,\widehat{\mathcal{N}}(Z_L)\big) \leqslant\ 4\langle -|\mathcal{N}^\dagger(\mathbb{1})|-\rangle_L. \quad (65)$$

The above theorem provides a more formal statement that generalizes the earlier statement Theorem 1 to trace-non-increasing maps and to subnormalized states. The metrological qubit construction also provides a clearer mathematical picture of the symmetric role of Bob and Eve in our uncertainty relation: Bob and Eve can be interchanged (i.e., $\mathcal{N} \leftrightarrow \widehat{\mathcal{N}}$) provided we correspondingly interchange $|\xi\rangle \leftrightarrow i|\xi\rangle$. For a state vector $|\psi\rangle$ evolving with respect to a Hamiltonian $\bar{H}$, the state $|\xi\rangle = \bar{H}|\psi\rangle$ is the derivative of $|\psi\rangle$ with respect to time, and $i|\xi\rangle$ can be thought of the derivative of $|\psi\rangle$ with respect to *imaginary time*. The symmetry in Theorem 2 between Bob and Eve, which involves the interchange $|\xi\rangle \leftrightarrow i|\xi\rangle$, is reproduced at the level of the parameters t and η by choosing η to parametrize the one-family parameter of state vectors $|\psi(\eta)\rangle$ in Eq. (21) governed by the imaginary-time evolution (24). The full proof of Theorem 2 is provided in Appendix E 2.

In Theorem 2 a different condition is stated for equality as in Theorem 1, where we require the rank of $\mathcal{N}(\psi(t))$ not to change. These conditions turn out to be equivalent, as shown in the following proposition. We defer the proof to Appendix E 2.

Proposition 1 (Conditions for equality in the uncertainty relation).—let $\{E_k\}$ be a set of Kraus operators for $\mathcal{N}_{A\to B}$ and $V_{A\to BE}$ be a Stinespring dilation of $\mathcal{N}$. The following conditions are equivalent:

(a) $(P^\perp_{\rho_B} \otimes P^\perp_{\rho_E})\, V|\xi\rangle = 0$;
(b) $\rho_B(t)$ does not change rank as a function of t locally at the point t_0;
(c) for any linear combination $E = \sum c_k E_k$ (with $c_k \in \mathbb{C}$) such that $E|\psi\rangle = 0$, then $P^\perp_{\rho_B} E|\xi\rangle = 0$.

In particular, it suffices that either $\rho_B = \mathcal{N}(\psi)$ or $\rho_E = \widehat{\mathcal{N}}(\psi)$ has full rank to ensure that these conditions are satisfied, and thereby that our uncertainty relation holds with equality [Eq. (64)].

As a consequence, the situations for which the conditions (1) do not hold, and correspondingly for which our uncertainty relation does not necessarily hold with equality, are edge cases that can be infinitesimally perturbed into situations where the corresponding conditions hold. Indeed, one can mix $\mathcal{N}$ with an infinitesimal amount of depolarizing noise to ensure that Bob's state is full rank, and therefore to ensure that equality holds in our uncertainty relation.

C. General uncertainty relation for any two parameters

The uncertainty relation between position and momentum can be generalized to any arbitrary pair of observables. The Robertson uncertainty relation states that for any two observables A, B, we have

$$\sigma_A\,\sigma_B \geqslant \frac{1}{2}\big|\big\langle i[A,B]\big\rangle\big|. \quad (66)$$

In the same spirit, we derive a generalization of (49) that is valid for any two observables. Suppose Alice prepares a pure state ψ that can evolve along two possible directions $\partial_a\psi$ and $\partial_b\psi$, and sends the state through the noisy channel $\mathcal{N}$ to Bob as in Fig. 1. We assume that the directions along a, b are generated by two Hermitian operators A, B acting on $\psi = |\psi\rangle\langle\psi|$ as

$$\partial_a\psi = -i[A,\psi], \quad \partial_b\psi = -i[B,\psi]. \quad (67)$$

Bob is tasked with estimating a deviation locally to first order around $\mathcal{N}(\psi)$ in the a direction, whereas Eve tries to distinguish $\widehat{\mathcal{N}}(\psi)$ from neighboring states along the b direction. The parameters a, b are analogous to the parameters t, η considered above, but the two directions $\partial_a\psi, \partial_b\psi$ can be arbitrary.

Theorem 3 (Bipartite uncertainty relation for any two parameters).—let ψ be a state vector and suppose that A, B are two Hermitian operators that generate evolutions locally at ψ in directions $\partial_a\psi, \partial_b\psi$ via (67). Suppose we apply a noisy channel as depicted in Fig. 1. Then

$$\frac{F_{\mathrm{Bob},a}}{F_{\mathrm{Alice},a}} + \frac{F_{\mathrm{Eve},b}}{F_{\mathrm{Alice},b}} \leqslant 1 + 2\sqrt{1 - \frac{\big\langle i[A,B]\big\rangle^2}{4\sigma_A^2\sigma_B^2}}, \quad (68)$$

where

$$F_{\mathrm{Alice},a} = F\big(\psi\,;\,\partial_a\psi\big), \quad F_{\mathrm{Bob},a} = F\big(\mathcal{N}(\psi)\,;\mathcal{N}(\partial_a\psi)\big),$$
$$F_{\mathrm{Alice},b} = F\big(\psi\,;\,\partial_b\psi\big), \quad F_{\mathrm{Eve},b} = F\big(\widehat{\mathcal{N}}(\psi)\,;\widehat{\mathcal{N}}(\partial_b\psi)\big).$$

Furthermore, assume that $\mathcal{N}[\psi(a)]$ does not change rank locally and that

$$\widehat{\mathcal{N}}\big(-i[B/\sigma_B,\psi]\big) = \pm\widehat{\mathcal{N}}\big(\{(A-\langle A\rangle)/\sigma_A,\psi\}\big). \tag{69}$$

Then

$$\frac{F_{\mathrm{Bob},a}}{F_{\mathrm{Alice},a}} + \frac{F_{\mathrm{Eve},b}}{F_{\mathrm{Alice},b}} = 1. \tag{70}$$

The proof of this statement is presented in Appendix E 4. The argument of the square root in Eq. (68) never becomes negative, thanks to the Robertson uncertainty relation (66) for A and B. The proof we present in Appendix E 4 considers in fact a more general statement in which the two sides of Eq. (69) are proportional to one another rather than differing only by a sign.

We can identify two extreme cases of interest to gain some intuition for the relation (68). First consider A, B to be two complementary observables in the sense that they saturate the Robertson inequality (66). Consider, for instance, the Pauli-Y and the Pauli-Z operators on a qubit. In this case the right-hand side of the inequality (68) equals one. There is a trade-off between the sensitivity losses associated with Bob sensing along the $\mathcal{N}(\partial_a\psi)$ direction and Eve sensing along the $\widehat{\mathcal{N}}(\partial_b\psi)$ direction, as both terms on the left-hand side of Eq. (68) cannot simultaneously be equal to one. On the other hand, we can consider two Hermitian generators A, B that commute. (Perhaps A, B act on different subsystems of Alice's noiseless clock.) In this case, the right-hand side of Eq. (68) evaluates to the constant 3. Our uncertainty relation no longer presents any obstruction to both Bob and Eve sensing along the respective directions a, b as well as Alice could, as there is room for both terms on the left-hand side of Eq. (68) to be equal to one. This is the case, for instance, if A, B act on different subsystems of Alice's clock, and the respective subsystems are sent to Bob and Eve via the noisy channel and its complementary channel.

We can recover our Theorem 1 if we consider the two generators $A = H$ and $B = -T$, with H, T defined in Sec. II B, leading to $\partial_a\psi = \partial_t\psi$ and $\partial_b\psi = \partial_\eta\psi$. To see this, we first compute

$$\begin{aligned}\big\langle i[H,T]\big\rangle &= \frac{1}{2\sigma_H^2}\big\langle i[H,-i[H,\psi]]\big\rangle \\ &= \frac{1}{2\sigma_H^2}\big(2\langle H^2\rangle - 2\langle H\rangle^2\big) = 1.\end{aligned} \tag{71}$$

Using Eq. (27) we further see that $4\sigma_H^2\sigma_T^2 = 1$. Therefore, the square root on the right-hand side of Eq. (68) vanishes and the entire right-hand side of the inequality evaluates to the constant 1. With the identifications $F_{\mathrm{Alice},t} = F_{\mathrm{Alice},a}$, $F_{\mathrm{Bob},t} = F_{\mathrm{Bob},a}$, $F_{\mathrm{Alice},\eta} = F_{\mathrm{Alice},b}$, $F_{\mathrm{Eve},\eta} = F_{\mathrm{Eve},b}$, we recover the expression (49) with an inequality instead of an equality. In this case, the additional condition (69) is in fact also satisfied, since $i[T,\psi] \propto \{H - \langle H\rangle, \psi\}$ [cf. Eqs. (25) and (26)]. We thus fully recover the equality statement of Theorem 1 subject to our additional condition on the absence of a rank change of the noisy state.

The strategy of the proof of Theorem 3 (Appendix E 4) is to first apply our main uncertainty relation (Theorem 2) between Bob's sensitivity to the parameter a and Eve's sensitivity to a parameter c that is complementary to a using the construction in Sec. II B and Fig. 5 identifying $t \to a, \eta \to c$. We then apply a general bound relating the quantum Fisher information with respect to two arbitrary evolution directions (Proposition 16 in Appendix C) to bound the difference between Eve's sensitivity to the parameters b and c.

One might have assumed that the equality (70) can only be achieved if the parameters a, b are complementary in the sense of Fig. 5. Yet it suffices for this property to hold on the support of the complementary channel, as seen in the condition (69). As a simple extreme example, consider $\mathcal{N} = \mathrm{id}$ and $\widehat{\mathcal{N}}(\cdot) = \mathrm{tr}(\cdot)\,\tau$ is a constant channel preparing some fixed quantum state τ. Then our uncertainty relation equality (70) necessarily holds for any parameters a, b, since Eve's sensitivity to any parameter b is zero and Bob's sensitivity to any parameter a is equal to Alice's. This example also illustrates how the right-hand side of Eq. (68) should necessarily be improved to depend on the channel $\mathcal{N}$ if we wanted the inequality to be tight for a fixed $\mathcal{N}$. Such an improvement can be obtained from our proof in Appendix E 4.

We furthermore provide a proof that the general uncertainty relation (68) also holds in infinite-dimensional Hilbert spaces, and even for unbounded operators. The details of this proof are given in Appendix F. The proof proceeds by considering a limiting case of the finite-dimensional setting for larger and larger system sizes, with additional care given to the definition of the Fisher information in the infinite-dimensional case and to the fact that the considered operators are not necessarily bounded.

It is expected that the bound (68) can be further tightened for observables that do not saturate the Robertson bound. For instance, consider two independent systems in a pure tensor product state, with one system evolving with a parameter t and the other with z: if we hand the first system to Bob and the second to Eve, then there is no sensitivity loss for either parties and the sum of the Fisher information ratios should be 2. But the right-hand side of our bound is 3.

IV. A SELECTION OF EXAMPLES

We now explore some examples illustrating the application of our main results.

A. Single qubit subject to partial dephasing

Consider the setup in Fig. 2 and described in Sec. II B, in which Alice prepares a pure qubit in the $|+\rangle$ state vector and lets it evolve according to the Hamiltonian $H = \omega Z/2$. At time t, the qubit is in the state $\psi(t)$ given in Eq. (30) and its derivative $\partial_t\psi$ is given by Eq. (31).

Suppose that at time t_0 we apply the partially dephasing noisy channel

$$\mathcal{N}_p = (1-p)\,\mathrm{id} + p\,\mathcal{D}_Z, \tag{72}$$

where

$$\mathcal{D}_Z(\cdot) = \langle\uparrow|\cdot|\uparrow\rangle\,|\uparrow\rangle\langle\uparrow| + \langle\downarrow|\cdot|\downarrow\rangle\,|\downarrow\rangle\langle\downarrow|. \tag{73}$$

In the following, we will verify that our uncertainty relation holds in this setting, by first computing directly Bob's Fisher information with respect to t, and then computing Eve's Fisher information with respect to η.

1. Direct computation of $F_{\mathrm{Bob},t}$

Using $\mathcal{D}_Z(X) = 0 = \mathcal{D}_Z(Y)$ we find from Eq. (30b) that Bob receives the state

$$\rho_B(t_0) = \frac{1}{2}\begin{bmatrix} 1 & (1-p)\,e^{-i\omega t_0} \\ (1-p)\,e^{i\omega t_0} & 1 \end{bmatrix}. \tag{74}$$

Using Eq. (30a) along with the fact that the superoperator action of $U_t = e^{-iHt}$ and $\mathcal{N}_p$ commute and that $\mathcal{N}_p(X) = (1-p)\,X$, we can alternatively write Bob's state as

$$\begin{aligned}\rho_B(t_0) &= U_{t_0}\,\frac{\mathbb{1} + (1-p)\,X}{2}\,U_{t_0}^\dagger \\ &= \left(1-\frac{p}{2}\right)|+_{t_0}\rangle\langle +_{t_0}| + \frac{p}{2}|-_{t_0}\rangle\langle -_{t_0}|,\end{aligned} \tag{75}$$

defining the rotated basis state vectors $|\pm_t\rangle := U_t|\pm\rangle$. For the time derivative, using Eq. (31a) along with $\mathcal{N}_p(Y) = (1-p)\,Y$ we find

$$\partial_t\rho_B\,(t_0) = \mathcal{N}_p\big(\partial_t\psi\,(t_0)\big) = \frac{\omega}{2}(1-p)\,U_{t_0}\,Y\,U_{t_0}^\dagger. \tag{76}$$

We may compute the Fisher information with the formula (12), using the eigendecomposition of $\rho_B(t_0)$ given by Eq. (75)

$$\begin{aligned}F_{\mathrm{Bob},t} &= \frac{\omega^2}{4}(1-p)^2\Big[\frac{1}{1-p/2}\,|\langle+|Y|+\rangle|^2 \\ &\quad + 2\,|\langle+|Y|-\rangle|^2 + 2\,|\langle-|Y|+\rangle|^2 \\ &\quad + \frac{1}{p/2}\,|\langle-|Y|-\rangle|^2\Big] \\ &= \omega^2(1-p)^2,\end{aligned} \tag{77}$$

using $\langle+|Y|-\rangle = \langle+|YZ|+\rangle = i\langle+|X|+\rangle = i$ and $\langle+|Y|+\rangle = 0 = \langle-|Y|-\rangle$.

Recalling Eq. (37), the ratio of the Fisher information of the noisy versus the noiseless clock is

$$\frac{F_{\mathrm{Bob},t}}{F_{\mathrm{Alice},t}} = (1-p)^2. \tag{78}$$

2. Computation of $F_{\mathrm{Eve},\eta}$

Now we turn to Eve's picture. We start with computing a complementary channel to $\mathcal{N}_p$. We can use Eq. (46) for this effect from any Kraus representation of $\mathcal{N}_p$. It is useful to choose a representation with the fewest possible Kraus operators to simplify our computation of $F_{\mathrm{Eve},\eta}$. From Eq. (72), and using $\mathcal{D}_Z(\cdot) = \mathbb{1}(\cdot)\mathbb{1}/2 + Z(\cdot)Z/2$ we can read off a representation of $\mathcal{N}_p$ with the two Kraus operators

$$E_0^{(p)} = \sqrt{1-\frac{p}{2}}\,\mathbb{1}, \quad E_1^{(p)} = \sqrt{\frac{p}{2}}\,Z. \tag{79}$$

The complementary channel constructed via Eq. (46) takes the form

$$\widehat{\mathcal{N}}_p(\cdot) = \begin{bmatrix} \left(1-\frac{p}{2}\right)\mathrm{tr}(\cdot) & \sqrt{\frac{p}{2}\left(1-\frac{p}{2}\right)}\,\mathrm{tr}[Z(\cdot)] \\ \sqrt{\frac{p}{2}\left(1-\frac{p}{2}\right)}\,\mathrm{tr}[Z(\cdot)] & \frac{p}{2}\,\mathrm{tr}(\cdot) \end{bmatrix}. \tag{80}$$

Hence Eve's state is

$$\rho_E(t_0) = \widehat{\mathcal{N}}_p\big(\psi(t_0)\big) = \begin{bmatrix} 1-\frac{p}{2} & 0 \\ 0 & \frac{p}{2} \end{bmatrix}. \tag{81}$$

The derivative in the η direction is given by the image of Eq. (36) under $\widehat{\mathcal{N}}_p$, namely

$$\partial_\eta\rho_E\,(t_0) = \widehat{\mathcal{N}}_p\big(\partial_\eta\psi\,(t_0)\big) = \frac{2}{\omega}\sqrt{\frac{p}{2}\left(1-\frac{p}{2}\right)}\,X. \tag{82}$$

We may now directly compute $F_{\mathrm{Eve},\eta}$ using Eq. (12),

$$\begin{aligned}F_{\mathrm{Eve},\eta} &= \frac{4}{\omega^2}\,\frac{p}{2}\left(1-\frac{p}{2}\right)\Big[0+2+2+0\Big] \\ &= \frac{4}{\omega^2}\left(2p-p^2\right).\end{aligned} \tag{83}$$

Using Eq. (37) we find that the ratio of Eve's Fisher information to Alice's Fisher information with respect to η is

$$\frac{F_{\mathrm{Eve},\eta}}{F_{\mathrm{Alice},\eta}} = 2p - p^2 = 1-(1-p)^2. \tag{84}$$

The fact that Eqs. (78) and (84) sum to unity is a manifestation of Theorem 1 in the present setting.

Consider now our Fisher information loss formula (55). Using Eq. (35) and $\bar{H} = H - \langle H \rangle = H$ we have

$$\widehat{\mathcal{N}}_p(\{\bar{H}, \psi\}) = \frac{\omega}{2}\widehat{\mathcal{N}}_p(Z) = \omega\sqrt{\frac{p}{2}\left(1 - \frac{p}{2}\right)}X. \tag{85}$$

Then we can compute $F\big(\rho_E\,;\widehat{\mathcal{N}}_p(\{\bar{H}, \psi\})\big)$ using Eq. (12) as

$$F\big(\rho_E\,;\widehat{\mathcal{N}}_p(\{\bar{H}, \psi\})\big) = \omega^2\left(2p - p^2\right). \tag{86}$$

We can then verify that the difference in Fisher information between the noiseless clock and the noisy clock is indeed

$$\Delta F_{\mathrm{Bob},t} = F_{\mathrm{Alice},t} - F_{\mathrm{Bob},t} = \omega^2\big(1 - (1-p)^2\big)$$
$$= F\big(\rho_E\,;\widehat{\mathcal{N}}_p(\{\bar{H}, \psi\})\big). \tag{87}$$

B. Single qubit subject to complete dephasing along a transversal axis

Now we consider a variant of the above single-qubit example: we replace the noisy channel by a complete dephasing along the X axis (Fig. 6). The qubit is initialized in the state vector $|\psi_{\mathrm{init}}\rangle = |+\rangle$, where $|\pm\rangle = [|\uparrow\rangle \pm |\downarrow\rangle]/\sqrt{2}$, with a Hamiltonian $H = \omega/(2)Z$. After a time t, the state is given by Eq. (29) and at all times we have $\langle H\rangle_{\psi(t)} = 0$ and $\sigma_H^2 = \omega^2/4$. At time $t \approx t_0$ the clock is completely dephased in the X basis, as described by the noisy channel

$$\mathcal{D}_X(\cdot) = \langle +|\cdot|+\rangle\,|+\rangle\langle +| + \langle -|\cdot|-\rangle\,|-\rangle\langle -|. \tag{88}$$

This completely dephasing map acts on the Pauli operator basis as $\mathcal{D}_X(\mathbb{1}) = \mathbb{1}$, $\mathcal{D}_X(X) = X$, and $\mathcal{D}_X(Y) = 0 = \mathcal{D}_X(Z)$. Bob receives the density matrix

$$\rho_B = \cos^2\left(\frac{\omega t}{2}\right)|+\rangle\langle +| + \sin^2\left(\frac{\omega t}{2}\right)|-\rangle\langle -|. \tag{89}$$

Now the complementary channel of $\mathcal{D}_X$ is again $\widehat{\mathcal{D}}_X = \mathcal{D}_X$, and so Eve gets the same density matrix as Bob.

1. Computation of $F_{\mathrm{Eve},\eta}$

Recalling Eq. (35), we find

$$\widehat{\mathcal{D}}_X(\{H - \langle H\rangle, \psi\}) = \frac{\omega}{2}\widehat{\mathcal{D}}_X(Z) = 0, \tag{90}$$

because $\mathcal{D}_X$ maps the Pauli-Y and Pauli-Z operators to zero. Therefore, Eve obtains zero information about η, i.e., $F_{\mathrm{Eve},\eta} = 0$. Therefore, there is no sensitivity loss for Bob regardless of the time $t \approx t_0$ at which the noisy channel is applied, as long as the rank of $\rho_B(t_0)$ does not change locally at t_0. The state ρ_B changes rank whenever either term of Eq. (89) vanishes, i.e., when t_0 is a multiple of π/ω. At those discrete points, we hit the edge cases where our main uncertainty relation does not hold with equality and we cannot deduce that Bob has maximal sensitivity at those points. However, at all other points t_0 the clock does not lose any sensitivity when sent to Bob.

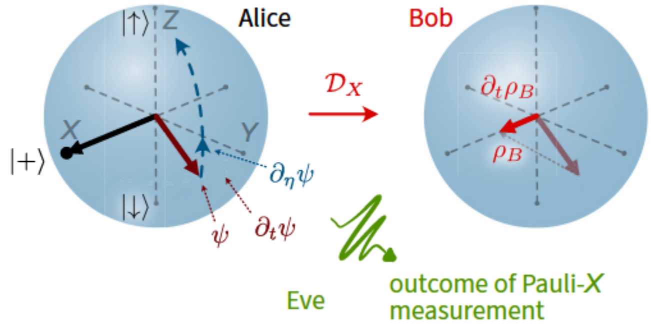

FIG. 6. Single-qubit probe evolving according to the Hamiltonian $H = \omega Z/2$ and subject to complete dephasing along the X direction at time close to t_0. For almost every t_0, the noisy probe remains maximally sensitive to time to first order around t_0. This property might sound surprising, because Bob's state can be very mixed. In the purified picture, Eve is given the outcome of a measurement of Alice's state along the X axis. Observe that in contrast to the setting in Fig. 2, this information does not reveal any information about the energy of Alice's state.

The same conclusions apply for any noisy channel that is a complete dephasing operation along an axis that lies in the equatorial plane, by rotational symmetry of the problem around the Z axis. (Any axis in the equatorial plane can be described as a rotation of the X axis that is equivalent to a time evolution of the system for some given time t_*. Because the Fisher information is invariant under unitary transformations, the calculation of Bob's Fisher information of this qubit after complete dephasing along that given axis at time t_0 is equivalent to calculating the Fisher information after a complete dephasing along the X axis at the time $t_0 \to t_0 - t_*$.)

2. Check by direct computation of $F_{\mathrm{Bob},t}$

We now compute $F_{\mathrm{Bob},t} = F\big(\rho_B\,;\mathcal{D}_Z(\partial_t\psi)\big)$ directly, by using the definition of the Fisher information. From Eq. (31b) we find

$$\mathcal{D}_X(\partial_t\psi) = -\frac{\omega}{2}\sin(\omega t_0)\,X. \tag{91}$$

If $\sin(\omega t_0) = 0$, which happens when t_0 is a multiple of π/ω, we find that Bob's state is locally stationary and Bob has no sensitivity to first order in t. (For this discrete set of points one could argue that the Fisher information no longer represents the relevant sensitivity for Bob, since the evolution should be considered to its leading order—here the second order—and no longer only to first order.)

We now compute $F_{\mathrm{Bob},t}$ for all times t_0 where $\sin(\omega t_0) \neq 0$. Observe that ρ_B and $\mathcal{D}_X(\partial_t\psi)$ commute. Using Eq. (15) and $X^2 = \mathbb{1}$, we find

$$F_{\mathrm{Bob},t} = \frac{\omega^2}{4}\sin^2(\omega t_0)\,\mathrm{tr}\big(\rho_B^{-1}\big) = \omega^2\left[\sin\left(\frac{\omega t_0}{2}\right)\cos\left(\frac{\omega t_0}{2}\right)\right]^2 \times\left[\frac{1}{\cos^2\left(\frac{\omega t_0}{2}\right)} + \frac{1}{\sin^2\left(\frac{\omega t_0}{2}\right)}\right] = \omega^2, \tag{92}$$

using $\sin(\omega t_0) = 2\sin(\omega t_0/2)\cos(\omega t_0/2)$ in the second equality.

Overall, we see that Bob still has maximal sensitivity even after application of the completely dephasing channel along the transversal X axis, for all times except for the discrete set of times t_0 where the rank of ρ_B changes. This conclusion matches our earlier conclusions obtained via considerations from Eve's perspective (except for a discrete set of times t_0).

It might appear counterintuitive that Bob's state still has as high a sensitivity as Alice's noiseless state for almost all t_0, especially as Bob's state can get arbitrarily mixed. Indeed, ρ_B coincides with the maximally mixed state for times t_0 that are midpoints between the multiples of π/ω. However, we see that $\rho_B(t)$ still varies with t sufficiently to enable optimal discrimination of nearby states to first order around t_0.

C. Probe in a GHZ state with one partial erasure

Consider as initial state an n-party GHZ state vector,

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}[|\uparrow\cdots\uparrow\rangle + |\downarrow\cdots\downarrow\rangle], \tag{93}$$

and let the system evolve according to the local Hamiltonian $H = \sum_i(\omega/2)Z^{(i)}$ where $Z^{(i)}$ denotes the Pauli-Z operator acting on the ith site. Suppose that the first qubit is lost with probability p. This is represented by the noisy channel

$$\mathcal{N}(\cdot) = p\,|\phi_\perp\rangle\langle\phi_\perp| \otimes \mathrm{tr}_1(\cdot) + (1-p)\,(\cdot), \tag{94}$$

where tr_1 traces out the first qubit and where $|\phi_\perp\rangle$ is a state vector in a new, orthogonal dimension that has no overlap with the input state. A Stinespring dilation of the first term in $\mathcal{N}$ is described as giving the first qubit of Alice's system to Eve, and the remaining qubits to Bob; any missing qubits on either Bob or Eve's side is replaced by $|\phi_\perp\rangle$. The complementary channel can thus be computed as

$$\widehat{\mathcal{N}}(\cdot) = p\ \mathrm{tr}_{2\ldots n}(\cdot) + (1-p)\,\mathrm{tr}(\cdot)\,|\phi_\perp\rangle\langle\phi_\perp|. \tag{95}$$

We compute the sensitivity loss associated with the noise according to Eq. (55). We have

$$H|\psi\rangle = \frac{n\omega}{2\sqrt{2}}[|\uparrow\cdots\uparrow\rangle - |\downarrow\cdots\downarrow\rangle] = P_\psi^\perp H|\psi\rangle, \tag{96}$$

noting that $H|\psi\rangle$ is already orthogonal to $|\psi\rangle$ since $\langle H\rangle_\psi = 0$. The optimal noiseless sensitivity is

$$F_{\mathrm{Alice},t} = 4\sigma_H^2 = 4\langle\psi|H^2|\psi\rangle = n^2\omega^2, \tag{97}$$

exhibiting the expected Heisenberg scaling for optimally entangled probe states. We write $\{\psi,\bar{H}\} = P_\psi^\perp H\psi +$ h.c. $= \sigma_H Z_L$, with Z_L defined in Eq. (63). The local reduced operator of $\{\psi,\bar{H}\}$ on a single site is

$$\mathrm{tr}_{\setminus i}(\{\psi,\bar{H}\}) = \frac{n\omega}{4}|\uparrow\rangle\langle\uparrow| - \frac{n\omega}{4}|\downarrow\rangle\langle\downarrow| + \text{h.c.} = \frac{n\omega}{2}Z^{(i)}, \tag{98}$$

where $\mathrm{tr}_{\setminus i}$ denotes the partial trace over all subsystems except the ith subsystem. Noting that $\mathrm{tr}(\{\psi,\bar{H}\}) = 0$, we obtain

$$\widehat{\mathcal{N}}(\{\psi,\bar{H}\}) = p\,\frac{n\omega}{2}Z. \tag{99}$$

On the other hand, the reduced state of ψ on a single site is simply the maximally mixed state $\mathbb{1}_2/2$ and thus

$$\rho_E = \widehat{\mathcal{N}}(\psi) = p\,\frac{\mathbb{1}_2}{2} + (1-p)\,|\phi_\perp\rangle\langle\phi_\perp|. \tag{100}$$

As ρ_E and $\widehat{\mathcal{N}}(\{\psi,\bar{H}\})$ commute, we can use Eq. (15) to see that

$$\Delta F_{\mathrm{Bob},t} = \mathrm{tr}\left[\frac{2}{p}\left(p\frac{n\omega}{2}Z\right)^2\right] = pn^2\omega^2. \tag{101}$$

If $p = 1$, Eve is maximally disturbing and completely blocks Bob's ability to measure time, if $p = 0$ there is no sensitivity loss. Any value in between interpolates between these two cases.

Note that while it might appear here that Heisenberg scaling ($F_{\mathrm{Bob},t} \propto n^2$) is achieved for $p > 0$, this is an artifact of the lack of scaling in n of our choice of noisy channel and does not contradict the findings of, e.g., Ref. [22,31].

D. Estimating a signal Hamiltonian term

In this subsection, we briefly comment on the case where the parameter to estimate is not time t itself, but a parameter f in the Hamiltonian that influences time evolution. In other words, we now account for possible other terms in the Hamiltonian that contribute to time evolution but that reveal nothing about the parameter of interest.

We assume that the noiseless probe evolves according to a Hamiltonian

$$H_f = H_0 + fG, \tag{102}$$

where H_0 does not depend on f, and where H_0 and G are time independent. References [35,36] have determined that the Fisher information with respect to f that one achieves by initializing the system in some initial state vector $|\psi_0\rangle$ and letting the system evolve according to H_f for some fixed time T. Let $U_f(T) = e^{-iH_f T}$ be the time-evolution operator, and define $|\psi_f\rangle = U_f(T)|\psi_0\rangle$. The question is, how much sensitivity does the family of state vectors $f \mapsto |\psi_f\rangle$ offer with respect to f? The derivative relevant for the Fisher information is given by [36]

$$\partial_f \psi_f = -i[K_f, \psi_f], \tag{103}$$

where

$$K_f = -iU_f^{-1}\frac{dU_f}{df} = T\sum_{k=0}^{\infty}\frac{(-iT)^k}{(k+1)!}\,\mathrm{ad}^k_{H_f}(G), \tag{104}$$

where $\mathrm{ad}_M(G) := [M, G]$ and

$$\mathrm{ad}^k_M(G) := [M,[M,\ldots,[M,G]] \tag{105}$$

is the kth commutator of M with G. The operator K_f can be thought of as an effective "Hamiltonian" for the parameter f, driving an "evolution" in $|\psi_f\rangle$ with respect to f according to Eq. (103).

If we send this probe state through a noisy channel following the setting in Fig. 1, then our uncertainty relation can be applied, where the complementary parameter evolution is generated by the operator $L = -i[K_f, \psi_f]/(2\sigma^2_{K_f})$. That is, Bob's sensitivity to f trades off with Eve's sensitivity to the parameter generated by L.

E. Symmetric codes against erasures via superpositions of Dicke states

Based on the relevance of Dicke states for metrology [37–41] and for quantum error correction [42–44], we can ask whether our uncertainty relation can guide a search for good clock states. To ensure good sensitivity even in the noiseless setting, we seek probe states with a large spread over energy eigenstates. So we consider a general superposition of Dicke states corresponding to different numbers of excitations. We note an important class of permutation-invariant codes are those developed in Refs. [41,42].

Consider the n-spin noninteracting Hamiltonian $H = \sum_{i=1}^{n}(\omega/2)\,Z^{(i)}$. A Dicke state is an eigenstate of H that is symmetric under permutations of the sites. Consider the Dicke state

$$|h^n_q\rangle := \binom{n}{q}^{-1/2}\sum_{\substack{s_i=\pm 1\\ \sum s_i = n-2q}} |s_1\ldots s_n\rangle, \tag{106}$$

where $s_i = \pm 1$ represents the eigenstates of Z and where $q = 0,\ldots,n$. We construct our probe states as a superposition of Dicke states of different values of q. In general, such a state vector can be written as

$$|\psi\rangle = \sum_{q=0}^{n}\psi_q|h^n_q\rangle, \tag{107}$$

for some arbitrary complex amplitudes $\{\psi_q\}$ that satisfy $\sum_q|\psi_q|^2 = 1$.

As a noise model, we assume that k systems chosen at random are entirely erased. Because the probe state is completely symmetric, it does not matter which subsystems are erased; we may assume that the first k sites are erased. The complementary channel to the erasure of k subsystems is a channel that provides those lost subsystems to Eve,

$$\widehat{\mathcal{N}}(\cdot) = \mathrm{tr}_{k+1\ldots n}(\cdot), \tag{108}$$

where $\mathrm{tr}_{k+1\ldots n}$ denotes the partial trace over sites $k+1$ to n.

We compute numerical values for the quantities $\Delta F_{\mathrm{Bob},t}$ and $4\sigma^2_H$, enabling us to infer $F_{\mathrm{Bob},t}$. Consider the probe state vector consisting of an even superposition of two Dicke states with associated parameters q_1, q_2

$$|\psi\rangle = [|h^n_{q_1}\rangle + |h^n_{q_2}\rangle]/\sqrt{2}. \tag{109}$$

The sensitivity of this probe state for $n = 100$ and subject to $k = 9$ erasures is plotted as a function of q_1, q_2 in Fig. 7 (with $\omega/2 = 1$). The sensitivity $F_{\mathrm{Bob},t}$ is obtained by computing $\Delta F_{\mathrm{Bob},t}$ and σ^2_H via Eq. (52). On the one hand, our trade-off relation facilitates the calculation of the remaining Fisher information after the erasures. On the other hand, the trade-off relation explains that the high sensitivity loss experienced for states with a broad spread in energy ($q_1 \to 0$ and $q_n \to n$) is directly related to the fact that the environment can well infer the energy of the state from few-site reduced states.

Because the noise is local, numerical computations only have to take place on a smaller system representing the local degrees of freedom. Because of permutation symmetry globally and also locally (the reduced state also lives in the local symmetric subspace), our computations run on $k+1$ dimensions and not on the full $(n+1)$-dimensional symmetric subspace. We will return to the example of permutation-invariant states on n spins in Sec. VIII, where we consider an IID amplitude-damping noise model instead of erasures.

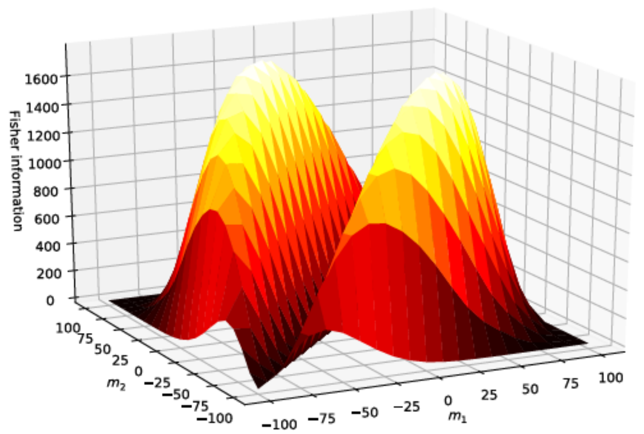

FIG. 7. Fisher information of an even superposition of two Dicke states of magnetizations $m_1 = n - 2q_1$ and $m_2 = n - 2q_2$ on a n-site noninteracting spin chain with local terms $H_i = (\omega/2)Z$. A good probe state has m_1, m_2 far from one another (for a large energy spread), but also far from the edges $-n$ and n (to avoid decoherence caused by the erasures). Here we set $n = 100$ total spins, $\omega/2 = 1$, and $k = 9$ spins are lost to the environment. Our trade-off relation facilitates the calculation of the Fisher information plotted above. It also gives an interpretation of the loss in sensitivity with respect to the noiseless case (where the GHZ state $m_1 = -m_2 = \pm n$ would be optimal; leftmost and rightmost edges of the plot) as the sensitivity that Eve gains with respect to the energy of the state.

V. BOUNDS ON THE FISHER INFORMATION

Because it might not always be simple to compute the Fisher information trade-off quantity $\Delta F_{\mathrm{Bob},t}$ in Eq. (55), we provide a few bounds that might be applicable to different settings, and that avoid the calculation of the symmetric logarithmic derivative on Eve's system.

A. Upper bound on Bob's sensitivity by postprocessing Eve's system

A useful bound for the Fisher information is the data processing inequality [19]. The inequality states that for any $\rho(t)$, and for any t-independent completely positive, trace-non-increasing map $\mathcal{E}$, the sensitivity after application of the channel can only decrease:

$$F(\rho(t)) \geqslant F(\mathcal{E}(\rho(t))). \tag{110}$$

A trace-non-increasing map can be used to describe only a subspace of interest of a larger Hilbert space while accounting for leakage outside of that subspace.

Consider our setup with Alice, Bob, and Eve as in Fig. 1. Suppose now that Eve sends her state to another agent, Eve′, through a trace-non-increasing, completely positive map $\mathcal{N}'$ as depicted in Fig. 8(a). The data-processing inequality ensures that $F_{\mathrm{Eve},\eta} \geqslant F_{\mathrm{Eve}',\eta}$. Combining this

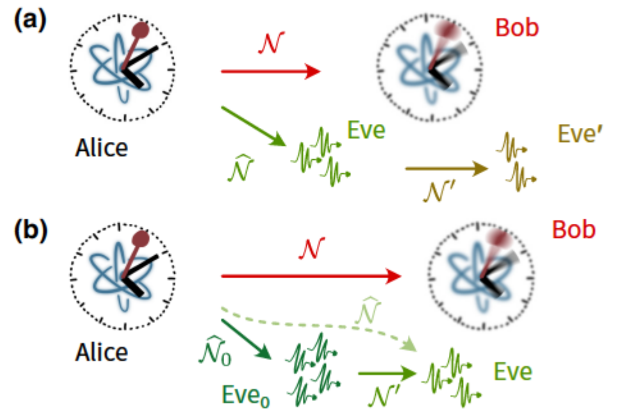

FIG. 8. Combining our uncertainty relation with the data processing inequality for the Fisher information yields new bounds for the Fisher information. (a) Suppose Eve applies a suitably chosen map $\mathcal{N}'$ to her system, resulting in a system we denote by Eve′, on which the sensitivity to energy might be significantly easier to compute. Eve′ can only have a worse sensitivity to energy than Eve, so our uncertainty relation gives an upper bound to Bob's sensitivity to time. (b) Suppose that Eve's output can be written as a composition of two maps $\widehat{\mathcal{N}}_0$ and $\mathcal{N}'$ via an intermediate system Eve$_0$. Then we obtain a lower bound on Bob's sensitivity to time by computing Eve$_0$'s sensitivity to energy.

with our uncertainty relation (49) yields

$$\frac{F_{\mathrm{Bob},t}}{F_{\mathrm{Alice},t}} + \frac{F_{\mathrm{Eve}',\eta}}{F_{\mathrm{Alice},\eta}} \leqslant 1. \tag{111}$$

We can also obtain this inequality by starting from the quantum Fisher information loss on Bob's end, Eq. (55),

$$\begin{aligned}\Delta F_{\mathrm{Bob},t} &= F\big(\rho_E\,;\widehat{\mathcal{N}}(\{\bar{H},\psi\})\big)\\ &\geqslant F\big(\mathcal{N}'(\rho_E)\,;\mathcal{N}'\big(\widehat{\mathcal{N}}(\{\bar{H},\psi\})\big)\big),\end{aligned} \tag{112}$$

which in turn provides an upper bound on Bob's Fisher information via Eq. (52) as

$$F_{\mathrm{Bob},t} \leqslant 4\sigma_H^2 - F\big(\mathcal{N}'(\rho_E)\,;\mathcal{N}'\big(\widehat{\mathcal{N}}(\{\bar{H},\psi\})\big)\big). \tag{113}$$

By choosing the map $\mathcal{N}'$ suitably, one can potentially significantly simplify the computation of the Fisher information. For instance, $\mathcal{N}'$ can be a dephasing map that ensures that $\mathcal{N}'(\rho_E)$ and $\mathcal{N}'(\widehat{\mathcal{N}}\{\psi, H\})$ commute, therefore enabling the use of Eq. (15) and removing the necessity of computing the symmetric logarithmic derivative. Alternatively $\mathcal{N}'$ can be chosen to enforce some symmetry that might be convenient for the computation of the Fisher information.

The bound (113) can be spelled out in the case of IID noise on a many-body probe state. Consider a single-site noisy channel $\mathcal{N}_1$ with Kraus operators $\{E_x\}$ for $x =$

$0, \ldots, m-1$. The full noisy channel is $\mathcal{N} = \mathcal{N}_1^{\otimes n}$. Its Kraus operators are $E_{\mathbf{x}}$, where $\mathbf{x} = (x_1, \ldots, x_n)$ is a collection of indices $x_i = 0, \ldots, m-1$ indicating which Kraus operator is applied on the ith site

$$E_{\mathbf{x}} = \bigotimes_{i=1}^{n} E_{x_i}. \tag{114}$$

The complementary channel $\widehat{\mathcal{N}}$ can then be written in terms of the Kraus operators of $\mathcal{N}$ as

$$\widehat{\mathcal{N}}(\cdot) = \sum_{\mathbf{x},\mathbf{x}'} \mathrm{tr}\big(E_{\mathbf{x}'}^{\dagger} E_{\mathbf{x}}\,(\cdot)\big)|\mathbf{x}\rangle\langle\mathbf{x}'|, \tag{115}$$

where $\{|\mathbf{x}\rangle\}$ is a basis of the Hilbert space of E.

Computing the Fisher information analytically on the output of either $\mathcal{N}$ or $\widehat{\mathcal{N}}$ might not be straightforward if the state and its derivative are mapped to operators whose eigenbases are not aligned in any obvious way, which would complicate the calculation of the symmetric logarithmic derivative when computing the expression (55). Here, we see that by completely dephasing the output of $\widehat{\mathcal{N}}$ in the computational basis, and projecting onto the subspace of the environment associated with low-weight Kraus operators of $\mathcal{N}$, we obtain a lower bound on $\Delta F_{\mathrm{Bob},t}$ which translates into a upper bound on $F_{\mathrm{Bob},t}$ that is easy to compute. Here, we assume that the first Kraus operator E_0 is close to the identity and that the other Kraus operators represent "jump terms." We mean by "weight" the number of Kraus operators that are jump terms.

We now choose a suitable completely positive, trace-non-increasing map $\mathcal{N}'$ in order to use Eq. (113) to obtain an upper bound on the Fisher information at Bob's end. In the following, we assume that $m = 2$, but the argument generalizes straightforwardly to noisy channels that have more Kraus operators. We design the map such that it (i) completely dephases the environment system in the computational basis, and (ii) projects its input onto the subspace associated with basis vectors $|\mathbf{x}\rangle$ with small Hamming weight $|\mathbf{x}|$. Fix $k > 0$ and let

$$\mathcal{N}'(\cdot) = \sum_{\mathbf{x}:\ |\mathbf{x}|\leqslant \mathbf{k}} |\mathbf{x}\rangle\langle\mathbf{x}|\,(\cdot)\,|\mathbf{x}\rangle\langle\mathbf{x}|. \tag{116}$$

Then we can see that

$$\big(\mathcal{N}' \circ \widehat{\mathcal{N}}\big)(\cdot) = \sum_{\mathbf{x}:\ |\mathbf{x}|\leqslant \mathbf{k}} \mathrm{tr}\big(E_{\mathbf{x}}^{\dagger} E_{\mathbf{x}}\,(\cdot)\big)|\mathbf{x}\rangle\langle\mathbf{x}|. \tag{117}$$

The upper bound on Bob's Fisher information with respect to time comes from Eq. (113). Starting from Eq. (112) and since the two arguments of the Fisher information commute, we can use Eq. (15) to find

$$\begin{aligned}\Delta F_{\mathrm{Bob},t} &\geqslant \mathrm{tr}\Big\{\big[\big(\mathcal{N}' \circ \widehat{\mathcal{N}}\big)(\psi)\big]^{-1}\big[\big(\mathcal{N}' \circ \widehat{\mathcal{N}}\big)(\{\psi, \bar{H}\})\big]^2\Big\} \\ &= \sum_{\mathbf{x}:\ |\mathbf{x}|\leqslant \mathbf{k}} \frac{\big[2\,\mathrm{Re}\langle\psi|\bar{H}E_{\mathbf{x}}^{\dagger}E_{\mathbf{x}}|\psi\rangle\big]^2}{\mathrm{tr}(E_{\mathbf{x}}^{\dagger}E_{\mathbf{x}}\,\psi)},\end{aligned} \tag{118}$$

where we have used the fact that the output of $\mathcal{N}' \circ \widehat{\mathcal{N}}$ is diagonal in the computational basis. The completely dephasing channel ensures that the expression (118) is a classical Fisher information, which is easier to compute than the quantum Fisher information in which the state and the derivative do not commute.

The number of terms in the above sum, which corresponds to the dimension of the subspace associated with basis vectors $|\mathbf{x}\rangle$ satisfying $|\mathbf{x}| \leqslant k$, is given by $\binom{n}{k} + \binom{n}{k-1} + \cdots + \binom{n}{0} = O(n^k)$. For fixed k, this number scales polynomially in n. The complexity of computing the numerator and denominator in Eq. (118) also scales only polynomially in n as long as $|\psi\rangle$ and $\bar{H}|\psi\rangle$ can be expressed using a representation that enables efficient computation of local expectation values, such as a superposition of a constant number of computational basis vectors, or alternatively as matrix-product states [45]. We discuss below the case of IID amplitude damping noise, where numerical evidence indicates that for small values of p (say $p \lesssim 0.1$), even for $n = 50$ it can suffice to set $k = 4$ to obtain meaningful bounds (see Sec. VIII).

B. Lower bound on Bob's Fisher information by preprocessing Eve's system

Let us return to the original setting with Alice, Bob, and Eve as in Fig. 1. Suppose now that we can find a completely positive, trace-preserving map $\widehat{\mathcal{N}}_0$ and a completely positive, trace-non-increasing $\mathcal{N}'$ such that $\widehat{\mathcal{N}} = \mathcal{N}' \circ \widehat{\mathcal{N}}_0$. That is, we suppose that Eve gets her state through an intermediary, which we call Eve_0 as shown in Fig. 8(b). The data-processing inequality now tells us that $F_{\mathrm{Eve},\eta} \leqslant F_{\mathrm{Eve}_0,\eta}$. Combining this with our uncertainty relation gives us

$$\frac{F_{\mathrm{Bob},t}}{F_{\mathrm{Alice},t}} + \frac{F_{\mathrm{Eve}_0,\eta}}{F_{\mathrm{Alice},\eta}} \geqslant 1. \tag{119}$$

A more explicit bound on $F_{\mathrm{Bob},t}$ can be obtained starting from Eq. (55) and writing

$$\begin{aligned}\Delta F_{\mathrm{Bob},t} &= F\big(\rho_E\,;\widehat{\mathcal{N}}(\{\bar{H},\psi\})\big) \\ &= F\big(\mathcal{N}'(\widehat{\mathcal{N}}_0(\psi))\,;\widehat{\mathcal{N}}'(\widehat{\mathcal{N}}_0(\{\bar{H},\psi\}))\big) \\ &\leqslant F\big(\widehat{\mathcal{N}}_0(\psi)\,;\widehat{\mathcal{N}}_0(\{\bar{H},\psi\})\big).\end{aligned} \tag{120}$$

We present two simple example uses of this type of bound. The first example applies to permutation-invariant systems. The second example applies to the setting where Eve's state is reasonably close to being diagonal.

1. Permutation-invariant system

Consider a permutation-invariant clock state ψ and Hamiltonian H. If the noise $\mathcal{N}$ acts only locally on at most k known sites (or $k/2$ unknown sites), then $\widehat{\mathcal{N}}$ can be written as the composition of a channel that traces out all but k sites, and another channel that completes the implementation of $\widehat{\mathcal{N}}$. To see this, observe that we can write $\widehat{\mathcal{N}}(\cdot) = \sum_{j',j} \operatorname{tr}\big(E_{j'}^\dagger E_j\,(\cdot)\big)\,|j'\rangle\langle j|$, where $\{E_j\}$ are the Kraus operators of $\mathcal{N}$. By assumption, $E_{j'}^\dagger E_j$ acts nontrivially on at most k sites. Therefore, the expression $\operatorname{tr}\big(E_{j'}^\dagger E_j\,(\cdot)\big)$ depends only on the k-site reduced state of the input. The full complementary channel can be written as the composition of a channel that traces all but k sites, and the channel $\rho_k \mapsto \sum_{j',j} \operatorname{tr}(E_{j'}^\dagger E_j\,\rho_k)\,|j'\rangle\langle j|$ (where here we reuse the notation $E_{j'}^\dagger E_j$ to denote the action of those operators on only the k sites where either operator acts nontrivially). Therefore, the sensitivity loss $\Delta F_{\mathrm{Bob},t}$ can be upper bounded, for any noisy channel consisting of Kraus operators of weight at most $k/2$, by the sensitivity loss associated with k located erasures.

2. If Eve's state is nearly diagonal

Computing useful expressions of the Fisher information when a diagonal representation of the state is not known can be tricky. The idea if ρ_E is reasonably close to being diagonal is to hope that one can essentially neglect the off-diagonal elements of ρ_E and still obtain a good approximation of the Fisher information via the formula (12).

Suppose we find an invertible matrix A (with hopefully $A \approx \mathbb{1}$) and a diagonal matrix $\tau = \operatorname{diag}(\tau_0, \ldots, \tau_{d_E}) \geqslant 0$ such that

$$\rho_E = A\,\tau\,A^\dagger. \tag{121}$$

Such a matrix is given, for instance, by the LDLT and Cholesky decomposition of ρ_E. [The eigendecomposition of ρ_E also gives such a matrix A, but if we can compute an eigendecomposition one might as well use Eq. (12) to compute the Fisher information directly.] Now we decompose $\widehat{\mathcal{N}}$ by including a scaling factor α as

$$\alpha\widehat{\mathcal{N}} = \mathcal{N}' \circ \widehat{\mathcal{N}}_0, \tag{122}$$

with $\alpha = \|A\|^{-2}\|A^{-1}\|^{-2}$ and with the two completely positive, trace-non-increasing maps

$$\widehat{\mathcal{N}}_0(\cdot) = \frac{1}{\|A^{-1}\|^2} A^{-1}\,\widehat{\mathcal{N}}(\cdot)\,A^{-\dagger}, \tag{123}$$

$$\mathcal{N}'(\cdot) = \frac{1}{\|A\|^2} A\,(\cdot)\,A^\dagger. \tag{124}$$

If A is close to $\mathbb{1}$ then we have $\alpha \approx 1$. Recalling the scaling property (14) of the quantum Fisher information, we find

$$\begin{aligned}
&\Delta F_{\mathrm{Bob},t} \\
&= F\big(\rho_E\,;\,\widehat{\mathcal{N}}(\{\bar{H},\psi\})\big) \\
&= \frac{1}{\alpha} F\big(\alpha\widehat{\mathcal{N}}(\psi)\,;\,\alpha\widehat{\mathcal{N}}(\{\bar{H},\psi\})\big) \\
&\leqslant \frac{1}{\alpha} F\big(\widehat{\mathcal{N}}_0(\psi)\,;\,\widehat{\mathcal{N}}_0(\{\bar{H},\psi\})\big) \\
&= \frac{1}{\alpha} F\left(\frac{1}{\|A^{-1}\|^2}\,\tau\,;\,\frac{1}{\|A^{-1}\|^2} A^{-1}\widehat{\mathcal{N}}\big(\{\bar{H},\psi\}\big)(A^{-1})^\dagger\right) \\
&= \|A\|^2\, F\Big(\tau\,;\,A^{-1}\widehat{\mathcal{N}}\big(\{\bar{H},\psi\}\big)(A^{-1})^\dagger\Big).
\end{aligned} \tag{125}$$

In the last expression, the Fisher information is evaluated on a state that is diagonal, so one can directly use Eq. (12). Furthermore, if A is determined by a LDLT and Cholesky decomposition then it is lower triangular and its inverse can be computed efficiently (matrix multiplication of the inverse with another matrix can be done by forward substitution).

C. Bound in terms of Eve's access to the probe's energy

In this section, a further bound on Bob's sensitivity to time is presented, which is given in terms of how well Eve can approximate a measurement of energy on the noiseless clock state. The properties that Eve can measure on the noiseless probe are given by the adjoint of the complementary channel: Eve applying an operator W on her system can equivalently be described as the operator $\widehat{\mathcal{N}}^\dagger(W)$ being applied onto Alice's system, because $\operatorname{tr}\big(\widehat{\mathcal{N}}(\psi)\,W\big) = \operatorname{tr}\big(\psi\,\widehat{\mathcal{N}}^\dagger(W)\big)$. One measure of how well Eve can approximate a measurement of the Hamiltonian around $|\psi\rangle$ with an observable S on her system is the minimum root-mean-squared error $\min_{S=S^\dagger}\big[\big\langle\big(\widehat{\mathcal{N}}^\dagger(S)-H\big)^2\big\rangle_\psi\big]^{1/2}$. It turns out that the minimum square of this quantity is a lower bound to Bob's Fisher information to time

$$F_{\mathrm{Bob},t} \geqslant \min_{S=S^\dagger} 4\Big\langle\big(\widehat{\mathcal{N}}^\dagger(S)-H\big)^2\Big\rangle_\psi. \tag{126}$$

While this bound is aesthetically interesting, finding the optimal S in this expression is not significantly easier than directly solving the semidefinite program (16a). Furthermore, a candidate for S in Eq. (16a) immediately provides an upper bound on F_{Bob}, whereas a candidate in Eq. (126) does not provide any useful bound on F_{Bob} because of the direction of the inequality.

The bound (126) is proven as follows. Starting from Eq. (56) and using Eq. (16a),

$$\begin{aligned}
&\frac{1}{4}F_{\mathrm{Bob},t}\\
&= \min_{S=S^\dagger}\left[\langle \bar{H}^2\rangle - \mathrm{tr}\big(\psi\{\bar{H},\widehat{\mathcal{N}}^\dagger(S)\}\big) + \langle \widehat{\mathcal{N}}^\dagger(S^2)\rangle\right]\\
&\geqslant \min_{S=S^\dagger}\left[\langle \bar{H}^2\rangle - \langle\{\bar{H},\widehat{\mathcal{N}}^\dagger(S)\}\rangle + \langle[\widehat{\mathcal{N}}^\dagger(S)]^2\rangle\right]\\
&= \min_{S=S^\dagger}\Big\langle(\bar{H}-\widehat{\mathcal{N}}^\dagger(S))^2\Big\rangle, \qquad (127)
\end{aligned}$$

where we have used $\widehat{\mathcal{N}}^\dagger(S^2) \geqslant [\widehat{\mathcal{N}}^\dagger(S)]^2$ (see Corollary 2 in Appendix A). Finally, we can replace $\bar{H}$ by H in Eq. (127) because any shifts of $\bar{H}$ by the identity can be canceled out by corresponding shifts of S by the identity.

D. If Eve can measure the probe's energy almost perfectly

If Eve has (approximate) access to the energy of the probe state, then this (approximately) kills sensitivity on Bob's end. Suppose we can find an observable S on Eve's system such that $\|\widehat{\mathcal{N}}^\dagger(S)-\bar{H}\| \leqslant \|\bar{H}\|\,\delta$ and $\|\widehat{\mathcal{N}}^\dagger(S^2)-\bar{H}^2\| \leqslant \|\bar{H}\|^2\,\delta$. Then

$$F_{\mathrm{Bob},t} \leqslant 12\delta\,\|\bar{H}\|^2. \qquad (128)$$

To show this inequality, we first write $\Delta = \widehat{\mathcal{N}}^\dagger(S)-\bar{H}$ and $\Delta' = \widehat{\mathcal{N}}^\dagger(S^2)-\bar{H}^2$, with $\|\Delta\| \leqslant \|\bar{H}\|\delta$ and $\|\Delta'\| \leqslant \|\bar{H}\|^2\delta$. Then, from Eq. (56) and using Eq. (16a) we obtain

$$\begin{aligned}
&\frac{1}{4}F_{\mathrm{Bob},t}\\
&= \min_{S=S^\dagger}\left[\langle \bar{H}^2\rangle - \mathrm{tr}\big(\{\psi,\bar{H}\}\,\widehat{\mathcal{N}}^\dagger(S)\big) + \langle \widehat{\mathcal{N}}^\dagger(S^2)\rangle\right]\\
&\leqslant \min_{S=S^\dagger}\left\{-\,\mathrm{tr}\big(\{\psi,\bar{H}\}\,\Delta\big) + \mathrm{tr}(\psi\,\Delta')\right\}\\
&\leqslant 2\|\bar{H}\|\,\|\Delta\| + \|\Delta'\| \leqslant 3\delta\,\|\bar{H}\|^2. \qquad (129)
\end{aligned}$$

E. Clock sensitivity loss for weak IID noise

Here, we consider an n-site system subject to weak IID noise, where each site is affected by a noisy channel $\mathcal{N}_\epsilon$ such that $\mathcal{N}_\epsilon \to \mathrm{id}$ if $\epsilon \to 0$. Clearly for $\epsilon = 0$ there is no sensitivity loss. For a given clock state and Hamiltonian, we develop a set of tools to understand and determine to which order m in ϵ the Fisher information loss is suppressed, $\Delta F_{\mathrm{Bob},t} = O(\epsilon^m)$.

The question is partly motivated by a similar question in the context of quantum error correction. A quantum error-correcting code of distance d can correct any $(d-1)/2$ arbitrary single-site errors. In the case of a weak IID noisy channel $\mathcal{N}_\epsilon^{\otimes n}$ affecting the n sites, a weight-$[(d-1)/2]$ error happens with probability of order $O(\epsilon^{(d-1)/2})$ if we assume that a single-site error happens with probability $O(\epsilon)$. This means that the chance of an uncorrectable error occurring is upper bounded by $O(\epsilon^{(d-1)/2})$. In this scenario, we see that the higher the distance of the code, the better robustness is achieved against weak IID noise. In the context of quantum metrology, we ask the following analogous question: can we determine the robustness of the sensitivity of the clock to time when affected by a weak IID noisy channel, a function of a certain feature (analogous to the code distance) of the clock state, the Hamiltonian, and the noisy channel?

There does not appear to be any obvious property of the setup (analogous to the code distance) that immediately determines the order m in the Fisher information loss $\Delta F_{\mathrm{Bob},t} = O(\epsilon^m)$. Instead, we explain a general procedure for how to obtain a bound on m when given a weak IID noisy channel, a clock state and a Hamiltonian.

The simplest case presents itself if the complementary channel $\widehat{\mathcal{N}}_\epsilon^{\otimes n}$ maps the clock state ψ onto a full-rank state $\rho_E = \sum p_{\mathbf{x}}|\mathbf{x}\rangle\langle\mathbf{x}|_E$ that is diagonal in the tensor product computational basis on E. (This is equivalent to all vectors $\{E_{\mathbf{x}}|\psi\rangle\}_{\mathbf{x}}$ being orthogonal on Bob's system.) In such a case we can use Eq. (12) to express the Fisher information loss as

$$\begin{aligned}
\Delta F_{\mathrm{Bob},t} &= \sum_{\mathbf{x},\mathbf{x}'}\frac{2}{p_{\mathbf{x}}+p_{\mathbf{x}'}}\left|\langle\mathbf{x}|\widehat{\mathcal{N}}_\epsilon^{\otimes n}\big(\{\bar{H},\psi\}\big)|\mathbf{x}'\rangle\right|^2\\
&= \sum_{\mathbf{x},\mathbf{x}'}\frac{O(\epsilon^{2q_{\mathbf{x},\mathbf{x}'}})}{\Omega(\epsilon^{\min(r_{\mathbf{x}},r_{\mathbf{x}'})})}\\
&= O(\epsilon^m) \qquad (130)
\end{aligned}$$

defining $r_{\mathbf{x}}$ and $q_{\mathbf{x},\mathbf{x}'}$ via $p_{\mathbf{x}} = \Omega(\epsilon^{r_{\mathbf{x}}})$ and $\left|\langle\mathbf{x}|\widehat{\mathcal{N}}_\epsilon^{\otimes n}\big(\{\bar{H},\psi\}\big)|\mathbf{x}'\rangle\right| = O(\epsilon^{q_{\mathbf{x},\mathbf{x}'}})$, and with

$$m = \min_{\substack{\mathbf{x},\mathbf{x}':\\ r_{\mathbf{x}}\leqslant r_{\mathbf{x}'}}}\left\{2q_{\mathbf{x},\mathbf{x}'} - r_{\mathbf{x}}\right\}. \qquad (131)$$

As we can see above, it is not obvious which $\mathbf{x},\mathbf{x}'$ minimizes the expression in the exponent above. One might have expected that events $\mathbf{x}$ whose probability of occurring vanish faster than other events (large $r_{\mathbf{x}}$ compared to other $r_{\mathbf{x}'}$) are less relevant and would not contribute significantly to the Fisher information loss. However, this is not the case; terms with high $r_{\mathbf{x}}, r_{\mathbf{x}'}$ can contribute to leading order to the sensitivity loss if the corresponding term $q_{\mathbf{x},\mathbf{x}'}$ is sufficiently small. If the state ρ_E is not diagonal, then it is unclear whether or not one can easily determine the order of the Fisher information loss.

VI. CLOCK SENSITIVITY IN THE PRESENCE OF CONTINUOUS NOISE

The setting presented in Fig. 1 is nonstandard in metrology, because in typical settings the noise and the signal both get imprinted on the state in the same physical time-evolution process. It is more common to consider, for instance, a Lindbladian master equation that governs the time evolution of the clock state, with terms that encode any noise processes via jump operators.

Here we consider the situation where the noise is described by a Lindbladian master equation. Under suitable conditions, we can decompose the time evolution into a pure unitary evolution followed by some effective noisy channel, and the time dependence of the effective noisy channel can be neglected. In this case our Theorem 1 can be applied to compute the sensitivity loss after some time t_0.

One can follow a similar procedure in the setting where the goal is to determine an unknown parameter in the Hamiltonian when the overall evolution is governed by a Lindbladian master equation. The full derivation is presented in Appendix G. We can carry out a similar decomposition in the case of a clock sensing an unknown parameter in the Hamiltonian, while subject to continuous noise described by a Lindblad evolution.

A. Decomposing a Lindbladian evolution of a clock into a pure unitary time evolution and an instantaneous noisy channel

Consider a clock initialized at time $t = 0$ in the state vector $|\psi_{\mathrm{init}}\rangle$. Suppose that the dynamics $\rho(t)$ of the clock are given by the Lindblad master equation

$$\partial_t\rho = \mathcal{L}_{\mathrm{tot}}[\rho], \tag{132a}$$

where

$$\mathcal{L}_{\mathrm{tot}} = \mathcal{L}_0 + \mathcal{L}_1, \quad \mathcal{L}_0(\rho) = -i[H,\rho], \tag{132b}$$

$$\mathcal{L}_1(\rho) = \sum_j\Big[L_j\,\rho L_j^\dagger - \frac{1}{2}\{L_j^\dagger L_j,\rho\}\Big]. \tag{132c}$$

Here we assume that the operators H and L_j are time independent. The evolution up to a time t is given by the completely positive, trace-preserving map

$$\mathcal{E}_t = e^{t(\mathcal{L}_0+\mathcal{L}_1)}. \tag{133}$$

The evolution driven by the Hamiltonian part $\mathcal{L}_0$ of the dynamics can be written as $e^{t\mathcal{L}_0}(\cdot) = e^{-iHt}\,(\cdot)\,e^{iHt}$.

We would like to compute the sensitivity of the clock at a given time t_0, meaning that the relevant quantity to compute is the Fisher information

$$F_{\mathrm{clock},t}(t_0) = F\big(\rho(t_0)\,;\,\partial_t\rho(t_0)\big). \tag{134}$$

We can decompose the evolution $\mathcal{E}_t$ as first a unitary evolution according to H for a time t followed by the instantaneous application of an effective noisy channel $\mathcal{N}_t$. Define

$$\mathcal{N}_t = \mathcal{E}_t\,e^{-t\mathcal{L}_0} = e^{t(\mathcal{L}_0+\mathcal{L}_1)}\,e^{-t\mathcal{L}_0}. \tag{135}$$

Here, $e^{-t\mathcal{L}_0}$ is the inverse of the unitary evolution $e^{t\mathcal{L}_0}$. By construction, if we apply $\mathcal{N}_t$ after applying $e^{t\mathcal{L}_0}$, then the overall effect is the same as letting the system evolve for time t under the full Lindbladian dynamics $\mathcal{L}_0 + \mathcal{L}_1$:

$$\mathcal{E}_t = \mathcal{N}_t\,e^{t\mathcal{L}_0}. \tag{136}$$

An alternative expression for $\mathcal{N}_t$ is obtained from Eq. (135) using the Baker-Campbell-Hausdorff formula,

$$\mathcal{N}_t = e^{t\mathcal{L}_1 - \frac{t^2}{2}[\mathcal{L}_1,\mathcal{L}_0]+\cdots}. \tag{137}$$

Observe that if $[\mathcal{L}_1,\mathcal{L}_0] = 0$, then we simply have $\mathcal{N}_t = e^{t\mathcal{L}_1}$. This situation is known as *phase-covariant dynamics* (cf., e.g., Refs. [46,47]). This is the case if $[L_j, H] = 0$ for all jump operators L_j. In other cases, the map can be determined from Eq. (135) directly if the superoperator $\mathcal{E}_t$ can be computed.

Let us introduce the family of states $\psi(t) = e^{-iHt}\,\psi_{\mathrm{init}}\,e^{iHt}$ associated with the (fictitious) pure unitary evolution of ψ_{init} if we artificially turn off the noise terms.

The derivative of the quantum state $\rho(t) = \mathcal{E}_t(\psi_{\mathrm{init}})$ can then be written as

$$\partial_t\rho(t) = \partial_t\Big[\mathcal{N}_t\big(\psi(t)\big)\Big] = \mathcal{N}_t\big(\partial_t\psi\,(t)\big) + \big(\partial_t\mathcal{N}_t\big)\big(\psi(t)\big). \tag{138}$$

Therefore, the derivative of the noisy state can be decomposed into a sum of two terms, the first associated with the unitary dynamics $\psi(t)$, and the other associated with the time dependence of the effective noisy channel $\mathcal{N}_t$. Plugging into Eq. (134), this gives us

$$F_{\mathrm{clock},t} = F\Big(\mathcal{N}(\psi)\,;\,\mathcal{N}(\partial_t\psi) + \big(\partial_t\mathcal{N}\big)(\psi)\Big), \tag{139}$$

where now $F_{\mathrm{clock},t}$, $\mathcal{N}$, $\partial_t\mathcal{N}$, ψ, and $\partial_t\psi$ are all implicitly evaluated at t_0.

In the following, we consider settings where the local time dependence of the state due to the time dependence of the effective noisy channel terms can be neglected when computing $F_{\mathrm{clock},t}$. (We will study in greater depth below when exactly this situation arises.) In other words, for now we assume that

$$F_{\mathrm{clock},t} \approx F\big(\mathcal{N}(\psi)\,;\mathcal{N}(\partial_t\psi)\big) =: F_{\mathrm{clock,U},t}. \tag{140}$$

Expanding $\partial_t\psi$, we obtain

$$F_{\mathrm{clock,U},t} = F\big(\mathcal{N}(\psi)\,;\mathcal{N}(-i[H,\psi])\big). \tag{141}$$

This quantity is what we defined as $F_{\mathrm{Bob},t}$ in the context of our main uncertainty relation.

The complementary channel $\widehat{\mathcal{N}}_{t_0}$ is directly determined by the complementary channel of the overall evolution up to that time $\widehat{\mathcal{E}}_{t_0}$, since the two channels differ only by a unitary evolution $e^{-t_0\mathcal{L}_0}$ on their input:

$$\widehat{\mathcal{N}}_t = \widehat{\mathcal{E}}_t\, e^{-t_0\mathcal{L}_0}. \tag{142}$$

This means that the Fisher information on Eve's end with respect to the complementary direction can be expressed entirely in terms of the complementary channel $\widehat{\mathcal{E}}_{t_0}$ to the entire evolution up to time t_0:

$$\Delta F_{\mathrm{clock,U},t} = F\big(\widehat{\mathcal{E}}_{t_0}(\psi_{\mathrm{init}})\,;\widehat{\mathcal{E}}_{t_0}\big(\{\bar{H},\psi_{\mathrm{init}}\}\big)\big), \tag{143}$$

with $\bar{H} = H - \langle H\rangle_{\psi(t_0)}$, and Theorem 1 states that

$$F_{\mathrm{clock,U},t} = 4\sigma_H^2 - \Delta F_{\mathrm{clock,U},t}. \tag{144}$$

Now we turn to discussing when the approximation (140) is a reasonable assumption, by characterizing the error induced on the Fisher information. First of all, the approximation is exact in the case of phase-covariant dynamics, where $[\mathcal{L}_1,\mathcal{L}_0]=0$ (e.g., Refs. [46,47]). In other settings, we can use a continuity bound of the Fisher information in its second argument (Proposition 17 in Appendix C) to try to get a handle on the error terms involved in the approximation (140). Denote by δ the error in the approximation (140),

$$\delta = F_{\mathrm{clock},t} - F_{\mathrm{clock,U},t}, \tag{145}$$

then we have

$$|\delta| \leqslant F\big(\rho\,;(\partial_t\mathcal{N})(\psi)\big) + \big[F\big(\rho\,;(\partial_t\mathcal{N})(\psi)\big)\,F_{\mathrm{clock,U},t}\big]^{1/2}. \tag{146}$$

That is, the relative error in the approximation (140) is demonstrably small if $F\big(\rho\,;(\partial_t\mathcal{N})(\psi)\big)$ is much smaller than $F_{\mathrm{clock,U},t}$. We can rewrite this term using (138) as

$$\begin{aligned}(\partial_t\mathcal{N})(\psi) &= \partial_t\rho(t) - \mathcal{N}_t(-i[H,\psi(t)])\\ &= \mathcal{L}_{\mathrm{tot}}[\rho(t)] - \mathcal{E}_t(-i[H,\psi_0]).\end{aligned} \tag{147}$$

The above expression is given in terms of the Lindbladian map and the overall evolution map, and can aid in determining an analytical or numerical upper bound to the quantity $F\big(\rho\,;(\partial_t\mathcal{N})(\psi)\big)$. In Appendix G, we study two single-qubit examples that are subject to continuous dephasing along various axes in order to illustrate the connections between the Lindbladian setting and the setting in Fig. 1.

VII. ERROR-CORRECTION CONDITIONS FOR ZERO SENSITIVITY LOSS

The uncertainty relation (55) enables us to provide a characterization of when the noise reduces a probe's sensitivity to time. In this section, we study the situation where the sensitivity loss $\Delta F_{\mathrm{Bob},t}$ introduced in Eq. (52) is equal to zero. This is a situation where the probe is chosen cleverly enough such that the noise has no effect on sensitivity. The main contribution of this section is a set of necessary and sufficient conditions for $\Delta F_{\mathrm{Bob},t}=0$, which bear resemblance to the Knill-Laflamme conditions for quantum error correction [20] and which are closely related to the Hamiltonian-not-in-Lindblad-span condition of Refs. [21,22].

A. Conditions for zero sensitivity leakage

In the following, we suppose that our uncertainty relation holds with equality, i.e., that the conditions given in Proposition 1 hold. Recall the expression for the Fisher information loss on Bob's end (55), and consider the expression (16b) for the Fisher information. If $\Delta F_{\mathrm{Bob},t} = 0$, then there exists an operator L such that $\mathrm{tr}(L^\dagger L)=0$ and $\rho^{1/2}L + L^\dagger\rho^{1/2} = \widehat{\mathcal{N}}(\{\psi,\bar{H}\})$; the former condition implies $L=0$ and thus the latter implies $\widehat{\mathcal{N}}(\{\psi,\bar{H}\})=0$. Therefore, we see that $\Delta F_{\mathrm{Bob},t}=0$ if and only if

$$\widehat{\mathcal{N}}(\{\psi,\bar{H}\}) = 0, \tag{148}$$

i.e., $\{\psi,\bar{H}\}$ must lie in the kernel of the superoperator $\widehat{\mathcal{N}}$. It is instructive to rewrite this condition in terms of the "virtual qubit" introduced in Sec. III B. With Z_L defined in Eq. (62), then Eq. (148) becomes

$$\widehat{\mathcal{N}}(Z_L) = 0. \tag{149}$$

Alternatively, the above condition is equivalent to requiring that for all operators O,

$$\mathrm{tr}\big[\widehat{\mathcal{N}}^\dagger(O)\,Z_L\big] = 0, \tag{150}$$

meaning that error operations of the form $\widehat{\mathcal{N}}^\dagger(O)$ should not have any overlap with the "logical" Z_L operator on the qubit subspace.

So the task of finding probe states that perfectly counter the noisy channel $\mathcal{N}$ can be formulated as ensuring the logical Z Pauli operator in the logical qubit subspace spanned by $|+\rangle = |\psi\rangle$ and $|-\rangle \propto |\xi\rangle = P_\psi^\perp H|\psi\rangle$ is in the kernel of the complementary channel to the noisy channel.

Note that simply looking for zero sensitivity loss is not sufficient to find the best probe states; we still need to make sure that $|\psi\rangle$ has as large energy variance as possible to ensure good sensitivity.

An alternative representation of the zero sensitivity loss condition can be obtained if we consider an operator-sum representation of the noisy channel in terms of Kraus operators $\{E_k\}$ as in Eq. (45). The condition (148) is then equivalent to the condition

$$\langle\psi|\, E_{k'}^\dagger E_k\,|\xi\rangle + \langle\xi|\, E_{k'}^\dagger E_k\,|\psi\rangle = 0 \qquad \text{for all } k, k'. \quad (151)$$

These may be interpreted as Knill-Laflamme-like conditions for optimal sensitivity. Whereas for a traditional quantum error-correcting code, we require any two code words $|\psi_i\rangle, |\psi_j\rangle$ to satisfy $\langle\psi_i|E_{k'}^\dagger E_k|\psi_j\rangle \propto \delta_{i,j}$, here we require that the error operator $E_k^\dagger E_{k'}$ cannot map the state $|\psi\rangle$ onto the vector $|\xi\rangle$, or at least not in a way that is not suitably antisymmetric. The weird antisymmetrization in Eq. (151) can be expressed in a more elegant form if we switch back to the picture of the logical qubit spanned by $|\psi\rangle$ and $|\xi\rangle$. Analogously to Eq. (149), we may rewrite the condition (151) as

$$\mathrm{tr}\big[Z_L\, \Pi_L\, E_{k'} E_k\, \Pi_L\big] = 0, \quad (152)$$

where $\Pi_L = |+\rangle\langle+|_L + |-\rangle\langle-|_L$ is the projector onto the virtual qubit subspace spanned by $|\psi\rangle$ and $|\xi\rangle$. The full Knill-Laflamme conditions applied to the subspace Π_L would require $\Pi_L E_{k'} E_k \Pi_L \propto \Pi_L$. The condition (152) is simply a weaker condition where only the corresponding projection onto the logical Pauli operator Z_L is considered and where the projection onto the other Pauli operators is unconstrained.

The form (152) also helps clarify that for zero sensitivity loss, the terms in Eq. (151) need not vanish individually. Indeed, only the Hilbert-Schmidt projection of $\Pi_L\, E_{k'} E_k\, \Pi_L$ onto Z_L is required to vanish, and not in principle on Y_L or X_L. An example below in Sec. VII H 1, consisting of a single-qubit subject to transversal noise, will illustrate this point.

The conditions (152) are reminiscent of quantum error correction for operator algebras, where we require a code to preserve the outcomes of any operator in a given algebra [48–50]. In fact, if the algebra associated with any choice of optimal sensing operator of the form (20) is preserved, then our conditions (152) are satisfied. Indeed, suppose that $[T, \widehat{\mathcal{N}}^\dagger(W)] = 0$ for any operator W on Eve and for a fixed choice of M in Eq. (20), meaning that the Abelian algebra generated by T is correctable [48–50]. Then taking the expectation value $\langle\cdot\rangle_\psi$ of this commutator we find $0 = \big\langle[T, \widehat{\mathcal{N}}^\dagger(W)]\big\rangle = \mathrm{tr}\big([\psi, T]\widehat{\mathcal{N}}^\dagger(W)\big) \propto \mathrm{tr}\big(Z_L\, \widehat{\mathcal{N}}^\dagger(W)\big)$, using Eq. (25), which holds for all W, and therefore our Knill-Laflamme-like condition (150) holds. The converse implication is unclear, in part because the optimal sensing operator is not unique and different choices can generate different algebras.

The conditions (151) are actually tightly related to the Hamiltonian-not-in-Kraus-span condition of Refs. [21,22,30,31,51–53]. There, it was shown that there exists a clock state vector $|\psi\rangle$ that achieves Heisenberg scaling in the presence of noise using quantum error correction if and only if the Hamiltonian signal term is not in the linear span of the Lindblad noise operators. Here we argue that the Hamiltonian-not-in-Kraus-span condition is in fact equivalent to the existence of a state $|\psi\rangle$ that satisfies our zero sensitivity-loss conditions (151). (In our setting, the clock state vector $|\psi\rangle$ is a given fixed state.) As we have a discrete noisy channel, we consider the Kraus operators $\{E_k\}$ of the noisy channel instead of Lindblad operators. If $H = \sum \alpha_{k',k} E_{k'}^\dagger E_k$, and supposing the conditions (151) are satisfied for some $|\psi\rangle$, then by taking a linear combination $\sum \alpha_{k',k}$ of the conditions (151) we obtain $0 = 2\langle\psi|H\, P_\psi^\perp H|\psi\rangle = 2\sigma_H^2$; therefore the conditions (151) cannot be satisfied by any ψ that has nonzero energy variance. Conversely, we know (see, e.g., Refs. [21,22,52]) that if the Hamiltonian is not in the span of the noisy channel's Kraus operators, then there is a code space Π, possibly involving an ancilla system, with $\Pi E_{k'}^\dagger E_k \Pi = c_{k',k}\Pi$ such that $[\Pi, H] = 0$ (i.e., Π is spanned by a subset of energy eigenvectors) and such that Π contains a state vector $|\psi\rangle$ with nonzero energy variance; then for any k, k' we have $\langle\psi|E_{k'}^\dagger E_k P_\psi^\perp H|\psi\rangle = \langle\psi|\Pi E_{k'}^\dagger E_k P_\psi^\perp H \Pi|\psi\rangle = \langle\psi|\Pi E_{k'}^\dagger E_k \Pi\, P_\psi^\perp H|\psi\rangle = c_{k',k}\langle\psi|P_\psi^\perp H|\psi\rangle = 0$ using the fact that $[P_\psi^\perp, \Pi] = [H, \Pi] = 0$, so the conditions (151) are satisfied. Therefore, if the Hamiltonian is not in the span of the Kraus operators, then there exists a clock state vector $|\psi\rangle$ that suffers no sensitivity loss after being exposed to the noise locally at t_0. This state is constructed in the above mentioned references using a quantum error-correcting code.

We can ask whether there is a relation between our conditions for no sensitivity loss and when the sensitivity can achieve Heisenberg scaling in the system size [2]. The Heisenberg scaling refers to situations where $F_{\mathrm{Bob},t}$ scales like n^2, where n is the number of systems that are jointly prepared in the clock state vector $|\psi\rangle_n$. (If no entanglement is present between the n systems, the best scaling that can be achieved is $F_{\mathrm{Bob},t} \propto n$.) We assume that the clock state vector $|\psi\rangle_n$ has a variance that scales quadratically in n, i.e., $[\sigma_H(\psi_n)]^2 \propto n^2$, as otherwise even the noiseless clock does not achieve Heisenberg scaling. Suppose the conditions (151) are satisfied: then $F_{\mathrm{Bob},t} = 4[\sigma_H(\psi_n)]^2 \propto n^2$ as there is no sensitivity loss, and the Fisher information displays Heisenberg scaling. On the other hand, even if there is some loss of sensitivity due to the noise, the

Heisenberg scaling might survive. Suppose, for example, that we consider two independent one-dimensional spin chains, each consisting of $n/2$ sites that are prepared in a GHZ state and that evolve according to an on-site Z Hamiltonian. Both spin chains are independent probes whose sensitivity each scales as approximately n^2, and therefore the overall probe state exhibits Heisenberg scaling. Now consider the noisy channel that erases one of the spin chains. Half the sensitivity is lost; because there is sensitivity loss our Knill-Laflamme-like conditions cannot be satisfied. However, the single spin chain that is left for Bob still exhibits Heisenberg scaling. This shows that Heisenberg scaling is guaranteed if the environment has zero sensitivity to energy (and the noiseless probe itself has Heisenberg scaling), but that there are also situations where the environment induces sensitivity loss without hindering the Heisenberg scaling of the probe. In the language of Refs. [21,22], this corresponds to a Hamiltonian that might have both a parallel component to the signal as well as a perpendicular component that can be exploited to achieve Heisenberg scaling. We see that zero sensitivity loss implies Heisenberg scaling for a family of state vectors $|\psi\rangle$ that are sufficiently entangled. But there are states that achieve the Heisenberg scaling even if some sensitivity is lost due to the noise.

When the zero sensitivity loss conditions (148) hold, then by definition there must exist a sensing observable for Bob to estimate the parameter t, whose sensitivity matches that of Alice. We can extract this optimal sensing observable from our technical analysis using semidefinite programming (see Appendix E 2). Namely, in Appendix E 3 we show that if the zero sensitivity-loss conditions hold, then the operator $i\rho\mathcal{N}(|\xi\rangle\langle\psi|)$ is Hermitian. Furthermore, the operator

$$R_B = -2i\mathcal{N}(|\xi\rangle\langle\psi|)\rho^{-1} + 2i\rho^{-1}\mathcal{N}(|\psi\rangle\langle\xi|)P_\rho^\perp \quad (153)$$

is also Hermitian and satisfies $1/2\{R_B, \rho_B\} = \mathcal{N}(Y_L)$, i.e., we obtain an explicit expression of the symmetric logarithmic derivative on Bob's end. The optimal sensing observable on Bob's system is then given via Eq. (13) as $T_b = [F_{\mathrm{Bob},t}]^{-1}R_B + t_0$. That is, when a clock state and associated Hamiltonian fulfill the metrological code conditions for a given noise channel, we obtain an explicit expression for the optimal measurement on Bob's end.

B. Metrological codes and metrological distance

We now introduce the concept of a *metrological code.* The idea is to study the qubit space spanned by the vectors $|\psi\rangle$ and $|\xi\rangle = \bar{H}|\psi\rangle = \big(H - \langle H\rangle\big)\,|\psi\rangle$. If the state loses no sensitivity upon the action of a noisy channel, one could expect these states to span some kind of quantum error-correcting code space. We can see that they do not necessarily form a full error-correcting code as follows. Consider the single-qubit state $|\psi\rangle = |+\rangle = \big[|0\rangle + |1\rangle\big]/\sqrt{2}$ evolving under the Hamiltonian $H = \omega\sigma_Z/2$, which we expose to an error channel whose Kraus operators are proportional to $\mathbb{1}$ and X. We see that the condition (151) is satisfied, given that $|\xi\rangle = |-\rangle = \big[|0\rangle - |1\rangle\big]/\sqrt{2}$ is orthogonal to $|+\rangle$ and that $|+\rangle$ is an eigenstate of both $\mathbb{1}$ and X. Yet a quantum state stored on this qubit would be corrupted by the noise, as the bit flips would be uncorrectable. We identify a concept that is weaker than a full error-correcting code, which applies precisely to states that satisfy the condition (151). Here, we assume that the setting is specified as a pair of orthogonal states $|\psi\rangle, |\xi\rangle$, whereby $|\xi\rangle$ is presumably obtained from a Hamiltonian H as $|\xi\rangle = \big(H - \langle H\rangle\big)|\psi\rangle$. Specifying the full Hamiltonian is not necessary as the relevant quantum Fisher information quantities can be fully expressed only in terms of $|\psi\rangle, |\xi\rangle$.

Metrological code. Let $\mathscr{E}$ be any set of operators. We say that the state vectors $|\psi\rangle$ and $|\xi\rangle$ form a *metrological code* against the errors $\mathscr{E}$ if for all $E, E' \in \mathscr{E}$, we have

$$\mathrm{tr}\big[E'^\dagger E\,\big(|\xi\rangle\langle\psi| + |\psi\rangle\langle\xi|\big)\big] = 0. \quad (154)$$

As a consequence of the zero sensitivity loss condition (151), a metrological code prevents sensitivity loss against any noise channel whose Kraus operators are linear combinations of elements in $\mathscr{E}$ (as long as the conditions of Proposition 1 are satisfied).

A natural class of errors to consider is the set of all operators that act on only a subset of n components of a composite quantum system $A = A_1 \otimes A_2 \otimes \cdots \otimes A_n$. The *weight* $\mathrm{wgt}(O)$ of an operator O acting on the n systems is defined as the number of systems on which O acts nontrivially. Specifically, if O is expanded in the Pauli operator basis (or in any tensor basis using a single-site operator basis that includes the identity matrix), all nonidentity elements in tensor products of basis operators that appear in the decomposition of O must be supported on a fixed set of $\mathrm{wgt}(O)$ sites. Equivalently, the expectation value of O on any state can be computed exactly even after tracing out all but a given set of $\mathrm{wgt}(O)$ sites.

We say that the pair of state vectors $|\psi\rangle$ and $|\xi\rangle$ form a *metrological code of distance d_m* if it is a metrological code against all operators of weight at most $d_m - 1$; in other words, for all operators O satisfying $\mathrm{wgt}(O) < d_m$, we have

$$\mathrm{tr}\big[O\big(|\xi\rangle\langle\psi| + |\psi\rangle\langle\xi|\big)\big] = 0. \quad (155)$$

Metrological codes of distance d_m have the property that for any noise channel $\mathcal{N}$ whose Kraus operators $\{E_k\}$ are such that $\mathrm{wgt}(E_{k'}^\dagger E_k) < d_m$ for all k', k, the associated sensitivity loss is zero (as long as Proposition 1 is satisfied).

Metrological codes are, roughly speaking, in between classical and quantum codes. On one hand, they are not full-blown classical codes because condition (155) requires protection against both X- and Z-type physical noise. Because of this, the pair $|\psi\rangle \propto |0\rangle^n + |1\rangle^n$ and $|\xi\rangle \propto |0\rangle^n - |1\rangle^n$ of GHZ states *is not* a metrological code of nontrivial distance because single-qubit Z errors cause a logical-X error, thereby violating Eq. (155). On the other hand, metrological codes are not full-blown quantum codes because the sensitivity conditions say nothing about other types of logical noise. In other words, noise can cause logical-Y and logical-Z errors for a metrological code, but not for a bona-fide error-correcting code.

C. Uncertainty relation equality and conditions for metrological codes

In order to deduce from Eve's lack of sensitivity to energy that Bob loses no sensitivity to time, it is necessary to ensure that the conditions of Proposition 1 hold. When we presented Proposition 1, we already noted that the situations where these conditions are not satisfied are edge cases that can be perturbed away. Here, we strengthen this statement for metrological codes: if a metrological code for a given noise channel happens not to satisfy the conditions of Proposition 1, then the noise channel can be infinitesimally perturbed to obtain a situation for which these conditions hold, and furthermore, the zero sensitivity loss conditions (148) are preserved.

Proposition 2 (Perturbation bound for noise channels consistent with a metrological code).—Let $V_{A\to BE}$ be an isometry, let $|\psi\rangle_A, |\xi\rangle_A$ with $\langle\psi|\xi\rangle_A = 0$ and let $\mathcal{N}(\cdot) = \mathrm{tr}_E\big(V(\cdot)V^\dagger\big)$, $\widehat{\mathcal{N}}(\cdot) = \mathrm{tr}_B\big(V(\cdot)V^\dagger\big)$. Suppose that $\widehat{\mathcal{N}}(|\xi\rangle\langle\psi| + |\psi\rangle\langle\xi|) = 0$. We furthermore assume that there exists a unitary operator G_B acting on the system B with the properties that $0 = P_{\rho_B} G_B P_{\rho_B} = P_{\zeta_B} G_B P_{\zeta_B} = P_{\rho_B} G_B P_{\zeta_B} = P_{\zeta_B} G_B P_{\rho_B}$, where $\zeta_B = \mathcal{N}(|\xi\rangle\langle\xi|)$. Then, for any $\epsilon > 0$, there exists an isometry $V'_{A\to BE}$ with $\|V' - V\| \leqslant \epsilon$ such that

$$\big(P^\perp_{\rho'_B} \otimes P^\perp_{\rho'_E}\big) V'|\xi\rangle = 0\,; \quad \text{and} \tag{156a}$$

$$\widehat{\mathcal{N}}'\big(|\xi\rangle\langle\psi| + |\psi\rangle\langle\xi|\big) = 0, \tag{156b}$$

where $\rho'_B = \mathrm{tr}_E\big\{V'\psi V'^\dagger\big\}$, $\rho'_E = \mathrm{tr}_B\big\{V'\psi V'^\dagger\big\}$, and $\widehat{\mathcal{N}}'(\cdot) = \mathrm{tr}_B\big\{V'(\cdot)V'^\dagger\big\}$.

The proof is presented as Proposition 22 in Appendix H. Note that the existence of such an operator G_B can always be ensured by augmenting the B system to include a qubit, which $\mathcal{N}$ prepares in a fixed pure state vector $|0\rangle$ for all inputs. The operator G_B can be chosen to flip the qubit to $|1\rangle$. The additional qubit can represent an additional "failure" flag such as, for instance, an additional photon that is emitted at the output of the noise process.

D. Sensitivity loss of metrological codes under weak IID noise

1. Sensitivity loss under weak IID noise

If we encode a logical quantum state using a quantum error-correcting code of a distance d, and each site has a small probability $O(\epsilon)$ of incurring an error, then we know that the errors that the code cannot correct occur with probability at most $O(\epsilon^{d/2})$. In turn, this implies that the infidelity of recovery of the logical information also scales as $O(\epsilon^{cd})$ with a constant c depending on which convention for the infidelity measure we choose. It is then natural to conjecture that if $|\psi\rangle$ and $|\xi\rangle$ form a metrological code of metrological distance d_m, then the loss in Fisher information must similarly be upper bounded by $O(\epsilon^{cd_m})$, for some universal constant c.

Interestingly, the order of the Fisher information loss in ϵ is not directly related to the metrological distance of a metrological code. In fact, there are examples of metrological codes with large metrological distance, but for which the Fisher information loss is always of order ϵ. This behavior appears to contradict the expectation that events of vanishing probability should not significantly influence observable properties of the system (such as its sensitivity to time). An explanation stems from the fact that the operational interpretation of the Fisher information via the Cramér-Rao bound involves an implicit averaging of the error over infinitely many samples. It might turn out in the present case that events with vanishing probability can contribute non-negligibly to the quantum Fisher information. To remedy this issue, it would be desirable to consider a measure of sensitivity that accounts for finite data acquisition. One such measure has been put forward in Ref. [54]. We refer to Appendix I for a more detailed discussion.

E. Clock states from time-covariant quantum error-correcting codes

Here, we explore a simple method to construct states that satisfy the zero sensitivity-loss condition, using time-covariant quantum error-correcting codes. A code is said to be time-covariant code with respect to a given Hamiltonian H if H (and hence also time evolution generated by H) is a nontrivial logical operator. In the following, Pauli operators X, Y, Z carry an index indicating the qubit on which the operator acts. This strategy is the one pursued by, e.g., Refs. [22,51–53].

1. Four nearest-neighbor interacting qubits in a square pattern

As a warm-up example, we first consider how to leverage the $[[4, 2, 2]]$ code for quantum metrology with a Hamiltonian on four qubits with ZZ interactions arranged in a square pattern. Consider four qubits arranged in a square as depicted in Fig. 9. The Hamiltonian is defined

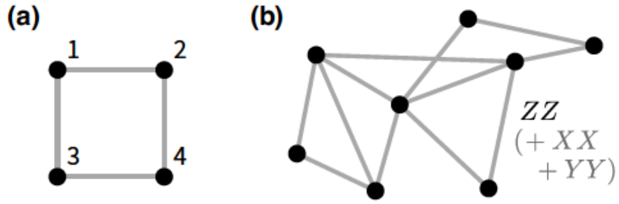

FIG. 9. Metrology with interacting qubits. (a) Consider four qubits in a square with nearest-neighbor ZZ Ising interactions (alternatively with additional XX and YY interactions). A clock state with maximal sensitivity and zero sensitivity loss under a single located erasure can be obtained via the time-covariant [[4, 2, 2]] code. (b) We can extend the construction based on the [[4, 2, 2]] code to any number of qubits interacting with respect to any graph of ZZ interactions (alternatively with additional XX and YY interactions), while offering protection against a single located erasure.

by placing a ZZ interaction on each side of the square,

$$H = \omega\big(Z_1Z_2 + Z_1Z_3 + Z_2Z_4 + Z_3Z_4\big). \tag{157}$$

The [[4, 2, 2]] code [55,56] has stabilizers $X_1X_2X_3X_4$ and $Z_1Z_2Z_3Z_4$. The logical operators X_1, Z_1 and X_2, Z_2 for the first and second logical qubits are $\overline{X}_1 = X_1X_3, \overline{X}_2 = X_1X_2$, $\overline{Z}_1 = Z_1Z_2$, and $\overline{Z}_2 = Z_1Z_3$.

Observe that the Hamiltonian is a logical operator: the second and fourth terms in Eq. (157) have the same action on the code space as the first and third terms, respectively, because they differ only by the stabilizer $Z_1Z_2Z_3Z_4$. When acting on the code space, we have

$$H\,\Pi = 2\omega\big(\overline{Z}_1 + \overline{Z}_2\big)\Pi. \tag{158}$$

Let us choose the clock state as a logical state with the largest possible energy spread under this Hamiltonian,

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|\overline{00}\rangle + |\overline{11}\rangle\right], \tag{159}$$

where $|\overline{00}\rangle$ and $|\overline{11}\rangle$ refer to logical state vectors with the first and second logical qubits in the given logical computational basis states.

Now we check our Knill-Laflamme-like condition. Having distance 2, the code can correct a single erasure at a known location. Crucially, the operator $|\xi\rangle = (H - \langle H\rangle)|\psi\rangle = 2\omega[|\overline{00}\rangle - |\overline{11}\rangle]/\sqrt{2}$ is still in the code space because H is a logical operator. Then from the Knill-Laflamme conditions we know that $\langle\xi|O_i|\psi\rangle = 0$ for any single-site operator O_i, because $|\xi\rangle$ and $|\psi\rangle$ are orthogonal vectors in the code space, and hence our conditions (151) are satisfied for single located errors.

If we have some freedom in engineering our Hamiltonian, there are other choices of logical operators to use in the Hamiltonian that would achieve a similar sensitivity while also offering protection against single located erasures. For instance, we could ignore the second logical qubit (or treat it as a gauge qubit) and the Hamiltonian could be chosen to act only on sites 1 and 2 as $H = 2\omega\overline{Z}_1 = 2\omega Z_1Z_2$.

We see that the probe state (159) does not lose any sensitivity to time if a system is erased at a known location. The variance of $|\psi\rangle$ is given by

$$\sigma_H^2 = \langle\psi|H^2|\psi\rangle = 16\omega^2. \tag{160}$$

Because we have not specified how this model scales with n, we cannot talk yet about achieving Heisenberg scaling.

In this example, the sensitivity is in fact as good as you can get without any noise at all, for any probe state: the state (159) is a superposition between two states that have extremal eigenvalues with respect to H, which is optimal in the absence of noise. What is special about the state vector $|\psi\rangle$ is that it retains its sensitivity even after a single located error, which is not in general the case of other probe states that would be optimal in the noiseless setting. For instance, the state vector $\left[|0\,0\,0\,0\rangle + |0\,1\,1\,0\rangle\right]/\sqrt{2}$ has the same sensitivity as $|\psi\rangle$ if no noise is applied, but it does not satisfy our conditions (151) and so is subject to sensitivity loss under single-site errors.

The above construction can also be applied if we include XX and YY interactions between the neighboring qubits on top of the existing ZZ interactions (enabling us to model, e.g., Heisenberg interactions):

$$H = \omega\sum_{\langle i,j\rangle}\left[s_xX_iX_j + s_yY_iY_j + Z_iZ_j\right], \tag{161}$$

with the additional coupling constants s_x, s_y allowing for some anisotropy in the interaction strengths. In this case, the interaction terms are again all logical operators, which can be seen from the fact that $X_1X_2X_3X_4$ and $Y_1Y_2Y_3Y_4 = (X_1X_2X_3X_4)(Z_1Z_2Z_3Z_4)$ are stabilizers. Our zero sensitivity-loss conditions are therefore still satisfied. To compute the variance of $|\psi\rangle$ under this new Hamiltonian, we need to determine the action of the additional terms on $|\psi\rangle$. The X terms give us again $\overline{X}_1 + \overline{X}_2$ when acting on the code space following the same argument as for the Z terms. Now $|\psi\rangle$ is a maximally entangled state vector between the two logical qubits, satisfying $(\overline{A}_1 \otimes \overline{\mathbb{1}})|\psi\rangle = (\overline{\mathbb{1}}_1 \otimes \overline{A}_2^T)|\psi\rangle$ where $(\cdot)^T$ denotes the matrix transpose in the (logical) computational basis, and where $\overline{A}_i$ is a logical operator acting on the ith logical qubit. For the Y terms, we then find

$$\begin{aligned}
Y_1Y_2|\psi\rangle &= -\overline{X}_2\overline{Z}_1|\psi\rangle = -\overline{Z}_1\overline{X}_1|\psi\rangle = i\overline{Y}_1|\psi\rangle,\\
Y_1Y_3|\psi\rangle &= -\overline{X}_1\overline{Z}_2|\psi\rangle = -\overline{X}_1\overline{Z}_1|\psi\rangle = -i\overline{Y}_1|\psi\rangle,\\
Y_2Y_4|\psi\rangle &= Y_1Y_3|\psi\rangle = -i\overline{Y}_1|\psi\rangle,\\
Y_3Y_4|\psi\rangle &= Y_1Y_2|\psi\rangle = i\overline{Y}_1|\psi\rangle.
\end{aligned} \tag{162}$$

Thus the sum of all four Y interacting terms vanishes when applied onto $|\psi\rangle$. The variance of H is hence given by

$$H|\psi\rangle = 2\omega\big[\overline{Z}_1 + \overline{Z}_2 + s_x\big(\overline{X}_1 + \overline{X}_2\big)\big]|\psi\rangle \\ = 4\omega\big[\overline{Z}_1 + s_x\overline{X}_1\big]|\psi\rangle, \tag{163}$$

using the fact that $|\psi\rangle$ is a maximally entangled state vector between the two logical qubits, and

$$\sigma_H^2 = \langle\psi|H^2|\psi\rangle = 4\omega^2\big(1+s_x^2\big). \tag{164}$$

The increase in the variance σ_H^2 when we switch on transversal interactions can be simply associated with the increased norm of the Hamiltonian. Had we defined the clock state (159) with a -1 relative phase, then the YY terms would contribute instead of the XX terms and we would get $\sigma_H^2 = 4\omega^2\big(1+s_y^2\big)$.

2. Time-covariant codes lead to states with no sensitivity loss

The construction above based on the $[[4,2,2]]$ code exploited a key property of that code with respect to the Hamiltonian, namely *time covariance* [44,57–60]. A time-covariant code with respect to a given Hamiltonian H is a code for which the time evolution generated by H is a (nontrivial) logical operator. If we can find a time-covariant code with respect to the system's Hamiltonian, then the clock state can be chosen to lie within the code space, so that errors that affect it can be corrected, all while evolving nontrivially in time and thus serving as a clock.

However, there are constraints on the possibility of constructing time-covariant codes. Consider a Hamiltonian that is a sum of terms of weight at most k, which we call a *k-local Hamiltonian*. Any code that can correct up to k arbitrary errors at known locations cannot be time covariant with respect to a k-local Hamiltonian, because the Hamiltonian would be a sum of correctable terms that cannot have a nontrivial action that preserves the code space. On the other hand, physical systems like spin chains and the anti-de Sitter–conformal field theory (AdS-CFT) correspondence as a model for quantum gravity offer natural examples of time-covariant codes that can approximately correct against low-weight errors [43]. The above example using the $[[4,2,2]]$ code is a concrete case of a time-covariant code with respect to a 2-local Hamiltonian and which can correct a single erasure at a known location.

We can see that whenever we can find a time-covariant code with respect to a given Hamiltonian, then we can construct from the code a clock state with zero sensitivity loss. Consider a code space Π and suppose that the Hamiltonian H is a nontrivial logical operator. We can choose $|\psi\rangle$ to be any logical state vector that has nonzero variance with respect to H. Let $|\xi\rangle = (H - \langle H\rangle)|\psi\rangle$, noting that $|\xi\rangle$ lies in the code space. Denoting by $\{E_k\}$ the Kraus operators of $\mathcal{N}$, we see that $\langle\psi|E_{k'}^\dagger E_k|\xi\rangle \propto \langle\psi|\xi\rangle = 0$ from the Knill-Laflamme conditions of the code, and therefore the conditions (151) are satisfied. Therefore, the following can be observed.

Observation 1 (Clock state from a time-covariant code).—Let Π be the projector onto a code space that corrects errors of the error channel $\mathcal{N}$. Assume that the code is time covariant with respect to the Hamiltonian H. Then any logical state vector $|\psi\rangle$ and associated $|\xi\rangle = (H - \langle H\rangle)|\psi\rangle$ satisfy the conditions (151). Furthermore, if Π defines a $[[n,1,d]]$ quantum code, then $|\psi\rangle$ and $P_\psi^\perp H|\psi\rangle$ define a metrological code of metrological distance d.

That is, *any* logical state of the code satisfies our Knill-Laflamme-like conditions for zero sensitivity loss. The sensitivity is maximized by picking the state with the largest energy variance.

If we are given an ϵ-approximate quantum error-correcting code that is time-covariant, that is, if the error-correction procedure is allowed to fail with some probability $\epsilon > 0$, then we can still use a state lying in the code space to construct a clock state with little sensitivity loss. Approximate quantum error-correcting codes can be characterized by the fact that the channel that maps the code space to the environment, $\widehat{\mathcal{N}}(\Pi(\cdot)\Pi)$, is close to a constant channel that always outputs a fixed state [61,62]. Specifically, $\widehat{\mathcal{N}}(\Pi(X)\Pi) \approx \operatorname{tr}(X)\,\tau_E$ for all X, for some fixed state τ_E. If we pick a logical state vector $|\psi\rangle$ with nonzero energy variance, then we have that $\{\psi,\bar{H}\} = \Pi\{\psi,\bar{H}\}\Pi$ is a logical operator and therefore $\widehat{\mathcal{N}}\big(\{\psi,\bar{H}\}\big) = \widehat{\mathcal{N}}\big(\Pi\{\psi,\bar{H}\}\Pi\big) \approx \operatorname{tr}\big(\{\psi,\bar{H}\}\big)\,\tau_E = 0$ since $\langle\bar{H}\rangle = 0$. Therefore, $\Delta F_{\mathrm{Bob},t}$ in Eq. (55) satisfies $\Delta F_{\mathrm{Bob},t} \approx 0$, and $F_{\mathrm{Bob},t} \approx F_{\mathrm{Alice},t} = 4\sigma_H^2$. This choice of a clock state is hence expected to lose little sensitivity under action of the noisy channel. Deriving a universal quantitative bound on $F_{\mathrm{Bob},t}$ in this scenario in terms of ϵ does not appear easy. In such a scenario, a direct use of our uncertainty relation (49) [or of a corresponding bound such as Eq. (120)] seems likely to be the most straightforward way to obtain useful quantitative expressions for $F_{\mathrm{Bob},t}$ in the case where the clock state is prepared using an approximate error-correcting code.

F. Clock state for interacting many-body systems

Consider now an arbitrary interaction graph, where each vertex is associated with a single qubit [Fig. 9(b)] and consider the Hamiltonian

$$H = \frac{J}{2}\sum_{\langle i,j\rangle}\big(Z_iZ_j + s_xX_iX_j + s_yY_iY_j\big), \tag{165}$$

where the sum ranges over all graph vertices i,j that are connected by an edge, and where s_x, s_y are arbitrary real coefficients. (In fact, the coefficients s_x, s_y may also vary for each pair of sites i,j, though we omit the dependence here for clarity.) We recover the Ising model with $s_x = s_y = 0$ and the Heisenberg model with $s_x = s_y = 1$. We denote by m the number of edges in the graph, which is also the number of terms in the sum.

We define the clock-state vector $|\psi\rangle$ as follows. Denote by $|0^n\rangle$ and $|1^n\rangle$ the all-zero and the all-one state. Choose any bit string $\mathbf{x}$ and let $|\mathbf{x}\rangle$ be the corresponding spin configuration, where each bit corresponds to one of the qubit basis vectors on the corresponding vertex. We assume that $\mathbf{x}$ violates a number c out of the m possible ZZ-interaction terms, i.e., we denote by c the number of pairs of bits in $\mathbf{x}$ that differ and that are connected by an edge in the graph. (It might not be possible to violate all the interaction terms simultaneously, as the graph might be frustrated.) An assumption we will need later is that the bit strings 0^n, 1^n, and $\mathbf{x}$ all differ on at least four sites. Now define

$$|\psi\rangle = \frac{1}{2}\big[|0^n\rangle + |1^n\rangle + |\mathbf{x}\rangle + |\widetilde{\mathbf{x}}\rangle\big], \tag{166}$$

where the bit string $\widetilde{\mathbf{x}}$ is obtained by flipping all the bits of $\mathbf{x}$. We then have

$$H|\psi\rangle = \frac{J}{4}\big[m|0^n\rangle + m|1^n\rangle + (m-2c)|\mathbf{x}\rangle + (m-2c)|\widetilde{\mathbf{x}}\rangle\big] + \frac{J}{2}\sum_{\langle i,j\rangle}\big(s_x X_i X_j + s_y Y_i Y_j\big)\,|\psi\rangle. \tag{167}$$

The XX and YY operators applied on $|\psi\rangle$ generate terms associated with new bit strings where, each time, two bits are flipped and a possible phase is acquired. These new configurations are all orthogonal to $|0^n\rangle$, $|1^n\rangle$, $|\mathbf{x}\rangle$, and $|\widetilde{\mathbf{x}}\rangle$ thanks to our assumption that the chosen configurations differ on at least four sites. So we have

$$\langle H\rangle_\psi = \frac{J}{8}\big[2\times m + 2\times(m-2c)\big] = \frac{J}{2}(m-c). \tag{168}$$

With $\bar{H} = H - J/2(m-c)\mathbb{1}$ and $|\xi\rangle = \bar{H}|\psi\rangle$, we see that

$$|\xi\rangle = \frac{J}{4}\big[c|0^n\rangle + c|1^n\rangle - c|\mathbf{x}\rangle - c|\widetilde{\mathbf{x}}\rangle\big] + \frac{J}{2}\sum_{\langle i,j\rangle}\big(s_x X_i X_j + s_y Y_i Y_j\big)\,|\psi\rangle. \tag{169}$$

To check the zero sensitivity-loss conditions (148), we compute the following expression for any single-site operator O_i,

$$\begin{aligned}\langle\psi|O_i|\xi\rangle &= \frac{J}{8}\big[c\,\langle 0|O_i|0\rangle + c\,\langle 1|O_i|1\rangle \\ &\quad - c\,\langle x_i|O_i|x_i\rangle - c\,\langle\widetilde{x}_i|O_i|\widetilde{x}_i\rangle\big] \\ &\quad + \frac{J}{2}\sum_{\langle i,j\rangle}\langle\psi|O_i\big(s_x X_i X_j + s_y Y_i Y_j\big)|\psi\rangle \\ &= \frac{J}{8}\big[c\,\mathrm{tr}(O_i) - c\,\mathrm{tr}(O_i)\big] + 0 = 0,\end{aligned} \tag{170}$$

where x_i (respectively $\widetilde{x}_i$) denote the value of the ith bit in $\mathbf{x}$ (respectively, $\widetilde{\mathbf{x}}$). The terms corresponding to XX and YY interactions vanish because all configurations 0^n, 1^n, $\mathbf{x}$, and $\widetilde{\mathbf{x}}$ differ on at least four sites, and XX and YY flip two bits of the basis vector state on which they are applied (with a possible phase). Therefore, the zero sensitivity-loss conditions (151) are satisfied, and the clock state can suffer a single located erasure while retaining full sensitivity.

The energy variance of the probe state is given by

$$\begin{aligned}\sigma_H^2 &= \langle\psi|(H-\langle H\rangle)^2|\psi\rangle = \langle\xi|\xi\rangle \\ &= \frac{1}{4}J^2c^2 + (\text{contrib. from } XX/YY\text{terms}).\end{aligned} \tag{171}$$

The contribution from XX and YY terms is zero if the configurations $0^n, 1^n, \mathbf{x}, \widetilde{\mathbf{x}}$ all differ on at least five sites (or in the case of Ising interactions with $s_x = s_y = 0$).

The question of whether this achieves n^2 scaling depends on how we choose the graph and the string $\mathbf{x}$ to grow with n. In the case of a square lattice with nearest-neighbor interactions, we have that the number of edges scales like the number of vertices ($m \sim 2n$) and we can simultaneously violate all ZZ interaction terms by choosing an alternating configuration of 0's and 1's. In this case $\sigma_H^2 \sim J^2n^2$, achieving Heisenberg scaling. For other graphs, the question of whether $\sigma_H^2 \sim n^2$ is determined by how the number of edges scales with the number of vertices in the graph, and how many of those ZZ-interaction terms can be simultaneously violated. If there is a linear relationship between these quantities then Heisenberg scaling is achieved, noting that only a single error at a known location can be incurred without sensitivity loss.

G. Metrological codes from stabilizer codes

In this section, we present a general scheme to construct metrological codes based on the stabilizer formalism [63] and study some simple examples. We show that our construction is strictly more general than constructing time-covariant error-correcting codes. Our aim is to study and illustrate our general construction; the Hamiltonians in our examples are not intended as practical schemes to be engineered with near-term technology.

Consider the Pauli group $\mathcal{G}_n$ on n qubits, defined as comprising all tensor product operators on n qubits of single-site Pauli operators and the identity operator, with all possible prefactors ± 1 and $\pm i$ [63]. Consider a subgroup $\mathcal{S} \subset \mathcal{G}_n$ presented as $\mathcal{S} = \langle S_1, \ldots, S_\ell \rangle$ with independent commuting generators $S_1, \ldots, S_\ell$ such that $-\mathbb{1} \notin \mathcal{S}$. The *normalizer* of $\mathcal{S}$ in $\mathcal{G}_n$ is $N(\mathcal{S}) = \{E \in \mathcal{G}_n : \forall g \in \mathcal{S},\ EgE^\dagger \in \mathcal{S}\}$. A state is said to be *stabilized* by $\mathcal{S}$ if it lies in the simultaneous $+1$ eigenspace of all $S \in \mathcal{S}$; the elements of $\mathcal{S}$ are called *stabilizers*. The code space associated with the Pauli stabilizer group $\mathcal{S}$ is the subspace spanned by all states that are stabilized by $\mathcal{S}$. If $\mathscr{E} \subset \mathcal{G}_n$ is a set of error operators such that for all $E, E' \in \mathscr{E}$ either $E'^\dagger E \notin N(\mathcal{S})$ or $E'^\dagger E$ lies in $\mathcal{S}$ up to a phase, then a fundamental theorem of quantum error-correction states that the subspace of all common $+1$ eigenstates of the operators $\{S_i\}$ forms a code space that can correct any error in $\mathscr{E}$. One defines the distance d of the code as the minimal weight of an element in $N(\mathcal{S}) \setminus \mathcal{S}$, i.e., of a nontrivial logical operation. Then, the code can correct any t errors at unknown locations as long as $2t + 1 \leqslant d$.

As a simple example, consider the n-qubit GHZ state vector $|\psi\rangle = \big[|\uparrow^n\rangle + |\downarrow^n\rangle\big]/\sqrt{2}$ and the Hamiltonian $H = \sum_{j=1}^n Z_j$. We have $|\xi\rangle \propto \big[|\uparrow^n\rangle - |\downarrow^n\rangle\big]/\sqrt{2}$. Suppose our error model consists of an arbitrary number of X errors. From Eq. (151), since acting with X operators on $|\psi\rangle$ can never generate any overlap with $|\xi\rangle$, we see that $|\psi\rangle, |\xi\rangle$ form a metrological code against any number of X errors. We now present an overview of our procedure using this example. In our procedure, we first find a set $\{S_i\}$ of independent commuting Pauli operators that stabilize $|\psi\rangle$. We fix a set of error operators $\mathscr{E}$, which we choose in our example to consist of all n-qubit Pauli operators that are a product of only $\mathbb{1}$'s and X's. Suppose that we are given an operator H with the following property: for any operators $E, E' \in \mathscr{E}$, there exists a $S \in \mathcal{S}$ such that $\{H, S\} = 0$ and $[E'^\dagger E, S] = 0$. The state vector $|\psi\rangle$ is stabilized by the choice of commuting Pauli operators $Z_1 Z_2$, $Z_2 Z_3$, $\ldots$, $Z_{n-1} Z_n, X^{\otimes n}$. Multiplying all but the last stabilizer by $X^{\otimes n}$, we obtain the following choice of independent stabilizer generators:

$$-Y_1 Y_2 X_3 X_4 \ldots X_n, \quad -X_1 Y_2 Y_3 X_4 \ldots X_n, \quad \ldots, \quad -X_1 \ldots X_{n-2} Y_{n-1} Y_n, \quad X^{\otimes n}. \tag{172}$$

For any site j, the operator Z_j anticommutes with all the above stabilizer generators. Our structural constraint turns out to apply in this case; it will be detailed later. Our construction then implies that the pair $(|\psi\rangle, H|\psi\rangle)$ is a metrological code. Here, $|\xi\rangle = H|\psi\rangle$ is in fact the state vector that is stabilized by all the operators $\{-S_i\}$.

1. Statement of the construction

Our construction is given by the following theorem.

Theorem 4 (Metrological codes from stabilizer states).—let $\mathcal{S} \subset \mathcal{G}_n$ be an abelian subgroup of the Pauli group with $-\mathbb{1} \notin \mathcal{S}$, and let $|\psi\rangle$ be stabilized by $\mathcal{S}$. Let H be any Hermitian operator such that $H|\psi\rangle \neq 0$ and let $\mathscr{E} \subset \mathcal{G}_n$ be any set of Pauli error operators. Assume that for all $E, E' \in \mathscr{E}$, there exists $S \in \mathcal{S}$ such that $\{H, S\} = 0$ and $[E'^\dagger E, S] = 0$. Then $|\psi\rangle, H|\psi\rangle$ form a metrological code against $\mathscr{E}$.

We recall the definition of a metrological code as satisfying the condition (154). On the other hand, a defining property of a time-covariant code (recall definition in Sec. VII E) is that the Hamiltonian H must be a logical operator, and thus, for a stabilizer code, must commute with all the stabilizers of the code. This is not in contradiction with Theorem 4 since the stabilizer group $\mathcal{S}$ in the theorem is not necessarily that of the code for which H is a logical operator. Below, we present examples of metrological codes; some are error-correcting time-covariant codes in disguise, yet others cannot be written as a time-covariant error-correcting code with similar distance as the metrological code.

Proof.—First, let $S_0 \in \mathcal{S}$ with $\{H, S_0\} = 0$; such a stabilizer must exist from our assumption. We then have

$$\langle\psi|H|\psi\rangle = \langle\psi|HS_0|\psi\rangle = -\langle\psi|S_0 H|\psi\rangle = -\langle\psi|H|\psi\rangle \tag{173}$$

and thus $\langle H\rangle_\psi = 0$. Let

$$|\xi\rangle = H|\psi\rangle, \tag{174}$$

with $\langle\xi|\psi\rangle = 0$ automatically satisfied. Let $E, E' \in \mathscr{E}$. We need to show that Eq. (154) holds. From our assumption there exists an $S \in \mathcal{S}$ with $\{H, S\} = 0$ and $[E'^\dagger E, S] = 0$. We have

$$\begin{aligned}\langle\xi|E'^\dagger E|\psi\rangle &= \langle\psi|HE'^\dagger ES|\psi\rangle \\ &= -\langle\psi|SHE'^\dagger E|\psi\rangle \\ &= -\langle\xi|E'^\dagger E|\psi\rangle,\end{aligned} \tag{175}$$

and thus $\langle\xi|E'^\dagger E|\psi\rangle = 0$, confirming that Eq. (154) holds and that $|\psi\rangle, |\xi\rangle$ indeed constitute a metrological code against $\mathscr{E}$. ■

In the remainder of this section we review some examples of codes resulting from the construction of Theorem 4. We begin by connecting our construction with error-correcting codes in which the Hamiltonian is a nontrivial logical operator, i.e., time-covariant error-correcting codes. We present an example of a time-covariant code based on the seven-qubit Steane code, and we then show that all time-covariant error-correcting codes are special cases of Theorem 4. We then show that there are metrological codes that cannot be formulated in terms of a corresponding time-covariant error-correcting code; in other words, there are schemes that enable the communication of a clock state through a noisy channel that achieve zero sensitivity loss without having to construct a full quantum error-correcting code.

2. Example based on the [[7, 1, 3]] Steane code

As an example of a time-covariant code, we consider an example deriving from the Steane stabilizer code [64]. The latter is given by the following generators and logical $\overline{X}, \overline{Z}$ operators:

$$\begin{aligned}
\hat{S}_1 &= X_4X_5X_6X_7,\\
\hat{S}_2 &= X_2X_3X_6X_7,\\
\hat{S}_3 &= X_1X_3X_5X_7,\\
\hat{S}_4 &= Z_4Z_5Z_6Z_7,\\
\hat{S}_5 &= Z_2Z_3Z_6Z_7,\\
\hat{S}_6 &= Z_1Z_3Z_5Z_7,\\
\overline{X} &= X_1X_2X_3X_4X_5X_6X_7,\\
\overline{Z} &= Z_1Z_2Z_3Z_4Z_5Z_6Z_7.
\end{aligned} \tag{176}$$

Let $|\psi\rangle = |\overline{+}\rangle$ be the state vector in the logical space associated with the $+1$ logical eigenspace of the $\overline{X}$ operator, and consider the Hamiltonian

$$H = \overline{Z}\hat{S}_4 = Z_1Z_2Z_3. \tag{177}$$

The Hamiltonian is a logical operator, being stabilizer-equivalent to the logical $\overline{Z}$ operator, and rotates the state vector $|\psi\rangle$ to $|\xi\rangle = H|\psi\rangle = |\overline{-}\rangle$. (The above choice of H was preferred to the choice $H = \overline{Z}$ because it has lower weight.) The code is therefore time covariant with respect to the action of H, and we for this reason already know that it is a metrological code of metrological distance 3. To illustrate our construction, we explain how the same conclusion can be reached by applying Theorem 4. We now define a set of stabilizer generators that serve to define the state vector $|\psi\rangle$ of the resulting metrological code. The new stabilizer generators $\{S_i\}$ are obtained by multiplying each of the $\hat{S}_i$ by $\overline{X}$, all while including $\overline{X}$ itself, as

$$\begin{aligned}
S_1 &= \overline{X}\hat{S}_1 = X_1X_2X_3,\\
S_2 &= \overline{X}\hat{S}_2 = X_1X_4X_5,\\
S_3 &= \overline{X}\hat{S}_3 = X_2X_4X_6,\\
S_4 &= \overline{X}\hat{S}_4 = X_1X_2X_3Y_4Y_5Y_6Y_7,\\
S_5 &= \overline{X}\hat{S}_5 = X_1Y_2Y_3X_4X_5Y_6Y_7,\\
S_6 &= \overline{X}\hat{S}_6 = Y_1X_2Y_3X_4Y_5X_6Y_7,\\
S_7 &= \overline{X} = X_1X_2X_3X_4X_5X_6X_7.
\end{aligned} \tag{178}$$

One can verify that H anticommutes with each S_i listed above. (For the application of Theorem 4, it is convenient to use a choice of stabilizer generators that anticommute with H.) Let $\mathscr{E}$ be the set of all single-site operators. For any $E, E' \in \mathscr{E}$, we will show that there is a $S \in \mathcal{S}$ with $\{H, S\} = 0$ and $[E'^{\dagger}E, S] = 0$. If one of the S_i has support outside of that of $E'^{\dagger}E$, it will do the job. Alternatively, any product of an odd number of the S_i will also do, for instance,

$$\begin{aligned}
S_1S_2S_3 &= X_3X_5X_6,\\
S_1S_2S_7 &= X_1X_6X_7,\\
S_2S_3S_7 &= X_3X_4X_7.
\end{aligned} \tag{179}$$

One can verify that for any two among the seven sites, at least one operator among $S_1, S_2, S_3, S_1S_2S_3, S_1S_2S_7, S_2S_3S_7$ has its support outside of those two sites. Since these operators all anticommute with H, we have that for all $E, E' \in \mathscr{E}$, there is a $S \in \langle S_1, \ldots, S_7\rangle$ such that $[E'^{\dagger}E, S] = 0$ and $\{S, H\} = 0$. From Theorem 4, we see that $|\psi\rangle$ and $|\xi\rangle = H|\psi\rangle$ must form a metrological code against $\mathscr{E}$, and is therefore a metrological code with metrological distance 3.

3. Time-covariant codes

In this paragraph, we show that the assumptions of Theorem 4 are in fact always satisfied for time-covariant stabilizer codes like the seven-qubit Steane code example above.

Let $\hat{\mathcal{S}} = \langle \hat{S}_1, \ldots, \hat{S}_\ell\rangle$ be a stabilizer code with a nontrivial logical operator $\overline{Z}$. Let $\overline{X}$ be a logical operator that anticommutes with $\overline{Z}$, and define the stabilizer group $\mathcal{S} = \langle \overline{X}\hat{S}_1, \overline{X}\hat{S}_2, \ldots, \overline{X}\hat{S}_\ell, \overline{X}\rangle$. Observe that $\overline{Z}$ anticommutes with all the chosen generators for $\mathcal{S}$. (Such an operator $\overline{X}$ must always exist, cf. e.g., [63, Proposition 10.4].)

We show the following: for any Pauli operator $A \notin N(\hat{\mathcal{S}}) \setminus \hat{\mathcal{S}}$, there exists $S \in \mathcal{S}$ such that $[S, A] = 0$ and $\{S, \overline{Z}\} = 0$. This property implies that for a given set of errors $\mathscr{E}$ that are correctable for $\hat{\mathcal{S}}$, i.e., if we have $E'^{\dagger}E \notin N(\hat{\mathcal{S}}) \setminus \hat{\mathcal{S}}$ for all $E, E' \in \mathscr{E}$, then the conditions of Theorem 4 are satisfied, where the Hamiltonian is $H = \bar{Z}$.

Suppose first that $A \in \hat{\mathcal{S}} \subset \mathcal{S}$. Then A commutes with all stabilizers in $\mathcal{S}$, including $\overline{X}$, which anticommutes with $\overline{Z}$. Now suppose that $A \in \mathcal{G}_n$ and $A \notin N(\hat{\mathcal{S}})$, i.e., there is a $\hat{S} \in \hat{\mathcal{S}}$ with $\{A, \hat{S}\} = 0$. If $[A, \overline{X}] = 0$, then the choice $S = \overline{X} \in \mathcal{S}$ satisfies $[S, A] = 0$ and $\{S, \overline{Z}\} = 0$. If, instead, we have $\{A, \overline{X}\} = 0$, we can set $S = \overline{X}\hat{S}$ to find $S\overline{Z} = \overline{X}\hat{S}\overline{Z} = \overline{X}\overline{Z}\hat{S} = -\overline{Z}\overline{X}\hat{S} = -\overline{Z}S$ and $AS = A\overline{X}\hat{S} = -\overline{X}A\hat{S} = \overline{X}\hat{S}A = SA$, and thus $\{S, \overline{Z}\} = 0$ and $[S, A] = 0$ as required.

4. Example: a [[4, 2, 2]] code state with an auxiliary qubit

Whereas in the earlier seven-qubit Steane code example the state vectors $|\psi\rangle$ and $|\xi\rangle$ both lie within a subspace of a distance $d = 3$ code, the following example illustrates a situation in which $|\psi\rangle$ and $|\xi\rangle$ cannot be contained in a code space that can correct the same errors against which the states form a metrological code. In other words, Theorem 4 can be used to construct metrological codes that cannot be formulated as time-covariant quantum error-correcting codes with respect to the same errors. Consider the five-qubit Pauli operators

$$\begin{aligned} S_1 &= X_1X_2, \\ S_2 &= X_3X_4, \\ S_3 &= X_1X_3, \\ S_4 &= X_5, \\ S_5 &= Z_1Z_2Z_3Z_4. \end{aligned} \tag{180}$$

We can see that the stabilizer group $\mathcal{S} = \langle S_1, \ldots, S_5\rangle$ is generated by

(i) the stabilizers for the [[4, 2, 2]] code on the first four qubits ($Z_1Z_2Z_3Z_4 = S_5$ and $X_1X_2X_3X_4 = S_1S_2$);
(ii) the logical X operators of the first and second logical qubits of that [[4, 2, 2]] code ($X_1X_3 = S_3$ and $X_1X_2 = S_1$); and
(iii) an independent stabilizer fixing the state of the fifth qubit ($X_5 = S_4$).

We choose the Hamiltonian

$$H = Y_1Z_4Y_5. \tag{181}$$

The Hamiltonian can be written as a product of three terms: a logical Z operator on both logical qubits of the [[4, 2, 2]] code (Z_1Z_4), a Y operation on the fifth physical qubit, and a single X_1 on the first physical qubit. The Hamiltonian is not a logical operator of the [[4, 2, 2]] code. Also, a suitable permutation of the qubits would make H geometrically local, should this property be desired.

We can verify that H anticommutes with each of the stabilizers $S_1, \ldots, S_5$. Furthermore, for any two sites i, j, one of the S_k acts as the identity on the sites i, j; therefore, for any two-site operator A, there always exists a stabilizer S with $[S, A] = 0$ and $\{S, H\} = 0$. We can apply Theorem 4 to deduce that $|\psi\rangle, |\xi\rangle$ define a distance-3 metrological code.

The state vector $|\psi\rangle$ can be expressed in terms of the logical +1 X eigenvectors $|\overline{++}\rangle$ of the [[4, 2, 2]] code, and in terms of the $|\pm\rangle_i$ physical state vectors, as

$$\begin{aligned} |\psi\rangle &= |\overline{++}\rangle_{1234} \otimes |+\rangle_5 \\ &= \frac{1}{\sqrt{2}}\big(|+++++\rangle + |----+\rangle\big). \end{aligned} \tag{182}$$

Recalling $Y|\pm\rangle = \mp i|\mp\rangle$ and $Z|\pm\rangle = |\mp\rangle$, we find

$$\begin{aligned} |\xi\rangle = H|\psi\rangle &= \frac{1}{\sqrt{2}}\big(-|-++--\rangle + |+--+-\rangle\big) \\ &= \frac{1}{\sqrt{2}}\big(|+--+\rangle - |-++-\rangle\big) \otimes |-\rangle_5. \end{aligned} \tag{183}$$

We see that

$$\langle\psi|X_5|\psi\rangle = 1, \quad \langle\xi|X_5|\xi\rangle = -1, \tag{184}$$

so it is not possible for $|\psi\rangle, |\xi\rangle$ to lie in the code space of a distance $d = 3$ quantum error-correcting code.

This example shows that metrological codes are a class of codes that is broader than traditional error-correction codes as there are certain errors that the former does not have to completely correct. Metrological codes might therefore offer additional possibilities to find noise-resilient schemes for communicating clock states across a noise channel.

5. Metrological toric code

A further example application of Theorem 4 is based on *Kitaev's toric code* [65,66]. We consider a two-dimensional square lattice of dimension $L \times L$ that wraps around a torus. We define star operators $A_{\mathbf{x}}$ and plaquette operators $B_{\mathbf{x}}$ as depicted in Fig. 10(a), where $\mathbf{x}$ ranges over all pairs of the lattice coordinates.

First, we can always use the toric code to form a time-covariant code, by choosing a state vector $|\psi\rangle$ in the code space (for instance, $|\overline{++}\rangle$), and choosing the Hamiltonian to be a logical operator (for instance, $\overline{Z}_1 + \overline{Z}_2$). This code being by construction time covariant, it is necessarily a metrological code with distance equal to the lattice side length L.

For the sake of the example, we construct here a metrological code from the toric code that cannot be written as a time-covariant error-correcting code of similar distance. Our example is meant to (i) illustrate our construction as combining states that lie either in the simultaneous +1 or simultaneous −1 eigenspaces of all the stabilizer generators of some given stabilizer code, (ii) furnish another example of a metrological code that cannot be phrased in terms of a time-covariant error-correcting code with similar distance, and (iii) illustrate how a metrological code can be a terrible quantum error-correcting code—any single

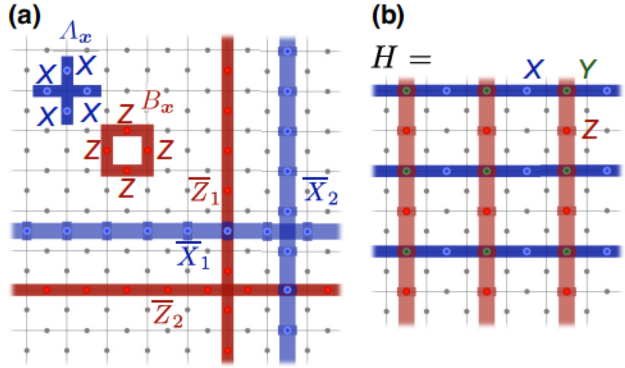

FIG. 10. Metrological code based on the toric code. (a) Star ($A_{\mathbf{x}}$) and plaquette ($B_{\mathbf{x}}$) operators generate the stabilizer group of the toric code, where $\mathbf{x}$ is a pair of integer coordinates on the two-dimensional lattice. Two encoded logical qubits have logical Pauli operators $\overline{X}_1$, $\overline{Z}_1$, $\overline{X}_2$, $\overline{Z}_2$ corresponding to strings of physical Pauli X or Z operators that wrap around the torus. (b) In our metrological code example based on the toric code, we map a state from the toric code to a state of a related code, which we call the *antitoric code*. The antitoric code is the subspace stabilized by all the operators $\{-A_{\mathbf{x}}\}$ and $\{-B_{\mathbf{x}}\}$. Assuming the lattice side length is even, the depicted operator H anticommutes with all star and plaquette operators, meaning that it maps a logical state of the toric code to a logical state of the antitoric code. Picking $|\psi\rangle$ in the code space of the toric code and choosing the depicted operator H as the corresponding Hamiltonian yields an example of a distance-$\Omega(L^2)$ metrological code. Interestingly, this metrological code cannot be phrased in terms of a time-covariant error-correcting code of similar distance, since $|\psi\rangle$ and $H|\psi\rangle$ can be distinguished by measuring a single star or plaquette operator. This example shows that there are additional possible schemes for sending a clock state through a noisy channel without any sensitivity loss, without resorting to a time-covariant quantum error-correcting code.

plaquette or star operator acts nontrivially on the subspace spanned by the state and its time-evolution state. Our example is more of a conceptual illustration than a practical proposal, as it requires a Hamiltonian that is highly nonlocal.

To better explain our example, we first define the *antitoric code* as the code whose code space is stabilized by all negative star $-A_{\mathbf{x}}$ and negative plaquette operators $-B_{\mathbf{x}}$. Being equivalent to the standard toric code, the antitoric code also has distance L and we can see it also has the logical operators $\overline{X}_1, \overline{X}_2, \overline{Z}_1, \overline{Z}_2$ defined as for the toric code.

As the state vector $|\psi\rangle$ of our metrological code, we simply choose a logical state vector of the toric code; we can conventionally fix it to be $|\overline{00}\rangle_{\text{toric}}$ stabilized by $\overline{Z}_1, \overline{Z}_2$ along with all the toric code stabilizers $\{A_{\mathbf{x}}\}$ and $\{B_{\mathbf{x}}\}$. For the Hamiltonian we choose an operator H that anticommutes with all star and all plaquette operators. Such an operator is depicted in Fig. 10(b); we assume for convenience that L is even. The operator H has the property that it maps a code word of the toric code (i.e., a state vector $|\psi\rangle$ satisfying $A_{\mathbf{x}}|\psi\rangle = |\psi\rangle = B_{\mathbf{x}}|\psi\rangle$) to a code word of the *antitoric code* (we have $A_{\mathbf{x}}H|\psi\rangle = -HA_{\mathbf{x}}|\psi\rangle = -H|\psi\rangle$ and similarly for $B_{\mathbf{x}}$). We can verify that the assumptions of Theorem 4 are satisfied. The operator H anticommutes with our choice of stabilizer generators for $|\psi\rangle$. Also, for any operator O of weight $< L^2/4$, there must be a star or plaquette operator that has disjoint support with, and therefore commutes with, O. [Indeed, there are $(L/2)^2$ disjoint plaquette operators that cover all qubits; an operator that has overlapping support with all plaquette operators must therefore have support on one qubit in each plaquette. The bound can presumably be improved by accounting for the star operators as well.] As a consequence of Theorem 4, the state vectors $(|\psi\rangle, |\xi\rangle = H|\psi\rangle)$ form a metrological code of distance $L^2/4$.

Is the space spanned by $(|\psi\rangle, |\xi\rangle)$ secretly a code space of a similar-distance code in which H acts as a logical operator? We can rule out this possibility because the state vectors $|\psi\rangle$ and $|\xi\rangle$ can easily be distinguished by measuring any single star or plaquette operator, recalling that $A_{\mathbf{x}}|\psi\rangle = B_{\mathbf{x}}|\psi\rangle = |\psi\rangle$ but that $A_{\mathbf{x}}|\xi\rangle = B_{\mathbf{x}}|\xi\rangle = -|\xi\rangle$. The environment only has to measure a weight-4 operator to determine whether $|\psi\rangle$ or $|\xi\rangle$ was encoded.

6. *Simultaneous* +1 *eigenspace and simultaneous* −1 *eigenspace of stabilizers*

The intuition behind the construction in Theorem 4 is that if we can choose $|\psi\rangle$ to be stabilized by $\mathcal{S} = \langle S_1, \ldots, S_\ell\rangle$, then we might want to pick $|\xi\rangle$ to be stabilized by the closely related stabilizer group $\mathcal{S}' = \langle -S_1, \ldots, -S_\ell\rangle$. This idea was already illustrated by the example above based on the toric code, where the Hamiltonian maps a code word of the toric code to a code word of the antitoric code. We now show in general that such a construction is a special case of Theorem 4.

Let $\mathcal{S} = \langle S_1, \ldots, S_\ell\rangle$ be a subgroup of the Pauli group with $-\mathbb{1} \notin \mathcal{S}$, where $S_1, \ldots, S_\ell$ are a choice of independent commuting stabilizer generators. Let $\mathcal{S}' = \langle -S_1, \ldots, -S_\ell\rangle$. Let $\mathscr{E}$ denote any set of Pauli operators with the following property: for any $E, E' \in \mathscr{E}$, there exists $S \in \mathcal{S}$ such that $-S \in \mathcal{S}'$ and such that $[E'^{\dagger}E, S] = 0$.

The two stabilizer groups $\mathcal{S}, \mathcal{S}'$ share many stabilizers, including S_1S_2, S_1S_3, ..., S_1S_ℓ. We can pick a Pauli operator H such that H anticommutes with S_1 and such that H commutes with each of the operators S_1S_2, S_1S_3, ..., S_1S_ℓ (see, e.g., Ref. [63, Proposition 10.4]). Observe that for all $i = 2, \ldots, \ell$, we have

$$HS_i = HS_1^2S_i = -S_1HS_1S_i = -S_1^2S_iH = -S_iH \quad (185)$$

and thus we have that $\{H, S_i\} = 0$ for all $i = 1, \ldots, \ell$. Suppose $|\psi\rangle$ is stabilized by $\mathcal{S}$. Then $H|\psi\rangle$ is stabilized by $\mathcal{S}'$, since

$$S_i H|\psi\rangle = -HS_i|\psi\rangle = -H|\psi\rangle. \tag{186}$$

Furthermore, supposing $E, E' \in \mathscr{E}$, by assumption we have $S \in \mathcal{S}$ such that $-S \in \mathcal{S}'$ and such that $[E'^{\dagger}E, S] = 0$. We can write $S = S_{i_1}S_{i_2}\cdots S_{i_m}$ in terms of our choice of independent generators S_i above. Writing $-S = (-1)^{m+1}(-S_{i_1})(-S_{i_2})\cdots(-S_{i_m})$, we see that m must be odd, as otherwise, we would have

$$-S = (-1)(-S_{i_1})(-S_{i_2})\cdots(-S_{i_m}) \notin \mathcal{S}'. \tag{187}$$

This observation implies that $\{H, S\} = 0$, because we can anticommute H through the product of an odd number of S_i's. Therefore there exists $S \in \mathcal{S}$ such that $[E'^{\dagger}E, S] = 0$ and $\{H, S\} = 0$. At this point, all the assumptions of Theorem 4 are satisfied, implying that $|\psi\rangle, H|\psi\rangle$ form a metrological code that can protect against the error set $\mathscr{E}$.

H. Further examples of metrological codes

We now present two additional examples of state vectors $|\psi\rangle, |\xi\rangle$ that satisfy the zero sensitivity loss conditions (152). These metrological codes serve to illustrate the sense in which the conditions (152) are weaker than the conditions for quantum error correction.

1. Single-qubit subject to complete X / Y dephasing

Consider the qubit example studied in Sec. IV B, where the clock state vector $|+\rangle$ evolves according to $H = \omega Z/2$ and is exposed to complete dephasing along the X axis around a given time t_0. From Eq. (90) we immediately see that the zero sensitivity loss condition (148) is satisfied for all t_0. In this setting, the clock state loses no sensitivity after complete dephasing in the X axis for any t_0, with the exception of possible discrete points where the rank of $\rho_B(t)$ changes (see Sec. IV B).

Alternatively one could also check the form (151) of the zero sensitivity-loss conditions. For any t_0, we have from Eq. (29) that

$$\begin{aligned} |\xi(t_0)\rangle &= H|\psi(t_0)\rangle \\ &= \frac{\omega}{2}\Big[\cos\Big(\frac{\omega t_0}{2}\Big)|-\rangle - i\sin\Big(\frac{\omega t_0}{2}\Big)|+\rangle\Big]. \end{aligned} \tag{188}$$

At this point, we can compute

$$\begin{aligned} &\langle\psi||+\rangle\langle+||\xi\rangle + \langle\xi||+\rangle\langle+||\psi\rangle \\ &= \frac{\omega}{2}\Big[-i\cos\Big(\frac{\omega t_0}{2}\Big)\sin\Big(\frac{\omega t_0}{2}\Big) + i\sin\Big(\frac{\omega t_0}{2}\Big)\cos\Big(\frac{\omega t_0}{2}\Big)\Big] \\ &= 0, \end{aligned} \tag{189}$$

and similarly for $\langle\psi||-\rangle\langle-||\xi\rangle + \langle\xi||-\rangle\langle-||\psi\rangle = 0$, showing that Eq. (151) are satisfied for all t_0.

An interesting aspect of this example is that there exists no recovery operation that can restore the noiseless clock state vector $|\psi(t_0 + dt)\rangle$ accurately to first order in dt. Let us consider for simplicity the point $t_0 = \pi/(2\omega)$. Using Eqs. (29), (31), and (89), we have at that point

$$\begin{aligned} &\psi(t_0) = |{+i}\rangle\langle{+i}|, \quad \rho_B(t_0) = \frac{\mathbb{1}}{2}, \\ &\partial_t\psi(t_0) = -\frac{\omega}{2}X, \quad \mathcal{D}_X(\partial_t\psi(t_0)) = -\frac{\omega}{2}X, \end{aligned} \tag{190}$$

where $|{+i}\rangle := \big[|\uparrow\rangle + i|\downarrow\rangle\big]/\sqrt{2}$. We seek a completely positive, trace-preserving map Rec such that $\mathsf{Rec}(\rho_B(t_0 + dt)) = \psi(t_0 + dt) + O(dt^2)$, which means that

$$\mathsf{Rec}\Big(\frac{\mathbb{1}}{2}\Big) = |{+i}\rangle\langle{+i}|, \quad \mathsf{Rec}(X) = X. \tag{191}$$

There is no completely positive map that satisfies these constraints. If there was such a Rec map, then we would have $\mathsf{Rec}(|+\rangle\langle+|) = \mathsf{Rec}(\mathbb{1}/2) + \mathsf{Rec}(X/2) = |{+i}\rangle\langle{+i}| + X/2 = \mathbb{1}/2 + Y/2 + X/2$. One can easily check that the final expression has a negative eigenvalue, contradicting the requirement that Rec be completely positive. We conclude that in general, a metrological code does not necessarily come with a recovery operation that enables an agent to recover the noiseless clock state, even if the agent can sense the parameter to the same precision as before the application of the noise.

2. A superposition of a simple state and a generic pure state

Consider a one-dimensional chain of n qubits. Consider a generic pure state vector $|\chi\rangle$, chosen, for instance, randomly from the Haar measure on the n-qubit system. For a given $d_m > 0$, let us perturb the state vector $|\chi\rangle$ to $|\tilde{\chi}\rangle$ by projecting it onto the subspace of all computational basis states that do not contain fewer than a number d_m of 1's,

$$|\tilde{\chi}\rangle = \tilde{\Pi}|\chi\rangle, \quad \tilde{\Pi} = \prod_{|\mathbf{x}|\geqslant d_m} |\mathbf{x}\rangle\langle\mathbf{x}|. \tag{192}$$

If $|\chi\rangle$ is generic in some suitable sense (e.g., chosen Haar randomly), then $|\tilde{\chi}\rangle \approx |\chi\rangle$ and $\||\tilde{\chi}\rangle\| \approx 1$. The present example metrological code is constructed by picking $|\psi\rangle = |0^n\rangle$, which is the computational basis all-zero state, and $|\xi\rangle = \||\tilde{\chi}\rangle\|^{-1}\,|\tilde{\chi}\rangle \approx |\chi\rangle$.

We proceed to check that the zero sensitivity loss conditions (148) are satisfied as long as operators of the form $E_{k'}^{\dagger}E_k$ have weight at most $d_m - 1$. If $\mu \subset \{1, \ldots, n\}$ denotes a subset of at most $|\mu| = d_m - 1$ systems, then the reduced operator of $|\tilde{\chi}\rangle\langle 0^n|$ on the sites labeled by μ can be written as

$$\mathrm{tr}_{\setminus\mu}\big[|\tilde{\chi}\rangle\langle 0^n|\big] = (\mathbb{1}_{d_m} \otimes \langle 0^{n-d_m}|)\,|\tilde{\chi}\rangle\langle 0^{d_m}| = 0, \tag{193}$$

because $|\tilde{\chi}\rangle$ has no overlap with bit strings that have $(n - d_m)$ or more zeros. Hence,

$$\mathrm{tr}_{\setminus\mu}\big(|\psi\rangle\langle\xi| + |\xi\rangle\langle\psi|\big) = 0. \tag{194}$$

For any operator O of weight $\mathrm{wgt}(O) < d_m$, the condition (155) is thus satisfied and $|\psi\rangle, |\xi\rangle$ form a metrological code of distance d_m.

An interesting observation is that this code does not form a quantum error-correcting code in the usual sense. The reason is that the environment, by receiving a few sites, can tell the difference between whether the state vector $|\psi\rangle$ or the state vector $|\xi\rangle$ was prepared. More precisely, the environment can test whether the received qubits are all in the state vector $|0\rangle$. If this is the case, it is much more likely that the original state vector was $|\psi\rangle$ and not $|\xi\rangle$, as long as $\mathrm{tr}_{\setminus\nu}(\tilde{\chi})$ is sufficiently distinct from $|0\rangle\langle 0|_\nu$ (which is the case for a Haar-random state).

For some choices of $|\chi\rangle$, the metrological code can be interpreted as a quantum error-correcting code that protects only against certain types of errors. For instance, we can choose $|\chi\rangle = |1\rangle^{\otimes n}$ in the above and our conclusions still hold; this choice corresponds to a classical repetition code that can correct bit flips but which is vulnerable to phase flips.

Yet, there are choices of $|\chi\rangle$ for which this interpretation appears more problematic. Consider for instance the choice $|\chi\rangle = |+\rangle^{\otimes n}$. From the above argument we have that $|\psi\rangle = |0\rangle^{\otimes n}$ and $|\xi\rangle \approx |+\rangle^{\otimes n}$ form again a metrological code. Again, the environment can distinguish $|\psi\rangle$ from $|\xi\rangle$ with access only to a few sites. Here, the environment can use either an X or a Z measurement to (imperfectly) distinguish between the two state vectors $|0\rangle$ and $|+\rangle$. It is hence not obvious how to interpret this code as an error-correcting code that is tailored to biased noise.

While this example might illustrate the conceptual differences between quantum error-correcting codes and metrological codes, we expect this construction of a metrological code to be of limited practical use as it would require a Hamiltonian that is extremely nonlocal.

VIII. MANY-BODY SYSTEM SUBJECT TO IID AMPLITUDE DAMPING NOISE

In this section we consider a system consisting of n spin-1/2 particles evolving under a many-body Hamiltonian H that is either noninteracting or that has Ising interaction terms. The system is exposed to IID amplitude damping noise. First, we consider a noninteracting Hamiltonian with an on-site magnetic field, and in the second part of this section we consider a Hamiltonian with Ising interactions.

We consider an IID amplitude damping noise model, meaning that each site is independently exposed to the noisy channel

$$\begin{aligned}\mathcal{N}_{\mathrm{AD}}^{(p)}(\cdot) &= E_0^{(p)}\,(\cdot)\,E_0^{(p)\dagger} + E_1^{(p)}\,(\cdot)\,E_1^{(p)\dagger},\\ E_0^{(p)} &= \begin{pmatrix}\sqrt{1-p} & 0\\ 0 & 1\end{pmatrix}, \quad E_1^{(p)} = \begin{pmatrix}0 & 0\\ \sqrt{p} & 0\end{pmatrix},\end{aligned} \tag{195}$$

sticking to the convention that the first basis vector is $|\uparrow\rangle$ and the second one is $|\downarrow\rangle$. The amplitude-damping noise is often also called the spontaneous emission channel.

As in Fig. 1, the system is initialized in a state $|\psi_{\mathrm{init}}\rangle$ and evolves according to H; at time t_0 we apply the noisy channel $[\mathcal{N}_{\mathrm{AD}}^{(p)}]^{\otimes n}$ to obtain Bob's state. We seek to characterize the Fisher information of Bob's state with respect to time.

In this section, we present simple numerical computations of the upper bound (118) for IID amplitude damping noise for different clock states. In the first part of this section, we suppose the spins are exposed to a uniform external magnetic field aligned along the Z axis. We present numerical calculations of Bob's Fisher information and our lower bound (118) for a choice of clock states, and we numerically optimize the initial state to achieve better output sensitivity. In the second part of this section, we place the spins on a 1D chain with strong Ising interactions. We present numerical calculations of Bob's Fisher information and our lower bound (118) for a choice of clock states; we numerically show that the sensitivity loss for the metrological code state given in Eq. (166) is suppressed to first order in the amplitude damping parameter.

A. Noninteracting Hamiltonians

The system of n spins is assumed to evolve under the Hamiltonian

$$H = \sum_{i=1}^{n} \frac{\omega}{2} Z_i. \tag{196}$$

We compute Bob's Fisher information with respect to time of a selection of states after exposure to the channel $\mathcal{N} = [\mathcal{N}_{\mathrm{AD}}^{(p)}]^{\otimes n}$. First we consider the GHZ state, which has the optimal sensitivity if no noise is present:

$$|\psi_{\mathrm{GHZ}}\rangle = \frac{1}{\sqrt{2}}\big[|\uparrow\uparrow\cdots\uparrow\rangle + |\downarrow\downarrow\cdots\downarrow\rangle\big]. \tag{197}$$

The GHZ state satisfies

$$F_{\mathrm{Alice},t}[\psi_{\mathrm{GHZ}}] = 4\langle H^2\rangle_{\mathrm{GHZ}} = n^2\omega^2. \tag{198}$$

We can also consider the product state vector of all spins pointing in the $+X$ direction,

$$|\psi_+\rangle = |+\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}}\big[|\uparrow\rangle + |\downarrow\rangle\big]^{\otimes n}. \tag{199}$$

Then

$$F_{\text{Alice},t}[\psi_+] = 4\langle H^2\rangle_+ = n\omega^2. \tag{200}$$

Our upper bound (118) on Bob's Fisher information for these states is presented for $n = 12$ and for $n = 50$ in Fig. 11. In Figs. 11(a) and 11(b) are also depicted an ad hoc lower bound for the state vectors $|\psi_{\text{GHZ}}\rangle$ and $|+^n\rangle$ for the same values of n and ω. We can see that for our choice of the amplitude damping noise model and at least for our choice of states, the upper bound on $F_{\text{Bob},t}$ provided by Eq. (118) is reasonably tight for $k = n$. The inset of Fig. 11(a) depicts the same bound for different choices of the value $k = 1, 4, 8, 12$. Recall that the bound includes a projection onto Eve's subspace associated with error operators of weight at most k. The probability that this projection fails is the total probability of observing an error with weight greater than k; this probability is of the order of p^{k+1}. In the inset of Fig. 11(a), for $k = 1, 4, 8$ we display gray lines identifying the values of p for which $p^{k+1} = 10^{-3}$. Values of p beyond the corresponding gray line represent situations in which the projection is expected to fail with probability greater than the order of approximately 10^{-3}. Here, we see that our bound is indeed reasonably tight up until the corresponding value of p. In the inset of Fig. 11(b), we determine more precisely the total weight of the events neglected by ignoring Kraus operators of weight greater than k. Namely, for $k = 1, 2, 5, 10$, we compute the smallest value of p for which $\sum_{|\mathbf{x}|>k} \operatorname{tr}\big(E_{\mathbf{x}}^\dagger E_{\mathbf{x}}\, |\psi_{\text{GHZ}}\rangle\langle\psi_{\text{GHZ}}|\big) > 10^{-3}$. We see that these values of p correspond approximately to where our upper bound (118) fails to accurately predict the value of the quantum Fisher information on the state vector $|\psi_{\text{GHZ}}\rangle$.

We can ask, which state vector $|\psi\rangle$ has the best sensitivity after application of the noisy channel for a given value of p? Here we use the understanding brought by our main Fisher information trade-off relation. In this case, Eve receives any photons emitted by spontaneous emission, which tell her exactly which sites suffered a decay. As a consequence, if Eve observes a number k of photons, then she can safely guess that the energy of Alice's state must have been at least the energy corresponding to k excitations. Eve has therefore obtained information about

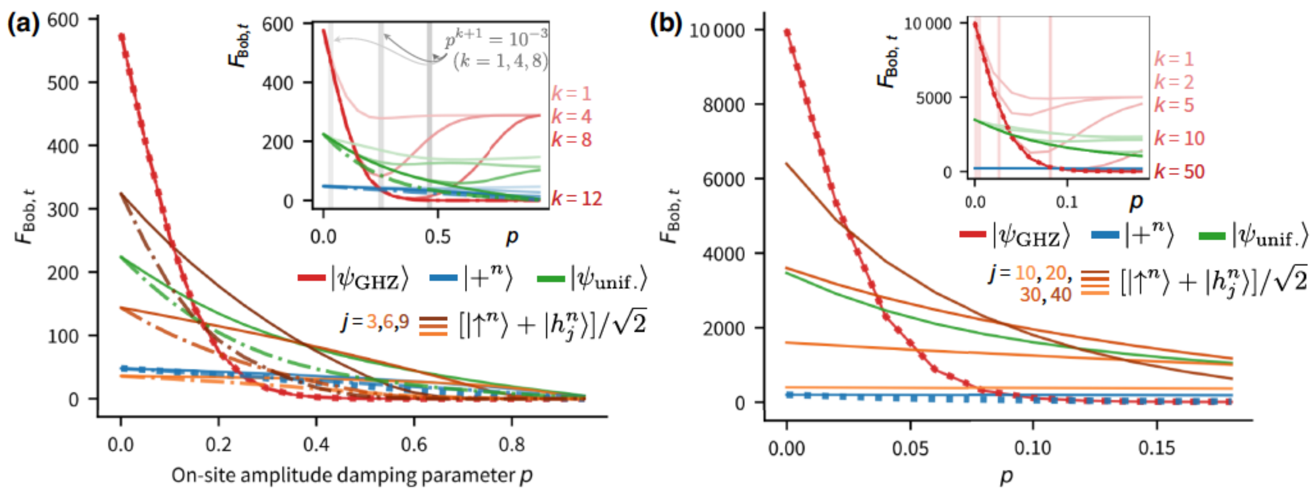

FIG. 11. Quantum Fisher information of a system of n spin-1/2 particles with the Hamiltonian $H = \sum_i \omega Z_i/2$ (with $\omega = 2$) after the application of an amplitude damping channel of parameter p on all sites. (a) Here $n = 12$. Solid lines depict our upper bound (118) with $k = n = 12$ for the state vectors $|\psi_{\text{GHZ}}\rangle$ (red), $|+^n\rangle$ (blue), and $|\psi_{\text{unif}}\rangle$ (green), which are defined in the main text, as well as for the family of states corresponding to an even superposition of the most excited state and a symmetric state (Dicke state) $|h_j^n\rangle$ with a fixed number $n - j$ of excitations (shades of orange) with $j = 3, 6, 9$. Dotted lines are corresponding ad hoc lower bounds for the state vectors $|\psi_{\text{GHZ}}\rangle$ and $|+^n\rangle$ (see main text). Dash-dotted lines are the corresponding exact values of the quantum Fisher information, which can still be directly computed for $n = 12$. The curves corresponding to the superpositions of pairs of Dicke states illustrate situations where the upper bound is not tight. The inset depicts our upper bound (118) for different values of $k = 1, 4, 8, 12$ for the state vectors $|\psi_{\text{GHZ}}\rangle$, $|+^n\rangle$ and $|\psi_{\text{unif}}\rangle$. The value of k corresponds to a projection onto the subspace on Eve's system associated with errors of weight less than or equal to k, which is included in our bound (118). The three vertical gray lines indicate values of p for which $p^{k+1} = 10^{-3}$ for $k = 1, 4, 8$; these lines roughly indicate the values of p beyond which the this projection is expected to fail with a probability exceeding approximately 10^{-3}. Indeed, for our choice of noisy channel and states, our bound (118) for each k is reasonably tight up until values of p for which p^{k+1} is no longer negligibly small. (b) The same computations are repeated for $n = 50$ and $\omega = 2$. Solid lines depict our upper bound with $k = n$ for the same states as in (a), and dotted lines depict an ad hoc lower bound for $|+^n\rangle$ and $|\psi_{\text{GHZ}}\rangle$. The red vertical lines in the inset depict values for p for which the total weight of the IID amplitude-damping Kraus operators $E_{\mathbf{x}}$ with $|\mathbf{x}| > k$ exceeds 10^{-3} for $k = 1, 2, 5, 10$ when applied onto the GHZ input state vector $|\psi_{\text{GHZ}}\rangle$. These values of p are where our upper bound (118) for the corresponding k value is expected to no longer be accurate.

the energy of Alice's state. This observation provides a simple explanation for why the GHZ state has a high Fisher information loss even for small values of p: when Alice exposes a GHZ state to the noise, then Eve can estimate the energy of Alice's GHZ state by noting whether or not she observes a photon. If Eve observes even a single photon, then she can safely guess the energy associated with the all-excited state, and if she observes no decay, she guesses the energy of the ground state. [Her guess is wrong with probability $(1-p)^n$, corresponding to the probability of the all-excited state suffering no decay.] Our trade-off relation thus tells us that we seek a state with a large energy spread, but for which a decay would not betray the value of the total energy of the state. As the effect of the decay becomes more significant with increasing p, some of the energy spread is sacrificed in order to make the state more resilient to Eve's probe.

Here we consider states that are invariant under permutations of the n spins, motivated by the fact that the Hamiltonan is permutation invariant (see also Refs. [41,42]). These states live in the symmetric subspace and can be written in the basis of Dicke states of the symmetric subspace. A Dicke state is a permutation-invariant state with a fixed number of excitations. More specifically, for $q = 0,\ldots,n$ we define

$$|h_q^n\rangle = \binom{n}{q}^{-1/2} \sum_{|\mathbf{x}|=q} |\mathbf{x}\rangle, \tag{201}$$

where the sum ranges over all strings $\mathbf{x}$ with $x_i = \uparrow,\downarrow$, and where $|\mathbf{x}|$ denotes the number of sites i where $x_i = \downarrow$. In the standard basis $\{|0\rangle = |\uparrow\rangle, |1\rangle = |\downarrow\rangle\}$ for spin-1/2 particles, the value $|\mathbf{x}|$ is the Hamming weight of the corresponding computational basis state $\mathbf{x}$.

A general pure symmetric state vector can therefore be written as

$$|\psi\rangle = \sum_{q=0}^{n} \psi_q\, |h_q^n\rangle. \tag{202}$$

We consider symmetric states for convenience, although the optimal state in such settings need not be symmetric [67].

We can consider an even superposition of two Dicke states (as in Sec. IV E). Namely, for $0 \leqslant q_1, q_2 \leqslant n$ we consider the state vector

$$|\widetilde{\psi}_{q_1;q_2}\rangle = \frac{1}{\sqrt{2}}\big[|h_{q_1}^n\rangle + |h_{q_2}^n\rangle\big]. \tag{203}$$

Our upper bound on the sensitivity of the state vector $|\widetilde{\psi}_{q_1;q_2}\rangle$ for all q_1, q_2 is depicted in Fig. 12 for $n = 50$ and the values of $p = 0.01, 0.05, 0.1, 0.25$. In contrast to the case of erasures (Sec. IV E), the states among this family where our bound is large have one of the terms being close to the maximally excited state ($q_1 = 0$ or $q_2 = 0$). [The bound is not necessarily expected to be tight, in light of the gap that is apparent for $n = 12$ in Fig. 11(a) between our bound and the exact value of the quantum Fisher information. The discussion that follows aims to identify states that can potentially have high sensitivity, while ruling out states that are certain to have low sensitivity.] This property can again be understood from our trade-off relation. In the case of erasures, Eve receives the entire reduced state of the systems that have been lost. If $q_1 = 0$ or $q_2 = 0$, then the reduced state on each subsystem is the pure state vector $|\uparrow\rangle$; since it is a pure state, it is easier for Eve to distinguish it from the reduced state of the other Dicke state vector $|h_{q_2}^n\rangle$. In the case of amplitude damping, Eve knows only whether a decay happened or not on each site and she cannot access the full reduced state. An alternative phrasing of this argument is to express erasures as a random operation X, Y, Z applied onto each site; equivalently, a random operation from the set $\{\sigma_+, \sigma_-, Z\}$ is applied on each site, where $\sigma_\pm = [X \pm iY]/\sqrt{2}$ are the creation and annihilation operators of the qubit excitation on a specific site. In the case of amplitude damping, the Kraus operators have no overlap with σ_+, meaning that physically, there is no event in which excitations are created in the system. Such events, however, happen in the case of erasures. If Eve receives the information that k such events have occurred, she can safely assert that the energy of Alice's state could not have exceeded the energy of the state that can still accommodate k further excitations. Thus, the state $q_1 = 0$ can easily be ruled out by Eve in the case of erasures if she receives a report of even a single σ_+ event.

Another interesting choice of state is the uniform superposition of all Dicke states, giving rise to

$$|\psi_{\mathrm{unif}}\rangle = \frac{1}{\sqrt{n+1}} \sum_{q=0}^{n} |h_q^n\rangle. \tag{204}$$

The intuitive reason we expect this state to achieve a good sensitivity after the noise is that if Eve observes emitted photons, she gains comparatively little information about the energy of Alice's state as opposed to if the state is a superposition of few spaced-out Dicke states. The Fisher information of this state after the application of the amplitude damping noisy channel is depicted in Fig. 11.

A more systematic, numerical optimization of $F_{\mathrm{Bob},t}$ by varying Alice's state using different Ansätze for the coefficients $\{\psi_q\}$ indicate that the Fisher information obtained by states of the form (203) can be marginally exceeded for specific values of p by states for which the amplitudes $\{\psi_q\}$ are concentrated around two values $q_1 = 0$ and some value q_2, but with some broadening to include some weight on neighboring Dicke states to q_2. Interestingly, there appears to be many states with very different profiles of $\{\psi_q\}$ that

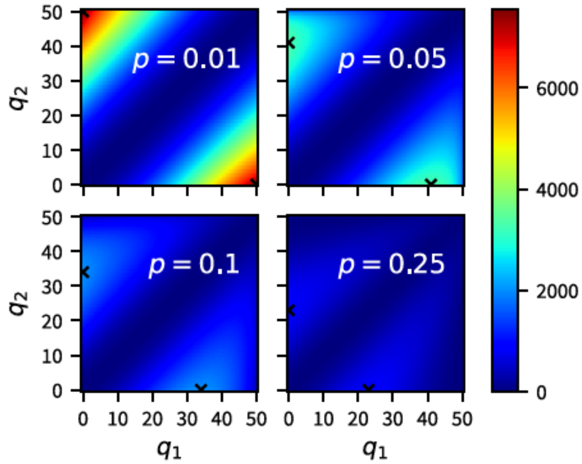

FIG. 12. Upper bound (118) on the quantum Fisher information of a superposition of two Dicke states on $n = 50$ spin-1/2 particles after being exposed to IID amplitude-damping noise. A Dicke state with q excitations is an even superposition of all states with exactly q excitations. For each q_1, q_2, the considered state is an even superposition of the two Dicke states with a number q_1 and q_2 of downwards-pointing spins, respectively. Each plot corresponds to a different value of the single-site amplitude-damping noise parameter p. Our bound attains its maximum on this family of states (black crosses) for states that are an even superposition of the highest excited state and of a weakly excited Dicke state that is separated from the ground state. This separation hinders Eve from accurately guessing the energy of Alice's state, which via our trade-off relation improves the state's sensitivity after application of the noisy channel. The bound is computed with $k = n = 50$.

achieve a very similar sensitivity after the application of the noisy channel.

One particular such state is the state vector $|\psi_{\text{half-Gauss}}\rangle$ of the form (202) where the ψ_q coefficients are a half-Gaussian centered on the all-excited state as

$$|\psi_{\text{half-Gauss}}\rangle = \sum_q \psi_q\, |h^n_q\rangle, \quad \psi_q = \frac{1}{c} e^{-\frac{(q/n)^2}{2w^2}}, \tag{205}$$

where c is determined from the normalization condition. Empirically, we find that this state with a value of $w = 0.4$ yields a sensitivity after application of the noisy channel that is competitive with respect to the other studied states. The half-Gaussian spreads over the entire Dicke basis. The amplitude of the ground state is $\psi_n = e^{-1/(2w^2)}\psi_0 \approx 0.04\psi_0$. Here again, the state vector $|\psi_{\text{half-Gauss}}\rangle$ balances a broad spread in energy values while still preventing Eve from easily finding out the energy of Alice's state.

For our numerical calculations, we employed the standard Python *NumPy* and *SciPy* toolboxes along with *QuTip* [68,69]. The permutation invariance of our setting greatly simplifies the calculation of terms of the form $\mathrm{tr}\big(E^\dagger_{\mathbf{x}'} E_{\mathbf{x}} \psi\big)$ and $\mathrm{tr}\big(E^\dagger_{\mathbf{x}'} E_{\mathbf{x}} \{\bar{H}, \psi\}\big)$ because $E_{\mathbf{x}}$ is a tensor product of single-site operators. Similarly, the reduced operator on a given number k of sites of any operator acting on the symmetric subspace can be computed easily by combinatorial considerations in a basis of the symmetric subspace [41–43]. Even for $n = 50$, our pinched bound is easy to compute even for $k \sim n$ in the permutation-invariant setting: to determine the diagonal matrix elements associated with $\widehat{\mathcal{N}}(\psi)$ and $\widehat{\mathcal{N}}(\{\bar{H}, \psi\})$, it suffices to compute terms of the form $\mathrm{tr}\big(E^\dagger_{\mathbf{x}} E_{\mathbf{x}} \psi\big)$ and $\mathrm{tr}\big(E^\dagger_{\mathbf{x}} E_{\mathbf{x}} \{\bar{H}, \psi\}\big)$ for operators $E_{\mathbf{x}}$ of the form $E_1^{\otimes w} \otimes E_0^{\otimes(n-w)}$ (the other terms are determined by symmetry).

B. Strongly interacting Ising Hamiltonian with a noisy channel

Consider a one-dimensional spin chain with nearest-neighbor ZZ couplings, with the Hamiltonian

$$H = \sum_{j=1}^{n-1} \frac{J}{2} Z_j Z_{j+1}. \tag{206}$$

Our upper bound on Bob's sensitivity to time, computed using the expression (118) for various states, is plotted in Fig. 13 for $n = 12$ and $n = 50$, with $J = 2$ in both plots. We first consider the state vector corresponding to an even superposition of a ferromagnetic all-zero state and an antiferromagnetic state vector

$$|\psi_{\text{F-AF}}\rangle = \frac{1}{\sqrt{2}}[|{\downarrow\downarrow\downarrow\downarrow}\ldots\rangle + |{\downarrow\uparrow\downarrow\uparrow}\ldots\rangle]. \tag{207}$$

Since $|{\downarrow\downarrow\downarrow\downarrow}\ldots\rangle$ and $|{\downarrow\uparrow\downarrow\uparrow}\ldots\rangle$ are energy eigenvectors of respective energies $\omega(n-1)$ and $-\omega(n-1)$, we see that the state vector $|\psi_{\text{F}-\text{AF}}\rangle$ has energy variance $\sigma^2_H(\psi_{\text{F-AF}}) = \omega^2(n-1)^2$. For comparison, we compute the true values of the Fisher information (for $n = 12$), plotted as dashed lines in Fig. 13(a), as well as an ad hoc lower bound, plotted as dotted lines. As can be seen in Fig. 13, our upper bound yields tight bounds on the time sensitivity of the many-body interacting probe, as witnessed by its proximity to the true value and to the ad hoc lower bound, provided p is not too large and k can be taken to be large enough.

The next probe state vector we consider is

$$|\psi_{\text{code-F-AF}}\rangle = \frac{1}{2}[|{\downarrow\downarrow\downarrow\downarrow}\ldots\rangle + |{\uparrow\uparrow\uparrow\uparrow}\ldots\rangle + |{\downarrow\uparrow\downarrow\uparrow}\ldots\rangle + |{\uparrow\downarrow\uparrow\downarrow}\ldots\rangle]. \tag{208}$$

The state vector $|\psi_{\text{code-F-AF}}\rangle$ is the state (166) using the antiferromagnetic configuration as the bit string $\mathbf{x}$. Recall that this state satisfies our Knill-Laflamme-like conditions for a single located error. Here we study how this state's

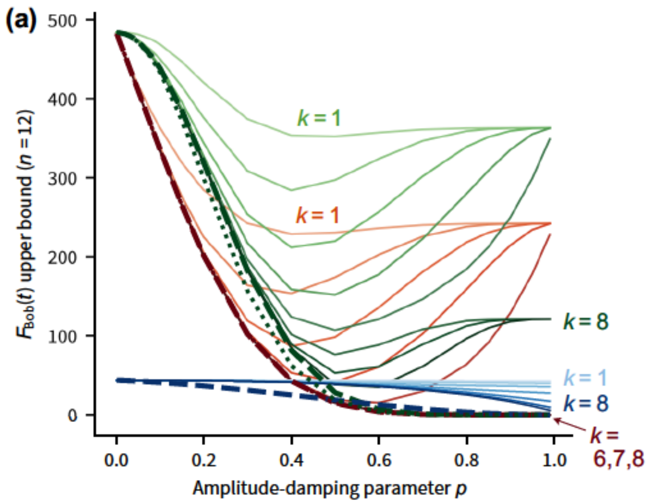

FIG. 13. Numerical calculation of the upper bound on the sensitivity of a many-qubit state after IID amplitude-damping noise, for different states evolving according to the one-dimensional Ising Hamiltonian $H = \sum_j (J/2) Z_j Z_{j+1}$. (a) Upper bound on Bob's Fisher information $F_{\text{Bob}}(t)$ for $n = 12$ qubits as a function of amplitude-damping parameter p, for the state $|\psi_{\text{F-AF}}\rangle$, which is an even superposition of a ferromagnet state and an antiferromagnet state (in red), for $|\psi_{\text{code-F-AF}}\rangle$, which satisfies our Knill-Laflamme-like conditions for a single located error (in green), and for the spin-coherent state vector $|+\rangle^{\otimes n}$ where $|+\rangle$ is the $+1$ eigenvector of X (in blue). Spin-coherent states are typically used in non-quantum-enhanced metrology. For each state, bounds are shown for various values of k, where k is a parameter in the additional noisy channel acting on Eve's system, which is used to derive the upper bound; bounds with larger values of k are harder to compute but are tighter. The upper bounds can increase again for $p \gtrsim 0.4$ because the bound accounts only for low-weight Kraus operators, and higher-weight errors cannot be ignored in this regime. Dashed curves indicate the true Fisher information values, determined by direct computation, and the dotted lines are lower bounds associated with $|\psi_{\text{F-AF}}\rangle$ and $|\psi_{\text{code-F-AF}}\rangle$. The red curves for $k = 6, 7, 8$, the associated true value, and the lower bound appear superimposed. (b) Upper bounds for the states $|\psi_{\text{F-AF}}\rangle$ and $|\psi_{\text{code-F-AF}}\rangle$ for $n = 50$ qubits, enlargement of low values of p. Dotted lines are lower bounds on the Fisher information for these states. Computing the true values of the Fisher information in this regime would require more advanced methods, such as tensor networks [28].

sensitivity is affected when exposed to IID amplitude-damping noise. Our upper bound on Bob's Fisher information via Eq. (118) is plotted in Fig. 13 for $n = 12$ and $n = 50$, alongside that of $|\psi_{\text{F-AF}}\rangle$. The probe state remains almost maximally sensitive when p is small, in contrast to the probe $|\psi_{\text{F-AF}}\rangle$, which immediately loses sensitivity at what appears to be a linear rate with p. This is a manifestation of the fact that the sensitivity of the probe state is unaffected by a single error, and only in the event that two simultaneous errors occur does the sensitivity decrease.

Finally, we consider for comparison the natural probe state given by an ensemble of independent spins, each pointing in the X direction

$$|+^n\rangle = |+\rangle \otimes |+\rangle \otimes \cdots \otimes |+\rangle, \tag{209}$$

where $|+\rangle = [|\uparrow\rangle + |\downarrow\rangle]/\sqrt{2}$ is the $+1$ eigenvector of X. Our upper bound computed for the spin-coherent state vector $|+^n\rangle$ is plotted in blue in Fig. 13. We can see that this probe state performs significantly worse than the entangled probe states for $p \lesssim 0.4$. This is expected, since such a probe's noiseless sensitivity scales only linearly in n, as opposed to the quadratic scaling of the sensitivity of the $|\psi_{\text{F-AF}}\rangle$ and $|\psi_{\text{code-F-AF}}\rangle$ probe states. However, the robustness of the spin-coherent state to the noise is significant. At larger values of the amplitude damping parameter ($p \sim 0.5$ for $n = 12$), the other probe states have all but lost their advantage in sensitivity.

For our numerical calculations, we employed the standard Python *NumPy* and *SciPy* toolboxes along with *QuTip* [68,69]. Our source code is published on *Github* [70]. To compute the trace terms in Eq. (118) we express $|\psi\rangle$ and $\bar{H}|\psi\rangle$ as superpositions of a small number of computational basis vectors over the n sites. The traces then factorize into tensor factors enabling their efficient computation. For the spin-coherent state we work with the local X basis instead of the Z basis, such that the spin-coherent state becomes a basis state in this picture. The direct computation of the Fisher information is performed via an eigenvalue decomposition of the full n-body noisy probe state ρ_B to transform the anticommutator equation $1/2\{\rho_B, R\} = \mathcal{N}(-i[H, \psi])$ in a basis where ρ_B is diagonal, and then solving elementwise to determine R. The ad hoc lower bound is computed by numerically solving the symmetric logarithmic derivative in a restricted subspace consisting of the computational basis vectors that appear in the decomposition of the probe state and those bit strings that are close by in Hamming distance. The resulting value is guaranteed to be a lower bound, because the map that projects the state down to any subspace of the state space is a trace-non-increasing, completely positive map for which one can apply the data-processing inequality satisfied by the Fisher information [19] (see Proposition 13 in Appendix C for details). It is likely the quantum Fisher information in this setting can also be computed based on existing techniques, such as those introduced in Refs. [21,31,53,71].

IX. CONCLUSIONS AND OUTLOOK

Our results present a new paradigm for characterizing the sensitivity of a quantum clock or sensor when exposed to noise, by establishing a quantitative trade-off between the quantum Fisher information of the noisy system with respect to the parameter of interest and the quantum Fisher information that the environment acquires with respect to a complementary parameter. Information trade-offs are interesting because they reveal properties of the mathematical structure of quantum theory, which in turn determine what tasks can be accomplished within the laws of quantum mechanics. Here, our results provide a guiding principle for finding noise-resilient clock states: in order to avoid sensitivity loss due to the application of a noise channel, clock states should hide their energy from the environment.

Energy-time uncertainty relations have historically been harder to formulate than position-momentum-type uncertainty principles, because there is no global time observable in quantum mechanics in the same sense as there is a position observable. Our work contributes an additional type of time-energy uncertainty relation, complementing existing uncertainty relations such as Mandelstamm-Tamm-type uncertainty relations [13], Fisher-based uncertainty relations with a single system [12], and entropic uncertainty relations [15]. Our relation exploits a type of complementarity between the local optimal sensing operator for time and the Hamiltonian (Fig. 5), in the same spirit as uncertainty relations derived in Refs. [1,12]. Our relation furthermore connects the estimation capabilities of two distinct parties (Bob and Eve); in this sense our results can be seen as a Fisher information counterpart of the entropic uncertainty relations for time and energy [15].

A. Summary and discussion

An overview of the results presented in this work can be found in Fig. 3.

1. Time-energy sensitivity trade-off

Our main result is a quantitative time-energy sensitivity trade-off relation in the setting of Fig. 1. If a quantum system is subjected to an instantaneous noisy channel, then the loss in sensitivity to time trades off exactly with the environment's ability to sense the energy of the system as laid out in Eq. (1).

The setting of our uncertainty relation (Fig. 1) is unconventional for quantum metrology: a quantum clock usually accumulates noise continuously as time evolves, much like a quantum probe usually accumulates noise continuously while sensing an unknown parameter. Our setting is instead the communication scenario studied in Ref. [29]: Alice possesses a noiseless quantum clock that already encodes some time value, and she sends it to Bob over a noisy communication channel. In this alternative setting one can analyze the quantum information that leaks to the environment, which is more challenging to do if we consider continuous noise.

An appealing feature of our trade-off relation is that Bob's time sensitivity and Eve's sensitivity to energy are related by an equality. Concretely, this feature means that not only does a gain in energy sensitivity imply a time sensitivity loss by Bob, but also a loss in time sensitivity for Bob automatically implies a gain in energy sensitivity by Eve. In contrast, uncertainty relations in quantum mechanics often relate two observable uncertainties or two entropic quantities via an inequality. For instance, a Schrödinger particle in one dimension that has a large variance in the momentum observable need not have a narrow variance in the position observable.

Our results furthermore hold for an arbitrary pure probe state vector $|\psi\rangle$ and Hamiltonian H. We evade the question of formally optimizing over the probe state vector $|\psi\rangle$ itself—a central question in quantum metrology that many contributions on using quantum error correction for metrology address [21,22,52,72,73]—by identifying instead what features a probe state vector $|\psi\rangle$ must exhibit to avoid being affected by the noise. The alternative expression of the Fisher information obtained by our trade-off relation can potentially facilitate the computation of the Fisher information when optimizing the clock or probe state, potentially improving state optimization schemes such as those in Refs. [67,74] in the presence of noise. Our trade-off relation also offers a guideline to seek good clock states, especially in settings where it might not be possible to reliably prepare the probe state that has the absolute best sensitivity: noise-resilient clock states need to hide their energy from the environment.

Importantly, it is not necessarily the probe state which is least affected by the noise that is the most sensitive. Another state might exhibit a better sensitivity after the noisy channel, even if its sensitivity loss is greater, by ensuring that it is initially sufficiently more sensitive. As an extreme case, this point is illustrated by the ground state of a qubit affected by amplitude-damping noise; the ground state trivially remains unaffected by the noise but has no sensitivity, whereas the $+X$ eigenstate has a better sensitivity, even if it is affected by the noise.

A key technique in our approach is the formulation of the quantum Fisher information as a semidefinite program [26,28] (see Appendix C). Semidefinite programming offers a versatile toolbox in which an alternate expression for an optimization (known as *dual problem*) can be derived and bounds on such optimizations can be proven more easily [75,76]. The technical proof of our main trade-off result (Appendix E 2) offers additional insight into the meaning of the dual problem associated with the semidefinite programming formulation of the quantum Fisher information.

2. The setting of continuous noise

In certain specific settings, the setup in Fig. 1 remains a good approximation of a quantum clock exposed to continuous noise described by a Lindbladian master equation (see Sec. VI). A sufficient condition that guarantees the accuracy of this approximation is to ensure, on one hand, that the Hamiltonian part $\mathcal{L}_0$ of the evolution commutes (as a superoperator) with the noise part $\mathcal{L}_1$ of the Lindbladian that contains all the noise operators, and, on the other hand, that the time derivative of the state is primarily driven by the Hamiltonian and not by the noise. More precisely, in the notation of Sec. VI, the sufficient condition consists in checking that $[\mathcal{L}_0, \mathcal{L}_1] = 0$ as well as ensuring that $\partial_t\mathcal{N}$ contributes only a negligible part of the Fisher information $F\big(\rho\,;\,\partial_t\rho\big)$ [for which a rigorous bound can, for instance, be computed in Eq. (146)]. The second condition is rarely expected to be violated, as sensors are typically designed to have their signal imprinted on their state through their Hamiltonian evolution; noise is usually a degrading process and is typically not the mechanism by which the signal is acquired. If the Hamiltonian of a many-body system consists only of single-site Z terms, then both IID dephasing noise and IID amplitude-damping noise commute (as a superoperator) with the Hamiltonian part of the Lindbladian. Furthermore if the Hamiltonian H commutes with the individual Lindblad jump operators, then the corresponding evolutions also commute as superoperators; this is the case, for instance, if H consists of arbitrary-weight terms containing only Z operators and in the presence of IID dephasing noise. In the case where $[\mathcal{L}_0, \mathcal{L}_1] \neq 0$ the setting can still formally be mapped onto the setting of Fig. 1, by defining the effective noise as the full evolution map with a unitary applied on the input, as long as the time dependence of the effective noisy channel can be neglected. In this case, determining the effective noisy channel in general might be difficult.

3. Trade-off with generalized parameters

The trade-off relation for time and energy can be extended to other parameter evolutions. First of all, there is a choice in how ψ evolves along the t and η parameters: any choice of $|\psi(t,\eta)\rangle$ such that Eq. (23) is satisfied at (t_0,η_0) (but not necessarily at other even neighboring points) leads to the same Fisher information quantities $F_{\mathrm{Alice},t}$, $F_{\mathrm{Alice},\eta}$, $F_{\mathrm{Bob},t}$, and $F_{\mathrm{Eve},\eta}$, so our trade-off relation directly applies. An alternative choice for the η parameter is an evolution generated by the Lindbladian master equation $\partial_\eta\psi = \mathcal{L}[\psi]$ with $\mathcal{L}[\rho] = \sum_k\big[L_k\rho L_k^\dagger - \{L_k^\dagger L_k, \rho\}/2\big]$, where $L_k = \sigma_H^{-1}\sqrt{(e_k+c)}|\psi\rangle\langle e_k|$, where $\{|e_k\rangle\}$ are eigenvectors of the Hamiltonian, where $H = \sum e_k|e_k\rangle\langle e_k|$, and where $c \geqslant 0$ is chosen large enough such that $e_k + c \geqslant 0$ for all k. (We have the opposite sign for $\partial_\eta\psi$, but this can be corrected by redefining $\eta \mapsto -\eta$, and this does not impact the Fisher information.) We can check that this choice of $\partial_\eta\psi$ satisfies Eq. (26) at (t_0,η_0), and therefore also Eq. (23). Another interesting choice for $\psi(t,\eta)$ is to set $|\psi(t_0,\eta)\rangle = e^{(\eta-\eta_0)\bar{H}/(2\sigma_H^2)}|\psi(t_0,\eta_0)\rangle$ for η in a neighborhood of η_0, recalling $\bar{H} = H - \langle H\rangle_\psi$. Again, we see that Eq. (26) is satisfied. This evolution is nonunitary, but one can check that it does preserve the trace of ψ locally to first order at η_0: we have $\partial_\eta\,\mathrm{tr}(\psi)\big|_{\eta_0} = \mathrm{tr}\big(\{\bar{H},\psi\}\big) = 0$. Either of these choices of evolution might be relevant depending on the specific application, though we expect the primary application of our trade-off relation is to help characterize Bob's Fisher information to time, in which case the specific choice of how the clock state is stated to evolve along η might not be important.

The trade-off relation can further be extended to an inequality that is valid for any two arbitrary parameters (Sec. III C). The trade-off between the Fisher information that Alice and Bob, respectively, have with respect to either parameter is then quantified by a value that depends on the commutator of the generators of the two parameters (Theorem 3). The appearance of the commutator in this expression reinforces its central role in quantifying the incompatibility of physical observable quantities. Our main time-energy trade-off relation can be recovered from the more general Theorem 3 by plugging in the local generators for time and energy. While Theorem 3 appears to be tight whenever the Robertson-Weyl uncertainty relation (66) is saturated for the two generators, the bound can likely be improved when considering two generators that have a small commutator. We also present a sufficient condition under which a trade-off relation for any two parameters can be obtained in the form of an equality, mirroring the equality statement in our main trade-off relation for the time and energy parameters. One might have thought that equality in our general uncertainty relation would happen only if the parameters are complementary in the sense of Sec. II B and Fig. 5; in fact, it suffices that the parameters obey some suitable complementarity relation on the support of the complementary channel. Therefore, equality in our general uncertainty relation does not simply depend on the structure of the parameters a, b, but also on the noisy channel $\mathcal{N}$.

4. Bounds on the quantum Fisher information

Computing the quantum Fisher information for general states involves the calculation of the symmetric logarithmic derivative in Eq. (9). This object is straightforward to determine for pure states, it is simple when represented in the diagonal basis of the state, and it can be computed using numerical methods such as the Bartels-Stewart algorithm [77]. However, in the absence of a simple diagonal representation of the state, it is in general difficult to characterize analytically the Fisher information or to derive useful bounds on the Fisher information that apply in general settings of mixed states, especially if the state is rank deficient or close to the boundary of state space. Our results provide an alternative expression for the Fisher information in the scenario of Fig. 1. Combined with the powerful semidefinite methods for the Fisher information reviewed in Appendix C, we provide a general toolbox to characterize the Fisher information for mixed states in a variety of situations. For instance, for an interacting many-body system subject to noise that acts locally, the noise process might be well approximated by an environment that is small relative to the full many-body system. In this case, the computation of the Fisher information on Eve's end happens on a smaller-dimensional system. This observation is, for instance, a main component of our bound (118).

By applying known Fisher information bounds on Eve's system, our trade-off relation enables us to straightforwardly obtain an opposite bound for the Fisher information of Bob's noisy state (see Sec. V). (Upper bounds on the Fisher information can be difficult to obtain; see, for instance, Refs. [31,78].) An example of such a bound to apply is the data-processing inequality for the Fisher information [19]: further processing of a state that has been exposed to the unknown parameter can only decrease the sensitivity with respect to that parameter. This procedure is useful when Eve obtains a state that is not diagonal in the computational basis, making the Fisher information harder to compute. In such cases, we can dephase Eve's state to set all the off-diagonal matrix elements to zero. The resulting Fisher information for Eve can only decrease; by our trade-off relation this immediately yields an upper bound on Bob's Fisher information. This bound, for instance, facilitates the computation of the sensitivity loss of a state exposed to weak amplitude-damping noise, as discussed in Sec. VIII.

5. Metrological codes

Our main uncertainty relation leads to necessary and sufficient conditions for when a clock state loses zero sensitivity when a given noisy channel is applied (Sec. VII). These conditions are a weaker version of the Knill-Laflamme conditions for quantum error correction. Given a clock state vector $|\psi\rangle$ and a Hamiltonian H, we can consider the virtual qubit L spanned by the vectors $|\psi\rangle$ and $H|\psi\rangle$. The clock state vector $|\psi\rangle$ loses no sensitivity under the application of a noisy channel with Kraus operators $\{E_k\}$ if and only if all operators of the form $E_{k'}^\dagger E_k$, when projected onto the virtual qubit, do not have any overlap with the Pauli-Z operator on the virtual qubit. It would be, in principle, possible to prove these zero sensitivity-loss conditions directly on Alice's and Bob's systems, without invoking our trade-off relation; however, characterizing when Eve's Fisher information is zero provides an immediate proof whose simplicity we have not been able to match with alternative techniques.

The zero sensitivity-loss conditions (148) bear similarities with classical codes, where there is only a commutative algebra of observables that one wishes to reproduce [48,50]. Intuitively, the conditions simply ensure that there is a measurement on Bob's system that will reveal the time parameter as well as the local time sensing observable on Alice's system. In contrast to fully quantum error correction, however, there is in general no recovery operation that will restore the pure clock state accurately to first order in the parameter (see Sec. VII H 1 for a simple counterexample). An intriguing aspect of the zero sensitivity-loss conditions are that they do not appear to be formally equivalent to quantum error correction with respect to specific set of noise operators. (The results of Refs. [22,52] appear to indicate that it might be possible to implement certain metrological codes as an error-correcting code involving ancillary systems.) In some cases, such as the qubit example of Sec. VII H 1, the clock state can be thought of as an error-correcting code that corrects only a certain type of error (X or Y Pauli errors). But this is not generally the case—there are examples of a clock state and a (highly nonlocal) Hamiltonian that fulfill the metrological code condition for low-weight errors, but that are not quantum error-correcting codes with respect to neither low-weight X errors nor low-weight Z errors (Sec. VII H 2).

The conditions for zero sensitivity loss are closely related to the recent series of works detailing how to use quantum error correction for metrology in the presence of noise [21,22,52,72,73,79,80]. The main difference with our results is the setting that is being considered. We ask which initial clock states one can prepare on the clock system such that no sensitivity is lost when a noisy channel is applied (and what the associated optimal sensing measurement after the application of the noisy channel is), whereas the mentioned references consider the setting where, during the time a probe system is exposed to the signal and the continuous noise, one can control the probe [81] to repeatedly apply the recovery procedure associated with the quantum error-correcting code.

Metrological codes might be useful for ancillary measurements of error syndromes [82]. Consider an ancillary qudit (of dimension greater than two) which extracts a bit-valued error syndrome via an entangling gate, correlating a pair of states $|\psi\rangle, |\xi\rangle$ with respective binary syndrome values 0, 1. If the ancillary subspace participating in syndrome extraction satisfies the zero sensitivity-loss conditions (148) against physical noise, then, by definition, a measurement in the $\{|\psi\rangle, |\xi\rangle\}$ basis will not be affected by such noise. In other words, the loss conditions ensure protection against X-type logical noise, yielding more robust syndrome extraction using a Z-type measurement. However, such conditions do not preclude any Y-type logical noise. They also do not guarantee fault tolerance, which would require that ancilla errors not spread to any logical encoding via backaction.

6. Numerics for many-body systems

Characterizing the quantum Fisher information of a state exposed to a noise channel using the bound presented in Sec. V A is convenient in the setting of a many-body system subject to noise that acts locally. In the case of n qubits prepared in a permutation-invariant state and exposed to an IID amplitude-damping noise channel, we empirically find that the bound (118) with $k = n$ appears reasonably tight for the states that we investigated and for small values of the noise parameter p; furthermore, for a selection of states including the GHZ state, the bound appears to remain tight even in the regime of high values of p. Our bound can be computed for systems of size $n \gtrsim 50$ on a standard desktop computer.

If instead of on-site terms we consider only Ising-type nearest-neighbor interactions, we can study the robustness of the example "metrological code" introduced in Sec. VII F to an IID amplitude-damping channel with local noise parameter p. This state retains its sensitivity after a single located error. Our numerics show that in the presence of IID noise, the decrease in the quantum Fisher information scales only as p^2, and not linearly in p as for the other studied states with similar sensitivity. We observe that if we expose the interacting system to continuous amplitude-damping noise, then the noise part and the unitary part of the Lindblad evolution do not commute as superoperators (i.e., the setting is not that of phase-covariant noise); it is then possible that the advantages of the metrological code state might not persist in the setting of continuous noise.

B. Outlook

Our trade-off relation is perhaps most relevant in an intermediate regime where the clock is exposed to a signal without the possibility for intermittent quantum control. In such cases, the noise is expected to spoil any Heisenberg scaling that could be achieved using quantum error-correcting schemes due to the lack of recovery operations during the evolution (see, e.g., Refs. [22,31]). Provided the setting can be modeled with a single noisy channel, our results present an alternative expression for the sensitivity of the noisy probe in this regime where the sensitivity is not yet dominated by the asymptotic scaling. Our results might therefore help identify which states present sufficient robustness to the noise to present an advantage in sensitivity with respect to commonly used states (such as a GHZ state or a spin-coherent state).

In the situation where the clock evolves according to a Lindbladian master equation, and the Hamiltonian and noise parts of the Lindbladian fail to commute as superoperators, one might expect in certain cases to still be able to consider time-dependent noise using the following trick. Let us identify the system A as a full copy of the bipartite system $B \otimes E$, and let the unitary evolution of $|\psi(t)\rangle$ cover both systems. A time-dependent channel $\mathcal{E}_t(\cdot)$ can be written as $\mathcal{E}_t(\cdot) = \mathrm{tr}_E\big[U(t)(\cdot)U^\dagger(t)\big]$ where all the time dependence is encoded in the unitary $U(t)$ and where the environment system E is chosen suitably. We then select the noisy channel $\mathcal{N}_{A\to B} = \mathrm{tr}_E$ that simply performs the partial trace over E; the complementary channel is correspondingly $\widehat{\mathcal{N}}_{A\to E} = \mathrm{tr}_B$. While writing a Markovian master equation in this form might require a huge environment system E with rapidly mixing internal dynamics, we expect that our formalism can still account for simple time dependence in the noisy channel in this way. Note also that the unitary $U(t)$ only has to approximate on B the noisy channel $\mathcal{E}_t$ locally to first order around a fixed value of the parameter (e.g., $t = 0$) in order to determine the Fisher information.

A potential domain of application of our main uncertainty relation is for *quantum thermometry* [83], where the goal is to estimate the temperature of a quantum system. In the simple setting of quantum thermometry where the temperature of the system is known to some approximation, and a measurement is performed in order to refine that knowledge, the optimal measurement to carry out is an energy measurement [83]. Since our main result (1) involves the sensitivity of a party with respect to a parameter representing the energy, which is optimally measured using the Hamiltonian of the noiseless system, we expect that one can leverage our main results to yield new sensitivity bounds for quantum thermometry.

Our results are also likely to be relevant in situations where only a restricted set of operators can be measured on a system. Such a restriction could be imposed by limitations in control for a given experimental platform. Suppose we prepare a clock state vector $|\psi\rangle$ evolving noiselessly according to a Hamiltonian H. We would like to measure the clock at time $t \approx t_0$, but we are only permitted to use a measurement from a given set of measurements. What is the optimal local sensitivity that we can achieve? Should the set of allowed measurement operators form an algebra, then the problem is equivalent to sending the clock through a channel that represents the projection onto that algebra. Our results then imply that the resulting sensitivity trades off exactly with the sensitivity that one can achieve with the set of measurements in the commutant of that algebra, with respect to the complementary parameter η.

It might be possible to extend our results to the multiparameter metrology regime where more than one parameter is estimated by Bob. There are known uncertainty relations that determine trade-offs between the precision to which individual parameters can be simultaneously estimated by a single party [18,72,84–87]. In fact, the t and η parameters form a so-called *D-invariant model* [86,88,89], the latter referring to a multiparameter quantum statistical model in which the tangent space is invariant under taking symmetric logarithmic derivatives of the possible generated state-evolution directions. D-invariant models are interesting in multiparameter quantum metrology, because different sensitivity bounds, which in general are difficult to relate, can be shown to coincide [86]. It seems plausible that known multiparameter uncertainty relations can be extended to the present bipartite setting, either where all parameters are simultaneously estimated by Bob while Eve simultaneously estimates a set of complementary parameters, or where a number of parties estimate each individual parameter, where each party might be part of the output or the environment.

Our main uncertainty relation might offer a connection between the setting of quantum metrology hindered by a noisy quantum channel and the setting of multiparameter, noiseless quantum metrology. It appears that a key ingredient for our main uncertainty relation is that the t and η parameters form a D-invariant model, in the sense of the preceding paragraph. By construction, our uncertainty relation applies to any pure state D-invariant model consisting of two complementary generators related by Eq. (20), given that we made no specific assumptions about the parameter t or its local Hermitian generator H.

However, it remains unclear whether our results extend to general multiparameter D-invariant models, as our proof seems to utilize the fact that the space is spanned by only two complementary generators T and H. D-invariant models have a rich geometric structure [88], which might prove an essential conceptual component of our results; such connections nevertheless remain to be better understood. A further connection to D-invariant models appears in Eq. (69), which appears to be a D-invariance condition restricted onto the support of the complementary channel $\widehat{\mathcal{N}}$. In fact, one can view Bob's and Eve's measurements T_B and E as measurement operators $\mathcal{N}^\dagger(T_B)$ and $\widehat{\mathcal{N}}^\dagger(E)$ on Alice's system through the action of the adjoint channels $\mathcal{N}^\dagger$ and $\widehat{\mathcal{N}}^\dagger$. We could ask whether our uncertainty relation translates into a trade-off in how the two parameters t and η can be estimated by Alice, if the estimation of t (respectively, η) is required to employ an observable in the set of operators that is specified as the image of $\mathcal{N}^\dagger$ (respectively, of $\widehat{\mathcal{N}}^\dagger$). It is not clear if this is the case, as the quantum Fisher information attained by the observable $\mathcal{N}^\dagger(T_B)$ [respectively, $\widehat{\mathcal{N}}^\dagger(E)$] on the state ψ is not necessarily expected to match the corresponding value of the quantum Fisher information of T_B on $\mathcal{N}[\psi]$ (respectively, of E on $\widehat{\mathcal{N}}[\psi]$). It is thus unclear if or how our main uncertainty relation is connected with general bounds that hold in the multiparameter regime, such as multiparameter versions of the quantum Cramér-Rao bound [86] or the Gill-Massar inequality [90,91]. We might expect that deeper connections can be developed between the setting of parameter estimation after the application of a noisy channel and noiseless multiparameter estimation.

Also, our results apply locally to first order around a given fixed value of the unknown parameter; whether similar results can be derived in the global parameter estimation regime [89,92–95] is unknown. Global parameter estimation might be more relevant for applications to atomic quantum clocks [96,97]. We also anticipate extensions of our results to the finite-sample regime where the quantum Fisher information might no longer accurately quantify the sensitivity of a quantum state to an unknown parameter [98].

Along a similar vein, there are settings where one seeks to compute different variants of the quantum Fisher information. For instance, the so-called *right-logarithmic derivative* (see, e.g., Ref. [87]) is often used to bound the standard quantum Fisher information. Alternative sensitivity measures include the truncated Fisher information [54], which not only give useful bounds on the standard quantum Fisher information but can be more relevant in the regime of limited measurement data. An interesting question would be to study whether our results extend to such generalized sensitivity measures.

Entropic uncertainty relations play a central role in quantum cryptography [11,15,16,99], and cryptographic schemes have been studied for quantum metrology [100]. It is possible that our parameter-estimation trade-off can similarly form the basis of cryptographic schemes in which a parameter encoded in a quantum state is to be shielded from a malevolent eavesdropper. Furthermore, the Fisher information is closely related to relative entropy measures [87,101,102]; we might expect our trade-off relation to translate into a statement about Rényi relative entropies.

The development of quantum atomic clocks as ultraprecise time references [103] makes it all the more important to achieve a thorough understanding of how noise can be prevented from spoiling sensitivity. We also anticipate that our results will be relevant for recently developed atomic clocks built with a lattice of interacting atoms [9] and correlated many-body sensing probes [6, 104], as these platforms will offer new possibilities for metrology by exploiting the strong interactions between the particles.

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APPENDIX A: NOTATION, PRELIMINARIES, AND AUXILIARY LEMMAS

We first introduce some preliminaries and notation that will be used throughout the Appendices. All Hilbert spaces are finite dimensional unless otherwise indicated, and all projectors are Hermitian. A pure quantum state is a vector $|\psi\rangle$ in the Hilbert space that is normalized to unit norm, and the terminology pure quantum state is also used by extension for the associated density operator $|\psi\rangle\langle\psi|$. Quantum states are positive semidefinite operators ρ with unit trace, $\mathrm{tr}(\rho) = 1$. A subnormalized quantum state is a positive semidefinite operator ρ satisfying $\mathrm{tr}(\rho) \leqslant 1$. States are normalized to unit trace unless explicitly specified as being subnormalized.

For any Hermitian operator O, we denote by P_O the projector onto the support of O, and by $P_O^{\perp} = \mathbb{1} - P_O$ its complement. For any positive semidefinite operator A, we denote by A^{-1} its Moore-Penrose pseudoinverse, i.e., the operator obtained by taking the inverse on the support of A.

We denote by $\|A\|$ the maximal singular value of an operator A. We also define the Schatten one-norm as $\|A\|_1 = \mathrm{tr}\sqrt{A^\dagger A}$.

It will prove convenient to "vectorize" operators by viewing them as vectors in Hilbert-Schmidt space using the following representation. The vector space of operators acting on a Hilbert space $\mathscr{H}$ is isomorphic to $\mathscr{H} \otimes \mathscr{H}$. Let $|1\rangle\rangle$ denote the element

$$\sum_{i=1}^{d} |i\rangle \otimes |i\rangle \in \mathscr{H} \otimes \mathscr{H}, \tag{A1}$$

where $\{|i\rangle\}_{i=1}^d$ is a fixed basis of $\mathscr{H}$. We define the "vectorized" representation of any operator A acting on $\mathscr{H}$ as $|A\rangle\rangle = (A \otimes \mathbb{1})|1\rangle\rangle$. Similarly, we define $\langle\langle 1| = \sum_{i=1}^d \langle i| \otimes \langle i|$ and $\langle\langle A| = \langle\langle 1|(A^\dagger \otimes \mathbb{1})$. We recall the useful identity

$$(X \otimes \mathbb{1})|1\rangle\rangle = (\mathbb{1} \otimes X^T)|1\rangle\rangle, \tag{A2}$$

and note that $|1\rangle\rangle = |\mathbb{1}\rangle\rangle$ is the vectorized operator representation of the identity matrix $\mathbb{1}$. We denote a rank-one operator $|\phi\rangle\langle\psi|$ in this representation as $|\phi, \psi\rangle\rangle$, with $|\phi, \psi\rangle\rangle = |\phi\rangle \otimes (|\psi\rangle)^*$ and $\langle\langle\phi, \psi| = \langle\phi| \otimes (\langle\psi|)^*$. The Hilbert-Schmidt inner product in this notation is simply $\mathrm{tr}(A^\dagger B) = \langle\langle A|B\rangle\rangle$. The matrix elements of A in any basis $\{|\ell\rangle\}$ are also simply given by $\langle\ell|A|\ell'\rangle = \langle\langle \ell, \ell'|A\rangle\rangle$. A superoperator $\mathcal{E}$ acting on an operator M is denoted by $\mathcal{E}|M\rangle\rangle$. The superoperator consisting of a left multiplication by A and a right multiplication by B, i.e., $M \mapsto AMB$, is represented by $|M\rangle\rangle \mapsto (A \otimes B^T)|M\rangle\rangle$. The identity superoperator id is represented by $\mathbb{1} \otimes \mathbb{1}$. Also, $\langle\langle A|\mathcal{E}|B\rangle\rangle = \langle\langle B|\mathcal{E}^\dagger|A\rangle\rangle^*$, where $\mathcal{E}^\dagger$ is the usual superoperator adjoint defined by $\mathrm{tr}(M\mathcal{E}^\dagger(N)) = \mathrm{tr}(\mathcal{E}(M)\, N)$. In the following and unless otherwise stated, superoperators are expressed in this representation, unless they are explicitly applied onto an operator with the notation $\mathcal{E}(\cdot)$.

We now compute a few quantities that often recur throughout these appendices. Let $|\psi\rangle$ be a state vector and consider the evolution $\partial_t\psi = -i[H, \psi]$, where H is any Hermitian operator. Let M be any Hermitian operator. We have

$$\begin{aligned}
\left\langle \left(\frac{d\psi}{dt}\right)^2 \right\rangle &= \langle (-i[H,\psi])^2 \rangle = -\mathrm{tr}\big[\psi(H\psi H\psi \\
&\quad - H\psi H - \psi H^2\psi + \psi H\psi H)\big] \\
&= \mathrm{tr}(\psi H^2) - [\mathrm{tr}(\psi H)]^2\,; \qquad \text{(A3)} \\
-i[i[M,\psi],\psi] &= (M\psi - \psi M)\psi - \psi(M\psi - \psi M) \\
&= \{M,\psi\} - 2\langle M\rangle\,\psi \\
&= \{M - \langle M\rangle, \psi\}. \qquad \text{(A4)}
\end{aligned}$$

The notion of Schur complement will serve multiple times in these appendices, so we state it here.

Theorem 5 (Positive semidefiniteness via Schur complement).—let $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{m\times m}$ be positive semidefinite matrices. Let $W \in \mathbb{C}^{n\times m}$ be an arbitrary complex matrix. The following statements are equivalent:

$$\text{(i)}\ \begin{bmatrix} A & W \\ W^\dagger & B \end{bmatrix} \geqslant 0\,,$$
$$\text{(ii)}\ WP_B^\perp = 0 \quad\text{and}\quad A \geqslant WB^{-1}W^\dagger\,,$$
$$\text{(iii)}\ P_A^\perp W = 0 \quad\text{and}\quad B \geqslant W^\dagger A^{-1} W\,.$$

Moreover, (ii) implies $P_A^\perp W = 0$ and (iii) implies $WP_B^\perp = 0$.

For a proof, see, e.g., Ref. [105]. With respect to the proof of similar statements in standard textbooks, we can see that (ii) implies $P_A^\perp W = 0$ as follows: hitting the inequality with $P_A^\perp(\cdot)P_A^\perp$ and noting that $WB^{-1}W^\dagger \geqslant 0$, we see that $P_A^\perp WB^{-1}W^\dagger P_A^\perp = 0$, which implies $P_A^\perp WB^{-1/2} = 0$. Therefore, $P_A^\perp W = 0$ using the fact that $WP_B^\perp = 0$. Similarly, (iii) implies $WP_B^\perp = 0$.

Now we present a simple method to relate operator inequalities before and after the application of a completely positive map.

Lemma 1 (Positive semidefiniteness of block matrices under completely positive maps).—let $A, B, W \in \mathbb{C}^{n\times n}$ be complex matrices and assume that

$$\begin{bmatrix} A & W \\ W^\dagger & B \end{bmatrix} \geqslant 0. \tag{A5}$$

Let Φ be any completely positive map that maps operators on $\mathbb{C}^n$ to operators on $\mathbb{C}^m$. Then

$$\begin{bmatrix} \Phi(A) & \Phi(W) \\ \Phi(W^\dagger) & \Phi(B) \end{bmatrix} \geqslant 0. \tag{A6}$$

Proof.—The matrix (A6) is obtained by applying the completely positive map $\mathrm{id}_2 \otimes \Phi$ onto Eq. (A5). ■

While the above lemma is fairly trivial, paired with Theorem 5 it enables us to show less obvious inequalities such as the following.

Corollary 2 (Image of matrix squared under a subunital, completely positive map).—let M be a Hermitian operator and let Φ be any completely positive map that satisfies $\Phi(\mathbb{1}) \leqslant \mathbb{1}$. Then $\Phi(M^2) \geqslant [\Phi(M)]^2$.

Proof.—Observe first that $\begin{bmatrix} M^2 & M \\ M & \mathbb{1} \end{bmatrix} = \begin{bmatrix} M & \mathbb{1} \end{bmatrix}^\dagger \begin{bmatrix} M & \mathbb{1} \end{bmatrix} \geqslant 0$. With the subunitality condition on Φ and Lemma 1, we have

$$\begin{bmatrix} \Phi(M^2) & \Phi(M) \\ \Phi(M) & \mathbb{1} \end{bmatrix} \geqslant \begin{bmatrix} \Phi(M^2) & \Phi(M) \\ \Phi(M) & \Phi(\mathbb{1}) \end{bmatrix} \geqslant 0. \tag{A7}$$

Thanks to Theorem 5, this implies $\Phi(M^2) \geqslant \Phi(M)\,\mathbb{1}^{-1}\,\Phi(M) = [\Phi(M)]^2$. ■

APPENDIX B: SOLUTIONS OF THE ANTICOMMUTATOR EQUATION

In this Appendix, we briefly review the solutions of the anticommutator equation

$$\frac{1}{2}\{\rho, M\} = N, \tag{B1}$$

where M is the unknown operator, N is a fixed operator, ρ is a subnormalized quantum state, and $\{A, B\} := AB + BA$ denotes the anticommutator.

For any subnormalized state ρ, it is convenient to define the Hermiticity-preserving superoperator $\mathcal{R}_\rho$ as

$$\mathcal{R}_\rho(\cdot) = \frac{1}{2}\big\{\rho, (\cdot)\big\}. \tag{B2}$$

Note that $\mathcal{R}_\rho$ is neither completely positive nor trace preserving. The operator $\mathcal{R}_\rho$ is self-adjoint, since $\mathrm{tr}(N\mathcal{R}_\rho(M)) = \frac{1}{2}\,\mathrm{tr}(N\{\rho, M\}) = \frac{1}{2}\,\mathrm{tr}(\{\rho, N\}M) = \mathrm{tr}\,(\mathcal{R}_\rho(N)\,M)$. It is interesting to study the superoperator $\mathcal{R}_\rho$ as a linear operator in Hilbert-Schmidt space. In vectorized operator space, it is represented as

$$\mathcal{R}_\rho = \frac{1}{2}\big(\rho \otimes \mathbb{1} + \mathbb{1} \otimes \rho^T\big). \tag{B3}$$

This matrix is Hermitian and positive, and it is positive definite if and only if ρ has full rank. The fact that the vectorized matrix representing $\mathcal{R}_\rho$ is positive is not to be confused with the usual notion of a superoperator being positive, which means preserving the positivity of its argument. Here, $\mathcal{R}_\rho$ has a positive semidefinite vectorized representation, which means that $\langle\langle M|\mathcal{R}_\rho|M\rangle\rangle \geqslant 0$ for all operators $|M\rangle\rangle$.

Suppose for a moment that ρ has full rank. Then the superoperator $\mathcal{R}_\rho$ can be inverted, because its vectorized operator matrix representation has full rank, and we denote the inverse by $\mathcal{R}_\rho^{-1}$. The operator $M = \mathcal{R}_\rho^{-1}(N)$ is then the unique solution to the anticommutator equation $\frac{1}{2}\{\rho, M\} = N$. If $\{|k\rangle\}$ is a basis of the Hilbert space that diagonalizes ρ as $\rho = \sum_k p_k |k\rangle\langle k|$, then Eq. (B3) provides a diagonal representation of $\mathcal{R}_\rho$, and we obtain the familiar expression of $\mathcal{R}_\rho^{-1}$ as

$$\mathcal{R}_\rho^{-1}|N\rangle\rangle = \sum_{k,k'} \frac{2}{p_k + p_{k'}} |k, k'\rangle\rangle\langle\langle k, k'|N\rangle\rangle,\ \text{i.e.,}$$
$$\mathcal{R}_\rho^{-1}(N) = \sum_{k,k'} \frac{2}{p_k + p_{k'}} \langle k|N|k'\rangle\ |k\rangle\langle k'|. \tag{B4}$$

If ρ is not full rank, then we define $\mathcal{R}_\rho^{-1}$ as the Moore-Penrose inverse of the superoperator $\mathcal{R}_\rho$, i.e., we take the inverse on its support. From Eq. (B3) we can identify the kernel $\ker \mathcal{R}_\rho$ of the superoperator $\mathcal{R}_\rho$ as the space spanned by operators of the form $|\phi, \psi\rangle\rangle$ where $P_\rho|\phi\rangle = P_\rho|\psi\rangle = 0$, where P_ρ is the projector onto the support of ρ. If $\{|k\rangle\}$ is a basis of the Hilbert space that diagonalizes ρ as $\rho = \sum_k p_k|k\rangle\langle k|$, then Eq. (B3) is diagonal in the basis $\{|k, k'\rangle\rangle\}$ and we see that the expression (B4) remains the correct expression for $\mathcal{R}_\rho^{-1}$, provided we only keep those terms in the sum for which $p_k + p_{k'} \neq 0$.

We may now state the following useful proposition that characterizes the full solution set of the anticommutator equation $1/2\{\rho, M\} = N$ for M.

Proposition 3 (Solutions to the anticommutator equation).—let ρ be any subnormalized quantum state, let N be any operator, and let $\mathcal{R}_\rho$ be given by Eq. (B2). Let P_ρ denote the projector onto the support of ρ and let $P_\rho^\perp = \mathbb{1} - P_\rho$. Then the set $\mathcal{S}$ of solutions of the equation $Y = \mathcal{R}_\rho(M)$ for the operator M is

$$\mathcal{S} = \begin{cases} \emptyset & \text{if } P_\rho^\perp N P_\rho^\perp \neq 0\,; \\ \left\{\mathcal{R}_\rho^{-1}(N) + P_\rho^\perp M' P_\rho^\perp \;:\; M' \text{ any operator}\right\} & \text{if } P_\rho^\perp N P_\rho^\perp = 0, \end{cases} \tag{B5}$$

where $\mathcal{R}_\rho^{-1}$ denotes as above the Moore-Penrose pseudoinverse of the superoperator $\mathcal{R}_\rho$. Furthermore, if Y is Hermitian, then the set $\mathcal{S}_H$ of Hermitian solutions of the equation $N = \mathcal{R}_\rho(M)$ for the operator M is

$$\mathcal{S}_H = \begin{cases} \emptyset & \text{if } P_\rho^\perp N P_\rho^\perp \neq 0\,; \\ \left\{\mathcal{R}_\rho^{-1}(N) + P_\rho^\perp M'' P_\rho^\perp \;:\; M'' \text{ any Hermitian operator}\right\} & \text{if } P_\rho^\perp N P_\rho^\perp = 0, \end{cases} \tag{B6}$$

where $\mathcal{R}_\rho^{-1}(Y)$ is always a Hermitian operator.

This proposition is essentially obvious if we think of superoperators as linear operators in Hilbert-Schmidt space. Indeed, it is well known that the general solution to a system of equations given in matrix form can be expressed by the matrix pseudoinverse, plus anything that is in the matrix kernel.

Proof.—Let $\mathcal{P}^\perp$ be the superoperator projector onto the kernel of $\mathcal{R}_\rho$. In the vectorized-operator representation, we have $\mathcal{P}^\perp := P_\rho^\perp \otimes (P_\rho^\perp)^T$ as can be seen from Eq. (B3). Let $\mathcal{P} = \mathrm{id} - \mathcal{P}^\perp$ be the superoperator projector onto the complementary operator subspace, which is the support of $\mathcal{R}_\rho$. Observe that $\mathcal{R}_\rho\mathcal{R}_\rho^{-1} = \mathcal{R}_\rho^{-1}\mathcal{R}_\rho = \mathcal{P}$ and that $\mathcal{R}_\rho$ $\mathcal{P}^\perp = 0$.

The claim we want to show is that if $\mathcal{P}^\perp|N\rangle\rangle \neq 0$, then there is no solution to the equation $|N\rangle\rangle = \mathcal{R}_\rho|M\rangle\rangle$; otherwise, then the equation is satisfied if and only if

$$|M\rangle\rangle = \mathcal{R}_\rho^{-1}|N\rangle\rangle + \mathcal{P}^\perp|M'\rangle\rangle \tag{B7}$$

for some operator $|M'\rangle\rangle$. The condition $\mathcal{P}^\perp|N\rangle\rangle = 0$ is necessary for any solution to the equation $|N\rangle\rangle = \mathcal{R}_\rho|M\rangle\rangle$ to exist, as otherwise $|N\rangle\rangle$ would not be in the range of $\mathcal{R}_\rho$. We can therefore assume for the rest of this proof that $\mathcal{P}^\perp|N\rangle\rangle = 0$.

Suppose M solves $\mathcal{R}_\rho|M\rangle\rangle = |N\rangle\rangle$. Applying $\mathcal{R}_\rho^{-1}$ on both sides, we have $\mathcal{P}|M\rangle\rangle = \mathcal{R}_\rho^{-1}|N\rangle\rangle$, which determines $|M\rangle\rangle$ on the operator space projected onto by $\mathcal{P}$. On the complementary space (associated with $\mathcal{P}^\perp$), the operator $|M\rangle\rangle$ can be arbitrary because this subspace is the kernel of $\mathcal{R}_\rho$. A general operator in this subspace can be written as $\mathcal{P}^\perp|M'\rangle\rangle$ for some operator M'. This proves that the solution M must have the form given in the claim. Conversely, if

$$|M\rangle\rangle = \mathcal{R}_\rho^{-1}|N\rangle\rangle + \mathcal{P}^\perp|M'\rangle\rangle \tag{B8}$$

for some operator $|M'\rangle\rangle$, then we see that $\mathcal{R}_\rho|M\rangle\rangle = \mathcal{R}_\rho\left(\mathcal{R}_\rho^{-1}|N\rangle\rangle + \mathcal{P}^\perp|X'\rangle\rangle\right) = \mathcal{P}|N\rangle\rangle = |N\rangle\rangle$, thus proving the claim.

If N is Hermitian, then $\mathcal{R}_\rho^{-1}(N)$ is Hermitian because $\mathcal{R}_\rho$, and hence $\mathcal{R}_\rho^{-1}$, is Hermiticity preserving. Any two Hermitian solutions M_0, M_1, as seen above, must differ by a term $P_\rho^\perp M' P_\rho^\perp$ for some arbitrary M'; because the difference $M_0 - M_1$ is Hermitian, M' can be chosen to be Hermitian as well [specifically, one can set $M'' = (M' + M'^\dagger)/2$]. ■

We now compute the map $\mathcal{R}_\psi^{-1}(\cdot)$ in the case the reference state is a pure (normalized) state vector $|\psi\rangle$.

Proposition 4 (Computing $\mathcal{R}_\rho^{-1}$ when ρ is a pure state).—let $|\psi\rangle$ be a (normalized) state vector and let $P_\psi^\perp = \mathbb{1} - |\psi\rangle\langle\psi|$. Then for any Hermitian O we have

$$\mathcal{R}_\psi^{-1}(O) = 2(O - P_\psi^\perp O P_\psi^\perp) - \langle O\rangle_\psi \psi. \tag{B9}$$

Proof.—Define $\bar{O} := O - P_\psi^\perp O P_\psi^\perp - \langle O\rangle_\psi \psi$. By linearity, we have

$$\mathcal{R}_\psi^{-1}(\bar{O}) = \mathcal{R}_\psi^{-1}(O) - \langle O\rangle_\psi\, \psi, \tag{B10}$$

noting that $1/2\{\psi,\psi\} = \psi$ and therefore $\mathcal{R}_\psi^{-1}(\psi) = \psi$. The operator $\bar{O}$ satisfies $P_\psi^\perp \bar{O} P_\psi^\perp = 0$ and $\langle\bar{O}\rangle_\psi = 0$, the latter implying that $\bar{O}\psi = P_\psi^\perp \bar{O}\psi$. Then

$$\bar{O} = (\psi + P_\psi^\perp)\bar{O}(\psi + P_\psi^\perp) = P_\psi^\perp \bar{O}\psi + \psi\bar{O}P_\psi^\perp = \{\psi, \bar{O}\},$$

and therefore $\mathcal{R}_\psi^{-1}(\bar{O}) = 2\bar{O}$. From Eq. (B10) we then find

$$\mathcal{R}_\psi^{-1}(O) = 2\bar{O} + \langle O\rangle\psi = 2(Z - P_\psi^\perp O P_\psi^\perp) - \langle O\rangle\psi.$$

■

APPENDIX C: SEMIDEFINITE PROGRAMMING METHODS FOR THE FISHER INFORMATION

In this Appendix, we review some methods based on semidefinite programming [75,76] for computing the Fisher information, and review some elementary properties of the Fisher information. Let ρ be any subnormalized quantum state, and let D be any Hermitian operator that satisfies $P_\rho^\perp D P_\rho^\perp = 0$ (recall $P_\rho^\perp$ is the projector onto the kernel of ρ). Define the quantity

$$F(\rho\,;D) := \mathrm{tr}(\rho R^2), \tag{C1}$$

where R is any solution to $(1/2)\{\rho, R\} = D$. For a normalized state ρ and for traceless D, the quantity $F(\rho\,;D)$ corresponds to the Fisher information associated with a one-parameter family of states $\lambda \mapsto \rho_\lambda$ taken at a value of λ where $\rho_\lambda = \rho$ and $d\rho_\lambda/d\lambda = D$. We allow subnormalized states ρ and operators D with nonzero trace in the definition (C1) for later technical convenience. We require that $P_\rho^\perp D P_\rho^\perp = 0$ as otherwise the anticommutator equation $(1/2)\{\rho, R\} = D$ has no solution for R.

The definition of $F(\rho\,;D)$ does not depend on the choice of R that solves $(1/2)\{\rho, R\} = D$. Indeed, Proposition 3 guarantees that any two solutions differ only by a term $P_\rho^\perp M' P_\rho^\perp$; such a term does not contribute to the trace in Eq. (C1). We may therefore write, using the notation of Appendix B,

$$F(\rho\,;D) = \mathrm{tr}\big(\rho\,[\mathcal{R}_\rho^{-1}(D)]^2\big). \tag{C2}$$

We now write this expression as a pair of convex optimizations. These expressions have been derived in Refs. [26,28]; we provide a proof using our notation for self-consistency.

Proposition 5 (Fisher information in terms of convex optimization problems).—let ρ be a subnormalized quantum state and D be a Hermitian operator that satisfies $P_\rho^\perp D P_\rho^\perp = 0$. The quantity $F(\rho\,;D)$ defined in Eq. (C1) is equivalently expressed as the following optimizations:

$$F(\rho\,;D) = \max_{S=S^\dagger} 4\big[\mathrm{tr}(DS) - \mathrm{tr}(\rho S^2)\big] \tag{C3a}$$

$$= \min\left\{4\,\mathrm{tr}(L^\dagger L)\ :\ \rho^{1/2}L + L^\dagger\rho^{1/2} = D\right\}, \tag{C3b}$$

where the first optimization ranges over all Hermitian operators S and where in the second optimization L is an arbitrary complex matrix. Optimal choices for the variables are $S = (1/2)\mathcal{R}_\rho^{-1}(D)$ and $L = \rho^{1/2}S$, noting that $\{\rho, S\} = D$. Furthermore, alternative forms for the minimization are

$$F(\rho\,;D) = 4\min\left\{\mathrm{tr}(N)\ :\ \begin{bmatrix}\rho & O\\ O^\dagger & N\end{bmatrix} \geqslant 0\quad\text{with}\quad O + O^\dagger = D,\ N \geqslant 0\right\} \tag{C3c}$$

$$= \min\left\{\mathrm{tr}(J)\ :\ \begin{bmatrix}\rho & D + iK\\ D - iK & J\end{bmatrix} \geqslant 0\quad\text{with}\quad K = K^\dagger,\ J \geqslant 0\right\}, \tag{C3d}$$

in which optimal choices are $O = \rho S$, $N = S\rho S$, $K = -i[\rho, S]$, and $J = \mathcal{R}_\rho^{-1}(D)\,\rho\,\mathcal{R}_\rho^{-1}(D)$.

Note that the condition in the optimization (C3d) implicitly enforces the fact that $P_\rho^\perp(D + iK) = 0$ (see Theorem 5); this can make it more complicated to guess a candidate for K in Eq. (C3d) if ρ does not have full rank, especially if $P_\rho^\perp D \neq 0$. Also, note that if there is any feasible choice of candidates in Eq. (C3c), then automatically $P_\rho^\perp O = 0$ and $O^\dagger P_\rho^\perp = 0$, such that $P_\rho^\perp D P_\rho^\perp = P_\rho^\perp(O + O^\dagger)P_\rho^\perp = 0$. Therefore, finding feasible candidates automatically enforces the condition in the definition (C1). A similar argument holds if feasible candidates are found in Eqs. (C3b) or (C3d).

Proof.—The maximization (C3a) is a quadratic optimization can be cast into a semidefinite program using Schur complements (Theorem 5). We stick closely to the formalism of Watrous [76,106]. We introduce a variable $Q \geqslant 0$ with the constraint $Q \geqslant S^2$ expressed as a Schur complement condition:

$$\frac{1}{4}\{\text{maximization in Eq. (C3a)}\}$$
$$= \quad \text{maximize} \quad [\operatorname{tr}(DS) - \operatorname{tr}(\rho Q)] \tag{C4}$$
$$\text{over variables} \quad S = S^\dagger, Q \geqslant 0$$
$$\text{subject to} \quad \begin{bmatrix} Q & -S \\ -S & \mathbb{1} \end{bmatrix} \geqslant 0.$$

(The sign of $-S$ in the last constraint is for later convenience.) We now determine the corresponding dual problem. Let $\left[\begin{smallmatrix} M & O \\ O^\dagger & N \end{smallmatrix}\right] \geqslant 0$ be the Lagrange dual variable corresponding to the primal constraint, with $M, N \geqslant 0$ and O arbitrary. The primal constraint can be written as

$$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \otimes Q + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes S \leqslant \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \mathbb{1}. \tag{C5}$$

The dual objective is obtained by collecting the constant terms of the constraints and taking the inner product with the corresponding dual variable. Here we only have the right-hand side of Eq. (C5) and we obtain the objective that is simply to minimize $\operatorname{tr}(N)$. There are two dual constraints, one for each primal variable S and Q; to a Hermitian variable corresponds an equality constraint and to a positive semidefinite variable corresponds a positive semidefinite constraint. The primal objective gives the constant terms for each constraint, which are $(\ldots) = D$ and $(\ldots) \geqslant -\rho$. For the left-hand side of the first constraint we obtain the term $\operatorname{tr}_1\left(\left[\begin{smallmatrix} M & O \\ O^\dagger & N \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]\right) = O + O^\dagger$. For the left-hand side of the second constraint, we find $-\operatorname{tr}_1\left(\left[\begin{smallmatrix} M & O \\ O^\dagger & N \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]\right) = -M$. We thus obtain the following dual problem:

$$(\text{C4}) = \quad \text{minimize} \quad \operatorname{tr}(N) \tag{C6}$$
$$\text{over variables} \quad M \geqslant 0,\ N \geqslant 0,\ O,$$
$$\text{subject to} \quad O + O^\dagger = D,$$
$$M \leqslant \rho,$$
$$\begin{bmatrix} M & O \\ O^\dagger & N \end{bmatrix} \geqslant 0.$$

Equality with the primal optimization problem holds thanks to strong duality, which is ensured by the Slater conditions [76,106]. We can further simplify the dual problem. First, the choice $M = \rho$ is optimal: indeed, for any optimal choices of variables with $M \leqslant \rho$, we can replace M by ρ while still achieving the same value. Therefore,

$$(\text{C6}) = \quad \text{minimize} \quad \operatorname{tr}(N) \tag{C7}$$
$$\text{over variables} \quad N \geqslant 0,\ O,$$
$$\text{subject to} \quad O + O^\dagger = D,$$
$$\begin{bmatrix} \rho & O \\ O^\dagger & N \end{bmatrix} \geqslant 0.$$

Using the Schur complement argument again (Theorem 5), we find that $n \geqslant O^\dagger \rho^{-1} O$, and for the same reason as above, there is an optimal choice of variables with $Y = O^\dagger \rho^{-1} O$. Hence

$$(\text{C7}) = \quad \text{minimize} \quad \operatorname{tr}(O^\dagger \rho^{-1} O) \tag{C8}$$
$$\text{over variables} \quad O \text{ arb.}$$
$$\text{subject to} \quad O + O^\dagger = D$$
$$P_\rho^\perp O = 0.$$

We may introduce the variable $L = \rho^{-1/2} O$, which yields

$$(\text{C8}) = \quad \text{minimize} \quad \operatorname{tr}(L^\dagger L) \tag{C9}$$
$$\text{over variables} \quad L \text{ arb.}$$
$$\text{subject to} \quad \rho^{1/2} L + L^\dagger \rho^{1/2} = D.$$

We recognize the optimization in Eq. (C3b). At this point we have shown that both optimizations in the claim, Eqs. (C3a) and (C3b), are equal thanks to semidefinite programming duality. It remains to show that the common optimal value is $F(\rho\,;D)$ as given by Eq. (C2).

To find optimal variables, we examine the complementary slackness conditions [106] corresponding to the primal-dual problem pair (C4) and (C6). Namely, taking the product of an inequality constraint with the corresponding dual variable turns the inequality into an equality for optimal primal and dual choices of variables. For the primal constraint this gives us the equalities

$$\begin{aligned} QM - SO^\dagger = 0\,; &\quad QO - SN = 0; \\ -SM + O^\dagger = 0\,; &\quad -SO + N = 0. \end{aligned} \tag{C10}$$

From $-SM + O^\dagger = 0$ along with the optimal $M = \rho$ we deduce that $\rho S = O$ and thus $\rho^{1/2} S = \rho^{-1/2} O = L$. Plugging this into the constraint in Eq. (C9) we find

$$\rho S + S\rho = D. \tag{C11}$$

The solutions of this anticommutator equation have been studied in Appendix B, leading us to the primal candidate

$$S = \frac{1}{2}\mathcal{R}_\rho^{-1}(D). \tag{C12}$$

Plugging this choice into Eq. (C4), along with the choice $Q = S^2$, we obtain

$$(\mathrm{C4}) \geqslant \frac{1}{2}\operatorname{tr}\big(D\mathcal{R}_\rho^{-1}(D)\big) - \frac{1}{4}\operatorname{tr}\big(\rho[\mathcal{R}_\rho^{-1}(D)]^2\big) = \frac{1}{4}F\big(\rho\,;D\big), \tag{C13}$$

where we have used the fact that $\operatorname{tr}\big(D\mathcal{R}_\rho^{-1}(D)\big) = \langle\langle D|\mathcal{R}_\rho^{-1}|D\rangle\rangle = \langle\langle D|\mathcal{R}_\rho^{-1}\mathcal{R}_\rho\mathcal{R}_\rho^{-1}|D\rangle\rangle = \langle\langle\mathcal{R}_\rho^{-1}(D)|\mathcal{R}_\rho|\mathcal{R}_\rho^{-1}(D)\rangle\rangle = \operatorname{tr}\big(\rho\,[\mathcal{R}_\rho^{-1}(D)]^2\big)$.

By construction, $L = \rho^{1/2}S$ satisfies the constraint in Eq. (C9), noting that we have used the assumption that $P_\rho^\perp D P_\rho^\perp = 0$ as in the proposition statement. The corresponding value attained in the dual problem is

$$(\mathrm{C9}) \leqslant \operatorname{tr}(L^\dagger L) = \frac{1}{4}\operatorname{tr}\big(\rho[\mathcal{R}_\rho^{-1}(D)]^2\big) = \frac{1}{4}F\big(\rho\,;D\big), \tag{C14}$$

noting that $\operatorname{tr}(L^\dagger L) = \operatorname{tr}(\rho S^2)$. Combining Eqs. (C13) and (C14) with the above statement that (C4) = (C9) proves the first part of the claim.

The alternative form (C3c) is nothing else than Eq. (C7). Now we show the alternative form (C3d). Consider the optimization (C7). Decompose $O = O_R + iO_I$ into Hermitian and anti-Hermitian parts with $O_R = (O + O^\dagger)/2 = O_R^\dagger$, $O_I = -i(O - O^\dagger)/2 = O_I^\dagger$. The constraint on O indicates that the Hermitian part O_R of O must satisfy $2O_R = D$. The second constraint then becomes

$$\begin{bmatrix} \rho & D/2 + iO_I \\ D/2 - iO_I & N \end{bmatrix} \geqslant 0. \tag{C15}$$

Conjugating by $\begin{bmatrix} \mathbb{1} & 0 \\ 0 & 2\mathbb{1} \end{bmatrix}$, we see that this condition is equivalent to

$$\begin{bmatrix} \rho & D + 2iO_I \\ D - 2iO_I & 4N \end{bmatrix} \geqslant 0. \tag{C16}$$

Now we set $K = 2O_I$ and $J = 4N$, showing that the optimization (C3d) is equivalent to Eq. (C7) (up to a factor of 4), and therefore equal to $F\big(\rho\,;D\big)$.

For completeness, we exhibit optimal choices for K, J. Choose K to be the anti-Hermitian part of $\rho\mathcal{R}_\rho^{-1}(D)$, i.e., $K = (\rho\mathcal{R}_\rho^{-1}(D) - \mathcal{R}_\rho^{-1}(D)\rho)/(2i)$. Then

$$D + iK = \frac{1}{2}\big(\rho\mathcal{R}_\rho^{-1}(D) + \mathcal{R}_\rho^{-1}(D)\rho\big) + \frac{1}{2}\big(\rho\mathcal{R}_\rho^{-1}(D) - \mathcal{R}_\rho^{-1}(D)\rho\big) = \rho\mathcal{R}_\rho^{-1}(D), \tag{C17}$$

and its Hermitian conjugate is $D - iK = \mathcal{R}_\rho^{-1}(D)\rho$. Now choose $J = (D - iK)\rho^{-1}(D + iK) = \mathcal{R}_\rho^{-1}(D)\rho\mathcal{R}_\rho^{-1}(D)$; the constraint in Eq. (C3d) is satisfied thanks to Theorem 5. The value reached by this choice of candidates is then the optimal value $\operatorname{tr}(J) = F\big(\rho\,;D\big)$. ■

The expressions in Proposition 5 lead to simple proofs of elementary properties of the Fisher information.

Proposition 6 (Simple bounds for the Fisher information).—let ρ be a subnormalized quantum state and D be a Hermitian operator that satisfies $P_\rho^\perp D P_\rho^\perp = 0$. Then we have

$$\|D\|^2 \leqslant F\big(\rho\,;D\big) \leqslant \operatorname{tr}(\rho^{-1}D'^2), \tag{C18}$$

where $D' = 2D - P_\rho D P_\rho$.

Proof.—First we show the lower bound. Let $|\phi\rangle$ be a (normalized) eigenvector associated with the largest eigenvalue of D (in magnitude), such that $\langle\phi|D|\phi\rangle = \|D\|$. For some $s \in \mathbb{R}$ to be determined later, we choose the optimization candidate $S = s|\phi\rangle\langle\phi|$ in Eq. (C3a). Then the corresponding objective value is

$$F\big(\rho\,;D\big) \geqslant 4\operatorname{tr}(DS) - 4\operatorname{tr}(\rho S^2) = 4s\|D\| - 4s^2\langle\phi|\rho|\phi\rangle. \tag{C19}$$

The latter expression is maximal when $0 = (d/ds)(\cdots) = 4\|D\| - 8s\langle\phi|\rho|\phi\rangle$, i.e., when $s = \|D\|/(2\langle\phi|\rho|\phi\rangle)$. We obtain the bound

$$F\big(\rho\,;D\big) \geqslant 2\frac{\|D\|^2}{\langle\phi|\rho|\phi\rangle} - \frac{\|D\|^2}{\langle\phi|\rho|\phi\rangle} = \frac{\|D\|^2}{\langle\phi|\rho|\phi\rangle} \geqslant \|D\|^2, \tag{C20}$$

recalling furthermore that $\langle\phi|\rho|\phi\rangle \leqslant 1$.

For the upper bound, consider the optimization problem (C3b) and choose the candidate $L = \rho^{-1/2}D'/2$. This is a feasible candidate because

$$\begin{aligned}\rho^{1/2}L + L^\dagger\rho^{1/2} &= P_\rho D' + (\text{h.c.}) \\ &= 2P_\rho D - P_\rho D P_\rho + (\text{h.c.}) \\ &= 2P_\rho D(P_\rho + P_\rho^\perp) - P_\rho D P_\rho + (\text{h.c.}) \\ &= P_\rho D P_\rho + 2P_\rho D P_\rho^\perp + (\text{h.c.}) \\ &= 2(P_\rho D P_\rho + P_\rho D P_\rho^\perp + P_\rho^\perp D P_\rho) = 2D,\end{aligned} \tag{C21}$$

where h.c. stands for the Hermitian conjugate of the entire preceding expression, and where we furthermore recall that $P_\rho^\perp D P_\rho^\perp = 0$. The objective value attained by this choice of candidate is $F\big(\rho\,;D\big) \leqslant 4\operatorname{tr}\big(L^\dagger L\big) = \operatorname{tr}\big(\rho^{-1}D'^2\big)$. ■

Proposition 7 (Right logarithmic derivative (RLD) bound [89]).—let ρ be a subnormalized quantum state and let D be a Hermitian operator satisfying $P_\rho^\perp D P_\rho^\perp = 0$. Let G be any operator (possibly non-Hermitian) that satisfies $\big(\rho G + G^\dagger\rho\big)/2 = D$. Then

$$F\big(\rho\,;D\big) \leqslant \operatorname{tr}(\rho G G^\dagger). \tag{C22}$$

Proof.—Use $L = \rho^{1/2}G/2$ in Eq. (C3b). ■

Proposition 8 (Fisher information under parameter rescaling).—let ρ be a sub-normalized quantum state and D be a Hermitian operator that satisfies $P_\rho^\perp D P_\rho^\perp = 0$. Then for any $\alpha \le 1$, $\beta \in \mathbb{R}$,

$$F(\alpha\rho\,;\,\beta D) = \frac{\beta^2}{\alpha} F(\rho\,;\,D). \tag{C23}$$

Proof.—Let S, L be optimal variables in Eq. (C3a) and (C3b) for $F(\alpha\rho\,;\,\beta D)$. Let $S' = (\alpha/\beta)S$ and $L' = (\sqrt{\alpha}/\beta)L$. Then

$$\frac{1}{4}F(\alpha\rho\,;\,\beta D) = \operatorname{tr}(\beta D S) - \operatorname{tr}(\alpha\rho S^2) = \frac{\beta^2}{\alpha}\left[\operatorname{tr}(DS') - \operatorname{tr}(\rho S'^2)\right] \leqslant \frac{\beta^2}{\alpha}\frac{1}{4}F(\rho\,;\,D); \tag{C24a}$$

$$\frac{1}{4}F(\alpha\rho\,;\,\beta D) = \operatorname{tr}(LL^\dagger) = \frac{\beta^2}{\alpha}\operatorname{tr}(L'L'^\dagger) \geqslant \frac{\beta^2}{\alpha}\frac{1}{4}F(\rho\,;\,D), \tag{C24b}$$

noting that L' is a valid choice of optimization candidate in (C3b) for $F(\rho\,;\,D)$ because $\rho^{1/2}L' + L'^\dagger\rho^{1/2} = (1/\beta)\big((\alpha\rho)^{1/2}L + L^\dagger(\alpha\rho)^{1/2}\big) = D$. ■

Proposition 9 (Fisher information bound for trace-decreasing maps).—let $|\psi\rangle$ be a (normalized) state vector and let $|\xi\rangle$ be any vector such that $\langle\psi|\xi\rangle = 0$. Let $\mathcal{N}$ be any completely positive, trace-non-increasing map and let $0 \leqslant \alpha \le 1$ such that $\mathcal{N}^\dagger(\mathbb{1}) \leqslant \alpha\mathbb{1}$. Then

$$F\big(\mathcal{N}(|\psi\rangle\langle\psi|)\,;\,\mathcal{N}(|\xi\rangle\langle\psi| + |\psi\rangle\langle\xi|)\big) \leqslant 4\alpha\langle\xi|\xi\rangle. \tag{C25}$$

Proof.—Let $O = \mathcal{N}(|\psi\rangle\langle\xi|)$ and $N = \mathcal{N}(|\xi\rangle\langle\xi|)$. These choices are feasible in Eq. (C3c) because applying the completely positive map $\mathcal{N} \otimes \mathrm{id}_2$ onto the positive semidefinite matrix

$$\begin{bmatrix} |\psi\rangle\langle\psi| & |\psi\rangle\langle\xi| \\ |\xi\rangle\langle\psi| & |\xi\rangle\langle\xi| \end{bmatrix} = \begin{bmatrix} |\psi\rangle \\ |\xi\rangle \end{bmatrix}\begin{bmatrix} \langle\psi| & \langle\xi| \end{bmatrix} \geqslant 0 \tag{C26}$$

gives again a positive semidefinite matrix. This choice of variable yields the objective value $\operatorname{tr}\big(\mathcal{N}(|\xi\rangle\langle\xi|)\big) = \operatorname{tr}\big(\mathcal{N}^\dagger(\mathbb{1})\,|\xi\rangle\langle\xi|\big) \leqslant \alpha\langle\xi|\xi\rangle$, proving the claim. ■

Proposition 10 (Joint convexity of the Fisher information).—let $\{\rho_k\}$ be a set of subnormalized states and $\{D_k\}$ be a set of Hermitian operators such that $P_{\rho_k}^\perp D_k P_{\rho_k}^\perp = 0$. Let $\{\alpha_k\}$ be a real positive coefficients such that $\sum_k \alpha_k \operatorname{tr}(\rho_k) \leqslant 1$. Then

$$F\left(\sum_k \alpha_k\rho_k\,;\,\sum_k \alpha_k D_k\right) \leqslant \sum_k \alpha_k F(\rho_k\,;\,D_k). \tag{C27}$$

Proof.—For each k, let K_k, J_k be optimal choices in Eq. (C3d) for $F(\rho_k\,;\,D_k)$. Set $K = \sum_k \alpha_k K_k$ and $J = \sum_k \alpha_k J_k$. Then

$$\begin{bmatrix} \rho & D + iK \\ D - iK & J \end{bmatrix} = \sum_k \alpha_k \begin{bmatrix} \rho_k & D_k + iK_k \\ D_k - iK_k & J_k \end{bmatrix} \geqslant 0, \tag{C28}$$

and so K, J are feasible candidates in the problem (C3d) for $F(\rho\,;\,D)$. The objective value achieved for this choice of variables gives the bound $F(\rho\,;\,D) \leqslant \operatorname{tr}(J) = \sum_k \alpha_k \operatorname{tr}(J_k) = \sum_k \alpha_k F(\rho_k\,;\,D_k)$. ■

Proposition 11 (Additivity of independent probes).—let ρ_A, ρ'_B be two subnormalized quantum states on two systems A, B, and let D_A, D'_B be two traceless Hermitian operators such that $P_{\rho_A}^\perp D_A P_{\rho_A}^\perp = 0$ and $P_{\rho'_B}^\perp D'_B P_{\rho'_B}^\perp = 0$. Then

$$F(\rho_A \otimes \rho'_B\,;\,D_A \otimes \rho'_B + \rho_A \otimes D'_B) = F(\rho_A\,;\,D_A) + F(\rho'_B\,;\,D'_B). \tag{C29}$$

Observe that the second argument on the left-hand side corresponds to the derivative of the state of a composite system that remains in a tensor product, $(d/dt)(\rho_A \otimes \rho'_B) = (d\rho_A/dt) \otimes \rho'_B + \rho_A \otimes (d\rho'_B/dt)$.

Proof.—Here we may directly guess a solution R to $\mathcal{R}_{\rho_A\otimes\rho'_B}(R) = D_A \otimes \rho'_B + \rho_A \otimes D'_B$. Compute first

$$\mathcal{R}_{\rho_A\otimes\rho'_B}(\mathbb{1}_A \otimes M_B) = \frac{1}{2}\left\{\rho_A \otimes \rho'_B, \mathbb{1}_A \otimes M_B\right\} = \frac{1}{2}\rho_A \otimes \left\{\rho'_B, M_B\right\}, \tag{C30}$$

so we see that, setting

$$R = \mathbb{1}_A \otimes \mathcal{R}_{\rho'_B}^{-1}(D'_B) + \mathcal{R}_{\rho_A}^{-1}(D_A) \otimes \mathbb{1}_B, \tag{C31}$$

we have $\mathcal{R}_{\rho_A\otimes\rho'_B}(R) = \rho_A \otimes D'_B + D_A \otimes \rho'_B$. Then

$$\begin{aligned} F(\rho_A \otimes \rho'_B\,;\,D_A \otimes \rho'_B + \rho_A \otimes D'_B) &= \operatorname{tr}\big((\rho_A \otimes \rho'_B)\,R^2\big) \\ &= \operatorname{tr}\Big((\rho_A \otimes \rho'_B)\Big(\mathbb{1}_A \otimes \big[\mathcal{R}_{\rho'_B}^{-1}(D'_B)\big]^2 + \big[\mathcal{R}_{\rho_A}^{-1}(D_A)\big]^2 \otimes \mathbb{1}_B \\ &\quad + 2\,\mathcal{R}_{\rho_A}^{-1}(D_A) \otimes \mathcal{R}_{\rho'_B}^{-1}(D'_B)\Big)\Big) \\ &= F(\rho_A\,;\,D_A) + F(\rho'_B\,;\,D'_B), \end{aligned} \tag{C32}$$

where in the last line we have used $\operatorname{tr}\big(\rho_A \mathcal{R}_{\rho_A}^{-1}(D_A)\big) = \operatorname{tr}\big(D_A - P_{\rho_A}^\perp D_A P_{\rho_A}^\perp\big) = \operatorname{tr}(D_A) = 0$. ■

Proposition 12 (Fisher information for pure states).—let $|\psi\rangle$ be a subnormalized state vector and let D be a Hermitian operator satisfying $\operatorname{tr}(D) = 0$ and $P_\psi^\perp D P_\psi^\perp = 0$. Then

$\langle D\rangle_\psi = 0$ and

$$F\big(\psi\,;D\big) = \frac{1}{(\operatorname{tr}\psi)^2}\Big[4\operatorname{tr}\big(\psi D^2\big)\Big]. \qquad \text{(C33)}$$

Furthermore, if $\operatorname{tr}(\psi) = 1$ and $D = -i[H,\psi]$ for some Hermitian operator H, then

$$F\big(\psi\,;D\big) = 4\sigma_H^2 = 4\big(\langle H^2\rangle_\psi - \langle H\rangle_\psi^2\big). \qquad \text{(C34)}$$

Proof.—First of all thanks to Proposition 8 we assume without loss of generality that $\operatorname{tr}(\psi) = 1$. Then, to see that $\langle D\rangle = 0$ we write

$$0 = \operatorname{tr}(D) = \operatorname{tr}\big[(\psi + P_\psi^\perp)D\big] = \langle D\rangle + \operatorname{tr}\big[P_\psi^\perp D P_\psi^\perp\big] = \langle D\rangle. \qquad \text{(C35)}$$

Using Eq. (C2) and Proposition 4, we then find

$$F\big(\psi\,;D\big) = \operatorname{tr}\big(\psi\,(2D)^2\big) = 4\operatorname{tr}(\psi D^2). \qquad \text{(C36)}$$

If furthermore $D = -i[H,\psi]$ for some Hermitian H, then we use Eq. (A3) to see that $\operatorname{tr}(\psi D^2) = \langle H^2\rangle_\psi - \langle H\rangle_\psi^2$. ■

Proposition 13 (Data-processing inequality for the Fisher information [19]).—let ρ be a subnormalized quantum state and D be a Hermitian operator that satisfies $P_\rho^\perp D P_\rho^\perp = 0$. Let $\mathcal{E}$ be any completely positive, trace-non-increasing map. Then

$$F\big(\rho\,;D\big) \geqslant F\big(\mathcal{E}(\rho)\,;\mathcal{E}(D)\big). \qquad \text{(C37)}$$

Proof.—First we show that $P_{\mathcal{E}(\rho)}^\perp \mathcal{E}(D) P_{\mathcal{E}(\rho)}^\perp = 0$, ensuring that the right-hand side in Eq. (C37) is well defined. Decompose $D = P_\rho D P_\rho + P_\rho^\perp D P_\rho + P_\rho D P_\rho^\perp = D_0 + D_0^\dagger$, defining $D_0 = (P_\rho D P_\rho)/2 + P_\rho D P_\rho^\perp$ such that $P_\rho^\perp D_0 = 0$. For $c > 0$ large enough, we have $\begin{bmatrix}\rho & D_0\\ D_0^\dagger & c\mathbb{1}\end{bmatrix} \geqslant 0$ thanks to Theorem 5. Applying the completely positive map $\mathrm{id}_2\otimes\mathcal{E}$ we obtain $\begin{bmatrix}\mathcal{E}(\rho) & \mathcal{E}(D_0)\\ \mathcal{E}(D_0^\dagger) & c\mathcal{E}(\mathbb{1})\end{bmatrix} \geqslant 0$, and therefore thanks to Theorem 5, $P_{\mathcal{E}(\rho)}^\perp \mathcal{E}(D_0) = 0$. Then $P_{\mathcal{E}(\rho)}^\perp \mathcal{E}(D) P_{\mathcal{E}(\rho)}^\perp = 0$ recalling $D = D_0 + D_0^\dagger$.

Let S be optimal in Eq. (C3a) for $F\big(\mathcal{E}(\rho)\,;\mathcal{E}(D)\big)$, thus satisfying $F\big(\mathcal{E}(\rho)\,;\mathcal{E}(D)\big) = 4\big[\operatorname{tr}\big(S\,\mathcal{E}(D)\big) - \operatorname{tr}\big(\mathcal{E}(\rho)\,S^2\big)\big]$. Choosing the candidate $\mathcal{E}^\dagger(S)$ in Eq. (C3a) for $F\big(\rho\,;D\big)$ we obtain

$$\begin{aligned} F\big(\rho\,;D\big) &\geqslant 4\big[\operatorname{tr}\big(D\,\mathcal{E}^\dagger(S)\big) - \operatorname{tr}\big(\rho\,[\mathcal{E}^\dagger(S)]^2\big)\big]\\ &\geqslant 4\big[\operatorname{tr}\big(\mathcal{E}(D)\,S\big) - \operatorname{tr}\big(\rho\,\mathcal{E}^\dagger(S^2)\big)\big]\\ &= 4\big[\operatorname{tr}\big(\mathcal{E}(D)\,S\big) - \operatorname{tr}\big(\mathcal{E}(\rho)\,S^2\big)\big]\\ &= F\big(\mathcal{E}(\rho)\,;\mathcal{E}(D)\big), \end{aligned} \qquad \text{(C38)}$$

where we have used Corollary 2 in the second inequality. ■

In the case of commuting state and differential, the symmetric logarithmic derivative reduces to a matrix inverse as described by the following proposition.

Proposition 14 (Fisher information for commuting state and derivative).—let ρ be any subnormalized quantum state and let D be a Hermitian operator that satisfies $P_\rho^\perp D P_\rho^\perp = 0$. Suppose that ρ and D commute. Then

$$F\big(\rho\,;D\big) = \operatorname{tr}\big(\rho^{-1}D^2\big). \qquad \text{(C39)}$$

Proof.—This can be shown from the properties of the symmetric logarithmic derivative, but we give a simple alternative proof using our convex optimizations for fun. Choose $S = \rho^{-1}D/2$ in Eq. (C3a), which we note is a Hermitian operator because ρ and D commute. This gives $F\big(\rho\,;D\big) \geqslant$ (D39)(C39). Similarly, the choice $L = \rho^{-1/2}D/2$ in Eq. (C3b) provides the opposite bound. ■

The following proposition interprets the Fisher information for subnormalized states according to the definition (C1) as the Fisher information of a normalized state that was projected onto a smaller subspace. This interpretation works as long as the subnormalized state does not change trace along its evolution, meaning that the derivative D has zero trace.

Proposition 15 (Fisher information for subnormalized and normalized states).—let ρ be any subnormalized quantum state and let D be any Hermitian operator that satisfies both $\operatorname{tr}(D) = 0$ and $P_\rho^\perp D P_\rho^\perp = 0$. Define ρ', D', with an additional new Hilbert space dimension, as

$$\rho' = \begin{pmatrix}\boxed{\rho} & 0\\ 0 & 1 - \operatorname{tr}(\rho)\end{pmatrix}\;;\quad D' = \begin{pmatrix}\boxed{\mathrm{D}} & 0\\ 0 & 0\end{pmatrix}. \qquad \text{(C40)}$$

Then

$$F\big(\rho\,;D\big) = F\big(\rho'\,;D'\big). \qquad \text{(C41)}$$

Proof.—Let P denote the projector onto the subspace of the Hilbert space on which the upper left block of ρ', D' acts. Let $R = \mathcal{R}_\rho^{-1}(D)$, and define

$$R' = \begin{pmatrix}\boxed{\mathrm{R}} & 0\\ 0 & 0\end{pmatrix}. \qquad \text{(C42)}$$

Multiplying together block-diagonal matrices preserves the block-diagonal structure, hence

$$\frac{1}{2}\{\rho', R'\} = \begin{pmatrix}\boxed{\frac{1}{2}\{\rho, R\}} & 0\\ 0 & 0\end{pmatrix} = D'. \qquad \text{(C43)}$$

Then with the definition (C1),

$$F\big(\rho'\,;D'\big) = \operatorname{tr}\big(\rho' R'^2\big) = \operatorname{tr}\big(\rho R^2\big) = F\big(\rho\,;D\big).$$

■

We can furthermore prove a relation between the Fisher information of two different directions in state space that might be associated with two different parametrized evolutions.

Proposition 16 (Relation between the Fisher information of two directions).—let ρ be a subnormalized quantum state and let D, D' be two Hermitian operators that satisfy $P_\rho^\perp D P_\rho^\perp = P_\rho^\perp D' P_\rho^\perp = 0$. Then

$$F(\rho\,;D) \leqslant F(\rho\,;D') + \Big[F(\rho\,;D+D')\,F(\rho\,;D-D')\Big]^{1/2}. \qquad (C44)$$

Consequently,

$$\big|F(\rho\,;D) - F(\rho\,;D')\big| \leqslant \Big[F(\rho\,;D+D')\,F(\rho\,;D-D')\Big]^{1/2}. \qquad (C45)$$

Furthermore, equality holds in Eq. (C45) if and only if D, D' are linearly dependent.

Proof.—Define the shorthand $\Delta_\pm = D \pm D'$. We compute

$$\begin{aligned} F(\rho\,;D) &= \mathrm{tr}\Big(\rho\Big[\mathcal{R}_\rho^{-1}(D') + \mathcal{R}_\rho^{-1}(\Delta_-)\Big]^2\Big) \\ &= F(\rho\,;D') + \mathrm{tr}\Big(\rho\Big[\big(\mathcal{R}_\rho^{-1}(\Delta_-)\big)^2 \\ &\quad + \Big\{\mathcal{R}_\rho^{-1}(D'),\, \mathcal{R}_\rho^{-1}(\Delta_-)\Big\}\Big]\Big) \\ &= F(\rho\,;D') + \mathrm{tr}\Big(\rho\,\Big\{\frac{1}{2}\mathcal{R}_\rho^{-1}(\Delta_-) \\ &\quad + \mathcal{R}_\rho^{-1}(D'),\, \mathcal{R}_\rho^{-1}(\Delta_-)\Big\}\Big), \end{aligned} \qquad (C46)$$

where in the last equality we have used $M^2 = \{\frac{1}{2}M, M\}$ for any operator M along with the linearity of the anticommutator in the first argument. Furthermore, we see from the definition of Δ_- that

$$D' + \frac{1}{2}\Delta_- = \frac{1}{2}(D + D') = \frac{1}{2}\Delta_+. \qquad (C47)$$

Then

$$\begin{aligned} (C46) &= F(\rho\,;D') + \frac{1}{2}\,\mathrm{tr}\Big(\rho\,\Big\{\mathcal{R}_\rho^{-1}(\Delta_+)\,,\, \mathcal{R}_\rho^{-1}(\Delta_-)\Big\}\Big) \\ &= F(\rho\,;D') + \mathrm{Re}\,\mathrm{tr}\Big(\rho\,\mathcal{R}_\rho^{-1}(\Delta_+)\,\mathcal{R}_\rho^{-1}(\Delta_-)\Big) \\ &=: F(\rho\,;D') + C_\rho(\Delta_+, \Delta_-), \end{aligned} \qquad (C48)$$

where $C_\rho(\Delta_+, \Delta_-)$ is defined as the second term in the above expression. From the Cauchy-Schwarz inequality,

$$|C_\rho(\Delta_+, \Delta_-)|^2 \leqslant \mathrm{tr}\Big(\rho\Big[\mathcal{R}_\rho^{-1}(\Delta_+)\Big]^2\Big)\,\mathrm{tr}\Big(\rho\Big[\mathcal{R}_\rho^{-1}(\Delta_-)\Big]^2\Big). \qquad (C49)$$

Hence,

$$(C48) \leqslant F(\rho\,;D) + \Big[F(\rho\,;\Delta_+)\,F(\rho\,;\Delta_-)\Big]^{1/2}. \qquad (C50)$$

Equation (C45) follows by repeating the argument while inverting the roles of D and D'.

Equality in Eq. (C45) is equivalent to the Cauchy-Schwarz inequality being tight. In turn is equivalent to the operators $\rho^{1/2}\mathcal{R}_\rho^{-1}(\Delta_+)$ and $\rho^{1/2}\mathcal{R}_\rho^{-1}(\Delta_-)$ being linearly dependent, i.e., there exist $\alpha_1, \alpha_2 \in \mathbb{R}$, $(\alpha_1, \alpha_2) \neq (0, 0)$, such that

$$\alpha_1\rho^{1/2}\mathcal{R}_\rho^{-1}(\Delta_+) + \alpha_2\rho^{1/2}\mathcal{R}_\rho^{-1}(\Delta_-) = 0. \qquad (C51)$$

Since the operator $\mathcal{R}_\rho^{-1}(\Delta_\pm)$ vanishes on the operator subspace spanned by $P_\rho^\perp(\cdot)P_\rho^\perp$, we have that Eq. (C51) is equivalent to

$$\alpha_1\mathcal{R}_\rho^{-1}(\Delta_+) + \alpha_2\mathcal{R}_\rho^{-1}(\Delta_-) = 0, \qquad (C52)$$

and therefore to

$$\mathcal{R}_\rho^{-1}\Big[\alpha_1\Delta_+ + \alpha_2\Delta_-\Big] = 0. \qquad (C53)$$

Because the kernel of the superoperator $\mathcal{R}_\rho^{-1}$ is spanned by $P_\rho^\perp\,(\cdot)\,P_\rho^\perp$, onto which $\Delta_\pm$ have no support by assumption, then Eq. (C53) is further equivalent to

$$\alpha_1\Delta_+ + \alpha_2\Delta_- = 0. \qquad (C54)$$

Therefore, equality in Eq. (C45) is achieved if and only if $\Delta_\pm$ are linearly dependent, which is equivalent to the linear dependence of D with D'. ■

Using a similar idea, we can also prove a continuity bound on the Fisher information with respect to its second argument.

Proposition 17 (A continuity bound of the Fisher information in its second argument).—let ρ be any subnormalized quantum state and let D, Δ be any Hermitian operators such that $P_\rho^\perp D P_\rho^\perp = 0 = P_\rho^\perp \Delta P_\rho^\perp$. Then

$$\Big|F(\rho\,;D+\Delta) - F(\rho\,;D) - F(\rho\,;\Delta)\Big| \leqslant 2[F(\rho\,;D)F(\rho\,;\Delta)]^{1/2}. \qquad (C55)$$

As a consequence,

$$\Big|F(\rho\,;D+\Delta) - F(\rho\,;D)\Big| \leqslant F(\rho\,;\Delta) + 2[F(\rho\,;D)F(\rho\,;\Delta)]^{1/2}. \qquad (C56)$$

Proof.—Using the formula $F(\rho\,;D') = \mathrm{tr}(D'\,\mathcal{R}_\rho^{-1}(D'))$ for the Fisher information, we write

$$F(\rho\,;D+\Delta) = \mathrm{tr}\big((D+\Delta)\,\mathcal{R}_\rho^{-1}(D+\Delta)\big) = F(\rho\,;D) + F(\rho\,;\Delta) + 2\,\mathrm{tr}\big(D\,\mathcal{R}_\rho^{-1}(\Delta)\big), \tag{C57}$$

recalling that $\mathcal{R}_\rho^{-1}$ is superoperator self-adjoint. The claim follows by bounding the last term in the above expression using the Cauchy-Schwarz inequality, to get

$$\big|\mathrm{tr}\big(D\,\mathcal{R}_\rho^{-1}(\Delta)\big)\big| \leqslant \sqrt{\mathrm{tr}\big(D\,\mathcal{R}_\rho^{-1}(D)\big)\,\mathrm{tr}\big(\Delta\,\mathcal{R}_\rho^{-1}(\Delta)\big)} = \sqrt{F(\rho\,;D)F(\rho\,;\Delta)}.$$

■

We can consider more precisely how $F_{\mathrm{Bob},t}$ behaves when seen as a function of the noise channel $\mathcal{N}$, for channels $\mathcal{N}$ that are close to the identity channel id. More specifically, we prove a continuity bound for the quantum Fisher information $F\big(\mathcal{N}(\psi)\,;\mathcal{N}(\partial_t\psi)\big)$ at the point $\mathcal{N} = \mathrm{id}$, when that quantity is seen as a function of $\mathcal{N}$.

Proposition 18.—let $|\psi\rangle$ be a pure state and let D be a Hermitian operator such that $\langle D\rangle_\psi = 0$ and $P_\psi^\perp D P_\psi^\perp = 0$. Let $\epsilon > 0$ and let $\mathcal{N}$ be a channel with $\|\mathcal{N} - \mathrm{id}\|_\diamond \leqslant \epsilon$. Then

$$F(\psi, D) \geqslant F(\mathcal{N}(\psi),\mathcal{N}(D)) \geqslant F(\psi, D) - 8\epsilon\,\|D\|_1\|D\|_\infty. \tag{C58}$$

Observe that the stated conditions on D are satisfied if $D = -i[H,\psi]$ for some Hermitian operator H.

Proof.—Let $\epsilon > 0$ and let $\mathcal{N} = \mathrm{id} + \Delta$ where Δ is a Hermiticity-preserving superoperator with $\|\Delta\|_\diamond \leqslant \epsilon$. The first claimed inequality immediately follows from the data processing inequality. We now prove the second inequality. Using Proposition 4, let $S = \frac{1}{2}\mathcal{R}_\psi^{-1}(D) = D$. Since this S is known to be optimal in Eq. (C3a) for $F(\psi\,;D)$, we can compute

$$\begin{aligned}
F(\psi\,;D) &= 4\Big\{\mathrm{tr}\big[(D)\,S\big] - \mathrm{tr}\big[\psi\,S^2\big]\Big\}\\
&= 4\Big\{\mathrm{tr}\big[\mathcal{N}(D)\,S\big] - \mathrm{tr}\big[\mathcal{N}(\psi)\,S^2\big]\Big\} - 4\Big\{\mathrm{tr}\big[\Delta(D)\,S\big] - \mathrm{tr}\big[\Delta(\psi)\,S^2\big]\Big\}\\
&\leqslant F(\mathcal{N}(\psi),\mathcal{N}(D)) + 4\big\|\Delta(D)\big\|_1\|S\|_\infty + 4\big\|\Delta(\psi)\big\|_1\|S\|_\infty^2\\
&\leqslant F(\mathcal{N}(\psi),\mathcal{N}(D)) + 8\epsilon\,\|D\|_1\|D\|_\infty,
\end{aligned} \tag{C59}$$

using $\|D\|_\infty \leqslant \|D\|_1$, and thus proving the claim. ■

APPENDIX D: OPTIMAL LOCAL SENSING AND THE CRAMÉR-RAO BOUND

Here we review which operators achieve the optimal variance in estimating an unknown parameter [1,12,23,24,89]. An unknown parameter t of an evolution ρ_t of a (normalized) quantum state is estimated locally around t_0 using an observable T, whose measurement outcomes are the estimates of the parameter. We ask for the observable to have the correct average and first order deviation, $\langle T\rangle_{\rho_{t_0+dt}} = t_0 + dt + O(dt^2)$; except in edge cases, this condition can be enforced by a suitable scaling factor and a suitable shift by the identity. The conditions then become $\langle T\rangle_{\rho_{t_0}} = t_0$ and $\mathrm{tr}\big\{(\partial_t\rho_t|_{t_0})\,T\big\} = 1$. We seek to minimize the operator T's variance $\langle T^2\rangle_{\rho_{t_0}} - \langle T\rangle_{\rho_{t_0}}^2$. We call such an operator with minimal variance an *optimal local sensing operator*, and the square root of the minimal variance is the *optimal estimation error* $\Delta t_{\mathrm{unc}}(t_0)$ locally at t_0. That is, the optimal estimation error locally at t_0, along with an optimal local sensing operator at t_0, are given by the following optimization problem:

$$\begin{aligned}
\Delta t_{\mathrm{unc}}^2(t_0) = \quad & \min_{T=T^\dagger} \quad \mathrm{tr}\Big\{\rho_{t_0}(T - t_0\mathbb{1})^2\Big\},\\
& \text{such that} \quad \mathrm{tr}\Big\{\rho_{t_0}T\Big\} = t_0,\ \mathrm{tr}\Big\{(\partial_t\rho_t|_{t_0})\,T\Big\} = 1.
\end{aligned} \tag{D1}$$

In the event that $\partial_t\rho_t|_{t_0} = 0$, there is no operator T that satisfies the given conditions. We conventionally set $\Delta t_{\mathrm{unc}} = \infty$, since the state is locally stationary and no observable is able to detect a first-order deviation in the parameter t.

A more general scheme would enable an agent to use a generalized measurement given by a POVM instead of an observable T. However, as shown in, e.g., Ref. [1], the optimal POVM can in fact be chosen to be a projective measurement. Therefore, one cannot sense a parameter more accurately using a POVM instead of an observable.

The following proposition fully characterizes the locally optimal sensing observables (cf., e.g., Ref. [1]). In the following, we write as a shorthand ρ and $\partial_t\rho$ instead of ρ_{t_0} and $\partial_t\rho_t|_{t_0}$.

Proposition 19 (Locally optimal sensing).—assume $\partial_t\rho \neq 0$. Then any operator T that is optimal in Eq. (D1) is of the form

$$T = t\mathbb{1} + (\Delta t^2_{\text{unc}})\,\mathcal{R}_\rho^{-1}\big(\partial_t\rho\big) + P_\rho^\perp M P_\rho^\perp, \tag{D2}$$

for some Hermitian operator M.

If $P_\rho^\perp(\partial_t\rho)P_\rho^\perp = 0$, then $\Delta t^2_{\text{unc}} = [F\big(\rho\,;\,\partial_t\rho\big)]^{-1}$ with the Fisher information defined in Eq. (C1), and M can be arbitrary.

If $P_\rho^\perp(\partial_t\rho)P_\rho^\perp \neq 0$, then $\Delta t^2_{\text{unc}} = 0$ and M satisfies $\text{tr}\big(M\,P_\rho^\perp\partial_t\rho P_\rho^\perp\big) = 1$.

Let us further note that if $\partial_t\rho = 0$, we have $F(\rho;\partial_t\rho) = 0$. Therefore, provided that $P_\rho^\perp\partial_t\rho P_\rho^\perp = 0$, we can in full generality write

$$\Delta t^2_{\text{unc}} = \frac{1}{F\big(\rho\,;\,\partial_t\rho\big)}, \tag{D3}$$

along with the convention that $\Delta t_{\text{unc}} = \infty$ if $F\big(\rho\,;\,\partial_t\rho\big) = 0$. In our setting, the optimal sensing scheme always achieves the value of the Cramér-Rao bound.

Proof of Proposition 19.—Without loss of generality, we assume $t_0 = 0$ throughout this proof; this is achieved by shifting the parameter to center it at zero, implying the corresponding shift $T \to T' = T - t_0\mathbb{1}$. We thus consider the optimization problem

$$\begin{aligned}\Delta t^2_{\text{unc}} = \quad &\min_{T=T^\dagger} \quad \text{tr}\big\{\rho_{t_0}T^2\big\},\\ &\text{such that} \quad \text{tr}\big\{\rho_{t_0}T\big\} = 0\,,\ \text{tr}\big\{(\partial_t\rho_t|_{t_0})\,T\big\} = 1.\end{aligned} \tag{D4}$$

First of all we observe that the first condition, $\text{tr}(\rho T) = 0$, can be ignored without changing the optimal value of the problem. Indeed, for any T that satisfies $\text{tr}\big((\partial_t\rho)T\big) = 1$ but with $\text{tr}(\rho T) \neq 0$, we can define $T' = T - \text{tr}(\rho T)\,\mathbb{1}$, with $\text{tr}(\rho T') = 0$ and $\text{tr}\big((\partial_t\rho)\,T\big) = \text{tr}\big((\partial_t\rho)\,T'\big)$ since $\text{tr}(\partial_t\rho) = \partial_t\,\text{tr}(\rho) = 0$; then $\text{tr}(\rho T'^2) = \text{tr}(\rho T^2) - [\text{tr}(\rho T)]^2 \leqslant \text{tr}(\rho T^2)$, meaning that T' not only satisfies $\text{tr}(\rho T') = 0$ in addition to the other condition, but it achieves a better objective function value.

We can recast this optimization as semidefinite problem, following Refs. [89,107], by using Schur complements (Theorem 5):

$$\begin{aligned}\Delta t^2_{\text{unc}} = \quad &\min_{Q\geq 0,\ T=T^\dagger} \quad \text{tr}(\rho Q)\\ &\text{such that} \quad \text{tr}\big((\partial_t\rho)\,T\big) = 1;\\ &\qquad\qquad\qquad \begin{bmatrix} Q & -T\\ -T & \mathbb{1}\end{bmatrix} \geqslant 0.\end{aligned} \tag{D5}$$

The associated dual problem takes the following form, noting that strong duality holds thanks to Slater's conditions [76,106].

$$\begin{aligned}\Delta t^2_{\text{unc}} = \quad &\max_{A,C\geq 0,\ B\,\text{arb.},\ \mu\in\mathbb{R}} \quad \mu - \text{tr}(C)\\ &\text{such that} \quad A \leqslant \rho\\ &\qquad\qquad B + B^\dagger = \mu\,\partial_t\rho\\ &\qquad\qquad \begin{bmatrix} A & B\\ B^\dagger & C\end{bmatrix} \geqslant 0\\ = \quad &\max_{B\,\text{arb.},\ \mu\in\mathbb{R}} \quad \mu - \text{tr}(B^\dagger\rho^{-1}B)\\ &\text{such that} \quad B + B^\dagger = \mu\,\partial_t\rho\\ &\qquad\qquad P_\rho B = B\end{aligned} \tag{D6}$$

$$\begin{aligned}= \quad &\max_{L\,\text{arb.},\ \mu\in\mathbb{R}} \quad \mu - \text{tr}(L^\dagger L)\\ &\text{such that} \quad \rho^{1/2}L + L^\dagger\rho^{1/2} = \mu\,\partial_t\rho\,,\end{aligned} \tag{D7}$$

using again Schur complements and where we introduced the variable L via $B = \rho^{1/2}L$, and where $P_\rho = \mathbb{1} - P_\rho^\perp$ is the projector onto the support of ρ.

A powerful characterization of the whole family of optimal solutions to a semidefinite problem with strong duality are the complementary slackness relations. An inequality constraint multiplied by the corresponding dual variable becomes an equality for any choice of primal and dual optimal solutions [76,106]. Here, this means that

$$\begin{bmatrix} Q & -T\\ -T & \mathbb{1}\end{bmatrix}\begin{bmatrix} A & B\\ B^\dagger & C\end{bmatrix} = 0. \tag{D8}$$

This gives us the following relations that must be satisfied for any choice of optimal variables:

$$Q\rho = TB^\dagger;\quad QB = TC;\quad B^\dagger = T\rho;\quad C = TB. \tag{D9}$$

The third equality ($B^\dagger = T\rho$) along with the dual constraint in Eq. (D6) implies that $\rho T + T\rho = \mu\,\partial_t\rho$. Proposition 3 asserts that the solutions are necessarily of the form $T = (\mu/2)\mathcal{R}_\rho^{-1}\big(\partial_t\rho\big) + P_\rho^\perp M P_\rho^\perp$ for some Hermitian M.

Now first suppose that $P_\rho^\perp\,(\partial_t\rho)\,P_\rho^\perp = 0$. The primal value achieved for a T of this form, and for any μ and M, is

$$\text{primal achieved} = \text{tr}(\rho T^2) = \frac{\mu^2}{4}F(t), \tag{D10}$$

with $F(t)$ as in Eq. (9). From complementary slackness we have $B^\dagger = T\rho$ and hence $\text{tr}(B^\dagger\rho^{-1}B) = \text{tr}(\rho T^2) = \mu^2F(t)/4$. The dual problem therefore reaches the value

$$\text{dual achieved} = \mu - \mu^2F(t)/4. \tag{D11}$$

Optimality implies that the primal and dual values are equal, $\mu^2F(t)/4 = \mu - \mu^2F(t)/4$ and therefore $\mu =$

$2/F(t)$ [note $\mu = 0$ is ruled out because the primal constraint $\mathrm{tr}\big((\partial_t\rho)\,T\big) = 1$ would be impossible to satisfy]. Therefore, the optimal solution to the problem is

$$\Delta t_{\mathrm{unc}}^2 = \frac{1}{F(t)}. \tag{D12}$$

Now suppose that $P_\rho^\perp(\partial_t\rho)P_\rho^\perp \neq 0$. Then there cannot be any solution for L in the constraint in Eq. (D7) unless $\mu = 0$ [the left-hand side vanishes entirely if we hit it with $P_\rho^\perp(\cdot)P_\rho^\perp$ but not the right-hand side if $\mu \neq 0$]. Then $T = P_\rho^\perp X P_\rho^\perp$, which implies $\mathrm{tr}(\rho T^2) = 0$, and furthermore M must satisfy $\mathrm{tr}\big((\partial_t\rho)P_\rho^\perp M P_\rho^\perp\big) = 1$ from the primal constraint. The dual candidate $L = 0$ yields objective value of zero in the dual problem, and therefore the optimal value of the optimization problem is zero, $\Delta t_{\mathrm{unc}}^2 = 0$. ■

APPENDIX E: PROOF OF THE SENSITIVITY UNCERTAINTY RELATION

The goal of this section is to prove the statements made in Sec. III. The setting is the one introduced in Sec. II. We provide two independent proofs of the uncertainty relation. The first proof is more intuitive and straightforward. The second proof is slightly more general and provides greater insight into some technicalities that underpin the uncertainty relation. The second proof directly relates the semidefinite characterizations of the quantities $F_{\mathrm{Bob},t}$ and $F_{\mathrm{Eve},\eta}$, making it easier to analyze edge cases, to gain insight on what choices of semidefinite variables are optimal, and to consider the more general situation where $\mathcal{N}$ is a trace-non-increasing map.

1. Proof via the second-order expansion of the fidelity

The strategy of our first proof of our uncertainty relation is to provide a direct proof of the statement presented as Corollary 1; we have already seen in the main text that the statement in Theorem 1 is equivalent to Corollary 1.

First observe that without loss of generality, we can assume that the Hamiltonian is time independent. This is because the Fisher information depends only on the state and its local time derivative at t, which is given by Eq. (42) and depends only on the value of the Hamiltonian at the fixed value t of interest.

Our proof proceeds in a similar fashion to that of the *channel-extension bound* developed in Refs. [30,31,78]. While our uncertainty relation could also be derived from the results in those references, we provide a self-contained proof for completeness and consistency of notation.

A remarkable property of the Fisher information is that it is directly related to the Bures distance and the fidelity of quantum states [1,12,25,108] according to

$$F_{\mathrm{Bob},t} = -4\,\frac{d^2}{dt'^2}\bigg|_{t'=t} F(\rho_B(t),\rho_B(t')), \tag{E1}$$

where $F(\rho,\rho') = \|\rho^{1/2}\rho'^{1/2}\|_1 = \mathrm{tr}\big[(\rho^{1/2}\rho'\rho^{1/2})^{1/2}\big]$ is the root fidelity between two quantum states [63], where $\|A\|_1$ denotes trace norm, i.e., the sum of the singular values of A. Note that at $t' = t$, the fidelity reaches its maximum value 1. We assume that $\rho(t)$ is does not change rank at $t' = t$, avoiding edge cases where the expression (E1) is incomplete [32–34].

By Uhlmann's theorem, and writing $|\rho(t)\rangle_{BE} = V_{A\to BE}|\psi(t)\rangle_A$ in terms of the Stinespring dilation $V_{A\to BE}$ of $\mathcal{N}$ given in Eq. (43), we have that

$$F(\rho_B(t),\rho_B(t')) = \max_{W_E\ \text{unitary}} \big|\langle\rho(t')|_{BE}\, W_E\, |\rho(t)\rangle_{BE}\big|, \tag{E2}$$

where W_E is a unitary operation on E. We therefore have the following equivalent expressions:

$$F(\rho_B(t),\rho_B(t')) = \max_{W_E} \mathrm{Re}\,\langle\psi(t')|_A\, V^\dagger\, W_E\, V_{A\to BE}\,|\psi(t)\rangle_A \tag{E3a}$$

$$= \max_{W_E} \mathrm{Re}\,\langle\psi(t')|_A\, \widehat{\mathcal{N}}^\dagger(W_E)\,|\psi(t)\rangle_A \tag{E3b}$$

$$= \max_{W_E} \mathrm{Re}\,\langle\psi(t)|_A\, e^{iH(t'-t)}\widehat{\mathcal{N}}^\dagger(W_E)\,|\psi(t)\rangle_A \tag{E3c}$$

$$= \max_{W_E} \mathrm{Re}\,\mathrm{tr}\Big(\widehat{\mathcal{N}}\big(\psi_A(t)\,e^{iH(t'-t)}\big)\,W_E\Big) \tag{E3d}$$

$$= \big\|\widehat{\mathcal{N}}\big(\psi\, e^{iH(t'-t)}\big)\big\|_1, \tag{E3e}$$

where the complementary channel $\widehat{\mathcal{N}}$ is given by Eq. (44). In the above expressions, the maximization can be taken over operators W_E that are unitary, or equivalently, it can be relaxed to all operators W_E satisfying $\|W_E\| \leqslant 1$.

The optimal unitary W_E is given by the polar decomposition of the operator $\widehat{\mathcal{N}}\big(\psi e^{iH(t'-t)}\big)$. For $t' = t + dt$ with a small dt, we have that the optimal W_E is close to the identity, which is the optimal for $t' = t$. Let us expand $W_E = \mathbb{1} - i\,dt\,S - (1/2)dt^2 S_2 + O(dt^3)$ for general matrices S and S_2 to be determined. The unitary constraint $W_E^\dagger W_E = \mathbb{1}_E$ for all dt implies that $S = S^\dagger$ and that $S_2 + S_2^\dagger = 2S^\dagger S = 2S^2$. Starting from Eq. (E3c) and expanding up to order dt^2 we find

$$F(\rho_B(t),\rho_B(t')) = \max_{S=S^\dagger,\,S_2} \operatorname{Re}\operatorname{tr}\left\{\psi\left(\mathbb{1}+idtH-\frac{H^2}{2}dt^2\right)\widehat{\mathcal{N}}^\dagger\left(\mathbb{1}-idtS-\frac{S_2}{2}dt^2\right)\right\}+O(dt^3)$$
$$=1+\max_{S=S^\dagger,\,S_2}\left\{dt\operatorname{Re}\operatorname{tr}\left[i\psi H-i\psi\widehat{\mathcal{N}}^\dagger(S)\right]+dt^2\operatorname{Re}\operatorname{tr}\left[-\frac{1}{2}\psi H^2-\frac{1}{2}\psi\widehat{\mathcal{N}}^\dagger(S_2)+\psi H\widehat{\mathcal{N}}^\dagger(S)\right]+O(dt^3)\right\}$$
$$=1+\frac{dt^2}{2}\max_{S=S^\dagger,\,S_2}\operatorname{Re}\operatorname{tr}\left\{-\psi H^2-\psi\widehat{\mathcal{N}}^\dagger(S_2)+2\psi H\widehat{\mathcal{N}}^\dagger(S)\right\}+O(dt^3), \tag{E4}$$

recalling that $\widehat{\mathcal{N}}^\dagger(\mathbb{1})=\mathbb{1}$, and where the first-order term vanishes because a product of two Hermitian operators has a real trace; with the factor i the term is killed by taking the real part. Continuing with only the second-order term we find

$$\left.\frac{d^2}{dt'^2}\right|_{t'=t}F(\rho_B(t),\rho_B(t'))=\max_{S=S^\dagger,\,S_2}\left\{-\operatorname{tr}(\psi H^2)-\frac{1}{2}\operatorname{tr}\left(\widehat{\mathcal{N}}(\psi)\,(S_2+S_2^\dagger)\right)+\operatorname{tr}\left[\{\psi,H\}\widehat{\mathcal{N}}^\dagger(S)\right]\right\}$$
$$=\max_{S=S^\dagger}\left\{-\operatorname{tr}(\psi H^2)-\operatorname{tr}\left(\widehat{\mathcal{N}}(\psi)\,S^2\right)+\operatorname{tr}\left[\widehat{\mathcal{N}}(\{\psi,H\})\,S\right]\right\}, \tag{E5}$$

where we have used the identity $2\operatorname{Re}\operatorname{tr}(AO)=\operatorname{tr}(A(O+O^\dagger))$ for Hermitian A, the identity $2\operatorname{Re}\operatorname{tr}(ABC)=\operatorname{tr}(\{A,B\}\,C)$ for Hermitian A,B,C, as well as the condition $S_2+S_2^\dagger=2S^2$ that came from enforcing the unitarity of W_E.

It is instructive to briefly comment on the situation of a time-dependent Hamiltonian. The derivation of the above expression, especially Eq. (E3c) and the expansion of the time-evolution operator leading up to Eq. (E4), looks like it necessitated the assumption of time independence of the Hamiltonian and that a time-dependent Hamiltonian might have led to a different result. In fact, we obtain the same result with a time-dependent Hamiltonian, which can be seen as follows. Write

$$H(t)=H+tH'+O(t^2) \tag{E6}$$

and expand the time-evolution operator via the time-ordered exponential as $U^\dagger(t'-t)=\mathcal{T}e^{i\int_t^{t'}dt''H(t'')}=1+i\int_t^{t'}dt''H(t'')-\int_t^{t'}dt''H(t'')\int_t^{t''}dt'''H(t''')+O(t^3)=1+idtH+(dt^2/2)(iH'-H^2)+O(dt^3)$, then we see that the only difference in the expressions leading up to Eq. (E4) is an additional term $\operatorname{Re}\operatorname{tr}\{\psi iH'dt^2\}$, which is equal to zero.

Now, we proceed to prove the uncertainty relation. With the definition $\Delta F_{\mathrm{Bob},t}=F_{\mathrm{Alice},t}-F_{\mathrm{Bob},t}$, we have

$$\Delta F_{\mathrm{Bob},t}=4\left(\operatorname{tr}(\psi H^2)-\left(\operatorname{tr}(\psi H)\right)^2\right)+4\left.\frac{d^2}{dt'^2}\right|_{t'=t}F(\rho_B(t),\rho_B(t'))$$
$$=\max_{S=S^\dagger}\left\{4\operatorname{tr}\left[\widehat{\mathcal{N}}(\{\psi,H\})\,S\right]-4\operatorname{tr}\left[\widehat{\mathcal{N}}(\psi)\,S^2\right]-4\left(\operatorname{tr}(\psi H)\right)^2\right\}, \tag{E7}$$

recalling that $F_{\mathrm{Alice},t}=4\sigma_H^2=4(\langle H^2\rangle-\langle H\rangle^2)$ and using the expression (E5). Observe that $\Delta F_{\mathrm{Bob},t}$ is necessarily invariant under a constant shift of the Hamiltonian $H\mapsto H+c\mathbb{1}$, because such a shift does not influence the evolution $\psi(t)$ and therefore both $F_{\mathrm{Alice},t}$ and $F_{\mathrm{Bob},t}$ are invariant under such shifts. [This invariance can also be checked explicitly by carrying out the corresponding transformations $H\mapsto H+c\mathbb{1}$ and $S\to S+c\mathbb{1}$ in Eq. (E7).] Applying the shift $H\mapsto H-\langle H\rangle_\psi$ yields

$$\text{(F7)}=\max_{S=S^\dagger}\left\{4\operatorname{tr}\left[\widehat{\mathcal{N}}(\{\psi,\bar{H}\})\,S\right]-4\operatorname{tr}\left[\widehat{\mathcal{N}}(\psi)\,S^2\right]\right\}, \tag{E8}$$

using the shorthand $\bar{H}:=H-\langle H\rangle_\psi$. At this point we recognize the expression of the Fisher information from Proposition 5, with $\rho=\widehat{\mathcal{N}}(\psi)$ and $D=\widehat{\mathcal{N}}(\{\psi,\bar{H}\})$. Let us briefly check that the requirement $P_\rho^\perp DP_\rho^\perp=0$ in Proposition 5 and in the definition of the Fisher information (C1) is satisfied. Thanks to Theorem 5, we have $\begin{bmatrix}\psi & \psi\bar{H}\\ \bar{H}\psi & \bar{H}\psi\bar{H}\end{bmatrix}\geqslant 0$, and furthermore, by Lemma 1, $\begin{bmatrix}\widehat{\mathcal{N}}(\psi) & \widehat{\mathcal{N}}(\psi\bar{H})\\ \widehat{\mathcal{N}}(\bar{H}\psi) & \widehat{\mathcal{N}}(\bar{H}\psi\bar{H})\end{bmatrix}\geqslant 0$; by Theorem 5 again, this implies that $P_\rho^\perp\widehat{\mathcal{N}}(\psi\bar{H})=0$. Therefore, $P_\rho^\perp\widehat{\mathcal{N}}(\{\psi,\bar{H}\})P_\rho^\perp=0$. It follows that

$$\Delta F_{\mathrm{Bob},t}=\text{(F8)}=F\left(\widehat{\mathcal{N}}(\psi)\,;\widehat{\mathcal{N}}(\{\psi,\bar{H}\})\right), \tag{E9}$$

as claimed.

2. Direct proof using the semidefinite characterization of the Fisher information

For this section, we fix $|\psi\rangle,|\xi\rangle$ be such that $\langle\psi|\psi\rangle=1$ and $\langle\psi|\xi\rangle=0$, and let $\mathcal{N}$ be a completely positive, trace-non-increasing map. Let $V_{A\to BE}$ be a Stinespring dilation of $\mathcal{N}$, i.e., $\mathcal{N}(\cdot) = \mathrm{tr}_E\big(V(\cdot)V^\dagger\big)$, and let $\widehat{\mathcal{N}}(\cdot) = \mathrm{tr}_B\big(V(\cdot)V^\dagger\big)$. Let

$$D_A^Y = -i\big(|\xi\rangle\langle\psi| - |\psi\rangle\langle\xi|\big)\ ; \quad D_A^Z = |\xi\rangle\langle\psi| + |\psi\rangle\langle\xi|. \tag{E10}$$

Suppose that $|\Phi_{B:E}\rangle$ is a maximally entangled ket between two suitable subspaces of B and E that are sufficiently large to ensure that there exist M, Λ matrices on B satisfying

$$V|\phi\rangle = (\Lambda\otimes\mathbb{1})|\Phi_{B:E}\rangle\ ; \quad V|\xi\rangle = (M\otimes\mathbb{1})|\Phi_{B:E}\rangle. \tag{E11}$$

(Alternatively, one can embed both B and E into larger systems B', E' with $B'\simeq E'$, on which one can consider the canonical maximally entangled ket $|\Phi'_{B':E'}\rangle = \sum|k\rangle_{B'}|k\rangle_{E'}$ with respect to the canonical bases of B', E'. We then define $|\Phi_{B:E}\rangle$ by projecting down $|\Phi'_{B':E'}\rangle$ onto $B\otimes E$.) Throughout the following, we only ever consider operators that are in the support of the reduced operators of $\Phi_{B:E}$ on B and E.

We define the operation $t_{B\to E}(\cdot) := \mathrm{tr}_B\big\{\Phi_{B:E}\,[(\cdot)\otimes\mathbb{1}_E]\big\}$, which is the partial transpose operation with respect to the bases used to define $|\Phi_{B:E}\rangle$. Equivalently, a defining property of this operation is that for any operator X_B, we have $(X_B\otimes\mathbb{1}_E)|\Phi_{B:E}\rangle = (\mathbb{1}_B\otimes t_{B\to E}(X_B))|\Phi_{B:E}\rangle$. Furthermore, for any M, we have $t_{B\to E}(M^\dagger) = \big[t_{B\to E}(M)\big]^\dagger$ and for any X, Y we have $t_{B\to E}(MN) = t_{B\to E}(N)\,t_{B\to E}(M)$. Similarly, we define the inverse operation $t_{E\to B}(\cdot) = \mathrm{tr}\big\{\Phi_{B:E}\,[\mathbb{1}_B\otimes(\cdot)]\big\}$, which has the same properties.

Observe that $\Lambda\Lambda^\dagger = \mathcal{N}(|\psi\rangle\langle\psi|) = \rho_B$ and $MM^\dagger = \mathcal{N}(|\xi\rangle\langle\xi|)$. Furthermore, we define W via the polar decomposition of $\Lambda = \rho^{1/2}W$, with

$$\begin{aligned}\Lambda &= \rho^{1/2}W; \quad \Lambda^\dagger = W^\dagger\rho^{1/2};\\ \Lambda^{-1} &= W^\dagger\rho^{-1/2}; \quad \Lambda^{-\dagger} = \rho^{-1/2}W.\end{aligned} \tag{E12}$$

The operators Λ^{-1} and $\Lambda^{-\dagger}$ are the Moore-Penrose pseudoinverses of Λ and $\Lambda^\dagger$, respectively, as can be seen by computing $\Lambda\Lambda^{-1} = P_\rho$ and $\Lambda^{-1}\Lambda = W^\dagger P_\rho W$ as well as $\Lambda^\dagger\Lambda^{-\dagger} = W^\dagger P_\rho W$ and $\Lambda^{-\dagger}\Lambda^\dagger = P_\rho$. Furthermore, we have

$$\mathcal{N}(D_A^Y) = -i(M\Lambda^\dagger - \Lambda M^\dagger), \tag{E13a}$$

$$\begin{aligned}\widehat{\mathcal{N}}(D_A^Z) &= \mathrm{tr}_B\big\{M_B\Phi_{B:E}\Lambda_B^\dagger + \Lambda_B\Phi_{B:E}M_B^\dagger\big\}\\ &= t_{B\to E}\big[\Lambda^\dagger M + M^\dagger\Lambda\big].\end{aligned} \tag{E13b}$$

We may also relate these objects to the state on Eve's system, via the partial transpose operation $t_{B\to E}$. Observe that $V|\psi\rangle = (\Lambda\otimes\mathbb{1})|\Phi_{B:E}\rangle = (\mathbb{1}\otimes t_{B\to E}(\Lambda))|\Phi_{B:E}\rangle$, and therefore $\rho_E = \mathrm{tr}_B(V\psi V^\dagger) = \big[t_{B\to E}(\Lambda)\big]\big[t_{B\to E}(\Lambda)\big]^\dagger = t_{B\to E}(\Lambda^\dagger\Lambda) = t_{B\to E}(W^\dagger\rho_B W)$. Then $P_{\rho_E} = t_{B\to E}(W^\dagger P_{\rho_B}W)$ and $P^\perp_{\rho_E} = t_{B\to E}(W^\dagger P^\perp_{\rho_B}W)$. We begin with a characterization of when our uncertainty relation holds with equality.

Proposition 20 (Conditions for uncertainty relation equality).—the following statements are equivalent:

(i) $(P^\perp_{\rho_B}\otimes P^\perp_{\rho_E})V|\xi\rangle = 0$.
(ii) We have $P^\perp_{\rho_B}MW^\dagger P^\perp_{\rho_B} = 0$.
(iii) We have $P^\perp_{\rho_B}\mathcal{N}(|\xi\rangle\langle\xi|)P^\perp_{\rho_B} = P^\perp_{\rho_B}\mathcal{N}(|\xi\rangle\langle\psi|)\rho_B^{-1}\mathcal{N}(|\psi\rangle\langle\xi|)P^\perp_{\rho_B}$.
(iv) We have $P^\perp_{\rho_E}\widehat{\mathcal{N}}(|\xi\rangle\langle\xi|)P^\perp_{\rho_E} = P^\perp_{\rho_E}\widehat{\mathcal{N}}(|\xi\rangle\langle\psi|)\rho_E^{-1}\widehat{\mathcal{N}}(|\psi\rangle\langle\xi|)P^\perp_{\rho_E}$.
(v) Let $\{E_k\}$ are Kraus operators for $\mathcal{N}$. For any linear combination $E = \sum_k c_kE_k$ with $c_k\in\mathbb{C}$ and such that $E|\psi\rangle = 0$, we have $P^\perp_{\rho_B}E|\xi\rangle = 0$.

Furthermore, consider the setting of Theorem 1 and suppose that $|\xi\rangle$ is defined as $|\xi\rangle = (H - \langle H\rangle)|\psi\rangle$. Then (i)–(v) are furthermore equivalent to the following.

(vi) For any eigenvalue $p_k(t)$ of $\mathcal{N}(\psi(t))$ such that $p_k(t_0) = 0$, we have $\partial_t^2 p_k\,(t_0) = 0$.

Observe that all the conditions above do not depend on the choice of Stinespring dilation and/or on the choice of the Kraus operator representation, as all such choices differ by a partial isometry on the E system. In other words, if the conditions above hold for particular choices of $V, \widehat{\mathcal{N}}$, and $\{E_k\}$, they hold for all other choices as well.

Proof of Proposition 20.—We have the following implications. *(i)* ⇔ *(ii)*: consider

$$\begin{aligned}(P^\perp_{\rho_B}\otimes P^\perp_{\rho_E})V|\xi\rangle &= \big((P^\perp_{\rho_B}M)\otimes P^\perp_{\rho_E}\big)|\Phi\rangle_{BE}\\ &= \big((P^\perp_{\rho_B}M\,t_{E\to B}(P^\perp_{\rho_E}))\otimes\mathbb{1}\big)|\Phi\rangle_{BE}.\end{aligned} \tag{E14}$$

Since $V|\psi\rangle = (\Lambda\otimes\mathbb{1})|\Phi\rangle = (\mathbb{1}\otimes t_{B\to E}(\Lambda))|\Phi\rangle$, we have $\rho_E = t_{B\to E}(\Lambda)\,t_{B\to E}(\Lambda)^\dagger = t_{B\to E}(\Lambda^\dagger\Lambda) = t_{B\to E}(W^\dagger\rho_B W)$. Then $P_{\rho_E} = t_{B\to E}(W^\dagger P_{\rho_B}W)$ and $P^\perp_{\rho_E} = t_{B\to E}(W^\dagger P^\perp_{\rho_B}W) = t_{B\to E}(\tilde{P}^\perp_{\rho_B})$, and

$$(\text{F14}) = \big((P^\perp_{\rho_B}M\,\tilde{P}^\perp_{\rho_B})\otimes\mathbb{1}\big)|\Phi\rangle_{BE}. \tag{E15}$$

Therefore, we have that $(P^\perp_{\rho_B}\otimes P^\perp_{\rho_E})V|\xi\rangle = 0$ is equivalent to $0 = P^\perp_{\rho_B}M\tilde{P}^\perp_{\rho_B}$.

(ii) ⇒ *(iii)*: let $K = P^\perp_\rho\mathcal{N}(|\xi\rangle\langle\psi|)\rho_B^{-1/2} = P^\perp_\rho\,\mathrm{tr}_E\big(M\Phi_{BE}\Lambda^\dagger\big)\rho_B^{-1/2} = P^\perp_\rho MW^\dagger P_\rho$. Now assume that (ii) holds; then $K = P^\perp_\rho MW^\dagger$ and we have $KK^\dagger = P^\perp_\rho MM^\dagger P^\perp_\rho = P^\perp_\rho\mathcal{N}(|\xi\rangle\langle\xi|)P^\perp_\rho$, showing (iii).

(iii) ⇒ *(ii)*: conversely, assuming (iii) and if $K = P^\perp_{\rho_B}\mathcal{N}(|\xi\rangle\langle\psi|)\rho_B^{-1/2} = P^\perp_\rho MW^\dagger P_\rho$, we have by assumption that $KK^\dagger = P^\perp_\rho MM^\dagger P^\perp_\rho = (P^\perp_\rho MW^\dagger P_\rho)(P_\rho MW^\dagger P_\rho)^\dagger + P^\perp_\rho MW^\dagger P^\perp_\rho WM^\dagger P^\perp_\rho$. This means that $0 = (P^\perp_\rho MW^\dagger P^\perp_\rho)(P^\perp_\rho WM^\dagger P^\perp_\rho)$. The latter equation can only hold if $P^\perp_\rho MW^\dagger P^\perp_\rho = 0$, showing (ii).

(i) ⇔ *(iv)*: condition (i) is symmetric if we replace $B \leftrightarrow E$ (and correspondingly $\mathcal{N} \leftrightarrow \widehat{\mathcal{N}}$), meaning that the condition holds if and only if the condition with B and E swapped also holds. Therefore, we can swap $B \leftrightarrow E$ in the other conditions and those will also hold if and only if (i) holds. Condition (iv) is obtained by performing this transformation on (iii).

(i) ⇒ *(v)*: we choose the representation $V = \sum E_k \otimes |k\rangle_E$ and assume (i), i.e., that we have

$$(P^{\perp}_{\rho_B} \otimes P^{\perp}_{\rho_E}) V|\xi\rangle = 0. \tag{E16}$$

Let $\{c_k\}$ with $c_k \in \mathbb{C}$ such that $\sum_k c_k E_k|\psi\rangle = 0$ and let $E = \sum_k c_k E_k$. Define $|\mathrm{e}\rangle_E = \sum c_k^* |k\rangle_E$. We have

$$\begin{aligned}\langle \mathrm{e}|\widehat{\mathcal{N}}(\psi)|\mathrm{e}\rangle &= \sum_{k,k'} \langle \mathrm{e}|k\rangle \operatorname{tr}\big(E_k \psi E_{k'}^\dagger\big) \langle k'|\mathrm{e}\rangle \\ &= \operatorname{tr}\Big[\Big(\sum c_k E_k\Big)\, \psi\, \Big(\sum c_k E_k\Big)^\dagger\Big] \\ &= \big\| E|\psi\rangle \big\|^2 = 0,\end{aligned} \tag{E17}$$

which implies that $|\mathrm{e}\rangle_E \in \ker \widehat{\mathcal{N}}(\psi)$, i.e., $P^{\perp}_{\rho_E}|\mathrm{e}\rangle_E = |\mathrm{e}\rangle_E$. Applying $(\mathbb{1} \otimes \langle \mathrm{e}|)$ onto Eq. (E16) we find

$$0 = \big(P^{\perp}_{\rho_B} \otimes \langle \mathrm{e}| P^{\perp}_{\rho_E}\big) V|\xi\rangle = \sum_k \big(P^{\perp}_{\rho_B} E_k|\xi\rangle\big)\, \langle \mathrm{e}|k\rangle = P^{\perp}_{\rho_B} E|\xi\rangle, \tag{E18}$$

showing that (v) holds.

(i) ⇐ *(v)*: we now suppose that condition (v) holds. Let $|\chi_j\rangle_E$ be a set of orthonormal states that span the support of $P^{\perp}_{\rho_E}$, i.e., $P^{\perp}_{\rho_E} = \sum_j |\chi_j\rangle\langle\chi_j|_E$. Fix any such $|\chi_j\rangle$ and define $E^{(j)} = \sum \langle \chi_j|k\rangle\, E_k$. We repeat Eq. (E17) by replacing $|\mathrm{e}\rangle \to |\chi_j\rangle$, $c_k \to \langle \chi_j|k\rangle$ to find

$$0 = \langle \chi_j|\widehat{\mathcal{N}}(\psi)|\chi_j\rangle = \ \dots\ = \big\|E^{(j)}|\psi\rangle\big\|^2, \tag{E19}$$

which implies that $E^{(j)}|\psi\rangle = 0$. We use the assumption that (v) holds to deduce that $P^{\perp}_{\rho_B} E^{(j)}|\xi\rangle = 0$; we note the latter expression holds for all j by repeating this argument for each j individually. Then

$$\begin{aligned}\big(P^{\perp}_{\rho_B} \otimes P^{\perp}_{\rho_E}\big)\, V|\xi\rangle &= \Big(P^{\perp}_{\rho_B} \otimes \sum |\chi_j\rangle\langle\chi_j|\Big)\, V|\xi\rangle \\ &= \sum_{k,j} (P^{\perp}_{\rho_B} E_k|\xi\rangle) \otimes (|\chi_j\rangle\langle\chi_j|k\rangle) \\ &= \sum_j (P^{\perp}_{\rho_B} E^{(j)}|\xi\rangle) \otimes |\chi_j\rangle = 0,\end{aligned} \tag{E20}$$

showing that (i) holds.

(iii) ⇔ *(vi)*: now consider the setting of Theorem 1 and suppose that $|\xi\rangle$ is defined as $|\xi\rangle = (H - \langle H\rangle)|\psi\rangle$. We invoke Ref. [33, Eq. (B15)], which in the present context reads

$$\operatorname{tr}\big(P^{\perp}_{\rho}\, \partial_t^2 \rho\big) = \sum_{k:\, p_k = 0} \partial_t^2 p_k + 2 \sum_{\substack{k,\ell:\\ p_k > 0\\ p_\ell = 0}} \frac{|\langle \lambda_k|\partial_t\rho|\lambda_\ell\rangle|^2}{p_k}, \tag{E21}$$

where $\{|\lambda_k\rangle\}$ is a complete eigenbasis of ρ with eigenvalues p_k. Using Eq. (11) one can check that the second term on the right-hand side satisfies

$$\begin{aligned}2 \sum_{\substack{k,\ell:\\ p_k > 0\\ p_\ell = 0}} \frac{|\langle \lambda_k|\partial_t\rho|\lambda_\ell\rangle|^2}{p_k} &= 2 \operatorname{tr}\big\{\rho^{-1}\, (\partial_t \rho)\, P^{\perp}_{\rho}\, (\partial_t \rho)\big\} \\ &= 2 \operatorname{tr}\big\{\rho^{-1}\, \mathcal{N}(|\psi\rangle\langle\xi|)\, P^{\perp}_{\rho} \mathcal{N}(|\xi\rangle\langle\psi|)\big\},\end{aligned} \tag{E22}$$

using the fact that $\partial_t \rho = \mathcal{N}(-i[H,\psi]) = \mathcal{N}(-i|\xi\rangle\langle\psi| + i|\psi\rangle\langle\xi|)$ and that $\mathcal{N}(X\psi) P^{\perp}_{\rho} = 0$ for any X. On the other hand, we can see that $\partial_t^2 \rho = \partial_t \mathcal{N}(-i[H,\psi]) = \mathcal{N}\big(-i[\partial_t H, \psi] - [H,[H,\psi]]\big)$, and recalling that $\mathcal{N}(X\psi)\, P^{\perp}_{\rho} = 0$ for any X we obtain

$$\begin{aligned}\operatorname{tr}\big(P^{\perp}_{\rho}\, \partial_t^2 \rho\big) &= \operatorname{tr}\big(P^{\perp}_{\rho}\, \mathcal{N}(2H\psi H)\big) = 2 \operatorname{tr}\big(P^{\perp}_{\rho}\, \mathcal{N}(\bar{H}\psi\bar{H})\big) \\ &= 2 \operatorname{tr}\big(P^{\perp}_{\rho}\, \mathcal{N}(|\xi\rangle\langle\xi|)\big),\end{aligned} \tag{E23}$$

writing $\bar{H} = H - \langle H\rangle_\psi$ and where $|\xi\rangle = \bar{H}|\psi\rangle$.

Now suppose that (iii) holds. Then

$$(\text{E23}) = 2 \operatorname{tr}\big(P^{\perp}_{\rho} \mathcal{N}(|\xi\rangle\langle\psi|)\, \rho^{-1} \mathcal{N}(|\psi\rangle\langle\xi|)\big) = (\text{E22}), \tag{E24}$$

and therefore the first term on the right-hand side of Eq. (E21) must vanish, and since $\partial_t^2 p_k \geqslant 0$ for all k for which $p_k = 0$ as p_k reaches a minimum at that point, we must necessarily have that $\partial_t^2 p_k = 0$ for all those k.

Conversely, if the first term on the right-hand side of Eq. (E21) vanishes, then we have

$$\operatorname{tr}\big(P^{\perp}_{\rho}\, \mathcal{N}(|\xi\rangle\langle\psi|)\, \rho^{-1}\, \mathcal{N}(|\psi\rangle\langle\xi|)\big) = \operatorname{tr}\big(P^{\perp}_{\rho} \mathcal{N}(|\xi\rangle\langle\xi|)\big). \tag{E25}$$

By applying the completely positive map $\mathrm{id}_2 \otimes \mathcal{N}$ onto the matrix $\begin{pmatrix} |\psi\rangle\langle\psi| & |\psi\rangle\langle\xi| \\ |\xi\rangle\langle\psi| & |\xi\rangle\langle\xi| \end{pmatrix}$ and further conjugating by

$\begin{pmatrix} \mathbb{1} & \\ & P_\rho^\perp \end{pmatrix}$ we find that

$$\begin{bmatrix} \rho & \mathcal{N}(|\psi\rangle\langle\xi|)P_\rho^\perp \\ P_\rho^\perp \mathcal{N}(|\xi\rangle\langle\psi|) & P_\rho^\perp \mathcal{N}(|\xi\rangle\langle\xi|)P_\rho^\perp \end{bmatrix} \geqslant 0. \tag{E26}$$

From the Schur complement (Theorem 5) we find that

$$P_\rho^\perp \Big[\mathcal{N}(|\xi\rangle\langle\xi|) - \mathcal{N}(|\xi\rangle\langle\psi|)\,\rho^{-1}\,\mathcal{N}(|\psi\rangle\langle\xi|)\Big] P_\rho^\perp \geqslant 0. \tag{E27}$$

But a positive semidefinite operator has trace zero if and only if it is identically equal to zero, so with Eq. (E25) we find that $P_\rho^\perp[\mathcal{N}(|\xi\rangle\langle\xi|) - \mathcal{N}(|\xi\rangle\langle\psi|)\,\rho^{-1}\,\mathcal{N}(|\psi\rangle\langle\xi|)]P_\rho^\perp = 0$, showing that (iii) holds. ■

Our main technical theorem is the following.

Theorem 6 (Time-energy uncertainty relation in the virtual metrological qubit picture).—let A, B, and E be finite-dimensional quantum systems. Let $\mathcal{N}_{A\to B}$ be a completely positive, trace-non-increasing map. Let $V_{A\to BE}$ be such that $\mathcal{N}_{A\to B}(\cdot) = \mathrm{tr}_E\big(V(\cdot)V^\dagger\big)$ and $V^\dagger V \leqslant \mathbb{1}$, i.e., V is a Stinespring dilation of $\mathcal{N}$. Let $\widehat{\mathcal{N}}_{A\to E}(\cdot) = \mathrm{tr}_B\big(V(\cdot)V^\dagger\big)$. Let $|\psi\rangle$ be any subnormalized state on A, and let $|\xi\rangle$ be any vector on A such that $\langle\psi|\xi\rangle = 0$. Define $D_A^Y = -i\big(|\xi\rangle\langle\psi| - |\psi\rangle\langle\xi|\big)$ and $D_A^Z = |\xi\rangle\langle\psi| + |\psi\rangle\langle\xi|$. Then

$$F\big(\mathcal{N}(\psi)\,;\mathcal{N}(D_A^Y)\big) + F\big(\widehat{\mathcal{N}}(\psi)\,;\widehat{\mathcal{N}}(D_A^Z)\big) \leqslant 4\langle\xi|\mathcal{N}^\dagger(\mathbb{1})|\xi\rangle. \tag{E28}$$

Furthermore, if $(P_{\rho_B}^\perp \otimes P_{\rho_E}^\perp)V|\xi\rangle = 0$, then equality holds.

First, we remark that both Fisher information expressions in Eq. (E28) are well defined in that we always have $P_{\mathcal{N}(\psi)}^\perp \mathcal{N}(D_A^Y) P_{\mathcal{N}(\psi)}^\perp = 0$ and $P_{\widehat{\mathcal{N}}(\psi)}^\perp \widehat{\mathcal{N}}(D_A^Z) P_{\widehat{\mathcal{N}}(\psi)}^\perp = 0$ as required in the definition (C1). These conditions can be verified by first noting that the following matrix is positive semidefinite:

$$\begin{bmatrix} |\psi\rangle\langle\psi| & |\psi\rangle\langle\xi| \\ |\xi\rangle\langle\psi| & |\xi\rangle\langle\xi| \end{bmatrix} = \begin{bmatrix} |\psi\rangle \\ |\xi\rangle \end{bmatrix} \begin{bmatrix} \langle\psi| & \langle\xi| \end{bmatrix} \geqslant 0, \tag{E29}$$

and applying either completely positive map $\mathrm{id}_2 \otimes \mathcal{N}$ or $\mathrm{id}_2 \otimes \widehat{\mathcal{N}}$ to obtain

$$\begin{bmatrix} \mathcal{N}(|\psi\rangle\langle\psi|) & \mathcal{N}(|\psi\rangle\langle\xi|) \\ \mathcal{N}(|\xi\rangle\langle\psi|) & \mathcal{N}(|\xi\rangle\langle\xi|) \end{bmatrix} \geqslant 0; \quad \begin{bmatrix} \widehat{\mathcal{N}}(|\psi\rangle\langle\psi|) & \widehat{\mathcal{N}}(|\psi\rangle\langle\xi|) \\ \widehat{\mathcal{N}}(|\xi\rangle\langle\psi|) & \widehat{\mathcal{N}}(|\xi\rangle\langle\xi|) \end{bmatrix} \geqslant 0. \tag{E30}$$

Then, Theorem 5 ensures that $P_{\mathcal{N}(\psi)}^\perp \mathcal{N}(|\psi\rangle\langle\xi|) = 0$ and therefore $P_{\mathcal{N}(\psi)}^\perp \mathcal{N}(D_A^Y) P_{\mathcal{N}(\psi)}^\perp = 0$; likewise $P_{\widehat{\mathcal{N}}(\psi)}^\perp \widehat{\mathcal{N}}(D_A^Z) P_{\widehat{\mathcal{N}}(\psi)}^\perp = 0$.

Proof of Theorem 6.—Let Λ, M be operators acting on B such that $V|\psi\rangle = (\Lambda \otimes \mathbb{1})|\Phi\rangle$ and $V|\xi\rangle = (M \otimes \mathbb{1})|\Phi\rangle$. We can write

$$\begin{aligned} D_B = \mathcal{N}(D_A^Y) &= \mathrm{tr}_E\big(-i\big(V|\xi\rangle\langle\psi|V^\dagger - V|\psi\rangle\langle\xi|V^\dagger\big)\big) \\ &= -i\big(M\Lambda^\dagger - \Lambda M^\dagger\big), \end{aligned} \tag{E31a}$$

$$\begin{aligned} \widehat{D}_E = \widehat{\mathcal{N}}(D_A^Z) &= \mathrm{tr}_B(V|\xi\rangle\langle\psi|V^\dagger + V|\psi\rangle\langle\xi|V^\dagger) \\ &= \mathrm{tr}_B(M\Phi\Lambda^\dagger + \Lambda\Phi M^\dagger); \end{aligned} \tag{E31b}$$

where in Eq. (E31b) the operators M, Λ act only on B with a tensor product with the identity on E implied but $\Phi = \Phi_{BE} = |\Phi\rangle\langle\Phi|_{BE}$. Now consider

$$\begin{aligned} \frac{1}{4}\Big\{4\langle\xi|\mathcal{N}^\dagger(\mathbb{1})|\xi\rangle - F\big(\rho_E\,;\widehat{D}_E\big)\Big\} &= \mathrm{tr}(MM^\dagger) \\ &\quad - \max_{S_E = S_E^\dagger}\Big\{\mathrm{tr}(\widehat{D}_E S_E) - \mathrm{tr}(\rho_E S_E^2)\Big\}, \end{aligned} \tag{E32}$$

using Eq. (C3a) and noting that $\langle\xi|\mathcal{N}^\dagger(\mathbb{1})|\xi\rangle = \mathrm{tr}(\mathcal{N}(|\xi\rangle\langle\xi|)) = \mathrm{tr}(MM^\dagger)$. Then, using Eq. (E31b), and writing $t(\cdot) = t_{E\to B}(\cdot)$ as a shorthand,

$$\begin{aligned} (\text{E32}) &= \min_{S_E = S_E^\dagger}\Big\{\mathrm{tr}(MM^\dagger) - \mathrm{tr}\big((M\Phi\Lambda^\dagger + \Lambda\Phi M^\dagger)S_E\big) + \mathrm{tr}\big(S_E\Lambda\Phi\Lambda^\dagger S_E\big)\Big\} \\ &= \min_{S_E = S_E^\dagger}\Big\{\mathrm{tr}(MM^\dagger) - \mathrm{tr}\big(M\,t(S_E)\,\Lambda^\dagger + \Lambda\,t(S_E)\,M^\dagger\big) + \mathrm{tr}\big(\Lambda(t(S_E))^2\Lambda^\dagger\big)\Big\} \\ &= \min_{S' = S'^\dagger}\Big\{\mathrm{tr}(MM^\dagger) - \mathrm{tr}\big(MS'\Lambda^\dagger + \Lambda S' M^\dagger\big) + \mathrm{tr}\big(\Lambda S'^2\Lambda^\dagger\big)\Big\}, \\ &= \min_{S' = S'^\dagger}\mathrm{tr}\Big((M - \Lambda S')(M - \Lambda S')^\dagger\Big), \end{aligned} \tag{E33}$$

where the optimization now ranges over all Hermitian operators S' acting on B. On the other hand, using Eq. (C3b),

$$\frac{1}{4}F(\rho\,;D) = \min\Big\{\mathrm{tr}(L^\dagger L)\ :\quad \rho^{1/2}L + L^\dagger\rho^{1/2} = D\Big\}, \tag{E34}$$

where ρ, D refer to operators on B. To prove the inequality (E28), which is the first part of our main theorem claim, our strategy is to show that for any candidate S' in Eq. (E33), there is a valid candidate L in Eq. (E34) that achieves the same value. This statement then implies that (E34) $\leq$ (E33) as desired.

Recall that $\Lambda\Lambda^\dagger = \rho$ (where $\rho \equiv \rho_B$ for short in this proof), and therefore the polar decomposition of Λ can be written as $\Lambda = \rho^{1/2}W$ for some unitary matrix W. Let S' be any Hermitian operator that is candidate in the optimization (E33), and let $L = iW(M^\dagger - S'\Lambda^\dagger)$. Then one can verify that

$$\begin{aligned}\rho^{1/2}L + L^\dagger\rho^{1/2} &= i\Lambda(M^\dagger - S'\Lambda^\dagger) - i(M - \Lambda S')\Lambda^\dagger \\ &= -i\big(M\Lambda^\dagger - \Lambda M^\dagger\big) = D,\end{aligned} \tag{E35}$$

and thus L is a feasible candidate in Eq. (E34). Furthermore it holds that $\mathrm{tr}(L^\dagger L) = \mathrm{tr}\big((M-\Lambda S')(M-\Lambda S')^\dagger\big)$, thus proving the inequality (E28).

We now show that, assuming $(P^\perp_{\rho_B}\otimes P^\perp_{\rho_E})V|\xi\rangle = 0$, the inequality becomes an equality. The proof strategy is to go in reverse direction above, starting with an optimal candidate L in Eq. (E34), and constructing a candidate S' in Eq. (E33) that achieves the same value. From Proposition 20 we see that $(P^\perp_{\rho_B}\otimes P^\perp_{\rho_E})V|\xi\rangle = 0$ is equivalent to

$$P^\perp_\rho M W^\dagger P^\perp_\rho = 0. \tag{E36}$$

Let L be an optimal candidate in Eq. (E34), i.e., such that $\rho^{1/2}L + L^\dagger\rho^{1/2} = D$ and $F\big(\rho\,;D\big) = 4\,\mathrm{tr}(L^\dagger L)$. Without loss of generality, we may assume that $P_\rho L = L$, since otherwise $P_\rho L$ would yield a better optimization candidate in Eq. (E34). Denoting by P_X^{supp} and P_X^{rng} the projectors onto the support and the range of an operator X, and defining $\tilde P_\rho = W^\dagger P_\rho W$, we have

$$\begin{aligned}&P^{\mathrm{rng}}_\Lambda = P^{\mathrm{supp}}_{\Lambda^\dagger} = P_\rho, \quad P^{\mathrm{supp}}_\Lambda = P^{\mathrm{rng}}_{\Lambda^\dagger} = W^\dagger P_\rho W = \tilde P_\rho, \\ &P^\perp_\rho = \mathbb{1} - P_\rho, \quad \tilde P^\perp_\rho = \mathbb{1} - \tilde P_\rho.\end{aligned} \tag{E37}$$

Let us compute the object $LP^\perp_\rho$:

$$\begin{aligned}LP^\perp_\rho &= P_\rho LP^\perp_\rho = \rho^{-1/2}\big(\rho^{1/2}L + L^\dagger\rho^{1/2}\big)P^\perp_\rho \\ &= \rho^{-1/2}\big(-i(M\Lambda^\dagger - \Lambda M^\dagger)\big)P^\perp_\rho \\ &= i\rho^{-1/2}\big(\Lambda M^\dagger\big)P^\perp_\rho = iP_\rho W M^\dagger P^\perp_\rho = iW\tilde P_\rho M^\dagger P^\perp_\rho \\ &= iWM^\dagger P^\perp_\rho,\end{aligned} \tag{E38}$$

where we have employed Eq. (E36) in the last equality.

Now let us get started with constructing S'. Our goal is to find a Hermitian matrix S' such that

$$L \stackrel{!}{=} iW(M^\dagger - S'\Lambda^\dagger). \tag{E39}$$

Indeed, this would ensure a valid candidate in Eq. (E33) reaching the same value as $\mathrm{tr}(L^\dagger L)$. The equality (E39) is equivalent to both simultaneous conditions

$$LP_\rho \stackrel{!}{=} iW(M^\dagger - S'\Lambda^\dagger)P_\rho; \quad LP^\perp_\rho \stackrel{!}{=} iW(M^\dagger - S'\Lambda^\dagger)P^\perp_\rho. \tag{E40}$$

The latter follows immediately from Eq. (E38), noting that $\Lambda^\dagger P^\perp_\rho = 0$. It suffices, therefore, to find a Hermitian matrix S' such that the first equality in Eq. (E40) is satisfied.

Let $\Lambda^{-1} = W^\dagger\rho^{-1/2}$ noting that $\Lambda^{-1}\Lambda = \tilde P_\rho$ and $\Lambda\Lambda^{-1} = P_\rho$. Define

$$\begin{aligned}S' = \Lambda^{-1}\Big[\Lambda M^\dagger + i\Lambda W^\dagger L\Big](\Lambda^{-1})^\dagger + \tilde P^\perp_\rho M^\dagger(\Lambda^{-1})^\dagger \\ + \Lambda^{-1}M\tilde P^\perp_\rho.\end{aligned} \tag{E41}$$

First we show that S' is Hermitian by proving that the term in brackets in the first term above is, in fact, Hermitian. Using $\Lambda W^\dagger = \rho^{1/2}$ we can compute

$$\begin{aligned}&\Big[\Lambda M^\dagger + i\Lambda W^\dagger L\Big] - \Big[\Lambda M^\dagger + i\Lambda W^\dagger L\Big]^\dagger \\ &\quad = \big(\Lambda M^\dagger - M\Lambda^\dagger\big) + i\big(\rho^{1/2}L + L^\dagger\rho^{1/2}\big) \\ &\quad = -iD + iD = 0,\end{aligned} \tag{E42}$$

using properties of L noted above and using Eq. (E31a). Therefore, S' is Hermitian. Then

$$\begin{aligned}iW(M^\dagger - S'\Lambda^\dagger)P_\rho &= iWM^\dagger P_\rho - iW\tilde P_\rho M^\dagger P_\rho \\ &+ P_\rho LP_\rho - iW\tilde P^\perp_\rho M^\dagger P_\rho = LP_\rho,\end{aligned} \tag{E43}$$

noting that $\tilde P^\perp_\rho\Lambda^\dagger = 0$, $(\Lambda^{-1})^\dagger\Lambda^\dagger = (\Lambda\Lambda^{-1})^\dagger = P_\rho$, and recalling that $P_\rho L = L$. With this choice of S', the first equality in Eq. (E40) is thus also satisfied, thereby completing the proof. ■

3. Additional equivalent conditions for zero sensitivity loss

The following theorem provides additional conditions under which zero sensitivity loss is achieved (see Sec. VII), leading to an explicit form of Bob's optimal sensing observable whenever these conditions are satisfied.

Theorem 7.—we use the notation of Appendix E 2. Suppose that the conditions for our uncertainty relation equality (Proposition 20) hold. Then the following statements are equivalent:

(i) We have $F\big(\mathcal{N}(\psi)\,;\mathcal{N}(D_A^Y)\big) = 4\langle\xi|\mathcal{N}^\dagger(\mathbb{1})|\xi\rangle$.
(ii) We have $\operatorname{tr}\big(E_{k'}^\dagger E_k D_A^Z\big) = 0$ for all k, k', where $\{E_k\}$ is any set of Kraus operators for $\mathcal{N}$.
(iii) We have $\widehat{\mathcal{N}}(D_A^Z) = 0$.
(iv) We have $\Lambda^\dagger M + M^\dagger\Lambda = 0$.
(v) The operator $i\rho^{1/2}MW^\dagger$ is Hermitian.
(vi) The operator $i\rho\mathcal{N}(|\xi\rangle\langle\psi|)$ is Hermitian and $\mathcal{N}(|\xi\rangle\langle\xi|) = \mathcal{N}(|\xi\rangle\langle\psi|)\rho^{-1}\mathcal{N}(|\psi\rangle\langle\xi|)$.
(vii) The operator $i\rho\mathcal{N}(|\xi\rangle\langle\psi|)$ is Hermitian and $\langle\xi|\mathcal{N}^\dagger(\mathbb{1})|\xi\rangle = \operatorname{tr}\big[\mathcal{N}(|\xi\rangle\langle\psi|)\rho^{-1}\mathcal{N}(|\psi\rangle\langle\xi|)\big]$.

Furthermore, if these conditions are satisfied then

$$\mathcal{R}_{\rho_B}^{-1}\big(\mathcal{N}(D_A^Y)\big) = -2i\mathcal{N}(|\xi\rangle\langle\psi|)\rho^{-1} + 2i\rho^{-1}\mathcal{N}(|\psi\rangle\langle\xi|)P_\rho^\perp. \tag{E44}$$

Proof.—The proof of (i)⇔(ii)⇔(iii) is presented in the main text (Sec. VII).

(iii) ⇒ *(iv)* : write $0 = \widehat{\mathcal{N}}(D_A^Z) = \operatorname{tr}_B\big\{\Phi_{B:E}\big[\Lambda^\dagger M + M^\dagger\Lambda\big]\big\}$. Observe that $\operatorname{tr}_B\big\{\Phi_{B:E}\,(\cdot)\big\}$ is the partial transpose map with respect to the bases used to define $\Phi_{B:E}$; therefore, $\Lambda^\dagger M + M^\dagger\Lambda = 0$.

(iv) ⇔ *(v)* : we compute

$$i\rho^{1/2}MW^\dagger - (i\rho^{1/2}MW^\dagger)^\dagger = iW\big(\Lambda^\dagger M + M^\dagger\Lambda\big)W^\dagger, \tag{E45}$$

which vanishes thanks to the assumption that (iv) holds. Conversely, because W is unitary we may only have Eq. (E45) = 0 if $\Lambda^\dagger M + M^\dagger\Lambda = 0$.

(iv) ⇒ *(vi)* : recall that $\rho = \Lambda\Lambda^\dagger$ and $\mathcal{N}(|\xi\rangle\langle\psi|) = \operatorname{tr}_E\big(V|\xi\rangle\langle\psi|V^\dagger\big) = M\Lambda^\dagger$. Then $i\rho\mathcal{N}(|\xi\rangle\langle\psi|) = i\Lambda\Lambda^\dagger M\Lambda^\dagger$. To check that $i\rho\mathcal{N}(|\xi\rangle\langle\psi|)$ is Hermitian we compute

$$\begin{aligned} &i\rho\mathcal{N}(|\xi\rangle\langle\psi|) - (i\rho\mathcal{N}(|\xi\rangle\langle\psi|))^\dagger \\ &\quad = i\Lambda\big(\Lambda^\dagger M + M^\dagger\Lambda\big)\Lambda^\dagger = 0, \end{aligned} \tag{E46}$$

using our assumption that (iv) holds. Furthermore, we have

$$0 = \rho^{-1/2}W\big(\Lambda^\dagger M + M^\dagger\Lambda\big)W^\dagger P_\rho^\perp = P_\rho MW^\dagger P_\rho^\perp; \tag{E47}$$

recalling point (ii) of Proposition 20, we find that

$$MW^\dagger P_\rho^\perp = 0. \tag{E48}$$

Then

$$\begin{aligned} \mathcal{N}(|\xi\rangle\langle\xi|) &= MM^\dagger = MW^\dagger(P_\rho + P_\rho^\perp)WM^\dagger \\ &= \mathcal{N}(|\xi\rangle\langle\psi|)\rho^{-1}\mathcal{N}(|\psi\rangle\langle\xi|). \end{aligned} \tag{E49}$$

(vi) ⇒ *(vii)* : this implication follows immediately from $\langle\xi|\mathcal{N}^\dagger(\mathbb{1})|\xi\rangle = \operatorname{tr}(\mathcal{N}(|\xi\rangle\langle\xi|))$.

(vii) ⇒ *(i)* : our proof strategy for this implication is to show that the expression of the symmetric logarithmic derivative in Eq. (E44) is correct, and that the corresponding Fisher information at Bob's end has no sensitivity loss. Let

$$R = -2i\mathcal{N}(|\xi\rangle\langle\psi|)\rho^{-1} + 2i\rho^{-1}\mathcal{N}(|\psi\rangle\langle\xi|)P_\rho^\perp. \tag{E50}$$

We can see that R is Hermitian by writing

$$\begin{aligned} R &= -2i\,(P_\rho + P_\rho^\perp)\mathcal{N}(|\xi\rangle\langle\psi|)\rho^{-1} + 2i\rho^{-1}\mathcal{N}(|\psi\rangle\langle\xi|)P_\rho^\perp \\ &= -2i\rho^{-1}\big[\rho\mathcal{N}(|\xi\rangle\langle\psi|)\big]\rho^{-1} + \big(-2iP_\rho^\perp\mathcal{N}(|\xi\rangle\langle\psi|)\rho^{-1} \\ &\quad + \text{h.c.}\big). \end{aligned} \tag{E51}$$

The first term is Hermitian by assumption and the second term is manifestly Hermitian. We note for convenience that $RP_\rho = -2i\mathcal{N}(|\xi\rangle\langle\psi|)\rho^{-1}$ and $P_\rho R = 2i\rho^{-1}\mathcal{N}(|\psi\rangle\langle\xi|)$. We can compute

$$\frac{1}{2}(\rho R + R\rho) = i\mathcal{N}(|\psi\rangle\langle\xi|) - i\mathcal{N}(|\xi\rangle\langle\psi|) = \mathcal{N}(D_A^Y). \tag{E52}$$

Combining with the fact that $P_\rho^\perp R P_\rho^\perp = 0$ we have that $\mathcal{R}_\rho^{-1}(\mathcal{N}(D_Y)) = R$ (see also Proposition 3), thus proving Eq. (E44). The Fisher information at the output of the mapping $\mathcal{N}$ is therefore

$$\begin{aligned} F\big(\mathcal{N}(\psi)\,;\mathcal{N}(D_A^Y)\big) &= \operatorname{tr}(\rho R^2) \\ &= \operatorname{tr}\big[\rho\,\big(2i\rho^{-1}\mathcal{N}(|\psi\rangle\langle\xi|)\big)\big(-2i\mathcal{N}(|\xi\rangle\langle\psi|)\rho^{-1}\big)\big] \\ &= 4\operatorname{tr}\big(\mathcal{N}(|\xi\rangle\langle\psi|)\rho^{-1}\mathcal{N}(|\psi\rangle\langle\xi|)\big) \\ &= 4\langle\xi|\mathcal{N}^\dagger(\mathbb{1})|\xi\rangle. \end{aligned} \tag{E53}$$

We conclude that (i) holds. ■

4. Proof of the generalized bipartite Fisher information uncertainty relation for any two parameters

In this Appendix, we prove the generalized uncertainty relation (68) that applies to any two parameters generated by unitary evolutions.

Proposition 21 (Uncertainty relation for any two parameters with associated generators).—let $|\psi\rangle$ be a state vector on Alice's system, and let A, B be two Hermitian operators. The latter generate two respective parametrized evolutions

$$\partial_a\psi = -i[A,\psi]; \quad \partial_b\psi = -i[B,\psi]. \tag{E54}$$

Consider the setting depicted in Fig. 1, where $\mathcal{N}$ can be any completely positive, trace-non-increasing map. Then

$$\frac{F_{\text{Bob},a}}{F_{\text{Alice},a}} + \frac{F_{\text{Eve},b}}{F_{\text{Alice},b}} \leqslant 1 + 2\sqrt{1 - \frac{\langle i[A,B]\rangle^2}{4\,\sigma_A^2\sigma_B^2}}. \tag{E55}$$

Furthermore, assume that $\mathcal{N}[\psi(a)]$ does not change rank locally as a function of a and that there exists $\beta \in \mathbb{R}, \beta \neq 0$ such that

$$\widehat{\mathcal{N}}\left(-i\left[\frac{B}{\sigma_B},\ \psi\right]\right) = \beta\,\widehat{\mathcal{N}}\left(\left\{\frac{A-\langle A\rangle}{\sigma_A},\ \psi\right\}\right). \tag{E56}$$

Then

$$\frac{F_{\text{Bob},a}}{F_{\text{Alice},a}} + \frac{1}{\beta^2}\frac{F_{\text{Eve},b}}{F_{\text{Alice},b}} = 1. \tag{E57}$$

Corollary 3 (Uncertainty relation for any two parameters).—let $|\psi(a,b)\rangle$ be any state vector depending on parameters a, b. Then

$$\frac{F_{\text{Bob},a}}{F_{\text{Alice},a}} + \frac{F_{\text{Eve},b}}{F_{\text{Alice},b}} \leqslant 1 + 2\sqrt{1 - \frac{\langle i[\partial_a\psi,\partial_b\psi]\rangle^2}{4\langle(\partial_a\psi)^2\rangle\langle(\partial_b\psi)^2\rangle}}. \tag{E58}$$

We first prove the following lemma.

Lemma 2.—let $|\psi\rangle$ be any state vector and let $\mathcal{M}$ be any completely positive, trace-non-increasing map. Consider two Hermitian operators C, B generating respective evolutions

$$\partial_c\psi = -i[C,\psi], \quad \partial_b\psi = -i[B,\psi]. \tag{E59}$$

We write $\rho = \mathcal{M}(\psi)$, $\partial_c\rho = \mathcal{M}(-i[C,\psi])$ and $\partial_b\rho = \mathcal{M}(-i[B,\psi])$. Then for any $x, y > 0$,

$$\frac{y}{\sigma_B^2}F(\rho\,;\partial_b\rho) \leqslant \frac{x}{\sigma_C^2}F(\rho\,;\partial_c\rho) + 4(x+y)\sqrt{1 - \frac{xy}{(x+y)^2}\frac{4\left[\text{Re}\langle\bar{C}\bar{B}\rangle\right]^2}{\sigma_C^2\sigma_B^2}}, \tag{E60}$$

where $\bar{C} = C - \langle C\rangle\mathbb{1}$ and $\bar{B} = B - \langle B\rangle\mathbb{1}$. In addition, suppose that C can be written as $C = i\alpha[A,\psi]$ for some Hermitian operator A and some $\alpha \in \mathbb{R}$. Then the above inequality takes the form

$$\frac{y}{\sigma_B^2}F(\rho\,;\partial_b\rho) \leqslant \frac{x}{\alpha^2\sigma_A^2}F(\rho\,;\partial_c\rho) + 4(x+y)\sqrt{1 - \frac{xy}{(x+y)^2}\frac{\langle i[A,B]\rangle^2}{\sigma_A^2\sigma_B^2}}. \tag{E61}$$

Furthermore, let $x, y > 0$. If there exists $s \in \{+1,-1\}$ such that

$$\mathcal{M}\left(-i\left[\frac{\sqrt{y}B}{\sigma_B} + s\frac{\sqrt{x}C}{\sigma_C},\psi\right]\right) = 0, \tag{E62}$$

then

$$\frac{y}{\sigma_B^2}F(\rho\,;\partial_b\rho) = \frac{x}{\sigma_C^2}F(\rho\,;\partial_c\rho). \tag{E63}$$

Proof of Lemma 2.—For any $x, y > 0$, define the shorthands

$$\tilde{C} = \frac{\sqrt{x}}{\sigma_C}\left(C - \langle C\rangle\right), \quad \tilde{B} = \frac{\sqrt{y}}{\sigma_B}\left(B - \langle B\rangle\right). \tag{E64}$$

Observe that $\sigma_{\tilde{C}}^2 = x$ and $\sigma_{\tilde{B}}^2 = y$. Furthermore, we define for convenience $D_{(\cdot)} = \mathcal{M}(-i[(\cdot),\psi])$, observing that $D_C = \partial_c\rho$ and $D_B = \partial_b\rho$. Then using Proposition 8 we see that

$$F(\rho\,;D_{\tilde{C}}) = \frac{x}{\sigma_C^2}F(\rho\,;D_C); \quad F(\rho\,;D_{\tilde{B}}) = \frac{y}{\sigma_B^2}F(\rho\,;D_B). \tag{E65}$$

Invoking Proposition 16,

$$F(\rho\,;D_{\tilde{B}}) \leqslant F(\rho\,;D_{\tilde{C}}) + \left[F(\rho\,;\Delta_+)F(\rho\,;\Delta_-)\right]^{1/2}, \tag{E66}$$

where $\Delta_\pm = D_{\tilde{C}} \pm D_{\tilde{B}} = D_{\tilde{C}\pm\tilde{B}}$. We proceed to compute the second term on the right-hand side of this inequality. The data-processing inequality (Proposition 13), along with Proposition 12, gives us

$$F(\rho\,;\Delta_\pm) \leqslant F(\psi\,;-i[\tilde{C}\pm\tilde{B},\psi]) = 4\text{Var}_\psi(\tilde{C}\pm\tilde{B}), \tag{E67}$$

where we write $\text{Var}_\rho(X) = \langle X^2\rangle_\rho - (\langle X\rangle_\rho)^2$. We find

$$\begin{aligned}4\text{Var}_\psi(\tilde{C}\pm\tilde{B}) &= 4\,\langle(\tilde{C}\pm\tilde{B})^2\rangle = 4\,\langle\tilde{C}^2 + \tilde{B}^2 \pm \{\tilde{C},\tilde{B}\}\rangle \\ &= 4(x+y) \pm 8\,\text{Re}\,\langle\tilde{C}\tilde{B}\rangle.\end{aligned} \tag{E68}$$

Then

$$4^2\mathrm{Var}_\psi\big(\tilde{C}+\tilde{B}\big)\mathrm{Var}_\psi\big(\tilde{C}-\tilde{B}\big) = 4^2\big(x+y\big)^2 - 8^2\frac{xy}{\sigma_C^2\sigma_B^2}\big[\mathrm{Re}\,\langle\bar{C}\bar{B}\rangle\big]^2, \tag{E69}$$

where $\bar{C} = C - \langle C\rangle$ and $\bar{B} = B - \langle B\rangle$. Combining the above,

$$\Big[F\big(\rho\,;\Delta_+\big)F\big(\rho\,;\Delta_-\big)\Big]^{1/2} \leqslant 4(x+y)\sqrt{1-\frac{xy}{(x+y)^2}\,\frac{4\big[\mathrm{Re}\,\langle\bar{C}\bar{B}\rangle\big]^2}{\sigma_C^2\sigma_B^2}}. \tag{E70}$$

Plugging this expression back into Eq. (E66), along with Eq. (E65), proves Eq. (E60). Now suppose that $C = i\alpha[A,\psi]$ for some Hermitian operator A and for a real number α. Then $\langle C\rangle = 0$ so $\bar{C} = C$ and

$$\begin{aligned}\mathrm{Re}\,\langle\bar{C}\bar{B}\rangle &= \alpha\,\mathrm{Re}\big\langle i[A,\psi]\,\bar{B}\big\rangle = \alpha\,\mathrm{Re}\,\big(\langle\psi|iH|\psi\rangle\langle\psi|\bar{B}|\psi\rangle \\ &\quad - \langle\psi|iA\bar{B}|\psi\rangle\big) \\ &= \alpha\,\mathrm{Re}\big(-i\langle A\bar{B}\rangle\big) = \frac{\alpha}{2}\big(-i\langle A\bar{B}\rangle + i\langle\bar{B}A\rangle\big) \\ &= \frac{\alpha}{2}\big\langle i[A,\bar{B}]\big\rangle = \frac{\alpha}{2}\big\langle i[A,B]\big\rangle.\end{aligned} \tag{E71}$$

Equation (E61) follows from this and using the fact that $\sigma_C^2 = \langle\bar{C}^2\rangle = \alpha^2\langle-(A\psi-\psi A)^2\rangle = \alpha^2\big(\langle A^2\rangle - \langle A\rangle^2\big) = \alpha^2\sigma_A^2$.

Now assume that Eq. (E62) is satisfied. Recalling that $\Delta_\pm = D_{\tilde{C}} \pm D_{\tilde{B}} = \mathcal{M}\big(-i[\tilde{C}\pm\tilde{B},\psi]\big)$, we find that condition (E62) immediately implies that either $\Delta_+ = 0$ or $\Delta_- = 0$ and therefore either $F\big(\rho\,;\Delta_+\big) = 0$ or $F\big(\rho\,;\Delta_-\big) = 0$. In this case, Proposition 16 immediately implies that $F\big(\rho\,;D_{\tilde{C}}\big) = F\big(\rho\,;D_{\tilde{B}}\big)$. We conclude that Eq. (E63) holds, recalling Eq,. (E65). ■

Proof of Proposition 21.—Consider the evolution $\psi(a,c)$, where the parameter a is generated by the first given Hermitian operator A and where the parameter c is generated by the complementary generator C (as per Fig. 5 in the main text) given by

$$\partial_c\psi = i[C,\psi], \quad C = \frac{1}{2\mathrm{Var}_\psi(A)}\big(-i[A,\psi]\big). \tag{E72}$$

Recall $F_{\mathrm{Alice},c} = 4\sigma_C^2 = \sigma_A^{-2}$ from Eq. (27) with $H\to A$ and $T\to C$. Our time-energy uncertainty relation, in its form of Theorem 6, asserts that

$$\frac{1}{4\sigma_A^2}F_{\mathrm{Bob},a} + \sigma_A^2 F_{\mathrm{Eve},c} \leqslant 1. \tag{E73}$$

Now we invoke Lemma 2, with $\mathcal{M} = \widehat{\mathcal{N}}$, c, b, $C = i\alpha[A,\psi]$, $\alpha = -(2\sigma_A^2)^{-1}$, B, and $x = y = 1/4$. From Eq. (E61) we find

$$\frac{1}{4\sigma_B^2}F_{\mathrm{Eve},b} \leqslant \sigma_A^2 F_{\mathrm{Eve},c} + 2\sqrt{1-\frac{\langle i[A,B]\rangle^2}{4\sigma_A^2\sigma_B^2}}. \tag{E74}$$

We find, applying Eqs. (E73) and (E74) in succession,

$$\begin{aligned}\frac{1}{4\sigma_A^2}F_{\mathrm{Bob},a} + \frac{1}{4\sigma_B^2}F_{\mathrm{Eve},b} &\leqslant 1 - \sigma_A^2F_{\mathrm{Eve},c} + \frac{1}{4\sigma_B^2}F_{\mathrm{Eve},b} \\ &\leqslant 1 + 2\sqrt{1-\frac{\langle i[A,B]\rangle^2}{4\sigma_A^2\sigma_B^2}}.\end{aligned} \tag{E75}$$

This shows the desired uncertainty relation.

Now assume that $\mathcal{N}[\psi]$ does not change rank locally as a function of a and that Eq. (E56) holds. Let $\mathcal{M} = \widehat{\mathcal{N}}$, $\rho_E = \widehat{\mathcal{N}}[\psi]$, $C = i\alpha[A,\psi]$, and $\alpha = -(2\sigma_A^2)^{-1}$. Then as computed above $\sigma_C = |\alpha|\sigma_A = 1/(2\sigma_A)$. Let us compute now

$$\begin{aligned}-i\left[\sqrt{x}\frac{C}{\sigma_C},\psi\right] &= -i\left[\sqrt{x}\frac{-i[A,\psi]}{\sigma_A},\psi\right] \\ &= -\sqrt{x}\left\{\frac{A-\langle A\rangle}{\sigma_A},\psi\right\},\end{aligned} \tag{E76}$$

recalling that $\big[[A,\psi],\psi\big] = \{A-\langle A\rangle,\psi\}$. Let $y=1$, $x = |\beta|^2$ and $s = \mathrm{sign}(\beta)$ such that $s\sqrt{x}/\sqrt{y} = \beta$. We then have

$$\begin{aligned}&\mathcal{M}\left(-i\left[\sqrt{y}\frac{B}{\sigma_B} + s\sqrt{x}\frac{C}{\sigma_C},\ \psi\right]\right) \\ &= \widehat{\mathcal{N}}\left(-i\left[\sqrt{y}\frac{B}{\sigma_B},\psi\right]\right) + s\widehat{\mathcal{N}}\left(-i\left[\sqrt{x}\frac{C}{\sigma_C},\psi\right]\right) \\ &= \frac{\sqrt{y}}{2}\left\{\widehat{\mathcal{N}}\left(-i\left[\frac{B}{\sigma_B},\psi\right]\right) - \beta\widehat{\mathcal{N}}\left(\left\{\frac{A-\langle A\rangle}{\sigma_A},\psi\right\}\right)\right\} \\ &= 0.\end{aligned} \tag{E77}$$

The latter expression then vanishes thanks to our assumption that Eq. (E56) holds. Thanks to Lemma 2 we find

$$\begin{aligned}\frac{F_{\mathrm{Eve},b}}{F_{\mathrm{Alice},b}} &= \frac{1}{4\sigma_B^2}F\big(\rho_E\,;\partial_b\rho_E\big) = \frac{x}{4\sigma_C^2}F\big(\rho_E\,;\partial_c\rho_E\big) \\ &= \beta^2\frac{F_{\mathrm{Eve},c}}{F_{\mathrm{Alice},c}}.\end{aligned} \tag{E78}$$

Thanks to our assumption that $\mathcal{N}[\psi]$ does not change rank locally as a function of a, we know that our main uncertainty relation (Theorem 6) holds with equality:

$$\frac{1}{4\sigma_A^2} F_{\text{Bob},a} + \sigma_A^2 F_{\text{Eve},c} = 1. \tag{E79}$$

We therefore find, recalling $\sigma_A^2 = 1/(4\sigma_C^2)$,

$$\frac{F_{\text{Bob},a}}{F_{\text{Alice},a}} + \frac{1}{\beta^2}\frac{F_{\text{Eve},b}}{F_{\text{Alice},b}} = \left[1 - \sigma_A^2 F_{\text{Eve},c}\right] + \frac{1}{\beta^2}\frac{F_{\text{Eve},b}}{4\sigma_B^2}$$
$$= 1 - \frac{1}{\beta^2}\frac{F_{\text{Eve},c}}{4\sigma_C^2} + \frac{1}{\beta^2}\frac{F_{\text{Eve},b}}{4\sigma_B^2} = 1, \tag{E80}$$

using Eq. (E78), thus proving the claim. ■

Proof of Corollary 3.—The main idea of this corollary is to note that the Fisher information depends only on the state and its first derivative with respect to the parameter, and that any derivative $\partial_a\psi$ can be written in the form $\partial_a\psi = -i[A,\psi]$ for some Hermitian generator A. Therefore, we seek Hermitian operators A, B such that $\partial_a\psi = -i[A,\psi]$ and $\partial_b\psi = -i[B,\psi]$, such that we can apply Proposition 21. We let $A = i\left[\partial_a\psi,\psi\right]$ and $B = i\left[\partial_b\psi,\psi\right]$, and we compute

$$-i[A,\psi] = -i\left[i[\partial_a\psi,\psi],\psi\right] = \left\{\partial_a\psi,\psi\right\}$$
$$= \partial_a(\psi^2) = \partial_a\psi, \tag{E81}$$

using Eq. (A4) and the fact that $\langle\partial_a\psi\rangle = \text{tr}[\partial_a\psi] = \partial_a\,\text{tr}(\psi) = 0$. Similarly,

$$-i[B,\psi] = \partial_b\psi. \tag{E82}$$

We can therefore apply Proposition 21. It remains to compute the quantities appearing in the right-hand side of Eq. (E55). We have

$$\langle i[A,B]\rangle = i\,\text{tr}\left\{\psi\left[i[\partial_a\psi,\psi], i[\partial_b\psi,\psi]\right]\right\}$$
$$= i\,\text{tr}\left\{\left[\psi\,,\, i[\partial_a\psi,\psi]\right]\left(i[\partial_b\psi,\psi]\right)\right\}$$
$$= \text{tr}\left\{-i\left[i[\partial_a\psi,\psi]\,,\,\psi\right]\left(i[\partial_b\psi,\psi]\right)\right\}$$
$$= \text{tr}\left\{(\partial_a\psi)\left(i[\partial_b\psi,\psi]\right)\right\}$$
$$= \left\langle i[\partial_a\psi,\partial_b\psi]\right\rangle, \tag{E83}$$

using the cyclicity of the trace and invoking Eq. (E81) for the fourth equality. Furthermore,

$$\sigma_A^2 = \langle A^2\rangle - \langle A\rangle^2 = \left\langle (i[\partial_a\psi,\psi])^2\right\rangle - \left\langle i[\partial_a\psi,\psi]\right\rangle^2$$
$$= -\left\langle\left((\partial_a\psi)\,\psi - \psi\,(\partial_a\psi)\right)\left((\partial_a\psi)\,\psi - \psi\,(\partial_a\psi)\right)\right\rangle$$
$$= \left\langle\psi\,(\partial_a\psi)\,(\partial_a\psi)\,\psi\right\rangle = \left\langle(\partial_a\psi)^2\right\rangle, \tag{E84}$$

where we have made use of $\psi\,(\partial_a\psi)\,\psi = 0$. Similarly $\sigma_B^2 = \left\langle(\partial_b\psi)^2\right\rangle$, which ends the proof. ■

APPENDIX F: GENERALIZATIONS TO INFINITE-DIMENSIONAL HILBERT SPACES

While the main text has put an emphasis on discussing notions of quantum metrology making use of finite-dimensional quantum systems, in this section, we generalize the above findings to the setting of infinite-dimensional Hilbert spaces. A specific attention is given to unbounded operators, as many physical systems of practical use fall under this category.

1. Uncertainty relation for any two parameters

We start with a generalisation of Theorem 3 to infinite dimensions (cf. Proposition 21).

Theorem 8 (Uncertainty relation for infinite-dimensional systems).—let $\mathtt{A}, \mathtt{B}$ be two self-adjoint operators (possibly unbounded) on a separable Hilbert space $\mathscr{H}_A$ with domains $\mathcal{D}(\mathtt{A})$ and $\mathcal{D}(\mathtt{B})$, respectively. Let $|\psi\rangle \in \mathcal{D}(\mathtt{A}) \cap \mathcal{D}(\mathtt{B})$ and $|\psi(a)\rangle \in \mathcal{D}(\mathtt{A})$, $|\psi(b)\rangle \in \mathcal{D}(\mathtt{B})$ for some $b, a \in \mathbb{R}$ where $|\psi(a)\rangle := e^{-ia\mathtt{A}}|\psi\rangle$, $|\psi(b)\rangle := e^{-ib\mathtt{B}}|\psi\rangle$. Let $V_{A\to BE}$ be any isometry $\mathscr{H}_A \to \mathscr{H}_B \otimes \mathscr{H}_E$, where the Hilbert spaces $\mathscr{H}_B, \mathscr{H}_E$ associated with Bob and Eve are also separable and possibly of infinite dimensions. Consider the two pure state evolutions given by Eq. (E54). Then Eq. (E55) holds, with the following quantities defined by

$$\left\langle i[\mathtt{A},\mathtt{B}]\right\rangle := i\left\langle\mathtt{A}\psi,\mathtt{B}\psi\right\rangle - i\left\langle\mathtt{B}\psi,\mathtt{A}\psi\right\rangle, \tag{F1}$$
$$\left\langle\mathtt{A}^2\right\rangle := \left\langle\mathtt{A}\psi,\mathtt{A}\psi\right\rangle \tag{F2}$$

and

$$F_M(y) := \liminf_{l\to\infty}\text{tr}\left[\rho_M^{(l)}(y)R^2\right] \in \mathbb{R}, \tag{F3}$$

where $M \in \{B, E\}$, $y \in \{a, b\}$ and $\rho_X^{(l)}$ is an l-dimensional subnormalised density operator and $R = R(l)$ is defined in Eq. (11) on an l-dimensional Hilbert space for $\rho_M^{(l)}$. Specifically,

$$\rho_M^{(l)}(y) := \text{tr}_{\backslash M}\left[P_{BE}^{(l)} V_{A\to BE}\rho_A(y)V_{A\to BE}^\dagger P_{BE}^{(l)}\right], \tag{F4}$$

where $\backslash E := B$, $\backslash B := E$, and $P_{BE}^{(l)}$ is the orthogonal projection onto the first l basis elements of a basis for $\mathscr{H}_B \otimes \mathscr{H}_E$. Furthermore, the derivative of $\rho_M^{(l)}(y)$ is defined via

$$\frac{d}{dy}\rho_M^{(l)}(y) := \text{tr}_{\backslash M}\left[P_{BE}^{(l)} V_{A\to BE}\frac{d}{dy}\rho_A(y)V_{A\to BE}^\dagger P_{BE}^{(l)}\right], \tag{F5}$$

where

$$\frac{d}{da}\rho_A(a) := i|\psi(a)\rangle\langle\psi(a)|\mathtt{A} - i\mathtt{A}|\psi(a)\rangle\langle\psi(a)|, \tag{F6}$$
$$\frac{d}{db}\rho_A(b) := i|\psi(b)\rangle\langle\psi(b)|\mathtt{B} - i\mathtt{B}|\psi(b)\rangle\langle\psi(b)|. \tag{F7}$$

Proof of Theorem 8.—The proof will proceed in two steps. First we will approximate B and A by bounded operators (if they are already bounded, then this first step is not necessary, although the approximation will nevertheless be well defined). Second, we will approximate these bounded operators by finite dimensional operators. Then we will apply Eq. (E55) before taking a sequence of limits in which the approximations vanish. We start with a few elementary definitions and results, which will be necessary for our proof.

Let A, $(A_n)_n$, be bounded operators on a Hilbert space $\mathscr{H}$. We define all bounded operators we consider to have domain equal to the entire Hilbert space. We say that A_n converges (as $n \to \infty$) to A in the strong limit if $A_n\Psi \to A\Psi$ as $n \to \infty$ for any $\Psi \in \mathscr{H}$. We denote this as $A_n \xrightarrow{s} A$. Some properties are the following.

(i) Let A, $(A_n)_n$, B, $(B_n)_n$, C, $(C_n)_n$, be bounded operators on a Hilbert space $\mathscr{H}$. $A_n \xrightarrow{s} A$, $B_n \xrightarrow{s} B$ and $C_n \xrightarrow{s} C$ imply $A_nB_n \xrightarrow{s} AB$ and $A_nB_nC_n \xrightarrow{s} ABC$.
Proof. $(A_nB_n - AB)\Psi = A_n(B_n - B)\Psi + (A_n - A)B\Psi$. By the uniform boundedness principle, $A_n \xrightarrow{s} A$ implies $\|A_n\| \leqslant c$ for some $c \in \mathbb{R}$ for all n. Therefore,

$$\|(A_nB_n - AB)\Psi\| \leqslant c\|(B_n - B)\Psi\| + \|(A_n - A)B\Psi\|, \tag{F8}$$

where the rhs tends to zero as $n \to \infty$. This proves the first claim. For the second, simply define $\bar{A}_n := A_nB_n$. Hence $\bar{A}_n \xrightarrow{s} AB$ and thus $\bar{A}_nC_n \xrightarrow{s} (AB)C$, hence proving the second claim.

(ii) $A_n \xrightarrow{s} A$ implies $e^{-iA_nt} \xrightarrow{s} e^{-iAt}$ for $t \in \mathbb{R}$.
Proof. $e^{-iA_nt} - e^{-iAt} = e^{-iA_ns}e^{-iA(t-s)}\Big|_{s=0}^{s=t} = -i\int_0^t ds\, e^{-iA_ns}(A_n - A)e^{-iA(t-s)}$. But we have $(A_n - A)e^{-iA(t-s)} \xrightarrow{s} \bar{0}$ pointwise in s, where $\bar{0}$ is the bounded operator mapping all vectors in $\mathscr{H}$ to the zero vector in $\mathscr{H}$. Thus via (i), $e^{-iAs}(A_n - A)e^{-iA(t-s)} \xrightarrow{s} \bar{0}$ pointwise in s and the result follows by dominated convergence.

(iii) Let A be self-adjoint and possibly unbounded. Let f, $(f_n)_n : \mathbb{R} \to \mathbb{C}$ be uniformly bounded functions with $f_n \to f$ as $n \to \infty$ point-wise. Then $f_n(A) \xrightarrow{s} f(A)$.
Proof. See Ref. [109].

We can now prove the theorem. Let $(P_N^{(n)})_n$ be the orthogonal projections onto the span of the first n basis elements of a separable Hilbert space $\mathscr{H}_N$. Consider two bounded operators $\tilde{\mathrm{A}}$ and $\tilde{\mathrm{B}}$ on $\mathscr{H}_A$ and define $\tilde{\mathrm{A}}_n$, $\tilde{\mathrm{B}}_n$ by

$$\tilde{\mathrm{A}}_n := P_A^{(n)}\tilde{\mathrm{A}}P_A^{(n)}, \quad \tilde{\mathrm{B}}_n := P_A^{(n)}\tilde{\mathrm{B}}P_A^{(n)}. \tag{F9}$$

Furthermore, consider the sequence of states $(\rho_B^{(n,l)})_{n,l}$ on $\mathscr{H}_B$, and $(\rho_E^{(n,l)})_{n,l}$ on $\mathscr{H}_E$, where

$$\begin{aligned}\rho_B^{(n,l)}(a) &:= \mathrm{tr}_E\left[P_{BE}^{(l)}V_{A\to BE}P_A^{(n)}\left(|\psi_n(a)\rangle\langle\psi_n(a)|\right)P_A^{(n)}V_{A\to BE}^{\dagger}P_{BE}^{(l)}\right],\\ \rho_E^{(n,l)}(b) &:= \mathrm{tr}_B\left[P_{BE}^{(l)}V_{A\to BE}P_A^{(n)}\left(|\psi_n(b)\rangle\langle\psi_n(b)|\right)P_A^{(n)}V_{A\to BE}^{\dagger}P_{BE}^{(l)}\right],\end{aligned} \tag{F10}$$

where $|\psi_n(a)\rangle := e^{-ia\tilde{\mathrm{A}}_n}|\psi\rangle$, $|\psi_n(b)\rangle := e^{-ib\tilde{\mathrm{B}}_n}|\psi\rangle$ and the sequences of derivatives, $(\frac{d}{da}\rho_B^{(n,l)}(t))_{n,l}$ on $\mathscr{H}_B$, and $(\frac{d}{da}\rho_E^{(n,l)}(a))_{n,l}$ on $\mathscr{H}_E$ are

$$\begin{aligned}\frac{d}{da}\rho_B^{(n,l)}(a) &= \mathrm{tr}_E\left[P_{BE}^{(l)}V_{A\to BE}P_A^{(n)}\left(i|\psi_n(a)\rangle\langle\psi_n(a)|\tilde{\mathrm{A}}_n - i\tilde{\mathrm{A}}_n|\psi_n(a)\rangle\langle\psi_n(a)|\right)P_A^{(n)}V_{A\to BE}^{\dagger}P_{BE}^{(l)}\right],\\ \frac{d}{db}\rho_E^{(n,l)}(b) &= \mathrm{tr}_B\left[P_{BE}^{(l)}V_{A\to BE}P_A^{(n)}\left(i|\psi_n(b)\rangle\langle\psi_n(b)|\tilde{\mathrm{B}}_n - i\tilde{\mathrm{B}}_n|\psi_n(b)\rangle\langle\psi_n(b)|\right)P_A^{(n)}V_{A\to BE}^{\dagger}P_{BE}^{(l)}\right].\end{aligned} \tag{F11}$$

We can use Eqs. (F10) and (F11) to construct the Fisher information for these states. Since $V_{A\to BE}V_{A\to BE}^{\dagger} = \mathbb{1}_{BE}$, where $\mathbb{1}_{BE}$ is the identity operator on $\mathscr{H}_{BE}$, it follows that

$$P_A^{(n)} - \left(P_{BE}^{(l)}V_{A\to BE}P_A^{(n)}\right)^{\dagger}\left(P_{BE}^{(l)}V_{A\to BE}P_A^{(n)}\right) \geqslant 0 \tag{F12}$$

for all l,n. Hence, by Kraus' theorem, Eq. (F10) are completely positive and trace nonincreasing maps evaluated on inputs $|\psi_n(a)\rangle\langle\psi_n(a)|$. Since Proposition 21 holds for any completely positive, trace nonincreasing map, we can apply it to our setup. This yields

$$\frac{F\left(\rho_B^{(n,l)}(a)\right)}{F\left(\rho_A^{(n)}(a)\right)} + \frac{F\left(\rho_E^{(n,l)}(b)\right)}{F\left(\rho_A^{(n)}(b)\right)} \leqslant 1 + 2\sqrt{1 - \frac{\langle i[\tilde{\mathrm{A}}_n, \tilde{\mathrm{B}}_n]\rangle^2}{4\,\tilde{\sigma}_{\tilde{\mathrm{A}},n}^2\tilde{\sigma}_{\tilde{\mathrm{B}},n}^2}}, \tag{F13}$$

recalling that the uncertainty relation also applies to subnormalized positive operators, and where

$$F\left(\rho_A^{(n)}(a)\right) = 4\,\tilde{\sigma}_{\mathrm{A},n}^2, \tag{F14}$$

$$F\left(\rho_A^{(n)}(b)\right) = 4\,\tilde{\sigma}_{\mathrm{B},n}^2, \tag{F15}$$

$$\tilde{\sigma}_{\mathrm{A},n} := \left(\langle \tilde{\mathrm{A}}_n\psi, \tilde{\mathrm{A}}_n\psi\rangle - \langle \psi, \tilde{\mathrm{A}}_n\psi\rangle^2\right)^{1/2}, \tag{F16}$$

$$\tilde{\sigma}_{\mathrm{B},n} := \left(\langle \tilde{\mathrm{B}}_n\psi, \tilde{\mathrm{B}}_n\psi\rangle - \langle \psi, \tilde{\mathrm{B}}_n\psi\rangle^2\right)^{1/2}. \tag{F17}$$

We can now take the limit $n \to \infty$ on both sides of Eq. (F13). Due to property (i), it follows

$$\frac{\lim_{n\to\infty} F\left(\rho_B^{(n,l)}(a)\right)}{F\left(\rho_A^{(\infty)}(a)\right)} + \frac{\lim_{n\to\infty} F\left(\rho_E^{(n,l)}(b)\right)}{F\left(\rho_A^{(\infty)}(b)\right)} \leqslant 1 + 2\sqrt{1 - \frac{\langle i[\tilde{\mathrm{A}}, \tilde{\mathrm{B}}]\rangle^2}{4\,\tilde{\sigma}_{\mathrm{A}}^2\tilde{\sigma}_{\mathrm{B}}^2}}, \tag{F18}$$

where

$$F\left(\rho_A^{(\infty)}(a)\right) := 4\,\tilde{\sigma}_{\mathrm{A}}^2, \tag{F19}$$

$$F\left(\rho_A^{(\infty)}(b)\right) := 4\,\tilde{\sigma}_{\mathrm{B}}^2, \tag{F20}$$

$$\tilde{\sigma}_{\mathrm{A}} := \left(\langle \tilde{\mathrm{A}}\psi, \tilde{\mathrm{A}}\psi\rangle - \langle \psi, \tilde{\mathrm{A}}\psi\rangle^2\right)^{1/2}, \tag{F21}$$

$$\tilde{\sigma}_{\mathrm{B}} := \left(\langle \tilde{\mathrm{B}}\psi, \tilde{\mathrm{B}}\psi\rangle - \langle \psi, \tilde{\mathrm{B}}\psi\rangle^2\right)^{1/2}. \tag{F22}$$

Observe that the quantities $\lim_{n\to\infty} F(\rho_B^{(n,l)}(a))$, $\lim_{n\to\infty} F(\rho_E^{(n,l)}(b))$ cannot diverge, since it would contradict the inequality (since the Fisher information is non-negative). This observation follows alternatively from applying the data processing inequality (110) to bound Bob's Fisher information in terms of Alice's, followed by talking the $n \to \infty$ limit. Similarly for Eve's Fisher information. By direct calculation, we observe that the Fisher information F of a state ρ on a d-dimensional Hilbert space, according to Eqs. (9) and (11), is given by

$$F = \sum_{\substack{k,k'=1 \\ \text{s.t. } p_k+p_{k'}>0}}^{d} \frac{p_k}{(p_k+p_{k'})^2} \left|\langle k|\frac{d\rho}{da}|k'\rangle\right|^2, \tag{F23}$$

where $\rho = \sum_{k=1}^d p_k |k\rangle\langle k|$. Hence

$$\lim_{n\to\infty} F\left(\rho_B^{(n,l)}(a)\right) = \lim_{n\to\infty} \sum_{\substack{k,k'=1 \\ \text{s.t. } p_k^{(n,l)}+p_{k'}^{(n,l)}>0}}^{d_B(l)} \frac{p_k^{(n,l)}}{(p_k^{(n,l)}+p_{k'}^{(n,l)})^2} \left|\langle k,n,l|\frac{d\rho_B^{(n,l)}(a)}{da}|k',n,l\rangle\right|^2, \tag{F24}$$

where $d_B(l)$ is the dimension of Bob's reduced system, which is l independent, and $\rho_B^{(n,l)}(a) = \sum_{k=1}^{d_B(l)} p_k^{(n,l)} |k,n,l\rangle\langle k,n,l|$. Observe that all terms in the summation must be finite in the limit, since they are all non-negative and we are guaranteed that the rhs of Eq. (F24) does not diverge. Observe that for terms in the summation for which $\lim_{n\to\infty} p_k^{(n,l)} + p_{k'}^{(n,l)} > 0$, the summation can be interchanged with the limit. However, while for terms such that $p_k^{(n,l)} + p_{k'}^{(n,l)} > 0$ for all n, but $\lim_{n\to\infty} p_k^{(n,l)} + p_{k'}^{(n,l)} = 0$, the summation and integration cannot be interchanged, the interchange of the limit and summation will result in the lower bound

$$\lim_{n\to\infty} F\left(\rho_B^{(n,l)}(a)\right) \geqslant \sum_{\substack{k,k'=1 \\ \text{s.t. } p_k^{(\infty,l)}+p_{k'}^{(\infty,l)}>0}}^{d_B(l)} \frac{p_k^{(\infty,l)}}{(p_k^{(\infty,l)}+p_{k'}^{(\infty,l)})^2} \left|\langle k,\infty,l|\frac{d\rho_B^{(\infty,l)}(a)}{da}|k',\infty,l\rangle\right|^2, \tag{F25}$$

where $\rho_B^{(\infty,l)}(a) = \sum_{k=1}^{d_B(l)} p_k^{(\infty,l)} |k,\infty,l\rangle\langle k,\infty,l|$, with

$$\rho_B^{(\infty,l)}(a) := \lim_{n\to\infty} \mathrm{tr}_E\left[P_{BE}^{(l)} V_{A\to BE} P_A^{(n)} \left(|\psi_n(a)\rangle\langle\psi_n(a)|\right) P_A^{(n)} V_{A\to BE}^\dagger P_{BE}^{(l)}\right] \tag{F26}$$

$$= \mathrm{tr}_E\left[P_{BE}^{(l)} V_{A\to BE} \left(|\tilde{\psi}(a)\rangle\langle\tilde{\psi}(a)|\right) V_{A\to BE}^\dagger P_{BE}^{(l)}\right], \tag{F27}$$

where $|\tilde{\psi}(a)\rangle := e^{-i\tilde{\mathrm{A}}a}|\psi\rangle$ and using properties (i) and (ii). Similarly, use properties (i) and (ii) again to obtain

$$\frac{d}{da}\rho_B^{(\infty,l)}(a) := \lim_{n\to\infty} \mathrm{tr}_E\left[P_{BE}^{(l)} V_{A\to BE} P_A^{(n)} \left(i|\psi_n(a)\rangle\langle\psi_n(a)|\tilde{\mathrm{A}}_n - i\tilde{\mathrm{A}}_n|\psi_n(a)\rangle\langle\psi_n(a)|\right) P_A^{(n)} V_{A\to BE}^\dagger P_{BE}^{(l)}\right] \tag{F28}$$

$$= \mathrm{tr}_E\left[P_{BE}^{(l)} V_{A\to BE} \left(i|\tilde{\psi}(a)\rangle\langle\tilde{\psi}(a)|\tilde{\mathrm{A}} - i\tilde{\mathrm{A}}|\tilde{\psi}(a)\rangle\langle\tilde{\psi}(a)|\right) V_{A\to BE}^\dagger P_{BE}^{(l)}\right]. \tag{F29}$$

Likewise, we obtain the same expression for $\lim_{n\to\infty} F(\rho_E^{(n,l)}(b))$ that we have achieved for

$$\lim_{n\to\infty} F\left(\rho_B^{(n,l)}(a)\right), \tag{F30}$$

but interchanging $a \mapsto b$, $\tilde{\mathrm{A}} \mapsto \tilde{\mathrm{B}}$ and $\mathrm{tr}_E \mapsto \mathrm{tr}_B$.

Now that we have an expression for the bound, which holds for bounded operators $\tilde{\mathrm{A}}$ and $\tilde{\mathrm{B}}$, our next step is to move to unbounded operators. For this task, we define sequences of bounded operators $(\mathrm{A}_m)_m$ and $(\mathrm{B}_m)_m$ as

$$\mathrm{A}_m := \frac{\mathrm{A}}{1+\mathrm{A}^2/m}, \quad \mathrm{B}_m := \frac{\mathrm{B}}{1+\mathrm{B}^2/m}. \tag{F31}$$

We now evaluate Eq. (F18) choosing $\tilde{\mathrm{A}}$ equal to $\tilde{\mathrm{A}}_m$ and $\tilde{\mathrm{A}}$ equal to $\tilde{\mathrm{A}}_m$, followed by taking the limit $m \to \infty$ on both sides of the equation. Since by (iii), it follows that $1/(1+\mathrm{A}^2/m) \xrightarrow{s} \mathbb{1}_A$ and $1/(1+\mathrm{B}^2/m) \xrightarrow{s} \mathbb{1}_A$, where $\mathbb{1}_A$ is the identity operator on $\mathscr{H}_A$, we have that $\mathrm{A}_m\psi \to \mathrm{A}\psi$ and $\mathrm{B}_m\psi \to \mathrm{B}\psi$ for all $\psi \in \mathcal{D}(\mathrm{A}) \cap \mathcal{D}(\mathrm{B})$, and we find

$$\frac{\lim_{m\to\infty}\lim_{n\to\infty} F\left(\rho_B^{(n,l)}(a)\right)}{F\left(\rho_A^{(\infty,\infty)}(a)\right)} + \frac{\lim_{m\to\infty}\lim_{n\to\infty} F\left(\rho_E^{(n,l)}(b)\right)}{F\left(\rho_A^{(\infty,\infty)}(b)\right)} \leqslant 1 + 2\sqrt{1 - \frac{\langle i[\mathrm{A},\mathrm{B}]\rangle^2}{4\,\sigma_{\mathrm{A}}{}^2\sigma_{\mathrm{B}}{}^2}}, \tag{F32}$$

where

$$F\left(\rho_A^{(\infty,\infty)}(a)\right) := 4\,\sigma_{\mathrm{A}}^2, \tag{F33}$$

$$F\left(\rho_A^{(\infty,\infty)}(b)\right) := 4\,\sigma_O^2, \tag{F34}$$

$$\sigma_{\mathrm{A}} := \left(\langle \mathrm{A}\psi, \mathrm{A}\psi\rangle - \langle \psi, \mathrm{A}\psi\rangle^2\right)^{1/2}, \tag{F35}$$

$$\sigma_{\mathrm{B}} := \left(\langle \mathrm{B}\psi, \mathrm{B}\psi\rangle - \langle \psi, \mathrm{B}\psi\rangle^2\right)^{1/2}. \tag{F36}$$

The rhs of this inequality is now of the form in the corollary statement. We continue with the lhs. First observe that

$$\lim_{m\to\infty}\lim_{n\to\infty} F\left(\rho_B^{(n,l)}(a)\right) \geqslant \tag{F37}$$

$$\sum_{\substack{k,k'=1 \\ \text{s.t. } p_k^{(\infty,\infty,l)}+p_{k'}^{(\infty,\infty,l)}>0}}^{d_B(l)} \frac{p_k^{(\infty,\infty,l)}}{(p_k^{(\infty,\infty,l)}+p_{k'}^{(\infty,\infty,l)})^2} \left|\langle k,\infty,\infty,l|\frac{d\rho_B^{(\infty,\infty,l)}(a)}{da}|k',\infty,\infty,l\rangle\right|^2, \tag{F38}$$

where $\rho_B^{(\infty,\infty,l)}(a) = \sum_{k=1}^{d_B(l)} p_k^{(\infty,\infty,l)}|k,\infty,\infty,l\rangle\langle k,\infty,\infty,l|$, with

$$\rho_B^{(\infty,\infty,l)}(a) := \lim_{m\to\infty} \mathrm{tr}_E\left[P_{BE}^{(l)} V_{A\to BE}\left(|\tilde{\psi}_m(a)\rangle\langle\tilde{\psi}_m(a)|\right) V_{A\to BE}^\dagger P_{BE}^{(l)}\right], \tag{F39}$$

$$\frac{d}{da}\rho_B^{(\infty,\infty,l)}(a) := \lim_{m\to\infty} \mathrm{tr}_E\left[P_{BE}^{(l)} V_{A\to BE}\left(i|\tilde{\psi}_m(a)\rangle\langle\tilde{\psi}_m(a)|\mathrm{A}_m - i\mathrm{A}_m|\tilde{\psi}_m(a)\rangle\langle\tilde{\psi}_m(a)|\right) V_{A\to BE}^\dagger P_{BE}^{(l)}\right], \tag{F40}$$

and $|\tilde{\psi}_m(a)\rangle = e^{-ia\mathrm{A}_m}|\psi\rangle$. To see that Eq. (F38) holds, observe that the same reasoning to why the limit and summation could be interchanged going from Eq. (F23) to Eq. (F24), holds for the limit $m \to \infty$ also. Now define $f_m(x) = e^{-iax/(1+x^2/m)}$ and $f(x) = e^{-iax}$. Assumptions in (iii) hold, thus $e^{-ia\mathrm{A}_m} \xrightarrow{s} e^{-ia\mathrm{A}}$, hence using (ii) $1/(1+\mathrm{A}^2/m)e^{-ia\mathrm{A}_m} \xrightarrow{s} e^{-ia\mathrm{A}}$. Furthermore, since, by definition $e^{-ia\mathrm{A}}|\psi\rangle \in \mathcal{D}(\mathrm{A})$, we have $H|\psi(a)\rangle \in \mathscr{H}_A$. Taking all these things into account, we conclude that

$$\rho_B^{(\infty,\infty,l)}(a) = \mathrm{tr}_E\left[P_{BE}^{(l)} V_{A\to BE}\left(|\psi(a)\rangle\langle\psi(a)|\right) V_{A\to BE}^\dagger P_{BE}^{(l)}\right], \tag{F41}$$

$$\frac{d}{da}\rho_B^{(\infty,\infty,l)}(a) = \mathrm{tr}_E\left[P_{BE}^{(l)} V_{A\to BE}\left(i|\psi(a)\rangle\langle\psi(a)|\mathrm{A} - i\mathrm{A}|\psi(a)\rangle\langle\psi(a)|\right) V_{A\to BE}^\dagger P_{BE}^{(l)}\right]. \tag{F42}$$

Likewise, we obtain the same expression for $\lim_{m\to\infty}\lim_{n\to\infty} F(\rho_E^{(n,l)}(b))$ that we have achieved for $\lim_{m\to\infty}\lim_{n\to\infty} F(\rho_B^{(n,l)}(a))$, but interchanging $a \mapsto b$, $\mathrm{A} \mapsto \mathrm{B}$ and $\mathrm{tr}_E \mapsto \mathrm{tr}_B$. Lastly, by comparing the rhs of Eq. (F38) with the rhs of Eq. (F23), one sees that $\lim_{m\to\infty}\lim_{n\to\infty} F(\rho_B^{(n,l)}(a))$ is given by evaluating the Fisher information for $\rho_B^{(\infty,\infty,l)}(a)$ [defined by Eq. (F41)] with derivative $\frac{d}{da}\rho_B^{(\infty,\infty,l)}(a)$ [defined by Eq. (F41)] according to Eqs. (9) and (11). The same observation holds for Eve's Fisher information. Hence to conclude the proof, we take $\liminf_{l\to\infty}$ on both sides of the equation. ■

2. Time-energy uncertainty equality in infinite dimensions

In fact, building on the previous result, we get the following statement in the case where the commutator in the previous theorem vanishes. This can be viewed as a generalization of Theorem 1 to the unbounded operator case.

Theorem 9 (Time-energy uncertainty relation for infinite-dimensional systems).—let $|\psi\rangle$ be a state vector in a separable Hilbert space $\mathscr{H}_A$ of possibly infinite dimensions, let H, X be self-adjoint operators (possibly unbounded) with domains $\mathcal{D}(H)$ and $\mathcal{D}(X)$, respectively, so that $|\psi\rangle \in \mathcal{D}(H) \cap \mathcal{D}(X)$. Define $\sigma_H := [\langle H\psi, H\psi\rangle - \langle\psi, H\psi\rangle^2]^{1/2}$, which is finite due to $|\psi\rangle \in \mathcal{D}(H)$, and $P_\rho^\perp := \mathbb{1} - P_\rho$, where P_ρ denotes the projector onto the support of ρ. Define analogously as before

$$T := t_0 - \frac{i[H,\psi]}{2\sigma_H^2} + P_\psi^\perp X P_\psi^\perp, \tag{F43}$$

where X captures the freedom left when defining the optimal local time sensing observable, and consider for real t, η and t_0, η_0 the two-parameter family $|\psi(t,\eta)\rangle$ with $|\psi(t_0,\eta_0)\rangle = |\psi\rangle$, again $\psi = |\psi\rangle\langle\psi|$ and

$$|\psi(t,\eta)\rangle = \exp\{-i[(t-t_0)H - (\eta-\eta_0)T]\}\,|\psi\rangle. \tag{F44}$$

Let, as in Theorem 8, $V_{A\to BE}$ be any isometry $\mathscr{H}_A \to \mathscr{H}_B \otimes \mathscr{H}_E$, where the Hilbert spaces $\mathscr{H}_B, \mathscr{H}_E$ associated with Bob and Eve are also separable and possibly of infinite dimensions, and define F_A, F_B, F_E analogously as in Theorem 8. Then the uncertainty principle

$$\frac{F_B(t)}{F_A(t)} + \frac{F_E(\eta)}{F_A(\eta)} \leqslant 1 \tag{F45}$$

holds.

Indeed, even in the infinite-dimensional setting for unbounded operators, the uncertainty principle can be attained with equality, so that

$$\frac{F_B(t)}{F_A(t)} + \frac{F_E(\eta)}{F_A(\eta)} = 1 \tag{F46}$$

still holds true.

Proof of Theorem 9.—The proof follows the same line of thought as that of Theorem 8, with some differences. To start with, consider the bounded operators $\tilde{H}$ and $\tilde{X}$ on $\mathscr{H}_A$ and define for a positive integer n the truncated operators $\tilde{H}_n$ as

$$\tilde{H}_n := P_A^{(n)} \tilde{H} P_A^{(n)} \tag{F47}$$

and

$$\tilde{T}_n := P_A^{(n)} \tilde{T} P_A^{(n)}, \tag{F48}$$

with $\tilde{T}$ being defined as in Eq. (F43) with T being replaced by $\tilde{T}$ and X by $\tilde{X}$. As above, one can define the time-evolved states as

$$|\psi_n(t,\eta)\rangle = \exp\{-i[(t-t_0)\tilde{H}_n - (\eta-\eta_0)\tilde{T}_n]\}\,|\psi\rangle, \tag{F49}$$

with $|\psi_n\rangle := |\psi_n(t_0,\eta_0)\rangle$. In the same way as before, for positive integers l (and n), we can consider the sequence of positive operators $(\rho_B^{(n,l)})_{n,l}$ on $\mathscr{H}_B$ defined as

$$\rho_B^{(n,l)}(t,\eta) := \mathrm{tr}_E\Big[P_{BE}^{(l)} V_{A\to BE} P_A^{(n)} \left(|\psi_n(t,\eta)\rangle\langle\psi_n(t,\eta)|\right) P_A^{(n)} V_{A\to BE}^\dagger P_{BE}^{(l)}\Big], \tag{F50}$$

and

$$\rho_E^{(n,l)}(t,\eta) := \mathrm{tr}_B\Big[P_{BE}^{(l)} V_{A\to BE} P_A^{(n)} \left(|\psi_n(t,\eta)\rangle\langle\psi_n(t,\eta)|\right) P_A^{(n)} V_{A\to BE}^\dagger P_{BE}^{(l)}\Big]. \tag{F51}$$

Using these quantities, and proceeding as in the proof of Theorem 8, since this is a valid finite-dimensional setting in which the above proof in terms of a semidefinite program holds true, one has

$$\frac{F_{\mathrm{Bob},t}}{4\langle\tilde{H}\psi_n, \tilde{H}_n\psi_n\rangle - \langle\psi_n, \tilde{H}_n\psi_n\rangle^2} + \langle\tilde{H}_n\psi_n, \tilde{H}_n\psi_n\rangle - \langle\psi_n, \tilde{H}_n\psi_n\rangle^2 F_{\mathrm{Eve},\eta} = 1, \tag{F52}$$

with equality, since $|\psi\rangle \in \mathcal{D}(H) \cap \mathcal{D}(X)$ and hence the state vector is in the domains of H and X. Here,

$$F_{\mathrm{Bob},t} := F\left(\rho_B^{(n,l)}(t_0)\,;\,\partial_t \rho_B^{(n,l)}(t_0)\right), \tag{F53}$$

with

$$\rho_B^{(n,l)}(.) := \rho_B^{(n,l)}(., \eta_0), \tag{F54}$$

and $F_{\mathrm{Eve},\eta}$ defined analogously based on $\rho_E^{(n,l)}(\eta)$ with $\rho_E^{(n,l)}(.) := \rho_E^{(n,l)}(t_0, .)$. The limit to the infinite-dimensional setting involving the suitable limit of $n \to \infty$ and $l \to \infty$ can be performed as in Theorem 8, while maintaining equality for each n and l. ■

APPENDIX G: CALCULATIONS FOR THE CASE OF CONTINUOUS LINDBLADIAN NOISE

1. Sensing an unknown parameter in the Hamiltonian

Consider a probe initialized in the state vector $|\psi_{\mathrm{init}}\rangle$ and subject to the Lindblad dynamics

$$\dot{\rho} = \mathcal{L}_{\mathrm{tot}}^{(\omega)}(\rho), \tag{G1}$$

with

$$\mathcal{L}_{\mathrm{tot}}^{(\omega)} = \mathcal{L}_{\mathrm{sig}}^{(\omega)} + \mathcal{L}_{\mathrm{rest}}\,; \quad \mathcal{L}_{\mathrm{sig}}^{(\omega)}(\rho) = -i[\omega G, \rho];$$
$$\mathcal{L}_{\mathrm{rest}}(\rho) = -i[H_{\mathrm{rest}}, \rho] + \sum_j \Big[L_j \rho L_j^\dagger - \frac{1}{2}\{L_j^\dagger L_j, \rho\}\Big]. \tag{G2}$$

Here, ω is the unknown parameter to be estimated. The overall evolution up to some total time T is given by

$$\mathcal{E}_T^{(\omega)} = e^{T[\mathcal{L}_{\rm sig}^{(\omega)} + \mathcal{L}_{\rm rest}]}. \tag{G3}$$

As we did earlier, we can decompose the overall evolution into the unitary evolution driven by the signal (which depends on the unknown parameter ω), followed by an effective instantaneous noisy channel $\mathcal{N}_{T,\omega}$:

$$\mathcal{E}_T^{(\omega)} = \mathcal{N}_{T,\omega}\, e^{T\mathcal{L}_{\rm sig}^{(\omega)}}, \tag{G4}$$

where $\mathcal{N}_{T,\omega}$ is given by

$$\mathcal{N}_{T,\omega} = \mathcal{E}_T^{(\omega)}\, e^{-T\mathcal{L}_{\rm sig}^{(\omega)}}. \tag{G5}$$

We are interested in the sensitivity of the probe to the parameter ω, locally around ω_0, after letting the probe evolve for some fixed time T. The sensitivity is given in terms of the Fisher information

$$F_{T,\omega}(\omega_0) = F\big(\rho_{T,\omega_0}\,;\,(\partial_\omega \rho_{T,\omega})\,(\omega_0)\big). \tag{G6}$$

Defining the (fictitious) family of states

$$\begin{aligned}\psi_{T,\omega} &= e^{-iT\omega G}\,\psi_{\rm init}\, e^{iT\omega G};\\ \partial_\omega \psi_{T,\omega} &= -iT[G, \psi_{T,\omega}],\end{aligned} \tag{G7}$$

we may write

$$F_{T,\omega}(\omega_0) = F\big(\rho_T\,;\,\mathcal{N}_T(\partial_\omega\psi_{T,\omega}) + (\partial_\omega\mathcal{N}_{T,\omega})(\psi_T)\big), \tag{G8}$$

where we omit the subscript $(\cdot)_{\omega_0}$ on all objects, which are ultimately evaluated at $\omega = \omega_0$.

Again as earlier we assume that we can neglect the second term in the derivative in Eq. (G8), and carry on with the approximation

$$F_{T,\omega} \approx F\big(\mathcal{N}_{T,\omega_0}(\psi_{T,\omega_0})\,;\,\mathcal{N}_{T,\omega_0}(\partial_\omega\psi_{T,\omega})\big) =: F_{T,\omega}^{\rm unit.}. \tag{G9}$$

As above we are now in the setting of our main uncertainty relation; we can identify the above quantity with $F_{{\rm Bob},t}$ in Theorem 1, where now the relevant evolution generator is TG. Theorem 1 then implies that

$$\begin{aligned}F_{T,\omega}^{\rm unit.} &= 4T^2\sigma_G^2 - \Delta F_{T,\omega}^{\rm unit.};\\ \Delta F_{T,\omega}^{\rm unit.} &= T^2 F\big(\widehat{\mathcal{N}}_{T,\omega_0}(\psi_{T,\omega_0})\,;\,\widehat{\mathcal{N}}_{T,\omega_0}(\{\bar{G}, \psi_{T,\omega_0}\})\big),\end{aligned} \tag{G10}$$

where $\widehat{\mathcal{N}}_{T,\omega_0}$ is a channel that is complementary to $\mathcal{N}_{T,\omega_0}$, and where $\bar{G} = G - \langle G\rangle$ with $\langle G\rangle = {\rm tr}[G\,\psi_{T,\omega_0}]$. As earlier, the complementary channel can be written $\widehat{\mathcal{N}}_{T,\omega_0} = \widehat{\mathcal{E}}_{T,\omega_0}\, e^{-T\mathcal{L}_{\rm sig}^{(\omega)}}$.

The absolute error δ in the approximation (G9) can be bounded as earlier using Proposition 17 in Appendix C as

$$\begin{aligned}|\delta| \leqslant\ & F\big(\rho\,;\,(\partial_\omega\mathcal{N}_{T,\omega})(\psi_{T,\omega_0})\big)\\ &+ \big[F\big(\rho\,;\,(\partial_\omega\mathcal{N}_{T,\omega})(\psi_{T,\omega_0})\big)\, F_{T,\omega}^{\rm unit.}\big]^{1/2}.\end{aligned} \tag{G11}$$

Similar arguments to those presented earlier apply when computing $\partial_\omega\mathcal{N}_{T,\omega}$ in order to bound δ; we have

$$\big(\partial_\omega\mathcal{N}_{T,\omega}\big)(\psi_{T,\omega}) = \partial_\omega\rho - \mathcal{E}_T^{(\omega)}(-iT[G, \psi_0]). \tag{G12}$$

Any numerical or analytical upper bound on $F\big(\rho\,;\,(\partial_\omega\mathcal{N}_{T,\omega})(\psi_{T,\omega_0})\big)$ then directly gives an upper bound to $|\delta|$ in Eq. (G11).

2. Example: continuous dephasing noise along the Z axis

A qubit is initialized in the state vector

$$|\psi_{\rm init}\rangle = |+\rangle = \frac{1}{\sqrt{2}}[|\uparrow\rangle + |\downarrow\rangle], \tag{G13}$$

and evolves according to the Hamiltonian $H = \omega Z/2$. Suppose that the qubit is subject to continuous dephasing along the Z axis. This noise is represented by the Lindbladian jump operators

$$L_0 = \sqrt{\gamma}|\uparrow\rangle\langle\uparrow|, \quad L_1 = \sqrt{\gamma}|\downarrow\rangle\langle\downarrow|. \tag{G14}$$

In vectorized operator notation (same conventions as in the appendices of our work, i.e., row-major convention), we have

$$\begin{aligned}\mathcal{L}_1 &= \sum\Big[L_j\otimes L_j^T - \frac{1}{2}[L_j^\dagger L_j\otimes \mathbb{1} + \mathbb{1}\otimes (L_j^\dagger L_j)^T]\Big]\\ &= \begin{bmatrix}0 &&& \\ & -\gamma && \\ && -\gamma & \\ &&& 0\end{bmatrix};\\ \mathcal{L}_0 &= (\ldots) = \begin{bmatrix}0 &&& \\ & -i\omega && \\ && i\omega & \\ &&& 0\end{bmatrix}, \qquad \mathcal{E}_t = e^{t(\mathcal{L}_0+\mathcal{L}_1)}\\ &= \begin{bmatrix}1 &&& \\ & e^{-\gamma t - i\omega} && \\ && e^{-\gamma t + i\omega} & \\ &&& 1\end{bmatrix}.\end{aligned}$$

The full evolution map, represented as an operator in terms of matrix elements $\rho_{ij} = \langle i|\rho|j\rangle$, is

$$\mathcal{E}_t(\rho) = \begin{bmatrix}\rho_{00} & \rho_{01}e^{-it\omega-\gamma t}\\ \rho_{10}e^{it\omega-\gamma t} & \rho_{11}\end{bmatrix}. \tag{G15}$$

The next steps for this example are (a) a direct computation of Bob's sensitivity; (b) a calculation of Eve's sensitivity to energy via our effective picture; and (c) an assessment of the error made in the approximation (140).

a. Direct computation of the sensitivity of the noisy probe

At a time t, the state is

$$\rho(t) = \frac{1}{2}\begin{bmatrix} 1 & e^{-it\omega-\gamma t} \\ e^{it\omega-\gamma t} & 1 \end{bmatrix}$$
$$= U_t\begin{bmatrix} 1 & e^{-\gamma t} \\ e^{-\gamma t} & 1 \end{bmatrix}U_t^\dagger = U_t\frac{\mathbb{1}+e^{-\gamma t}X}{2}U_t^\dagger$$
$$= \frac{1+e^{-\gamma t}}{2}U_t|+\rangle\langle+|U_t^\dagger + \frac{1-e^{-\gamma t}}{2}U_t|-\rangle\langle-|U_t^\dagger, \tag{G16}$$

where we use the shorthand $U_t = e^{-iHt}$. The last expression in Eq. (G16) provides a diagonal form for ρ, which will serve in the calculation of the Fisher information. The derivative of the state is

$$\mathcal{L}_{\text{tot}}[\rho(t)] = \dot\rho(t) = \frac{1}{2}$$
$$\begin{bmatrix} 0 & (-i\omega-\gamma)e^{-it\omega-\gamma t} \\ (i\omega-\gamma)e^{it\omega-\gamma t} & 0 \end{bmatrix}$$
$$= \frac{e^{-\gamma t}}{2}\Big[-\gamma\, U_tXU_t^\dagger + \omega\, U_tYU_t^\dagger\Big], \tag{G17}$$

noting that $U_tXU_t^\dagger = \begin{bmatrix} 0 & e^{-it\omega} \\ e^{it\omega} & 0\end{bmatrix}$ and $U_tYU_t^\dagger = \begin{bmatrix} 0 & -ie^{-it\omega} \\ ie^{it\omega} & 0\end{bmatrix}$. We can interpret this derivative in terms of two different dynamics: one $\propto \omega\, U_tYU_t^\dagger$, which drives the rotation around the Bloch sphere and one $\propto -\gamma\, U_tXU_t^\dagger$, which drives decoherence (Fig. 14). The matrix elements of the derivative in the eigenbasis $\{U_t|\pm\rangle\}$ of ρ are

$$\langle+|U_t^\dagger\,\dot\rho\,U_t|+\rangle = -\gamma\frac{e^{-\gamma t}}{2}, \quad \langle+|U_t^\dagger\,\dot\rho\,U_t|-\rangle = i\omega\frac{e^{-\gamma t}}{2},$$
$$\langle-|U_t^\dagger\,\dot\rho\,U_t|+\rangle = -i\omega\frac{e^{-\gamma t}}{2}, \quad \langle-|U_t^\dagger\,\dot\rho\,U_t|-\rangle = \gamma\frac{e^{-\gamma t}}{2}, \tag{G18}$$

where $\langle+|Y|-\rangle = \langle+|YZ|+\rangle = i\langle+|X|+\rangle = i$. Now we compute the Fisher information using Eq. (12) as

$$F_{\text{clock},t} = F\big(\rho(t_0)\,;\,\dot\rho(t_0)\big) = \frac{2}{1+e^{-\gamma t_0}}\left|\gamma\frac{e^{-\gamma t_0}}{2}\right|^2$$
$$+2\left|i\omega\frac{e^{-\gamma t_0}}{2}\right|^2 + 2\left|i\omega\frac{e^{-\gamma t_0}}{2}\right|^2$$
$$+\frac{2}{1-e^{-\gamma t_0}}\left|\gamma\frac{e^{-\gamma t_0}}{2}\right|^2$$
$$= \omega^2 e^{-2\gamma t_0} + \gamma^2\frac{2e^{-2\gamma t_0}}{1-e^{-2\gamma t_0}}. \tag{G19}$$

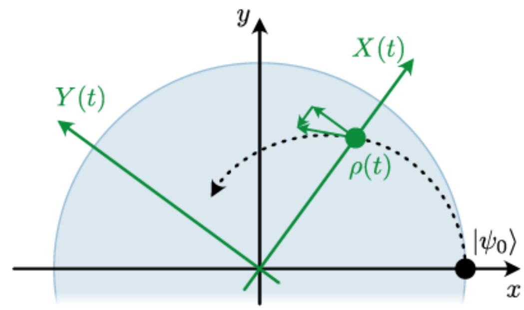

FIG. 14. Top view of the Bloch sphere for a single qubit prepared in the $+X$ eigenstate, evolving under the Hamiltonian $H = \omega Z/2$ and subject to continuous dephasing along the Z axis. The derivative of the state can be decomposed into a "longitudinal part" along $\sigma_Y(t)$ associated with the Hamiltonian dynamics, and a "radial part" along $-X(t)$ associated with the noise terms. The assumption that enables the mapping from the Lindblad setting to our bipartite uncertainty relation setting is that the noise component ("radial" component) contributes negligibly to the overall time sensitivity of the clock.

b. Eve's Fisher information with respect to energy

Now we turn to using the methods of our paper to characterize the sensitivity of the noisy probe. As described in Sec. VI A, we turn to computing

$$F_{\text{clock,U},t} = F\big(\rho(t_0)\,;\,\mathcal{N}_{t_0}(\partial_t\psi(t_0))\big), \tag{G20}$$

for the instantaneous effective noisy channel $\mathcal{N}_t$ and fictitious unitary evolution $\psi(t)$ defined in Sec. VI A. We will then later discuss how good of an approximation $F_{\text{clock,U},t}$ is to the original desired quantity $F_{\text{clock},t}$.

We decompose the full evolution $\mathcal{E}_t$ as in Eq. (136). Since $[H, L_j] = 0$, we have

$$\mathcal{N}_t = e^{t\mathcal{L}_1} = \begin{bmatrix} 1 & & & \\ & e^{-\gamma t} & & \\ & & e^{-\gamma t} & \\ & & & 1 \end{bmatrix} \to$$
$$\mathcal{N}_t(\rho) = \begin{bmatrix} \rho_{00} & \rho_{01}\,e^{-\gamma t} \\ \rho_{10}\,e^{-\gamma t} & \rho_{11} \end{bmatrix}. \tag{G21}$$

This channel can be described by the two Kraus operators

$$E_0^{(t)} = \sqrt{\frac{1+e^{-t\gamma}}{2}}\,\mathbb{1}; \quad E_1^{(t)} = \sqrt{\frac{1-e^{-t\gamma}}{2}}\,Z. \tag{G22}$$

The (fictitious) pure unitary evolution of the initial state vector $|\psi_{\text{init}}\rangle = |+\rangle$ is

$$\psi(t) = U_t\,\psi_{\text{init}}\,U_t^\dagger = \frac{1}{2}\begin{bmatrix} 1 & e^{-it\omega} \\ e^{it\omega} & 1 \end{bmatrix}. \tag{G23}$$

We compute Eve's Fisher information with respect to energy, which characterizes the sensitivity loss of the noisy probe. For any t, a complementary channel to Eq. (G21) is given by

$$\widehat{\mathcal{N}}_t(\rho) = \begin{bmatrix} \frac{1+e^{-\gamma t}}{2}\,\mathrm{tr}(\rho) & \frac{\sqrt{1-e^{-2\gamma t}}}{2}\,\mathrm{tr}(Z\rho) \\ \frac{\sqrt{1-e^{-2\gamma t}}}{2}\,\mathrm{tr}(Z\rho) & \frac{1-e^{-\gamma t}}{2}\,\mathrm{tr}(\rho) \end{bmatrix}. \tag{G24}$$

We would like to compute

$$F\big(\widehat{\mathcal{N}}_t(\psi)\,;\widehat{\mathcal{N}}_t(\{H-\langle H\rangle,\psi\})\big). \tag{G25}$$

Noting that $\langle H\rangle_{\psi(t)} = 0$ for all t and that $\psi_{\text{init}} = (\mathbb{1} + X)/2$, we can compute

$$\begin{aligned}\{H-\langle H\rangle,\psi\} &= \left\{\frac{\omega}{2}Z, U_t\,\psi_{\text{init}}\,U_t^\dagger\right\} \\ &= \frac{\omega}{2}\,U_t\left\{Z,\frac{1+X}{2}\right\}U_t^\dagger = \frac{\omega}{2}Z.\end{aligned} \tag{G26}$$

We then see that

$$\begin{aligned}\widehat{\mathcal{N}}_t(\psi) &= \begin{bmatrix} \frac{1+e^{-\gamma t}}{2} & 0 \\ 0 & \frac{1-e^{-\gamma t}}{2} \end{bmatrix};\quad \widehat{\mathcal{N}}_t\Big(\frac{\omega}{2}Z\Big) \\ &= \frac{\omega}{2}\begin{bmatrix} 0 & \sqrt{1-e^{-2\gamma t}} \\ \sqrt{1-e^{-2\gamma t}} & 0 \end{bmatrix}.\end{aligned} \tag{G27}$$

Then using Eq. (12) we find

$$\begin{aligned}&F\big(\widehat{\mathcal{N}}(\psi)\,;\widehat{\mathcal{N}}(\{H-\langle H\rangle,\psi\})\big) \\ &= 0 + 2\left[\frac{\omega^2}{4}\big(1-e^{-2\gamma t_0}\big)\right] + (\text{same term}) + 0 \\ &= \omega^2\,\big(1-e^{-2\gamma t_0}\big).\end{aligned} \tag{G28}$$

In the present picture of the effective noisy channel being applied instantly after unitary evolution of duration t_0, we see that Eve obtains no information about the energy direction for $t \approx 0$. However, for large t Eve obtains near-perfect information, which hinders Bob's sensitivity. Since the noiseless Fisher information is ω^2, we have via our uncertainty relation that

$$F_{\text{Bob},t} = \omega^2 e^{-2\gamma t_0}. \tag{G29}$$

Our method therefore correctly gives us the first term in Eq. (G19). We can also check by direct calculation that the first term in Eq. (G19) is indeed the Fisher information of the noisy clock state if we neglect the term in the derivative that is associated with the time derivative of the effective noise channel itself. First observe that

$$\partial_t\psi = \frac{1}{2}\begin{bmatrix} 0 & -i\omega e^{-it\omega} \\ i\omega e^{it\omega} & 0 \end{bmatrix}, \tag{G30}$$

$$\mathcal{N}(\partial_t\psi) = \frac{1}{2}\begin{bmatrix} 0 & -i\omega e^{-it\omega-\gamma t} \\ i\omega e^{it\omega-\gamma t} & 0 \end{bmatrix} = \frac{\omega e^{-\gamma t}}{2}\,U_t\,Y\,U_t^\dagger. \tag{G31}$$

We see that the object $\mathcal{N}(\partial_t\psi)$ is exactly the part of the derivative $\dot\rho$ with respect to the full dynamics that is associated with the Hamiltonian evolution of ρ, i.e., it is the "longitudinal" component of the derivative depicted in Fig. 14.

We use again Eq. (12) of our manuscript, recalling the diagonal form for ρ given in Eq. (G16):

$$\begin{aligned}F_{\text{clock,U},t} = F\big(\rho\,;\mathcal{N}(\partial_t\psi)\big) &= 0 + 2\left|\frac{\omega e^{-\gamma t_0}}{2}\right| \\ &+ 2\left|\frac{\omega e^{-\gamma t_0}}{2}\right| + 0 = \omega^2 e^{-2\gamma t_0}.\end{aligned} \tag{G32}$$

The difference between $F_{\text{clock,U},t}$ and $F_{\text{clock},t}$ is

$$\delta = F_{\text{clock},t} - F_{\text{clock,U},t} = \gamma^2\frac{2e^{-2\gamma t_0}}{1-e^{-2\gamma t_0}}. \tag{G33}$$

The relative error of the approximation is

$$\frac{\delta}{F_{\text{clock,U},t}} = \frac{\gamma^2}{\omega^2}\,\frac{1}{1-e^{-2\gamma t_0}}. \tag{G34}$$

(We computed the relative error with respect to $F_{\text{clock,U},t}$ because it is simpler.) We can see that δ is small relative to $F_{\text{clock,U},t}$ if the ratio γ/ω of the loss rate to the qubit's energy gap is small.

Numerical plots for $\omega = 1, \gamma = 0.1$ are presented in Fig. 15.

c. Error bound for the mapping from the Lindblad master equation to our setting

As a sanity check we compute the error bound (146). We have

$$\partial_t\mathcal{N} = \begin{bmatrix} 0 & & & \\ & -\gamma e^{-\gamma t} & & \\ & & -\gamma e^{-\gamma t} & \\ & & & 0 \end{bmatrix}, \tag{G35}$$

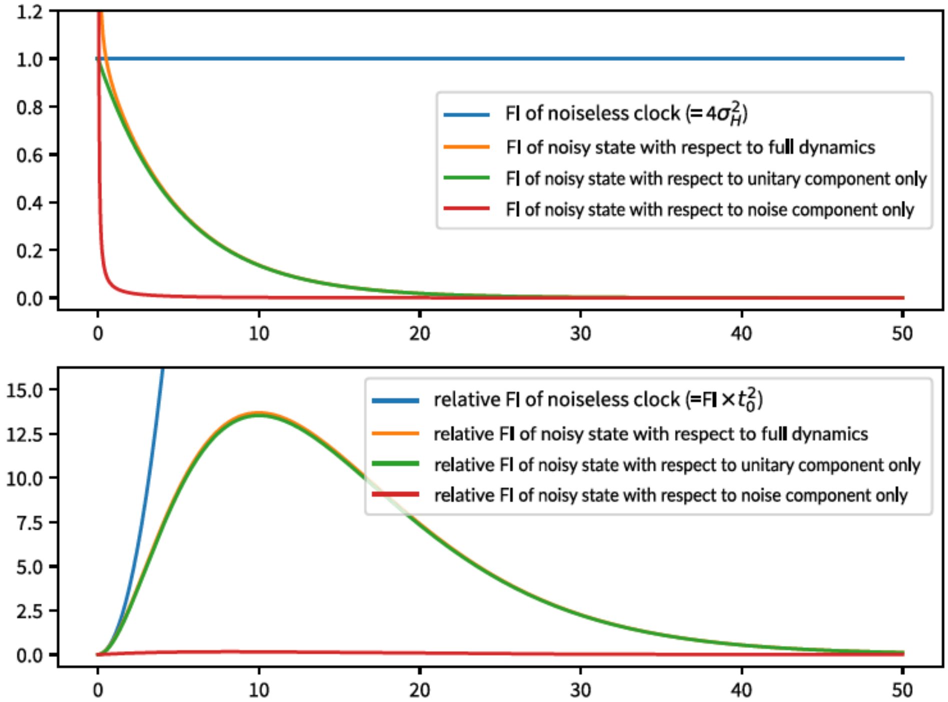

FIG. 15. Fisher information (FI) of a single qubit prepared in a $+X$ eigenstate evolving according to the Hamiltonian $H = \omega Z/2$ and subject to continuous dephasing along the Z axis. The horizontal axis represents the time t_0 at which we consider the clock sensitivity, and the vertical axis is the value of the different versions of the Fisher information (top plot) and relative Fisher information (bottom plot). The relative Fisher information is the Fisher information times t_0^2, which is relevant if we are interested in the relative sensitivity to time. In these plots we have set $\omega = 1$ and $\gamma = 0.1$ (see main text). We verify from these plots that the time dependency $\partial_t\mathcal{N}_t$ of the effective lnoisy channel contributes negligibly to the overall Fisher information; this example in the setting of continuous noise can therefore be reduced to a setting as in Fig. 1.

and thus

$$(\partial_t\mathcal{N})(\psi(t)) = \frac{1}{2}\begin{bmatrix} 0 & -\gamma e^{-\gamma t}e^{-i\omega t} \\ -\gamma e^{-\gamma t}e^{i\omega t} & 0 \end{bmatrix} = -\gamma e^{-\gamma t}U_t X U_t^\dagger. \tag{G36}$$

The matrix elements in the state's eigenbasis are

$$\langle +|U_t^\dagger(\partial_t\mathcal{N})U_t|+\rangle = -\gamma e^{-\gamma t}, \quad \langle +|U_t^\dagger(\partial_t\mathcal{N})U_t|-\rangle = 0,$$
$$\langle -|U_t^\dagger(\partial_t\mathcal{N})U_t|+\rangle = 0, \quad \langle -|U_t^\dagger(\partial_t\mathcal{N})U_t|-\rangle = \gamma e^{-\gamma t}. \tag{G37}$$

Then we can compute

$$F\big(\rho\,;\partial_t\mathcal{N}(\psi)\big) = \frac{2}{1+e^{-\gamma t_0}}|\gamma e^{-\gamma t_0}|^2 + \frac{2}{1-e^{-\gamma t_0}}|\gamma e^{-\gamma t_0}|^2 = 4\gamma^2\frac{e^{-2\gamma t_0}}{1-e^{-2\gamma t_0}}. \tag{G38}$$

Our bound (146) on the error δ becomes

$$\delta \leqslant 4\gamma^2\frac{e^{-2\gamma t_0}}{1-e^{-2\gamma t_0}} + 2\gamma\omega\frac{e^{-2\gamma t_0}}{\sqrt{1-e^{-2\gamma t_0}}}. \tag{G39}$$

The bound is consistent with our computed value of δ. However, in this case our bound is loose: The second term in our bound would suggest that the relative error with respect to $F_{\text{clock,U},t_0}$ behaves only as γ/ω (if $\gamma \ll \omega$), whereas we know from our explicit calculation of δ that the behavior of this relative error is γ^2/ω^2.

3. Example: continuous dephasing noise along the transversal X axis

Consider the qubit state vector

$$|\psi\rangle = |+\rangle = \frac{1}{\sqrt{2}}[|\uparrow\rangle + |\downarrow\rangle]. \tag{G40}$$

Suppose that the evolution of the qubit is given by the Lindbladian (132) with

$$H = \frac{\omega}{2}Z, \quad L_0 = \sqrt{\gamma}|+\rangle\langle+|, \quad L_1 = \sqrt{\gamma}|-\rangle\langle-|. \tag{G41}$$

One checks that the action of $\mathcal{L}_{\text{tot}}$ on the Pauli operators and the identity are

$$\mathcal{L}_{\text{tot}}(\mathbb{1}) = 0, \quad \mathcal{L}_{\text{tot}}(X) = \omega Y, \\ \mathcal{L}_{\text{tot}}(Y) = -\omega X - \gamma Y, \quad \mathcal{L}_{\text{tot}}(Z) = -\gamma Z. \tag{G42}$$

Therefore, $\mathcal{L}_{\text{tot}}$ can be represented in the orthonormal basis $\{|\mathbb{1}\rangle\rangle/\sqrt{2}, |X\rangle\rangle/\sqrt{2}, |Y\rangle\rangle/\sqrt{2}, |Z\rangle\rangle/\sqrt{2}\}$ of Pauli operators (denoted with subscript P) as

$$[\mathcal{L}_{\text{tot}}]_P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\omega & 0 \\ 0 & \omega & -\gamma & 0 \\ 0 & 0 & 0 & -\gamma \end{pmatrix}_P . \tag{G43}$$

One can verify that this matrix is diagonalized as

$$[\mathcal{L}_{\text{tot}}]_P = S \begin{pmatrix} 0 & & & \\ & \lambda_+ & & \\ & & \lambda_- & \\ & & & -\gamma \end{pmatrix} S^{-1}, \tag{G44}$$

$$\lambda_\pm = -\frac{\gamma}{2} \pm i\alpha, \quad \alpha = \frac{1}{2}\sqrt{4\omega^2 - \gamma^2},\ \lambda_+\lambda_- \\ = \omega^2,\ \lambda_+ + \lambda_- = -\gamma, \\ S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{\lambda_-}{\omega} & -\frac{\lambda_+}{\omega} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ S^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{-i\omega}{2\alpha} & \frac{-i\lambda_+}{2\alpha} & 0 \\ 0 & \frac{i\omega}{2\alpha} & \frac{i\lambda_-}{2\alpha} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{G45}$$

We can solve the dynamics analytically using this diagonal representation to compute the matrix exponential as

$$\mathcal{E}_t = [e^{t\mathcal{L}_{\text{tot}}}]_P = S \begin{pmatrix} 1 & & & \\ & e^{t\lambda_+} & & \\ & & e^{t\lambda_-} & \\ & & & e^{-\gamma t} \end{pmatrix} \\ S^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e_{xx} & e_{xy} & 0 \\ 0 & e_{yx} & e_{yy} & 0 \\ 0 & 0 & 0 & e^{-\gamma t} \end{pmatrix}_P ,$$

$$e_{xx} = e^{-\frac{\gamma t}{2}}\left[\cos(\alpha t) + \frac{\gamma}{2\alpha}\sin(\alpha t)\right], \quad e_{xy} = -e_{yx}, \\ e_{yx} = \frac{\omega}{\alpha} e^{-\frac{\gamma t}{2}} \sin(\alpha t), \quad e_{yy} = e_{xx}. \tag{G46}$$

This gives us a useful expression of the linear operator $\mathcal{E}_t$ acting on the operator basis of Pauli operators. If we let the initial state $\psi_{\text{init}} = |+\rangle\langle+|$ evolve for a time t, we obtain

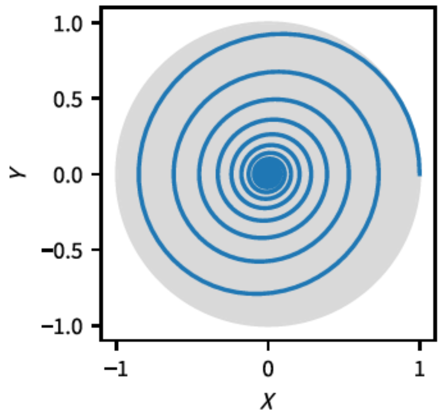

FIG. 16. Trajectory on the equatorial slice of the Bloch sphere of the state of a qubit initialized in the state vector $|+\rangle$, evolving under the Hamiltonian $H = (\omega/2)Z$ and subject to continuous dephasing along the X axis. Here $\omega = 1$ and $\gamma = 0.1$.

$$\rho(t) = \mathcal{E}_t(\psi_{\text{init}}) = \mathcal{E}_t\left(\frac{1+X}{2}\right) = \frac{\mathbb{1}}{2} \\ + \frac{e^{-\frac{\gamma t}{2}}}{2}\left[\left(\cos(\alpha t) + \frac{\gamma}{2\alpha}\sin(\alpha t)\right) X + \frac{\omega}{\alpha}\sin(\alpha t)\, Y\right]. \tag{G47}$$

See Fig. 16 for a plot of the trajectory of the state $\rho(t)$ in the X-Y plane of the Bloch sphere.

We can compute the derivative $\partial_t\rho$ by directly differentiating the expression (G47) or by simply applying the Lindbladian since we have determined its action in the Pauli basis:

$$\partial_t\rho = \frac{e^{-\frac{\gamma t}{2}}}{2}\left[\left(\cos(\alpha t) + \frac{\gamma}{2\alpha}\sin(\alpha t)\right)\omega Y \right. \\ \left. + \frac{\omega}{\alpha}\sin(\alpha t)\left(-\omega X - \gamma Y\right)\right] \\ = \frac{\omega}{2} e^{-\frac{\gamma t}{2}}\left[-\frac{\omega}{\alpha}\sin(\alpha t)\, X + \left(\cos(\alpha t) - \frac{\gamma}{2\alpha}\sin(\alpha t)\right) Y\right]. \tag{G48}$$

The approximation we make to apply our uncertainty relation is to replace this expression for $\partial_t\rho$ by

$$\mathcal{E}_t(-i[H, \psi_{\text{init}}]) = \mathcal{E}_t\left(-i\left[\frac{\omega}{2}Z, \frac{1+X}{2}\right]\right) = \frac{\omega}{2}\,\mathcal{E}_t(Y) \\ = \frac{\omega}{2} e^{-\frac{\gamma t}{2}}\left[-\frac{\omega}{\alpha}\sin(\alpha t)\, X + (\cos(\alpha t) \right. \\ \left. + \frac{\gamma}{2\alpha}\sin(\alpha t)\right) Y\bigg]. \tag{G49}$$

We see that the two expressions (G48) and (G49) differ by a term $(2\alpha)^{-1}\gamma\omega e^{-\gamma t/2}\sin(\alpha t) Y$, which is small as long as $\gamma \ll \omega$.

The Fisher information $F_{\mathrm{clock},t}$ given by Eq. (134) and $F_{\mathrm{clock,U},t}$ given by Eq. (140) are plotted in Fig. 17 as a function of t_0. Our approximation matches the exact Fisher information well, except for an out-of-phase oscillation of relatively small amplitude. This error to the contribution of the phase damping is expected to be attributable to the difference in sign of the smaller terms in Eqs. (G48) and (G49).

APPENDIX H: PERTURBING THE NOISY CHANNEL TO RESTORE EQUALITY IN THE UNCERTAINTY RELATION FOR METROLOGICAL CODES

In this Appendix, we study how to perturb a noisy channel $\mathcal{N}$ in order to restore uncertainty relation equality for a metrological code. We prove Proposition 2 of the main text, which shows that equality in the uncertainty relation can be restored by an infinitesimal perturbation of the Stinespring isometry, all while preserving the zero sensitivity loss conditions (148) (it might be necessary to enlarge Bob's system with an auxiliary qubit). The proposition is slightly reformulated to emphasize the fact that we can apply the same construction also without regards to the zero sensitivity loss condition.

Proposition 22.—let $V_{A\to BE}$ be an isometry, let $|\psi\rangle_A, |\xi\rangle_A$ with $\langle\psi|\xi\rangle_A = 0$ and let $\mathcal{N}(\cdot) = \mathrm{tr}_E\big(V(\cdot)\,V^\dagger\big)$, $\widehat{\mathcal{N}}(\cdot) = \mathrm{tr}_B\big(V(\cdot)\,V^\dagger\big)$. For any $\epsilon > 0$, there exists an isometry $V'_{A\to BE}$ with $\|V' - V\| \leqslant \epsilon$ and such that $\big(P^\perp_{\rho'_B} \otimes P^\perp_{\rho'_E}\big)V'|\xi\rangle = 0$, where $\rho'_B = \mathrm{tr}_E\big\{V'\psi V'^\dagger\big\}$ and $\rho'_E = \mathrm{tr}_B\big\{V'\psi V'^\dagger\big\}$.

Furthermore, assume that $\widehat{\mathcal{N}}(|\xi\rangle\langle\psi| + |\psi\rangle\langle\xi|) = 0$ and assume that there exists a unitary operator G_B acting on the system B with the properties that $P_{\rho_B}G_BP_{\rho_B} = 0$, $P_{\zeta_B}G_BP_{\zeta_B} = 0$, $P_{\rho_B}G_BP_{\zeta_B} = 0$, and $P_{\zeta_B}G_BP_{\rho_B} = 0$, where $\zeta_B = \mathcal{N}(|\xi\rangle\langle\xi|)$ and $\zeta_E = \widehat{\mathcal{N}}(|\xi\rangle\langle\xi|)$. Then the perturbed isometry V' can be chosen to also satisfy $\widehat{\mathcal{N}}'\big(|\xi\rangle\langle\psi| + |\psi\rangle\langle\xi|\big) = 0$, where $\widehat{\mathcal{N}}'(\cdot) = \mathrm{tr}_B\big\{V'\,(\cdot)\,V'^\dagger\big\}$.

Proof.—Write $\mathcal{N}(\cdot) = \mathrm{tr}_E\big(V(\cdot)\,V^\dagger\big)$ and let

$$\rho_B = \mathcal{N}(\psi); \quad \rho_E = \widehat{\mathcal{N}}(\psi). \tag{H1}$$

The strategy to perturb V is to include an infinitesimal rotation that rotates the state $V|\psi\rangle$ into the direction of another suitably chosen state $|\chi\rangle_{BE}$. We first compute some properties of a general such rotation, and then we will prove the stated claims.

Let $\epsilon > 0$. Let $\alpha > 0$ such that $4\sin^2(\alpha/2) \leqslant \epsilon$. Let $|\chi\rangle_{BE}$ be a state with the property that the reduced state on B lies in a subspace that is orthogonal to the reduced state ρ_B of $V|\psi\rangle$, i.e., $P_{\rho_B}|\chi\rangle = 0$, or equivalently, $|\chi\rangle_{BE}$ lies in the support of $P^\perp_{\rho_B} \otimes \mathbb{1}$. The state $|\chi\rangle_{BE}$ will be fixed later. Let $\big\{|\mu^{(j)}\rangle\big\}_j$ be a basis of BE with $|\mu^{(1)}\rangle_{BE} = V|\psi\rangle$ and $|\mu^{(2)}\rangle_{BE} = |\chi\rangle_{BE}$. Let

$$\begin{aligned} W_{BE\to BE} = &\Big(\cos(\alpha)|\mu^{(1)}\rangle + \sin(\alpha)|\mu^{(2)}\rangle\Big)\langle\mu^{(1)}| \\ &+ \Big(\cos(\alpha)|\mu^{(2)}\rangle - \sin(\alpha)|\mu^{(1)}\rangle\Big)\langle\mu^{(2)}| \\ &+ \sum_{j=3,\dots} |\mu^{(j)}\rangle\langle\mu^{(j)}|, \end{aligned} \tag{H2}$$

and note that $W_{BE\to BE}$ is a unitary close to the identity, effecting the rotation $W_0 = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$ between $|\mu^{(1)}\rangle_{BE}$ and $|\mu^{(2)}\rangle_{BE}$. The eigenvalues θ_1, θ_2 of W_0 are determined from $\theta_1 + \theta_2 = \mathrm{tr}(W_0) = 2\cos(\alpha)$ and $\theta_1\theta_2 = \det(W_0) = 1$ as $\theta_1 = \theta_2^* = e^{i\alpha}$. As the operator norm is the maximal singular value, we find $\|W_0 - \mathbb{1}\|_\infty = \max\big\{|e^{i\alpha} - 1|, |e^{-i\alpha} - 1|\big\} = (1 - \cos\alpha)^2 + (\sin\alpha)^2 = 2 - 2\cos(\alpha) = 4\sin^2(\alpha/2) \leqslant \epsilon$, and $\big\|W - \mathbb{1}\big\|_\infty \leqslant \epsilon$. Now let $V' = W_{BE}V$, with

$$\|V' - V\|_\infty \leqslant \|W - \mathbb{1}\|_\infty \|V\|_\infty \leqslant \epsilon. \tag{H3}$$

We find

$$\begin{aligned} \rho'_E &= \mathrm{tr}_B\big(V'\psi V'^\dagger\big) \\ &= \mathrm{tr}_B\Big[\cos^2(\alpha)\,|\mu_1\rangle\langle\mu_1| + \cos(\alpha)\sin(\alpha) \\ &\quad \times \big(|\mu_1\rangle\langle\mu_2| + |\mu_2\rangle\langle\mu_1|\big) \\ &\quad + \sin^2(\alpha)\,|\mu_2\rangle\langle\mu_2|\Big] \\ &= \cos^2(\alpha)\,\rho_E + \cos(\alpha)\sin(\alpha)\,\mathrm{tr}_B \\ &\quad \times \Big[V|\psi\rangle\langle\chi| + |\chi\rangle\langle\psi|V^\dagger\Big] + \sin^2(\alpha)\,\chi_E. \\ &= \cos^2(\alpha)\,\rho_E + \sin^2(\alpha)\,\chi_E. \end{aligned} \tag{H4}$$

The last equality holds thanks to our assumption that $P_{\rho_B}|\chi\rangle = 0$.

We now prove the first part of the proposition. We can assume without loss of generality that $\mathrm{rank}(P^\perp_{\rho_E}) \leqslant \mathrm{rank}(P^\perp_{\rho_B})$, by exchanging the roles of the B and E systems if necessary. Let $\{|\chi_k\rangle_B\}_{k=1}^K$, $\{|\chi'_k\rangle_E\}_{k=1}^K$ be two orthonormal families of states lying in the support of $P^\perp_{\rho_B}$ and $P^\perp_{\rho_E}$, respectively, with $K = \min\big\{\mathrm{rank}(P^\perp_{\rho_B}), \mathrm{rank}(P^\perp_{\rho_E})\big\} = \mathrm{rank}(P^\perp_{\rho_E})$. Define $|\chi\rangle_{BE} = (1/\sqrt{K})\,\sum_k |\chi_k\rangle_B \otimes |\chi_k\rangle_E$. By construction, we have that $\chi_E = (1/K)\sum_{k=1}^K |k\rangle\langle k|_E = P^\perp_{\rho_E}/\,\mathrm{tr}(P^\perp_{\rho_E})$. It follows that the state (H4) has full rank, and therefore our conditions for our uncertainty relation equality are fulfilled.

Now we prove the second part of the proposition, and we assume that $\widehat{\mathcal{N}}(D^Z_A) = 0$, with $D^Z_A = |\xi\rangle\langle\psi| + |\psi\rangle\langle\xi|$. The proof strategy is similar to above, to introduce a small "rotation" to fix the support of the state ρ_E all while preserving the zero sensitivity loss conditions (148).

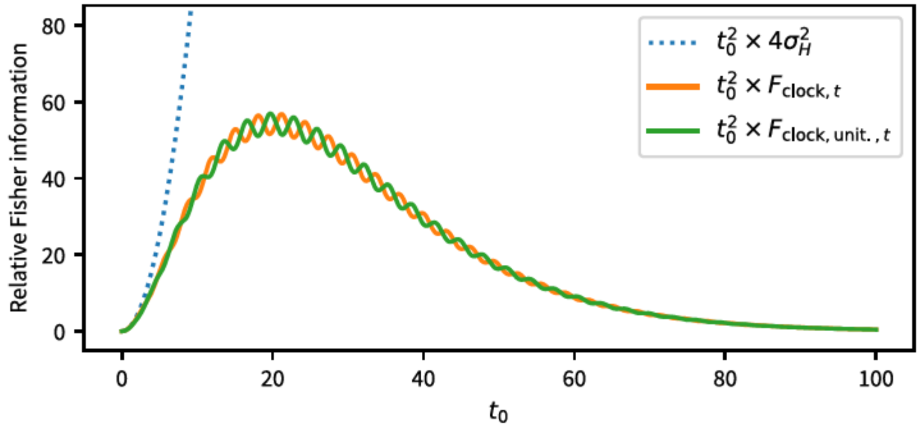

FIG. 17. Relative Fisher information with respect to time of a single qubit prepared in $|+\rangle$ and evolving according to the Hamiltonian $(\omega/2)Z$ and exposed to continuous dephasing along the X axis at a rate γ. The blue curve shows the sensitivity as a function of time t_0 of the probe to the signal if we turn off the noise. In orange, the exact Fisher information $F\big(\rho\,;\,\partial_\omega\rho\big)$ is computed directly. In green, an approximation to the desired Fisher information ignores the contribution of the time dependency $\partial_t\mathcal{N}_t$ of the effective noisy channel. This approximation is the quantity that appears in our trade-off relation in the alternative setting where Alice sends a noiseless quantum clock over a noisy channel to Bob. Because the unitary and noise parts of the Lindbladian do not commute as superoperators, invoking our the trade-off relation requires the channel $\mathcal{N}_t$ to be determined via Eq. (135). Here $\omega = 1, \gamma = 0.1$.

Without loss of generality, we may assume that $\big\||\xi\rangle\big\| = 1$. We define for later convenience

$$Z_L = |\xi\rangle\langle\psi| + |\psi\rangle\langle\xi|;\quad \Pi_L = |\psi\rangle\langle\psi| + |\xi\rangle\langle\xi|;\qquad \widetilde{Z}_L = Z_L + (\mathbb{1} - \Pi_L), \tag{H5}$$

noting that $\widetilde{Z}_L$ is the unitary operator that flips the normalized states $|\psi\rangle$ and $|\xi\rangle$ and acts as the identity on the subspace that is orthogonal to $|\psi\rangle, |\xi\rangle$.

As stated in the claim, we assume that there exists a unitary operator G_B with the properties that $P_{\rho_B}G_BP_{\rho_B} = 0$, $P_{\zeta_B}G_BP_{\zeta_B} = 0$, $P_{\rho_B}G_BP_{\zeta_B} = 0$, and $P_{\zeta_B}G_BP_{\rho_B} = 0$.

Let $0 < \epsilon \leqslant 1$. Let $\alpha = \epsilon/2$ with $0 < \alpha \leqslant 1/2$ and let

$$V' = \big(\cos(\alpha)V + \sin(\alpha)\, G_B\, V\widetilde{Z}_L\big). \tag{H6}$$

Then

$$\begin{aligned}\|V' - V\| &= \big\|(\cos(\alpha) - 1)\, V + \sin(\alpha)\, G_B V\widetilde{Z}\big\| \\ &\leqslant (1 - \cos(\alpha))\big\|V\big\| + \sin(\alpha)\, \big\|G_B V\widetilde{Z}\big\| \\ &\leqslant 2\sin^2(\alpha/2) + \sin(\alpha) \leqslant 2|\alpha|,\end{aligned} \tag{H7}$$

using $\sin(\alpha) \leqslant |\alpha|$ and with $|\alpha| \leqslant 1/2$.

We first show that the perturbed isometry V' also satisfies the zero sensitivity-loss conditions. Let $\widehat{\mathcal{N}}'(\cdot) = \mathrm{tr}_B\Big\{V'\,(\cdot)\,V'^\dagger\Big\}$ and we compute

$$\begin{aligned}\widehat{\mathcal{N}}'(Z_L) &= \mathrm{tr}_B\Big\{V'\,Z_L\,V'^\dagger\Big\} \\ &= \mathrm{tr}_B\Big\{\cos^2(\alpha)\,V Z_L\,V^\dagger \\ &\quad + \cos(\alpha)\sin(\alpha)\Big[V Z_L\,\widetilde{Z}_L\,V^\dagger G_B^\dagger + G_B V\widetilde{Z}_L\,Z_L\,V^\dagger\Big] \\ &\quad + \sin^2(\alpha)\, G_B V\widetilde{Z}_L Z_L\widetilde{Z}_L\,V^\dagger G_B^\dagger\Big\} \\ &= 0,\end{aligned} \tag{H8}$$

using $\mathrm{tr}_B\big\{VZ_LV^\dagger\big\} = \widehat{\mathcal{N}}(Z_L) = 0$ and $\widetilde{Z}_L Z_L\widetilde{Z}_L = Z_L$, as well as the fact that

$$\begin{aligned}\mathrm{tr}_B\big[G_B V\widetilde{Z}_L Z_L V^\dagger\big] &= \mathrm{tr}_B\big[G_B V[|\psi\rangle\langle\psi| + |\xi\rangle\langle\xi|]V^\dagger\big] \\ &= \mathrm{tr}_B\big[G_B P_{\rho_B} V\psi\,V^\dagger P_{\rho_B} \\ &\quad + G_B P_{\zeta_B} V\xi\,V^\dagger P_{\zeta_B}\big] = 0,\end{aligned} \tag{H9}$$

using the fact that $P_{\rho_B}G_BP_{\rho_B} = 0 = P_{\zeta_B}G_BP_{\zeta_B}$.

We then have

$$\begin{aligned}\rho_E' &= \mathrm{tr}_B\Big\{V'\,\psi\,V'^\dagger\Big\} \\ &= \mathrm{tr}_B\Big\{\cos^2(\alpha)\,V\psi\,V^\dagger + \cos(\alpha)\sin(\alpha) \\ &\qquad \times\Big[V|\psi\rangle\langle\xi|V^\dagger G_B^\dagger + G_B V|\xi\rangle\langle\psi|V^\dagger\Big] \\ &\qquad + \sin^2(\alpha)\, G_B V\xi\, V G_B^\dagger\Big\} \\ &= \cos^2(\alpha)\,\rho_E + \sin^2(\alpha)\,\zeta_E,\end{aligned} \tag{H10}$$

where the two middle terms in the long expression vanish because $\mathrm{tr}_B\big\{G_B V|\xi\rangle\langle\psi|V^\dagger\big\} = \mathrm{tr}_B\big\{P_{\rho_B}G_BP_{\zeta_B}V|\xi\rangle\langle\psi|V^\dagger\big\} = 0$.

Similarly,

$$\begin{aligned}\zeta'_E &= \mathrm{tr}_B\Big\{V'\,\xi\,V'^\dagger\Big\}\\ &= \mathrm{tr}_B\Big\{\cos^2(\alpha)\,V\xi V^\dagger + \cos(\alpha)\sin(\alpha)\\ &\quad\times\Big[V|\xi\rangle\langle\psi|V^\dagger G_B^\dagger + G_B V|\psi\rangle\langle\xi|V^\dagger\Big]\\ &\quad+\sin^2(\alpha)\,G_B V\psi V^\dagger G_B^\dagger\Big\}\\ &= \cos^2(\alpha)\,\zeta_E + \sin^2(\alpha)\,\rho_E. \end{aligned} \tag{H11}$$

Any state $|c\rangle_E$ that lies in the kernel of ρ'_E must satisfy

$$0 = \langle c|\rho'_E|c\rangle = \cos^2(\alpha)\,\langle c|\rho_E|c\rangle + \sin^2(\alpha)\,\langle c|\zeta_E|c\rangle, \tag{H12}$$

which in turn implies $0 = \langle c|\rho_E|c\rangle = \langle c|\zeta_E|c\rangle$. We then find

$$\begin{aligned}\big\|(\mathbb{1}_B\otimes\langle c|_E)\,V'|\xi\rangle\big\|^2 &= \langle c|_E\ \mathrm{tr}_B\big(V'\xi V'^\dagger\big)\,|c\rangle_E\\ &= \langle c|\zeta'_E|c\rangle\\ &= \Big\langle c|_E\Big[\cos^2(\alpha)\,\zeta_E + \sin^2(\alpha)\,\rho_E\Big]|c\Big\rangle_E\\ &= 0. \end{aligned} \tag{H13}$$

Therefore, $\big(\mathbb{1}_B\otimes P^\perp_{\rho'_E}\big)\,V'|\xi\rangle = 0$, implying that $\big(P^\perp_{\rho'_B}\otimes P^\perp_{\rho'_E}\big)\,V'|\xi\rangle = 0$ and our uncertainty relation equality conditions are satisfied. ■

APPENDIX I: BEHAVIOR OF METROLOGICAL CODES FOR WEAK IID NOISE; METROLOGICAL CODES, UNCERTAINTY RELATION EQUALITY, AND DISCONTINUITIES OF THE QUANTUM FISHER INFORMATION

In this Appendix, we consider a metrological code $(|\psi\rangle, |\xi\rangle)$ on n qubits, with a metrological distance $d_m > 1$. For any noise channel that acts on fewer than d_m qubits, we have seen in Sec. VII that $\Delta F_{\mathrm{Bob},t} = 0$. Instead of noise acting on few qubits, we now consider examples of IID noise channels $[\mathcal{N}_1^{(p)}]^{\otimes n}$, where each channel $\mathcal{N}_1^{(p)}$ acts on a single qubit and depends on a noise parameter p such that $\mathcal{N}_1^{(p=0)} = \mathrm{id}$. We ask, for constant n, to what order in p is the loss in quantum Fisher information $\Delta F_{\mathrm{Bob},t}$ suppressed?

Let us first consider a similar question in the conventional setting of quantum error correction, where a logical state is encoded into a physical state, is exposed to a noise channel, and is subsequently decoded to attempt to recover the initial state. If a state $|\psi\rangle$, encoded with a distance-d quantum error-correcting code, is exposed to a weak IID noise channel in which a single-site error happens with probability p, then after a subsequent decoding operation, the fidelity of the state with respect to the original state differs with the ideal value one by at most $O(p^{d/2})$. In other words, the fidelity loss is suppressed by the quantum error-correction procedure to an order in the noise parameter that is proportional to the distance of the code. This suppressed fidelity loss is explained by a fundamental principle in quantum information: Two states (respectively, two channels) that are ϵ close in trace distance (respectively, diamond distance) may not be distinguished by any physical operation, except with probability of the order at most $O(\epsilon)$. In the case of weak IID noise, any error operator whose weight is larger than $(d-1)/2$ occurs only with probability at most $O(p^{d/2})$. Consequently, no experiment should be able to distinguish the weak IID noise from a noise operator with only weight-$[(d-1)/2]$ operators with probability better than $O(p^{d/2})$, for which the quantum error-correction scheme enables perfect recovery.

By analogy, it is natural to expect that the quantum Fisher information loss $\Delta F_{\mathrm{Bob},t}$ should scale as approximately p^{cd_m}, where d_m is the metrological distance of the metrological code, and where c is some constant. However, this is not the case, as we will see in the remainder of this Appendix. While $\Delta F_{\mathrm{Bob},t}$ exhibits the expected behavior for certain examples of metrological codes, we can find counterexamples in which the quantum Fisher information loss scales as $\Delta F_{\mathrm{Bob},t} \sim p$ despite the state forming a metrological code of an arbitrarily large, but fixed, metrological distance d_m. This counterexample shows that when measuring the accuracy of Bob's estimate to the time parameter in terms of the quantum Fisher information, the code distance is not necessarily related to the loss in sensitivity of the state. This might be worrying, since the metrological distance of the metrological code would not be related to the degree of protection offered by such codes in suppressing the sensitivity loss. We argue, however, that the quantum Fisher information might not be the relevant sensitivity measure to study in such regimes. More specifically, we know that there are regimes in which we should question the operational relevance of the quantum Fisher information, because infinitesimal perturbations in the state or the noise channel result in observable consequences in the purported sensitivity as reported by the quantum Fisher information. We attribute this behavior to the fact that it ignores the error associated with the estimation of the expectation value of the optimal sensing observable from a finite number of measurement repetitions. Based on our examples, we hypothesize that the settings where $\Delta F_{\mathrm{Bob},t} \not\lesssim O(p^{d_m/2})$ fall into this regime. While confirming this hypothesis would invalidate known counterexamples in which a high metrological distance can still lead to a high accuracy loss, a full proof of the protection offered by metrological codes in the general setting remains elusive. Such a result would further require (a) establishing a measure of sensitivity that is robust to perturbations of the physical setting by accounting for limits on the number of available measurement repetitions and (b) showing that its loss is suppressed as a function of the metrological distance of the metrological code.

In the following, we first compute the quantum Fisher information loss of some states that form metrological codes after exposure to weak IID noise. In order to explore the cause of the behavior of some examples that appear problematic, we study more closely some properties of the quantum Fisher information: we argue that there are regimes in which the quantum Fisher information, being discontinuous, cannot be a representative measure of sensitivity, and we attribute this problematic behavior to the failure to account for the number of finite available measurement repetitions. Finally, we consider a restricted setting with additional assumptions on the state and the noise channel, in which we prove the expected bound on the quantum Fisher information loss $\Delta F_{\mathrm{Bob},t} \leqslant O(p^{d_m/2})$.

1. Examples of metrological codes exposed to weak IID noise

We now consider three single-site noise channels: the amplitude-damping channel, the dephasing channel in the Z basis, and the bit-flip channel. In the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$, the single-qubit amplitude-damping channel has Kraus operators

$$E^{(p)}_{\mathrm{AD},0} = \begin{pmatrix} \sqrt{1-p} & 0 \\ 0 & 1 \end{pmatrix}; \quad E^{(p)}_{\mathrm{AD},1} = \begin{pmatrix} 0 & 0 \\ \sqrt{p} & 0 \end{pmatrix}. \tag{I1}$$

The second noise channel we consider is the dephasing channel in the Z basis, described by the Kraus operators

$$E^{(p)}_{\mathrm{dephas.},0} = \sqrt{1-\frac{p}{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad E^{(p)}_{\mathrm{dephas.},1} = \sqrt{\frac{p}{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{I2}$$

Finally, the bit-flip channel is described by the Kraus operators

$$E^{(p)}_{\text{bit-flip},0} = \sqrt{1-\frac{p}{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad E^{(p)}_{\text{bit-flip},1} = \sqrt{\frac{p}{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{I3}$$

a. Four-qubit code state based on the [[4, 2, 2]] code

Consider the state vector introduced in Secs. VII E and VII F,

$$|\psi_{\mathrm{code}}\rangle = \frac{1}{2}\Big[|\uparrow\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\downarrow\rangle + |\uparrow\downarrow\uparrow\downarrow\rangle + |\downarrow\uparrow\downarrow\uparrow\rangle\Big]. \tag{I4}$$

Consider the Hamiltonian consisting of ZZ terms on the edges connecting the four qubits when they are arranged in a square, as in Fig. 9(a); with a suitable normalization we obtain

$$|\xi_{\mathrm{code}}\rangle = \frac{1}{2}\Big[|\uparrow\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\downarrow\rangle - |\uparrow\downarrow\uparrow\downarrow\rangle - |\downarrow\uparrow\downarrow\uparrow\rangle\Big]. \tag{I5}$$

We have seen that $(|\psi_{\mathrm{code}}\rangle, |\xi_{\mathrm{code}}\rangle)$ forms a metrological code of metrological distance 2.

Let us consider how the quantum Fisher information of this state drops when exposed to IID amplitude-damping noise and to IID dephasing noise. The quantum Fisher information loss $\Delta F_{\mathrm{Bob},t}$ is plotted in a log-log plot as a function of p in Fig. 18. We fit the computed values for points with $p < 0.1$ to the model $\ln(y) = a\ln(p) + b$ in order to determine the quantum Fisher information loss order (as $y \propto p^a$). We observe that while the quantum Fisher information loss is indeed affected only to second order in p for amplitude-damping noise, it is directly affected to first order for dephasing noise. The behavior of this small-scale example is not necessarily surprising, although it rules out an optimistic conjecture that states of the form (I4) could have their loss in quantum Fisher information be protected to second order in p against any IID noise channel, as could have been suggested from Fig. 13.

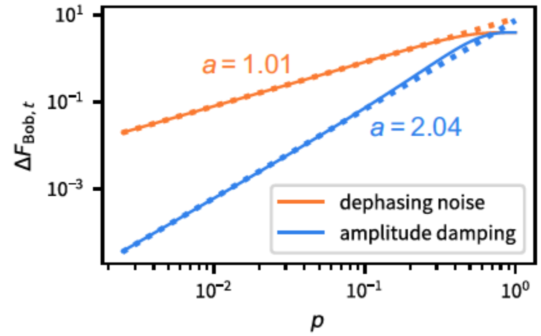

FIG. 18. Quantum Fisher information loss $\Delta F_{\mathrm{Bob},t}$ after exposure of $|\psi_{\mathrm{code}}\rangle$ [cf. Eq. (I4)] to IID amplitude-damping or dephasing noise in the Z basis, as a function of the noise parameter p. Based on our intuition of standard error-correcting codes, we might have expected that $\Delta F_{\mathrm{Bob},t}$ depends only on an order in p (for $p \to 0$) that is directly related to d_m (or $d_m/2$). In the case of either noise model, we fit the data points where $p < 0.1$ to $\ln(y) = a\ln(p) + b$ (which corresponds to a power law $y \propto p^a$) to obtain the order in p to which $\Delta F_{\mathrm{Bob},t}$ is affected. We see that for amplitude-damping noise, the loss in quantum Fisher information is suppressed to depend only on p to second order; for dephasing noise, the loss is affected to first order in p. The quantum Fisher information loss due to an IID bit-flip noise channel (not shown) behaves very similarly to the dephasing noise.

b. Repetition code in the +/− basis

Now we investigate a larger example that shows that the metrological distance is not always indicative of the order of quantum Fisher information loss in the noise parameter. On n qubits, let

$$|\psi\rangle = |+\rangle^{\otimes n}; \quad |\xi\rangle = |-\rangle^{\otimes n}. \tag{I6}$$

Here, the Hamiltonian corresponding to these states is the nonlocal operator $H = Z^{\otimes n}$. (Note that this example differs starkly from a standard ensemble of n spins where the Hamiltonian is as a sum of Z terms on each site. In that case, $|\xi\rangle$ would be a superposition of strings that consist of all $|+\rangle$ state vectors and a single $|-\rangle$ state vector.) The $(|\psi\rangle, |\xi\rangle)$ given above form a metrological code of distance $d_m = n$. Indeed, any operator O with $\mathrm{wgt}(O) < n$ cannot make $|\psi\rangle$ nonorthogonal to $|\xi\rangle$, and the conditions (155) are satisfied.

We show that if we expose this state to IID dephasing noise along the Z axis, the quantum Fisher information loss is indeed suppressed to the order $O(p^{n/2})$, as we would expect. On the other hand, if we expose the state to IID bit-flip noise, which can be seen as dephasing noise along the X axis, then the quantum Fisher information loss is not suppressed as expected and we find $\Delta F_{\mathrm{Bob},t} \sim p$.

Let us first consider IID dephasing noise along the Z axis. We show that the quantum Fisher information loss is indeed suppressed to order $O(p^{n/2})$ for this noise channel. We now prove this statement. We may choose for the noise channel $\mathcal{N}^{(p)}_{\mathrm{dephas}}(\cdot) = (1-p/2)(\cdot) + (p/2)Z(\cdot)Z$ the Stinespring isometry

$$\begin{aligned} V_{A\to BE} &= \sqrt{1-\frac{p}{2}}\,\mathbb{1}\otimes|0\rangle_E + \sqrt{\frac{p}{2}}\,Z\otimes|1\rangle_E \\ &= |p_+\rangle_E\langle\uparrow|_A\otimes|\uparrow\rangle_B + |p_-\rangle_E\langle\downarrow|_A\otimes|\downarrow\rangle_B, \end{aligned} \tag{I7}$$

with respect to some basis $|0\rangle, |1\rangle$ on E, and with

$$|p_\pm\rangle = \sqrt{1-\frac{p}{2}}\,|0\rangle \pm \sqrt{\frac{p}{2}}\,|1\rangle. \tag{I8}$$

This choice leads to the complementary channel

$$\widehat{\mathcal{N}}^{(p)}_{\mathrm{dephas}}(\cdot) = \langle\uparrow|\cdot|\uparrow\rangle_A\,|p_+\rangle\langle p_+|_E + \langle\downarrow|\cdot|\downarrow\rangle_A\,|p_-\rangle\langle p_-|_E. \tag{I9}$$

We find

$$\begin{aligned} \rho_{E_1} &= \widehat{\mathcal{N}}^{(p)}_{\mathrm{dephas}}(|+\rangle\langle+|) = \frac{1}{2}\Big[|p_+\rangle\langle p_+|_E + |p_-\rangle\langle p_-|_E\Big] = \begin{pmatrix} 1-p & 0 \\ 0 & p\end{pmatrix}; \\ &\widehat{\mathcal{N}}^{(p)}_{\mathrm{dephas}}(|+\rangle\langle-|) = \frac{1}{2}\Big[|p_+\rangle\langle p_+|_E - |p_-\rangle\langle p_-|_E\Big] = \sqrt{\frac{p}{2}\Big(1-\frac{p}{2}\Big)}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}. \end{aligned} \tag{I10}$$

We can then compute

$$\begin{aligned} \Delta F_{\mathrm{Bob},t} &= F\Big(\big[\widehat{\mathcal{N}}^{(p)}_{\mathrm{dephas}}(|+\rangle\langle+|)\big]^{\otimes n}; \big[\widehat{\mathcal{N}}^{(p)}_{\mathrm{dephas}}\big]^{\otimes n}\big([|+\rangle\langle-|]^{\otimes n} + [|-\rangle\langle+|]^{\otimes n}\big)\Big) \\ &= F\left(\begin{pmatrix} 1-p & 0 \\ 0 & p\end{pmatrix}^{\otimes n}; \left[\sqrt{\frac{p}{2}\Big(1-\frac{p}{2}\Big)}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\right]^{\otimes n} + \mathrm{h.c.}\right) \\ &= 4\Big[\frac{p}{2}\Big(1-\frac{p}{2}\Big)\Big]^n F\left(\begin{pmatrix} 1-p & 0 \\ 0 & p\end{pmatrix}^{\otimes n}; X^{\otimes n}\right) \\ &= 4\,p^n\,2^{-n}\Big(1-\frac{p}{2}\Big)^n \sum_{\mathbf{x},\mathbf{x}'} \frac{2}{\lambda_{\mathbf{x}} + \lambda_{\mathbf{x}'}}\Big|\langle\mathbf{x}|X^{\otimes n}|\mathbf{x}'\rangle\Big|^2, \end{aligned} \tag{I11}$$

where $\mathbf{x}, \mathbf{x}'$ are bit strings and where $\lambda_{\mathbf{x}} = (1-p)^{|\mathbf{x}|}p^{n-|\mathbf{x}|}$ is the eigenvalue of $\rho_{E_1}^{\otimes n}$ associated with the eigenvector $|\mathbf{x}\rangle$. Observe that $\langle\mathbf{x}|X^{\otimes n}|\mathbf{x}'\rangle = \delta_{\tilde{\mathbf{x}},\mathbf{x}'}$, where $\tilde{\mathbf{x}}$ is the bit string obtained by flipping all the bits of $\mathbf{x}$. Then

$$\begin{aligned} \lambda_{\mathbf{x}} + \lambda_{\tilde{\mathbf{x}}} &= (1-p)^{|\mathbf{x}|}p^{n-|\mathbf{x}|} + (1-p)^{n-|\mathbf{x}|}p^{|\mathbf{x}|} \\ &= \Omega\big(p^{\min(|\mathbf{x}|,n-|\mathbf{x}|)}\big) = \Omega(p^{n/2}), \end{aligned} \tag{I12}$$

noting that $\min(|\mathbf{x}|, n-|\mathbf{x}|) \leqslant n/2$. Therefore,

$$(\mathrm{J11}) = p^n \sum_{\mathbf{x}} O\big(p^{-n/2}\big) = O(p^{n/2}). \tag{I13}$$

In other words, the quantum Fisher information on Bob's end after exposure of the state to IID dephasing noise along the Z axis is well protected, in that the loss is suppressed to the order $O(p^{n/2})$. Observe that ρ_E is full rank, and therefore our uncertainty relation holds with equality in this setting.

Consider now the IID bit-flip noise channel $[\mathcal{N}^{(p)}_{\text{bit-flip}}]^{\otimes n}$ determined by the single-site Kraus operators $E^{(p)}_{\text{bit-flip},0}$ and $E^{(p)}_{\text{bit-flip},1}$. We find

$$\mathcal{N}^{(p)}_{\text{bit-flip}}(|+\rangle\langle+|) = |+\rangle\langle+|; \quad \mathcal{N}^{(p)}_{\text{bit-flip}}(|+\rangle\langle-|) \\ = (1-p)|+\rangle\langle-|. \tag{I14}$$

We would like to compute

$$\begin{aligned} F_{\text{Bob},t} &= F\Big([\mathcal{N}^{(p)}_{\text{bit-flip}}]^{\otimes n}(|+\rangle\langle+|^{\otimes n})\,;\,[\mathcal{N}^{(p)}_{\text{bit-flip}}]^{\otimes n} \\ &\quad\times\big(-i[|-\rangle\langle+|]^{\otimes n} + i[|+\rangle\langle-|]^{\otimes n}\big)\Big) \\ &= F\Big(|+\rangle\langle+|^{\otimes n}\;;\;-i[(1-p)|-\rangle\langle+|]^{\otimes n} + \text{h.c.}\Big) \\ &= 4\,\langle+^n|\Big[-i[(1-p)|-\rangle\langle+|]^{\otimes n} + \text{h.c.}\Big]^2|+^n\rangle, \end{aligned} \tag{I15}$$

where the last equality follows from Proposition 12. With

$$\langle+^n|\Big[-i[(1-p)|-\rangle\langle+|]^{\otimes n} + \text{h.c.}\Big] = i(1-p)^n\langle-^n|, \tag{I16}$$

we find

$$(\text{I15}) = 4(1-p)^{2n} = 4 - 8np + O(p^2). \tag{I17}$$

Therefore, for bit-flip IID noise, we have

$$\Delta F_{\text{Bob},t}[\text{bit-flip}] = 8np + O(p^2), \tag{I18}$$

meaning that the quantum Fisher information loss is linear in p despite the high metrological distance d_m.

Note that, in the case of IID bit-flip noise, our uncertainty relation equality conditions are not satisfied, since the rank of ρ_B changes locally as a function of time. In other words, we should not expect our uncertainty relation to hold with equality. This fact does not impact our calculation of the quantum Fisher information loss (I18), since we determined this value by direct computation on Bob's side. However, based on this example, we are tempted to hypothesize that settings in which a high metrological distance does not inhibit a high accuracy loss under weak IID noise coincide with the settings in which our uncertainty relation does not hold with equality. In the remainder of this Appendix, we provide additional indications in favor of this hypothesis.

2. Discontinuities of the quantum Fisher and uncertainty relation equality conditions

We briefly return to study the behavior of the quantum Fisher information in a simple example in which our uncertainty relation equality conditions are not satisfied. In such cases, the state on Bob's side changes rank, and it is known that the quantum Fisher information can be discontinuous [33,34,110].

The definition of the quantum Fisher information that we use [Eq. 9], which can differ from the expression stemming from the second-order expansion of the Bures metric [33,34,110], directly expresses the accuracy to which one can sense an unknown parameter via an observable that reveals the true value of the parameter locally in expectation value (see Proposition 19 in Appendix D).

It is a fundamental principle in quantum information that a quantity that is measurable in a physical setting should be robust to infinitesimal perturbations of the quantum state. Yet, how is it possible that the quantum Fisher information is discontinuous, if it directly corresponds to the physically operational sensitivity to which one can estimate an unknown parameter locally? We attribute this discontinuity to the assumption, in Proposition 19 in Appendix D, that the sensing observable reveals the true parameter value *in expectation value*. An expectation value needs to be estimated using multiple rounds of measurements, and depending on the outcome distribution of the observable, an arbitrary large number of measurements might be required to accurately estimate its expectation value. In the following example, we study how the optimal sensing observable diverges close to discontinuity points of the quantum Fisher information; namely, the discontinuity can be associated with diverging eigenvalues of the observable associated with eigenstates that are outside the support of the state at the discontinuity point.

Overall, this example indicates that the operational relevance of the quantum Fisher information might break down in certain regimes where it is not possible to accurately estimate the expectation value of the optimal sensing observable.

The following example is based on Refs. [33,34,110]. Consider the example of Sec. IV B: a qubit state evolving along the equator of the Bloch sphere is collapsed by the noise channel along the X axis of the Bloch sphere. Bob's quantum Fisher information is constant and equal to ω^2 almost all the time, except when the state is exactly a $\pm X$ eigenstate, in which case Bob's quantum Fisher information is equal to zero. The state on Bob's end is given by Eq. (89) as

$$\rho_B = p_+|+\rangle\langle+| + p_-|-\rangle\langle-|; \\ p_+ = \cos^2\Big(\frac{\omega t_0}{2}\Big); \quad p_- = \sin^2\Big(\frac{\omega t_0}{2}\Big). \tag{I19}$$

When Bob's quantum Fisher information $F_{\mathrm{Bob},t}$ is nonzero, there is always an observable O whose expectation value reveals the true parameter value locally, i.e., $\langle O\rangle_{\rho(t_0+dt)} = t_0 + dt + O(dt^2)$, and whose variance is $\langle O^2\rangle - \langle O\rangle^2 = 1/\omega^2$ (cf. Appendix D). The optimal sensing observable is given by the suitably normalized symmetric logarithmic derivative (Proposition 19) and can be computed, when ωt_0 is not a multiple of π, as follows:

$$\begin{aligned} O - t_0\mathbb{1} &= \frac{1}{\omega^2}\mathcal{R}^{-1}_{\rho_{t_0}}(\partial_t\rho) = \frac{1}{\omega^2} \\ &\quad\times \sum_{k,k'=\pm}\frac{2}{p_k+p_{k'}}\langle k|(\partial_t\rho)|k'\rangle|k\rangle\langle k'| \\ &= \frac{2}{2\omega^2\cos^2(\frac{\omega t_0}{2})}\langle +|(\partial_t\rho)|+\rangle|+\rangle\langle +| \\ &\quad+\frac{2}{2\omega^2\sin^2(\frac{\omega t_0}{2})}\langle -|(\partial_t\rho)|-\rangle|-\rangle\langle -| \\ &= -\frac{1}{\omega}\tan\Big(\frac{\omega t_0}{2}\Big)|+\rangle\langle +| + \frac{1}{\omega}\Big[\tan\Big(\frac{\omega t_0}{2}\Big)\Big]^{-1}|-\rangle\langle -|, \end{aligned} \tag{I20}$$

using the relation $(\partial_t\rho) = -(\omega/2)\sin(\omega t_0)\,X$ [cf. Eq. (91)], which implies $\langle\pm|(\partial_t\rho)|\pm\rangle = \mp(\omega/2)\sin(\omega t_0) = \mp\omega \sin(\omega t_0/2)\cos(\omega t_0/2)$.

As a sanity check, we can verify that O satisfies

$$\langle O\rangle_{\rho(t_0+dt)} = t_0 + dt + O(dt^2), \tag{I21}$$

as well as

$$\sigma_O^2 = \langle O^2\rangle_{\rho_{t_0}} - \langle O\rangle^2_{\rho_{t_0}} = \frac{1}{\omega^2}. \tag{I22}$$

As the state gets closer to a discontinuity (for instance, at $t_0 = 0$), this optimal sensing observable has one eigenvalue that diverges (for $t_0 = 0$, this eigenvalue is associated with the eigenvector $|-\rangle$). At the discontinuous point, the derivative is zero locally, so no observable will ever be able to correctly reveal the true value of the parameter to first order locally. The state does not change to first order in t at all! We can attribute the discontinuity to the fact that an optimal sensing observable for one state might turn out to no longer be an acceptable sensing observable for a neighboring point. In other words, while the variance of an observable is continuous both as a function of the state and of the observable, the *optimal* variance in the local sensing scenario is discontinuous because the conditions of the optimization (D1) are discontinuous.

At the discontinuity $t_0 = 0$, the derivative $\partial_t\rho$ vanishes locally, and it is impossible to find an observable O such that $\langle O\rangle_{\rho(t_0+dt)} = t_0 + dt + O(dt^2)$. By convention we set the corresponding quantum Fisher information to be zero; first, it is convenient because we do not have to modify the definition of the quantum Fisher information, and second, it expresses the fact that we cannot have any sensitivity locally to first order in the parameter by measuring the expectation value of an observable. If the quantum Fisher information is defined starting from the Bures distance, a mismatch will be observed; this mismatch could be interpreted as a failure of the Cramér-Rao bound.

Operationally, even for t_0 not at one of the discontinuities, the use of the expectation value as the way of reading out the parameter in the estimation process might be problematic. Estimating the expectation value of O to good accuracy, for $t_0 \approx 0$, requires that we observe sufficiently many times the $|-\rangle$ outcome, even though the latter only appears with the vanishing probability $\sin^2(\omega(t_0+dt)/2)$. If we do not repeat the measurement on enough copies, we would only empirically observe $|+\rangle$ events and we would erroneously estimate the expectation value of O to be equal to $-[\tan(\omega t_0/2)]/\omega$, and that its variance is zero. Not only this result would be wrong as it does not depend on the actual value dt that we wanted to measure, but the variance is certainly incorrect since the optimal variance when an infinite number of measurements is available is $1/\omega^2$. There might be opportunities for defining and investigating refined measures of sensitivity that can account for the finite amount of measurement outcomes that can be collected in the estimation process.

The above example illustrates that the quantum Fisher information can be problematic to interpret in certain regimes close to points where the rank of the state can change. This type of regime can occur for metrological codes, if the noise happens to fix the state vector $|\psi\rangle$ while not fixing other states that are infinitesimally close to $|\psi\rangle$, resulting in a rank change for Bob and Eve's states. We observe that in the context of metrological codes exposed to weak IID noise, the quantum Fisher information is not actually discontinuous as a function of the noise parameter; rather, it is the order in p of the Fisher information loss that can behave unexpectedly. That the quantum Fisher information loss must be suppressed at least to the order $O(p)$ follows from our continuity bound Proposition 18, noting that the weak IID noise channel is $O(p)$ close to the identity channel.

3. Suppression of quantum Fisher information loss in a restricted setting

Here we show that, when considering a metrological code exposed to weak IID noise in a restricted setting with additional assumptions, the quantum Fisher information loss $\Delta F_{\mathrm{Bob},t}$ is suppressed to the expected order $O(p^{d_m/2})$, where d_m is the metrological distance of the metrological code.

Proposition 23.—let $|\psi\rangle, |\xi\rangle$ define a metrological code of metrological distance d_m. Let $\mathcal{N}_1$ be a single-site noise operator with a Kraus representation $\{E_1^{(k)}\}_{k=1}^{K}$ that is such that $\|E_1^{(k')}\|_\infty = O(\sqrt{p})$ for $k' \neq 1$. Furthermore, if

$\mathbf{x}$ denotes a string of Kraus operator labels with $x_i \in \{1, \ldots, K\}$, and if $E_{\mathbf{x}} = \big(\bigotimes_{i=1}^{n} E_1^{(x_i)}\big)$, we assume that the states $\{E_{\mathbf{x}}|\psi\rangle\}_{\mathbf{x}}$ are all nonzero and orthogonal, and that $\|E_{\mathbf{x}}|\psi\rangle\| \geqslant \Omega(p^{|\mathbf{x}|/2})$. Then $\Delta F_{\text{Bob},t} = O(p^{d_m/2})$.

This result follows fairly straightforwardly from Eq. (130) in Sec. V E.

Proof.—Using the notation in Eq. (130), with $\epsilon = p$, we have that $\Delta F_{\text{Bob},t} = O(p^m)$ with

$$m = \min_{\mathbf{x},\mathbf{x}'} \Big\{ 2q_{\mathbf{x},\mathbf{x}'} - \min\big(r_{\mathbf{x}}, r_{\mathbf{x}'}\big) \Big\}, \tag{I23}$$

where $r_{\mathbf{x}}$ and $q_{\mathbf{x},\mathbf{x}'}$ are defined via

$$\langle\psi| E_{\mathbf{x}}^\dagger E_{\mathbf{x}} |\psi\rangle = \Omega(p^{r_{\mathbf{x}}});$$
$$\mathrm{tr}\big\{E_{\mathbf{x}'}^\dagger E_{\mathbf{x}}\big(|\xi\rangle\langle\psi| + |\psi\rangle\langle\xi|\big)\big\} = O\big(p^{q_{\mathbf{x},\mathbf{x}'}}\big), \tag{I24}$$

setting by convention $q_{\mathbf{x},\mathbf{x}'} = \infty$ whenever we have $\mathrm{tr}\big\{E_{\mathbf{x}'}^\dagger E_{\mathbf{x}}\big(|\xi\rangle\langle\psi| + |\psi\rangle\langle\xi|\big)\big\} = 0$. From our assumption that $E_{\mathbf{x}}|\psi\rangle \neq 0$, we see that $r_{\mathbf{x}}$ is always finite.

We now consider different cases for $\mathbf{x}, \mathbf{x}'$. Suppose first that $|\mathbf{x}| + |\mathbf{x}'| < d_m$. Then, since $|\psi\rangle, |\xi\rangle$ form a metrological code of metrological distance d_m, we have $q_{\mathbf{x},\mathbf{x}'} = \infty$. Now suppose instead that $|\mathbf{x}| + |\mathbf{x}'| \geqslant d_m$, implying that either $|\mathbf{x}| \geqslant d_m/2$ or $|\mathbf{x}'| \geqslant d_m/2$. Then, since $\|E_{\mathbf{x}}|\psi\rangle\| = \Omega\big(p^{|\mathbf{x}|/2}\big)$, we find

$$\langle\psi| E_{\mathbf{x}}^\dagger E_{\mathbf{x}} |\psi\rangle = \|E_{\mathbf{x}}|\psi\rangle\|^2 = \Omega\big(p^{|\mathbf{x}|}\big), \tag{I25}$$

so we can pick $r_{\mathbf{x}} = |\mathbf{x}|$. Since $E_1^{(x_i)} = O(\sqrt{p})$ for each $x_i \neq 0$, we have

$$\begin{aligned}
&\mathrm{tr}\big\{E_{\mathbf{x}'}^\dagger E_{\mathbf{x}}\big(|\xi\rangle\langle\psi| + |\psi\rangle\langle\xi|\big)\big\} \\
&\quad = \langle\xi| E_{\mathbf{x}'}^\dagger E_{\mathbf{x}} |\psi\rangle + \langle\psi| E_{\mathbf{x}'}^\dagger E_{\mathbf{x}} |\xi\rangle \\
&\quad = (\sqrt{p})^{|\mathbf{x}'|+|\mathbf{x}|}\, O(1) = O\big(p^{(|\mathbf{x}'|+|\mathbf{x}|)/2}\big),
\end{aligned} \tag{I26}$$

so we can pick $q_{\mathbf{x},\mathbf{x}'} = (|\mathbf{x}'| + |\mathbf{x}|)/2$. Then

$$\begin{aligned}
2q_{\mathbf{x},\mathbf{x}'} - \min\{r_{\mathbf{x}}, r_{\mathbf{x}'}\} &= |\mathbf{x}| + |\mathbf{x}'| - \min\{|\mathbf{x}|, |\mathbf{x}'|\} \\
&= \max\{|\mathbf{x}|, |\mathbf{x}'|\} \geqslant d_m/2.
\end{aligned} \tag{I27}$$

In all cases, we have $2q_{\mathbf{x},\mathbf{x}'} - \min\{r_{\mathbf{x}}, r_{\mathbf{x}'}\} \geqslant d_m/2$ and thus

$$\Delta F_{\text{Bob},t} \leqslant O\big(p^{d_m/2}\big), \tag{I28}$$

as claimed. ■

There are two strong assumptions made in the above proposition. First, we assume that the Kraus operator representation satisfies $\mathrm{tr}\{E_{\mathbf{x}'}^\dagger E_{\mathbf{x}}\, \psi\} \propto \delta_{\mathbf{x},\mathbf{x}'}$, or equivalently, that ρ_E is diagonal; such a representation always exists but might be difficult to find. Second, the state on Eve must not be rank-deficient, or equivalently, there is no Kraus operator $E_{\mathbf{x}}$ that has zero probability of occurring when the channel is applied onto the state ψ. It is not immediately clear to us how to generalize the above proposition to weaken either of these assumptions.

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