Interval Observers for Hybrid Dynamical Systems with Known Jump Times

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***Abstract*— This paper proposes a novel asymptotically stable interval estimator design for hybrid systems with nonlinear dynamics and observations under the assumption of known jump times. The proposed architecture leverages the concepts of mixed-monotone decompositions to construct a hybrid interval observer that is guaranteed to frame the true states (i.e., is correct) by construction. Moreover, using Lyapunov analysis and the positive system property of the framer error dynamics, we propose two approaches for computing the observer gains to achieve asymptotic stability of the error system based on mixed-integer semidefinite and linear programs. Further, our observer design incorporates additional degrees of freedom that may provide some advantages similar to coordinate transformations. Finally, we demonstrate the efficacy of the proposed hybrid observer design using two illustrative examples.**

I. Introduction

Hybrid systems, which combine continuous and discrete system dynamics, are prevalent in cyber-physical systems (CPS) due to their ability to model complex behavior and interactions between physical and computational elements. State estimation is a crucial problem in these CPS applications such as autonomous vehicles and power systems, either for the purpose of monitoring, fault diagnosis or control and decision making. Interval observers are one such class of state estimators that provide interval-valued state estimates, which are especially useful when the distributions of the initial state uncertainty and/or noise signals are unknown.

Literature Review. The design of set-valued/interval observers for various system classes, including linear, nonlinear, mixed-monotone, and cooperative/monotone/Metzler dynamics, has been extensively researched, e.g., in [1]–[3]. The primary idea in most interval observers is to design appropriate observer gains to ensure that the observer error dynamics are both Schur/Hurwitz stable and positive/cooperative. For certain classes of systems, interval observers were designed by leveraging interval arithmetic approaches [4], by applying state transformations [2] or by transformation to positive systems [5]. Moreover, for more general nonlinear systems, bounding functions have be leveraged to pose the observer design as a semidefinite program [2], [6]–[10]. Further, to tackle infeasibility problems in such observer designs, coordinate transformations or additional degrees of freedom were introduced in [11]–[13].

On the other hand, the design of observers for hybrid systems is more challenging because hybrid systems combine both continuous/flow dynamics and discrete/jump dynamics. An asymptotic observer design framework was introduced for hybrid systems with (approximately) known jump times in [14], [15]. Moreover, hybrid interval observers were designed for specific classes of hybrid systems such as linear impulsive systems [16]–[19], switched linear systems [19]–[21] and switched nonlinear systems [22] under various potentially conservative simplifying assumptions.

Contributions. Inspired by the design of interval observers for general nonlinear continuous- and discrete-time systems in [11] and the construction of (non-interval) observers for certain hybrid systems with approximately known jump times in [14], [15], we propose a novel interval observer design for general nonlinear hybrid systems with known jump times. Specifically, we leverage the concept of mixed-monotone decompositions [6] in the design framework to construct a hybrid interval observer whose framers are guaranteed to upper and lower bound the true hybrid system state, i.e., the hybrid interval observer is *correct* by construction without imposing any additional positivity constraints. Additionally, in this framework, we propose additional degrees of freedom that can be simultaneously synthesized with the observer gains. Further, we utilize Lyapunov analysis with quadratic and linear Lyapunov functions to design two observer variants for computing the observer gains to guarantee asymptotic stability of the error dynamics of the proposed hybrid interval observer using mixed-integer semidefinite and mixed-integer linear programs, respectively, which can be solved using off-the-shelf optimization solvers.

II. Preliminaries

Notation. The p-norm of a vector $v \in \mathbb{R}^n$ is given by $\|v\|_p \triangleq (\sum_{i=1}^n |v_i|^p)^{\frac{1}{p}}$, and for a matrix $M \in \mathbb{R}^{n\times p}$, the element in i-th row and j-th column is denoted by M_{ij}. The element-wise signum function of a matrix M is denoted by $\mathrm{sgn}(M)$, and $M^{\oplus} \triangleq \max(M, \mathbf{0}_{n\times p})$, $M^{\ominus} \triangleq M^{\oplus} - M$, and $|M| \triangleq M^{\oplus} + M^{\ominus}$ is the element-wise absolute value of M. The diagonal matrix with the diagonal elements of a square matrix $M \in \mathbb{R}^{n\times n}$ is denoted by M^{d}, while $M^{\mathrm{nd}} \triangleq M - M^{\mathrm{d}}$ is the matrix with only its off-diagonal elements. The "Metzlerized" matrix $M^{\mathrm{m}} \triangleq M^{\mathrm{d}} + |M^{\mathrm{nd}}|$ is a square matrix in which all the off-diagonal components are non-negative. All matrix and vector inequalities are element-wise inequalities, and the matrices of ones and zeros of dimension $n\times p$ are denoted by $\mathbf{1}_{n\times p}$ and $\mathbf{0}_{n\times p}$, respectively. Further, an interval $\mathcal{I} \triangleq [\underline{z}, \overline{z}] \subset \mathbb{R}^{n_z}$ is a set of vectors $z \in \mathbb{R}^{n_z}$ satisfying $\underline{z} \le z \le \overline{z}$. A corresponding definition applies to intervals of matrices.

First, we review some mixed-monotonicity theory basics that will be leveraged in our interval observer design.

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Proposition 1 (Jacobian Sign-Stable Decomposition [9, Proposition 2]). *If a mapping $f : \mathcal{Z} \subset \mathbb{R}^{n_z} \to \mathbb{R}^p$ has Jacobian matrices satisfying $J^f(z) \in [\underline{J}^f, \overline{J}^f]$ for all $z \in \mathcal{Z}$ with $\underline{J}^f, \overline{J}^f \in \mathbb{R}^{p \times n_z}$, f can be decomposed into the additive remainder-form*

$$f(z) = Hz + \mu(z), \qquad \forall z \in \mathcal{Z}, \tag{1}$$

where the matrix $H \in \mathbb{R}^{p \times n_z}$ satisfies

$$H_{ij} = \underline{J}^f_{ij} \text{ or } H_{ij} = \overline{J}^f_{i,j}, \quad \forall (i,j) \in \mathbb{N}_p \times \mathbb{N}_{n_z}, \tag{2}$$

and $\mu(\cdot)$ and Hz are nonlinear and linear Jacobian sign-stable (JSS) mappings, respectively, i.e., the signs of each element of their Jacobian matrices do not change within their domains ($J^{\nu}_{ij}(\cdot) \geq 0$ or $J^{\nu}_{ij}(\cdot) \leq 0$, $\nu(z) \in \{\mu(z), Hz\}$).

Definition 1 (Mixed-Monotonicity and Decomposition Functions). *[23, Definition 1], [24, Definition 4] Given $g : \mathcal{X} \to \mathbb{R}^n$ with $\mathcal{X} \subset \mathbb{R}^n$, a function $g_\delta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^n$ is a mixed-monotone decomposition mapping of g for*

1) *the discrete-time (DT) system $x^+ = g(x)$ if i) $g_\delta(x, x) = g(x)$ for all $x \in \mathcal{X}$, ii) g_δ is monotone increasing in its first argument, i.e., $\hat{x} \geq x \Rightarrow g_\delta(\hat{x}, x') \geq g_\delta(x, x')$ for all x', and iii) g_δ is monotone decreasing in its second argument, i.e., $\hat{x} \geq x \Rightarrow g_\delta(x', \hat{x}) \leq g_\delta(x', x)$ for all x'.*
2) *the continuous-time (CT) system $\dot{x} = g(x)$ if i) and iii) hold as in item 1, and ii') g_δ is monotone increasing in its first argument with respect to "off-diagonal" arguments, i.e., $\forall (i,j) \in \mathbb{N}_n \times \mathbb{N}_n$ such that $i \neq j$, $\hat{x}_j \geq x_j, \hat{x}_i = x_i \Rightarrow g_{\delta,i}(\hat{x}, x') \geq g_{\delta,i}(x, x')$ for all x'.*

Proposition 2 (Tight and Tractable Decomposition Functions for JSS Mappings [9, Proposition 4]). *Suppose $\mu : \mathcal{Z} \subset \mathbb{R}^{n_z} \to \mathbb{R}^p$ is a JSS mapping on its domain. Then, for each $\mu_i(\cdot)$, $i \in \mathbb{N}_p$, its tight decomposition function[a] is given by:*

$$\mu_{\delta,i}(z_1, z_2) = \mu_i(D^i z_1 + (I_{n_z} - D^i) z_2), \tag{3}$$

for any ordered $z_1, z_2 \in \mathcal{Z}$, with a binary diagonal matrix D^i that is determined by the vertex of the interval $[z_1, z_2]$ that minimizes the function μ_i (if $z_1 < z_2$) or the vertex of the interval $[z_2, z_1]$ that maximizes μ_i (if $z_2 \leq z_1$), given by $D^i = \text{diag}(\max(\text{sgn}(\overline{J}^{\mu}_i), \mathbf{0}_{1,n_z}))$.

Consequently, by applying Proposition 2 to the Jacobian sign-stable decomposition obtained using Proposition 1, a tight and tractable remainder-form decomposition function can be obtained. Further details can be found in [9].

Definition 2 (Embedding System). *[10, Definition 6] Given $g : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}^n$ and a pair of mixed-monotone decomposition functions, $\underline{g}_\delta$ and $\overline{g}_\delta$, respectively, the embedding systems associated with the CT system $\dot{x} = g(x)$ and the DT system $x^+ = g(x)$ are $2n$-dimensional systems given by, respectively,*

$$\begin{bmatrix} \dot{\underline{x}} \\ \dot{\overline{x}} \end{bmatrix} = \begin{bmatrix} \underline{g}_\delta(\underline{x}, \overline{x}) \\ \overline{g}_\delta(\overline{x}, \underline{x}) \end{bmatrix} \text{(CT)}, \quad \begin{bmatrix} \underline{x}^+ \\ \overline{x}^+ \end{bmatrix} = \begin{bmatrix} \underline{g}_\delta(\underline{x}, \overline{x}) \\ \overline{g}_\delta(\overline{x}, \underline{x}) \end{bmatrix} \text{(DT)}. \tag{4}$$

[a]A decomposition function μ_δ of the function μ is tight if $\max_{z \in \{z \in \mathcal{Z} | \underline{z} \leq z \leq \overline{z}\}} \mu(z) = \mu_\delta(\overline{z}, \underline{z})$ and $\min_{z \in \{z \in \mathcal{Z} | \underline{z} \leq z \leq \overline{z}\}} \mu(z) = \mu_\delta(\underline{z}, \overline{z})$.

Then, by [6, Proposition 3], the embedding systems in (4) have a *state framer property*, i.e., the solution $t \mapsto x(t)$ to the CT system in (4) starting from $\begin{bmatrix} \underline{x}(0)^\top & \overline{x}(0)^\top \end{bmatrix}^\top$ satisfies $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ for all $t \geq 0$.

III. Problem Formulation

Consider a hybrid system $\mathcal{H}$ as in [25] with flow and jump dynamics characterized by mappings $f_c : \mathbb{R}^n \to \mathbb{R}^n$ and $f_d : \mathbb{R}^n \to \mathbb{R}^n$, respectively and output mappings $h_c : \mathbb{R}^n \to \mathbb{R}^{l_c}$ and $h_d : \mathbb{R}^n \to \mathbb{R}^{l_d}$, as follows:

$$\mathcal{H} \begin{cases} \dot{x} \ = f_c(x) & x \in \mathcal{C}, \\ x^+ = f_d(x) & x \in \mathcal{D}, \\ y_c \ = h_c(x) & x \in \mathcal{C}, \\ y_d \ = h_d(x) & x \in \mathcal{D}, \end{cases} \tag{5}$$

with an *uncertain* initial state $x(0,0)$ satisfying $x(0,0) \in \mathcal{X}_0 \triangleq [\underline{x}(0,0), \overline{x}(0,0)] \subset \mathcal{X}$, where $\underline{x}(0,0)$ and $\overline{x}(0,0)$ are known, while $x \in \mathbb{R}^n$ is the state and $y = (y_c, y_d)$ is the output with $y_c \in \mathbb{R}^{l_c}$ and $y_d \in \mathbb{R}^{l_d}$. The sets $\mathcal{C} \subset \mathbb{R}^n$ and $\mathcal{D} \subset \mathbb{R}^n$ represent the flow set and the jump set of $\mathcal{H}$, respectively.

Additionally, a solution x of the hybrid system $\mathcal{H}$ is given by a function defined on a hybrid time domain denoted by dom $x \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ such that for any $(T, J) \in$ dom x, $\exists$ $0 = t_0 \leq t_1 \leq \cdots \leq t_J$ that satisfy:

$$\text{dom } x \cap ([0,T] \times \{0, 1, ..., J\}) = \textstyle\bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j).$$

Further, $\text{dom}_t(x)$ and $\text{dom}_j(x)$ represent the projection of dom x in its first and second dimension, respectively. We will also require the following assumption:

Assumption 1. *The system jump times, as well as the outputs y_c during flows and/or y_d at jumps, are known.*

We are ready to state the hybrid interval observer synthesis problem as follows:

Problem 1. *Given the hybrid system $\mathcal{H}$ in* (5)*, under the assumption of known jump times (cf. Assumption 1), design a hybrid interval observer that computes the lower and upper framers, $\underline{x}(t,j), \overline{x}(t,j)$, that frame each solution x to $\mathcal{H}$ from $x(0,0) \in \mathcal{X}_0$ (with known $\underline{x}(0,0), \overline{x}(0,0)$), i.e., $\underline{x}(t,j) \leq x(t,j) \leq \overline{x}(t,j)$ for all $(t,j) \in$ dom $x =$ dom $\overline{x} =$ dom $\underline{x}$ (such an observer is called* correct*) such that the framer error dynamics for $\varepsilon(t,j) \triangleq \overline{x}(t,j) - \underline{x}(t,j)$ is asymptotically stable, i.e., $\exists$ $\theta > 0$ and $\gamma > 0$ such that for any given $\varepsilon(0,0) \triangleq \overline{x}(0,0) - \underline{x}(0,0)$,*

$$|\varepsilon(t,j)| \leq \gamma |\varepsilon(0,0)| e^{-\theta(t+j)}, \quad \forall (t,j) \in \text{dom } x. \tag{6}$$

IV. Hybrid Interval Observer

In this section, we describe the construction of the proposed hybrid interval observer as well as analyze its correctness and asymptotic stability properties. Note that in the rest of the paper, for brevity we drop the explicit dependence on the hybrid time (t,j) unless explicitly necessary.

A. Interval Observer Design

Inspired by our work on interval observers for discrete- and continuous-time systems in [11], we begin by deriving an equivalent representation of the system dynamics for the hybrid system $\mathcal{H}$ in (5), under the following assumption.

Assumption 2. *The mappings f_c, f_d, h_c, h_d are known and differentiable w.r.t. x with a priori known lower and upper bounds for their Jacobian matrices w.r.t. x, $\underline{J}^{f_c}, \overline{J}^{f_c}, \underline{J}^{f_d}, \overline{J}^{f_d} \in \mathbb{R}^{n\times n}$, $\underline{J}^{h_c}, \overline{J}^{h_c} \in \mathbb{R}^{l_c\times n}$ and $\underline{J}^{h_d}, \overline{J}^{h_d} \in \mathbb{R}^{l_d\times n}$, respectively.*

Lemma 1 (Equivalent System)**.** *Consider the hybrid system $\mathcal{H}$ in* (5) *and suppose that Assumption 2 hold. Let $L_c, N_c \in \mathbb{R}^{n\times l_1}$, $L_d, N_d \in \mathbb{R}^{n\times l_2}$ and $T_c, T_d \in \mathbb{R}^{n\times n}$ be arbitrary matrices that satisfy $T_c + N_c H_c = I_n$ and $T_d + N_d H_d = I_n$. Then, the hybrid system dynamics* (5) *can be equivalently written as*

$$\begin{aligned}
&\left.\begin{aligned}
x &= \xi + N_c y_c,\\
\zeta &= T_d x - N_d \psi_d(x),\\
\dot{\xi} &= (T_cA_c - L_cH_c - N_cA_{2c})(\xi + N_c y_c)\\
&\quad + T_c\phi_c(x) - N_c\rho_c(x) - L_c\psi_c(x) + L_c y_c,
\end{aligned}\right\} \quad x \in \mathcal{C},\\
&\left.\begin{aligned}
x &= \zeta + N_d y_d,\\
\zeta^+ &= (T_dA_d - L_dH_d - N_dA_{2d})(\zeta + N_d y_d)\\
&\quad + T_d\phi_d(x) - N_d\rho_d(x) - L_d\psi_d(x) + L_d y_d,\\
\xi^+ &= T_c(A_dx + \phi_d(x)) - N_c\psi_c(A_dx + \phi_d(x)),
\end{aligned}\right\} \quad x \in \mathcal{D},
\end{aligned} \tag{7}$$

where ξ, ζ are auxiliary states of the equivalent system, while y_c, y_d and x can be viewed as its inputs and output, respectively. Further, $A_c, A_d \in \mathbb{R}^{n\times n}$, $C_c, A_{2c} \in \mathbb{R}^{l_1\times n}$, and $C_d, A_{2d} \in \mathbb{R}^{l_2\times n}$ are chosen such that the following decompositions hold for all $x \in \mathcal{X}$ (cf. Proposition 1):

$$\begin{aligned}
f_c(x) = A_cx + \phi_c(x),&\ \ f_d(x) = A_dx + \phi_d(x),\\
h_c(x) = H_cx + \psi_c(x),&\ \ h_d(x) = H_dx + \psi_d(x),\\
\tfrac{\partial\psi_c}{\partial x}(x)f_c(x) = A_{2c}x + \rho_c(x),&\ \ \psi_d(f_d(x)) = A_{2d}x + \rho_d(x),
\end{aligned} \tag{8}$$

such that $\phi_c, \phi_d, \psi_c, \psi_d, \rho_c, \rho_d$ are JSS.

Proof. We begin by defining auxiliary states $\xi \triangleq x - N_c(H_cx + \psi_c(x))$ and $\zeta \triangleq x - N_d(H_dx + \psi_d(x))$, where, from (5) and (8), $\xi = x - N_cy_c$ when $x \in \mathcal{C}$ and $\zeta = x - N_dy_d$ when $x \in \mathcal{D}$. Moreover, since N_c and N_d satisfy $T_c + N_cH_c = T_d + N_dH_d = I_n$, we obtain $\xi = T_cx - N_c\psi_c(x)$ and $\zeta = T_dx - N_d\psi_d(x)$ that have the following dynamics:

$$\begin{aligned}
\dot{\xi} &= T_c(A_cx + \phi_c(x)) - N_c(A_{2c}x + \rho_c(x)),\\
\zeta^+ &= T_d(A_dx + \phi_d(x)) - N_d(A_{2d}x + \rho_d(x)),
\end{aligned}$$

where we applied the decompositions in (8). Finally, adding 'zero terms' $L_c(y_c - H_cx - \psi_c(x)) = 0$ and $L_d(y_d - H_dx - \psi_d(x)) = 0$ (cf. (5) and (8)) to each of the above, respectively, yields (7), where x can be recovered from the definitions of ξ and ζ, while ξ^+ can be found from the definition of $\xi = x - N_c(H_cx + \psi_c(x)) = T_cx - N_c\psi_c(x)$:

$$\xi^+ = T_cx^+ - N_c\psi_c(x^+),$$

with $x^+ = A_dx + \phi_d(x)$. ■

Then, using the equivalent system in (7), we propose a hybrid interval observer $\hat{\mathcal{H}}$ based on the construction of an embedding system (cf. Definition 2; see Appendix for details on the construction) to address Problem 1:

$$\hat{\mathcal{H}}\begin{cases}
\left.\begin{aligned}
\underline{x} &= \underline{\xi} + N_cy_c, \quad \overline{x} = \overline{\xi} + N_cy_c,\\
\underline{\zeta} &= T_d^{\oplus}\underline{x} - T_d^{\ominus}\overline{x}\\
&\quad - N_d^{\oplus}\psi_{d,\delta}(\overline{x},\underline{x}) + N_d^{\ominus}\psi_{d,\delta}(\underline{x},\overline{x}),\\
\overline{\zeta} &= T_d^{\oplus}\overline{x} - T_d^{\ominus}\underline{x}\\
&\quad - N_d^{\oplus}\psi_{d,\delta}(\underline{x},\overline{x}) + N_d^{\ominus}\psi_{d,\delta}(\overline{x},\underline{x}),\\
\dot{\underline{\xi}} &= (M_c^{\mathrm{d}} + M_c^{\mathrm{nd},\oplus})\underline{\xi} - M_c^{\mathrm{nd},\ominus}\overline{\xi}\\
&\quad + (L_c + M_cN_c)y_c\\
&\quad + T_c^{\oplus}\phi_{c,\delta}(\underline{x},\overline{x}) - T_c^{\ominus}\phi_{c,\delta}(\overline{x},\underline{x})\\
&\quad - L_c^{\oplus}\psi_{c,\delta}(\overline{x},\underline{x}) + L_c^{\ominus}\psi_{c,\delta}(\underline{x},\overline{x})\\
&\quad - N_c^{\oplus}\rho_{c,\delta}(\overline{x},\underline{x}) + N_c^{\ominus}\rho_{c,\delta}(\underline{x},\overline{x}),\\
\dot{\overline{\xi}} &= (M_c^{\mathrm{d}} + M_c^{\mathrm{nd},\oplus})\overline{\xi} - M_c^{\mathrm{nd},\ominus}\underline{\xi}\\
&\quad + (L_c + M_cN_c)y_c\\
&\quad + T_c^{\oplus}\phi_{c,\delta}(\overline{x},\underline{x}) - T_c^{\ominus}\phi_{c,\delta}(\underline{x},\overline{x})\\
&\quad - L_c^{\oplus}\psi_{c,\delta}(\underline{x},\overline{x}) + L_c^{\ominus}\psi_{c,\delta}(\overline{x},\underline{x})\\
&\quad - N_c^{\oplus}\rho_{c,\delta}(\underline{x},\overline{x}) + N_c^{\ominus}\rho_{c,\delta}(\overline{x},\underline{x}),
\end{aligned}\right\} & \text{when } \mathcal{H} \text{ flows,}\\
\left.\begin{aligned}
\underline{x} &= \underline{\zeta} + N_dy_d, \quad \overline{x} = \overline{\zeta} + N_dy_d,\\
\underline{\zeta}^+ &= M_d^{\oplus}\underline{\zeta} - M_d^{\ominus}\overline{\zeta} + (L_d + M_dN_d)y_d\\
&\quad + T_d^{\oplus}\phi_{d,\delta}(\underline{x},\overline{x}) - T_d^{\ominus}\phi_{d,\delta}(\overline{x},\underline{x})\\
&\quad - L_d^{\oplus}\psi_{d,\delta}(\overline{x},\underline{x}) + L_d^{\ominus}\psi_{d,\delta}(\underline{x},\overline{x})\\
&\quad - N_d^{\oplus}\rho_{d,\delta}(\overline{x},\underline{x}) + N_d^{\ominus}\rho_{d,\delta}(\underline{x},\overline{x}),\\
\overline{\zeta}^+ &= M_d^{\oplus}\overline{\zeta} - M_d^{\ominus}\underline{\zeta} + (L_d + M_dN_d)y_d\\
&\quad + T_d^{\oplus}\phi_{d,\delta}(\overline{x},\underline{x}) - T_d^{\ominus}\phi_{d,\delta}(\underline{x},\overline{x})\\
&\quad - L_d^{\oplus}\psi_{d,\delta}(\underline{x},\overline{x}) + L_d^{\ominus}\psi_{d,\delta}(\overline{x},\underline{x})\\
&\quad - N_d^{\oplus}\rho_{d,\delta}(\underline{x},\overline{x}) + N_d^{\ominus}\rho_{d,\delta}(\overline{x},\underline{x}),\\
\underline{\xi}^+ &= (T_cA_d)^{\oplus}\underline{x} - (T_cA_d)^{\ominus}\overline{x}\\
&\quad + T_c^{\oplus}\phi_{d,\delta}(\underline{x},\overline{x}) - T_c^{\ominus}\phi_{d,\delta}(\overline{x},\underline{x})\\
&\quad - N_c^{\oplus}\psi_{c,\delta}(\overline{z},\underline{z}) + N_c^{\ominus}\psi_{c,\delta}(\underline{z},\overline{z}),\\
\overline{\xi}^+ &= (T_cA_d)^{\oplus}\overline{x} - (T_cA_d)^{\ominus}\underline{x}\\
&\quad + T_c^{\oplus}\phi_{d,\delta}(\overline{x},\underline{x}) - T_c^{\ominus}\phi_{d,\delta}(\underline{x},\overline{x})\\
&\quad - N_c^{\oplus}\psi_{c,\delta}(\underline{z},\overline{z}) + N_c^{\ominus}\psi_{c,\delta}(\overline{z},\underline{z}),
\end{aligned}\right\} & \text{when } \mathcal{H} \text{ jumps,}
\end{cases} \tag{9}$$

where $\overline{x}, \underline{x} \in \mathbb{R}^n$ are upper and lower framers of the state x, respectively, $\overline{\xi}, \underline{\xi}, \overline{\zeta}, \underline{\zeta} \in \mathbb{R}^n$ are auxiliary framers and

$$\begin{aligned}
M_c &\triangleq T_cA_c - L_cH_c - N_cA_{2c},\\
M_d &\triangleq T_dA_d - L_dH_d - N_dA_{2d},\\
\underline{z} &\triangleq A_d^{\oplus}\underline{x} - A_d^{\ominus}\overline{x} + \phi_{d,\delta}(\underline{x},\overline{x}),\\
\overline{z} &\triangleq A_d^{\oplus}\overline{x} - A_d^{\ominus}\underline{x} + \phi_{d,\delta}(\overline{x},\underline{x}).
\end{aligned}$$

The hybrid Interval observer $\hat{\mathcal{H}}$ is initialized with $\overline{\xi}(0,0) = \overline{x}(0,0) - N_cy_c(0,0)$, $\underline{\xi}(0,0) = \underline{x}(0,0) - N_cy_c(0,0)$ when $\mathcal{H}$ begins with a flow and $\overline{\zeta}(0,0) = \overline{x}(0,0) - N_dy_d(0,0)$, $\underline{\zeta}(0,0) = \underline{x}(0,0) - N_dy_d(0,0)$ when $\mathcal{H}$ begins with a jump. Note that we know when $\mathcal{H}$ flows or jumps by Assumption 1; hence, $\hat{\mathcal{H}}$ is well defined.

Further, $\phi_{c,\delta}, \phi_{d,\delta} : \mathbb{R}^{2n} \to \mathbb{R}^n$, $\psi_{c,\delta}, \rho_{c,\delta} : \mathbb{R}^{2n} \to \mathbb{R}^{l_c}$ and $\psi_{d,\delta}, \rho_{d,\delta} : \mathbb{R}^{2n} \to \mathbb{R}^{l_d}$ are tight mixed-monotone decomposition functions of ϕ_c, ϕ_d, ψ_c, ρ_c, ψ_d, and ρ_d, respectively (cf. (5), Definition 1), which are JSS and thus, can be computed using (3). Finally, $N_c, L_c \in \mathbb{R}^{n\times l_c}$, $N_d, L_d \in$

$\mathbb{R}^{n\times l_d}$, and $T_c, T_d \in \mathbb{R}^{n\times n}$ are the observer gain matrices to be designed that satisfies $T_c + N_c H_c = T_d + N_d H_d = I_n$. The detailed properties of correctness and asymptotic stability of the observer design $\hat{\mathcal{H}}$ are proven in the next subsections.

Remark 1. *Note that the above observer design is inspired by [7] and [16] and involves six observer gains, N_c, T_c, L_c, N_d, T_d and L_d, which provide additional degrees of freedom when compared to existing hybrid observer designs, e.g., in [14], which often only have two degrees of freedom with gains L_c and L_d. These extra gains function as surrogates for coordinate transformation, but as discussed in [11, Remark 1], coordinate transformations can still be helpful to make the observer gain design problem in Theorem 2 and 3 feasible. This can be done in a straightforward manner (omitted for brevity), as detailed in [2, Section V] and [26].*

B. Hybrid Observer Correctness

In this section, we demonstrate that by construction, the hybrid interval estimator $\hat{\mathcal{H}}$ proposed in (9) for the hybrid system $\mathcal{H}$ is *correct*, i.e., its framers bound the true states.

Theorem 1 (Correctness)**.** *Suppose Assumptions 1-2 hold for the hybrid system $\mathcal{H}$ in* (5) *and let $\hat{\mathcal{H}}$ be its corresponding hybrid interval observer built according to* (9)*. Then, their respective solutions x and $[\underline{x}^\top, \overline{x}^\top]^\top$ satisfy $\underline{x}(t,j) \leq x(t,j) \leq \overline{x}(t,j), \forall (t,j) \in \text{dom } x = \text{dom } \overline{x} = \text{dom } \underline{x}$, i.e., the hybrid interval observer $\hat{\mathcal{H}}$ functions as a correct interval framer for the hybrid system $\mathcal{H}$.*

Proof. We start by considering the base case, $\underline{x}(0,0) \leq x(0,0) \leq \overline{x}(0,0)$, which is trivially true because of the assumption on the initial condition $x(0,0) \in \mathcal{X}_0 \triangleq [\underline{x}(0,0), \overline{x}(0,0)] \subset \mathcal{X}$. Next, assuming that $\underline{x}(t,j) \leq x(t,j) \leq \overline{x}(t,j)$ holds for some $(t,j) \in \text{dom } x$ with $t_j \leq t \leq t_{j+1}$, we will show that $\underline{x}(t,j+1) \leq x(t,j+1) \leq \overline{x}(t,j+1)$ holds for $(t,j+1) \in \text{dom } x$ with $t_{j+1} \leq t \leq t_{j+2}$. By construction, the continuous-time embedding system during flow in (9) guarantees the framer properties by [6, Proposition 3], i.e., $\underline{x}(t',j) \leq x(t',j) \leq \overline{x}(t',j),\ \forall t' : t \leq t' \leq t_{j+1}$. Then, by the construction of the discrete-time embedding system during jumps in (9) and [6, Proposition 3], we have $\underline{x}(t_{j+1}, j+1) \leq x(t_{j+1}, j+1) \leq \overline{x}(t_{j+1}, j+1),\ (t_{j+1}, j+1) \in \text{dom } x$. Finally, by [6, Proposition 3] again for the flow, we obtain $\underline{x}(t,j+1) \leq x(t,j+1) \leq \overline{x}(t,j+1)$ for each $(t,j+1) \in \text{dom } x$ with $t_{j+1} \leq t \leq t_{j+2}$. Thus, by the principle of mathematical induction, the theorem holds. ∎

C. Stable Observer Design

In addition to proving correctness, it is essential to ensure the stability of the proposed hybrid framer. Thus, we propose two variants for designing the observer gains T_c, T_d, N_c, N_d, L_c, and L_d to asymptotically stabilize (cf. (6)) the error dynamics of the hybrid interval observer under the following assumption:

Assumption 3. *There exists a closed subset $\mathcal{I}$ of $\mathbb{R}_{\geq 0}$ such that any maximal solution to $\mathcal{H}$ in* (5) *satisfies:*

- $t_{j+1}(x) - t_j(x) \in \mathcal{I},\ \forall j \in \{1, ..., \mathcal{J}(x) - 1\}$ *if* $\mathcal{J}(x) < +\infty$ *or* $\forall j \in \mathbb{N}_{\geq 0}$ *if* $\mathcal{J}(x) = +\infty$,
- $0 \leq t - t_j(x) \leq \sup \mathcal{I},\ \forall (t,j) \in \text{dom } x$,

where $t_j(x)$ is the time stamp corresponding to the j-th jump and $\mathcal{J}(x) \triangleq \sup \text{dom}_j x$.

1) Q-Hybrid Interval Observer: First, we outline the first variant that is based on the use of a *quadratic* Lyapunov function to prove asymptotic stability.

Theorem 2 (Q-Hybrid Interval Observer)**.** *The hybrid interval observer $\hat{\mathcal{H}}$ in* (9) *for the hybrid system $\mathcal{H}$ in* (5) *is such that the framer error dynamics for $\varepsilon \triangleq \overline{x} - \underline{x}$ is asymptotically stable if Assumptions 1-2 hold and there exist $a_c, a_d \in \mathbb{R}$, $\tilde{T}_c, \tilde{T}_d \in \mathbb{R}^{n\times n}$, $\tilde{N}_c, \tilde{L}_c \in \mathbb{R}^{n\times l_c}$, $\tilde{N}_d, \tilde{L}_d \in \mathbb{R}^{n\times l_d}$ and a diagonal matrix $P \succ 0$ such that:*

$$\Gamma^T + \Gamma \preceq a_c P, \tag{10a}$$

$$\begin{bmatrix} P & \Omega \\ \Omega^T & e^{a_d} P \end{bmatrix} \succeq 0, \tag{10b}$$

$$a_c \tau + a_d < 0, \quad \forall \tau \in \mathcal{I}, \tag{10c}$$

$$\tilde{T}_c + \tilde{N}_c H_c = P, \tag{10d}$$

$$\tilde{T}_d + \tilde{N}_d H_d = P, \tag{10e}$$

where $\Gamma \triangleq (\tilde{T}_c A_c - \tilde{L}_c H_c - \tilde{N}_c A_{2c})^{\text{m}} + |\tilde{T}_c|\overline{F}_{\phi_c} + |\tilde{L}_c|\overline{F}_{\psi_c} + |\tilde{N}_c|\overline{F}_{\rho_c}$, $\Omega \triangleq |\tilde{T}_d A_d - \tilde{L}_d H_d - \tilde{N}_d A_{2d}| + |\tilde{T}_d|\overline{F}_{\phi_d} + |\tilde{L}_d|\overline{F}_{\psi_d} + |\tilde{N}_d|\overline{F}_{\rho_d}$, $\mathcal{I}$ satisfies Assumption 3 and $\forall \mu \in \{\phi_c, \psi_c, \rho_c, \phi_d, \psi_d, \rho_d\}$, $\overline{F}_\mu$ are computed from the JSS functions μ with Jacobian matrices $J^\mu \in [\underline{J}^\mu, \overline{J}^\mu]$ as follows: $\overline{F}_\mu \triangleq (\overline{J}^\mu)^\oplus + (\underline{J}^\mu)^\ominus$.

Furthermore, the observer gains in (9) *can be obtained as $T_c \triangleq P^{-1}\tilde{T}_c$, $N_c \triangleq P^{-1}\tilde{N}_c$, $L_c \triangleq P^{-1}\tilde{L}_c$, $T_d \triangleq P^{-1}\tilde{T}_d$, $N_d \triangleq P^{-1}\tilde{N}_d$ and $L_d \triangleq P^{-1}\tilde{L}_d$.*

Proof. From (9), the dynamics of the framer error $\varepsilon \triangleq \overline{x} - \underline{x}$ is given by $\dot{\varepsilon} = \dot{\overline{x}} - \dot{\underline{x}} = \dot{\overline{\xi}} - \dot{\underline{\xi}}$ and $\varepsilon^+ = \overline{x}^+ - \underline{x}^+ = \overline{\xi}^+ - \underline{\xi}^+$. Then, the hybrid framer error dynamics $\tilde{\mathcal{H}}$ and its comparison system can be obtained as

$$\tilde{\mathcal{H}} \begin{cases} \left.\begin{aligned} \dot{\varepsilon} &= (T_c A_c - L_c H_c - N_c A_{2c})^{\text{m}} \varepsilon \\ &\quad + |T_c|\Delta_\delta^{\phi_c} + |L_c|\Delta_\delta^{\psi_c} + |N_c|\Delta_\delta^{\rho_c} \\ &\leq E_c \varepsilon \end{aligned}\right\} \text{when } \mathcal{H} \text{ flows}, \\ \left.\begin{aligned} \varepsilon^+ &= |T_d A_d - L_d H_d - N_d A_{2d}| \varepsilon \\ &\quad + |T_d|\Delta_\delta^{\phi_d} + |L_d|\Delta_\delta^{\psi_d} + |N_d|\Delta_\delta^{\rho_d} \\ &\leq E_d \varepsilon \end{aligned}\right\} \text{when } \mathcal{H} \text{ jumps}, \end{cases} \tag{11}$$

where we define $\Delta_\delta^\mu \triangleq \mu_\delta(\overline{x}, \underline{x}) - \mu_\delta(\underline{x}, \overline{x})$ for all $\mu \in \{\phi_c, \phi_d, \rho_c, \rho_d, \psi_c, \psi_d\}$. By [9, Lemma 3], Δ_δ^μ satisfies $\Delta_\delta^\mu \leq \overline{F}_\mu \varepsilon$; hence, $\dot{\varepsilon} \leq E_c \varepsilon$ and $\varepsilon^+ \leq E_d \varepsilon$ with $\varepsilon \geq 0$ by the correctness property in Theorem 1 as well as $E_c \triangleq (T_c A_c - L_c H_c - N_c A_{2c})^{\text{m}} + |T_c|\overline{F}_{\phi_c} + |L_c|\overline{F}_{\psi_c} + |N_c|\overline{F}_{\rho_c}$ and $E_d \triangleq |T_d A_d - L_d H_d - N_d A_{2d}| + |T_d|\overline{F}_{\phi_d} + |L_d|\overline{F}_{\psi_d} + |N_d|\overline{F}_{\rho_d}$.

Consequently, by applying [14, Theorem 3.1] (that uses a *quadratic* Lyapunov function) to the linear comparison hybrid system in (11), the error dynamics is global asymptotically stable if there exists a positive definite matrix $P \in$

$\mathbb{R}^{n\times n}$ and scalars a_d and a_c such that:

$$E_c^T P + PE_c \preceq a_c P, \tag{12a}$$
$$E_d^T PE_d \preceq e^{a_d} P, \tag{12b}$$
$$a_c\tau + a_d < 0, \quad \forall \tau \in \mathcal{I}. \tag{12c}$$

Moreover, since P is diagonal (by assumption), then it can be trivially shown that $PM^{\rm m} = (PM)^{\rm m}$ and $P|M| = |PM|$ for any matrix M. Given this, as well as defining Γ and Ω as in Theorem 2, (12a) can be written as in (10a), while (12b) can be represented as

$$\Omega^T P^{-1}\Omega \leq e^{a_d} P. \tag{13}$$

Then, by applying the Schur complement, (13) is equivalent to (10b). The conditions in (10d) and (10e) can be recovered by left multiplying the two linear transformations $T_c + N_cH_c = T_d + N_dH_d = I_n$ introduced in the hybrid observer design in (9) with the matrix P and defining $\tilde{M} \triangleq PM$, $\forall M \in \{T_c, T_c, N_c, N_d\}$. Hence, the feasibility of the constraints (10a)-(10e) prove the asymptotic stability of the error comparison linear hybrid system in (11). Consequently, the hybrid framer error dynamics in (11) of the hybrid interval observer in (9) are asymptotically stable according to the Comparison Lemma [27, Lemma 3.4]. ■

2) L-Hybrid Interval Observer: Next, we introduce the second variant that is based on the use of a *linear* Lyapunov function by leveraging the fact that our hybrid framer error dynamics are positive systems (i.e., the set of non-negative states is left invariant by the dynamics [28]) by design.

Theorem 3 (L-Hybrid Interval Observer). *The hybrid interval observer $\hat{\mathcal{H}}$ in* (9) *for the hybrid system $\mathcal{H}$ in* (5) *is such that the framer error dynamics for $\varepsilon \triangleq \overline{x}-\underline{x}$ is asymptotically stable if Assumptions 1-2 hold and there exist $a_c, a_d \in \mathbb{R}$, $\tilde{T}_c, \tilde{T}_d \in \mathbb{R}^{n\times n}$, $\tilde{N}_c, \tilde{L}_c \in \mathbb{R}^{n\times l_c}$, $\tilde{N}_d, \tilde{L}_d \in \mathbb{R}^{n\times l_d}$ and a positive vector $z \in \mathbb{R}^n_{>0}$ such that:*

$$\Gamma^\top \mathbf{1}_{n\times 1} \leq a_c z, \tag{14a}$$
$$\Omega^\top \mathbf{1}_{n\times 1} \leq e^{a_d} z, \tag{14b}$$
$$a_c\tau + a_d < 0, \quad \forall \tau \in \mathcal{I}, \tag{14c}$$
$$\tilde{T}_c + \tilde{N}_c H_c = P, \tag{14d}$$
$$\tilde{T}_d + \tilde{N}_d H_d = P, \tag{14e}$$

where $P \triangleq diag(z)$ (a diagonal matrix with z as its diagonal elements), $\Gamma \triangleq (\tilde{T}_c A_c - \tilde{L}_c H_c - \tilde{N}_c A_{2c})^{\rm m} + |\tilde{T}_c|\overline{F}_{\phi_c} + |\tilde{L}_c|\overline{F}_{\psi_c} + |\tilde{N}_c|\overline{F}_{\rho_c}$, $\Omega \triangleq |\tilde{T}_d A_d - \tilde{L}_d H_d - \tilde{N}_d A_{2d}| + |\tilde{T}_d|\overline{F}_{\phi_d} + |\tilde{L}_d|\overline{F}_{\psi_d} + |\tilde{N}_d|\overline{F}_{\rho_d}$, $\mathcal{I}$ satisfies Assumption 3 and $\forall \mu \in \{\phi_c, \psi_c, \rho_c, \phi_d, \psi_d, \rho_d\}$, $\overline{F}_\mu$ are computed as described in Theorem 2.

Furthermore, the observer gains in (9) *can be obtained as $T_c \triangleq P^{-1}\tilde{T}_c$, $N_c \triangleq P^{-1}\tilde{N}_c$, $L_c \triangleq P^{-1}\tilde{L}_c$, $T_d \triangleq P^{-1}\tilde{T}_d$, $N_d \triangleq P^{-1}\tilde{N}_d$ and $L_d \triangleq P^{-1}\tilde{L}_d$.*

Proof. The proof follows similar steps as the proof of Theorem 2. Since the flow dynamics and jump dynamics of our proposed hybrid interval observer in (9) are correct by construction according to Theorem 1, it can be easily seen that the linear comparison hybrid system of the hybrid framer error dynamics in (11) are positive systems (i.e., the framer errors remain non-negative [28]) by construction. Hence, by [28, Propositions 1 and 2], a *linear* Lyapunov function $V(\varepsilon) = z^\top \varepsilon$ can be considered. Consequently, by applying [25, Proposition 3.29] to the linear comparison hybrid system in (11), the error dynamics is globally asymptotically stable if there exists a vector $z > 0$ and scalars a_d and a_c such that:

$$z^\top E_c \leq a_c z^\top, \tag{15a}$$
$$z^\top E_d \leq e^{a_d} z^\top, \tag{15b}$$
$$a_c\tau + a_d < 0, \quad \forall \tau \in \mathcal{I}. \tag{15c}$$

Moreover, defining $P \triangleq diag(z)$ and therefore, $z = P\mathbf{1}_{n\times 1}$, it can be trivially shown that for any matrix M, we have $M^{\rm m}P = (MP)^{\rm m}$ and $|M|P = |MP|$, and defining Γ and Ω as in Theorem 3, inequalities (14a) and (14b) can be obtained from (15a) and (15b), respectively. Further, similar to the proof of Theorem 2, the constraints (14d) and (14e) are consequences of left multiplying the linear transformations $T_c + N_cH_c = T_d + N_dH_d = I_n$ with the matrix P and defining $\tilde{M} \triangleq PM$, $\forall M \in \{T_c, T_c, N_c, N_d\}$. Hence, the linear hybrid system in (11) and by the Comparison Lemma [27, Lemma 3.4], the framer error dynamics in (11) of the hybrid interval observer in (9) are asymptotically stable. ■

Remark 2. *Note that the presence of absolute value terms[b] $|M|$ and "Metzlerization" $M^{\rm m} = M^{\rm d} + |M^{\rm nd}|$ results in mixed-integer optimization problems in Theorems 2 and 3. Additionally, due to the presence of the term e^{a_d} in inequalities* (14b) *and* (10b)*, the optimizations in Theorems 2 and 3 are non-trivial. However, by (line) searching over a_c and a_d that satisfy $a_c\tau + a_d < 0, \forall \tau \in \mathcal{I}$, the optimization problems in Theorems 2 and 3 can be simplified to mixed-integer semidefinite programs (MISDP) and mixed-integer linear programs (MILP), respectively, which can be solved using off-the-shelf tools. If desired, extra positivity constraints can be imposed (i.e., by setting $M \geq 0$, $M^{\rm nd} \geq 0$ and replacing $|M|$, $|M^{\rm nd}|$ with M, $M^{\rm nd}$), then the MISDP in Theorem* (2) *and the MILP in Theorem* (3) *can be further simplified to semidefinite programs (SDP) and linear programs (LP), respectively, that are often more computationally amenable.*

V. Illustrative Examples

A. Bouncing Ball

Consider a bouncing ball with gravity coefficient $g > 0$, restitution coefficient $\lambda > 0$ and (nonlinear) drag coefficient $\beta \geq 0$, modeled as system (5) with

$$f_c(x) = \begin{bmatrix} x_2 \\ -g - \beta x_2|x_2| \end{bmatrix}, f_d(x) = \begin{bmatrix} x_1 \\ -\lambda x_2 \end{bmatrix},$$
$$\mathcal{C} = \mathbb{R}_{\geq 0}\times\mathbb{R}, \mathcal{D} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0, x_2 \leq 0\},$$

with output $h_c(x) = h_d(x) = x_1$, where state x_1 represents the position (above ground) and x_2 represents the velocity. Next, we consider two cases where the minimum dwell time τ_m is zero ($\lambda < 1$) or non-zero (sufficiently large λ).

[b]Note that absolute values are internally converted into a mixed-integer formulation in off-the-shelf tools, e.g., YALMIP [29], where a binary variable is introduced to indicate if $|x| = x$ *or* $|x| = -x$.

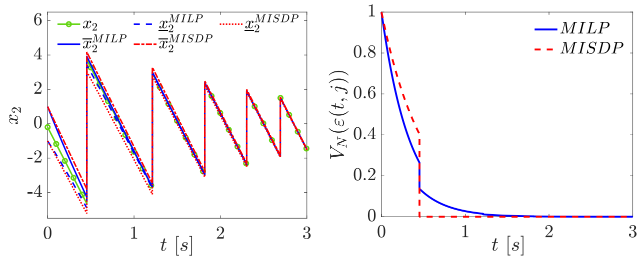

Fig. 1: Linear Bouncing Ball ($\beta = 0$) with $\tau_m = 0$: State x_2, its lower and upper framers $\underline{x}_2, \overline{x}_2$ and normalized Lyapunov function $V_N(\varepsilon(t,j)) \triangleq V(\varepsilon(t,j))/V(\varepsilon(0,0))$ of both observer variants.

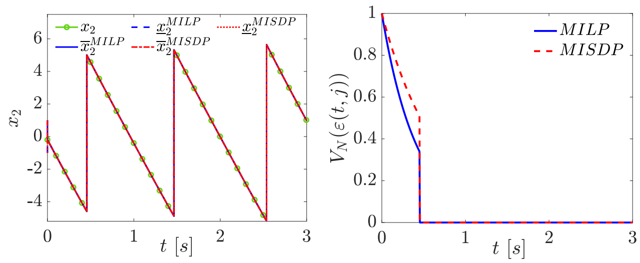

Fig. 2: Nonlinear Bouncing Ball with $\tau_m \neq 0$: State x_2, its lower and upper framers $\underline{x}_2, \overline{x}_2$ and normalized Lyapunov function $V_N(\varepsilon(t,j)) \triangleq V(\varepsilon(t,j))/V(\varepsilon(0,0)))$ of both observer variants.

1) Zero minimum dwell time: First, we consider the case of the linear ($\beta = 0$) bouncing ball problem with $\lambda = 0.8$, i.e., the overall the system loses energy and exhibits Zeno behavior. Hence, for this case, we have minimum dwell time $\tau_m = 0$. In this scenario, from Fig. 1, it can be observed that the MILP approach (using Theorem 3) estimates the unmeasured velocity x_2 faster than the MISDP approach (using Theorem 2). Moreover, from the analysis of normalized Lyapunov function values (normalized by the initial value) from Fig. 1, both the observers have comparable performance in asymptotically stabilizing the framer errors.

2) Non-zero minimum dwell time: Next, we consider the case of a nonlinear bouncing ball problem with $\beta = 0.02$, i.e., the system loses energy during flow due to nonlinear drag forces, but due to a sufficiently large coefficient of restitution $\lambda = 1.09$ (such a system can be realized in practice by considering an actuated table), the system gains more energy than it loses during flows. Hence, the system has a non-zero minimum dwell time τ_m (no Zeno behavior). From Fig. 2, similar to the zero dwell time case, the velocity x_2 using the MILP approach converges faster to the true values than the MISDP approach (using Theorems 3 and 2, respectively). Moreover, from Fig. 2, it is evident that both variants have comparable performances in the sense of their normalized Lyapunov function values.

B. Power Control with a Thyristor

Consider a electric thyristor circuit from [25] with resistance R, inductance L, capacitance C_0 and capacitor impedance R_0, modeled as system (5) with

$$f_c(x) = \left[wz_2, -wz_1, q\tfrac{v_0 - Ri_L}{L}, -\tfrac{v_0}{C_0R_0} + \tfrac{z_1}{C_0R_0} - \tfrac{i_L}{C_0}, 0, 1\right]^\top,$$
$$f_d(x) = \left[z_1, z_2, i_L, v_0, 1-q, 0\right]^\top,$$

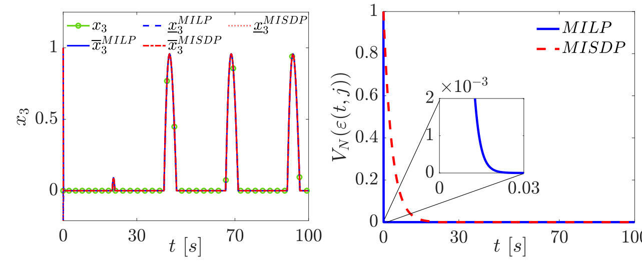

Fig. 3: Power Control with a Thyristor: State x_3, its lower and upper framers $\underline{x}_3, \overline{x}_3$ and normalized Lyapunov function $V_N(\varepsilon(t,j)) \triangleq V(\varepsilon(t,j))/V(\varepsilon(0,0)))$ of both observer variants.

$$h_c(x) = h_d(x) = \left[z_1, z_2, v_0, q, \tau\right]^\top,$$
$$\mathcal{C} \triangleq \{x : q = 0, \tau < \alpha/\omega, i_L = 0\} \cup \{x : q = 1, i_L \geq 0\},$$
$$\mathcal{D} \triangleq \{x : q = 0, \tau \geq \alpha/\omega, v_0 > 0\} \cup \{x : q = 1, i_L \leq 0, v_0 < 0\},$$

with state $x = [z_1, z_2, i_L, v_0, q, \tau]^\top$, sinusoidal input voltage z_1, output voltage z_2, load current i_L, capacitor voltage v_0, binary variable $q \in \{0, 1\}$ (to indicate whether the thyristor is on ($q = 1$) or off ($q = 0$)), and trigger state τ (used to model the firing event). Additionally, the parameter values used for simulation are angular frequency $\omega = 0.5$, firing angle $\alpha = 20\omega$, $R = 0.5$, $L = 0.5$, $C_0 = 10$ and $R_0 = 0.25$.

In this example, if we only compare the framers of the unmeasured load current $x_3 = i_L$ in Fig. 3, they converge exponentially fast to the true value in both MISDP and MILP variants (using Theorems 2 and 3), i.e., in this case, they have comparable performance in estimating the unmeasured state. On the other hand, if we look at the overall performance of the observers by comparing the normalized Lyapunov function $V_N(\varepsilon(t,j))$ values in Fig. 3, the MILP variant converges much faster than the MISDP variant.

In summary, both observer variants in both examples show comparable performance (note that we only computed feasible gains). However, the main advantage of the MILP variant over the MISDP one is that MILP is a much "easier" class of optimization problem to solve than MISDP in terms scalability to larger systems and availability of fast solvers.

VI. Conclusion and Future Work

In this work, we proposed a novel interval observer design framework for hybrid systems with known jump times and nonlinear dynamics and observations. Specifically, by construction, the error system of our proposed observer is a positive system; thus, the observer is correct by construction without any additional positivity constraints. This is achieved by leveraging the ideas of mixed-monotone decompositions and hybrid embedding systems. Additionally, using Lyapunov analysis, we introduced two observer variants that are asymptotically stable. Our proposed observer designs involve solving mixed-integer semidefinite programs or mixed-integer linear programs (leveraging the positive system property of the error dynamics) to compute the observer gains, including the additional degrees of freedom from a system transformation. Finally, we successfully demonstrated the effectiveness of our hybrid interval observer framework using simulation examples of a bouncing ball and an electric thyristor. In our future work, we will extend our proposed framework to consider noisy/uncertain hybrid systems with unobserved discrete modes and unknown jump times.

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APPENDIX

A. Construction of Hybrid Interval Observer

From the equivalent system in (7), we can design an embedding system by formulating the corresponding decomposition functions $\underline{g}^c_\delta, \overline{g}^c_\delta$ and $\underline{g}^d_\delta, \overline{g}^d_\delta$ for flow and jump dynamics, respectively. First, note that the component of the flow/jump dynamics that is affine in x (with known y_c and y_d), i.e., $g^\mu_a(x) = (T_\mu A_\mu - L_\mu H_\mu - N_\mu A_{2\mu})x + (M_\mu N_\mu + L_\mu)y_\mu \triangleq M_\mu x + (M_\mu N_\mu + L_\mu)y_\mu$, $\mu \in \{c, d\}$, admits a tight continuous-time decomposition function (cf. Definition 1 and [11] for details):

$$g^\mu_{a,\delta}(x_1, x_2) = M^\uparrow_\mu x_1 - M^\downarrow_\mu x_2 + (M_\mu N_\mu + L_\mu)y_\mu,$$

where $M^\downarrow_c \triangleq M^{\text{nd},\ominus}_c$, $M^\uparrow_c \triangleq M^{\text{d}}_c + M^{\text{nd},\oplus}_c$ and $M^\downarrow_d \triangleq M^\ominus_d$, $M^\uparrow_d \triangleq M^\oplus_d$. Next, the nonlinear flow/jump dynamics $g^\mu_{nl}(x) = T_\mu\phi_\mu(x) - N_\mu\rho_\mu(x) - L_\mu\psi_\mu(x)$, $\mu \in \{c, d\}$ (treated as a disturbance signal) admits the following decomposition function:

$$\begin{aligned} g^\mu_{nl,\delta}(x_1,x_2) = {} & T^\oplus_\mu\phi_{\mu,\delta}(x_1,x_2) - T^\ominus_\mu\phi_{\mu,\delta}(x_2,x_1) - N^\oplus_\mu\rho_{\mu,\delta}(x_2,x_1) \\ & + N^\ominus_\mu\rho_{\mu,\delta}(x_1,x_2) - L^\oplus_\mu\psi_{\mu,\delta}(x_2,x_1) + L^\ominus_\mu\psi_{\mu,\delta}(x_1,x_2), \end{aligned}$$

based on the discrete-time decomposition functions for $\phi_\mu(x)$, $\psi_\mu(x)$, $\rho_\mu(x)$ and non-negativity of $T^\oplus_\mu$, $T^\ominus_\mu$, $N^\oplus_\mu$, $N^\ominus_\mu$, $L^\oplus_\mu$, and $L^\ominus_\mu$. Then, it can be easily demonstrated that the sum of the decomposition functions of individual constituents is also a decomposition function of the sum of the constituents. Hence, $\underline{g}^\mu_\delta(\underline{x}, \overline{x}) = g^\mu_{a,\delta}(\underline{x}, \overline{x}) + g^\mu_{nl,\delta}(\underline{x}, \overline{x})$ and $\overline{g}^\mu_\delta(\overline{x}, \underline{x}) = g^\mu_{a,\delta}(\overline{x}, \underline{x}) + g^\mu_{nl,\delta}(\overline{x}, \underline{x})$, $\mu \in \{c, d\}$.

Moreover, the embedding system for $\xi^+ = T_c(A_d x + \phi_d(x)) - N_c\psi_c(A_d x + \phi_d(x))$ in (1) can be similarly constructed using both its affine and nonlinear components (omitted for brevity) using discrete-time decomposition functions to obtain its framers $\underline{\zeta}^+$ and $\overline{\zeta}^+$ as in (9). Finally, from $x = \xi + N_c y_c$ and $\zeta = T_d x - N_d\psi_d(x)$ when $x \in \mathcal{C}$ and $x = \zeta + N_d y_d$ when $x \in \mathcal{D}$ in (7), we can obtain $\overline{x} = \overline{\xi} + N_c y_c$, $\underline{x} = \underline{\xi} + N_c y_c$ as well as $\underline{\zeta} = T^\oplus_d\underline{x} - T^\ominus_d\overline{x} - N^\oplus_d\psi_{d,\delta}(\overline{x}, \underline{x}) + N^\ominus_d\psi_{d,\delta}(\underline{x}, \overline{x})$, $\overline{\zeta} = T^\oplus_d\overline{x} - T^\ominus_d\underline{x} - N^\oplus_d\psi_{d,\delta}(\underline{x}, \overline{x}) + N^\ominus_d\psi_{d,\delta}(\overline{x}, \underline{x})$ when $\mathcal{H}$ flows and $\overline{x} = \overline{\zeta} + N_d y_d$, $\underline{x} = \underline{\zeta} + N_d y_d$ when $\mathcal{H}$ jumps, by leveraging discrete-time decomposition functions.