Entropic and Operational Characterizations of Dynamic Quantum Resources

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***Abstract*—Dynamic quantum resource theories study the manipulation of quantum channels by means of a restricted set of free superoperations. In this paper, we formulate general dynamic resource theories using a "top-down" framework, and we provide systematic characterizations for closed and convex resource theories from both information-theoretic and operational perspectives. Our results are summarized as follows. First, we propose and investigate a branch of resource-induced measures of uncertainty, called the free conditional min-entropy (FCME), generalizing the conditional min-entropy and its dynamic extension to scenarios where information processing is subject to variable operational restriction. We provide a complete set of entropic conditions in terms of the FCME for characterizing channel convertibility via free superoperations in any closed and convex resource theory. We also find that the resource global robustness of channels can be equivalently cast as a mutual-information-like quantity derived from the FCME, thereby offering the resource global robustness an information-theoretic interpretation. Apart from the entropic approach, we also study closed and convex resource theories in the contexts of various operational tasks. These tasks are formulated such that each of them induces a complete set of operationally meaningful resource monotones, and therefore they can be used to faithfully test free convertibility between channels. We also systematically study the quantitative relations between the operational advantage of channels in these tasks and the resource robustness measures of channels. In particular, we prove that *every* well-defined robustness-based measure can be operationally interpreted as some kind of advantage in a task called semiquantum partial preprocessing. Ultimately, our results provide both entropic and operational characterizations for general dynamic quantum resources with a closed and convex structure.**

I. Introduction

A quantum resource theory (QRT) is a framework to systematically study one or more feature of quantum mechanics [1], [2], [3]. For example, quantum entanglement is a powerful property of multi-party quantum systems whose operational utility can be formally captured within a resource theory [4], [5]. A QRT takes the perspective of an experimenter who wishes to perform some protocol or task in an operationally-restricted environment. While there are "free" operations that the experimenter can perform, their limitations prohibit the experimenter from preparing certain objects. Such objects are said to possess resource, whereas the objects prepared by the allowed operations are similarly called "free". In the example of entanglement, the free operations are local operations and classical communication (LOCC), and whatever quantum objects cannot be generated by LOCC possesses entanglement.

A variety of other QRTs have been studied in the past few years that characterize a wide range of quantum phenomena [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18]. Despite their physical differences, these phenomena have a remarkably similar structure when viewed through the lens of a resource theory. Resource measures can be defined in analogous ways, and the allowed resource manipulations can have a similar form across the different theories. This has motivated researchers to study the general structure of QRTs, independent of any specific physically-constrained scenario [1], [19], [2], [20], [21], [22], [23], [24], [3], [25], [26], [27], [28], [29], [30], [31], [32]. Through an axiomatic approach, one can identify common structural features of different QRTs that emerge after assuming a minimal set of physical properties.

One of the central questions in any resource theory is when one object can be converted to another using the free operations of the theory. In *static* resource theories, the underlying objects are states (i.e. density operators), and the free operations are certain completely positive trace-preserving (CPTP) maps (i.e. channels) that transform density operators to density operators. In *dynamic* resource theories [33], [34], [35], [36], [37], the objects embodying resource are quantum channels themselves, and the free operations consist of superchannels [38], [39], which are devices that transform channels to channels. In fact, every static QRT is a special case of a dynamic QRT since every density matrix can be seen as a quantum channel with a one-dimensional input. When one state or channel can be transformed to another using the free operations, the former is no less resourceful than the latter. This relationship can be quantified using resource monotones, which are real-valued functions that monotonically decrease under the free operations of the theory. A *complete* set of resource monotones provide both necessary and sufficient convertibility conditions: one object can be freely converted to another if and only if the first has a value no less than the second for all monotones in the family. Recently, such complete monotones have been obtained in general (closed and convex) resource theories of states and measurements in GPTs [27] and in general (closed and convex) dynamic QRTs [32]. However, in the dynamic setting, the physical significance of the monotones in Ref. [32]

was left unclear.

A. Summary of Results

Our main contribution to the problem in this paper is two-fold. First, in Section V, we simplify the conditions of free channel convertibility given in Ref. [32] and establish information-theoretic interpretations for these conditions. We arrive at these results by proposing a generalized entropic quantity called the free conditional min-entropy (FCME), which we develop in Section IV. The FCME can be viewed as quantifying the uncertainty about a physical system from the perspective of an experimenter whose operational capacity is limited by a resource theory. Based on the free min-entropy, we further provide an information-theoretic interpretation of the resource global robustness of channels as a mutual-information-like quantity in this generalized entropic sense. These results reveals a new angle underpinning the inseparable relationship between quantum resources and quantum information theory.

Second, we characterize general convex quantum resources through new designs of operational tasks. We show in Section VI that the maximum expected score the performer can achieve constitutes a complete set of monotones with respect to the channel Λ held by the performer. We investigate the operational advantage the channel Λ brings, over when such channel is absent, and show that such advantage is intimately related to various resource robustness measures of Λ. These results provide intuitive operational interpretations of the resource monotones, and thus endow an abstract QRT with a physical bearing.

II. Preliminaries

We use subscripted Latin capitals, such as A_0, A_1, B_0, etc., to label *static quantum systems*. Each static system represents a quantum information carrier at a certain spot in spacetime. Given a system A_0, we use $\mathbb{H}^{A_0}$ to denote its associated space of Hermitian operators, and $\mathbb{H}_+^{A_0} \subset \mathbb{H}^{A_0}$ the cone of positive-semidefinite (in short, positive) operators therein. We use the symbol # to denote the trivial static system, meaning the unique system whose dimensionality equals 1. We write $\mathbb{R} := \mathbb{H}^{\#}$ and $\mathbb{R}_+ := \mathbb{H}_+^{\#}$ to denote the sets of real and non-negative real numbers respectively.

A *quantum state* in A_0 is a positive operator $\rho \in \mathbb{H}_+^{A_0}$ with a unit trace. We denote the set of states in A_0 by $\mathsf{D}^{A_0} \subset \mathbb{H}_+^{A_0}$, and the set of states in an arbitrary system by $\mathsf{D} := \bigcup_{A_0} \mathsf{D}^{A_0}$.

Given any two static systems A_0 and A_1, the tuple $A_0 \to A_1$ is referred to as the *dynamic quantum system* which starts with A_0 and ends with A_1 (in short, from A_0 to A_1). For compactness, we use the shorthand $A := A_0 \to A_1$ whenever convenient. The trivial dynamic system is denoted by $\# := \# \to \#$. We use $\mathbb{L}^A := \mathbb{L}^{A_0 \to A_1}$ to denote the space of Hermiticity-preserving (HP) linear maps from $\mathbb{H}^{A_0}$ to $\mathbb{H}^{A_1}$. Given a map $\Lambda \in \mathbb{L}^A$, its Choi operator $J_\Lambda \in \mathbb{H}^{A_0 A_1}$ is defined as

$$J_\Lambda^{A_0 A_1} := \left(\mathrm{id}^{A_0} \otimes \Lambda^{\tilde{A}_0 \to A_1}\right)\left[\hat{\phi}_+^{A_0 \tilde{A}_0}\right], \tag{1}$$

where $\mathrm{id}^{A_0} \in \mathbb{L}^{A_0 \to A_0}$ is an identity map, $\tilde{A}_0$ is a replica of A_0, and $\hat{\phi}_+^{A_0 \tilde{A}_0} := \sum_{i,j} |i\rangle\langle j|^{A_0} \otimes |i\rangle\langle j|^{\tilde{A}_0} \in \mathbb{H}_+^{A_0 \tilde{A}_0}$ is a maximally entangled operator.

A *quantum channel* from A_0 to A_1 is a linear map $\Lambda \in \mathbb{L}^A$ being both completely positive (CP) and trace preserving (TP). In other words, quantum channels are maps that preserve the set of quantum states even when in tensor product with any identity map. We denote the set of channels from A_0 to A_1 by $\mathsf{C}^A := \mathsf{C}^{A_0 \to A_1} \subset \mathbb{L}^A$, and the set of channels between arbitrary systems by $\mathsf{C} := \cup_{A_0, A_1} \mathsf{C}^A$.

Linear transformations between linear maps are called *supermaps*. Given any two dynamic systems A and $B := B_0 \to B_1$, we use $\mathbb{S}^{A \twoheadrightarrow B}$ to denote the space of HP-preserving supermaps from $\mathbb{L}^A$ to $\mathbb{L}^B$. Given a supermap $\Theta \in \mathbb{S}^{A \twoheadrightarrow B}$, its Choi operator $J_\Theta \in \mathbb{H}^{A_0 A_1 B_0 B_1}$ is defined [39] as

$$J_\Theta^{A_0 A_1 B_0 B_1} := J_{\Delta_\Theta}^{A_0 A_1 B_0 B_1}, \tag{2}$$

where

$$\Delta_\Theta^{AB} := \left(\mathrm{Id}^A \otimes \Theta^{\tilde{A} \twoheadrightarrow B}\right)\left\{\Phi_+^{A\tilde{A}}\right\}, \tag{3}$$

where $\mathrm{Id}^A \in \mathbb{S}^{A \twoheadrightarrow A}$ is an identity supermap, $\tilde{A} := \tilde{A}_0 \to \tilde{A}_1$ is a replica of A, and $\Phi_+^{A\tilde{A}}[\cdot] := \mathrm{tr}[\hat{\phi}_+^{A_0 \tilde{A}_0}[\cdot]]\hat{\phi}_+^{A_1 \tilde{A}_1} \in \mathbb{L}^{A\tilde{A}}$ is a maximally entangled map.

A *quantum superchannel* from A to B is a supermap $\Theta \in \mathbb{S}^{A \twoheadrightarrow B}$ that is both completely CP-preserving (CCPP) and TP preserving (TPP). We denote the set of superchannels from A to B by $\mathsf{S}^{A \twoheadrightarrow B} \subset \mathbb{S}^{A \twoheadrightarrow B}$, and the set of superchannels between arbitrary systems by $\mathsf{S} := \cup_{A_0, A_1, B_0, B_1} \mathsf{S}^{A \twoheadrightarrow B}$.

III. A "Top-Down" Framework for Quantum Resource Theories (QRTs)

In this section, we reformulate the framework of quantum resource theories (QRTs) by adopting a "top-down" viewpoint. Our formulation unifies both static and dynamic QRTs.

Definition 1. *Let $\mathcal{S} \subseteq \mathsf{S}$ be a subset of superchannels. For any two dynamic systems A and B, define the following components:*
(1) a set of superchannels $\mathcal{S}^{A \twoheadrightarrow B} := \mathcal{S} \cap \mathsf{S}^{A \twoheadrightarrow B}$;
(2) a set of channels $\mathcal{C}^B := \mathcal{S}^{\# \twoheadrightarrow B} \subseteq \mathsf{C}^B$.
Define $\mathcal{C} := \bigcup_{B_0, B_1} \mathcal{C}^B \subseteq \mathsf{C}$. Then the tuple $(\mathcal{S}, \mathcal{C})$ is called a ***dynamic quantum resource theory*** *whenever the following conditions hold:*
(1) for any dynamic system B, it holds that $\mathrm{Id}^B \in \mathcal{S}^{B \twoheadrightarrow B}$;
(2) for any three dynamic systems A, B, and C, if $\Theta \in \mathcal{S}^{A \twoheadrightarrow B}$ and $\Theta' \in \mathcal{S}^{B \twoheadrightarrow C}$, then $\Theta' \circ \Theta \in \mathcal{S}^{A \twoheadrightarrow C}$.
In this case, superchannels in $\mathcal{S}$ are called ***free superoperations****, and channels in $\mathcal{C}$ are called* ***free channels****. We say that $(\mathcal{S}, \mathcal{C})$ is* ***closed*** *and* ***convex*** *whenever $\mathcal{S}^{A \twoheadrightarrow B}$ is a closed and convex set for all A and B.*

Following the top-down approach, every static QRT is reduced from a dynamic QRT, being the "sub-theory" that only includes dynamic systems with a trivial input.

Definition 2. *Let $(\mathcal{S}, \mathcal{C})$ be a dynamic QRT. For any two static systems A_1 and B_1, define the following components:*

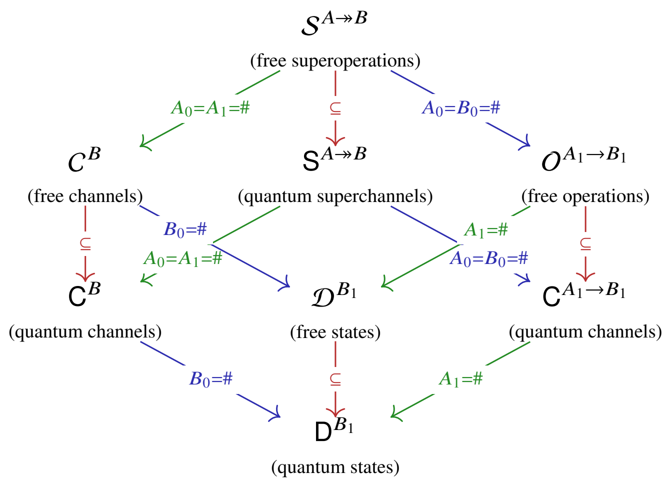

Fig. 1. Relationships between different components of a dynamic QRT $(\mathcal{S}, \mathcal{C})$, its reduced static QRT $(\mathcal{O}, \mathcal{D})$, and the standard quantum theory $(\mathsf{S}, \mathsf{C}, \mathsf{D})$.

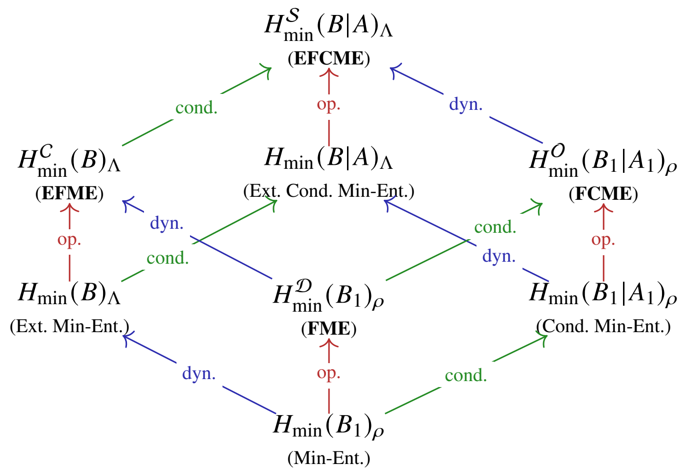

Fig. 2. Relationships between members of the free conditional min-entropy (FCME) branch and the (conventional) conditional min-entropy branch. Green arrows represent generalization through conditioning.

(1) a set of channels $\mathcal{O}^{A_1\to B_1} := \mathcal{S}^{(\#\to A_1)\twoheadrightarrow(\#\to B_1)} \subseteq \mathsf{C}^{A_1\to B_1}$*;*
(2) a set of states $\mathcal{D}^{B_1} := \mathcal{O}^{\#\to B_1} \subseteq \mathsf{D}^{B_1}$.

Define $\mathcal{O} := \bigcup_{A_1,B_1} \mathcal{O}^{A_1\to B_1} \subseteq \mathsf{C}$ *and* $\mathcal{D} := \bigcup_{B_1} \mathcal{D}^{B_1} \subseteq \mathsf{D}$. *Then the tuple* $(\mathcal{O}, \mathcal{D})$ *is called the* ***static quantum resource theory*** *reduced from* $(\mathcal{S}, \mathcal{C})$*. In this case, channels in* $\mathcal{O}$ *are called* ***free operations****, and states in* $\mathcal{D}$ *are called* ***free states****. We say that* $(\mathcal{O}, \mathcal{D})$ *is* ***closed*** *and* ***convex*** *whenever* $\mathcal{O}^{A_1\to B_1}$ *is a closed and convex set for all* A_1 *and* B_1*.*

We denote the set of Choi operators of free superoperations from A to B by $\mathcal{S}_{\mathrm{J}}^{A\twoheadrightarrow B} := \{J_\Theta : \Theta \in \mathcal{S}^{A\twoheadrightarrow B}\} \subset \mathbb{H}_{+}^{A_0A_1B_0B_1}$, and the set of Choi operators of free operations from A_1 to B_1 by $\mathcal{O}_{\mathrm{J}}^{A_1\to B_1} := \mathcal{S}_{\mathrm{J}}^{(\#\to A_1)\twoheadrightarrow(\#\to B_1)} \subset \mathbb{H}_{+}^{A_1B_1}$. We summarize in Fig. 1 the relationships between different components of a dynamic QRT, its reduced static QRT, and the standard quantum theory. The dynamic-to-static reduction implies that any property possessed by a dynamic QRT is conveniently inherited by the static QRT reduced from it.

IV. Free Conditional Min-Entropy (FCME)

In this section, we generalize two important entropic measures in quantum information theory, the quantum min-entropy and the quantum conditional min-entropy, to scenarios where variable operational restriction may apply. These generalized entropies can be viewed as measures of quantum uncertainty in operationally restricted scenarios, i.e., resource theories.

The basic idea of entropy generalization in this paper is that, we relax the default assumption that the observer is able to use arbitrary quantum operations for information processing. Instead, the observer's usable operations are restricted to the free (super-)operations of a dynamic QRT $(\mathcal{S}, \mathcal{C})$, whose reduced static QRT is $(\mathcal{O}, \mathcal{D})$.

In the rest of this paper, we always assume that the set of free superoperations is topologically *closed* and *convex*, and is closed under complex conjugation with respect to a pre-specified computational basis.

Definition 3. *Given a state* $\rho \in \mathsf{D}^{B_1}$*, the* ***free min-entropy (FME)*** *of the system* B_1 *is defined as*

$$H^{\mathcal{D}}_{\min}(B_1)_{\rho} := -\log \min_{\left\{\substack{r\,\in\,\mathbb{R} \\ r\hat{I}^{B_1} - \rho^{B_1} \,\in\, \mathrm{cone}^*(\mathcal{D}^{B_1})}\right\}} r. \tag{4}$$

Given a bipartite state $\rho \in \mathsf{D}^{A_1B_1}$*, the* ***free conditional min-entropy (FCME)*** *of the system* B_1 *conditioned on* A_1 *is defined as*

$$H^{\mathcal{O}}_{\min}(B_1|A_1)_{\rho} := -\log \min_{\left\{\substack{\gamma\,\in\,\mathbb{H}^{A_1} \\ \gamma^{A_1}\otimes\hat{I}^{B_1} - \rho^{A_1B_1} \,\in\, \mathrm{cone}^*(\mathcal{O}_{\mathrm{J}}^{A_1\to B_1})}\right\}} \mathrm{tr}[\gamma]. \tag{5}$$

Given a channel $\Lambda \in \mathsf{C}^{B}$*, the* ***extended free min-entropy (EFME)*** *of the dynamic system* B *is defined by*

$$H^{\mathcal{C}}_{\min}(B)_{\Lambda} := H^{\mathcal{C}}_{\min}(B_1|B_0)_{\frac{1}{d_{B_0}}J_{\Lambda}}, \tag{6}$$

Given a bipartite channel $\Lambda \in \mathsf{C}^{AB}$*, the* ***extended free conditional min-entropy (EFCME)*** *of the dynamic system* B *conditioned on* A *is defined as*

$$H^{\mathcal{S}}_{\min}(B|A)_{\Lambda} := \log(d_{A_0}d_{B_0}) - \log \min_{\left\{\substack{\gamma\,\in\,\mathbb{H}^{A_0A_1B_0} \\ \gamma^{A_0A_1B_0}\otimes\hat{I}^{B_1} - J_{\Lambda}^{A_0A_1B_0B_1} \,\in\, \mathrm{cone}^*(\mathcal{S}_{\mathrm{J}}^{A\twoheadrightarrow B}) \\ \gamma^{A_0B_0} = \hat{\pi}^{A_0}\otimes\gamma^{B_0}}\right\}} \mathrm{tr}[\gamma]. \tag{7}$$

The FME, FCME, EFME, and EFCME are all defined by convex conic programs. They reduce to the (conventional) min-entropy measures [39] in the special case where the underlying QRT is the standard quantum theory. We summarize in Fig. 2 the relationships between all entropic functions introduced so far.

V. Resource Characterization with the FCME

In this section, we make use of the free conditional min-entropy branch to provide information-theoretic characterization for general closed and convex quantum resources.

A. *Deterministic free convertibility*

It is of fundamental concern in any resource theory to develop a systematic approach for deciding whether a given object is convertible to another via free transformations. Recently, complete sets of monotones have been obtained in general convex resource theories of states and measurements [27] and of channels [32] (i.e., dynamic QRTs). However, in the dynamic setting, the monotones in Ref. [32] lacks a clear physical interpretation. We close this gap by framing these monotones as entropic conditions in terms of the FCME branch, and in addition provide simpler alternatives for them.

Theorem 1. *Let $\Lambda \in \mathsf{C}^A$ and $\Psi \in \mathsf{C}^B$ be two channels. Let $\mathbb{N} := \{N_n\}_{n\in\mathbb{N}}$ be an arbitrary informationally complete POVM on B_0. Then the following statements are equivalent.*

(1) There exists a free superoperation $\Theta \in \mathcal{S}^{A\twoheadrightarrow B}$ such that $\Theta\{\Lambda\} = \Psi$.

(2) For every channel $\Omega \in \mathsf{C}^B$,

$$H^{\mathcal{S}}_{\min}(\tilde{B}|A)_{\Lambda^A\otimes\Omega^{\tilde{B}}} \leq H^{\mathcal{S}}_{\min}(\tilde{B}|B)_{\Psi^B\otimes\Omega^{\tilde{B}}}. \tag{8}$$

(3) Equation (8) holds for every measure-and-prepare channel $\Omega \in \mathsf{C}^B$ of the form $\Omega[\cdot] = \sum_n \mathrm{tr}[N_n[\cdot]]\omega_n$, where $\omega_n \in \mathsf{D}^{B_1}$ is a variable state for all n.

The entropic conditions in Theorem 1 (2) are mathematically equivalent to the complete set of resource monotones provided in Ref. [32, Theorem 4]. And yet, their information-theoretic interpretation is disclosed only after being recast in terms of the EFCME. Furthermore, Theorem 1 (3) simplifies their conditions and reduces the size of the set of convertibility conditions is significantly.

Corollary 1 ([26, Corollary 11])**.** *Let $\rho \in \mathsf{D}^{A_1}$ and $\tau \in \mathsf{D}^{B_1}$ be two states. Then there exists a free operation $\Psi \in \mathcal{O}^{A_1\to B_1}$ such that $\Psi[\rho] = \tau$ if and only if*

$$H^{\mathcal{O}}_{\min}(\tilde{B}_1|A_1)_{\rho^{A_1}\otimes\omega^{\tilde{B}_1}} \leq H^{\mathcal{O}}_{\min}(\tilde{B}_1|B_1)_{\tau^{B_1}\otimes\omega^{\tilde{B}_1}} \tag{9}$$

for every state $\omega \in \mathsf{D}^{B_1}$.

B. *Resource global robustness*

Resource robustness measures are effective and meaningful functions that reflect the proximity of an object to the set of free objects in the object space. One representative of such measures is the *resource global robustness* [40], [19], which quantifies the amount of "global" noise (within the object set) that an object can tolerate before losing its identify of being resourceful. The resource global robustness of a channel $\Lambda \in \mathsf{C}^A$ in a dynamic QRT $(\mathcal{S}, \mathcal{C})$ is defined as

$$R^{\mathcal{C}}_{\mathrm{glob}}(\Lambda) = \min_{\left\{\substack{r\in\mathbb{R}_+\\ \Psi\in\mathsf{C}^A\\ \frac{\Lambda+r\Psi}{1+r}\in\mathcal{C}^A}\right\}} r. \tag{10}$$

In what follows, we show that the resource global robustness of a channel, albeit as a geometrically motivated function, has an entropic formulation in terms of the FCME branch.

Let $(\mathcal{S}, \mathcal{C})$ be a closed and convex static QRT. Likewise, given a bipartite channel $\Lambda \in \mathsf{C}^{AB}$, we define the *extended free min-mutual information* between the two dynamic systems A and B as

$$I^{\mathcal{S}}_{\min}(A;B)_\Lambda := H^{\mathcal{C}}_{\min}(B)_\Lambda - H^{\mathcal{S}}_{\min}(B|A)_\Lambda, \tag{11}$$

where $H^{\mathcal{C}}_{\min}(B)_\Lambda$ is defined on the marginal channel $\Lambda^B[\cdot] := \mathrm{tr}_{A_1} \circ \Lambda^{AB}[\hat{\pi}^{A_0} \otimes [\cdot]^{B_0}]$.

Theorem 2. *Let $\Lambda \in \mathsf{C}^A$ be a channel. Then*

$$\log\left(1 + R^{\mathcal{C}}_{\mathrm{glob}}(\Lambda)\right) = \sup_{\Omega\in\mathsf{C}} I^{\mathcal{S}}_{\min}(A;B)_{\Lambda^A\otimes\Omega^B}. \tag{12}$$

The right-hand side of Eq. (12) represents the supremum amount information that Λ could potentially provide about any system B independent of it, and this quantity may as well be termed the "independent free-informativeness" of Λ.

Corollary 2. *Let $\rho \in \mathsf{D}^{A_1}$ be a state. Then*

$$\log\left(1 + R^{\mathcal{D}}_{\mathrm{glob}}(\rho)\right) = \max_{\omega\in\mathsf{D}^{A_1}} I^{\mathcal{O}}_{\min}(A_1;\tilde{A}_1)_{\rho^{A_1}\otimes\omega^{\tilde{A}_1}}. \tag{13}$$

VI. Resource Characterization with Operational Tasks

In this section, we characterize general closed and convex quantum resources through new designs of operational tasks.

A. *Operational tasks as free convertibility tests*

We first introduce the operational tasks and show how they give rise to complete sets of resource monotones.

1) One-shot classical communication:

Task 1. *A* ***one-shot classical communication*** *task for Alice is specified by a state ensemble $\upsilon := \{\nu_n\}_{n\in\mathbb{N}}$ in a system B_0 and a POVM $\mathbb{N} := \{N_{\hat{n}}\}_{\hat{n}\in\mathbb{N}}$ on a system B_1. It has the following steps.*

1) An index $n \in \mathbb{N}$ is generated with a probability $\mathrm{tr}[\nu_n]$. The state $\nu_n/\mathrm{tr}[\nu_n] \in \mathsf{D}^{B_0}$ is given to Alice without letting her know n.

2) Alice submits a state in B_1.

3) The system B_1 is measured with the POVM $\mathbb{N}$, producing an outcome $\hat{n}$. Alice succeeds whenever $\hat{n} = n$.

We assume that Alice is operationally restricted by a closed and convex dynamic QRT $(\mathcal{S}, \mathcal{C})$ and can access a one-shot oracle to a channel $\Lambda \in \mathsf{C}^A$. Then her most general strategy for

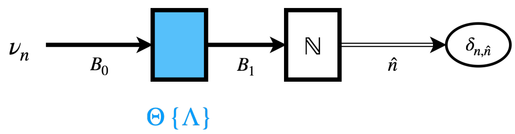

Fig. 3. One-shot classical communication over an encoding ensemble $\upsilon := \{\nu_n\}_n$ and a decoding POVM $\mathbb{N} := \{N_{\hat{n}}\}_{\hat{n}}$. Solid arrows stand for quantum systems, and hollow arrows for classical systems.

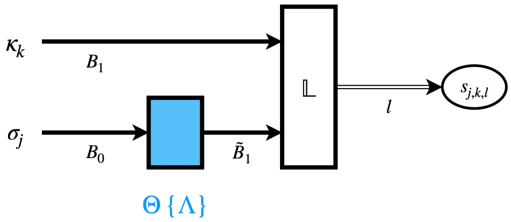

Fig. 4. Semiquantum partial preprocessing of two state ensembles $\varsigma := \{\sigma_j\}_j$ and $\varkappa := \{\kappa_k\}_k$ before a POVM $\mathbb{L} := \{L_l\}_l$ under a score function $\{s_{j,k,l}\}_{j,k,l}$. Solid arrows stand for quantum systems, and hollow arrows for classical systems.

Task 1 is transforming Λ with a free superoperation $\Theta \in \mathcal{S}^{A \twoheadrightarrow B}$ and using $\Theta\{\Lambda\}$ to conduct data transmission between Steps 1 and 2 of Task 1. We show in Fig. 3 an illustration of the task and Alice's strategy. Her maximum correct transmission probability is thus given by

$$P_{\mathrm{OCC}}^{\mathcal{S}}(\Lambda; \upsilon, \mathbb{N}) := \max_{\Theta \in \mathcal{S}^{A \twoheadrightarrow B}} \sum_n \operatorname{tr}[N_n \Theta\{\Lambda\}[\nu_n]]. \tag{14}$$

The following theorem indicates that if the POVM $\mathbb{N}$ is informationally complete, then $P_{\mathrm{OCC}}^{\mathcal{S}}(\Lambda; \upsilon, \mathbb{N})$ forms a complete set of resource monotones with respect to Λ.

Theorem 3. *Let $\Lambda \in \mathsf{C}^A$ and $\Psi \in \mathsf{C}^B$ be two channels. Let $\mathbb{N} := \{N_{\hat{n}}\}_{\hat{n} \in \mathtt{N}}$ be an arbitrary informationally complete POVM on B_1. Then there exists a free superoperation $\Theta \in \mathcal{S}^{A \twoheadrightarrow B}$ such that $\Theta\{\Lambda\} = \Psi$ if and only if $P_{\mathrm{OCC}}^{\mathcal{S}}(\Lambda; \upsilon, \mathbb{N}) \geq P_{\mathrm{OCC}}^{\mathcal{S}}(\Psi; \upsilon, \mathbb{N})$ for every state ensemble $\upsilon := \{\nu_n\}_{n \in \mathtt{N}}$ in B_0.*

2) Semiquantum partial preprocessing:

Task 2. *A **semiquantum partial preprocessing** task for Alice is specified by two state ensembles $\varsigma := \{\sigma_j\}_{j \in \mathtt{J}}$ and $\varkappa := \{\kappa_k\}_{k \in \mathtt{K}}$, respectively on systems B_0 and B_1, a distributed POVM $\mathbb{L} := \{L_l\}_{l \in \mathtt{L}}$ on $B_1 \tilde{B}_1$, and a score function $s\colon \mathtt{J} \times \mathtt{K} \times \mathtt{L} \to [-1, 1]$. It has the following steps.*

1) *Bob generates an index $j \in \mathtt{J}$ with a probability $\operatorname{tr}[\sigma_j]$, prepares the state $\sigma_j / \operatorname{tr}[\sigma_j] \in \mathsf{D}^{B_0}$, and sends the state to Alice without letting her know j.*
2) *Alice submits a state in $\tilde{B}_1$ to Bob.*
3) *Bob generates another index $k \in \mathtt{K}$ with a probability $\operatorname{tr}[\kappa_k]$ and prepares an independent state $\kappa_k / \operatorname{tr}[\kappa_k] \in \mathsf{D}^{B_1}$. He measures $B_1 \tilde{B}_1$ with $\mathbb{L}$ and obtains an outcome l. Alice's score equals $s_{j,k,l}$.*

We still assume that Alice is operationally restricted by a closed and convex QRT $(\mathcal{S}, \mathcal{C})$ and that she can access a one-shot oracle to a channel $\Lambda \in \mathsf{C}^A$. We show in Fig. 4 an illustration of the task and Alice's strategy. Her maximum expected score is thus given by

$$S_{\mathrm{SPP}}^{\mathcal{S}}(\Lambda; s, \varsigma, \varkappa, \mathbb{L}) := \max_{\Theta \in \mathcal{S}^{A \twoheadrightarrow B}} \sum_{j,k,l} s_{j,k,l} \operatorname{tr}\big[L_l\big(\kappa_k \otimes \Theta\{\Lambda\}\big[\sigma_j\big]\big)\big]. \tag{15}$$

We say that a state ensemble is *informationally complete* whenever it is proportional to some informationally complete POVM.

Theorem 4. *Let $\Lambda \in \mathsf{C}^A$ and $\Psi \in \mathsf{C}^B$ be two channels. Let $\varsigma := \{\sigma_j\}_{j \in \mathtt{J}}$ and $\varkappa := \{\kappa_k\}_{k \in \mathtt{K}}$ be two arbitrary informationally complete state ensembles in B_0 and B_1 respectively, and $\mathbb{L} := \{L_l\}_{l \in \mathtt{L}}$ be an arbitrary POVM on $B_1 \tilde{B}_1$ satisfying $L_0 = \hat{\psi}_+$. Then there exists a free superoperation $\Theta \in \mathcal{S}^{A \twoheadrightarrow B}$ such that $\Theta\{\Lambda\} = \Psi$ if and only if $S_{\mathrm{OPP}}^{\mathcal{S}}(\Lambda; s, \varsigma, \varkappa, \mathbb{L}) \geq S_{\mathrm{OPP}}^{\mathcal{S}}(\Psi; s, \varsigma, \varkappa, \mathbb{L})$ for every score function $s\colon \mathtt{J} \times \mathtt{K} \times \mathtt{L} \to [0, 1]$.*

B. Relation to resource robustness measures

Apart from testing free convertibility between different channels, we can also use these operational tasks to characterize the resourcefulness of a single channel Λ. This is done by investigating the extent to which the channel oracle Λ can boost Alice's performance in these tasks.

We show that *every* well-defined resource robustness measure $R^{\mathcal{V},\mathcal{C}}(\Lambda)$ of a channel Λ can be interpreted as an operational advantage of Λ for certain type of semiquantum processing task. Here $R^{\mathcal{V},\mathcal{C}}(\Lambda)$ is the resource robustness of $\Lambda \in \mathsf{C}^A$ against a subset of channels $\mathcal{V} \subseteq \mathsf{C}$ (thereby $\mathcal{V}^A := \mathcal{V} \cap \mathsf{C}^A$) [3], defined as

$$R^{\mathcal{V},\mathcal{C}}(\Lambda) := \min_{\left\{\substack{r \in \mathbb{R}_+ \\ \Psi \in \mathcal{V}^A \\ \frac{\Lambda + r\Psi}{1+r} \in \mathcal{C}^A}\right\}} r. \tag{16}$$

For now, we consider a different scenario where Alice's freedom of manipulating her channel oracle Λ using free superoperations is deprived when performing Task 2. Her expected score in this case is given by

$$S_{\mathrm{SPP}}^{\{\mathrm{Id}\}}(\Lambda; s, \varsigma, \varkappa, \mathbb{L}) := \sum_{j,k,l \in \mathtt{J} \times \mathtt{K} \times \mathtt{L}} s_{j,k,l} \operatorname{tr}\big[L_l\big(\kappa_k \otimes \Lambda\big[\sigma_j\big]\big)\big]. \tag{17}$$

We also define

$$S_{\mathrm{SPP}}^{\mathcal{V}}(s, \varsigma, \varkappa, \mathbb{L}) := \max_{\Psi \in \mathcal{V}} S_{\mathrm{SPP}}^{\{\mathrm{Id}\}}(\Psi; s, \varsigma, \varkappa, \mathbb{L}) \tag{18}$$

for any subset of channels $\mathcal{V} \subseteq \mathsf{C}$.

Theorem 5. *Let $\mathcal{V} \subseteq \mathsf{C}$ be a subset of channels satisfying the following conditions:*
(1) the set $\mathcal{V}^A$ is closed and convex;
(2) the value of $R^{\mathcal{V},\mathcal{C}}(\Psi)$ is bounded for all $\Psi \in \mathsf{C}^A$.
Let $\Lambda \in \mathsf{C}^A$ be a channel. Let $\varsigma := \{\sigma_j\}_{j \in \mathtt{J}}$ and $\varkappa := \{\kappa_k\}_{k \in \mathtt{K}}$ be two arbitrary informationally complete state ensembles in A_0 and A_1 respectively, and $\mathbb{L} := \{L_l\}_{l \in \mathtt{L}}$ be an arbitrary POVM on $A_1 \tilde{A}_1$ satisfying $L_0 = \hat{\psi}_+$. Then

$$1 + R^{\mathcal{V},\mathcal{C}}(\Lambda) = \max_{\left\{\substack{s\colon \mathtt{J} \times \mathtt{K} \times \mathtt{L} \to [-1,1] \\ S_{\mathrm{SPP}}^{\mathcal{V}}(s, \varsigma, \varkappa, \mathbb{L}) \geq 0}\right\}} \frac{S_{\mathrm{SPP}}^{\{\mathrm{Id}\}}(\Lambda; s, \varsigma, \varkappa, \mathbb{L})}{S_{\mathrm{SPP}}^{\mathcal{C}}(s, \varsigma, \varkappa, \mathbb{L})}. \tag{19}$$

As a corollary of Theorem 5, both the *resource free robustness* [41] and *resource random robustness* [41] can be characterized with Task 2.

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