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Perspective

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Noncommuting conserved charges in quantum thermodynamics and beyond

Shayan Majidy[1,2] ✉, **William F. Braasch Jr.**[3], **Aleksander Lasek**[3], **Twesh Upadhyaya**[3,4], **Amir Kalev**[5,6] & **Nicole Yunger Halpern**[3,7] ✉

Abstract

Thermodynamic systems typically conserve quantities (known as charges) such as energy and particle number. The charges are often assumed implicitly to commute with each other. Yet quantum phenomena such as uncertainty relations rely on the failure of observables to commute. How do noncommuting charges affect thermodynamic phenomena? This question, upon arising at the intersection of quantum information theory and thermodynamics, spread recently across many-body physics. Noncommutation of charges has been found to invalidate derivations of the form of the thermal state, decrease entropy production, conflict with the eigenstate thermalization hypothesis and more. This Perspective surveys key results in, opportunities for and work adjacent to the quantum thermodynamics of noncommuting charges. Open problems include a conceptual puzzle: evidence suggests that noncommuting charges may hinder thermalization in some ways while enhancing thermalization in others.

Sections

[1]Institute for Quantum Computing, University of Waterloo, Waterloo, ON, Canada. [2]Perimeter Institute for Theoretical Physics, Waterloo, ON, Canada. [3]Joint Center for Quantum Information and Computer Science, NIST and University of Maryland, College Park, MD, USA. [4]Department of Physics, University of Maryland, College Park, MD, USA. [5]Information Sciences Institute, University of Southern California, Arlington, VA, USA. [6]Department of Physics and Astronomy, University of Southern California, Los Angeles, CA, USA. [7]Institute for Physical Science and Technology, University of Maryland, College Park, MD, USA. ✉e-mail: smajidy@uwaterloo.ca; nicoleyh@umd.edu

Introduction

A simple story from undergraduate statistical physics motivates this Perspective: throughout thermodynamics, we consider a small system $\mathcal{S}$ exchanging quantities with a large environment $\mathcal{E}$ (Fig. 1). Suppose that the systems are quantum. If they exchange only heat, $\mathcal{S}$ may thermalize to the canonical state $\rho_{\text{can}} \propto e^{-\beta H^{(\mathcal{S})}}$, where β is the inverse temperature of the environment and $H^{(\mathcal{S})}$ denotes the Hamiltonian of the system of interest. If $\mathcal{S}$ and $\mathcal{E}$ exchange heat and particles, $\mathcal{S}$ may thermalize to the grand canonical state $\rho_{\text{GC}} \propto e^{-\beta(H^{(\mathcal{S})} - \mu \mathcal{N}^{(\mathcal{S})})}$, where μ is the chemical potential and $\mathcal{N}^{(\mathcal{S})}$ denotes the system-of-interest particle-number operator. This pattern extends to many exchanged quantities (such as electric charge and magnetization) and many thermal states.

The exchanged quantities are conserved globally (across $\mathcal{SE}$), so we call them charges. The Hermitian operator Q_a represents the conserved quantities: $\mathcal{S}$ has an operator $Q_a^{(\mathcal{S})}$, $\mathcal{E}$ has $Q_a^{(\mathcal{E})}$ and the global system has $Q_a^{\text{tot}} := Q_a^{(\mathcal{S})} + Q_a^{(\mathcal{E})} \equiv Q_a^{(\mathcal{S})} \otimes \mathbb{1}^{(\mathcal{E})} + \mathbb{1}^{(\mathcal{S})} \otimes Q_a^{(\mathcal{E})}$. The index $a = 0, 1, \ldots, c$, where c is the dimension of the algebra of the charges; the Hamiltonian $H = Q_0$.

Across thermodynamics, we often assume implicitly that the charges commute with each other: $[Q_a, Q_{a'}] = 0 \;\; \forall\, a, a'$. This assumption is almost never mentioned (recall your undergraduate statistical physics course). However, the assumption underlies derivations of the form of thermal states[1,2], linear-response coefficients[3] and more. However, observables can fail to commute, and this possibility enables quintessentially quantum phenomena: the Einstein–Podolsky–Rosen paradox[4], uncertainty relations[5] and measurement disturbance[6], among others. Quantum physics, therefore, compels us to ask: what happens to thermodynamic phenomena under dynamics that conserve charges Q_a^{tot} that fail to commute with each other?

This question arose in recent years at the intersection of quantum information theory and quantum thermodynamics, then seeped into many-body physics. This Perspective surveys results, opportunities and adjacent work. In the rest of the introduction, we first present a simple physical example. Then, we sample example phenomena transformed by the noncommutation of charges. We later establish notation, followed by the outline of this Perspective.

A simple physical example was proposed theoretically[7,8] and realized with trapped ions[9]: a chain (Fig. 2), formed from qubits (quantum two-level systems). A few (say, two) qubits form $\mathcal{S}$, and the other qubits form $\mathcal{E}$. The chain constitutes a closed quantum many-body system, of the sort whose internal thermalization has recently been studied theoretically and experimentally[10–13]. Denote by σ_a the Pauli-a operator, for $a = x, y, z$; by $\sigma_a^{(j)}$, a spin component of qubit j; and by $\sigma_a^{\text{tot}} := \sum_j \sigma_a^{(j)}$, a total spin component. One can construct a Hamiltonian that overtly transports quanta of each σ_a locally while conserving the σ_a^{tot}s globally[7,8]. Denote the σ_z ladder operators by $\sigma_{\pm z} := \frac{1}{2}(\sigma_x \pm i\sigma_y)$. The operator $\sigma_{+z}^{(j)}\sigma_{-z}^{(j+1)} + \text{h.c.}$ transports one σ_z quantum from qubit $j+1$ to qubit j and vice versa, in superposition. Define ladder operators and couplings analogously for σ_x and σ_y. The Hamiltonian

$$H_{\text{Heis}}^{\text{tot}} = \sum_j \sum_{\alpha = x,y,z} (\sigma_{+\alpha}^{(j)}\sigma_{-\alpha}^{(j+1)} + \text{h.c.}) = \sum_j \boldsymbol{\sigma}^{(j)} \cdot \boldsymbol{\sigma}^{(j+1)} \tag{1}$$

transports the σ_as locally, while conserving them globally. $H_{\text{Heis}}^{\text{tot}}$ is the Heisenberg model, theoretically well-known and experimentally realizable[7,9]. Yet $H_{\text{Heis}}^{\text{tot}}$ is rarely, if ever, expressed outside the literature on the thermodynamics of noncommuting charges as it is in equation (1) – as locally transporting and globally conserving three noncommuting charges. Furthermore, we can extend $H_{\text{Heis}}^{\text{tot}}$ to nonintegrable models (which promote thermalization), to subsystems beyond qubits and to charges beyond spin components[8]. (For more on quantum thermodynamics of spin exchanges, see refs. 14–17).

Having exhibited a thermodynamic system whose Q_as fail to commute, we sample the physics altered by the noncommutation of charges. This sample only dips into the known results (detailed in the section 'New physics') but demonstrates why noncommuting thermodynamic charges merit study. If the Q_as fail to commute, the extent to which $\mathcal{S}$ can thermalize is unclear for several reasons. First, noncommuting charges impede two derivations of the form of the thermal state[1,2]. (An information-theoretic derivation holds regardless of the noncommutation of charges, but this derivation provides less physical insight; see the section 'Early work'.) Second, noncommuting charges force degeneracies on H^{tot}. The reason is Schur's lemma, a group-theoretic result[7,18,19] (Supplementary Information). Nondegenerate Hamiltonians underlie arguments supporting thermalization, mixing and equilibration[20–23] – arguments, thus, challenged by noncommuting charges. Third, noncommuting charges conflict with the eigenstate thermalization hypothesis (ETH)[24]. The ETH is a theoretical and conceptual toolkit for explaining how quantum many-body systems thermalize internally[20,21,25]. Myriad numerical and experimental observations support the ETH, which has been applied across many-body physics. Yet noncommuting charges preclude the ETH (as discussed in the section 'New physics'). Hence, noncommuting charges challenge expectations about thermalization in a third manner.

We encounter even more manners across this Perspective. Resistance to thermalization matters both fundamentally and for applications. Most systems thermalize, so resistance is unusual; few mechanisms for resistance in quantum many-body systems are known[26,27]. Additionally, thermalization disperses information about initial conditions, so thermalization resistance can promote memory storage.

Noncommuting thermodynamic charges present several research opportunities, three of which we flesh out in this Perspective. First, a major opportunity is to identify the extent to which noncommuting charges hinder thermalization. Another opportunity is to re-examine every thermodynamic or chaotic phenomenon, determining the extent to which it changes if charges fail to commute. The phenomena known to change, as reviewed in this Perspective, include the ETH, microcanonical states, thermodynamic-entropy production, entanglement entropy and the ability of local interactions to effect global evolutions. A third opportunity is to marry recent noncommuting-charge work, based largely on quantum thermodynamics and information theory, with adjacent work in other fields, such as high-energy theory. Other fields favour the terminology non-Abelian symmetries;

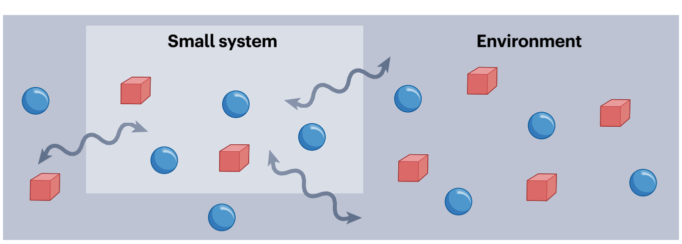

Fig. 1 | Common thermodynamic paradigm. A small system and large environment locally exchange quantities (indicated by spheres and cubes) that are conserved globally. Common quantities include energy and particles of different species.

fundamental thermodynamics motivates our noncommuting-charge language. A Hamiltonian conserves noncommuting charges, $[H^{\mathrm{tot}}, Q_a^{\mathrm{tot}}] = 0 \;\; \forall\, a$, if and only if it has a continuous non-Abelian symmetry: $U_a^\dagger H^{\mathrm{tot}} U_a = H^{\mathrm{tot}} \;\; \forall\, a$. The charges generate the unitary operators: $U_a = e^{-iQ_a t}$, for $t \in \mathbb{R}$. The Q_as form a Lie algebra, and the U_as form the associated Lie group[28].

Across the Perspective, we use the notation introduced above, as well as the following. We call a closed quantum many-body system (such as the composite $\mathcal{SE}$ in Fig. 1) a global system. N denotes the number of degrees of freedom (DOFs) in a global system. Often, $\mathcal{SE}$ consists of N copies of $\mathcal{S}$. For example, N denotes the number of qubits in Fig. 2. Large but finite N – the mesoscale – interests us: as $N \to \infty$, $\mathcal{SE}$ grows classical, according to the correspondence principle[2]. Noncommutation enables non-classical phenomena, so we should expect the noncommutation of charges to influence thermodynamic phenomena at finite N. Continuing with notation, we ascribe to the Pauli operators σ_a eigenstates $|a\pm\rangle$ associated with the eigenvalues ±1. Subscripts index charges (as in $\sigma_a^{(j)}$), whereas superscripts index sites or other subsystems (such as $\mathcal{S}$ and $\mathcal{E}$). We often denote commuting charges, or other observables that might commute, by $\widetilde{Q}_a$s.

The rest of this Perspective is organized as follows. First, we introduce the earliest work on noncommuting thermodynamic charges. We, then, survey new physics that arises from the noncommutation of charges. Next, we discuss bridges with related research. Finally, we point out the richest-looking avenues for future work.

Early work

Edwin Thompson Jaynes was the first, to our knowledge, to address anything like noncommuting thermodynamic charges[29]. He formalized the principle of maximum entropy, which pinpoints the state $\rho = \sum_k p_k \; |k\rangle\langle k|$ most reasonably attributable to a system about which one knows little. Imagine knowing about ρ only the expectation values $\langle \widetilde{Q}_a \rangle$ of observables $\widetilde{Q}_a$. The state obeys the constraints $\mathrm{Tr}(\rho \widetilde{Q}_a) = \widetilde{Q}_a^{\mathrm{avg}} \in \mathbb{R}$, plus the normalization condition $\mathrm{Tr}(\rho) = 1$. Whichever constraint-obeying state maximizes the von Neumann entropy $S_{\mathrm{vN}}(\rho) := -\mathrm{Tr}(\rho \log(\rho))$ is the most reasonable, according to the maximum-entropy principle. (The logarithms of this subsection are base-e.) The entropy maximization encapsulates our ignorance of everything except the constraints. The function maximized is

$$
\begin{aligned}
S_{\mathrm{vN}}(\rho) - 0 &= S_{\mathrm{vN}}(\rho) - \lambda[\mathrm{Tr}(\rho) - 1] - \sum_a \widetilde{\mu}_a \left[\mathrm{Tr}(\rho \widetilde{Q}_a) - \widetilde{Q}_a^{\mathrm{avg}} \right] \\
&=: \mathcal{L}(\rho, \lambda, \{\widetilde{\mu}_a\})
\end{aligned} \tag{2}
$$

Lagrange multipliers are denoted by $\lambda, \widetilde{\mu}_a \in \mathbb{R}$.

The state's eigenbasis, $\{|k\rangle\langle k|\}$, equals an eigenbasis of $\sum_a \widetilde{\mu}_a \widetilde{Q}_a$: if $\{|k\rangle\langle k|\}$ did not, decohering ρ with respect to any $\sum_a \widetilde{\mu}_a \widetilde{Q}_a$ eigenbasis would raise S_{vN} monotonically, while preserving the other terms in equation (2) (ref. 30). Hence, the decohered state would achieve at least as large an $\mathcal{L}$ value as ρ. Maximizing $\mathcal{L}(\rho, \lambda, \{\widetilde{\mu}_a\})$ with respect to p_k yields

$$
\rho = \frac{1}{Z} e^{-\sum_a \widetilde{\mu}_a \widetilde{Q}_a} \tag{3}
$$

Maximizing with respect to λ fixes the partition function: $Z = \mathrm{Tr}(e^{-\sum_a \widetilde{\mu}_a \widetilde{Q}_a})$. Maximizing with respect to $\widetilde{\mu}_a$ yields $\widetilde{Q}_a^{\mathrm{avg}} = -\frac{\partial}{\partial \widetilde{\mu}_a} \log(Z)$. This procedure works, Jaynes noted, even if the $\widetilde{Q}_a$s do not commute. He showed how to measure the $\widetilde{Q}_a^{\mathrm{avg}}$s of such observables using extra systems. One can derive his results alternatively via analytic properties of Z (ref. 31).

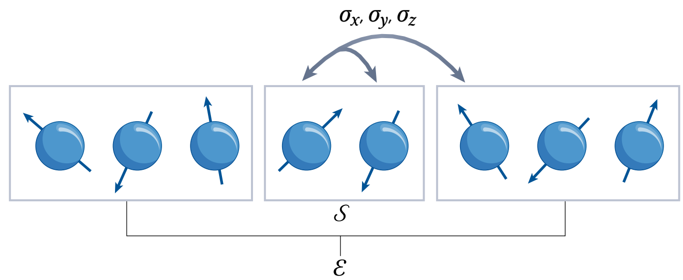

Fig. 2 | Example thermodynamic system that conserves noncommuting charges. Two qubits form the system $\mathcal{S}$ of interest, and the rest form the environment $\mathcal{E}$. A qubit's three-spin components, $\sigma_{a=x,y,z}$, form the local noncommuting charges. The dynamics locally transport and globally conserve the charges.

The work of Jaynes, though pioneering, left threads hanging. First, the maximum-entropy principle invokes only information theory and quantum physics, not thermodynamics or charge conservation. Connecting ρ to equilibrium requires arguments that are more physical in nature. Second, Jaynes wrote only one paragraph about noncommuting $\widetilde{Q}_a$s. What could they represent physically? Which systems might have them? When might they impact thermodynamics?

Still, the known-about observables $\widetilde{Q}_a$ of Jaynes can be noncommuting charges Q_a. Hence, Jaynes wrote what was later termed the non-Abelian thermal state (NATS)[2],

$$
\rho_{\mathrm{NATS}} := \frac{1}{Z} e^{-\beta(H - \sum_a \mu_a Q_a)} \tag{4}
$$

β denotes an inverse temperature. The μ_as denote generalized chemical potentials. ρ_{NATS} has the form of equation (3), which has been called the generalized Gibbs ensemble (GGE), regardless of whether the charges commute[32–35]. We use the term NATS for three reasons. First, the term GGE was initially introduced for integrable systems, whereas the NATS is a thermal state. Second, most GGE literature concerns commuting charges (for exceptions, see the section 'Adjacent work in other fields'). Third, justifying the form of the NATS physically (not merely information-theoretically) is more difficult than justifying the form of a typical (commuting-charge) GGE, as explained later in this section.

Later researchers expanded upon the work of Jaynes[36–39]. One extension[39] focused on seeking a more physical justification for the form of ρ_{NATS}. Consider N copies of the system of interest, in the ensemble tradition of thermodynamics. In thermodynamics, we regard all copies except one ($\mathcal{S}$) as forming an effective environment ($\mathcal{E}$) (ref. 40). Imagine $\mathcal{S}$ exchanging energy and particles with $\mathcal{E}$. How can we prove that $\mathcal{S}$ is in a canonical state ρ_{can}? We assume that $\mathcal{SE}$ has a fixed particle number and an energy in a small window – that $\mathcal{SE}$ is in a microcanonical subspace. Tracing out $\mathcal{E}$ from the microcanonical state yields the state of $\mathcal{S}$, which equals ρ_{can} if $\mathcal{S}$ and $\mathcal{E}$ couple weakly[41]. Suppose that $\mathcal{S}$ and $\mathcal{E}$ exchange several commuting charges $\widetilde{Q}_a$. The microcanonical subspace is an eigenspace shared by the $\widetilde{Q}_a^{\mathrm{tot}}$s (except $\widetilde{Q}_0^{\mathrm{tot}} = H^{\mathrm{tot}}$, the time-evolution generator). What if $\mathcal{S}$ and $\mathcal{E}$ exchange noncommuting charges Q_a? The Q_a^{tot}s share no eigenbasis, so they might share no eigenspaces. Microcanonical subspaces might not exist.

An attempt to overcome this challenge is presented by ref. 39. The charge density Q_a^{tot}/N commutes in the infinite-N limit: $\lim_{N\to\infty} \frac{1}{N^2}[Q_a^{\mathrm{tot}}, Q_{a'}^{\mathrm{tot}}] = 0 \;\; \forall\, a, a'$. This property was hoped to enable a generalization of microcanonical subspaces for noncommuting charges. But a well-justified generalization would wait for the maturation of quantum information theory.

For decades afterward, no literature, to our knowledge, addressed the ability of thermodynamic charges to not commute. The topic gained attention again in the 2010s, when two publications showed that noncommuting charges can violate thermodynamic expectations[1,42].

The first of these publications showed that noncommuting charges overturn an expectation about free energy[42]. Consider a system $\mathcal{S}$ of interest, in a state $\rho^{(\mathcal{S})}$. Let $\mathcal{S}$ begin uncorrelated with its environment $\mathcal{E}$, which consists of c subsystems: $\rho^{(\mathcal{S})} \otimes \rho^{(\mathcal{E})}$, wherein $\rho^{(\mathcal{E})}$ has the GGE form of equation (3). $\mathcal{S}$ has c commuting charges $\widetilde{Q}_a^{(\mathcal{S})}$, and $\mathcal{E}$ has $\widetilde{Q}_a^{(\mathcal{E})}$s. One can attribute to $\mathcal{S}$ a 'free energy' $F_a^{(\mathcal{S})}(\rho) := -\frac{1}{\beta\mu_a} S(\rho) + \mathrm{Tr}(\widetilde{Q}_a \rho)$. (The quotation marks reflect the controversy surrounding free energies defined information-theoretically, for out-of-equilibrium states). Let any charge-conserving unitary U evolve $\mathcal{SE}$ to a state $\rho_{\mathrm{f}}^{(\mathcal{SE})}$: $[U, \widetilde{Q}_a^{\mathrm{tot}}] = 0 \ \forall\, a$. The subscript f distinguishes the final states of $\mathcal{S}$ and $\mathcal{E}$. The jth subsystem of $\mathcal{E}$ ends up in $\rho_{\mathrm{f}}^{(\mathcal{E},j)} := \mathrm{Tr}_{\bar{j}}(\rho_{\mathrm{f}}^{(\mathcal{E})})$. Three more quantities change: the von Neumann entropy of $\mathcal{S}$, by $\Delta S_{\mathrm{vN}}^{(\mathcal{S})} := S_{\mathrm{vN}}(\rho_{\mathrm{f}}^{(\mathcal{S})}) - S_{\mathrm{vN}}(\rho^{(\mathcal{S})})$; the environmental expectation value of ath charge, by $\Delta\langle \widetilde{Q}_a^{(\mathcal{E})} \rangle := \mathrm{Tr}(\widetilde{Q}_a^{(\mathcal{E})} [\rho_{\mathrm{f}}^{(\mathcal{E})} - \rho^{(\mathcal{E})}])$; and the ath 'free energy' of the system, by $\Delta F_a^{(\mathcal{S})}$. U redistributes charges and information contents according to

$$
\begin{aligned}
&-\Delta S_{\mathrm{vN}}^{(\mathcal{S})} + D\left((\rho_{\mathrm{f}}^{(\mathcal{SE})} \| \rho_{\mathrm{f}}^{(\mathcal{S})} \otimes \rho_{\mathrm{f}}^{(\mathcal{E})}) \right) \\
&= \beta \sum_a \mu_a \left((\Delta\langle \widetilde{Q}_a^{(\mathcal{E})} \rangle - \Delta F_a^{(\mathcal{S})}) \right) - D\left((\rho_{\mathrm{f}}^{(\mathcal{E})} \| \otimes_j \rho_{\mathrm{f}}^{(\mathcal{E},j)}) \right)
\end{aligned}
\tag{5}
$$

The relative entropy $D(\sigma_1 \| \sigma_2) = -\mathrm{Tr}(\sigma_1[\log\sigma_1 - \log\sigma_2])$ quantifies the distance between quantum states $\sigma_{1,2}$[43]. Equation (5) relates that the growth in the information content of $\mathcal{S}$, plus the final nonseparability of global state, depends on the average changes in the charges of $\mathcal{E}$, the 'free-energy' change of $\mathcal{S}$, and the correlations formed in $\mathcal{E}$. If charges fail to commute, the derivation breaks down. No term decomposes into $\Delta F_a^{(\mathcal{S})}$ of distinct charges. Hence, noncommuting charges undermine an expectation about free energy.

The second of these publications reasoned about noncommuting charges in thermodynamic resource theories[1]. Resource theories are information-theoretic models for contexts in which restrictions constrain the operations performable and the systems accessible[44]. Using a resource theory, one calculates the optimal efficiencies with which an agent, subject to the restrictions, can perform tasks such as extracting work from non-equilibrium quantum systems. In thermodynamics, the first law constrains agents to conserve energy. Every unitary U performable on a closed, isolated system conserves the total Hamiltonian: $[U, H^{\mathrm{tot}}] = 0$ (ref. 45). Suppose that U must conserve commuting global charges $\widetilde{Q}_a^{\mathrm{tot}}$: $[U, \widetilde{Q}_a^{\mathrm{tot}}] = 0$ (ref. 46). Which systems, if freely accessible to a thermodynamic agent, would render the model non-trivial – would not enable the agent to, say, perform work for free, achieving a perpetuum mobile? The answer is, systems in the equilibrium state shown in equation (3) (refs. 46,47). The proof fails, however, if the charges fail to commute[1].

Three groups to apply quantum-information-theoretic thermodynamics to noncommuting charges are spurred by refs. 1,42, leading to a trio of papers[2,30,48] (as overviewed in ref. 49). (Ref. 50 predates refs. 2,30,48; it lacks thermodynamic motivations but has thermodynamic implications.) A quantification of trade-offs among work and charges is presented by ref. 30, using a resource theory similar to that in refs. 1,42. Consider a system $\mathcal{S}$ in an out-of-equilibrium state $\rho^{(\mathcal{S})}$. $\mathcal{S}$ begins uncorrelated with an environment $\mathcal{E}$ in a GGE $\rho^{(\mathcal{E})}$. Each of $\mathcal{S}$ and $\mathcal{E}$ has charges Q_a that might or might not commute. (In this section, only in this paragraph do Q_as denote possibly commuting charges.) The set-up also includes batteries, systems in which work or charges can be reliably stored and from which the resources can be reliably retrieved. A 'free entropy' can be ascribed to every state ρ of $\mathcal{S}$:

$$\widetilde{F}^{(\mathcal{S})}(\rho) := \beta \sum_a \mu_a \mathrm{Tr}(\rho Q_a^{(\mathcal{S})}) - S_{\mathrm{vN}}(\rho^{(\mathcal{S})}). \tag{6}$$

($\widetilde{F}^{(\mathcal{S})}(\rho)$ is the negative of a non-equilibrium extension of a Massieu function[40].) Let $\mathcal{SE}$ evolve under any unitary U that conserves each total (system–environment–battery) charge. For all $X = \mathcal{S}, \mathcal{E}$, the state $\rho^{(X)}$ evolves to $\rho_{\mathrm{f}}^{(X)}$. The average a-charge of X changes by $\Delta\langle Q_a^{(X)} \rangle := \mathrm{Tr}(Q_a^{(X)} [\rho_{\mathrm{f}}^{(X)} - \rho^{(X)}])$. The free entropy of $\mathcal{S}$ changes by $\Delta\widetilde{F}^{(\mathcal{S})}$. The batteries perform on $\mathcal{SE}$ the average 'a-charge work' $W_a = -(\Delta\langle Q_a^{(\mathcal{S})} \rangle + \Delta\langle Q_a^{(\mathcal{E})} \rangle)$. This charge work obeys a generalized second law of thermodynamics,

$$\beta \sum_a \mu_a W_a \le -\Delta\widetilde{F}^{(\mathcal{S})}. \tag{7}$$

If $\mathcal{S}$ is trivial (corresponds to a 0-dimensional Hilbert space) or is unchanged by U, $\Delta\widetilde{F}^{(\mathcal{S})} = 0$. Consequently, one can extract any amount of any charge from $\mathcal{E}$ by paying a price in the other charges: $\mu_a W_a \le -\sum_{a' \neq a} \mu_{a'} W_{a'}$. Under what conditions equation (7) is saturable depends on whether the charges commute. Trade-off conclusions are also drawn by ref. 48, which were applied to the erasure of qubits.

Rounding out the trio[2,30,48], ref. 2 has presented multiple physical justifications for the form of ρ_{NATS}, beginning with a microcanonical-like derivation. Noncommuting charges prevent microcanonical subspaces from existing (in abundance). Hence, microcanonical subspaces were generalized to approximate microcanonical (AMC) subspaces in ref. 2. In an AMC subspace $\mathcal{M}$, every Q_a^{tot} has a fairly well-defined value: measuring any Q_a^{tot} has a high probability of yielding a value near the expectation value $\langle Q_a^{\mathrm{tot}} \rangle$. The existence of $\mathcal{M}$ was proven under certain conditions[2]. Denote by $\Pi_{\mathcal{M}}$ the projector onto $\mathcal{M}$. Consider ascribing the AMC state $\Pi_{\mathcal{M}}/\mathrm{Tr}(\Pi_{\mathcal{M}})$ to the global system, formed from N copies of the system $\mathcal{S}$ of interest. Trace out all copies except the ℓ^{th}: $\rho^{(\ell)} = \mathrm{Tr}_{\bar{\ell}}(\Pi_{\mathcal{M}}/\mathrm{Tr}(\Pi_{\mathcal{M}}))$. Compare $\rho^{(\ell)}$ with ρ_{NATS} using the relative entropy. Average over ℓ. This average distance is upper-bounded as

$$\left\langle D(\rho^{(\ell)} \| \rho_{\mathrm{NATS}}) \right\rangle_{\ell} \le \frac{\theta}{\sqrt{N}} + \theta' \tag{8}$$

The constants θ and θ' are independent of N but depend on parameters (the number c of charges, their expectation values and so on). As the global system grows ($N \to \infty$), the $1/\sqrt{N} \to 0$, so the distance shrinks. Hence, a physical argument, based on an ensemble in an AMC subspace, complements the information-theoretic derivation of ρ_{NATS} by Jaynes. Furthermore, the AMC subspace enabled later noncommuting-charge work[7–9,24] (described later in this section and in the section 'New physics').

The form of ρ_{NATS} was justified also with the principle of complete passivity[2,48,51]. A state ρ is completely passive if no work can be extracted from $\rho^{\otimes N}$ adiabatically, for any $N \in \mathbb{Z}_{\ge 0}$ (refs. 52,53). Complete passivity extends Carnot's version of the second law of thermodynamics: no engine can extract work from one fixed-temperature heat bath. Consider work extraction that evolves the state of a battery from $\rho^{(\mathcal{B})}$ to $\rho_{\mathrm{f}}^{(\mathcal{B})}$. Let $Q_a^{(\mathcal{B})}$ denote the ath charge of a battery. One can define the (average) work performed on the battery as $\mathrm{Tr}(\rho_{\mathrm{f}}^{(\mathcal{B})}[\sum_a \mu_a Q_a^{(\mathcal{B})}]) - \mathrm{Tr}(\rho^{(\mathcal{B})}[\sum_a \mu_a Q_a^{(\mathcal{B})}])$. Using this definition and a resource theory, one can prove that ρ_{NATS} is completely passive[2].

This result resolved the problem spotlighted in ref. 1: noncommuting charges invalidate a resource-theoretic derivation of the form of thermal state. (By invalidate, we mean that, if the charges fail to commute, the derivation becomes unsound. We use unsound in the technical sense of the study of logic: a deductive argument is sound if, provided that the input premises are true, the conclusion is true.) Yet work is traditionally a change in the energy charge[48]. Indeed, ρ_{NATS} is completely passive if work is defined as a change in energy alone, if the extracted-from system obeys charge conservation[51]. Furthermore, one can define an analogue W_a of work for each charge Q_a (ref. 48), as in ref. 30. Consider substituting different W_as into the definition of complete passivity. Different W_as distinguish different thermal states as passive because the Q_as fail to commute[48]. Hence, complete passivity is another thermodynamic principle complicated by noncommuting charges.

The trio[2,30,48] have galvanized noncommuting-charge activity at the intersection of quantum information and quantum thermodynamics. Several publications have elaborated on resource theories[54–58]. For example, more-detailed 'second laws' were proven using the mathematical tools of matrix majorization[54]. These second laws stipulate necessary and sufficient conditions for any quantum state σ_1 to transform into any state σ_2. Other literature has shown how different batteries could store different Q_as despite the noncommutation of charges[50,56,57,59]. (Batteries are called reference frames when used to implement otherwise forbidden transformations. Consider transforming a system-and-environment composite $\mathcal{SE}$ with unitaries U restricted to conserving each $Q_a^{(\mathcal{S})} + Q_a^{(\mathcal{E})}$. What if a desired U breaks the conservation law? A reference frame may supply the charge needed, or accept the charge forfeited, by $\mathcal{SE}$.)

The above work, steeped in information theory, is theoretical and abstract. Yet it inspired an experimental demonstration that real-world systems exhibit the thermodynamics of noncommuting charges[9]. In the demonstration, 6–21 trapped ions form a linear Paul chain. Two qubits form $\mathcal{S}$, and the rest form $\mathcal{E}$ (Fig. 1). The global system is prepared in $|\psi_0\rangle = (|y+\rangle\ |x+\rangle\ |z+\rangle)^{\otimes N/3}$, in an AMC subspace. (The definition of AMC subspace was adapted by ref. 7 from the information-theoretic ref. 2 to many-body physics. In ref. 2, if $\mathcal{SE}$ occupies an AMC subspace, measuring any Q_a^{tot} has a high probability of yielding a value near $\langle Q_a^{\text{tot}}\rangle$. Constant-in-$N$ expressions quantify what 'high' and 'near' mean. In refs. 7–9, the probability distribution has a variance of $O(N^r)$, wherein $r \le 1$.) The native interactions are long-range Ising couplings $(\sigma^{(j)} \cdot \sigma^{(k)})$. However, Heisenberg interactions (also long-range) are simulated via Trotterization with rotations. The dynamics conserve the charges $\sigma_{x,y,z}^{\text{tot}}$ globally while transporting $\sigma_{x,y,z}$ locally. Quantum state tomography reveals the long-time state ρ_{f} of $\mathcal{S}$.

The distance of the state from ρ_{NATS} is calculated using the relative entropy $D(\rho_{\text{f}}||\rho_{\text{NATS}})$, following ref. 2. Averaging $D(\rho_{\text{f}}||\rho_{\text{NATS}})$ over the qubit pairs $\mathcal{S}$ that form the chain produces $\langle D(\rho_{\text{f}}||\rho_{\text{NATS}})\rangle$. $\langle D(\rho_{\text{f}}||\rho_{\text{GC}})\rangle$ and $\langle D(\rho_{\text{f}}||\rho_{\text{can}})\rangle$ are calculated similarly. The NATS predicted the experimental results best: $\langle D(\rho_{\text{f}}||\rho_{\text{NATS}})\rangle < \langle D(\rho_{\text{f}}||\rho_{\text{GC}})\rangle < \langle D(\rho_{\text{f}}||\rho_{\text{can}})\rangle$. Despite the plurality of the conservation laws that decoherence could break, the effects of noncommuting charges were experimentally observable; dynamical-decoupling pulse sequences mitigated the noise sufficiently. This experiment, a realization of the proposal in ref. 7, opens the door to experimentally testing the many theoretical results about noncommuting thermodynamic charges (discussed in the section 'New physics'). Along with trapped ions, feasible platforms include superconducting qudits, ultracold atoms, nitrogen-vacancy centres and quantum dots[7].

New physics

Noncommuting thermodynamic charges engender new physics. They conflict with the ETH, allow for multiple stationary states and violate the dichotomy of energy-level statistics. The noncommutation of charges also constrains the global dynamics implementable with local charge-conserving unitaries. Additionally, noncommuting charges decrease thermodynamic entropy production, yet increase average entanglement entropy.

Conflict with the eigenstate thermalization hypothesis

The ETH explains how reversible dynamics thermalize closed quantum many-body systems internally[20,21,25]. Consider a system with N DOFs governed by a nondegenerate Hamiltonian H^{tot}. Let $\mathcal{O}$ denote any operator, which we represent as a matrix relative to the energy eigenbasis. The ETH is an ansatz for the forms of matrix elements. Suppose that the system begins in a microcanonical state $|\psi(0)\rangle$ – in this context, a pure state with a small H^{tot} variance. The ETH implies thermalization: the time-averaged expectation value of $\mathcal{O}$ approximately equals the canonical expectation value

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \mathrm{d}t'\ \langle\psi(t')|\mathcal{O}|\psi(t')\rangle = \mathrm{Tr}(\mathcal{O}\rho_{\text{can}}) + O(N^{-1}) \tag{9}$$

Throughout this subsection, the O notation means 'scales as'. Noncommuting charges impede three assumptions behind the argument for thermalization. First, noncommuting charges prevent microcanonical states from existing in abundance (as discussed in the section 'Early work'). Second, noncommuting charges force degeneracies on the Hamiltonian (as discussed in the section 'Introduction'). Third, noncommuting charges lead the Wigner–Eckart theorem to supersede the ETH[60]. To review the Wigner–Eckart theorem briefly, we consider N qubits whose global spin components $S_{x,y,z}^{\text{tot}}$ are conserved, as in the section 'Introduction'. Denote by $\{|\alpha, m\rangle\}$ the eigenbasis shared by H^{tot}, $(\mathbf{S}^{\text{tot}})^2$ and S_z^{tot}: if $\hbar = 1$, then $H^{\text{tot}}\ |\alpha, m\rangle = E_\alpha\ |\alpha, m\rangle$, $(\mathbf{S}^{\text{tot}})^2|\alpha, m\rangle = s_\alpha(s_\alpha + 1)|\alpha, m\rangle$ and $S_z^{\text{tot}}\ |\alpha, m\rangle = m\ |\alpha, m\rangle$. The Wigner–Eckart theorem governs spherical tensor operators formed from components $T_q^{(k)}$ (ref. 60). The $T_q^{(k)}$s form a basis for the space of operators defined on the Hilbert space of the system. For example, consider an atom absorbing a photon (of spin $k = 1$), gaining $q = 1$ quantum of z-type angular momentum. $T_{q=1}^{(k=1)}$ represents the effect of the photon on the state of the atom. Consider representing $T_q^{(k)}$ as a matrix relative to the energy eigenbasis. That matrix obeys the Wigner–Eckart theorem[60]:

$$\langle\alpha, m|T_q^{(k)}|\alpha', m'\rangle = \langle s_\alpha, m|s_{\alpha'}, m'; k, q\rangle\langle\alpha||T^{(k)}||\alpha'\rangle \tag{10}$$

$\langle s_\alpha, m|s_{\alpha'}, m'; k, q\rangle$ denotes a Clebsch–Gordan coefficient, a conversion factor between the product state $|s_{\alpha'}, m'; k, q\rangle \equiv |s_{\alpha'}, m'\rangle|k, q\rangle$ and the total-spin eigenstate $|s_\alpha, m\rangle$. $\langle\alpha||T^{(k)}||\alpha'\rangle$ is a reduced matrix element – the part of $\langle\alpha, m|T_q^{(k)}|\alpha', m'\rangle$ that does not depend on magnetic spin quantum numbers. The theorem (10) conflicts with the ETH, an ansatz for the left-hand side. First, the two constraints are fundamentally different; the Wigner–Eckart theorem concerns SU(2) symmetry, whereas the ETH concerns randomness. Second, the ETH states that off-diagonal elements $\langle\alpha, m|T_q^{(k)}|\alpha', m'\rangle$ are exponentially small in N. The Wigner–Eckart theorem implies that these elements may be $O(1)$.

Therefore, a non-Abelian ETH is posited by ref. 24. This ansatz depends on the average energy $\mathcal{E} := \frac{1}{2}(E_\alpha + E_{\alpha'})$, energy difference $\omega := E_\alpha - E_{\alpha'}$, average spin quantum number $\mathcal{S} := \frac{1}{2}(s_\alpha + s_{\alpha'})$ and difference $\nu := s_\alpha - s_{\alpha'}$. Denote by $S_{\text{th}}(\mathcal{E}, \mathcal{S})$ the thermodynamic entropy at energy $\mathcal{E}$ and spin $\mathcal{S}$. The observable $T_q^{(k)}$ and Hamiltonian H^{tot} satisfy the non-Abelian ETH if, for smooth real functions $\mathcal{T}^{(k)}(\mathcal{E},\mathcal{S})$ and $f_\nu^{(k)}(\mathcal{E},\mathcal{S},\omega)$,

$$\langle\alpha||T^{(k)}||\alpha'\rangle = \mathcal{T}^{(k)}(\mathcal{E},\mathcal{S})\,\delta_{\alpha,\alpha'} + e^{-S_{\rm th}(\mathcal{E},\mathcal{S})/2} f_\nu^{(k)}(\mathcal{E},\mathcal{S},\omega)\,R_{\alpha,\alpha'} \tag{11}$$

$\mathcal{T}^{(k)}$ parallels a microcanonical average in the conventional ETH. $R_{\alpha,\alpha'}$ resembles a normalized $O(1)$ random variable[61–63]. The matrix element (10) deviates from the ordinary ETH through $\mathcal{S}$-dependent functions and a Clebsch–Gordan coefficient. Equation (11) has withstood preliminary numerical checks with a Heisenberg Hamiltonian on a 2D qubit lattice[64]. More testing is needed, however.

The non-Abelian ETH predicts thermalization to the usual extent in some, but not all, contexts. Consider preparing the system in a state $|\psi(0)\rangle$ in an AMC subspace (as discussed in the section 'Early work'). Suppose that $|\psi(0)\rangle$ has an extensive magnetization along an axis that we call $\hat{z}$: $\langle\psi(0)|S_z^{\rm tot}|\psi(0)\rangle = O(N)$. According to the non-Abelian ETH,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t {\rm d}t'\,\langle\psi(t')|T_q^{(k)}|\psi(t')\rangle = {\rm Tr}(T_q^{(k)}\rho_{\rm NATS}) + O(N^{-1}) \tag{12}$$

However, if $\langle\psi(0)|S_z^{\rm tot}|\psi(0)\rangle = 0$, the correction can become $O(N^{-1/2})$ – polynomially larger. This result relies on an assumption argued to be physically reasonable: the smooth function $\mathcal{T}^{(k)}$ in equation (11) can contain a non-zero term of $O(s_a/N)$. The unusually large, analytically expected correction constitutes further evidence that the noncommutation of charges can alter thermalization.

The non-Abelian ETH invites further studies. First, numerical and experimental tests shall be conducted. Second, from the final term of non-Abelian ETH, one can infer about fluctuations around the time average as in equation (9). Third, the time required for equilibration merits study: in quantum many-body physics, most thermalization-blocking mechanisms slow thermalization in time (as discussed in the section 'Adjacent work in other fields'). By contrast, the $O(N^{-1/2})$ 'slows' the time-averaged expectation value in N, in the approach to a thermal expectation value. How long does $\langle\psi(t')|T_q^{(k)}|\psi(t')\rangle$ take to reach its long-time value? Fourth, the ETH can be used to demonstrate the way classicality, Markovianity and local detailed balance emerge from the pure-state dynamics of a system with commuting charges[65]. Such results might be extended to noncommuting-charge systems, with the non-Abelian ETH and AMC subspaces. Finally, one might store information in a system that thermalizes – forgets its initial conditions – less than usual. This memory-storage opportunity resonates with the work that we explain now.

Stationary-state multiplicity

Consider an open quantum system with a Liouvilian superoperator $\mathcal{L}$. $\rho_{\rm stat}$ is a stationary state of $\mathcal{L}$ if $\mathcal{L}\rho_{\rm stat} = 0$. Consider an arbitrary basis for the stationary subspace. One might associate the jth basis element, $\rho_{\rm stat}^{(j)}$, with the jth letter, L_j, of a classical alphabet: $\rho_{\rm stat}^{(j)} \leftrightarrow L_j$. To encode L_j in the system, one would prepare any state that thermalizes to $\rho_{\rm stat}^{(j)}$. The larger the stationary subspace, the more classical information the system may store. Denote by $n_{\rm stat}$ the dimensionality of the stationary subspace. A lower bound $n_{\rm NC}$ on $n_{\rm stat}$ for a system with noncommuting charges[66] is $n_{\rm NC} = \sum_j \mathcal{D}_j^2$, wherein $\mathcal{D}_j$ denotes the jth irreducible representation of a symmetry group. For example, $n_{\rm NC} \approx N^3$ for a Heisenberg model (as discussed in the section 'Introduction') coupled to an environment that conserves the system-of-interest charges $\sigma_{x,y,z}^{(\mathcal{S})}$. If the charges commute, the lower bound, $n_{\rm C}$, scales as the number of simultaneous eigenspaces shared by all the $Q_a^{(\mathcal{S})}$. Because $n_{\rm NC}$ and $n_{\rm C}$ scale differently, noncommuting charges could alter the dimensionality of the stationary subspace and, thus, the amount of information storable.

Hybrid energy-level statistics

Energy-level statistics diagnose quantum chaos and integrability[67]. Denote by $\omega = E_{j+1} - E_j$ the spacing between consecutive eigenenergies of a many-body Hamiltonian. Any given spacing (near the centre of the spectrum) has a probability density $P(\omega)$ of having the size ω. A Poissonian $P(\omega)$ diagnoses integrability[67]. A Wigner–Dyson distribution, $A_\gamma\,\omega^\gamma \exp(-B_\gamma\omega^2)$, signals chaos[68,69]. The parameter $\gamma \in \{1,2,4\}$ depends on the time-reversal and rotational symmetries of the Hamiltonian. Normalization and the mean ω determine the coefficients A_γ and B_γ. Noncommuting charges defy the Poisson-vs-Wigner–Dyson dichotomy in a way that commuting charges cannot. The charges generate a non-Abelian algebra, which has multidimensional irreducible representations, which force degeneracies on the Hamiltonian. These representations induce statistics that interpolate between the two distributions[64,70,71]. Noh observed such statistics numerically, using a 2D Heisenberg model[64].

Constraints on charge-conserving dynamics

Natural interactions are spatially local, motivating a quantum-computational result: every N-qubit unitary decomposes into gates on pairs of qubits[72–74]. Can charge-conserving local unitaries effect every charge-conserving global unitary U? Perhaps surprisingly, the answer is no[75]. As proven using Lie theory, locality-constrained charge-conserving unitaries cannot even approximate Us. The reason is that the two types of unitaries form Lie groups of different dimensions[75–78]. Moreover, noncommuting charges impose four types of constraints on the implementable global unitaries, whereas commuting charges impose only two[78]. The extra constraints may restrict chaos, which enables thermalization. Hence, this result suggests that noncommuting charges may restrict thermalization.

Decreased thermodynamic-entropy production

Noncommuting charges reduce entropy production, which quantifies irreversibility[3]. Throughout this paragraph and the next, entropy refers to thermodynamic entropy, not entanglement entropy. Let systems $X = A, B$ have charges $Q_a^{(X)}$ that might or might not commute (in no other subsection of this section do $Q_a^{(X)}$s denote possibly commuting charges). Each system begins in a GGE $\rho_{\mu^{(X)}} \propto \exp\left(\sum_a \mu_a^{(X)} Q_a^{(X)}\right)$, wherein $\boldsymbol{\mu}^{(X)} = (\mu_0^{(X)}, \mu_1^{(X)}, \ldots)$ (ref. 79) (Fig. 3). Hence, AB begins in $\rho(0) := \rho_{\mu^{(A)}}^{(A)} \otimes \rho_{\mu^{(B)}}^{(B)}$. Let us specialize to the linear-response regime: $\boldsymbol{\mu}^{(A)} \approx \boldsymbol{\mu}^{(B)}$. A charge-conserving unitary U can shuttle charges between the systems, producing entropy. Noncommuting charges decrease the entropy production[3]. Denote by $\delta\mu_a := \mu_a^{(A)} - \mu_a^{(B)}$ the difference between the a-type chemical potentials of the two systems. The a-type charge system A changes, in the Heisenberg picture, by $U^\dagger Q_a^{(A)} U - Q_a^{(A)}$. (Tensored-on $\mathbb{1}$s are implicit where necessary to make operators act on the appropriate Hilbert spaces.) We combine the foregoing two quantities into $\tilde{\Sigma} := \sum_a \delta\mu_a (U^\dagger Q_a^{(A)} U - Q_a^{(A)})$. Taking the expectation value of $\tilde{\Sigma}$ in the initial state, we obtain the net entropy production, $\Sigma = {\rm Tr}(\tilde{\Sigma}\,\rho(0))$. The initial state lies near the fixed point $\pi := \rho_{\mu^{(A)}}^{(A)} \otimes \rho_{\mu^{(A)}}^{(B)}$ of U, by the linear-response assumption. (In π, unlike in $\rho(0)$, A and B have the same lists of chemical potentials.) π and $\tilde{\Sigma}$ have a Wigner–Yanase–Dyson skew information $I_y(\pi,\tilde{\Sigma}) := -\frac{1}{2}{\rm Tr}([\pi^y,\tilde{\Sigma}][\pi^{1-y},\tilde{\Sigma}])$, parameterized by $y \in (0,1)$. $I_y(\pi,\tilde{\Sigma})$ quantifies the coherence that $\tilde{\Sigma}$ has relative to the eigenbasis of π. The Wigner–Yanase–Dyson skew information contributes to the entropy production: $\Sigma = \frac{1}{2}{\rm var}(\tilde{\Sigma}) - \frac{1}{2}\int_0^1 {\rm d}\,y\, I_y(\pi,\tilde{\Sigma})$ (ref. 3). I_y is always ≥ 0 and is positive if and only if the charges fail to commute. Therefore, noncommuting charges lower Σ. Because entropy production accompanies thermalization, this result indicates that noncommutation may inhibit thermalization.

Two extensions support ref. 3. The first consists of numerical simulations of an optomechanical system interacting with a squeezed thermal bath[80]. The second extension is a progression beyond the linear-response regime[81]. Even there, the noncommutation of charges can decrease entropy production.

How ref. 3 impacts engine efficiencies remains an open question. Efficiencies suffer from waste and so may benefit from the lowering of Σ of noncommuting charges. However, the noncommutation of charges can decrease the amount W_a of a-charge (analogous to work) extractable by an engine[82]. If using finite-size NATS baths, an engine can extract $W_a \lesssim W_a|_{\text{infinite-size bath}} - O(C/\xi)$. C captures the correlations between the charges in ρ_{NATS}; if the charges commute, $C = 0$. ξ, encoding the sizes of baths, diverges in the infinite-bath limit. Extractable work, here lowered by noncommutation of charges, tends to trade off with efficiency. A precise relationship between the two figures of merit, in the presence of noncommuting charges, merits calculation.

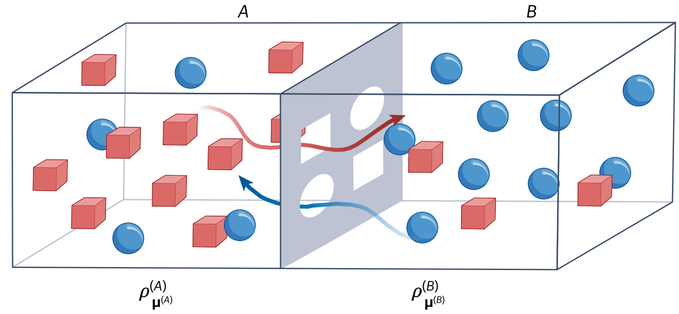

Fig. 3 | Two thermal reservoirs (A and B) exchange charges, producing entropy. Blue spheres represent charges of one type, and red cubes represent charges of another. For each of $X = A, B$, system X has a list $\mu^{(X)}$ of effective chemical potentials. This list parameterizes the NATS $\rho^{(X)}_{\mu^{(X)}}$.

Increased average entanglement entropy

Quantum many-body thermalization entails entanglement, which can be enhanced by the noncommutation of charges[8]. Consider initializing an isolated N-qubit system in a pure state. Divide the system into subsystems A and B with sizes N_A and N_B. A and B can share entanglement quantified by the entanglement entropy, S_{AB}. S_{AB} is the von Neumann entropy of the reduced state ρ_A of A, $S_{\text{vN}}(\rho_A)$. Plotting S_{AB} against N_A yields the Page curve[83].

Page curves for two comparable N-qubit models are reported by ref. 8 (Fig. 4), one with commuting charges only and the other with noncommuting charges. They involve no dynamics, so the observables of interest are not technically charges. However, the modelled states result from chaotic dynamics: Haar-random states were sampled from (approximate, when necessary)[2,7,9] microcanonical subspaces. Page curves were estimated numerically via exact diagonalization and analytically via large-N approximations of combinatorics. The noncommuting-charge Haar-averaged Page curves lay above the commuting-charge analogues. The difference was of $O(N_A^2/(N^2 N_B))$, in the simplest comparison. A possible reason centres on the least entangled basis for the subspace of each model. If the local charges $\widetilde{Q}_a$ commute, they share an eigenbasis. Hence, the global commuting $\widetilde{Q}_a^{\text{tot}}$s share a tensor-product eigenbasis for the subspace. If the local charges fail to commute, this argument breaks, and the least entangled basis of the subspace is entangled. One might, therefore, expect more entanglement of the noncommuting-charge model on average across the subspace.

Other works concern the effects of non-Abelian symmetries on entanglement entropy but focus less on the changes induced by the noncommutation of charges. For example, a non-Abelian symmetry raises the entanglement in Wess–Zumino–Witten models, which are (1 + 1)-dimensional conformal field theories with Lie-group symmetries[84,85]. Second, holographic calculations highlight another correction that non-Abelian symmetries introduce into entanglement entropy[86,87]. (This correction appears to be negative. However, refs. 86,87 concern symmetry-resolved Page curves, in contrast to the conventional Page curves of ref. 8. A symmetry-resolved Page curve models the entanglement, averaged over time, of a system whose charges move only within A and within B, not between the subsystems. Conventional Page curves model less-restricted thermalization.) Third, algebraic quantum field theory calculations agree that non-Abelian symmetries raise Page curves[88].

These works suggest several research opportunities. The increase in entanglement entropy merits checking with more comparable models analogous to those depicted in Fig. 4. Additionally, one might adjust the Page curve calculations following the revelation that local charge-conserving unitaries constrain global Us tightly[75]. Under locality constraints, the Haar distribution may model chaotic dynamics inaccurately. Finally, the entanglement entropy of refs. 8,84–88 differs from the thermodynamic entropy of ref. 3. Even so, the increase of the former by noncommuting charges conceptually conflicts, somewhat, with their decrease of the latter.

Adjacent work in other fields

Noncommuting charges have arisen in other thermodynamics-related contexts, usually under the guise of non-Abelian symmetries. We discuss four contexts. First, integrable systems have more charges, which might fail to commute, than the thermodynamic systems previously discussed. Second, noncommuting charges destabilize a thermalization-avoidant phase of quantum matter: many-body localization (MBL). Third, gauge theories naturally have non-Abelian symmetries, albeit local ones. Fourth, hydrodynamics describes the flow of some non-Abelian charges, including in heavy-ion collisions.

We briefly mention three more connections. Noncommuting charges may impact quantum information scrambling[89]. Also, ρ_{NATS} helps explain phase transitions[90–92]. Finally, the resource theory of asymmetry quantifies the noninvariance of a state under operation by the elements of a symmetry group[19]. The group may be non-Abelian, so the mathematical tools of the theory merit application to noncommuting thermodynamic charges.

Conventional integrable systems

Garden-variety integrable systems fail to thermalize. Each has extensively many non-trivial charges, which constrain the dynamics substantially. By contrast, our thermodynamic set-up (described in the section 'Introduction') entails a number of charges, $c \ll N$, much smaller than the number of DOFs. Integrable and near-integrable systems with noncommuting charges have been studied. Models include the GGE equal to the ρ_{NATS} in equation (4). Several works assess how the accuracy of GGE grows with the number of Q_as included in the state[92–95]. Outside of GGE studies, an integrable system can exhibit anomalous transport – diffusion with a diffusion constant $D \sim \sqrt{N}$ (refs. 96,97). The system studied in ref. 96 is the 1D nearest-neighbour Heisenberg model shown in equation (1). The anomalous transport can be explained with Bethe-ansatz calculations and the z-axis magnetization[98–100]. Can all three noncommuting charges ($\sigma_{x,y,z}$) predict anomalous transport

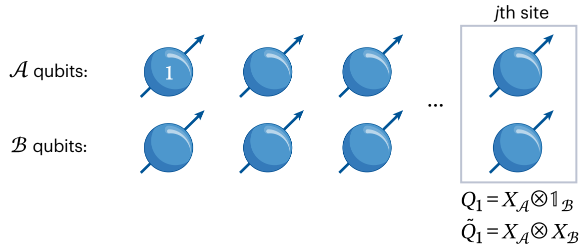

Fig. 4 | Analogous noncommuting-charge and commuting-charge models. Each model consists of N sites, each formed from $\mathcal{A}$ and $\mathcal{B}$ qubits. Local observables distinguish the models. The noncommuting observables are $Q_1 = X_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{B}}$, $Q_2 = Y_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{B}}$ and $Q_3 = Z_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{B}}$. The commuting observables are $\tilde{Q}_1 = X_{\mathcal{A}} \otimes X_{\mathcal{B}}$, $\tilde{Q}_2 = Y_{\mathcal{A}} \otimes Y_{\mathcal{B}}$ and $\tilde{Q}_3 = Z_{\mathcal{A}} \otimes Z_{\mathcal{B}}$. The local charges, summed across the chain, form global observables that define the (approximate) microcanonical subspaces. Figure adapted with permission from ref. 8, APS.

alternatively? Another open question follows from increasingly breaking the integrability of a noncommuting-charge system[101]. How do the behaviours of a system transmute into thermodynamic behaviours described in this Perspective?

Many-body localization

Disordered, interacting quantum systems exhibit MBL. Examples include a qubit chain subject to the disordered Heisenberg Hamiltonian $H_{\text{MBL}}^{\text{tot}} = J\sum_{j=1}^{N}(\boldsymbol{\sigma}^{(j)} \cdot \boldsymbol{\sigma}^{(j+1)} + h_j\sigma_z^{(j)})$ The disorder term, $\sum_{j=1}^{N} h_j\sigma_z^{(j)}$, acts as an external field whose magnitude h_j varies randomly across sites. Denote by h the standard deviation of the disorder term. If the disorder is much stronger than the interaction, $h \gg J$, the system localizes. Imagine measuring the σ_z of each qubit. The qubits approximately maintain the measured configuration long afterward. This behaviour contrasts with how thermalizing systems, such as classical gases, change configurations quickly. Hence, MBL resists thermalization for long times. The reason is that the Hamiltonian decomposes as a linear combination of quasilocal DOFs[26].

Noncommuting charges destabilize MBL[102]. Consider forcing a non-Abelian symmetry on $H_{\text{MBL}}^{\text{tot}}$. The resulting Hamiltonian, $H_{\text{MBL}'}^{\text{tot}}$, will have degeneracies, by Schur's lemma (Supplementary Information). So will the quasilocal DOFs, which can therefore become 'excited' at no energy cost. Consider adding to $H_{\text{MBL}'}^{\text{tot}}$ an infinitesimal field that violates the symmetry. The resulting Hamiltonian, $H_{\text{MBL}''}^{\text{tot}}$, can map $H_{\text{MBL}'}^{\text{tot}}$ eigenstates $|\psi\rangle$ to same-energy eigenstates $|\tilde{\psi}\rangle$: $\langle\tilde{\psi}|\ H_{\text{MBL}''}^{\text{tot}}\ |\psi\rangle \neq 0$. Two such eigenstates can be zero-energy excited states of neighbouring quasilocal DOFs. Hence, $H_{\text{MBL}''}^{\text{tot}}$ can transport zero-energy 'excitations' between quasilocal DOFs – across the system. Such transport is inconsistent with MBL. Therefore, non-Abelian symmetries promote a thermalizing behaviour.

Gauge theories

Classical electrodynamics exemplifies a gauge theory, a model that contains more DOFs than does the physical system it represents[103]. Choosing a gauge eliminates the extra DOFs. The transformations between gauges form a Lie group $\mathcal{G}$. The elements $U \in \mathcal{G}$ preserve the theory's action, $\mathcal{S}$, that is, $U: \mathcal{S} \mapsto \mathcal{S}$. Gauge theories model elementary-particle physics and condensed matter, both of which can have non-Abelian groups $\mathcal{G}$. For example, quantum chromodynamics, which describes the strong force, has SU(3) symmetry. Hence, elementary-particle physics should realize noncommuting-charge thermodynamics naturally; ρ_{NATS} might be observable in high-energy and nuclear systems[104]. Granted, confinement prevents quantum-chromodynamic systems from having non-zero $\langle Q_a^{\text{tot}}\rangle$s. Still, noncommuting charges raise the average entanglement entropy in spin systems with $\langle Q_a^{\text{tot}}\rangle = 0\ \ \forall\, a$[8]. Furthermore, subsystems j can contain positive and negative charges, $\langle Q_a^{(j)}\rangle \neq 0$, that can undergo dynamics[28]. Gauge symmetries are local, though, unlike the global symmetries covered in this Perspective. The contrast raises the question: how much noncommuting-charge quantum thermodynamics ports over into gauge theories?

Hydrodynamics and heavy-ion collisions

Hydrodynamics models long-wavelength properties of fluids that are in equilibrium locally[105]. The theory describes condensed matter and certain stages of heavy-ion collisions[106], among others. Noncommuting charges can flow similarly to the more-often studied energy and particles[107]. A few effects of the noncommutation of charges have been isolated; examples include non-Abelian contributions to conductivity[108] and entropy currents[109]. In addition, non-Abelian symmetries can shorten charge-neutralization times in heavy-ion collisions[110]. More such effects may be discoverable. Also, quantum thermodynamics might assist with long-standing questions about heavy-ion collisions: By what process does the system thermalize? How should the initial state be modelled? Does the (non-)Abelian ETH explain the thermalization? And why is the thermalization time so short[111]?

Outlook

The discoveries discussed in this Perspective suggest rich research opportunities, of which we detail five.

First, the predictions merit experimental testing. The first test of noncommuting-charge thermodynamics was performed with trapped ions[9]. Other potential platforms include superconducting qubits, quantum dots, ultracold atoms, quantum optics and optomechanics[3,7,28].

Second, existing results present a conceptual puzzle. Evidence suggests that noncommuting charges hinder thermalization to an extent: they invalidate derivations of the form of a thermal state[1,2], decrease thermodynamic-entropy production[3], clash with the ETH[24] and uniquely restrict the global unitaries implementable via local interactions[78]. However, other evidence suggests that noncommuting charges enhance thermalization: they destabilize MBL[102] and increase average entanglement entropy[8]. These results do not conflict with each other technically, as they stem from different set-ups. Yet the results clash conceptually, suggesting that the noncommutation of charges hinders thermalization in some ways and enhances it in others. Does the hindrance or enhancement win out overall? Reconciling these results presents a challenge.

Third, to what extent can classical mechanics reproduce noncommuting-charge thermodynamics? The noncommutation of observables underlies quintessentially quantum phenomena including uncertainty relations, measurement disturbance and the Einstein–Podolsky–Rosen paradox. Yet classical mechanics features quantities that fail to commute with each other — for example, rotations about different axes. How non-classical is noncommuting-charge thermodynamics (achievable only outside of classical physics), beyond being merely quantum (achievable within quantum physics)?

Fourth, every chaotic or thermodynamic phenomenon merits re-examination. To what extent does it change under dynamics that conserve noncommuting charges? Example phenomena include diffusion coefficients, transport relations, thermalization times, monitored circuits[112], out-of-time-ordered correlators[113], operator spreading[114], frame potentials[115] and quantum-complexity growth[116].

Finally, noncommuting-charge thermodynamics merits bridging to similar topics in neighbouring fields. Non-Abelian gauge theories, non-Abelian hydrodynamics, GGE studies and dynamical phase transitions overlap with noncommuting thermodynamic charges. To what extent can these areas inform each other? Do gauge theories realize noncommuting thermodynamic charges naturally?

For decades, conserved thermodynamic quantities were assumed implicitly to commute with each other. Noncommutation, however, is a trademark of quantum theory. The identification and elimination of the assumption, though initiated where quantum information theory meets quantum thermodynamics, have potential ramifications across quantum many-body physics.

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