Hierarchy of multipartite correlations based on concentratable entanglement

Louis Schatzki [*]
Department of Electrical and Computer Engineering, Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801, USA;
Illinois Quantum Information Science and Technology (IQUIST) Center, University of Illinois Urbana-Champaign, Urbana, Illinois 61801, USA;
and Information Sciences, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA

Guangkuo Liu
JILA, University of Colorado/NIST, Boulder, Colorado 80309, USA
and Department of Physics, University of Colorado, Boulder CO 80309, USA

M. Cerezo
Information Sciences, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA
and Quantum Science Center, Oak Ridge, Tennessee 37931, USA

Eric Chitambar [†]
Department of Electrical and Computer Engineering, Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801, USA
and Illinois Quantum Information Science and Technology (IQUIST) Center, University of Illinois Urbana-Champaign, Urbana, Illinois 61801, USA

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Multipartite entanglement is one of the hallmarks of quantum mechanics and is central to quantum information processing. In this work we show that concentratable entanglement (CE), an operationally motivated entanglement measure, induces a hierarchy upon pure states from which different entanglement structures can be experimentally certified. In particular, we find that nearly all genuine multipartite entangled states can be verified through the CE. Interestingly, GHZ states prove to be far from maximally entangled according to this measure. Instead we find the exact maximal value and corresponding states for up to 18 qubits and show that these correspond to extremal quantum error correcting codes. The latter allows us to unravel a deep connection between CE and coding theory. Finally, our results also offer an alternative proof, on up to 31 qubits, that absolutely maximally entangled states do not exist.

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I. INTRODUCTION

Entanglement is one of the defining properties of quantum mechanics [1], and serves as a fundamental resource for quantum information processing, from cryptography to computation and quantum sensing [2–11]. As such, characterizing the entanglement in a state is fundamentally connected to its utility for information processing.

Bipartite pure states are said to be entangled if they cannot be written as a tensor product $|\psi_A\rangle \otimes |\psi_B\rangle$ [12]. Multipartite systems are more complex, as entanglement can arise within subsystems but not necessarily across the entire state. For example, the four-party biseparable state $|\psi_{AB}\rangle \otimes |\psi_{CD}\rangle$ lacks any entanglement across the partition $AB|CD$. States for which no biseparable partition can be drawn are said to have genuine multipartite entanglement (GME) [13,14], and a significant amount of work has been put forward toward quantifying and characterizing GME [7,15–25]. Yet, even among the collection of non-GME states there is a great deal of complexity, and a non-GME state $|\psi\rangle$ can be classified according to the number of separable cuts it possesses, as well as the number of entangled systems within each cut.

With the rapid growth of quantum technologies [26–28] and hardware capable of generating multipartite entangled states, there is a natural demand for methods to decide whether a given n-partite state is GME, and if it is not, for how to determine its product structure. Ideally, one would like to answer these questions using some entanglement measure that is

[*]louisms2@illinois.edu
[†]echitamb@illinois.edu

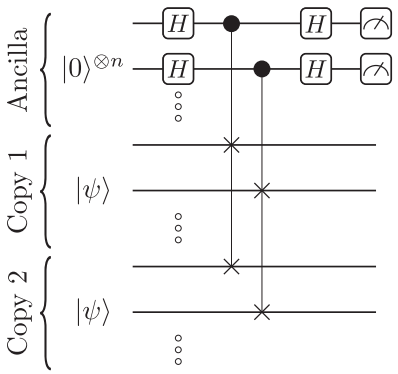

FIG. 1. Parallelized controlled SWAP test for measuring CE. After preparing two copies of the state of interest, one performs a cSWAP test on each triplet of qubits from the ancilla and the two copies of $|\psi\rangle$. The resulting probabilities of bitstrings on the ancilla registers yield the CE.

accessible through simple protocols. In this work we show that the concentratable entanglement can be used for identifying entanglement structures as well as other interesting features of multipartite entanglement.

A. Concentratable entanglement

The concentratable entanglements (CE) are a family of entanglement measures introduced in Ref. [29]. For an n-qubit pure state $|\psi\rangle$ and subset $S \subseteq [n]$ of the qubits, the corresponding CE is given by

$$\mathcal{C}_S(|\psi\rangle) = 1 - \frac{1}{2^{|s|}} \sum_{\alpha \in 2^S} \mathrm{Tr}\big[\psi_\alpha^2\big], \tag{1}$$

where α runs over all sets in the power set 2^S and ψ_α is the reduced density matrix of $|\psi\rangle$ on the parties in α. The CE captures entanglement in the system by averaging the reduced state purities across all partitions localized to some subset of the qubits. Note that $\emptyset \in 2^S$, and we take $\mathrm{Tr}[\psi_\emptyset^2]$ to be one by definition. With this convention, operationally, CE is the probability that at least one SWAP test should fail when n of them are applied in parallel across two copies of $|\psi\rangle$ (see Fig. 1). Let $\mathbf{z} \in \mathbb{F}_2^n$ be a bitstring with one corresponding to a swap test failing. Then, as noted in Ref. [29], we can write CE as

$$\mathcal{C}_S(|\psi\rangle) = 1 - \sum_{\mathbf{z} \in \mathcal{Z}(S)} p(\mathbf{z}), \tag{2}$$

where $\mathcal{Z}(S) = \{\mathbf{z} \in \mathbb{F}_2^n \mid z_i = 0,\ \forall i \in S\}$.

Since a failed SWAP test leads to the generation of a Bell pair, the CE also quantifies how well entanglement can be concentrated using two copies of the state and SWAP tests [29]. The controlled SWAP (cSWAP) used to measure CE is a basic building block in quantum communication protocols [30], and it has been recognized as an experimentally accessible tool for measuring and witnessing entanglement [29,31,32].

B. Overview of results

In this work we take a closer look at CE and how it relates to various entanglement properties of quantum states. Our starting motivation is if CE tells us anything about the entanglement between subsystems of a given state. To that end, we upper bound CE when considering n-qubit states having a fixed product state structure. These maximal values naturally induce a hierarchy on the set of all pure states, separating classes of states with different numbers of separable cuts or different numbers of qubits within each cut (for example, see Table II). Hence, using the CE measured for a given state, one can certify that entanglement must exist between at least a certain number of parties.

The rest of this paper is devoted to developing this hierarchy and exploring applications. Our first significant technical challenge is to compute the maximal value of CE among all n-qubit states. While we do not have a general solution, we provide a linear programming upper bound that is tight for at least up to 18 qubits (except $n \in \{7, 13, 15, 16\}$ where we know of states coming within a few decimal places of the bound). Interestingly, this linear program closely matches those arising in the study of quantum error correcting codes, which we later exploit to unravel deep connections between CE and coding theory. From the Haar statistics of the CE, we then show that most states have near maximal CE and our hierarchy is tight enough to certify GME in nearly all such states. Finally, we provide rigorous connections between CE and cSWAP to other entanglement measures in the Supplemental Material (SM). All results and discussion are presented below, with detailed proofs delayed to the SM [33].

II. PRELIMINARIES

We will adopt the following conventions. Unless stated otherwise, we work with pure states on n qubits, i.e., $|\psi\rangle$ denotes a unit vector in $\mathbb{C}_2^{\otimes n}$. As previously defined, $\mathcal{C}_S(|\psi\rangle) = 1 - \frac{1}{2^{|s|}} \sum_{\alpha \in 2^S} \mathrm{Tr}[\psi_\alpha^2]$, where ψ_α indicates the reduced density matrix on parties contained in the subset α. Of primary interest is the case where $S = [n]$ as this captures entanglement across all parties. We use the shorthand $\mathcal{C}(|\psi\rangle)$ for $S = [n]$. We write $\mathcal{C}_S^*(n)$ to denote the maximal value of CE on n qubits, i.e.,

$$\mathcal{C}_S^*(n) = \max_{|\psi\rangle \in \mathbb{C}_2^{\otimes n}} \mathcal{C}_S(|\psi\rangle). \tag{3}$$

For $S = [n]$ we will simply write $\mathcal{C}^*(n)$.

The swap operator will be denoted by $\mathbb{F}$. The n-dimensional boolean finite field will be denoted by $\mathbb{F}_2^n$. Partial transposition is indicated by X^{Γ_B} (here with respect to system B).

III. HIERARCHIES OF MULTIPARTITE STATES

A. Concentratable entanglement of product states

In this section we demonstrate that CE induces hierarchies upon pure states from which we can deduce product structures. This begins with the following observation:

Proposition 1. For any biseparable state $|\psi\rangle = |\psi_A\rangle|\psi_B\rangle$,

$$\begin{aligned}\mathcal{C}_S(|\psi\rangle) = &\mathcal{C}_{S\cap A}(|\psi_A\rangle) + \mathcal{C}_{S\cap B}(|\psi_B\rangle) \\ &- \mathcal{C}_{S\cap A}(|\psi_A\rangle)\mathcal{C}_{S\cap B}(|\psi_B\rangle).\end{aligned} \tag{4}$$

TABLE I. Maximal values of CE and thresholds for detecting GME as a function of system size. All $\mathcal{C}^*(n)$ are achievable except for $n = 7$. In this case, values of up to 0.7739 have been found numerically. $\zeta(n)$ is the threshold value above which the state must be GME. The third column is the upper bound from assuming that marginals are maximally mixed, i.e., that the state is AME.

Number of Qubits	$\mathcal{C}^*(n)$	$\zeta(n)$	AME Bound
2	0.25	0	0.25
3	0.375	0.25	0.375
4	0.5	0.4375	0.53125
5	0.625	0.53125	0.625
6	0.71875	0.625	0.71875
7	0.779296875*	0.71875	0.779296875
8	0.828125	0.7890625	0.83447265625
9	0.8671875	0.83447265625	0.87158203125
10	0.8984375	0.87109375	0.9036865234375
11	0.923828125	0.900390625	0.92584228515625
12	0.94287109375	0.923828125	0.9443817138671875

For the case of $S = [n]$, we can rewrite this as

$$\mathcal{C}(|\psi\rangle) = \mathcal{C}(|\psi_A\rangle) + \mathcal{C}(|\psi_B\rangle) - \mathcal{C}(|\psi_A\rangle)\mathcal{C}(|\psi_B\rangle). \quad (5)$$

An immediate corollary is as follows.

Corollary 1. For k copies of a state, we have that

$$\mathcal{C}(|\psi\rangle^{\otimes k}) = 1 - (1 - \mathcal{C}(|\psi\rangle))^k. \quad (6)$$

Note that this implies that CE goes to one with $k \to \infty$ if it is nonzero for a single copy.

From Proposition 1 it follows that if $|A| = k$ and $|B| = n - k$, then

$$\mathcal{C}_S(|\psi_A\rangle|\psi_B\rangle) \leqslant \mathcal{C}^*_{S\cap A}(k) + \mathcal{C}^*_{S\cap B}(n-k) - \mathcal{C}^*_{S\cap A}(k)\mathcal{C}^*_{S\cap B}(n-k). \quad (7)$$

Consequently, if a state has large enough CE, we know that it cannot be written as a product of pure states on k and n-k qubits. By proceeding iteratively, one can obtain inequalities similar to that in Eq. (7) for more separable cuts. We now focus on the case where $S = [n]$ for reasons we will return to shortly. For any product state structure, one thus obtains a bound ζ^* on CE, and if $\mathcal{C}(|\psi\rangle) > \zeta^*$, then $|\psi\rangle$ defies any of the product structures bounded by ζ^* (e.g., see Table II). Furthermore, by finding the largest CE possible across all bipartations, we obtain a threshold above which GME is certified,

$$\zeta(n) = \max_{1\leqslant k<n} \mathcal{C}^*(k) + \mathcal{C}^*(n-k) - \mathcal{C}^*(k)\mathcal{C}^*(n-k). \quad (8)$$

To find ζ^* and $\zeta(n)$ we must know $\mathcal{C}^*(n)$ for arbitrary n. Before discussing the optimization problem of Eq. (3), we observe that a simple upper bound on $\mathcal{C}^*(n)$ follows from assuming that all reduced density matrices are maximally mixed. A bound like this was considered for a similar task of verifying GME in qudit states [34], and numerics from a recent work suggest that graph states come close to saturating this upper bound [35]. Pure states for which all bipartitions have maximally mixed marginals are called absolutely maximally entangled (AME), and exist only for two, three, five, and six qubits [36–38]. Hence, this bound will necessarily be loose for any other system size.

TABLE II. Hierarchy for five qubit states based on CE. Notations such as $2 \otimes 2 \otimes 1$ denote the class of states having the form $|\psi_{AB}\rangle|\psi_{CD}\rangle|\psi_E\rangle$ for any labeling of parties. A state cannot have a product state structure according to any partitioning lying lower in the hierarchy than its CE.

Structure	Max CE ζ^*
5	0.625
$3 \otimes 2$	0.53125
$4 \otimes 1$	0.5
$2 \otimes 2 \otimes 1$	0.4375
$3 \otimes 1 \otimes 1$	0.375
$2 \otimes 1 \otimes 1 \otimes 1$	0.25
$1 \otimes 1 \otimes 1 \otimes 1 \otimes 1$	0

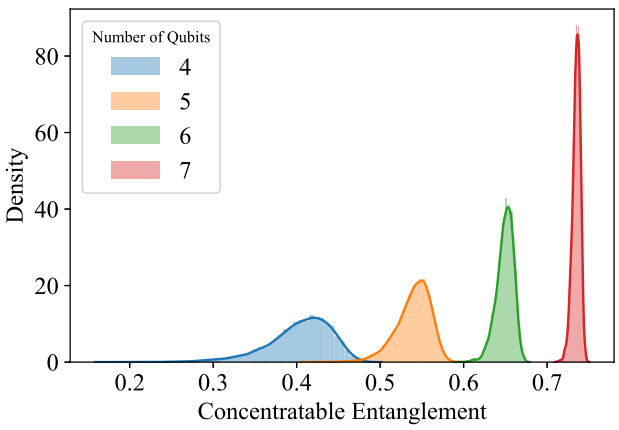

FIG. 2. Haar distribution of concentratable entanglement for four to seven qubits. As expected from Eq. (14), the mean grows with the system size while the distribution grows tighter. Data comes from 6000 random samples for each system size.

B. Upper bounding concentratable entanglement

To obtain better bounds on $\mathcal{C}^*(n)$, we first note that Eq. (3) can be equivalently expressed as

$$\mathcal{C}^*(n) = 1 - \min_{|\psi\rangle \in \mathbb{C}_2^{\otimes n}} \mathrm{Tr}[M|\psi\rangle\langle\psi|^{\otimes 2}], \tag{9}$$

where $M = \otimes_{i=1}^n \Pi_+^{(i)}$, and with $\Pi_+^{(i)} = \frac{1}{2}(\mathbb{1}^{(i)} + \mathbb{F}^{(i)})$ the projector onto the symmetric subspace for the ith qubit and its copy. While this is not a convex optimization problem, one can relax to the convex set of states with a positive partial transpose (PPT). Further, since $\mathbb{F}|\psi\rangle\langle\psi|^{\otimes 2} = |\psi\rangle\langle\psi|^{\otimes 2}\mathbb{F} = |\psi\rangle\langle\psi|^{\otimes 2}$ we enforce left and right swap invariance. Explicitly, we consider the relaxation

$$\begin{aligned} 1 - \min_{X \in \mathbb{C}^{2^{2n}\times 2^{2n}}} \ & \mathrm{Tr}[MX] \\ \text{subject to} \ \ & (i)\ \mathbb{F}X = X, \\ & (ii)\ X\mathbb{F} = X, \\ & (iii)\ X \geqslant 0, \\ & (iv)\ \mathrm{Tr}[X] = 1, \\ & (v)\ X^\dagger = X, \\ & (vi)\ X^{\Gamma_B} \geqslant 0. \end{aligned} \tag{10}$$

This results in a semidefinite program on $O(4^n)$ variables, far too many to readily evaluate for more than a few qubits. To obtain an efficient optimization problem, we exploit symmetry. In particular, note that M is invariant under twirling with respect to both the unitary group and the symmetry group S_n. That is,

$$M = \int U^{\otimes 2} M (U^{\otimes 2})^\dagger dU, \quad M = \frac{1}{n!}\sum_{\sigma \in S_n} V_\sigma M V_\sigma, \tag{11}$$

where V_σ is the defining representation on n qubits [39]. As discussed in the SM, using these symmetries we can reduce the PPT relaxation to the following linear program:

$$\begin{aligned} 1 - \min_{\mathbf{y} \in \mathbb{R}^{\lfloor \frac{n}{2} \rfloor + 1}} \ & 3^n y_0 \\ \text{subject to} \ \ & (i)\ \mathbf{y} \geqslant 0, \\ & (ii)\ K\mathbf{y} \geqslant 0, \\ & (iii)\ \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} y_w 3^{n-w} = 1, \end{aligned} \tag{12}$$

where $\mathbf{y} = (y_0, y_1, \cdots, y_{\lfloor \frac{n}{2} \rfloor})^T$ and K is a $(n+1) \times (\lfloor \frac{n}{2} \rfloor + 1)$ matrix whose elements are $(K)_{ij} = K_i(2j)$, the quaternary Krawtchouk polynomials [40]. The Krawtchouk polynomials play an important role in classical coding theory, and we return to this curious connection later in the paper.

While this just provides an upper bound on $\mathcal{C}^*(n)$, we have found states, corresponding to extremal quantum error correcting codes [37,41], for up to $n = 18$ qubits achieving this upper bound, except for $n \in \{7, 13, 15, 16\}$. We suspect that the results of our LP yield the exact maximum of $\mathcal{C}^*(n)$ for other system sizes as well.

Table I presents our results for up to 12 qubits (see the SM for up to $n = 31$). For $n \notin \{2, 3, 5, 6, 7\}$, the value $\mathcal{C}^*(n)$ lies below the CE of an AME state, thereby providing an alternative proof for the nonexistence of AME states on systems with these numbers of qubits. Table II provides an example of our CE-based hierarchy for $n = 5$. Here, there are seven classes of product state structures, and we compute exact values for the maximal CE obtainable within each class. Similar tables for up to $n = 12$ can be found in the SM.

IV. RANDOM STATES

Knowing the maximal possible values of CE, it is natural to ask how CE is distributed for random states and whether most GME states achieve $\mathcal{C}(|\psi\rangle) > \zeta(n)$. If the latter inequality holds with high probability, then the CE provides a good entanglement measure for certifying GME. In fact, we find that the average CE quickly goes to one and that most states lie above the GME threshold. Specifically, under the Haar measure, CE is distributed as follows.

Proposition 2. The Haar average of concentratable entanglement is

$$\langle \mathcal{C}_S \rangle_{\mathrm{Haar}} = 1 - \frac{3^{|S|}(2^{n-|S|}+1)}{2^{|S|}(2^n+1)}, \tag{13}$$

$$\langle \mathcal{C} \rangle_{\mathrm{Haar}} = 1 - 2\frac{3^n}{4^n + 2^n}, \tag{14}$$

with variance $\mathrm{Var}(\mathcal{C})_{\mathrm{Haar}} = O((\frac{3}{16})^n)$. For $S = [n]$, the Haar distribution is illustrated in Fig. 2.

We give an exact expression for the Haar variance in the SM. In the SM we also establish the bound $\mathcal{C}^*(n) \leqslant 1 - (\frac{3}{4})^n$ (while compiling this manuscript, the same upper bound was shown in [35]). Combined with the Haar statistics above, this proves the following corollary.

Corollary 2. For $S = [n]$, the maximal value of concentratable entanglement on n qubits goes to *one* like

$$1 - \mathcal{C}^*(n) = \Theta\left(\left(\frac{3}{4}\right)^n\right). \tag{15}$$

Interestingly, for as few as five qubits the majority of states have CE greater than that of the GHZ state as $\mathcal{C}(|\mathrm{GHZ}\rangle) = \frac{1}{2} - \frac{1}{2^n}$. As non-GME states measure zero under the Haar measure [42], an upper bound on the probability that a GME state cannot be verified via its CE can be easily found by applying Chebyshev's inequality. This bound seems to go to zero exponentially quickly, as is illustrated in Fig. S2 of the SM (along with other scaling properties of CE). Even tighter concentration bounds can be obtained via Levy's lemma and the following proposition.

Proposition 3. For any pure states $|\psi\rangle$ and $|\phi\rangle$, we have that the concentratable entanglements satisfy

$$|\mathcal{C}_S(|\psi\rangle) - \mathcal{C}_S(|\phi\rangle)| \leqslant \min\left\{\|\psi_S - \phi_S\|_1, \frac{\sqrt{2}}{2}\|\psi - \phi\|_1\right\}. \tag{16}$$

V. SWAP TEST ENTANGLEMENT

A. Distortion free entanglement

As noted in [29], CE quantifies the probability of creating at least one Bell pair via the cSWAP procedure. This offers an interpretation of CE/cSWAP as a form of universal distortion-free entanglement concentration [43]. Thus, it is interesting to ask how well a given state performs at concentrating Bell pairs using the cSWAP method. While CE corresponds to the probability of creating any Bell pairs whatsoever, it may be informative to consider other performance metrics such as the expected number of Bell pairs generated. We show that this value is precisely the average linear entropy, $1-\frac{1}{n}\sum_i \mathrm{Tr}[\rho_i^2]$, another entanglement measure.

Proposition 4. Given an n-qubit state $|\psi\rangle$, the expected number of Bell pairs from running the parallelized SWAP test on $|\psi\rangle^{\otimes 2}$ is

$$\mathcal{B}(|\psi\rangle)=\frac{1}{2}\left(n-\sum_i \mathrm{Tr}\left[\rho_i^2\right]\right), \tag{17}$$

with variance

$$var(\mathcal{B}(|\psi\rangle))=\frac{n}{4}-\frac{1}{4}\left(\sum_i \mathrm{Tr}\left[\rho_i^2\right]\right)^2+\frac{1}{2}\sum_{i\neq j}\mathrm{Tr}\left[\rho_{ij}^2\right].$$

Recall that a state is called k uniform if all reduced density matrices consisting of k or fewer parties are maximally mixed. We thus have the following.

Corollary 3. The maximum number of expected Bell pairs using the parallelized SWAP test is $\frac{n}{4}$, and it is achieved by any one-uniform state. Further, any two-uniform state achieves the maximal expected number while minimizing the variance.

While writing this manuscript it was brought to our attention that the expected value formula was independently derived in [44] but without an explicit proof. For completion we give a detailed proof in the SM.

B. Entanglement verification from swap tests

In practice, a sort of hybrid method may be the best approach to verifying GME. While determining $\mathcal{C}(|\psi\rangle)$ one measures a set of bitstrings $\{\mathbf{z}_i\}$.

Proposition 5. Given a measurement of a bitstring $\mathbf{z}$ in the cSWAP test, the state cannot be a product with respect to any partition such that the Hamming weight of the substring of $\mathbf{z}$ restricted to any set in the partition is odd.

Corollary 4. A measurement of bitstring $\mathbf{z}\neq\mathbf{0}$ implies that $|\psi\rangle$ cannot be a product state with respect to half of all possible bipartitions. Further, measuring k linearly independent bitstrings implies that a state could only be biseparable with respect to 2^{n-1-k} bipartitions.

For example, say one measures the bitstring 1100. Then it is not possible for the state to be biseparable with respect to the partitions: $\{1\}:\{2,3,4\}$, $\{2\}:\{1,3,4\}$, $\{1,3\}:\{2,4\}$, or $\{1,4\}:\{2,3\}$. While verifying GME may require exponential shots, just a handful of bitstrings removes most possibilities. In practice one could do the following: run cSWAP to determine $\mathcal{C}(|\psi\rangle)$ up to some resolution, in the process measuring a set of bitstrings, and then proceed to verify entanglement between the remaining possible 2^{n-1-k} bipartitions.

VI. CONNECTIONS TO CODING THEORY

While not obvious at first, our work hints at deep connections between CE and quantum coding theory. Previous works have related error correction to similar entanglement measures [37] and AME states [45,46]. In this work we identify connections involving the states achieving $\mathcal{C}^*(n)$ and the LP of Eq. (12). In particular, we show that Eq. (12) can be used to rule out the existence of codes with certain properties. We also show that the expected number of Bell pairs $\mathcal{B}(|\psi\rangle)$ can be cast in this framework as well and recover bounds on codes from this expression.

To briefly review (see [12,47,48] for more details), a quantum error correcting code is a subspace of an n-qubit Hilbert space with corresponding projector P_Q. To encode information we apply a map from a k-qubit space into this subspace. We say that the weight of an error operator E is the number of parties it acts nontrivially on. For example, $X\mathbb{1}Z$ has weight two. A code has distance d if all errors of weight less than d are detectable.

Among all codes, stabilizer codes [12,47] are ubiquitous. Here the codespace is the joint eigenspace of an abelian subgroup of the Pauli group. If this subgroup has n-k generators, the codespace is of dimension 2^k and we denote the code by $[[n,k,d]]$. These have a natural correspondence with codes over GF(4) [49] and can be represented as strings on GF(4). A code is said to be self-dual if it is equal to its orthogonal complement [over GF(4)]. Note that $[[n,0,d]]$ codes (pure states) are self-dual. Self-dual stabilizer codes are type II if all codewords have even weight, and type I otherwise. Type-dependent bounds on distance can be found in [37,41]. A code achieving these bounds is said to be extremal. We list these bounds below for clarity:

$$d\leqslant\begin{cases}2\lfloor\frac{n}{6}\rfloor+1 & \text{type I, } n\%6=0\\ 2\lfloor\frac{n}{6}\rfloor+3 & \text{type I, } n\%6=5\\ 2\lfloor\frac{n}{6}\rfloor+2 & \text{type I, } n\%6\notin\{0,5\}\\ 2\lfloor\frac{n}{6}\rfloor+2 & \text{type II}\end{cases}. \tag{18}$$

Based on numerical searching and constructing solutions to the LP in Eq. (12), we found that all states maximizing CE correspond to extremal codes (see the SM). We thus conjecture that if a $[[n,0,d]]$ stabilizer code achieves the maximal concentratable entanglement on n qubits it must be an extremal code. Note that the converse direction does not hold. For example, the ten qubit extremal code in Table 1 of [37] does not achieve $\mathcal{C}^*(n)$.

We now consider another property of stabilizer codes known as enumerators. Classically, these arise in linear programming bounds on codes [40]. In analogy to the weights of classical codes, quantum codes can be described through their Shor-Laflamme enumerators [50]:

$$A_i=\frac{1}{k^2}\sum_{\sigma:w(\sigma)=i}\mathrm{Tr}[\sigma P_Q]\mathrm{Tr}[\sigma^\dagger P_Q], \tag{19}$$

$$B_i=\frac{1}{k}\sum_{\sigma:w(\sigma)=i}\mathrm{Tr}[\sigma P_Q\sigma^\dagger P_Q], \tag{20}$$

where σ is in the Pauli group and $w(\sigma)$ is the corresponding weight. The quantum MacWilliams identity [50]

connects these: $B_i = 2^{k-n}\sum_j K_i(j)A_j$, where $K_i(j)$ are again the quaternary Krawtchouk polynomials. This is identical to a constraint in the LP for maximizing CE [see Eq. (12)]. As any enumerator $\{A_i\}$ is in the feasible set (up to multiplicative factors), this LP also yields a bound on quantum codes. Denoting the solution to the LP in Eq. (12) by $L(n)$ we find the following bounds.

Proposition 6. A stabilizer code such that $A_{2i+1} = 0$ must satisfy the inequality

$$B_n \leqslant \frac{1}{L(n)}\frac{3^n}{2^{n-k}}. \tag{21}$$

Note that when the LP yields an achievable value of CE, this can be rewritten as $B_n \leqslant \frac{1}{1-\mathcal{C}^*(n)}\frac{3^n}{2^{n-k}}$.

Now consider the problem of finding the maximal value of the expected number of Bell pairs. While in Corollary 3 we found this value to be $\frac{n}{4}$, we could, in a similar manner to CE, solve this problem via linear programming. This yields another bound on codes.

Proposition 7. A stabilizer code such that $A_{2i+1} = 0$ must satisfy the inequality

$$\sum_{i=0}^{n} i\cdot 3^{-i}\cdot A_i \leqslant \frac{n}{4}\frac{2^{n-k}}{3^n}B_n. \tag{22}$$

If the code is self-dual then $A_i = B_i$. Thus, we recover the fact that type II codes do not exist for odd n. Combining these bounds yields $\sum_{i=0}^{n} i\cdot 3^{-i}\cdot A_i \leqslant \frac{n}{4L(n)}$. Both of these bounds are special cases of more general inequalities holding for any stabilizer code, which can be found in the SM. These bounds could be used to prove that certain error correcting codes do not exist. We leave doing so as a problem for future work.

VII. DISCUSSION

In this work we showed that the CE and the parallelized SWAP test can be used to determine various multipartite entanglement properties of pure states. Particularly, CE induces a hierarchy upon pure states, separating genuine multipartite entangled states from biseparable. For even modest system sizes, most states fall neatly into these hierarchies and thus CE can verify GME for most states.

While our analysis has focused on pure states, the hierarchy has some robustness for certifying different entangled structures for mixed states. The key relation needed is a generalization of Proposition 1. As shown in the SM, every n-qubit product density matrix $\rho = \rho^A \otimes \rho^B$ with $|A| = k$ and $|B| = n-k$ satisfies the bound

$$\mathcal{C}(\rho) \leqslant \mathcal{C}^*(k) + \mathcal{C}^*(n-k) - \mathcal{C}^*(k)\mathcal{C}^*(n-k) + 2\sqrt{S_L(\rho)},$$

where $S_L(\rho) = 1 - \mathrm{Tr}[\rho^2]$ is the linear entropy of ρ. Consequently, entanglement structures can still be verified using the value ζ^*, provided ρ has sufficiently high purity.

We end with two open problems. First, it appears that states saturating CE are extremal codes. Can this be formalized? Lastly, numerics indicated that the maximal CE for a biseparable state is always achieved by taking one subsystem to be two qubits and the other n-2. We leave proving this as an open problem.

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