Maximal qubit violations of n-locality in star and chain networks

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The nonlocal correlations of noisy quantum systems are important for both understanding nature and developing quantum technology. We consider the correlations of star and chain quantum networks in which noisy entanglement sources are measured by nonsignaling parties. We derive the necessary and sufficient conditions for when a broad family of star and chain n-locality inequalities achieve their maximal quantum violation assuming that each party's dichotomic observables are separable across qudit systems. When pairs of local dichotomic observables are considered on qubit systems, we derive maximal n-local violations that are larger and more robust to noise than the maximal n-locality violations reported previously. To obtain these larger values, we consider observables that we have not found in previous studies. Thus, we gain insights into self-testing entanglement sources and measurements in star and chain networks.

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I. INTRODUCTION

When two or more nonsignaling parties share an entangled quantum state, the entanglement can be used to generate nonclassical correlations that defy local realism [1–4]. This phenomenon, often referred to as *Bell nonlocality*, provides operational advantages in information security [5–12], multipartite information processing [13–17], and testing quantum devices [18–25]. In quantum networks, many entangled states link parties into a complex topology [26–28]. Consequently, a wide range of theoretical frameworks have been developed to characterize the nonlocal network correlations and their operational advantages [25,29–36].

In particular, we consider the framework of non-n-locality in which a network consists of n independent sources that correlate a collection of nonsignaling parties. Note that the local ($n = 1$) case corresponds to the standard setting of Bell nonlocality in which all parties share a single source [see Fig. 1(a)]. Therefore, non-n-locality is a natural extension of Bell nonlocality. In this framework, a network is characterized by its correlations where classical sources emit shared randomness and quantum sources emit entanglement. Classical network correlations are bound by n-locality inequalities, however, quantum network correlations can violate these inequalities to demonstrate non-n-locality [35]. Quantum violations of n-locality are known for many network topologies including stars, chains, rings, and trees [37–53]. Moreover, non-n-local quantum correlations have been demonstrated experimentally [54–60].

We focus on star and chain network topologies [see Figs. 1(b) and 1(c)]. In quantum networks, these topologies have important applications in entanglement swapping [61,62] and long-distance quantum communication [63–65]. A broad family of n-locality inequalities are known to bound the correlations of star and chain classical networks [40,41,45,53]. We study the maximal quantum violations of these n-locality inequalities in the presence of noise.

Since quantum network hardware is noisy, it is crucial to understand how non-n-local correlations deteriorate in the presence of noise. The noise robustness of non-n-locality is typically investigated with respect to a particular noise model [39–41,66–68]. However, a more general approach gives the maximal n-local violation for any ensemble of mixed states prepared by the sources. For instance, in the local $n = 1$ case, the maximal violation of the Clauser-Horne-Shimony-Holt (CHSH) inequality [69] is known for any two-qubit mixed state [70]. Likewise, in star and chain networks, maximal qubit violations of n-locality have been derived when measuring separable multiqubit observables obtained by coarse-graining GHZ measurements [71–73]. However, the resulting n-locality violations are not globally maximal. Recently, numerical variational optimization methods have shown that larger qubit violations of n-locality can be achieved using qubit separable measurements [67].

Our goal is to formalize the discrepancy between the "maximal" violations using local qubit observables derived in Refs. [71–73] and the larger violations observed in Ref. [67]. Notably, we find that Ref. [45] misses an important condition for when a star network achieves its maximal n-locality violation when multiqudit observables are separable across qudits. For a family of n-locality inequalities, we correct this error by deriving the necessary and sufficient conditions for achieving the maximal violation of n-locality. Moreover, when parties have binary inputs and outputs, we prove that the upper bound on the n-locality violation can be achieved for all ensembles of two-qubit mixed states, showing that quantum non-n-locality has a greater noise robustness than previously known. Our main observation is that the maximal violation requires the *external* parties in a star network to measure observables in mutually unbiased bases, e.g., X and Z, while previous works

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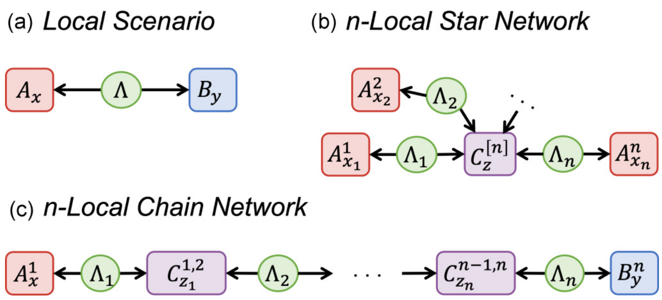

FIG. 1. Star and chain networks of n sources (green ellipses) and $m = n + 1$ nonsignaling parties (rectangles). External parties are labeled as A (red) or B (blue). Central parties are labeled as C (purple). The input to each party is specified in the subscript while the superscript indexes the linked sources. Parties A, B, and C have the respective classical inputs and alphabets $x \in \mathcal{X}$, $y \in \mathcal{Y}$, and $z \in \mathcal{Z}$. All parties are assumed to output binary values.

assumed that the *central* parties measure observables in mutually unbiased bases.

We first formalize the framework of quantum non-n-locality for general networks of nonsignaling parties. Then, we lay the foundation for our main results by introducing an important family of full correlation locality inequalities, and the maximal qubit violations of the CHSH inequality. Next, we present our results regarding the maximal quantum violations of n-locality in star and chain networks where we assume that measurements are separable across local qudit subsystems. Finally, we discuss the application of our results to self-testing, network nonlocality, and maximizing n-local violations.

II. METHODS

A. Quantum non-n-locality

We consider networks that consist of m nonsignaling parties $A^1, \dots, A^m$ that are correlated by n independent sources $\Lambda_1, \dots, \Lambda_n$. Each party is modeled as a black-box device that maps a classical input $x_j \in \mathcal{X}_j$ to a binary output $a_j \in \mathbb{B} \equiv \{0, 1\}$. Furthermore, each source links two parties together forming a network. A network can be characterized by its correlations $\mathbf{C}^{\mathrm{Net}} \equiv \{\langle O^{\mathrm{Net}}_{\vec{x}}\rangle\}_{\vec{x}\in\mathcal{X}^{\mathrm{Net}}}$ where $\mathcal{X}^{\mathrm{Net}} \equiv \mathcal{X}_1 \times \cdots \mathcal{X}_m$, the correlator is

$$\langle O^{\mathrm{Net}}_{\vec{x}}\rangle = \sum_{a_1,\dots,a_m} (-1)^{a_1+\cdots+a_m} P(\vec{a}|\vec{x}), \tag{1}$$

and $\vec{a} \in \mathbb{B}^m$ is the m-bit classical output. Note that we label parties as A^j for generality, however, it will later be convenient to label parties as B and C having inputs $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$, respectively.

In the classical case, sources can distribute an unbounded amount of shared randomness to neighboring parties. As a result, the network's transition probabilities decompose as [35,43,44]

$$P(\vec{a}|\vec{x}) = \sum_{\lambda_1,\dots,\lambda_n} \prod_{i=1}^{n} P(\lambda_i) \prod_{j=1}^{m} P(a_j|x_j, \vec{\lambda}_j), \tag{2}$$

where $\vec{\lambda}_j = (\lambda_i)_{i\in L_j}$ is the set of random values received by party A^j and $L_j \subseteq [n]$ indexes sources linked to A^j. In Eq. (2), the nonsignaling constraint is enforced by the product of $P(a_j|x_j, \vec{\lambda}_j)$ and the product of $P(\lambda_i)$ enforces the independence of sources.

For a given network topology, we define the set of classical network correlations as $\mathcal{L}^{\mathrm{Net}}$ where $\mathbf{C}^{\mathrm{Net}} \in \mathcal{L}^{\mathrm{Net}}$ if the network's transition probabilities decompose as in Eq. (2). The independence between sources causes $\mathcal{L}^{\mathrm{Net}}$ to be nonconvex, meaning that $\mathcal{L}^{\mathrm{Net}}$ is bound tightly by nonlinear n-locality inequalities [35], $S_{\mathrm{Net}}(\mathbf{C}^{\mathrm{Net}}) \leqslant \beta$ where $S_{\mathrm{Net}}(\mathbf{C}^{\mathrm{Net}})$ is a nonlinear function and β is its classical upper bound. These n-locality inequalities are satisfied by all $\mathbf{C}^{\mathrm{Net}} \in \mathcal{L}^{\mathrm{Net}}$, hence their violation can witness non-n-local correlations.

In the quantum case, each source emits a two-qudit mixed state, which is represented as the density operator $\rho_i \in D(\mathcal{H}^{A_i}_d \otimes \mathcal{H}^{B_i}_d)$ where $\mathcal{H}^{A_i}_d$ and $\mathcal{H}^{B_i}_d$ denote the respective Hilbert spaces for the qudit subsystems of state ρ_i. Each party measures their local qudits using a dichotomic Hermitian observable $O^j_{x_j}$ that has ± 1 eigenvalues and is conditioned upon the classical input x_j. In aggregate, the network prepares the state $\rho_{[n]} \equiv \bigotimes_{i=1}^n \rho_i$ and measures the observable $O^{\mathrm{Net}}_{\vec{x}} = \bigotimes_{j=1}^m O^j_{x_j}$ where the tensor product enforces independence between sources and parties, respectively. We then define the set of quantum network correlations as $\mathcal{Q}^{\mathrm{Net}}$ where $\mathbf{C}^{\mathrm{Net}} \in \mathcal{Q}^{\mathrm{Net}}$ if the network correlator can be expressed as

$$\langle O^{\mathrm{Net}}_{\vec{x}}\rangle_{\rho_{[n]}} = \mathrm{Tr}\big[O^{\mathrm{Net}}_{\vec{x}} \rho_{[n]}\big], \tag{3}$$

where care is taken to ensure that each party measures the qudits from the appropriate sources.

A quantum network can produce non-n-local correlations that cannot be reproduced by a classical network having the same topology. Formally, quantum correlation $\mathbf{C}^{\mathrm{Net}} \in \mathcal{Q}^{\mathrm{Net}}$ are non-n-local if $\mathbf{C}^{\mathrm{Net}} \notin \mathcal{L}^{\mathrm{Net}}$ and non-n-local correlations are witnessed via the violation of an n-locality inequality $S_{\mathrm{Net}}(\mathbf{C}^{\mathrm{Net}}) > \beta$ [4,35]. It is then our main goal to find the maximal n-local network score that can be achieved for a given state

$$S^{\star}_{\mathrm{Net}}(\rho_{[n]}) \equiv \max_{\{O^{\mathrm{Net}}_{\vec{x}}\}_{\vec{x}\in\mathcal{X}^{\mathrm{Net}}}} S_{\mathrm{Net}}(\mathbf{C}^{\mathrm{Net}}), \tag{4}$$

where $\mathbf{C}^{\mathrm{Net}} = \{\mathrm{Tr}[O^{\mathrm{Net}}_{\vec{x}}\rho_{[n]}]\}_{\vec{x}\in\mathcal{X}^{\mathrm{Net}}}$ and the optimization is over all observables $O^{\mathrm{Net}}_{\vec{x}} = \bigotimes_{j=1}^m O^j_{x_j}$.

Similarly to previous works [71–73], we consider the setting in which multiqudit observables are separable across qudit subsystems as $O^j_{x_j} = \bigotimes_{i\in L_j} O^i_{y_i=x_j}$ where the ± 1 eigenvalues of $O^j_{x_j}$ correspond to the parity of the $|L_j|$-qudit measurement result and $L_j \subseteq [n]$ indexes the linked sources. As a consequence, each two-qudit mixed state ρ_i is measured using dichotomic qudit observables $A^i_{x_i}$ and $B^i_{y_i}$ and the network correlator factors across all two-qudit pairs as

$$\langle O^{\mathrm{Net}}_{\vec{x}}\rangle_{\rho_{[n]}} = \prod_{i=1}^{n} \langle A^i_{x_i} \otimes B^i_{y_i}\rangle_{\rho_i}, \tag{5}$$

where $\langle A^i_{x_i} \otimes B^i_{y_i}\rangle_{\rho_i} = \mathrm{Tr}[A^i_{x_i} \otimes B^i_{y_i}\rho_i]$ and the inputs x_i and y_i are set as their corresponding party's input (see Fig. 2). The setting of separable multiqudit measurements is important

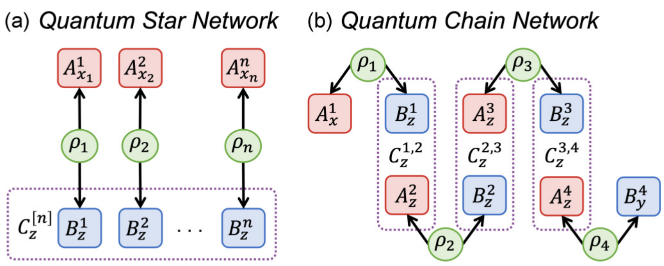

FIG. 2. Quantum sources (ellipses) distribute entangled qudits to linked parties (rectangles). Qudit observables are labeled using A (red) or B (blue), while multiqudit observables are labeled using C group together separable qudit observables (purple dashed). The n-locality inequalities bounding the chain network assume that all multiqudit observables are conditioned on the same value, $z = z_j \in \mathcal{Z}$, which is passed to each local qudit observable contained by the multiqubit observable, e.g., $C^{1,2}_z = B^1_z \otimes A^2_z$.

because it is both experimentally tractable and leads to convenient theoretical simplifications [45,58,72]. However, a more general analysis would also consider nonseparable multiqudit observables.

B. Full-correlation locality inequalities and the maximal qubit violation of the Clauser-Horne-Shimony-Holt inequality

As an important example, we consider the local ($n = 1$) case where one source links two parties A and B [see Fig. 1(a)]. The resulting local set $\mathcal{L}^{AB}$ forms a convex polytope, which is tightly bound by linear *locality inequalities*, also referred to as Bell inequalities [4]. These locality inequalities and their quantum violations form the basis of our quantum non-n-locality results in star and chain networks.

We consider a family of bipartite full-correlation locality inequalities having the form [45]

$$G(\mathbf{C}^{AB}) \equiv \sum_{y\in\mathcal{Y}} \left\langle O^{AB}_y \right\rangle_\rho \leqslant \gamma, \tag{6}$$

$$\text{where } \left\langle O^{AB}_y \right\rangle_\rho = \sum_{x\in\mathcal{X}} G_{x,y} \langle A_x \otimes B_y \rangle_\rho, \tag{7}$$

where $\gamma \geqslant 0$ is the classical bound and $G_{x,y} \in \mathbb{R}$. When maximizing the observables A_x and B_y to obtain $G^\star(\rho)$, observable B_y can always be selected such that $\langle O^{AB}_y \rangle_\rho \geqslant 0$. In the qubit case, the two observables are $A_x = \vec{\alpha}_x \cdot \vec{\sigma}$ and $B_y = \vec{\beta}_y \cdot \vec{\sigma}$ where $|\vec{\alpha}_x| = |\vec{\beta}_y| = 1$, $\vec{\sigma} = (X, Y, Z)$ are the Pauli observables. Entangled qudit states are sufficient to achieve the maximal violation when each party receives a binary input, however, the maximal violation may require entangled qudit states having dimension larger than two when $|\mathcal{X}| \geqslant 3$ or $|\mathcal{Y}| \geqslant 3$ [52].

The CHSH inequality is an important example of a full-correlation locality inequality where each party has binary inputs $\mathcal{X} = \mathcal{Y} = \mathbb{B}$ and the maximal quantum violation can be achieved with a two-qubit state. Then, the local correlations $\mathbf{C}^{AB} \in \mathcal{L}^{AB}$ satisfy the CHSH inequality [69]

$$G_{\text{CHSH}}(\mathbf{C}^{AB}) \equiv \sum_{y\in\mathbb{B}} \left\langle O^{\text{CHSH}}_y \right\rangle \leqslant 1, \tag{8}$$

$$O^{\text{CHSH}}_y = \frac{1}{2}[A_0 + (-1)^y A_1] \otimes B_y, \tag{9}$$

where $G_{x,y} = \frac{1}{2}(-1)^{x\wedge y}$ and $\gamma = 1$. Note that, without loss of generality, we have scaled the CHSH inequality by a factor of $\frac{1}{2}$ from its typical local bound of $\gamma = 2$. Furthermore, the $G_{\text{CHSH}}(\mathbf{C}^{AB})$ quantifies the performance in a game where the goal is to maximize the likelihood that the inputs and outputs satisfy $a \oplus b = x \wedge y$ [4].

The nonlocal content of a two-qubit mixed state ρ is found in its correlation matrix, $T_\rho \in \mathbb{R}^{3\times 3}$, that has elements $T^{(k,\ell)}_\rho = \text{Tr}[\sigma_k \otimes \sigma_\ell \rho]\ \forall\ \sigma_k, \sigma_\ell \in \vec{\sigma}$. For an arbitrary mixed state $\tilde{\rho} \in D(\mathcal{H}^A_2 \otimes \mathcal{H}^B_2)$, the correlation matrix can be diagonalized as [74]

$$T_\rho = \text{diag}(\vec{\tau}) = R^A T_{\tilde{\rho}} (R^B)^T, \tag{10}$$

where $R^A, R^B \in \text{SO}(3)$ and $\vec{\tau} \in \mathbb{R}^3$ where $1 \geqslant \tau_0 \geqslant \tau_1 \geqslant |\tau_2| \geqslant 0$ are the singular values of $T_{\tilde{\rho}}$. If R^A and R^B are chosen freely, the values $\tau_j \in \vec{\tau}$ can be permuted and/or sign-flipped as long as the constraint holds that $\det(T_{\tilde{\rho}}) = \det(T_\rho) = \prod_{j=1}^3 \tau_j$. We choose to order the singular values as $\vec{\tau} = (\tau_1, \tau_2, \tau_0)$ so that our results are consistent with standard examples of non-n-locality in star and chain networks where the optimal measurements are expressed in the xz plane of Bloch sphere. Note that a homomorphism maps SU(2) to SO(3) [75] such that the rotations R^A and R^B correspond to qubit unitaries V^A and $V^B \in \text{SU}(2)$, hence Eq. (10) becomes $\rho = V^A \otimes V^B \tilde{\rho} (V^A \otimes V^B)^\dagger$. Since we allow measurements to vary freely over local qubit unitaries, we assume without loss of generality, that ρ satisfies $T_\rho = \text{diag}(\vec{\tau})$.

In the CHSH scenario, all quantum correlations $\mathbf{C}^{AB} \in \mathcal{Q}^{AB}$ satisfy $G_{\text{CHSH}}(\mathbf{C}^{AB}) \leqslant \sqrt{2}$ [76] and, for any two-qubit mixed state ρ, the maximal CHSH score is [70]

$$G^\star_{\text{CHSH}}(\rho) = \sqrt{\tau_0^2 + \tau_1^2}, \tag{11}$$

where τ_j are the singular values of T_ρ. Note that ρ can be used to generate nonlocal correlations if and only if $\tau_0^2 + \tau_1^2 > 1$. Hence we define a classical state σ where $T_\sigma = \text{diag}(0, 0, 1)$ such that $G^\star_{\text{CHSH}}(\sigma) = 1$. Examples include the product of two pure qubit states, $\sigma = |\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|$, or a shared coin flip, $\sigma = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$.

To obtain $G^\star_{\text{CHSH}}(\rho)$, there are two choices of optimal qubit observables,

$$\left\{ A^\star_x = (1-x)Z + xX,\ B^\star_y = \frac{\tau_0 Z + \tau_1(-1)^y X}{\sqrt{\tau_0^2 + \tau_1^2}} \right\}, \text{ or} \tag{12}$$

$$\left\{ \overline{A}^\star_x = \frac{\tau_0 Z + \tau_1(-1)^x X}{\sqrt{\tau_0^2 + \tau_1^2}},\ \overline{B}^\star_y = (1-y)Z + yX \right\}, \tag{13}$$

where the choices yield the respective expectations

$$\left\langle O^{\text{CHSH}}_y \right\rangle^\star_\rho = \frac{1}{2}\sqrt{\tau_0^2 + \tau_1^2} = \frac{1}{2} G^\star_{\text{CHSH}}(\rho), \tag{14}$$

$$\left\langle \overline{O}^{\text{CHSH}}_y \right\rangle^\star_\rho = \frac{1}{\sqrt{\tau_0^2 + \tau_1^2}} \left[\tau_0^2(1-y) + \tau_1^2 y\right]. \tag{15}$$

To achieve $G^\star_{\text{CHSH}}(\rho)$, one party must measure observables in mutually unbiased bases, e.g., X and Z, while the other party's observables might be nonorthogonal, e.g., $|\text{Tr}[B_0 B_1]| > 0$.

Moreover, Eqs. (12) and (13) shows that the optimal measurements can be swapped between the two parties. A fact that follows from T_ρ being symmetric and the two parties being indistinguishable. From a self-testing perspective, this symmetry implies that a maximal CHSH score, $G^\star_{\text{CHSH}}(\rho)$, is insufficient on its own to determine with certainty which of the two parties measured in mutually unbiased bases. As we will see, this symmetry is broken in the star and chain networks.

III. RESULTS

A. Maximal qubit violations of star *n*-locality

Consider the star network as depicted in Fig. 2(a) where each party outputs a binary value. We introduce a family of star network n-locality inequalities

$$S_{\{G^i\}_{i=1}^n}(\mathbf{C}^{\text{Net}}) \equiv \sum_{z\in\mathcal{Z}} |I_{n,z}(\mathbf{C}^{\text{Net}})|^{\frac{1}{n}} \leqslant \beta, \tag{16}$$

$$I_{n,z}(\mathbf{C}^{\text{Net}}) \equiv \sum_{\vec{x}\in\mathcal{X}^{\text{Net}}} \left(\prod_{i=1}^n G^i_{x_i,z}\right)\langle O^{\text{Star}}_{\vec{x},z}\rangle_{\rho_{[n]}}, \tag{17}$$

where $O^{\text{Star}}_{\vec{x}} = (\bigotimes_{i=1}^n A^i_{x_i}) \otimes C_z^{[n]}$ and $C_z^{[n]}$ is an n-qubit observable having ± 1 eigenvalues. The inequality in Eq. (16) is constructed from a set of bipartite full-correlation locality inequalities $\{G^i\}_{i=1}^n$ and the n-local bound is $\beta = \prod_{i=1}^n (\gamma^i)^{\frac{1}{n}}$, the geometric mean of the upper bounds across all locality inequalities in $\{G^i\}_{i=1}^n$.

When considering local qudit observables, the central party's observable factors as $C_z^{[n]} = \bigotimes_{i=1}^n B^i_{y_i=z}$ where the ± 1 eigenvalues of $C_z^{[n]}$ correspond to the parity of the n-qudit measurement result. The star network observable is then $O^{\text{Star}}_{\vec{x}} \equiv \bigotimes_{i=1}^n A^i_{x_i} \otimes B^i_z$ where the vector $\vec{x} \in \mathcal{X}^{\text{Net}} \equiv \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \times \mathcal{Z}$ contains each party's input. The star network correlator then factors as in Eq. (5), allowing Eq. (17) to be rewritten as [45]

$$I_{n,z}(\mathbf{C}^{\text{Net}}) = \prod_{i=1}^n \langle O_z^{A_iB_i}\rangle_{\rho_i}, \tag{18}$$

where $\langle O_z^{A_iB_i}\rangle_{\rho_i}$ is defined in Eq. (7). Next, inserting Eq. (18) into Eq. (16), we obtain the star n-locality inequality for local qudit observables

$$S_{\{G^i\}_{i=1}^n}(\mathbf{C}^{\text{Net}}) = \sum_{z\in\mathcal{Z}}\prod_{i=1}^n \left|\langle O_z^{A_iB_i}\rangle_{\rho_i}\right|^{\frac{1}{n}} \leqslant \beta. \tag{19}$$

Furthermore, for the set $\{G^i\}_{i=1}^n$, the geometric mean of bipartite full-correlation locality scores must satisfy

$$\prod_{i=1}^n G^i(\mathbf{C}^{A_iB_i})^{\frac{1}{n}} = \prod_{i=1}^n \left(\sum_{z\in\mathcal{Z}} \left|\langle O_z^{A_iB_i}\rangle_{\rho_i}\right|\right)^{\frac{1}{n}} \leqslant \beta, \tag{20}$$

leading to the following inequality that holds for any fixed set of correlators $\langle O_z^{A_iB_i}\rangle_{\rho_i}$ [40,45]:

$$\sum_{z\in\mathcal{Z}}\prod_{i=1}^n \left|\langle O_z^{A_iB_i}\rangle_{\rho_i}\right|^{\frac{1}{n}} \leqslant \prod_{i=1}^n \left(\sum_{z\in\mathcal{Z}} \left|\langle O_z^{A_iB_i}\rangle_{\rho_i}\right|\right)^{\frac{1}{n}}. \tag{21}$$

We now prove the conditions for which equality is obtained in Eq. (21).

Lemma 1. Consider a matrix $M \in \mathbb{R}^{m\times n}$ where $M_{z,i} \geqslant 0$. Matrix M satisfies the following inequality,

$$0 \leqslant \sum_{z=1}^m \left(\prod_{i=1}^n M_{z,i}\right)^{\frac{1}{n}} \leqslant \prod_{i=1}^n \left(\sum_{z=1}^m M_{z,i}\right)^{\frac{1}{n}}, \tag{22}$$

where equality is obtained if and only if either $\text{Rank}(M) = 1$, or for some $i \in [n]$, $M_{z,i} = 0$ for all $z \in [m]$.

Proof. Let $M_{z,i} = a_z b_i$ where $a_z, b_i \geqslant 0$, then using Lemma 1 of Ref. [40], the inequality in Eq. (22) holds. We show the left-hand side (LHS) and right-hand side (RHS) of Eq. (22) to be equal as follows:

$$\sum_{z=1}^m \left(\prod_{i=1}^n M_{z,i}\right)^{\frac{1}{n}} = \sum_{z=1}^m \left(\prod_{i=1}^n a_z b_i\right)^{\frac{1}{n}} \tag{23}$$

$$= \left(\sum_{z=1}^m a_z\right)\left(\prod_{i=1}^n b_i\right)^{\frac{1}{n}} \tag{24}$$

$$= \prod_{i=1}^n \left(\sum_{z=1}^m a_z b_i\right)^{\frac{1}{n}} \tag{25}$$

$$= \prod_{i=1}^n \left(\sum_{z=1}^m M_{z,i}\right)^{\frac{1}{n}}. \tag{26}$$

Since equality holds when $M = \vec{a}\cdot\vec{b}^T$ for non-negative vectors $\vec{a} \in \mathbb{R}^m$ and $\vec{b} \in \mathbb{R}^n$, any matrix M that has $\text{Rank}(M) = 1$ achieves equality. As an edge case, equality is also obtained when, for some $i \in [n]$, $M_{z,i} = 0$ for all $z \in [m]$ because the upper bound on the RHS becomes zero, $\sum_{z=1}^m M_{z,i} = 0$. Finally, if $M_{z,i} \neq a_z b_i$ for all $z \in [m]$ and $i \in [n]$, then the inequality in Eq. (22) is strict because the sum and product cannot be interchanged. ■

Note that Ref. [45] presents a similar result to Lemma 1, however, the authors incorrectly state that equality is obtained *if and only if* $M_{z,i} = M_{z,i'}$ for all $i, i' \in [n]$. While the example matrix M is rank-one, the authors' equality condition neglects a large number of rank-one matrices. Indeed, these matrices correspond to correlations that lead to greater violations of star n-locality than previously known.

Theorem 1. For any ensemble of two-qudit mixed states $\rho_{[n]} = \bigotimes_{i=1}^n \rho_i$ where $\rho_i \in D(\mathcal{H}_d^{A_i} \otimes \mathcal{H}_d^{B_i})$, the maximal star n-locality score obtained using local qudit observables satisfies

$$S^\star_{\{G^i\}_{i=1}^n}(\rho_{[n]}) \leqslant \prod_{i=1}^n G^{i\star}(\rho_i)^{\frac{1}{n}}, \tag{27}$$

where equality occurs if and only if the conditions in Lemma 1 are satisfied by the matrix $M \in \mathbb{R}^{|\mathcal{Z}|\times n}$ where $M_{z,i} = \langle O_z^{A_iB_i}\rangle^\star_{\rho_i} \geqslant 0$ are the correlators that maximize $G^{i\star}(\rho_i)$ for all $i \in [n]$.

Proof. For any set of correlators $\{\langle O_z^{A_iB_i}\rangle_{\rho_i}\}_{i\in[n],z\in\mathcal{Z}}$ where $\langle O_z^{A_iB_i}\rangle_{\rho_i}$ is defined in Eq. (7), the inequality in Eq. (21) must hold. Thus, if the set of correlators $\{\langle O_z^{A_iB_i}\rangle^\star_{\rho_i}\}_{i\in[n],z\in\mathcal{Z}}$

maximize the star n-locality score, then

$$S^{\star}_{\{G^i\}_{i=1}^n}(\rho_{[n]}) \leqslant \prod_{i=1}^{n}\left(\sum_{z\in\mathcal{Z}}\langle O_z^{A_iB_i}\rangle^{\star}_{\rho_i}\right)^{\frac{1}{n}}, \tag{28}$$

where we may assume that $\langle O_z^{A_iB_i}\rangle^{\star}_{\rho_i} \geqslant 0$ because the qubit observable B_z^i may be selected to ensure nonnegativity. However, this set of observables is not guaranteed to maximize each respective locality inequality, implying that $S^{\star}_{\{G^i\}_{i=1}^n}(\rho_{[n]}) \leqslant \prod_{i=1}^n G^{i\star}(\rho_i)^{\frac{1}{n}}$. Now let M be a matrix with elements $M_{z,i} = \langle O_z^{A_iB_i}\rangle^{\star}_{\rho_i} \geqslant 0$. Then, using Lemma 1, $S^{\star}_{\{G^i\}_{i=1}^n}(\rho_{[n]}) = \prod_{i=1}^n G^{i\star}(\rho_i)^{\frac{1}{n}}$ if and only if either $\mathrm{Rank}(M) = 1$ or, for some $i \in [n]$, $\langle O_z^{A_iB_i}\rangle^{\star}_{\rho_i} = 0$ for all $z \in \mathcal{Z}$. ■

While Theorem 1 applies broadly to a family of star n-locality inequalities, it fails to dictate whether there exist local qudit observables that achieve the upper bound. However, we make an interesting observation when all parties are given binary inputs where $\mathcal{X}_i = \mathcal{Z} = \mathbb{B}$ for all $i \in [n]$. In this setting, the CHSH inequality in Eq. (8) is considered for each source such that $\prod_{i=1}^n G^i_{x_i,z} = \prod_{i=1}^n \frac{1}{2}(-1)^{x_i\wedge z}$ and $\beta = 1$. Inserting these values into Eq. (17) and considering local qubit observables, we find [40]

$$I_{n,z}(\mathbf{C}^{\mathrm{Net}}) = \prod_{i=1}^{n}\langle O_z^{\mathrm{CHSH}_i}\rangle_{\rho_i}, \tag{29}$$

where $O_z^{\mathrm{CHSH}_i} = \frac{1}{2}[A_0^i + (-1)^z A_1^i] \otimes B_z^i$. Thus we define the star n-locality CHSH inequality

$$S_{n\text{-CHSH}}(\mathbf{C}^{\mathrm{Net}}) \equiv \sum_{z\in\mathbb{B}}\left|\prod_{i=1}^{n}\langle O_z^{\mathrm{CHSH}_i}\rangle_{\rho_i}\right|^{\frac{1}{n}}, \tag{30}$$

where Eq. (30) quantifies the likelihood that $\bigoplus_{i=1}^n(a_i \oplus b_i) = \bigoplus_{i=1}^n(x_i \wedge z)$, which is the XOR of CHSH games each played using an independent source. Note that $S_{1\text{-CHSH}}(\mathbf{C}^{\mathrm{Net}}) = G_{\mathrm{CHSH}}(\mathbf{C}^{\mathrm{Net}})$ [69], and that $S_{2\text{-CHSH}}(\mathbf{C}^{\mathrm{Net}})$ is the bilocal score [37,39].

Theorem 2. For any ensemble of two-qubit mixed states $\rho_{[n]} = \bigotimes_{i=1}^n$ where $\rho_i \in D(\mathcal{H}_2^{A_i} \otimes \mathcal{H}_2^{B_i})$, the maximal CHSH n-locality star score obtained using local qubit observables is

$$S^{\star}_{n\text{-CHSH}}(\rho_{[n]}) = \prod_{i=1}^{n} G^{\star}_{\mathrm{CHSH}}(\rho_i)^{\frac{1}{n}}, \tag{31}$$

where $G^{\star}_{\mathrm{CHSH}}(\rho_i) = (\tau_{i,0}^2 + \tau_{i,1}^2)^{1/2}$ as in Eq. (11).

Proof. By Theorem 1, $S^{\star}_{n\text{-CHSH}}(\rho_{[n]}) \leqslant \prod_{i=1}^n G^{\star}_{\mathrm{CHSH}}(\rho_i)$. To prove equality, consider the optimal CHSH observables $A_{x_i}^{i\star}$ and $B_z^{i\star}$ from Eq. (12). Since the expectations in Eq. (14) satisfy $\langle O_0^{\mathrm{CHSH}_i}\rangle^{\star}_{\rho_i} = \langle O_1^{\mathrm{CHSH}_i}\rangle^{\star}_{\rho_i}$ for all $i \in [n]$, the matrix M with elements $M_{z,i} = \langle O_z^{\mathrm{CHSH}_i}\rangle^{\star}_{\rho_i} \geqslant 0$ has $\mathrm{Rank}(M) = 1$. Thus, Lemma 1 is satisfied, implying that $S^{\star}_{n\text{-CHSH}}(\rho_{[n]}) = \prod_{i=1}^n G^{\star}_{\mathrm{CHSH}}(\rho_i)^{\frac{1}{n}}$. ■

Theorem 2 contrasts with previous results, which assume that the multiqubit observables are in the mutually unbiased bases

$$\widehat{C}_z^{[n]} = (1-z)\bigotimes_{i=1}^{n} Z^{B_i} + z\bigotimes_{i=1}^{n} X^{B_i}. \tag{32}$$

This choice of observables has physical significance because they can be obtained by coarse-graining the outcomes of the GHZ measurement, which is necessary for generalized entanglement swapping the star network [62]. The observables $\widehat{C}_z^{[n]}$ lead to the maximal n-local star score [71–73]

$$\widehat{S}^{\star}_{n\text{-CHSH}}(\rho_{[n]}) = \sqrt{\prod_{i=1}^{n}\tau_{i,0}^{2/n} + \prod_{i=1}^{n}\tau_{i,1}^{2/n}}, \tag{33}$$

where each external party measures the qubit observables

$$\hat{A}_{x_i}^i = \frac{\prod_{i=1}^n \tau_{i,0}^{1/n} Z + (-1)^{x_i}\prod_{i=1}^n \tau_{i,1}^{1/n} X}{\sqrt{\prod_{i=1}^n \tau_{i,0}^{2/n} + \prod_{i=1}^n \tau_{i,1}^{2/n}}}. \tag{34}$$

In general, $\widehat{S}^{\star}_{n\text{-CHSH}}(\rho_{[n]}) \leqslant S^{\star}_{n\text{-CHSH}}(\rho_{[n]})$ where taking the square of both sides and substituting Eqs. (33) and (31) yields the inequality in Eq. (22):

$$\prod_{i=1}^{n}\tau_{i,0}^{2/n} + \prod_{i=1}^{n}\tau_{i,1}^{2/n} \leqslant \prod_{i=1}^{n}\left(\tau_{i,0}^2 + \tau_{i,1}^2\right)^{\frac{1}{n}}. \tag{35}$$

Applying Lemma 1, we derive the condition for equality between these two contrasting n-local star scores.

Corollary 1. $S^{\star}_{n\text{-CHSH}}(\rho_{[n]}) = \widehat{S}^{\star}_{n\text{-CHSH}}(\rho_{[n]})$ if and only if either $\mathrm{Rank}(M) = 1$ where $M_{z,i} = \tau_{i,z}^2$, or $\tau_{i,0} = 0$ for some $i \in [n]$.

Proof. Let $M_{j,i} = \tau_{i,j}^2 \geqslant 0$, then using Lemma 1 equality holds in Eq. (35) when $\mathrm{Rank}(M) = 1$, or when $\tau_{i,0} = 0$ because $\tau_{i,0} \geqslant \tau_{i,1} = 0$. ■

In theoretical works, the conditions in Corollary 1 are often assumed to hold. For instance, equality holds when all sources emit the same mixed state, or when white noise is modeled on sources. However, these examples of equality are exceptions that would rarely occur in general ensembles of two-qubit mixed states.

While $\widehat{S}^{\star}_{n\text{-CHSH}}(\rho_{[n]})$ is not globally maximal, it is maximal under the assumption that the central party measures local qubit observables in Eq. (32), which are in mutually unbiased bases. On the contrary, $S^{\star}_{n\text{-CHSH}}(\rho_{[n]})$ is globally maximal for all mixed state ensemble where $S^{\star}_{n\text{-CHSH}}(\rho_{[n]})$ is achieved when the *external* parties each measure their qubit observables in mutually unbiased bases, e.g., $A_{x_i}^i \in \{X, Z\}$. As noted earlier, a CHSH violation $G^{\star}_{\mathrm{CHSH}}(\rho) > 1$ is alone insufficient to determine which qubit was measured in mutually unbiased bases. However, this is not the case in the star network because the maximal violation $S^{\star}_{n\text{-CHSH}}(\rho_{[n]}) > \widehat{S}^{\star}_{n\text{-CHSH}}(\rho_{[n]})$ for a known state $\rho_{[n]}$ requires the external parties to measure observables in mutually unbiased bases. This fact could be used to self-test whether the external parties are measuring in mutually unbiased bases.

Overall, the n-local violations of $S^{\star}_{n\text{-CHSH}}(\rho_{[n]})$ are more robust to noise than $\widehat{S}^{\star}_{n\text{-CHSH}}(\rho_{[n]})$. In Fig. 3, we illustrate the separation between $S^{\star}_{n\text{-CHSH}}(\rho_{[n]})$ and $\widehat{S}^{\star}_{n\text{-CHSH}}(\rho_{[n]})$. In the left plot, we consider k sources to be affected by noise that dampens $\tau_{i,1}$ but preserves $\tau_{i,0}$, while the remaining sources are noiseless. As $\tau_{i,1}$ becomes small, a large separation exists. In the right plot, we consider a case where Corollary 1 does not hold. That is, for all $i \in [n]$, $G^{\star}_{\mathrm{CHSH}}(\rho_i)$ is constant, but the pair $\tau_{i,0}$ and $\tau_{i,1}$ are unique. We thus construct examples where $S^{\star}_{n\text{-CHSH}}(\rho_{[n]}) > 1 > \widehat{S}^{\star}_{n\text{-CHSH}}(\rho_{[n]})$.

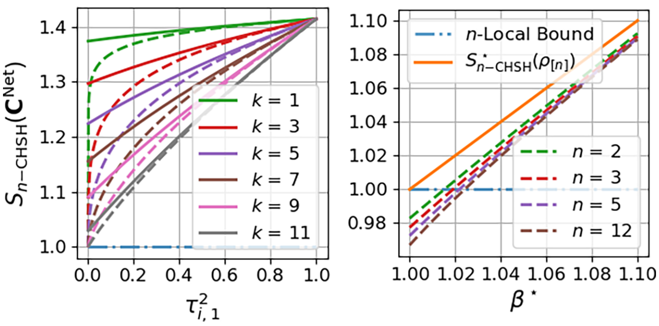

FIG. 3. We compare $S^\star_{n\text{-CHSH}}(\rho_{[n]})$ (solid) with $\widehat{S}^\star_{n\text{-CHSH}}(\rho_{[n]})$ (dashed). In both plots, the n-local bound is shown by the blue dash-dotted line and the solid or dashed curves are ordered from top to bottom as shown by the legend. (left) We consider $k < n = 12$ noisy sources that have $\tau_{i,0} = 1$ and $\tau_{i,1} \in [0, 1]$. The remaining $(n - k)$ sources have $\tau_{i,0} = \tau_{i,1} = 1$. Note that, for each k, each solid line has a corresponding dashed line that lies just below it. (right) For all $i \in [n]$, we consider $\frac{1}{2}G^\star_{\text{CHSH}}(\rho_i) = \beta^\star \in [1.0, 1.1]$ while $\tau^2_{i,1} = (\beta^\star)^2 - \tau^2_{i,0}$ where $\tau^2_{i,0}$ are evenly spaced in $[\frac{1}{2}(\tau^2_{i,0} + \tau^2_{i,1}), \min\{1, \tau^2_{i,0} + \tau^2_{i,1}\}]$. Note that, for all n, $S^\star_{n\text{-CHSH}}(\rho_{[n]})$ achieves the solid orange line while the dashed lines plotting $\widehat{S}^\star_{n\text{-CHSH}}(\rho_{[n]})$ achieve smaller scores.

In Ref. [67], an extreme example is noted where $S^\star_{n\text{-CHSH}}(\rho_{[n]}) > 1 \geqslant \widehat{S}^\star_{n\text{-CHSH}}(\rho_{[n]})$ where $k \in [1, n)$ sources each emit the classical state $\sigma_i = |00\rangle\langle 00|$ while the remaining sources each emit maximally entangled states. In this case,

$$S^\star_{n\text{-CHSH}}(\rho_{[n]}) = 2^{\frac{n-k}{2n}} > 1 = \widehat{S}^\star_{n\text{-CHSH}}(\rho_{[n]}), \tag{36}$$

thus showing an example where significant n-local violations occurs where no violation is predicted by previous results [72]. We now use Theorem 2 to generalize this example.

Corollary 2. For two-qubit states $\sigma_i, \rho_i \in D(\mathcal{H}_2^{A_i} \otimes \mathcal{H}_2^{B_i})$, consider the ensemble of k classical states $\sigma_{[k]} = \bigotimes_{i=1}^k \sigma_i$ where $T_{\sigma_i} = \text{diag}(0, 0, 1)$, and the ensemble of general mixed states $\rho_{[k+1,n]} = \bigotimes_{i=k+1}^n \rho_i$. Then, $\widehat{S}^\star_{n\text{-CHSH}}(\sigma_{[k]} \otimes \rho_{[k+1,n]}) \leqslant 1 \leqslant S^\star_{n\text{-CHSH}}(\sigma_{[k]} \otimes \rho_{[k+1,n]}) = S^\star_{(n-k)\text{-CHSH}}(\rho_{[k+1,n]})^{\frac{n-k}{n}}$.

Corollary 2 leads to bounds whose violation witnesses *full quantum network nonlocality*, in which all sources are verified to be nonclassical [34]. Namely, if $k = 1$, then the maximal n-local violation is bounded as $S^\star_{n\text{-CHSH}}(\rho_{[n]}) \leqslant 2^{\frac{n-1}{2n}}$. Thus, if $S_{n\text{-CHSH}}(\mathbf{C}^{\text{Net}}) > 2^{\frac{n-1}{2n}}$, then all sources are nonclassical. Similarly, if the central party measures the observable $\widehat{C}_z^{[n]}$ such that $\widehat{S}^\star_{n\text{-CHSH}}(\rho_{[n]})$ is maximal, then the violation $\widehat{S}^\star_{n\text{-CHSH}}(\rho_{[n]}) > 1$, witnesses all sources to be nonclassical. Thus for either choice of observables, a sufficiently large n-local violation asserts that no classical sources are present.

B. Maximal qubit violations of chain *n*-locality

Consider the n-local chain network as depicted in Fig. 2(b) where each party outputs a binary value. We introduce a family of chain network n-locality inequalities:

$$K_{\{G^1,G^n\}}(\mathbf{C}^{\text{Net}}) = \sum_{z\in\mathcal{Z}} \left|J_{n,z}(\mathbf{C}^{\text{Net}})\right|^{\frac{1}{2}} \leqslant \beta, \tag{37}$$

$$J_{n,z}(\mathbf{C}^{\text{Net}}) = \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} G^1_{x,z}G^n_{y,z}\langle O^{\text{Chain}}_{x,y,z}\rangle, \tag{38}$$

where $O^{\text{Chain}}_{x,y,z} = A^1_x \otimes (\bigotimes_{i=1}^{n-1} C_z^{i,i+1}) \otimes B^n_y$ with $C_z^{i,i+1} = B^i_z \otimes A^{i+1}_z$ and the coefficients for the bipartite full-correlation locality inequalities are $G^1_{x,z}, G^n_{y,z} \in \mathbb{R}$. The n-local bound of $\beta = (\gamma^1\gamma^n)^{1/2}$ is derived by assuming that each party's correlator satisfies $A^1_x, B^n_y, C_z^{i,i+1} \in \{\pm 1\}$ and is separable from the other parties [41,45,53].

When considering local qudit observables, the central observables in the chain network are $C_z^{i,i+1} = B^i_z \otimes A^{i+1}_z$. Then, using Eq. (5), we rewrite Eq. (38) as

$$J_{n,z}(\mathbf{C}^{\text{Net}}) = \prod_{i\in\{1,n\}} \left\langle O_z^{A_iB_i}\right\rangle_{\rho_i} \prod_{i=2}^{n-1} \left\langle A^i_z \otimes B^i_z\right\rangle_{\rho_i}, \tag{39}$$

where $\langle O_z^{A_1B_1}\rangle_{\rho_1} = \sum_{x\in\mathcal{X}} G^1_{x,z}\langle A^1_x \otimes B^1_z\rangle_{\rho_1}$ and $\langle O_z^{A_nB_n}\rangle_{\rho_n} = \sum_{y\in\mathcal{Y}} G^n_{y,z}\langle A^n_z \otimes B^n_y\rangle_{\rho_n}$ are full-correlation observables defined in Eq. (7). Next, we insert Eq. (39) into Eq. (37) and note that $\prod_{i\in\{1,n\}}\langle O_z^{A_iB_i}\rangle_{\rho_i} = I_{2,z}(\mathbf{C}^{1,n})$ to find

$$K_{\{G^1,G^n\}}(\mathbf{C}^{\text{Net}}) = \sum_{z\in\mathcal{Z}} \left| I_{2,z}(\mathbf{C}^{1,n}) \prod_{i=2}^{n-1} \left\langle A^i_z \otimes B^i_z\right\rangle_{\rho_i} \right|^{\frac{1}{2}}, \tag{40}$$

where $\mathbf{C}^{1,n}$ denotes the correlations of sources 1 and n.

We now derive an upper bound on the maximal chain n-locality score in Eq. (40) for any quantum correlations $\mathbf{C}^{\text{Net}} \in \mathcal{Q}^{\text{Net}}$. First, since $\langle A^i_z \otimes B^i_z\rangle_{\rho_i} \leqslant \tau_{i,0}$ for all $z \in \mathcal{Z}$, it follows that

$$\begin{aligned} K_{\{G^1,G^n\}}(\mathbf{C}^{\text{Net}}) &\leqslant \sum_{z\in\mathcal{Z}} \left|I_{2,z}(\mathbf{C}^{1,n})\right|^{\frac{1}{2}} \prod_{i=2}^{n-1}\sqrt{\tau_{i,0}} \\ &\leqslant S^\star_{\{G^1,G^n\}}(\rho_1 \otimes \rho_n)\prod_{i=2}^{n-1}\sqrt{\tau_{i,0}}, \end{aligned} \tag{41}$$

where in the second line $\sum_{z\in\mathbb{B}}|I_{2,z}(\mathbf{C}^{1,n})|^{\frac{1}{2}} = S_{\{G^1,G^n\}}(\mathbf{C}^{1,n}) \leqslant S^\star_{\{G^1,G^n\}}(\rho_1 \otimes \rho_n)$. When the number of inputs for a party exceeds three, sources $i \in \{1, n\}$ may need to prepare bipartite entangled states of dimension larger than two to obtain the maximal quantum violation [53]. However, as shown in the following theorem, the maximal violation of the chain n-locality inequalities can be obtained when the central sources $i \in [2, n - 1]$ prepare classical two-qubit states.

Theorem 3. For any ensemble of two-qudit mixed states $\rho_{[n]} = \bigotimes_{i=1}^n \rho_i$ where $\rho_i \in D(\mathcal{H}_d^{A_i} \otimes \mathcal{H}_d^{B_i})$, the maximal n-local chain score obtained by measuring local dichotomic observables is

$$K^\star_{\{G^1,G^n\}}(\rho_{[n]}) = S^\star_{\{G^1,G^n\}}(\rho_1 \otimes \rho_n)\prod_{i=2}^{n-1}\sqrt{\tau_{i,0}}, \tag{42}$$

where $S^\star_{\{G^1,G^n\}}(\rho_1 \otimes \rho_n)$ is the maximal bilocal star score using local qudit observables.

Proof. For all sources $i \in [2, n-1]$, let $\rho_i \in D(\mathcal{H}_2^{A_i} \otimes \mathcal{H}_2^{B_i})$ be measured by the observable $A^i_z \otimes B^i_z = Z \otimes Z$ such that $\langle A^i_z \otimes B^i_z\rangle_{\rho_i} = \tau_{i,0}$ for all $z \in \mathcal{Z}$. Then, for the two sources $i \in \{1, n\}$, consider the optimal full-correlation observables $\langle O_z^{A_iB_i}\rangle^\star_{\rho_i}$ that maximize the bilocal star score to obtain $K^\star_{\{G^1,G^n\}}(\rho_{[n]}) = S^\star_{\{G^1,G^n\}}(\rho_1 \otimes \rho_n)$. Inserting these observables into Eq. (40) yields the upper bound on the n-local chain score in Eq. (41). ■

Theorem 3 shows that to obtain the maximal n-local chain score, it is sufficient to measure all central sources $i \in [2, n-1]$ using the fixed local qubit observable $Z \otimes Z$. Only the measurements that share a source with an external party depend on z. Thus, the maximal n-local chain score can be achieved even when all sources $i \in [2, n-1]$ emit a classical state such as γ where $T_\gamma = \mathrm{diag}(0, 0, 1)$. Similarly, consider the single-outcome observables $A_z^i \otimes B_z^i = \mathbb{I}_d \otimes \mathbb{I}_d$ measured on sources $i \in [2, n-1]$, then for any mixed state ρ_i, $\langle A_z^i \otimes B_z^i\rangle_{\rho_i} = 1$ for all $z \in \mathcal{Z}$ and

$$K^\star_{\{G^1,G^n\}}(\rho_{[n]}) = S^\star_{\{G^1,G^n\}}(\rho_1 \otimes \rho_n). \tag{43}$$

That is, the maximal violation of chain n-locality is achieved by disregarding all sources $i \in [2, n-1]$ and replacing their measurements with a constant output.

Consider now the special case where all inputs are binary $\mathcal{X} = \mathcal{Y} = \mathcal{Z} = \mathbb{B}$. The n-local chain inequality then becomes a generalization of the CHSH inequality in Eq. (8). Since all inputs are binary, the maximal n-local chain score can be obtained using maximally entangled two-qubit states [53]. In this setting, $G^1_{x,z}G^n_{y,z} = \frac{1}{4}(-1)^{x\wedge z}(-1)^{y\wedge z}$ and $\beta = 1$. Inserting these values into Eq. (38) and considering local qubit observables we find

$$J_{n,z}(\mathbf{C}^{\mathrm{Net}}) = \prod_{i\in\{1,n\}} \left\langle O_z^{\mathrm{CHSH}_i}\right\rangle_{\rho_i} \prod_{i=2}^{n-1} \left\langle A_z^i \otimes B_z^i\right\rangle_{\rho_i}, \tag{44}$$

where $O_z^{\mathrm{CHSH}_i} = \frac{1}{2}[A_0^i + (-1)^z A_1^i] \otimes B_z^i$ as in Eq. (9). It follows that

$$K_{n\text{-CHSH}}(\mathbf{C}^{\mathrm{Net}}) \equiv \sum_{z\in\mathbb{B}} \prod_{i\in\{1,n\}} \left|\left\langle O_z^{\mathrm{CHSH}_i}\right\rangle_{\rho_i}\right|^{\frac{1}{2}} \times \left|\prod_{i=2}^{n-1}\left\langle A_z^i \otimes B_z^i\right\rangle_{\rho_i}\right|^{\frac{1}{2}}. \tag{45}$$

Theorem 4. For any ensemble of two-qubit mixed states $\rho_{[n]} = \bigotimes_{i=1}^n \rho_i$ where $\rho_i \in D(\mathcal{H}_2^{A_i} \otimes \mathcal{H}_2^{B_i})$, the maximal n-local chain score obtained by local qubit observables is

$$K^\star_{n\text{-CHSH}}(\rho_{[n]}) = \prod_{i\in\{1,n\}} \sqrt{G^\star_{\mathrm{CHSH}}(\rho_i)} \prod_{i=2}^{n-1} \sqrt{\tau_{i,0}}. \tag{46}$$

Proof. By direct application of Theorem 3, $K^\star_{n\text{-CHSH}}(\rho_{[n]}) = S^\star_{2\text{-CHSH}}(\rho_1 \otimes \rho_n)\prod_{i=2}^{n-1}\sqrt{\tau_{i,0}}$, then using Theorem 2 we recover Eq. (46) because $S^\star_{2\text{-CHSH}}(\rho_1 \otimes \rho_n) = \prod_{i\in\{1,n\}}\sqrt{G_{\mathrm{CHSH}}(\rho_i)}$. To achieve the maximal score, the central sources $i \in [2, n-1]$ are measured by the observables $A_z^i \otimes B_z^i = Z \otimes Z$ for all $z \in \mathbb{B}$ while sources $i \in \{1, n\}$ are measured by the observables described in the proof of Theorem 2. That is, the external parties measure $A_0^1 = B_0^n = Z$ and $A_1^1 = B_1^n = X$, while $B_z^1 = [\tau_{1,0}Z + \tau_{1,1}(-1)^z X]/(\tau_{1,0}^2 + \tau_{1,1}^2)^{1/2}$ and $A_z^n = [\tau_{n,0}Z + \tau_{n,1}(-1)^z X]/(\tau_{n,0}^2 + \tau_{n,1}^2)^{1/2}$. ■

The maximal n-local chain score $K^\star_{n\text{-CHSH}}(\rho_{[n]})$ derived in Theorem 4 is distinct from previous results that assume the central parties to measure the observables [73]

$$\widehat{C}_z^{i,i+1} = (1-z)Z \otimes Z + zX \otimes X. \tag{47}$$

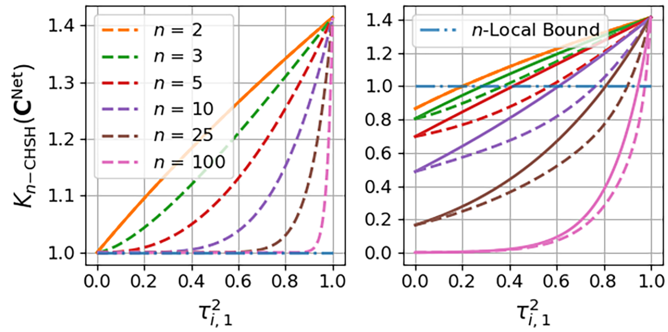

FIG. 4. For various n, $K^\star_{n\text{-CHSH}}(\rho_{[n]})$ (solid) is compared with $\widehat{K}^\star_{n\text{-CHSH}}(\rho_{[n]})$ (dashed). In both plots, the n-local bound is given by the dash-dotted blue line and, when $n = 2$, $K^\star_{2\text{-CHSH}}(\rho_{[n]}) = \widehat{K}^\star_{2\text{-CHSH}}(\rho_{[n]})$ both follow the solid orange line plotting the largest n-local violation. Furthermore, the curves are ordered from top to bottom as shown by the legend. (left) We set $\tau_{i,0}^2 = 1$ and vary $\tau_{i,1}^2 \in [0, 1]$. Note that for all n, $K^\star_{n\text{-CHSH}}(\rho_{[n]})$ achieves the solid orange line while $\widehat{K}^\star_{n\text{-CHSH}}(\rho_{[n]})$ is necessarily smaller for all $n > 2$. (right) We set $\tau_{i,0}^2 = \frac{3}{4} + \frac{1}{4}\tau_{i,1}^2$ and vary $\tau_{i,1}^2 \in [0, 1]$. Note that for each n, the dashed line lies just below the solid line while having the same value at $\tau_{i,1}^2 = \{0, 1\}$.

This choice of observable is physically motivated because it can be implemented as a coarse-graining on the outputs of a Bell state measurement, which is important for entanglement swapping [61]. However, these observables lead to a suboptimal n-local chain score

$$\widehat{K}^\star_{n\text{-CHSH}}(\rho_{[n]}) \equiv \sqrt{\prod_{i=1}^n \tau_{i,0} + \prod_{i=1}^n \tau_{i,1}}, \tag{48}$$

where the two external parties measure the observables

$$\hat{A}_x^1 = \frac{\prod_{i=1}^n \sqrt{\tau_{i,0}}Z + (-1)^x \prod_{i=1}^n \sqrt{\tau_{i,1}}X}{\sqrt{\prod_{i=1}^n \tau_{i,0} + \prod_{i=1}^n \tau_{i,1}}}, \tag{49}$$

and similarly for $\widehat{B}_y^n$. In general, $K^\star_{n\text{-CHSH}}(\rho_{[n]}) \geqslant \widehat{K}^\star_{n\text{-CHSH}}(\rho_{[n]})$ where equality occurs only in special cases.

Corollary 3. $K^\star_{n\text{-CHSH}}(\rho_{[n]}) = \widehat{K}^\star_{2\text{-CHSH}}(\rho_{[n]})$ if $\tau_{i,0} = \tau_{i,1}$ for all $i \in [2, n-1]$ and $S^\star_{\{G^1,G^n\}}(\rho_1 \otimes \rho_n) = \widehat{S}^\star_{2\text{-Star}}(\rho_1 \otimes \rho_n)$.

The result Corollary 3 was observed in Ref. [67] where significant chain n-locality observations were obtained when $\widehat{K}_{n\text{-CHSH}}(\mathbf{C}^{\mathrm{Net}}) \leqslant 1$. For instance, let $\rho_{[n]} = \rho_1 \otimes (\bigotimes_{i=2}^{n-1}\gamma_i) \otimes \rho_n$ where $T_{\gamma_i} = \mathrm{diag}(0, 0, 1)$, then $K^\star_{n\text{-CHSH}}(\rho_{[n]}) = S^\star_{2\text{-CHSH}}(\rho_1 \otimes \rho_n)$ and $\widehat{K}^\star_{n\text{-CHSH}}(\rho_{[n]}) \leqslant 1$. However, if the central parties measure the observables $\widehat{C}_z^{(i,i+1)}$ as in Eq. (47), then an n-local violation $\widehat{K}_{n\text{-CHSH}}(\mathbf{C}^{\mathrm{Net}}) > 1$ asserts that each source is nonclassical. As a consequence, full quantum network nonlocality [34] cannot be witnessed in the chain network with respect to local qubit observables but can if central parties measure their qubits in mutually unbiased bases.

Overall, the n-local violations of $K^\star_{n\text{-CHSH}}(\rho_{[n]})$ are more robust to noise than the n-local violations of $\widehat{K}^\star_{n\text{-CHSH}}(\rho_{[n]})$. In Fig. 4, we illustrate cases where $K^\star_{n\text{-CHSH}}(\rho_{[n]}) > \widehat{K}^\star_{n\text{-CHSH}}(\rho_{[n]})$. In both plots, we consider uniform noise on all sources such that Corollary 1 is satisfied such that $S^\star_{2\text{-CHSH}}(\rho_1 \otimes \rho_n) = S^\star_{2\text{-CHSH}}(\rho_1 \otimes \rho_n)$. Thus, we ensure that the separation between $K^{\star}_{n\text{-CHSH}}(\rho_{[n]})$ and $\widehat{K}^{\star}_{n\text{-CHSH}}(\rho_{[n]})$ is inherent to the chain network. In the left plot, we consider noise where $\tau_{i,0}$ is preserved but $\tau_{i,1}$ is damped. For all $n > 2$, we find $K^{\star}_{n\text{-CHSH}}(\rho_{[n]}) = S^{\star}_{2\text{-CHSH}}(\rho_1 \otimes \rho_n) \geqslant \widehat{K}^{\star}_{n\text{-CHSH}}(\rho_{[n]}) \geqslant 1$ where the separation increases with n. In the right plot, we consider noise such that $\tau_{i,0}$ is also damped, but the bias $\tau_{i,0} > \tau_{i,1}$ is preserved. For all $n > 2$, we find that $S^{\star}_{2\text{-CHSH}}(\rho_1 \otimes \rho_n) > K^{\star}_{n\text{-CHSH}}(\rho_{[n]}) \geqslant \widehat{K}^{\star}_{n\text{-CHSH}}(\rho_{[n]})$ and we find examples where $K^{\star}_{n\text{-CHSH}}(\rho_{[n]}) > 1 > \widehat{K}^{\star}_{n\text{-CHSH}}(\rho_{[n]})$. Hence the local qubit measurement strategy discussed in Theorem 4 lead to n-locality violations in noisy settings when the results of previous works predict the correlations to be local [73].

IV. DISCUSSION

In this work, we investigate the non-n-local correlations that form in noisy star and chain quantum networks when separable multiqubit observables are measured. In Lemma 1, we present a general condition for equality between n-locality scores and their upper bounds, broadening significantly the optimality condition derived in Ref. [45]. We apply Lemma 1 in Theorems 1 and 3 to derive general conditions that assert when local qudit observables achieve their maximal n-locality violation for a given state $\rho_{[n]}$. In Theorems 2 and 4, we derive the maximal n-local star and chain scores $S^{\star}_{n\text{-CHSH}}$ and $K^{\star}_{n\text{-CHSH}}$ for any ensemble of two-qubit mixed states measured using local qubit observables. The derived maximal n-local scores are larger and more robust to noise than $\widehat{S}^{\star}_{n\text{-CHSH}}$ and $\widehat{K}^{\star}_{n\text{-CHSH}}$ derived in previous works [72,73].

The optimal observables for these distinct n-local violations can be used to test the sources and measurements of star and chain networks. In particular, a maximal n-local violation satisfying $S^{\star}_{n\text{-CHSH}}(\rho_{[n]}) > \widehat{S}^{\star}_{n\text{-CHSH}}(\rho_{[n]})$ requires the external parties to apply qubit observables in mutually unbiased bases, X and Z. A fact that could be used in self-testing measurements. Furthermore, our results relate to the framework of full network nonlocality [34] where a sufficiently large violation indicates that no classical sources are present.

An advantage arises when optimizing a network's observables for the maximal n-local violation given uncharacterized sources. Namely, to achieve $\widehat{S}^{\star}_{n\text{-CHSH}}$ and $\widehat{K}^{\star}_{n\text{-CHSH}}$, the qubit observables of external parties depend on the states emitted from all sources, as shown in Eqs. (34) and (49). Thus, when optimizing a single qubit observable, the network must be considered as a whole. In contrast, to achieve $S^{\star}_{n\text{-CHSH}}$ and $S^{\star}_{n\text{-CHSH}}$, the optimal qubit observables depend only on the state they measure, allowing the observables on each source to be optimized as independent CHSH violations. This distinction simplifies the practical task of optimizing nonlocal correlations on quantum hardware [67,77,78].

Although $\widehat{S}^{\star}_{n\text{-CHSH}}$ and $\widehat{K}^{\star}_{n\text{-CHSH}}$ are only maximal in special cases, we find that most results applying these quantities still hold. Mainly, either the central parties are explicitly assumed to measure in mutually unbiased bases, or Corollaries 1 or 3 hold inadvertently due to the presence of uniform state preparations or white noise. Regardless, the fact that $S^{\star}_{n\text{-CHSH}}(\rho_{[n]}) \geqslant \widehat{S}^{\star}_{n\text{-CHSH}}(\rho_{[n]})$ and $S^{\star}_{n\text{-CHSH}}(\rho_{[n]}) \geqslant \widehat{K}^{\star}_{n\text{-CHSH}}(\rho_{[n]})$ is important to networking applications that rely upon non-n-locality such as information security [11,12], self-testing [21–25], nonlocality sharing [79–82], and quantum steering [83]. In future works, it would be interesting to consider explicitly the optimal observables of n-locality inequalities where parties have more than two inputs or outputs, or for topologies beyond stars and chains. Additionally, it is still an open question whether there exist nonseparable observables leading to larger violations of n-locality in noisy star and chain networks. Finally, we expect that variational optimization techniques for quantum networks [67] will lead to further theoretical insights, and should be applied more broadly.

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