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Local Dirac's condition on the existence of 2-factor ☆

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ABSTRACT

For a vertex u in a graph and a given positive integer k, let $M_k(u)$ denote the set of vertices whose distance from u is at most k. A graph satisfies the local Dirac's condition if the degree of each vertex u in it is at least $\frac{|M_2(u)|}{2}$. Asratian et al. disproved that a connected graph G on at least three vertices is Hamiltonian if G satisfies the local Dirac's condition. In this paper, we prove that if a connected graph G on at least three vertices satisfies the local Dirac's condition, then G contains a 2-factor. Our result is best possible.

1. Introduction

We consider finite and simple graphs in this paper. For notation and terminology not defined here, readers are referred to [6]. We denote the minimum degree of a graph G by $\delta(G)$. Let $N_G(v)$ denote the set of neighbors of v and $N_G[v] = N_G(v) \cup \{v\}$ in a graph G with $v \in V(G)$; moreover, if there is no confusion, we use $N(v)$ and $N[v]$ to denote $N_G(v)$ and $N_G[v]$, respectively. For a graph G with $v \in V(G)$ and $S \subseteq V(G)$, let $N_S(v) = S \cap N_G(v), d_S(v) = |N_S(v)|$ and $d(v)$ denote $d_G(v)$ for brevity if there is no confusion. For a graph G with $A, B \subseteq V(G)$, let $e_G(A, B)$ denote the number of edges with one end in A and the other end in B, and we use $e_G(v, B)$ to denote $e_G(\{v\}, B)$ for brevity. Let $\alpha(G)$ and $\kappa(G)$ denote the independence number and the connectivity of a graph G, respectively. For a vertex u in a graph and a positive integer k, let $M_k(u)$ denote the set of vertices whose distance from u is at most k. Let $N_2(u) = M_2(u) \setminus N[u]$ and $d_2(u) = |N_2(u)|$.

The following is a well-known result on the Hamiltonicity of graphs due to Dirac [9].

Theorem 1.1. ([9]) *Let G be a graph with at least three vertices. If $\delta(G) \geq \frac{|V(G)|}{2}$, then G is Hamiltonian.*

Asratian and Khachatryan [2] gave the following local criteria for the Hamiltonicity of graphs, which is a generalization of Dirac's Theorem.

Theorem 1.2. ([2]) *Let G be a connected graph with at least three vertices. If $d(u) \geq \frac{|M_3(u)|}{2}$ for each vertex $u \in V(G)$, then G is Hamiltonian.*

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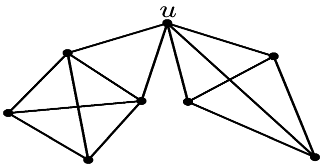

Fig. 1. A graph with a cut vertex u satisfying the local Dirac's condition.

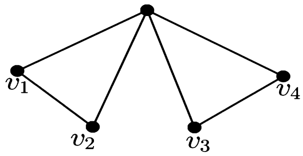

Fig. 2. A graph G without a 2-factor.

Moreover, in [3–5], the authors developed local criteria for the existence of Hamilton cycles in finite connected graphs, which are analogues of the classical global criteria due to Ore [11] and Chvátal-Erdő [8]. Readers are referred to [10,12,14] for more local conditions on Hamiltonicity of graphs.

We say a graph G satisfies the *local Dirac's condition* if for each vertex $u \in V(G), d(u) \geq \frac{|M_2(u)|}{2}$. Asratian et al. [5] gave a 2-connected graph as a counterexample to disprove that a connected graph G on at least three vertices is Hamiltonian if G satisfies the local Dirac's condition. The graph in Fig. 1 satisfies the local Dirac's condition and is not 2-connected, which implies the local Dirac's condition is neither a sufficient condition on the Hamiltonicity nor 1-toughness of a graph.

Though the local Dirac's condition can not guarantee the Hamiltonicity of graphs, in this paper, we prove that it is a sufficient condition for a graph to contain a 2-factor, as follows.

Theorem 1.3. *Let G be a connected graph with at least three vertices. If $d(u) \geq \frac{|M_2(u)|}{2}$ for each vertex $u \in V(G)$, then G contains a 2-factor.*

The graph G in Fig. 2 contains no 2-factor and $d(v_i) = \frac{|M_2(v_i)|-1}{2}$ for each vertex v_i in $G, i \in \{1, 2, 3, 4\}$. Thus the bound in Theorem 1.3 is sharp.

2. Preliminaries

Suppose (S, T) is an ordered pair of disjoint vertex sets in a graph G. For a component C of $G - (S \cup T)$, C is called an odd component with respect to (S, T) (resp., even component) if $e_G(C, T) \equiv 1 \pmod 2$ (resp., $e_G(C, T) \equiv 0 \pmod 2$). Let $\mathcal{H}_G(S, T)$ denote the set of the odd components of $G - (S \cup T)$ and $h_G(S, T) = |\mathcal{H}_G(S, T)|$; moreover, let $\delta_G(S, T) = 2|S| - 2|T| + \sum\limits_{x \in T} d_{G-S}(x) - h_G(S, T)$. The following sufficient and necessary conditions on the existence of a 2-factor are derived from Tutte's f-factor theorem in [13].

Theorem 2.1. ([13]) *A multigraph G contains a 2-factor if and only if $\delta_G(S, T) \geq 0$ for every $S, T \subseteq V(G)$ with $S \cap T = \emptyset$.*

By the definition of $\delta_G(S, T)$, we can obtain $\delta_G(S, T) \equiv 0 \pmod 2$ for every $S, T \subseteq V(G)$ with $S \cap T = \emptyset$; moreover, by Theorem 2.1, if G contains no 2-factor, then G has an ordered pair (S, T) with $S \cap T = \emptyset$ and $\delta_G(S, T) \leq -2$. In a multigraph G, an ordered pair (S, T) is called a *barrier* if $S \cap T = \emptyset$ and $\delta_G(S, T) \leq -2$. A barrier (S, T) is called a *minimal barrier* if $|S \cup T|$ is minimized among all the barriers of G.

In this paper, for a barrier (S, T) of a graph G without a 2-factor, let $\mathcal{C}_{2k+1}$ denote the union of each component C in $\mathcal{H}_G(S, T)$ with $e_G(C, T) = 2k + 1, k \geq 0$. The following result gives the characterizations of a graph G without a 2-factor, in which (1)-(3) are obtained from [1], and (4)-(5) from [7].

Lemma 2.1. ([1,7]) *Let G be a graph without a 2-factor, and (S, T) be a minimal barrier of G. Then*

(1) T is independent;
(2) if C is an even component with respect to (S, T), then $e_G(T, C) = 0$;
(3) if C is an odd component with respect to (S, T), then $e_G(v, C) \leq 1$ for every $v \in T$;
(4) for every $v \in S$, $|\{C \in \mathcal{H}_G(S, T) : e_G(v, C) \geq 1\}| + e_G(v, T) \geq 4$;
(5) $|T| > |S| + \sum_{k \geq 1} k \cdot |\mathcal{C}_{2k+1}|$.

The following result is an important tool in the proof of our main result.

Lemma 2.2. *Let $G[X, Y]$ be a bipartite graph without isolated vertices such that each edge $xy \in E(G)$ with $x \in X$, $y \in Y$ satisfies the following conditions:*

(1) $d(x) \geq d(y) - 1$ and y has at most one neighbor with degree $d(y) - 1$;
(2) if $x_1 \in N(y)$ with $d(x_1) = d(y) - 1$, then $d(y) \geq 3$ and each vertex in $N(y) \setminus \{x_1\}$ has degree at least $d(y) + 1$.

Then, $|X| \leq |Y|$.

Proof. We give each edge xy of G a weight with $w(xy) = \frac{1}{d(x)}$, where $w(xy)$ denotes the weight of $xy, x \in X, y \in Y$. For each vertex $y_0 \in Y$, by the hypothesis of Lemma 2.2, we can obtain the following inequalities (1)-(2):

$$\sum_{x \in N(y_0)} w(xy_0) = \sum_{x \in N(y_0)} \frac{1}{d(x)} \leq \frac{|N(y_0)|}{d(y_0)} = 1, \text{if } d(x) \geq d(y_0) \text{ for every } x \in N(y_0); \tag{1}$$

$$\sum_{x \in N(y_0)} w(xy_0) = \sum_{x \in N(y_0)} \frac{1}{d(x)} \leq \frac{|N(y_0)| - 1}{d(y_0) + 1} + \frac{1}{d(y_0) - 1} \leq 1, \text{otherwise}. \tag{2}$$

By the above inequalities (1)-(2), we have the following inequality (3), which is as we claimed.

$$|X| = \sum_{e \in E(G)} w(e) = \sum_{y \in Y} \sum_{\substack{x \in X \\ xy \in E(G)}} \frac{1}{d(x)} \leq \sum_{y \in Y} 1 = |Y| \quad \square \tag{3}$$

Note that $M_2(v) = N[v] \cup N_2(v)$ and $|M_2(v)| = d(v) + d_2(v) + 1$ for each vertex v in a graph. Then it is easy to obtain the following result.

Remark 1. If G satisfies the local Dirac's condition, then $d(v) \geq d_2(v) + 1$ for each vertex v of G.

3. Proof of Theorem 1.3

Suppose on the contrary, there exists a connected graph G with at least three vertices satisfying the local Dirac's condition, but G contains no 2-factor. Let $E(\mathcal{H}_G(S, T))$ denote the union of the edge sets of all the components in $\mathcal{H}_G(S, T)$, and for a vertex subset W of $S \cup T$, let $e_G(W, \mathcal{H}_G(S, T))$ denote the number of edges between W and all the components in $\mathcal{H}_G(S, T)$. Then, by Theorem 2.1, we choose a barrier (S, T) of G such that

(1) (S, T) is a minimal barrier of G;
(2) subject to (1), $|E(\mathcal{H}_G(S, T))|$ is maximized;
(3) subject to (1) and (2), $e_G(S, \mathcal{H}_G(S, T))$ is maximized.

Claim 1. $\delta(G) \geq 2$.

Proof. We have either $N[v] = V(G)$ or $d_2(v) \geq 1$ for each vertex $v \in V(G)$. If $N[v] = V(G)$, then $d(v) \geq |V(G)| - 1 \geq 2$. Since G satisfies the local Dirac's condition, $d(v) \geq d_2(v) + 1 \geq 2$ if $d_2(v) \geq 1$. $\square$

Claim 2. *For each $v \in T$, if $N_C(v) \neq \emptyset$ for some component $C \in \mathcal{C}_1$ with $|C| = 1$, then $d_{G-S}(v) = 1$.*

Proof. Suppose on the contrary, there exists a vertex $v \in T$ and a component $C \in \mathcal{C}_1$ with $|C| = 1$ and $d_{G-S}(v) \geq 2$. Let $C = \{u\}$. By Lemma 2.1 (1)-(3), there are $d_{G-S}(v)$ components of $\mathcal{H}_G(S, T)$, which contains exactly one neighbor of v. Thus $e_G(\{v\}, \mathcal{H}_G(S, T)) = d_{G-S}(v) \geq 2$. Let $T' := (T \cup \{u\}) \setminus \{v\}$. Clearly, $|S \cup T| = |S \cup T'|$. We have $h_G(S, T') = h_G(S, T) - d_{G-S}(v) + 1$ by Lemma 2.1 (2)-(3), and $\sum_{w \in T'} d_{G-S}(w) = \sum_{w \in T} d_{G-S}(w) - d_{G-S}(v) + 1$ by $C \in \mathcal{C}_1$. Thus, $\delta_G(S, T') = \delta_G(S, T)$. Since $|C| = 1$ and $d_{G-S}(v) \geq 2$, we have $|E(\mathcal{H}_G(S, T'))| > |E(\mathcal{H}_G(S, T))|$, a contradiction to the choice of (S, T). Thus $d_{G-S}(v) = 1$. $\square$

For each vertex $v \in T$, we define a mapping f_v from $\mathcal{H}_G(S, T)$ to $\mathcal{P}(N_2(v))$ such that $f_v(C) = N_2(v) \cap V(C)$ for $C \in \mathcal{H}_G(S, T)$, where $\mathcal{P}(N_2(v)) = \{S : S \subseteq N_2(v)\}$. Clearly, $f_v(C) \cap f_v(C') = \emptyset$ if C and C' are two distinct components in $\mathcal{H}_G(S, T)$.

By Claim 2 and Lemma 2.1 (3), we can obtain the following result.

Claim 3. *For each vertex $v \in T$, if $N_C(v) \neq \emptyset$ for some component $C \in \bigcup_{k \geq 1} \mathcal{C}_{2k+1}$, then $|C'| \geq 2$ and $f_v(C') \neq \emptyset$ for each component C' in $\mathcal{C}_1$ with $N_{C'}(v) \neq \emptyset$.*

By Lemma 2.1 (3), each vertex $v \in T$ has at most one neighbor in each component of $\mathcal{H}_G(S,T)$. Then we have the following result.

Claim 4. *For any vertex $v \in T$ and any component $C \in \mathcal{H}_G(S,T)$ with $|C| \geq 2$, if $N_C(v) \neq \emptyset$, then $f_v(C) \neq \emptyset$.*

Claim 5. *Given an edge uv with $v \in T$, $u \in V(C)$, and $C \in \bigcup_{k\geq 1} \mathcal{C}_{2k+1}$, if $d_{G-\mathcal{C}_1}(v) \leq 2$, then $N_T(u) = \{v\}$, $d_C(u) = 1$ and $d_S(v) = 1$.*

Proof. By Lemma 2.1 (3), $N_C(v) = \{u\}$. Let $\mathcal{C}'_1 = \{C \in \mathcal{C}_1 : N_C(v) \neq \emptyset\}$. By Claim 3, $|C'| \geq 2$ and $f_v(C') \neq \emptyset$ for each component C' in $\mathcal{C}'_1$ provided $\mathcal{C}'_1 \neq \emptyset$. Thus, $|\mathcal{C}'_1| \leq \sum\limits_{C' \in \mathcal{C}'_1} |f_v(C')| \leq d_2(v)$. We have $d_2(v) \leq |\mathcal{C}'_1| + 1$. Otherwise, $d_2(v) \geq |\mathcal{C}'_1| + 2$, and then $d(v) \geq d_2(v) + 1 \geq |\mathcal{C}'_1| + 3$, which implies $d_{G-\mathcal{C}_1}(v) \geq 3$, giving a contradiction.

Suppose $|C| = 1$, i.e., $C = \{u\}$. Then, $d_T(u) \geq 3$ since $C \in \bigcup_{k\geq 1} \mathcal{C}_{2k+1}$. Note that $N_T(u) \setminus \{v\} \subseteq N_2(v)$ by Lemma 2.1 (1). It follows that $d_2(v) \geq \sum\limits_{C' \in \mathcal{C}'_1} |f_v(C')| + (d_T(u) - 1) \geq |\mathcal{C}'_1| + (d_T(u) - 1) \geq |\mathcal{C}'_1| + 2$, giving a contradiction. Thus, $|C| \geq 2$ and then $|f_v(C)| \geq 1$ by Claim 4, which implies $d_2(v) \geq |\mathcal{C}'_1| + |f_v(C)| \geq |\mathcal{C}'_1| + 1$. Note that $d_C(u) = |f_v(C)|$. If $|f_v(C)| \geq 2$ or $N_T(u) \setminus \{v\} \neq \emptyset$, then we have $d_2(v) \geq |\mathcal{C}'_1| + 2$, giving a contradiction. Thus, $N_T(u) = \{v\}$ and $d_C(u) = 1$. Moreover, we have $d_2(v) = |\mathcal{C}'_1| + 1$ and hence $d(v) \geq |\mathcal{C}'_1| + 2$, which implies $d_{G-\mathcal{C}_1}(v) = 2$ by $d_{G-\mathcal{C}_1}(v) \leq 2$. Suppose $N_S(v) = \emptyset$. Then, by $d_{G-\mathcal{C}_1}(v) = 2$ and Lemma 2.1 (1), there is a component $C' \in (\bigcup_{k\geq 1} \mathcal{C}_{2k+1}) \setminus \{C\}$ with $N_{C'}(v) \neq \emptyset$. As the preceding proof for C, $|C'| \geq 2$ and $|f_v(C')| = 1$. It follows that $d_2(v) \geq |\mathcal{C}'_1| + |f_v(C')| + |f_v(C)| = |\mathcal{C}'_1| + 2$, giving a contradiction. Thus $N_S(v) \neq \emptyset$, and $d_S(v) = 1$ by $d_{G-\mathcal{C}_1}(v) = 2$. $\square$

Claim 6. *For $v \in T$, if $N_C(v) \neq \emptyset$ for some $C \in \bigcup_{k\geq 1} \mathcal{C}_{2k+1}$, then $d_{G-\mathcal{C}_1}(v) \geq 3$.*

Proof. Let $N_C(v) = \{u\}$ by Lemma 2.1 (3). Suppose on the contrary, $d_{G-\mathcal{C}_1}(v) \leq 2$. Then, $d_S(v) = 1$ by Claim 5. Let $N_S(v) = \{w\}$. Clearly, $w \neq u$. By $d_{G-\mathcal{C}_1}(v) \leq 2$, we have $N_{G-\mathcal{C}_1}(v) = \{w, u\}$. By Claim 5, $d_C(u) = 1$, which implies $|C| \geq 2$ and $|f_v(C)| = 1$. Let $N_C(u) = \{u_1\}$. Then, $f_v(C) = \{u_1\}$. Suppose $|N_T(w)| \geq 2$ and let $w_1 \in N_T(w) \setminus \{v\}$. Then, $w_1 \in N_2(v)$ by Lemma 2.1 (1). Let $\mathcal{C}'_1 = \{C \in \mathcal{C}_1 : N_C(v) \neq \emptyset\}$. By Claim 3, $f_v(C') \neq \emptyset$ for each component C' in $\mathcal{C}'_1$ provided $\mathcal{C}'_1 \neq \emptyset$. Clearly, $u_1 \neq w_1$ and $\{u_1, w_1\} \subseteq N_2(v)$. Moreover, $\{u_1, w_1\} \cap f_v(C') = \emptyset$ for each $C' \in \mathcal{C}'_1$. It follows that $d_2(v) \geq |\mathcal{C}'_1| + 2$, which implies $d_{G-\mathcal{C}_1}(v) \geq 3$, giving a contradiction. Thus $N_T(w) = \{v\}$.

By Lemma 2.1 (4), there are at least three components of $\mathcal{H}_G(S,T)$ in which w has a neighbor. Suppose $N_{C^*}(w) \neq \emptyset$ and $N_{C^*}(v) = \emptyset$ for a component $C^* \in \mathcal{H}_G(S,T)$. Let $w^* \in N_{C^*}(w)$. Then, $w^* \in N_2(v)$. Clearly, $w^* \neq u_1$ and $\{w^*, u_1\} \subseteq N_2(v)$. Moreover, $\{u_1, w^*\} \cap f_v(C') = \emptyset$ for each $C' \in \mathcal{C}'_1$. Thus $d_2(v) \geq |\mathcal{C}'_1| + 2$, and then $d_{G-\mathcal{C}_1}(v) \geq 3$, a contradiction. Thus $N_{C^*}(v) \neq \emptyset$ for each component $C^* \in \mathcal{H}_G(S,T)$ with $N_{C^*}(w) \neq \emptyset$. It follows that there are at least three components of $\mathcal{H}_G(S,T)$ in which v has a neighbor. Moreover, by $d_S(v) = 1$, we have $d_G(v) \geq 4$, which implies $|\mathcal{C}'_1| \geq 2$ by $d_{G-\mathcal{C}_1}(v) = 2$. Suppose C_1, C_2 are two distinct components in $\mathcal{C}'_1$. Then $(N_{C_1}(v) \cup N_{C_2}(v)) \subseteq N_2(u)$. Recall that $f_v(C) = \{u_1\}$. Since $C \in \bigcup_{k\geq 1} \mathcal{C}_{2k+1}$ and $d_T(u) = d_C(u) = 1$ by Claim 5, there is some vertex $u' \in V(C) \setminus \{u\}$ with $N_T(u') \neq \emptyset$, which implies $N(u_1) \cap N_2(u) \neq \emptyset$. Suppose $u^* \in N(u_1) \cap N_2(u)$. Clearly, $\{u^*\} \cap N_{C_i}(v) = \emptyset, i = 1, 2$, and hence $d_2(u) \geq 3$. Thus $d(u) \geq d_2(u) + 1 \geq 4$, which implies $d_S(u) \geq 2$ by $d_C(u) = 1$ and $d_T(u) = 1$. Let $u_2 \in N_S(u) \setminus \{w\}$. By $N_S(v) = \{w\}$, we have $u_2 \in N_2(v)$. Clearly, $u_1 \neq u_2$ and $\{u_1, u_2\} \cap f_v(C') = \emptyset$ for each $C' \in \mathcal{C}'_1$. Thus, $d_2(v) \geq |\mathcal{C}'_1| + |\{u_1, u_2\}| = |\mathcal{C}'_1| + 2$, and then $d_{G-\mathcal{C}_1}(v) \geq 3$, giving a contradiction. $\square$

Let H be the resulting graph obtained by doing the following operations on G:

(1) Remove all the even components;
(2) Remove all the components in $\mathcal{C}_1$;
(3) Remove all the edges in $G[S]$;
(4) For each component $C \in \bigcup_{k\geq 1} \mathcal{C}_{2k+1}$, suppose $N_T(C) = \{v_0^C, v_1^C, \cdots, v_{2k}^C\}$. Firstly, replace C by an independent set $U^C = \{u_1^C, u_2^C, \cdots, u_k^C\}$. Secondly, join u_i^C to v_{2i-1}^C and v_{2i}^C, respectively, and moreover, join u_1^C to v_0^C, $1 \leq i \leq k$.

Clearly, the vertices in $S \cup T$ of G are not changed in H, and we still use S and T to denote the two vertex sets in H. Since T is an independent set in G by Lemma 2.1 (1), by the above operations, H is a bipartite graph. In the following proof, let $H = H[Y, T]$ and $Y_1 = Y \setminus S$, where $Y = S \cup (\bigcup\limits_{k\geq 1} \bigcup\limits_{C \in \mathcal{C}_{2k+1}} U^C)$. By the above operations, we can obtain the following two results.

Claim 7. $|Y| = |S| + \sum_{k\geq 1} k \cdot |\mathcal{C}_{2k+1}|$.

Claim 8. *$d_H(y) \leq 3$ for each vertex $y \in Y_1$.*

Claim 9. *For each vertex $v \in T$, $d_H(v) = d_{G-\mathcal{C}_1}(v) \geq 1$.*

Proof. By Lemma 2.1 (1)-(2), $N_G(v) \subseteq S \cup (\bigcup_{k\geq 0} \mathcal{C}_{2k+1})$ for each vertex $v \in T$. Thus we have $d_H(v) = d_{G-\mathcal{C}_1}(v)$ from the operations on G. Suppose on the contrary that H contains an isolated vertex v in T. Then, $N_G(v) \subseteq \bigcup_{C\in\mathcal{C}_1} C$. Let $\mathcal{C}_1' = \{C \in \mathcal{C}_1 : N_C(v) \neq \emptyset\}$. Then, $|\mathcal{C}_1'| \geq 2$ since $d_G(v) \geq 2$ by Claim 1 and Lemma 2.1 (3), and hence $|C| \geq 2$ by Claim 2, for each component $C \in \mathcal{C}_1'$. Moreover, each C in $\mathcal{C}_1'$ contains at least one vertex in $N_2(v)$ in G by Claim 4. Thus $d_2(v) \geq \sum_{C\in\mathcal{C}_1'} |f_v(C)| \geq |\mathcal{C}_1'|$, which implies $d_G(v) \geq |\mathcal{C}_1'| + 1$. It follows that $N_G(v)$ contains a vertex not in any component of $\mathcal{C}_1'$, a contradiction. $\square$

Claim 10. *For any $v \in T$, if $N_G(v) \cap V(C) = \emptyset$ for each component $C \in \mathcal{C}_1$ with $|C| = 1$, then $d_H(v) \geq d_H(u)$ for each vertex $u \in N_H(v)$.*

Proof. Clearly, $N_H(v) \cap Y_1 \neq \emptyset$ if and only if $N_G(v)$ has a neighbor in some component of $\bigcup_{k\geq 1} \mathcal{C}_{2k+1}$. Suppose $N_G(v)$ has a neighbor in some component of $\bigcup_{k\geq 1} \mathcal{C}_{2k+1}$. Then, $d_{G-\mathcal{C}_1}(v) \geq 3$ by Claim 6, and hence $d_H(v) \geq 3 \geq d_H(y)$ by Claim 8 and Claim 9 for each $y \in Y_1 \cap N_H(v)$.

By the operations on G, $S \cap N_H(v) = S \cap N_G(v)$. Suppose $w \in S \cap N_H(v)$. Let $\mathcal{C}_1' = \{C \in \mathcal{C}_1 : N_C(v) \neq \emptyset\}$. By the hypothesis of the claim, $|C'| \geq 2$ for each $C' \in \mathcal{C}_1'$ provided $\mathcal{C}_1' \neq \emptyset$, and hence $|f_v(C')| \geq 1$ by Claim 4. Since $N_H(w) \subseteq T$, we have $N_H(w) \setminus \{v\} \subseteq N_2(v)$ by Lemma 2.1 (1). Clearly, $(\bigcup_{C'\in\mathcal{C}_1'} f_v(C')) \cap (N_H(w) \setminus \{v\}) = \emptyset$. Then $d_G(v) \geq d_2(v) + 1 \geq \sum_{C'\in\mathcal{C}_1'} |f_v(C')| + d_H(w) \geq |\mathcal{C}_1'| + d_H(w)$. Thus, $d_H(v) = d_{G-\mathcal{C}_1}(v) \geq d_H(w)$. $\square$

Claim 11. *For any vertex $v \in T$, if there exists a vertex $u \in N_H(v)$ with $d_H(u) > d_H(v)$, then $d_H(u) \geq 3, d_H(v) = d_H(u) - 1$, and $d_H(v') \geq d_H(u) + 1$ for each vertex $v' \in N_H(u) \setminus \{v\}$.*

Proof. By $d_H(u) > d_H(v)$ and Claim 10, v has a neighbor in some component $C \in \mathcal{C}_1$ with $|C| = 1$. Suppose $C = \{w\}$. Then, $N_G(v) \subseteq S \cup \{w\}$ by Claim 2, and so $u \in S$ and $d_H(v) = d_G(v) - 1$. Since $H = H[Y, T]$ is a bipartite graph, $N_H(u) \subseteq T$. By Lemma 2.1 (1), $(N_H(u) \setminus \{v\}) \subseteq N_2(v)$. Thus, $d_G(v) \geq d_2(v) + 1 \geq d_H(u)$, which implies $d_H(v) = d_G(v) - 1 \geq d_H(u) - 1$. By $d_H(u) > d_H(v)$, we have $d_H(v) = d_H(u) - 1$, which implies $N_2(v) = N_H(u) \setminus \{v\}$. Thus, $(N_G(u) \setminus T) \subseteq N_G(v)$, and u has at most w as a neighbor in the components of $\mathcal{H}_G(S, T)$. It follows that $d_H(u) = |N_G(u) \cap T| \geq 3$ by Lemma 2.1 (4). Since $N_2(v) = N_H(u) \setminus \{v\} \subseteq T$ and $N_G(w) \subseteq S \cup \{v\}$, we have $N_G[w] \subseteq N_G[v]$. We have $N_G[w] = N_G[v]$. Otherwise, $N_G(w) \setminus \{v\}$ is a proper subset of $N_G(v) \setminus \{w\}$, which implies $|e_G(w, S)| < |e_G(v, S)|$. Let $T' := (T \cup \{w\}) \setminus \{v\}$ and $C' := \{v\}$. By $|C| = 1$ and $d_{G-S}(v) = 1$, it is easy to see that $\mathcal{H}_G(S, T') = (\mathcal{H}_G(S, T) \setminus \{C\}) \cup \{C'\}$ and $\delta_G(S, T') = \delta_G(S, T)$. By $|e_G(v, S)| > |e_G(w, S)|$, we have $e_G(S, \mathcal{H}_G(S, T')) > e_G(S, \mathcal{H}_G(S, T))$, giving a contradiction to the choice of (S, T). Thus, we have $w \in N(u)$ by $N_G[w] = N_G[v]$. By $N_T(w) = \{v\}$ and Lemma 2.1 (1), we have $\{w\} \cup (N_H(u) \setminus \{u'\}) \subseteq N_2(u')$ for each vertex $u' \in N_H(u) \setminus \{v\}$. Thus, $d_G(u') \geq d_2(u') + 1 \geq d_H(u) + 1$.

Let $u_1 \in N_H(u) \setminus \{v\}$. Suppose u_1 has no neighbor in any component of $\mathcal{C}_1$. Then $d_H(u_1) = d_G(u_1) \geq d_H(u) + 1$ by $u_1 \in T$. Suppose $N_{C'}(u_1) \neq \emptyset$ for some component $C' \in \mathcal{C}_1$ with $|C'| = 1$. By $N_T(w) = \{v\}$ and $w \in N(u)$, we have $w \in N_2(u_1)$. Let $C' = \{w'\}$. Then, $N_{G-S}(u_1) = \{w'\}$ by Claim 2. Clearly, $w \neq w'$ and $N_G(w') \subseteq S \cup \{u_1\}$. Suppose there is a vertex $u_2 \in N_G(w') \setminus \{u_1\}$ with $u_1u_2 \notin E(G)$. Then, $u_2 \in S$ and hence $u_2 \neq w$. Thus $\{u_2, w\} \cup (N_H(u) \setminus \{u_1\}) \subseteq N_2(u_1)$ and $d_2(u_1) \geq d_H(u) + 1$, which implies $d_G(u_1) \geq d_2(u_1) + 1 \geq d_H(u) + 2$. Since $N_{G-S}(u_1) = \{w'\}$, we have $d_H(u_1) = d_G(u_1) - 1 \geq d_H(u) + 1$. Suppose $N_G(w') \subseteq N_G[u_1]$. Then, $N_G[w'] = N_G[u_1]$. Otherwise, $|e_G(u_1, S)| > |e_G(w', S)|$. Let $T^* := (T \cup \{w'\}) \setminus \{u_1\}$. As the preceding proof for v and w, we have $\delta_G(S, T^*) = \delta_G(S, T)$, and $e_G(S, \mathcal{H}_G(S, T^*)) > e_G(S, \mathcal{H}_G(S, T))$, giving a contradiction to the choice of (S, T). Thus $uw' \in E(G)$ by $u \in N_G(u_1)$, which implies $w' \in N_2(v)$, giving a contradiction with $N_2(v) = N_H(u) \setminus \{v\}$. Suppose $|C''| \geq 2$ for each component $C'' \in \mathcal{C}_1$ with $N_{C''}(u_1) \neq \emptyset$. Then, by Claim 4, $|f_{u_1}(C'')| \geq 1$ for each component $C'' \in \mathcal{C}_1$ with $N_{C''}(u_1) \neq \emptyset$. Note that $\bigcup_{C''\in\mathcal{C}_1} f_{u_1}(C'') \cup (N_H(u) \setminus \{u_1\}) \cup \{w\} \subseteq N_2(u_1)$. Then $d_G(u_1) \geq \sum_{C''\in\mathcal{C}_1} |f_{u_1}(C'')| + d_H(u) + 1$. Thus $d_H(u_1) \geq d_H(u) + 1$. $\square$

By Claim 9, T contains no isolated vertex in H. Note that Y may contain some isolated vertex y in H if and only if $y \in S$ with $N_G(y) \cap T = \emptyset$. Let $Y' = N_H(T)$ and $H' := H[Y', T]$ be a subgraph of $H[Y, T]$. By Claim 10 and Claim 11, each edge in H' satisfies the hypothesis of Lemma 2.2, and hence $|T| \leq |Y'| \leq |Y|$ by Lemma 2.2. By Lemma 2.1 (5) and Claim 7, we have $|T| > |Y|$, giving a contradiction. Thus Theorem 1.3 is true.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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