Applied Mathematics and Computation 445 (2023) 127842

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Spanning trees with at most k leaves in 2-connected $K_{1,r}$-free graphs

Guantao Chen[a,1], Yuan Chen[b,2], Zhiquan Hu[c,3], Shunzhe Zhang[d,4,*]

[a] Department of Mathematics and Statistics, Georgia State University, Atlanta, GA 30303, United States
[b] Research Center of Nonlinear Science, School of Mathematical and Physical Sciences, Wuhan Textile University, Wuhan 430073, PR China
[c] School of Mathematics and Statistics, Hubei Key Laboratory of Mathematical Sciences, Central China Normal University, Wuhan 430079, PR China
[d] Hubei Key Laboratory of Applied Mathematics, Faculty of Mathematics and Statistics, Hubei University, Wuhan 430062, PR China

ARTICLE INFO

Article history:
Received 30 May 2022
Revised 18 November 2022
Accepted 3 January 2023

MSC:
05C05
05C07
05C69

Keywords:
Spanning tree
Leaf
Independence number
$K_{1,r}$-free

ABSTRACT

A vertex with degree one and a vertex with degree at least three are called a leaf and a branch vertex in a tree, respectively. In this paper, we obtain that every 2-connected $K_{1,r}$-free graph G contains a spanning tree with at most k leaves if $\alpha(G) \le k + \lceil \frac{k+1}{r-3} \rceil - \lfloor \frac{1}{|r-k-3|+1} \rfloor$, where $k \ge 2$ and $r \ge 4$. The upper bound is best possible. Furthermore, we prove that if a connected $K_{1,4}$-free graph G satisfies that $\alpha(G) \le 2k+5$, then G contains either a spanning tree with at most k branch vertices or a block B with $\alpha(B) \le 2$. A related conjecture for 2-connected claw-free graphs is also posed.

1. Introduction

In this paper, we only consider simple and undirected graphs. Let G be a graph and $v \in V(G)$. We denote the degree of v by $deg_G(v)$ and the vertices which are adjacent to v by $N_G(v)$. For a set $S \subseteq V(G)$, the subgraph induced by S and $V(G) \setminus S$ are denoted by $G[S]$ and $G - S$, respectively. We denote the number of vertices in S by $|S|$.

A subset X is *independent* in G if $G[X]$ has no edge. The *independence number* of G is denoted by $\alpha(G)$, which means the maximum number of vertices in an independent set of G. Define $\sigma_k(G) = \min\{\sum_{x \in X} deg_G(x) \mid X$ is independent in G and $|X| = k\}$. G is called *$K_{1,r}$-free* if $K_{1,r}$ is not an induced subgraph of G. We write claw-free graph for the $K_{1,3}$-free graph. The *center of a claw* refers to the vertex of degree 3 in $K_{1,3}$ and x-claw refers to a claw with center x.

* Corresponding author.
E-mail addresses: gchen@gsu.edu (G. Chen), cheny@wtu.edu.cn (Y. Chen), hu_zhiq@aliyun.com (Z. Hu), shunzhezhang@hubu.edu.cn (S. Zhang).

[1] This author's work was partially supported by NSFC (No. 11871239).
[2] This author's work was partially supported by NSFC (No. 12201472).
[3] This author's work was partially supported by NSFC (No. 11971196).
[4] This author's work was partially supported by NSFC (No. 11971158), the Youth Project Funds of Hubei Provincial Department of Education (No. 202010401301002) and the General Project Funds of Hubei Provincial Department of Science and Technology (No. 2021CFB476).

https://doi.org/10.1016/j.amc.2023.127842

We call v a *leaf* of tree T if $deg_T(v) = 1$ and denote $L(T)$ the set of leaves of T. A vertex v with $deg_T(v) \geq 3$ is called a *branch vertex* of tree T and define $B(T)$ the set of branch vertices of T.

There are some well-known results such as Ore's Theorem [8] and Chvátal–Erdős's Theorem [4] related to conditions of degree sum and independence number ensuring a Hamiltonian path in G, respectively. Note that a Hamiltonian path is a spanning tree with two leaves. With this viewpoint, researchers gave several results concerning about such two types of conditions to guarantee the existence of spanning tree with bounded leaves(see the survey paper [9]).

The following two results generalize Ore's Theorem [8] and Chvátal–Erdős's Theorem [4], respectively.

Theorem 1.1 (Broerma and Tuinstra [1]). *Let $k \geq 2$. If G is a connected graph of order n such that $\sigma_2(G) \geq n - k + 1$, then G has a spanning tree with at most k leaves.*

Theorem 1.2 (Win [10]). *Let $k \geq 2$. If G is an m-connected graph such that $\alpha(G) \leq m + k - 1$, then G has a spanning tree with at most k leaves.*

Since there are many researches on Hamiltonian path problem in $K_{1,r}$-free graphs, it is also natural for us to search for conditions for $K_{1,r}$-free graphs to ensure the existence of spanning trees with bounded leaves. Here are some related results on $K_{1,r}$-free graphs.

Theorem 1.3 (Kano et al. [6]). *Let $k \geq 2$. If G is a connected claw-free graph of order n such that $\sigma_{k+1}(G) \geq n - k$, then G has a spanning tree with at most k leaves.*

Theorem 1.4 (Kyaw [7]). *Let G be a connected $K_{1,4}$-free graph of order n.*

(i) *If $\sigma_3(G) \geq n$, then G has a Hamiltonian path.*
(ii) *If $\sigma_{k+1}(G) \geq n - \frac{k}{2}$ for some integer $k \geq 3$, then G has a spanning tree with at most k leaves.*

Theorem 1.5 (Chen et al. [2]). *Let $m \geq 2$. If G is an m-connected $K_{1,4}$-free graph of order n such that $\sigma_{m+3}(G) \geq n + 2m - 2$, then G has a spanning tree with at most 3 leaves.*

Theorem 1.6 (Chen et al. [3]). *If G is a connected $K_{1,5}$-free graph of order n such that $\sigma_5(G) \geq n - 1$, then G has a spanning tree with at most 4 leaves.*

Theorem 1.7 (Hu and Sun [5]). *If G is a connected $K_{1,5}$-free graph of order n such that $\sigma_6(G) \geq n - 1$, then G has a spanning tree with at most 5 leaves.*

In this paper, we consider $\alpha(G)$ for a 2-connected $K_{1,r}$-free graph with $r \geq 4$ to guarantee the existence of a spanning tree with bounded leaves.

Theorem 1.8. *Let $k \geq 2$ and $r \geq 4$. If G is a 2-connected $K_{1,r}$-free graph such that $\alpha(G) \leq k + \lceil \frac{k+1}{r-3} \rceil - \lfloor \frac{1}{|r-k-3|+1} \rfloor$, then G has a spanning tree with at most k leaves.*

By taking $r = 4$ in Theorem 1.8, we have the following corollary.

Corollary 1.9. *Let $k \geq 2$. If G is a 2-connected $K_{1,4}$-free graph such that $\alpha(G) \leq 2k + 1$, then G has a spanning tree with at most k leaves.*

Note that a tree with at most k leaves contains at most $k - 2$ branch vertices. We can easily obtain the following corollary.

Corollary 1.10. *Let $k \geq 0$. If G is a 2-connected $K_{1,4}$-free graph such that $\alpha(G) \leq 2k + 5$, then G has a spanning tree with at most k branch vertices.*

With the same independence number condition of Corollary 1.10, we further provide the following result for connected $K_{1,4}$-free graphs.

Theorem 1.11. *Let $k \geq 0$. If G is a connected $K_{1,4}$-free graph such that $\alpha(G) \leq 2k + 5$, then one of the following two statements holds:*

(i) *G has a spanning tree with at most k branch vertices;*
(ii) *there exists a block B in G with $\alpha(B) \leq 2$.*

We provide the following conjecture for connected claw-free graphs to end this section.

Conjecture 1.12. *Let $k \geq 2$. If G is a 2-connected claw-free graph such that $\alpha(G) \leq 2k + 2$, then G has a spanning tree with at most k leaves.*

In next section, we show that the upper bounds of $\alpha(G)$ are sharp in Theorem 1.8 and Conjecture 1.12 if it is true. We prove Theorem 1.8 and Theorem 1.11 in Sections 3 and 4, respectively.

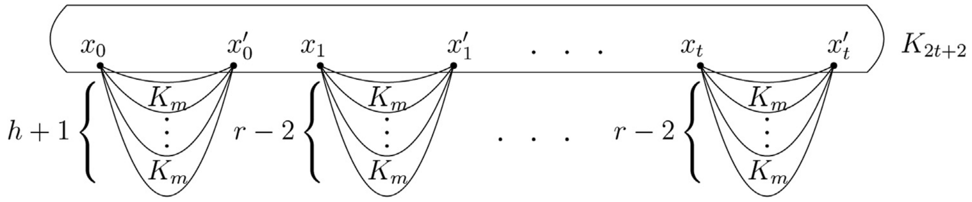

Fig. 1. Graph G_1.

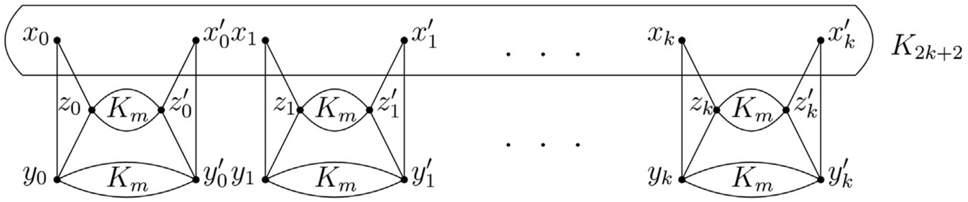

Fig. 2. Graph G_3.

2. Sharpness of Theorem 1.8 and Conjecture 1.12

First, we show that the upper bound of $\alpha(G)$ in Theorem 1.8 is sharp. This is shown in the following examples G_1 and G_2.

Denote $t = \lfloor \frac{k+1}{r-3} \rfloor$ and $h = k+1-t(r-3)$.

Case 1. $r \neq k+3$.

In this case, $\lfloor \frac{1}{|r-k-3|+1} \rfloor = 0$.

If $h \neq 0$, we construct a graph G_1 from a complete graph K_{2t+2} with $V(K_{2t+2}) = \{x_0, x'_0, x_1, x'_1, \ldots, x_t, x'_t\}$ and $(r-2)t + h+1$ complete graphs $K_m (m \geq 3)$ by identifying $r-2$ complete graphs K_m with every pair of $\{x_i, x'_i\}$ for $1 \leq i \leq t$ and by identifying $h+1$ complete graphs K_m with $\{x_0, x'_0\}$ (see Fig. 1). Then G_1 is 2-connected $K_{1,r}$-free and $\alpha(G_1) = t(r-2) + h+1 = t(r-2) + k+1-t(r-3)+1 = k+1+t+1 = k+1+\lceil \frac{k+1}{r-3} \rceil$. However, for every spanning tree T_1 of G_1, we have $|L(T_1)| \geq t(r-3)+h = k+1$. **Case 2.** $r = k+3$.

In this case, $\lceil \frac{k+1}{r-3} \rceil = 2$ and $\lfloor \frac{1}{|r-k-3|+1} \rfloor = 1$.

We construct a graph G_2 from a pair of vertex set $\{x_0, x'_0\}$ and $r-1$ complete graphs K_m $(m \geq 3)$ by identifying $r-1$ complete graphs K_m with $\{x_0, x'_0\}$. Then G_2 is 2-connected $K_{1,r}$-free and $\alpha(G_2) = r-1 = k+2$, but G_2 has no spanning tree with at most k leaves.

Next, we show that the upper bound $2k+2$ in Conjecture 1.12 is sharp if it is true. For $0 \leq i \leq k$, let T_i and T'_i be two triangles with $V(T_i) = \{x_i, y_i, z_i\}$ and $V(T'_i) = \{x'_i, y'_i, z'_i\}$, respectively. Consider a graph G_3 constructed from a complete graph K_{2k+2} with $V(K_{2k+2}) = \{x_0, x'_0, x_1, x'_1, \ldots, x_k, x'_k\}$ and $2k+2$ complete graphs K_m $(m \geq 3)$ by identifying $2k+2$ complete graphs K_m with every pair of $\{y_i, y'_i\}$ and $\{z_i, z'_i\}$ for $0 \leq i \leq k$, respectively (see Fig. 2). Then G_3 is 2-connected claw-free with $\alpha(G_3) = 2k+3$, but G_3 has no spanning tree with at most k leaves.

3. Proof of Theorem 1.8

We begin with some additional notations. Let x and y be two vertices of G, we denote the distance between x and y in G by $d_G(x,y)$. Let u and v be two vertices in a spanning tree T of G, the unique path from u to v in T is denoted by $T[u,v]$. We write $T[u,v]-\{u,v\}$, $T[u,v]-\{u\}$, $T[u,v]-\{v\}$ by $T(u,v)$, $T(u,v]$ and $T[u,v)$, respectively. Set $I(T) = V(T) - L(T)$ and $f(T) = \max_{v \in L(T)} f(T,v)$, where $f(T,v) = \sum_{z \in I(T)} (deg_T(z)-2) d_T(v,z)$. Note that $(deg_T(z)-2)d_T(v,z) = 0$ if $deg_T(z) = 2$. Set $g(T) = \sum_{x \in L(T)} g(T,x)$, where $g(T,x) = \max\{d_T(x,y) | y \in N_G(x)\}$.

Proof of Theorem 1.8.. Suppose that G is a 2-connected $K_{1,r}$-free graph and every spanning tree has at least $k+1$ leaves in G. We choose a spanning tree T of G satisfying that

- $(C1)$ $|L(T)|$ is as small as possible;
- $(C2)$ Subject to $(C1)$, $f(T)$ is as large as possible;
- $(C3)$ Subject to $(C1)$ and $(C2)$, $g(T)$ is as large as possible.

Assume that $L(T) = \{x_0, x_1, \ldots, x_t\}$ and $f(T) = f(T,x_0)$. Then $t \geq k$. T is considered as a rooted tree and x_0 is the root of T. For $1 \leq i \leq t$, r_i is the last branch vertex of T on $T[x_0, x_i]$ and r_i^+ is the successor of r_i on $T[x_0, x_i]$. For $v \in V(T) - \{x_0\}$, the predecessor of v is denoted by v^- on $T[x_0, v]$. □

Claim 3.1. *$L(T)$ is independent in G.*

Proof. Assume that $x_ix_j \in E(G)$ for some i and j with $0 \le i \ne j \le t$. Then $T^* = T - \{r_ir_i^+\} + \{x_ix_j\}$ is a spanning tree with $L(T^*) = (L(T) - \{x_i, x_j\}) \cup \{r_i^+\}$, contradicting $(C1)$. This proves Claim 3.1. $\square$

Remark 3.1. From the proof of Claim 3.1, we know that for every spanning tree T^* of G with $|L(T^*)| \le |L(T)|$, then $L(T^*)$ is independent in G with $|L(T^*)| = |L(T)|$.

Claim 3.2. *For $1 \le i \le t$, there is no neighbour of x_0 on $T(r_i, x_i)$.*

Proof. Assume that $y \in N_G(x_0)$ with $y \in V(T(r_i, x_i))$ for some $1 \le i \le t$. Then $T^* = T - \{yy^-\} + \{x_0y\}$ is a spanning tree of G. If $y^- = r_i$, then $|L(T^*)| < |L(T)|$, contrary to $(C1)$; if $y^- \ne r_i$, then T^* satisfies $(C1)$. Note that $B(T^*) = B(T)$. Then $d_{T^*}(z, x_i) = d_{T^*}(x_0, x_i) + d_T(z, x_0)$ for any $z \in B(T)$. Since $d_{T^*}(x_0, x_i) > 1$, we have $d_{T^*}(z, x_i) > d_T(z, x_0)$. Thus $f(T^*, x_i) > f(T, x_0)$. Then we have $f(T^*) > f(T)$, contrary to $(C2)$. $\square$

For $1 \le i_1 < \ldots < i_l \le t$, denote by $r_{i_1 \ldots i_l}$ the last common vertex of the paths $T[x_0, x_{i_1}], \ldots, T[x_0, x_{i_l}]$. We denote the successor of r_{ij} on $T[r_{ij}, x_i]$ and $T[r_{ij}, x_j]$ by r_{ij}^+ and r_{ji}^+, respectively. Denote the predecessor of r_{ij} on $T[x_0, r_{ij}]$ by r_{ij}^-. The predecessor of y on $T(r_{ij}, x_j)$ is denoted by y^-.

Claim 3.3. *$N_G(x_i) \subseteq V(T(x_0, x_i))$ for $1 \le i \le t$.*

Proof. Assume that there exists $x_j \in L(T) - \{x_0, x_i\}$ satisfying that x_i has a neighbour y on $T(r_{ij}, x_j)$. Obviously, $r_i \in V(T[r_{ij}, x_i))$ and $r_j \in V(T[r_{ij}, x_j))$.

Set $T^* = T - \{r_ir_i^+\} + \{x_iy\}$. Then T^* is a spanning tree with $L(T^*) = (L(T) - \{x_i\}) \cup \{r_i^+\}$. Then $I(T^*) = (I(T) - \{r_i^+\}) \cup \{x_i\}$. Note that $d_{T^*}(x_0, r_i) = d_T(x_0, r_i)$, $deg_{T^*}(r_i) = deg_T(r_i) - 1$, $d_{T^*}(x_0, y) = d_T(x_0, y)$ and $deg_{T^*}(y) = deg_T(y) + 1$. Note that $deg_{T^*}(x_i) = 2$, $deg_T(r_i^+)$=2 and $(deg_T(z) - 2)d_T(x_0, z) = (deg_{T^*}(z) - 2)d_{T^*}(x_0, z)$ for all $z \in I(T^*) \cap I(T) - \{r_i, y\}$. Hence,

$$\begin{aligned}
f(T^*, x_0) - f(T, x_0) &= \sum_{z \in I(T^*)} (deg_{T^*}(z) - 2)d_{T^*}(x_0, z) - \sum_{z \in I(T)} (deg_T(z) - 2)d_T(x_0, z) \\
&= \sum_{z \in I(T^*)\setminus\{x_i\}} (deg_{T^*}(z) - 2)d_{T^*}(x_0, z) - \sum_{z \in I(T)\setminus\{r_i^+\}} (deg_T(z) - 2)d_T(x_0, z) \\
&= \sum_{z \in \{r_i, y\}} (deg_{T^*}(z) - 2)d_{T^*}(x_0, z) - \sum_{z \in \{r_i, y\}} (deg_T(z) - 2)d_T(x_0, z) \\
&= \sum_{z \in \{r_i, y\}} (deg_{T^*}(z) - deg_T(z))d_T(x_0, z) \\
&= d_T(x_0, y) - d_T(x_0, r_i).
\end{aligned}$$

This together with $(C2)$ implies that $d_T(x_0, r_i) \ge d_T(x_0, y)$. (1)

If $y \in V(T(r_{ij}, r_j])$, we set $T' = T - \{yy^-\} + \{x_iy\}$. Then T' is a spanning tree and $I(T') = (I(T) - \{y^-\}) \cup \{x_i\}$. If $deg_T(y^-) \ge 3$, we have $L(T') = L(T) - \{x_i\}$, contradicting $(C1)$. So $deg_T(y^-) = 2$. Note that $(deg_T(z) - 2)d_T(x_0, z) = (deg_{T'}(z) - 2)d_{T'}(x_0, z)$ for all $z \in I(T^*) \cap I(T) - V(T[y, r_j])$. We have

$$\begin{aligned}
f\left(T', x_0\right) - f(T, x_0) &= \sum_{z \in I(T')} (deg_{T'}(z) - 2)d_{T'}(x_0, z) - \sum_{z \in I(T)} (deg_T(z) - 2)d_T(x_0, z) \\
&= \sum_{z \in I(T')\setminus\{x_i\}} (deg_{T'}(z) - 2)d_{T'}(x_0, z) - \sum_{z \in I(T)\setminus\{y^-\}} (deg_T(z) - 2)d_T(x_0, z) \\
&= \sum_{z \in V(T[y, r_j])} (deg_{T'}(z) - 2)d_{T'}(x_0, z) - \sum_{z \in V(T[y, r_j])} (deg_T(z) - 2)d_T(x_0, z) \\
&= \sum_{z \in V(T[y, r_j])} (deg_T(z) - 2)(d_{T'}(x_0, z) - d_T(x_0, z)) \\
&\ge \sum_{z \in V(T[y, r_j])} (deg_T(z) - 2)[d_T(x_0, r_i) - d_T(x_0, y) + 2].
\end{aligned}$$

This together with (1) implies that $f(T', x_0) - f(T, x_0) \ge 2 \sum_{z \in V(T[y, r_j])} (deg_T(z) - 2)$. Noting that $r_j \in V(T[y, r_j])$, we get $f(T') - f(T) \ge f(T', x_0) - f(T, x_0) \ge 2[deg_T(r_j) - 2] \ge 2$, contrary to $(C2)$.

If $y \in V(T(r_j, x_j))$, we set $T'' = T - \{r_jr_j^+\} + \{x_iy\}$. Then T'' is a spanning tree and $I(T'') = (I(T) - \{r_j^+\}) \cup \{x_i\}$. If $y^- = r_j$, then $L(T'') = L(T) - \{x_i\}$, contrary to $(C1)$. Thus, $y^- \ne r_j$. Note that $deg_{T''}(r_j) = deg_T(r_j) - 1$, $d_{T''}(x_0, r_j) = d_T(x_0, r_j)$,

$deg_T(y) = 2$, $deg_{T''}(y) = deg_T(y) + 1 = 3$. From (1), we have $d_{T''}(x_0, y) \geq d_T(x_0, r_i) + 2 \geq d_T(x_0, y) + 2$. By the similar discussion to that in the proof of (1),

$$\begin{aligned} f\left(T'\prime, x_0\right) - f(T, x_0) &= \textstyle\sum_{z\in I(T'\prime)} (deg_{T'\prime}(z) - 2)d_{T'\prime}(x_0, z) - \sum_{z\in I(T)} (deg_T(z) - 2)d_T(x_0, z) \\ &= \textstyle\sum_{z\in I(T'\prime)\setminus\{x_i\}} (deg_{T'\prime}(z) - 2)d_{T'\prime}(x_0, z) - \sum_{z\in I(T)\setminus\left\{r_j^+\right\}} (deg_T(z) - 2)d_T(x_0, z) \\ &= \textstyle\sum_{z\in\left\{r_j, y\right\}} (deg_{T'\prime}(z) - 2)d_{T'\prime}(x_0, z) - \sum_{z\in\left\{r_j, y\right\}} (deg_T(z) - 2)d_T(x_0, z) \\ &= \left(deg_{T'\prime}\left(r_j\right) - deg_T\left(r_j\right)\right)d_T\left(x_0, r_j\right) + d_{T'\prime}(x_0, y) \\ &= d_{T'\prime}(x_0, y) - d_T\left(x_0, r_j\right) \\ &> d_{T'\prime}(x_0, y) - d_T(x_0, y). \\ &\geq 2. \end{aligned}$$

This implies that $f(T'') - f(T) \geq f(T'', x_0) - f(T, x_0) > 2$, also contradicting $(C2)$. This proves Claim 3.3. □

Claim 3.4. *Let $1 \leq i \neq j \leq t$. Then $r_{ij}^- \notin N_G(x_i)$ and $r_{ij}^+ \notin N_G(x_0)$.*

Proof. Suppose that Claim 3.4 is false. Set

$$T^* = \begin{cases} T - \left\{r_{ij}^- r_{ij}\right\} + \left\{x_i r_{ij}^-\right\}, & \text{if } x_i r_{ij}^- \in E(G) \\ T - \left\{r_{ij} r_{ij}^+\right\} + \left\{x_0 r_{ij}^+\right\}, & \text{if } x_0 r_{ij}^+ \in E(G). \end{cases}$$

Then T^* is a spanning tree with $|L(T^*)| < |L(T)|$, contrary to $(C1)$. □

For $0 \leq i \leq t$, let y_i be the neighbour of x_i such that $d_T(x_i, y_i) = g(T, x_i)$. According to Claim 3.3, $y_i \in V(T(x_0, x_i))$ for $1 \leq i \leq t$. We denote the successor of y_i on $T[x_0, x_i]$ by y_i^+. Set $I_1 = \{i \in [1, t] : y_i \in V(T[r_i, x_i))\}$ and $I_2 = \{i \in [1, t] : y_i \in V(T(x_0, r_i))\}$. Obviously, $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 = [1, t]$.

Claim 3.5. *For $i \in I_1$, there exists $z_i \in V(T[y_i, x_i))$ satisfying that $z_i^+ \notin N_G[x_i]$ and $V(T[y_i, z_i]) \subseteq N_G(x_i)$ where $N_G[x_i] = N_G(x_i) \cup \{x_i\}$ and z_i^+ is the successor of z_i on $T[x_0, x_i]$.*

Proof. Suppose that Claim 3.5 is false. Then there is an integer $i \in I_1$ such that $N_G[x_i] \cap V(T[r_i, x_i]) = V(T[y_i, x_i])$. Since G is 2-connected, $G - y_i$ is connected. There is $u_i \in V(T(y_i, x_i))$ such that u_i has a neighbour v_i in $T - T[y_i, x_i]$. Set $T^* = T - \{u_i^- u_i\} + \{u_i^- x_i\}$. Then T^* is a spanning tree with $L(T^*) = (L(T) - \{x_i\}) \cup \{u_i\}$ that satisfies $(C1)$ and $(C2)$. Noting that $d_{T^*}(u_i, y_i) = d_T(x_i, y_i)$, we have $d_{T^*}(u_i, v_i) > d_T(x_i, y_i)$, which implies that $g(T^*, u_i) > g(T, x_i)$. On the other hand, by Claims 3.2 and 3.3, we have $N_G(x_j) \cap V(T(r_i, x_i)) = \emptyset$ for $0 \leq j \neq i \leq t$. Hence, $g(T^*, x_j) = g(T, x_j)$ for $0 \leq j \neq i \leq t$. We have $g(T^*) > g(T)$, contrary to $(C3)$. □

By Claim 3.5, there exists $z_i \in V(T[y_i, x_i))$ satisfying that $z_i^+ \notin N_G[x_i]$, $V(T[y_i, z_i]) \subseteq N_G(x_i)$ and let $L_1'(T) = \{z_i : i \in I_1\}$. Denote $z_j = y_j$ for $j \in I_2$ and let $L_2'(T) = \{z_j : j \in I_2\}$. For $h = 1, 2$, define $X^h = \{x_i : i \in I_h\}$ and $L_h(T) = \{z_i^+ : z_i \in L_h'(T)\}$.

By the choice of z_i for $i \in I_1 \cup I_2$, we define two surjections $\theta_h : X^h \to L_h'(T)$ for $h = 1, 2$. Note that $z_i \in V(T(y_i, x_i))$ for $i \in I_1$. Since $V(T(y_i, x_i)) \cap V(T(y_j, x_j)) = \emptyset$ for $i \neq j \in I_1$, θ_1 is a bijection. Thus $|L_1(T)| = |L_1'(T)| = |I_1|$.

Claim 3.6. $|L_2(T)| \geq |L_2'(T)| \geq \lceil \frac{|I_2|}{r-3} \rceil$.

Proof. Let $\theta_2^{-1}(z_i)$ be the preimage of z_i in X^2 for $z_i \in L_2'(T)$. Suppose that $\theta_2^{-1}(z_i) = \{x_{i_s} : s \geq 1\}$ for $i \in I_2$ and $z_i = z_{i_1} = \ldots = z_{i_s}$. By Claim 3.3, we have $z_i \in V(T(x_0, x_{i_j}))$ for $1 \leq j \leq s$. Hence, $z_i \in V(T(x_0, r_{i_1 \ldots i_s}])$. We claim that

$$\{z_i^-, x_{i_1}, \ldots, x_{i_s}\} \cup \{z_{i_j}^+ : 1 \leq j \leq s\} \text{ is independent in } G. \qquad (*)$$

Suppose to the contrary that $(*)$ is false. By Claim 3.1, $\{x_{i_1}, \ldots, x_{i_s}\}$ is independent. Then one of the following cases occurs.

- $z_i^- z_{i_j}^+ \in E(G)$ for some $j \in [1, s]$. If $deg_T(z_i) \geq 3$, then $T^* = T - \{z_i^- z_i, z_i z_{i_j}^+\} + \{z_i x_{i_j}, z_i^- z_{i_j}^+\}$ is a spanning tree with $L(T^*) = L(T) - \{x_{i_j}\}$, contrary to $(C1)$. Hence, $deg_T(z_i) = 2$. For any $h \in [1, s] \setminus \{j\}$, $T^{(1)} = T - \{z_i^- z_i, z_i z_{i_j}^+, r_{i_h} r_{i_h}^+\} + \{z_i x_{i_j}, z_i^- z_{i_j}^+, z_i x_{i_h}\}$ is a spanning tree with $L(T^{(1)}) = (L(T) - \{x_{i_j}, x_{i_h}\}) \cup \{r_{i_h}^+\}$, contrary to $(C1)$.
- $z_i^- x_{i_j} \in E(G)$ for some $j \in [1, s]$. It follows that $d_T(x_{i_j}, z_i^-) = d_T(x_{i_j}, z_i) + 1 > g(T, x_{i_j})$, contrary to the choice of z_i.
- $x_{i_j} z_{i_h}^+ \in E(G)$ for some $j \neq h \in [1, s]$. If $z_{i_j}^+ = z_{i_h}^+$, then $T^{(2)} = T - \{z_i z_{i_j}^+, x_{i_j}^- x_{i_j}\} + \{z_i x_{i_j}, z_{i_j}^+ x_{i_j}\}$ is a spanning tree with $L(T^{(2)}) \subseteq (L(T) - \{x_{i_j}\}) \cup \{x_{i_j}^-\}$. It is straight to check that $f(T^{(2)}, x_0) > f(T, x_0)$, which indicates that $f(T^{(2)}) > f(T)$, contrary to $(C2)$. Hence $z_{i_j}^+ \neq z_{i_h}^+$. Then $T^{(3)} = T - \{z_i z_{i_h}^+\} + \{x_{i_j} z_{i_h}^+\}$ is a spanning tree with $L(T^{(3)}) = L(T) - \{x_{i_j}\}$, contrary to $(C1)$.
- $z_{i_j}^+ z_{i_h}^+ \in E(G)$ for some $j \neq h \in [1, s]$. Then $T^{(4)} = T - \{z_i z_{i_j}^+, z_i z_{i_h}^+\} + \{z_{i_j}^+ z_{i_h}^+, z_i x_{i_j}\}$ is a spanning tree with $L(T^{(4)}) = L(T) - \{x_{i_j}\}$, contrary to $(C1)$.

Therefore, $(*)$ is true. Since G is $K_{1,r}$-free and z is adjacent to each vertex in $\{z_i^-, x_{i_1}, \dots, x_{i_s}\} \cup \{z_{i_j}^+ : 1 \le j \le s\}$, we have $s \le r-3$. This implies that $|L_2(T)| \ge |L_2'(T)| \ge \lceil \frac{|I_2|}{r-3} \rceil$. □

Set $U = L(T) \cup L_1(T) \cup L_2(T)$. By the definitions of $L(T)$, $L_1(T)$ and $L_2(T)$, three vertex sets $L(T)$, $L_1(T)$ and $L_2(T)$ are disjoint. Thus $|U| = |L(T)| + |L_1(T)| + |L_2(T)|$.

Claim 3.7. *U is independent in G.*

Proof. First, we show that $L_1(T) \cup L_2(T)$ is independent. Set $T_a = T - \{z_iz_i^+, z_jz_j^+\} + \{z_ix_i, z_jx_j\}$ for $z_i \ne z_j \in L_1'(T) \cup L_2'(T)$. By Claim 3.3, $z_i \in V(T(x_0, x_i))$ and $z_j \in V(T(x_0, x_j))$. Then T_a is a spanning tree with $L(T_a) \subseteq (L(T) - \{x_i, x_j\}) \cup \{z_i^+, z_j^+\}$. By Remark 3.1, $L(T_a)$ is independent in G. Hence, $z_i^+z_j^+ \notin E(G)$.

Next, we show that both $L(T) \cup L_1(T)$ and $L(T) \cup L_2(T)$ are independent sets. Set $T_b = T - \{z_iz_i^+\} + \{z_ix_i\}$ for $z_i \in L_1'(T) \cup L_2'(T)$. Then T_b is a spanning tree with $L(T_b) \subseteq (L(T) - \{x_i\}) \cup \{z_i^+\}$. By Remark 3.1, $L(T_b)$ is independent in G. Hence, $z_i^+x_j \notin E(G)$ for $j \in [0,t] - \{i\}$. On the other hand, by Claims 3.1 and 3.5, $L(T)$ is independent in G and $z_i^+ \notin N_G(x_i)$ for $z_i \in L_1'(T) \cup L_2'(T)$.

Therefore, U is independent in G. □

Claim 3.8. *$\alpha(G) = k + 1 + \lceil \frac{k}{r-3} \rceil$, $|I_1| + |I_2| = t = k$, $|I_1| + \lceil \frac{|I_2|}{r-3} \rceil = \lceil \frac{k}{r-3} \rceil$, and $k = p(r-3)$ for some integer $p > 1$.*

Proof. Recall that $|I_1| + |I_2| = t \ge k$ and $|L_1(T)| = |L_1'(T)| = |I_1|$. By Claim 3.6, $|L_2(T)| \ge |L_2'(T)| \ge \lceil \frac{|I_2|}{r-3} \rceil$. This together with Claim 3.7 and the assumption $\alpha(G) \le k + \lceil \frac{k+1}{r-3} \rceil - \lfloor \frac{1}{|r-k-3|+1} \rfloor$, we have

$$\begin{aligned}\alpha(G) \ \ge \ |U| &= |L(T)| + |L_1(T)| + |L_2(T)| \\ &\ge t + 1 + |I_1| + \left\lceil \frac{|I_2|}{r-3} \right\rceil \\ &\ge t + 1 + \left\lceil \frac{t}{r-3} \right\rceil \\ &\ge k + 1 + \left\lceil \frac{k}{r-3} \right\rceil,\end{aligned}$$

which implies $\alpha(G) = k + 1 + \lceil \frac{k}{r-3} \rceil$, $|I_1| + |I_2| = t = k$, $|I_1| + \lceil \frac{|I_2|}{r-3} \rceil = \lceil \frac{k}{r-3} \rceil$, and $k = p(r-3)$ for some integer $p > 1$. □

Recall that $L_2(T) = \{z_i^+ : z_i \in L_2'(T)\}$. By Claim 3.8, we have $p = \frac{k}{r-3}$ with $p > 1$ and there is a partition $\{X_1, \dots, X_p\}$ of $L(T) - \{x_0\}$ satisfying that $|X_i| = 1$ for $i \in I_1$, and $|X_i| = r-3$ for $i \in I_2$. By relabeling $x_1, \dots, x_k$ (if necessary), we may assume that for each $i \in [1,p]$, $x_i \in X_i$. For $i \in I_2$, let $X_i = \{x_{i_1}, \dots, x_{i_{r-3}}\}$, where $i_1 = i$. Then $z_{i_1}^+ = \dots = z_{i_{r-3}}^+ = z_i^+$. We denote $F^* = \{z_i : i \in [1,p],\ z_i \in V(T(x_0, y_0))\}$. By Claim 3.2, we assume that $y_0 \in V(T(x_0, r_{i_1}])$ for some $i_1 \in [1,k]$. Denote by r_0 the first branch vertex of T on $T[x_0, r_{i_1}]$ (possible $r_0 = r_{i_1}$) and r_0^+ the successor of r_0 on $T[x_0, x_{i_1}]$.

Case 1 $F^* = \emptyset$.

Claim 3.9. *There exists $z_0 \in V(T(x_0, y_0))$ such that $z_0 \notin N_G[x_0]$ and $V(T[z_0^+, y_0]) \subseteq N_G(x_0)$, where z_0^+ is the successor of z_0 on $T[x_0, x_{i_1}]$.*

Proof. Suppose that Claim 3.9 is false. Then $N_G[x_0] \cap V(T[x_0, r_{i_1}]) = V(T[x_0, y_0])$ and r_0, r_{i_1} and y_0 are all on the path $T[x_0, x_{i_1}]$. If $y_0 \in V(T(r_0, r_{i_1}])$, then $x_0r_0^+ \in E(G)$, a contradiction to Claim 3.4. If $y_0 \in V(T(x_0, r_0])$, since $G - y_0$ is a connected graph, there exists $u_0 \in V(T(x_0, y_0))$ satisfying that u_0 has a neighbour v_0 in $T - T[x_0, y_0]$. Set $T^* = T - \{u_0u_0^+\} + \{x_0u_0^+\}$. Then T^* is a spanning tree with $L(T^*) = (L(T) - \{x_0\}) \cup \{u_0\}$ that satisfies $(C1)$ and $(C2)$. Noting that $d_{T^*}(u_0, y_0) = d_T(x_0, y_0)$, we have $d_{T^*}(u_0, v_0) > d_T(x_0, y_0)$ and $g(T^*, u_0) > g(T, x_0)$. On the other hand, since $F^* = \emptyset$, we have $N_G(x_j) \cap V(T(x_0, y_0)) = \emptyset$ and $g(T^*, x_j) = g(T, x_j)$ for $1 \le j \le k$. Hence $g(T^*) > g(T)$, contrary to $(C3)$. □

Claim 3.10. *$\{z_0\} \cup U$ is independent in G.*

Proof. Recall that U is independent in G.

First, we prove that $\{z_0\} \cup L(T)$ is independent in G. We have $z_0 \notin N_G(x_0)$ by Claim 3.9. Set $T_a = T - \{z_0z_0^+\} + \{x_0z_0^+\}$. Then T_a is a spanning tree with $L(T_a) = (L(T) - \{x_0\}) \cup \{z_0\}$. By Remark 3.1, z_0 is a leaf of T_a and $L(T_a)$ is independent in G. Hence, $z_0x_i \notin E(G)$ for $i \in [1,k]$.

Next, we show that $\{z_0\} \cup L_1(T) \cup L_2(T)$ is independent in G. Set $T_b = T - \{z_0z_0^+, z_iz_i^+\} + \{x_0z_0^+, z_ix_i\}$ for $z_i \in L_1'(T) \cup L_2'(T)$. Since $F^* = \emptyset$ and $z_i \in V(T(x_0, r_i))$, T_b is a spanning tree with $L(T_b) \subseteq (L(T) - \{x_0, x_i\}) \cup \{z_0, z_i^+\}$. By Remark 3.1, both z_0 and z_i^+ are leaves of T_b and $L(T_b)$ is independent. Hence, $z_0z_i^+ \notin E(G)$ for $z_i \in L_1'(T) \cup L_2'(T)$.

Therefore, $\{z_0\} \cup U$ is independent in G. □

By Claim 3.10, we have $\alpha(G) \ge |\{z_0\} \cup U| \ge k + 1 + \lceil \frac{k}{r-3} \rceil + 1$, contrary to Claim 3.8. Hence Theorem 1.8 holds for Case 1.

Case 2 $F^* \ne \emptyset$.

Choose $z_j \in V(T(x_0, y_0))$ such that $d_T(x_0, z_j)$ is as large as possible for $z_j \in F^*$. Denote the successor of z_j on $T(x_0, x_{j_1})$ and $T(x_0, x_{i_1})$ by z_j^+ and z_j^*, respectively. By Claim 3.3, we have $r_{i_1 j_1} \in V(T[z_j, r_{i_1}])$.

Claim 3.11. *$z_j{}^+, z_j^* \notin N_G(x_0)$ and there exists $u_0 \in V(T(z_j, y_0))$ (possible $u_0 = z_j^*$) satisfying that $u_0 \notin N_G(x_0)$ and $V(T[u_0^+, y_0]) \subseteq N_G(x_0)$.*

Proof. If $x_0 z_j^+ \in E(G)$, then $T_a = T - \{z_j z_j^+, r_{j_1} r_{j_1}^+\} + \{x_0 z_j^+, z_j x_{j_1}\}$ with $L(T_a) = (L(T) - \{x_0, x_{j_1}\}) \cup \{r_{j_1}^+\}$, contrary to $(C1)$. Then $z_j^+ \notin N_G(x_0)$. If $r_{i_1 j_1} \neq z_j$, then $z_j^+ = z_j^*$ and thus $z_j^* \notin N_G(x_0)$. If $r_{i_1 j_1} = z_j$, then $T_b = T - \{z_j z_j^*\} + \{x_0 z_j^*\}$ with $L(T_b) = L(T) - \{x_0\}$, contrary to $(C1)$. So $z_j^* \notin N_G(x_0)$. Therefore, there exists $u_0 \in V(T(z_j, y_0))$ (possible $u_0 = z_j^*$) satisfying that $u_0 \notin N_G(x_0)$ and $V(T[u_0^+, y_0]) \subseteq N_G(x_0)$. □

Set $L^*(T) = (L(T) - \{x_{j_1}\}) \cup \{r_{j_1}^+\}$ and $U^* = L^*(T) \cup L_1(T) \cup L_2(T)$.

Claim 3.12. *U^* is independent in G.*

Proof. Note that U is independent in G.

First, we show that $L^*(T)$ is independent. Set $T_a = T - \{r_{j_1} r_{j_1}^+\} + \{z_j x_{j_1}\}$. Then T_a is a spanning tree with $L(T_a) = (L(T) - \{x_{j_1}\}) \cup \{r_{j_1}^+\}$. By Remark 3.1, $L(T_a)$ is independent. Hence, $r_{j_1}^+ x_h \notin E(G)$ for $h \in [0, k] - \{j_1\}$.

Next, we prove that $\{r_{j_1}^+\} \cup (L_1(T) \cup L_2(T) - \{z_j^+\})$ is independent in G. Set $T_b = T - \{r_{j_1} r_{j_1}^+, z_h z_h^+\} + \{z_j x_{j_1}, z_h x_h\}$ for $z_h \in L_1'(T) \cup L_2'(T) - \{z_j\}$. Then by Claim 3.3 and the maximality of $d_T(x_0, z_j)$, T_b is a spanning tree with $L(T_b) \subseteq (L(T) - \{x_{j_1}, x_h\}) \cup \{r_{j_1}^+, z_h^+\}$. By Remark 3.1, both $r_{j_1}^+$ and z_h^+ are leaves of T_b and $L(T_b)$ is independent in G. Hence, $r_{j_1}^+ z_h^+ \notin E(G)$ for $z_h \in L_1'(T) \cup L_2'(T) - \{z_j\}$.

At last, we may consider that $z_j^+ r_{j_1}^+ \notin E(G)$. In fact, if $z_j^+ r_{j_1}^+ \in E(G)$, then $T_c = T - \{r_{j_1} r_{j_1}^+, z_j z_j^+\} + \{z_j x_{j_1}, z_j^+ r_{j_1}^+\}$ with $L(T_c) = L(T) - \{x_{j_1}\}$, contrary to $(C1)$.

Therefore, U^* is independent in G. □

Claim 3.13. *$r_{i_1 j_1} \notin T[r_0, y_0)$.*

Proof. Assume that $r_{i_1 j_1} \in T[r_0, y_0)$. This together with Claim 3.2 and $r_{i_1 j_1} \in V(T[z_j, r_{i_1}])$ implies that $r_{i_1 j_1} \in V(T[z_j, y_0))$. Let u_0 be the vertex in Claim 3.11, we have $V(T[u_0^+, y_0]) \subseteq N_G(x_0)$. By Claim 3.4, $u_0 \in V(T(r_{i_1 j_1}, y_0))$. Then it follows that $r_{i_1 j_1} \in V(T[z_j, u_0))$. Thus $u_0 \notin N_G(z_j^+)$. Otherwise, if $r_{i_1 j_1} \in V(T(z_j, u_0))$, then $T' = T - \{z_j z_j^+, u_0 u_0^+, r_{i_1 j_1} r_{i_1 j_1}^+\} + \{x_0 u_0^+, u_0 z_j^+, x_{j_1} z_j\}$ is a spanning tree with $L(T') \subseteq L(T) - \{x_0, x_{j_1}\} + \{r_{i_1 j_1}^+\}$, contrary to $(C1)$. If $r_{i_1 j_1} = z_j$, then $T'' = T - \{z_j z_j^+, u_0 u_0^+\} + \{x_0 u_0^+, u_0 z_j^+\}$ is a spanning tree with $L(T'') \subseteq L(T) - \{x_0\}$, contrary to $(C1)$.

Now we show that $\{u_0\} \cup U^*$ or $\{r_{i_1 j_1}^+\} \cup U^*$ is independent in G.

Note that U^* is independent. Assume that $w \in \{u_0, r_{i_1 j_1}^+\}$.

First, we show that $\{w\} \cup L^*(T)$ is independent. Set

$$T_a := \begin{cases} T - \left\{u_0 u_0^+, r_{j_1} r_{j_1}^+\right\} + \left\{x_0 u_0^+, z_j x_{j_1}\right\}, & \text{if } w = u_0; \\ T - \left\{r_{i_1 j_1} r_{i_1 j_1}^+, r_{j_1} r_{j_1}^+\right\} + \left\{x_0 u_0^+, z_j x_{j_1}\right\}, & \text{if } w = r_{i_1 j_1}^+. \end{cases}$$

Then by Claim 3.3, T_a is a spanning tree with $L(T_a) \subseteq (L(T) - \{x_0, x_{j_1}\}) \cup \{w, r_{j_1}^+\}$. By Remark 3.1, both w and $r_{j_1}^+$ are leaves of T_a and $L(T_a)$ is independent in G. By Claims 3.4 and 3.11, $w \notin N_G(x_0)$. Hence, $w r_{j_1}^+ \notin E(G)$ and $w x_h \notin E(G)$ for $h \in [0, k] - \{j_1\}$.

Next, we prove that $\{u_0\} \cup L_1(T) \cup L_2(T)$ or $\{r_{i_1 j_1}^+\} \cup L_1(T) \cup L_2(T)$ is independent in G. Note that $w \notin N_G(z_j^+)$. For $z_h \in L_1'(T) \cup L_2'(T) - \{z_j\}$, we set

$$T_b := \begin{cases} T - \left\{u_0 u_0^+, z_h z_h^+\right\} + \left\{x_0 u_0^+, z_h x_h\right\}, & \text{if } z_h^+ \notin F^*; \\ T - \left\{r_{i_1 j_1} r_{i_1 j_1}^+, z_j z_j^+, z_h z_h^+\right\} + \left\{z_h^+ r_{i_1 j_1}^+, z_h x_{h_1}, z_j x_{j_1}\right\}, & \text{if } z_h^+ \in F^* \text{ and } r_{i_1 j_1}^+ z_h^+ \in E(G). \end{cases}$$

- If $z_h^+ \notin F^*$, then by Claim 3.3, T_b is a spanning tree with $L(T_b) \subseteq (L(T) - \{x_0, x_h\}) \cup \{u_0, z_h^+\}$. By Remark 3.1, both u_0 and z_h^+ are leaves of T_b and $L(T_b)$ is independent. Hence, $u_0 z_h^+ \notin E(G)$ for $z_h \in L_1'(T) \cup L_2'(T)$.
- If $z_h^+ \in F^*$ and $r_{i_1 j_1}^+ z_h^+ \in E(G)$, then by Claim 3.3, T_b is a spanning tree of G with $L(T_b) = (L(T) - \{x_{h_1}, x_{j_1}\}) \cup \{z_j^+\}$, contrary to $(C1)$. Thus $r_{i_1 j_1}^+ z_h^+ \notin E(G)$. Hence, $r_{i_1 j_1}^+ z_h^+ \notin E(G)$ for $z_h \in L_1'(T) \cup L_2'(T)$.

Therefore, $\{u_0\} \cup U^*$ or $\{r_{i_1 j_1}^+\} \cup U^*$ is independent in G. Thus $\alpha(G) \geq |U^*| + 1 \geq k + 1 + \lceil \frac{k}{r-3} \rceil + 1$, contrary to Claim 3.8. This proves Claim 3.13. □

By Claim 3.13, $r_{i_1 j_1} \in V(T[y_0, r_{i_1}]) \cup V(T[y_0, r_{j_1}])$. Without loss of generality, assume that $r_{i_1 j_1} \in V(T[y_0, r_{j_1}])$.

Claim 3.14. *One of the following two statements holds.*

(i) *$u_0 \notin N_G(z_j^+)$ or there exists $w_0 \in V(T(z_j^+, u_0))$ satisfying that $w_0 \notin N_G(z_j^+)$ and $V(T[w_0^+, u_0]) \subseteq N_G(z_j^+)$;*
(ii) *$u_0 = z_j^+$ or $V(T[z_j^+, u_0]) \subseteq N_G[z_j^+]$.*

Proof. Suppose that Claim 3.14(ii) is false. Then $|V(T[z_j^+, u_0])| \geq 3$ and $V(T[z_j^+, u_0]) \nsubseteq N_G[z_j^+]$. If $u_0 \in N_G(z_j^+)$, then since $V(T[z_j^+, u_0]) \nsubseteq N_G[z_j^+]$, there is $w_0 \in V(T(z_j^+, u_0))$ satisfying that $w_0 \notin N_G(z_j^+)$ and $V(T[w_0^+, u_0]) \subseteq N_G(z_j^+)$. □

Subcase 2.1 Claim 3.14*(i) holds.*

In this subcase, $w_0 \notin N_G(z_j^+) \cup N_G(x_0)$. In fact, suppose that $x_0w_0 \in E(G)$. Then by Claim 3.3, $T^* = T - \{r_{j_1}r_{j_1}^+, w_0w_0^+, z_jz_j^+\} + \{z_jx_{j_1}, x_0w_0, z_j^+w_0^+\}$ is a spanning tree with $L(T^*) = (L(T) - \{x_0, x_{j_1}\}) \cup \{r_{j_1}^+\}$, contrary to $(C1)$.

Claim 3.15. *If $u_0 \notin N_G(z_j^+)$, then $\{u_0\} \cup U^*$ is independent in G.*

Proof. Note that U^* is independent in G.

First, we show that $\{u_0\} \cup L^*(T)$ is independent in G. Set $T_a = T - \{u_0u_0^+, r_{j_1}r_{j_1}^+\} + \{x_0u_0^+, z_jx_{j_1}\}$. Then by Claim 3.3, T_a is a spanning tree of G with $L(T_a) \subseteq (L(T) - \{x_0, x_{j_1}\}) \cup \{u_0, r_{j_1}^+\}$. By Remark 3.1, both u_0 and $r_{j_1}^+$ are leaves of T_a and $L(T_a)$ is an independent set. By Claim 3.11, $u_0 \notin N_G(x_0)$. Hence, $u_0r_{j_1}^+ \notin E(G)$ and $u_0x_h \notin E(G)$ for $h \in [0, k] - \{j_1\}$.

Next, we prove that $\{u_0\} \cup L_1(T) \cup L_2(T)$ is independent in G. Note that $u_0 \notin N_G(z_j^+)$. For $z_h \in L_1'(T) \cup L_2'(T) - \{z_j\}$, we set

$$T_b := \begin{cases} T - \left\{u_0u_0^+, z_hz_h^+\right\} + \left\{x_0u_0^+, z_hx_{h_1}\right\}, & \text{if } z_h^+ \notin F^*; \\ T - \left\{u_0u_0^+, z_hz_h^+, r_hr_h^+\right\} + \left\{x_0u_0^+, z_hx_{h_1}, x_{j_1}z_j\right\}, & \text{if } z_h^+ \in F^*. \end{cases}$$

Then by Claim 3.3, T_b is a spanning tree of G with

$$L(T_b) \subseteq \begin{cases} (L(T) - \{x_0, x_h\}) \cup \left\{u_0, z_h^+\right\}, & \text{if } z_h^+ \notin F^*; \\ \left(L(T) - \left\{x_0, x_{h_1}, x_{j_1}\right\}\right) \cup \left\{u_0, z_h^+, r_h^+\right\}, & \text{if } z_h^+ \in F^*. \end{cases}$$

By Remark 3.1, both u_0 and z_h^+ are leaves of T_b and $L(T_b)$ is independent in G. Hence, $u_0z_h^+ \notin E(G)$ for $z_h \in L_1'(T) \cup L_2'(T)$.

Therefore, $\{u_0\} \cup U^*$ is independent in G. $\square$

Claim 3.16. *If $u_0 \in N_G(z_j^+)$, then $\{w_0\} \cup U^*$ is independent in G.*

Proof. Note that U^* is independent in G.

First, we show that $\{w_0\} \cup L^*(T)$ is independent in G. Set $T_a = T - \{w_0w_0^+, z_jz_j^+, r_{j_1}r_{j_1}^+\} + \{x_0u_0^+, z_j^+w_0^+, z_jx_{j_1}\}$. Then by Claim 3.3, T_a is a spanning tree of with $L(T_a) \subseteq (L(T) - \{x_0, x_{j_1}\}) \cup \{w_0, r_{j_1}^+\}$. By Remark 3.1, w_0 and $r_{j_1}^+$ are two leaves of T_a and $L(T_a)$ is independent in G. If $x_0w_0 \in E(G)$, then by Claim 3.3, $T^* = T - \{r_{j_1}r_{j_1}^+, w_0w_0^+, z_jz_j^+\} + \{z_jx_{j_1}, x_0w_0, z_j^+w_0^+\}$ is a spanning tree with $L(T^*) = (L(T) - \{x_0, x_{j_1}\}) \cup \{r_{j_1}^+\}$, contrary to (C1). Thus $w_0 \notin N_G(x_0)$. Hence, $w_0r_{j_1}^+ \notin E(G)$ and $w_0x_h \notin E(G)$ for $h \in [0, t] - \{j_1\}$.

Next, we prove that $\{w_0\} \cup L_1(T) \cup L_2(T)$ is independent in G. Note that $w_0 \notin N_G(z_j^+)$. For $z_h \in L_1'(T) \cup L_2'(T) - \{z_j\}$, we set

$$T_b := \begin{cases} T - \left\{w_0w_0^+, z_jz_j^+, z_hz_h^+\right\} + \left\{z_j^+w_0^+, z_hx_{h_1}, z_jx_{j_1}\right\}, & \text{if } z_h^+ \notin F^*; \\ T - \left\{w_0w_0^+, z_jz_j^+, z_hz_h^+, r_hr_h^+\right\} + \left\{z_j^+w_0^+, x_0u_0^+, z_hx_{h_1}, z_jx_{j_1}\right\}, & \text{if } z_h^+ \in F^*. \end{cases}$$

Then by Claim 3.3, T_b is a spanning tree with

$$L(T_b) \subseteq \begin{cases} \left(L(T) - \left\{x_{h_1}, x_{j_1}\right\}\right) \cup \left\{w_0, z_h^+\right\}, & \text{if } z_h^+ \notin F^*; \\ \left(L(T) - \left\{x_0, x_{h_1}, x_{j_1}\right\}\right) \cup \left\{w_0, z_h^+, r_h^+\right\}, & \text{if } z_h^+ \in F^*. \end{cases}$$

By Remark 3.1, both w_0 and z_h^+ are leaves of T_b and $L(T_b)$ is independent in G. Hence, $w_0z_h^+ \notin E(G)$ for $z_h \in L_1'(T) \cup L_2'(T)$.

Therefore, $\{w_0\} \cup U^*$ is independent in G. $\square$

Subcase 2.2 *Claim 3.14(ii) holds and $y_0 \neq r_{j_h}$ for some $1 \leq h \leq r - 3$.*

Claim 3.17. *$deg_T(x) = 2$ for any $x \in V(T[z_j^+, y_0^-])$.*

Proof. Suppose that $deg_T(x) \geq 3$ for some $x \in V(T[z_j^+, y_0^-])$. Denote the successor of x on $T[x_0, y_0]$ by x^+. If $x \in V(T[u_0, y_0^-])$, then $x^+ \in N_G(x_0)$. Set $T^* = T - \{xx^+\} + \{x_0x^+\}$. Then T^* is a spanning tree with $L(T^*) = L(T) - \{x_0\}$, contrary to (C1). If $x \in V(T[z_j^+, u_0^-])$, then if $|V(T[z_j^+, u_0])| = 2$, then $x = z_j^+$. Set $T' = T - \{z_jz_j^+, r_{j_1}r_{j_1}^+\} + \{x_0u_0^+, z_jx_{j_1}\}$. Then T' is a spanning tree with $L(T') = (L(T) - \{x_0, x_{j_1}\}) \cup \{r_{j_1}^+\}$, contrary to (C1). If $|V(T[z_j^+, u_0])| \geq 3$, then set $T'' = T - \{xx^+, z_jz_j^+, r_{j_1}r_{j_1}^+\} + \{x_0u_0^+, z_jx_{j_1}, z_j^+x^+\}$. Then T'' is a spanning tree with $L(T'') = (L(T) - \{x_0, x_{j_1}\}) \cup \{r_{j_1}^+\}$, contrary to (C1). $\square$

By Claims 3.3 and 3.17, $V(T[x_0, r_{i_1j_1}]) \supseteq V(T[x_0, y_0])$. Denote the successor of y_0 on $T[x_0, x_{j_1}]$ by y_0^+.

Claim 3.18. *$\{{y_0}^+\} \cup U^*$ is independent in G.*

Proof. Note that U^* is independent in G.

First, we show that $\{{y_0}^+\} \cup L^*(T)$ is independent in G. Set $T_a = T - \{y_0{y_0}^+, z_jz_j^+\} + \{x_0y_0, z_jx_{j_1}\}$. Then by Claim 3.3, T_a is a spanning tree with $L(T_a) \subseteq (L(T) - \{x_0, x_{j_1}\}) \cup \{y_0^+, z_j^+\}$. By Remark 3.1, both y_0^+ and z_j^+ are two leaves of T_a and $L(T_a)$ is independent in G. So ${y_0}^+x_h \notin E(G)$ for $h \in [1, t] - \{j_1\}$. By the choice of y_0, ${y_0}^+x_0 \notin E(G)$. If ${y_0}^+r_{j_1}^+ \in E(G)$, then set $T_b = T_a - \{r_{j_1}r_{j_1}^+\} + \{{y_0}^+r_{j_1}^+\}$. Thus $L(T_b) = (L(T) - \{x_0, x_{j_1}\}) \cup \{z_j^+\}$, contrary to (C1).

Next, we prove that $\{y_0{}^+\} \cup L_1(T) \cup L_2(T)$ is independent. If $y_0^+ = z_h^+$ for some $z_h \in L_1'(T) \cup L_2'(T) - \{z_j\}$, then we need to consider the graph $G[y_0, x_0, y_0^-, y_0^+, x_{h_1}, \ldots, x_{h_{r-3}}]$. Since G is $K_{1,r}$-free, we have $y_0^+ y_0^- \in E(G)$ or $x_0^+ y_0^- \in E(G)$ or $y_0^+ x_{h_l} \in E(G)$ for some $1 \le l \le r-3$.

- $y_0^+ y_0^- \in E(G)$. Then $T' = T - \{y_0 y_0^+, y_0 y_0^-, r_{h_1} r_{h_1}^+\} + \{x_0 y_0, y_0^+ y_0^-, y_0 x_{h_1}\}$ is a spanning tree with $L(T') = (L(T) - \{x_0, x_{h_1}\}) \cup \{r_{h_1}^+\}$, contrary to $(C1)$.
- $x_0 y_0^- \in E(G)$. Then $T'' = T - \{y_0 y_0^-, z_j z_j^+, r_{h_1} r_{h_1}^+\} + \{x_0 y_0^-, z_j x_{j_1}, y_0 x_{h_1}\}$ is a spanning tree with $L(T'') = (L(T) - \{x_0, x_{j_1}, x_{h_1}\}) \cup \{z_j^+, r_{h_1}^+\}$, contrary to $(C1)$.
- $y_0^+ x_{h_l} \in E(G)$ for some $1 \le l \le r-3$. Then $T''' = T - \{y_0 y_0^+, z_j z_j^+, r_{h_l} r_{h_l}^+\} + \{x_0 y_0, z_j x_{j_l}, y_0^+ x_{h_l}\}$ is a spanning tree with $L(T''') = (L(T) - \{x_0, x_{j_l}, x_{h_l}\}) \cup \{z_j^+, r_{h_l}^+\}$, contrary to $(C1)$.

So $y_0^+ \neq z_h^+$ for any $z_h \in L_1'(T) \cup L_2'(T) - \{z_j\}$. Set $T_c = T_a - \{z_h z_h^+\} + \{z_h x_{h_1}\}$ for $z_h \in L_1'(T) \cup L_2'(T) - \{z_j\}$. Then T_c is a spanning tree with $L(T_c) \subseteq (L(T) - \{x_0, x_{j_1}, x_h\}) \cup \{y_0^+, z_j^+, z_h^+\}$. By Remark 3.1, y_0^+, z_j^+ and z_h^+ are leaves of T_c and $L(T_c)$ is independent in G. Hence, $y_0^+ z_h^+ \notin E(G)$ for $z_h \in L_1'(T) \cup L_2'(T)$.

Therefore, $\{y_0{}^+\} \cup U^*$ is independent in G. □

Subcase 2.3 Claim 3.14*(ii) holds and* $y_0 = r_{j_h}$ *for any* $1 \le h \le r-3$.

Claim 3.19. $\{r_{j_1}^+, \ldots, r_{j_{r-3}}^+, x_0, y_0^-\}$ *is an independent set and* $deg_T(y_0) = r-2$.

Proof. We first show that $\{r_{j_1}^+, \ldots, r_{j_{r-3}}^+, x_0\}$ is independent in G. By Claim 3.4, $\{r_{j_s}^+, x_0\}$ is an independent set for $1 \le s \le r-3$. If $r-3 = 1$, then $\{r_{i_1}^+, x_0\}$ is independent in G. If $r-3 \ge 2$, set $T^* = T - \{y_0 r_{j_p}^+, y_0 r_{j_q}^+\} + \{z_j x_{j_p}, z_j x_{j_q}\}$ for $1 \le p \ne q \le r-3$. Then by Claim 3.3, T^* is a spanning tree with $L(T^*) = (L(T) - \{x_{j_p}, x_{j_q}\}) \cup \{r_{j_p}^+, r_{j_q}^+\}$. By Remark 3.1, $L(T^*)$ is independent in G. Hence, $r_{j_p}^+ r_{j_q}^+ \notin E(G)$.

Next, if $x_0 y_0^- \in E(G)$, then by Claim 3.3, $T' = T - \{y_0^- y_0, z_j z_j^+\} + \{x_0 y_0^-, z_j x_{j_1}\}$ is a spanning tree with $L(T') = (L(T) - \{x_0, x_{j_1}\}) \cup \{z_j^+\}$, contrary to $(C1)$; if $y_0^- r_{j_s}^+ \in E(G)$ for some $1 \le s \le r-3$, then by Claim 3.3, $T'' = T - \{y_0^- y_0, y_0 r_{j_s}^+\} + \{x_0 y_0, r_{j_s}^+ y_0^-\}$ is a spanning tree with $L(T'') = (L(T) - \{x_0, x_{j_s}\}) \cup \{r_{j_s}^+\}$, contrary to $(C1)$. Therefore, $\{r_{j_1}^+, \ldots, r_{j_{r-3}}^+, x_0, y_0^-\}$ is independent in G.

Now we prove that $deg_T(y_0) = r-2$. Assume that $deg_T(y_0) \ge r-1$ and $y_0 x \in E(T)$ for $x \notin \{r_{j_1}^+, \ldots, r_{j_{r-3}}^+, y_0^-\}$. Since G is $K_{1,r}$-free and $\{y_0, r_{j_1}^+, \ldots, r_{j_{r-3}}^+, x_0, y_0^-\}$ is an induced $K_{1,r-1}$, we have $xy \in E(G)$ for some $y \in \{r_{j_1}^+, \ldots, r_{j_{r-3}}^+, x_0, y_0^-\}$. By Claim 3.4, $x_0 x \notin E(G)$. If $x r_{j_s}^+ \in E(G)$ for some $1 \le s \le r-3$, then $T_a = T - \{y_0 r_{j_s}^+, y_0 x\} + \{x r_{j_s}^+, z_j x_{j_s}\}$ is a spanning tree with $L(T_a) = L(T) - \{x_{j_s}\}$, contrary to $(C1)$; if $x y_0^- \in E(G)$, then $T_b = T - \{y_0^- y_0, y_0 x\} + \{x_0 y_0, x y_0^-\}$ is a spanning tree with $L(T_b) = L(T) - \{x_0\}$, contrary to $(C1)$. □

By Claim 3.19, we have $deg_T(y_0) = r-2$. Let T_f be a connected component $T - z_j^+$ such that $z_j \in V(T_f)$. Then by Claim 3.17, $B(T) - \{y_0\} = B(T_f)$. Denote $B^* = B(T_f) - \{z_j\}$. Then $T^* = T - \{y_0 r_{j_1}^+, \ldots, y_0 r_{j_{r-3}}^+, z_j z_j^+\} + \{x_{j_1} z_j, \ldots, x_{j_{r-3}} z_j, x_0 y_0\}$ is a spanning tree with $|L(T^*)| = |L(T)|$. Assume that $d_T(x_0, z_j) = a$ and $d_T(z_j^+, y_0) = b$. Note that $deg_{T^*}(z) = deg_T(z)$, $d_{T^*}(z_j^+, z) = d_T(x_0, z) + b + 1$ for any $z \in B^*$ and $deg_{T^*}(y_0) = 2$, $deg_T(y_0) = r-2$, $d_{T^*}(z_j^+, y_0) = b$, $d_T(x_0, y_0) = a+b+1$ and $deg_{T^*}(z_j) = deg_T(z_j) + r - 4$, $d_{T^*}(z_j^+, z_j) = a+b+1$. Hence,

$$
\begin{aligned}
f(T^*, z_j^+) - f(T, x_0) &= \sum_{z \in I(T^*)} (deg_{T^*}(z) - 2) d_{T^*}(z_j^+, z) - \sum_{z \in I(T)} (deg_T(z) - 2) d_T(x_0, z) \\
&= \{\sum_{z \in B^*} (deg_{T^*}(z) - 2) d_{T^*}(z_j^+, z) + \sum_{z \in \{z_j, y_0\}} (deg_{T^*}(z) - 2) d_{T^*}(z_j^+, z)\} \\
&\quad - \{\sum_{z \in B^*} (deg_T(z) - 2) d_T(x_0, z) + \sum_{z \in \{z_j, y_0\}} (deg_T(z) - 2) d_T(x_0, z)\} \\
&= \sum_{z \in B^*} (deg_T(z) - 2)(b+1) + (deg_T(z_j) + r - 4 - 2)(a+b+1) \\
&\quad - \{(r-2-2)(a+b+1) + (deg_T(z_j) - 2)a\} \\
&= \sum_{z \in B^*} (deg_T(z) - 2)(b+1) + (deg_T(z_j) - 2)(b+1) \\
&= \sum_{z \in B^* \cup \{z_j\}} (deg_T(z) - 2)(b+1)
\end{aligned}
$$

This together with $b+1 > 0$ and $(C2)$ implies that $\sum_{z \in B^* \cup \{z_j\}} (deg_T(z) - 2) \le 0$. Thus $deg_T(z) = 2$ for $z \in B^* \cup \{z_j\}$. By Claim 3.17, we have $B(T) = \{y_0\}$. In this subcase, $k = r-3$ and thus, $p = \frac{k}{r-3} = 1$, contrary to Claim 3.8.

By Claims 3.15, 3.16 and 3.18, we have $\alpha(G) \ge |U^*| + 1 \ge k + 1 + \lceil \frac{k}{r-3} \rceil + 1$, contrary to Claim 3.8. This completes the proof of Case 2.

4. Proof of Theorem 1.11

We define (G_1, G_2, x) a *separation* of a connected graph G if G can be decomposed into two nonempty connected subgraphs G_1 and G_2 with $V(G_1) \cap V(G_2) = \{x\}$. We call a path P an x-path if P has an end vertex x. An (x, Y)-path is a path starting at x and ending at a vertex of Y, where the internal vertices are not in $\{x\} \cup Y$. An (x, Y, t)-fan is a set of t internally disjoint (x, Y)-paths with distinct terminal vertices in Y.

Lemma 4.1. *Let G be a connected $K_{1,4}$-free graph and (G_1, G_2, x) be a separation of G. If G_i is a block and $\alpha(G_i) \le 3$, then G_i has a Hamiltonian x-path for $i = 1, 2$.*

Proof. For convenience, we can only take G_1 into consideration. Assume that G_1 has no Hamiltonian x-path. Choose an x-path P in G_1 such that

$(C4)$ P is as long as possible.

Suppose that x and y are the end vertices of P. Obviously, $N_{G_1}(y) \subseteq V(P)$ as $(C4)$ and G_1 has no x-claw as G being $K_{1,4}$-free. We set a direction from x to y in P. Since P is not a hamiltonian x-path and G_1 is 2-connected, there exists a $(z, P, 2)$-fan such that zQ_1u_1 and zQ_2u_2 are two disjoint (z, P) paths, where $z \in V(G_1 - P)$ and $u_1, u_2 \in V(P)$. Let y_0 be a neighbour of y in G_1 such that $d_P(y, y_0) = \max\limits_{v \in N_{G_1}(y)} d_P(y, v)$. Obviously, $y \ne u_2$.

By the choice of $(C4)$ and y_0, it is easy for us to check the following claim. $\square$

Claim 4.2.

(1) $d(u_1, u_2) \ge 2$;
(2) $\{z, {u_1}^+, {u_2}^+\}$ *and* $\{z, {u_1}^+, y\}$ *are two independent sets;*
(3) *if* ${u_1}^-$ *exists, then* $\{z, {u_1}^-, {u_2}^-\}$ *is also an independent set;*
(4) ${u_1}^-, {u_1}^+, {u_2}^- \notin N_{G_1}(y)$.

Next, we will consider two assumptions:

We first assume that $x = u_1$. By Claim 4.2, $\{x^+, z, y\}$ is independent. Since G_1 has no x-claw, we have $x \notin N_{G_1}(y)$. Note that $y_0 \ne x^+, {u_2}^-$ and $\delta(G) \ge 2$.

If $y_0 \in V(x^{++}P{u_2}^{--})$, then $\{y_0^+, z, x^+, {u_2}^+\}$ is an independent set. In fact, we set

$$P' = \begin{cases} xPy_0y\overleftarrow{P}y_0^+z & \text{if } zy_0^+ \in E(G_1); \\ xQ_1zQ_2u_2Pyy_0\overleftarrow{P}x^+y_0^+P{u_2}^- & \text{if } x^+y_0^+ \in E(G_1); \\ xPy_0y\overleftarrow{P}u_2^+y_0^+Pu_2Q_2z & \text{if } {u_2}^+y_0^+ \in E(G_1). \end{cases}$$

Then P' is an x-path in G_1 with $|V(P')| > |V(P)|$, which contradicts $(C4)$. By Claim 4.2(1), $\{z, x^+, {u_2}^+\}$ is independent. Hence, $\{y_0^+, z, x^+, {u_2}^+\}$ is independent in G_1, a contradiction to $\alpha(G_1) \le 3$.

If $y_0 \in V(u_2Py)$, then we can utilize the similar discussion to Claim 3.5 in Theorem 1.8 to find $z_0 \in V(y_0Py)$ such that $z \in N_{G_1}(y)$ for all $z \in V(y_0Pz_0)$ and $y{z_0}^+ \notin E(G_1)$. Set

$$P'' = \begin{cases} xQ_1zQ_2u_2Pz_0y\overleftarrow{P}{z_0}^+x^+P{u_2}^- & \text{if } x^+{z_0}^+ \in E(G_1); \\ xPz_0y\overleftarrow{P}{z_0}^+z & \text{if } z{z_0}^+ \in E(G_1). \end{cases}$$

Then P'' is an x-path in G_1 with $|V(P'')| > |V(P)|$, which contradicts $(C4)$. Note that $\{x^+, z, y\}$ is independent. Hence, $\{x^+, z, {z_0}^+, y\}$ is independent in G_1, a contradiction to $\alpha(G_1) \le 3$.

We now assume that $x \ne u_1$. By Claim 4.2(4), ${u_1}^-, {u_1}^+, {u_2}^- \notin N_{G_1}(y)$.

If $y_0 \in V(xP{u_1}^{--})$, then $\{y_0^+, z, {u_1}^+, {u_2}^+\}$ is an independent set. In fact, if $y_0^+{u_1}^+ \in E(G_1)$, then $P' = xPy_0y\overleftarrow{P}{u_1}^+y_0^+Pu_1Q_1z$ is an x-path in G_1, which contradicts $(C4)$. By the similar discussion as above, we have $y_0^+{u_2}^+, y_0^+z \notin E(G_1)$. Note that $\{z, {u_1}^+, {u_2}^+\}$ is an independent set by Claim 4.2(1). Hence, $\{y_0^+, z, {u_1}^+, {u_2}^+\}$ is an independent set, a contradiction to $\alpha(G_1) \le 3$.

If $y_0 \in V(u_1P{u_2}^{--})$, then we can easily see that $\{y_0^+, z, {u_1}^-, {u_2}^+\}$ is an independent set, a contradiction to $\alpha(G_1) \le 3$; if $y_0 \in V(u_2Py)$, then it is easy to check that $\{y_0^+, z, {u_1}^-, {u_2}^-\}$ is an independent set, a contradiction to $\alpha(G_1) \le 3$.

Hence, G_1 has a Hamiltonian x-path. With the similar argument in G_1, G_2 also has a Hamiltonian x-path. Then Lemma 4.1 holds.

Proof of Theorem 1.11.. If G is 2-connected, then the result holds by Corollary 1.10. If G is not 2-connected, suppose that $\alpha(B) \ge 3$ for every block B in G and G is a minimal counterexample to Theorem 1.11. Let x be a cut vertex in G and (G_1, G_2, x) be a separation of G. Obviously, $\alpha(G_1) + \alpha(G_2) - 1 \le \alpha(G) \le \alpha(G_1) + \alpha(G_2)$ and $\alpha(G_i) \ge 3$.

Case 1 *$\alpha(G_1) > 5$ and $\alpha(G_2) > 5$.*

Let k_i be an integer such that $k_i = \lfloor \frac{\alpha(G_i) - 4}{2} \rfloor$ for $i = 1, 2$. Then $k_i \ge 1$.

On one hand, $2k + 5 \ge \alpha(G) \ge \alpha(G_1) + \alpha(G_2) - 1 \ge 2(k_1 + k_2 + 1) + 5$. Hence, $k_1 + k_2 + 1 \le k$. G_i satisfies the condition in Theorem 1.11 and the independence number of every block in G_i is also no less than 3. On the other hand, since G

is a minimal counterexample to Theorem 1.11, G_i has a spanning tree with at most k_i branch vertices. Then $|B(T_1 \cup T_2)| \le |B(T_1) \cup B(T_2) \cup \{x\}| \le |B(T_1)| + |B(T_2)| + 1 \le k_1 + k_2 + 1$.

Hence, $T_1 \cup T_2$ is a spanning tree of G with at most k branch vertices, a contradiction with G being a counterexample.

Case 2 $\alpha(G_1) > 5$ *and* $3 \le \alpha(G_2) \le 5$.

Let k_1 be an integer such that $k_1 = \lfloor \frac{\alpha(G_1)-4}{2} \rfloor$ and $k_2 = 0$. Then $k_1 \ge 1$ and $\alpha(G_2) \le 5 = 2k_2 + 5$.

On one hand, $2k + 5 \ge \alpha(G) \ge \alpha(G_1) + \alpha(G_2) - 1 \ge 2k_1 + 4 + 3 - 1$. Hence, $k_1 + 1 \le k$. G_i satisfies the condition in Theorem 1.11 and the independence number of every block in G_i is also no less than 3. On the other hand, since G is a minimal counterexample to Theorem 1.11, G_i has a spanning tree with at most k_i branch vertices. Then $|B(T_1 \cup T_2)| \le |B(T_1) \cup B(T_2) \cup \{x\}| \le |B(T_1)| + |B(T_2)| + 1 \le k_1 + k_2 + 1 = k_1 + 1$.

Therefore, $T_1 \cup T_2$ is a spanning tree of G with at most k branch vertices, a contradiction with G being a counterexample.

Case 3 $3 \le \alpha(G_1) \le 5$ *and* $3 \le \alpha(G_2) \le 5$.

Let $k_i = 0$. Then $\alpha(G_i) \le 5 = 2k_i + 5$.

On one hand, $\alpha(G) \ge \alpha(G_1) + \alpha(G_2) - 1 \ge 5$. G_i satisfies the condition in Theorem 1.11 and the independence number of every block in G_i is also no less than 3. On the other hand, since G is a minimal counterexample to Theorem 1.11, G_i has a spanning tree with at most k_i branch vertices. Then $|B(T_1 \cup T_2)| \le |B(T_1) \cup B(T_2) \cup \{x\}| \le |B(T_1)| + |B(T_2)| + 1 \le k_1 + k_2 + 1 = 1$. In fact, $\alpha(G) \le 5$. Otherwise, $2k + 5 \ge \alpha(G) \ge 6$. That is, $k \ge 1 \ge |B(T_1 \cup T_2)|$. Then $T_1 \cup T_2$ is a spanning tree of G with at most k branch vertices, a contradiction with G being a counterexample.

Therefore, $\alpha(G) = 5$ and $\alpha(G_1) = \alpha(G_2) = 3$. By Lemma 4.1, G_i has a Hamiltonian x-path P_i for $i = 1, 2$. Then $P_1 \cup P_2$ is a Hamiltonian path in G, a contradiction with G being a counterexample. Hence Theorem 1.11 holds. □

Data availability

No data was used for the research described in the article.

Acknowledgments

We are grateful to the anonymous referees for their valuable comments and suggestions of this paper.

References

[1] H. Broersma, H. Tuinstra, Independence trees and hamilton cycles, J. Graph Theory 29 (1998) 227–237.
[2] Y. Chen, G. Chen, Z. Hu, Spanning 3-ended trees in k-connected $k_{1,4}$-free graphs, Sci. China Math. 57 (2014) 1579–1586.
[3] Y. Chen, P.H. Ha, D.D. Hanh, Spanning trees with at most 4 leaves in $k_{1,5}$-free graphs, Discrete Math. 342 (2019) 2342–2349.
[4] V. Chvátal, P. Erdős, A note on Hamiltionian circuits, Discrete Math. 2 (1972) 111–113.
[5] Z. Hu, P. Sun, Spanning 5-ended trees in $k_{1,5}$-free graphs, Bull. Malays. Math. Sci. Soc 43 (2020) 2565–2586.
[6] M. Kano, A. Kyaw, H. Matsuda, K. Ozeki, A. Saito, T. Yamashita, Spanning trees with a bounded number of leaves in a claw-free graph, Ars Combin. 103 (2012) 137–154.
[7] A. Kyaw, Spanning trees with at most k leaves in $k_{1,4}$-free graphs, Discrete Math. 311 (2011) 2135–2142.
[8] O. Ore, Hamilton connected graphs, J. Math. Pures Appl. 42 (1963) 21–27.
[9] K. Ozeki, T. Yamashita, Spanning trees: a survey, Graphs Combin. 27 (2011) 1–26.
[10] S. Win, On a conjecture of las Vergnas concerning certain spanning trees in graphs, Results Math. 2 (1979) 215–224.