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On the coequal values of total chromatic number and chromatic index

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ABSTRACT

The **chromatic index** $\chi'(G)$ of a graph G is the least number of colors assigned to the edges of G such that no two adjacent edges receive the same color. The **total chromatic number** $\chi''(G)$ of a graph G is the least number of colors assigned to the edges and vertices of G such that no two adjacent edges receive the same color, no two adjacent vertices receive the same color and no edge has the same color as its two endpoints. The chromatic index and the total chromatic number are two of few fundamental graph parameters, and their correlation has always been a subject of intensive study in graph theory.

By definition, $\chi'(G) \leq \chi''(G)$ for every graph G . In 1984, Goldberg conjectured that for any multigraph G , if $\chi'(G) \geq \Delta(G) + 3$ then $\chi''(G) = \chi'(G)$. In this paper, we show that Goldberg's conjecture is asymptotically true. More specifically, we prove that for a multigraph G with maximum degree Δ sufficiently large, $\chi''(G) = \chi'(G)$ provided $\chi'(G) \geq \Delta + 10\Delta^{35/36}$. When $\chi'(G) \geq \Delta(G) + 2$, the chromatic index $\chi'(G)$ is completely determined by the fractional chromatic

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index. Consequently, the total chromatic number $\chi''(G)$ can be computed in polynomial-time in this case.

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1. Introduction

By a (multi)graph G , we mean a finite undirected graph without loops, but possibly with multiple edges. A **total-coloring** of a graph G is an assignment of colors to the edges and vertices of G such that no two adjacent edges receive the same color, no two adjacent vertices receive the same color and no edge has the same color as its two endpoints. The **total chromatic number** of a graph G , denoted by $\chi''(G)$, is the least number of colors required for a total-coloring of G . Similarly, **vertex-coloring** and **chromatic number** $\chi(G)$, and **edge-coloring** and **chromatic index** $\chi'(G)$ of a graph G are defined, respectively. Compared with vertex-coloring and edge-coloring, the theory of total-coloring is less studied with fewer results. This may be due to the fact that finding a total-coloring is much more difficult than finding a vertex-coloring or an edge-coloring separately.

Let $\Delta(G)$ and $\mu(G)$ denote the maximum degree and the maximum number of edges between any two distinct vertices of G , respectively. Behzad [2] in 1965 conjectured that $\chi''(G) \leq \Delta(G) + 2$ if G is a simple graph. The best known upper bound for total chromatic number is due to Molloy and Reed [9] who showed that there is a universal constant c such that $\chi''(G) \leq \Delta(G) + c$ for every simple graph G . They provided a proof for $c = 10^{26}$, so long as $\Delta(G) \geq \Delta_0$ for a particular constant Δ_0 .

It is a common belief that the total-coloring is strongly related to the edge-coloring. Vizing [14], and independently, Gupta [7] proved that $\chi'(G) \leq \Delta(G) + \mu(G)$. Vizing [15] in 1968 proposed that $\chi''(G) \leq \Delta(G) + \mu(G) + 1$. A slightly stronger version is that $\chi''(G) \leq \chi'(G) + 1$. (See Stiebitz et al. [13], page 262.) Goldberg [6] in 1984 conjectured that $\chi''(G) = \chi'(G)$ provided $\chi'(G) \geq \Delta(G) + 3$. We are not aware of any nontrivial sufficient conditions such that the total chromatic number and the chromatic index are the same besides the following one due to Cao, Chen and Jing [3]: if $\chi'(G) \geq \max\{\Delta(G) + 2, |V(G)| + 1\}$ then $\chi''(G) = \chi'(G)$. The condition $|V(G)| < \chi'(G)$ may be too strong. In this paper, we confirm Goldberg's conjecture asymptotically as follows.

Theorem 1.1. *Let G be a graph with maximum degree Δ sufficiently large. If $\chi'(G) \geq \Delta + 10\Delta^{35/36}$, then $\chi''(G) = \chi'(G)$.*

For the condition “ Δ sufficiently large”, we actually only need $e^{\Delta^{1/18}} > 4\Delta^3$ and $\Delta \geq 1$ from our proof. In the study of vertex-coloring and edge-coloring, there is an essential and powerful re-coloring tool – Kempe change (vertex version and edge version). However, this tool has yet been developed for total-coloring. By applying the probabilistic method (Chernoff bounds (various forms) and the Local Lemma), some upper bounds of χ'' are

obtained. (See Molloy and Reed [10].) In our proof, we develop a technique to give a total-coloring of “dense graphs” and extend this coloring to the whole graph. We then combine this technique with the probabilistic method to carry out the proof. This hybrid approach may shed some light on tackling other total-coloring problems.

Apart from the maximum degree $\Delta(G)$, there is another trivial lower bound for $\chi'(G)$, called the **density** of G , defined by

$$\omega(G) = \max \left\{ \frac{2|E(H)|}{|H| - 1} : H \subseteq G, |H| \geq 3 \text{ odd} \right\},$$

where $|H| = |V(H)|$ is the order of H . The following result, conjectured independently by Goldberg [5] and Seymour [11] in the 1970s and recently confirmed by Chen, Jing and Zang [4], shows that the density represents the chromatic index in some common cases.

Theorem 1.2. *Let G be a graph. If $\chi'(G) \geq \Delta(G) + 2$, then $\chi'(G) = \lceil \omega(G) \rceil$.*

Denote by $\chi_f(G)$ the **fractional chromatic index** of a graph G . Seymour [11] showed that $\chi_f(G) = \max\{\Delta(G), \omega(G)\}$. Since the fractional chromatic index can be computed in polynomial-time, $\chi'(G)$ can be determined in polynomial-time provided $\chi'(G) \geq \Delta(G) + 2$. Hence, Theorem 1.1 implies that for graphs G with maximum degree Δ sufficiently large and $\chi'(G) \geq \Delta + 10\Delta^{35/36}$, the total chromatic number can be computed in polynomial-time.

The rest of this paper is organized as follows. In Section 2, we will give some notation and terminology, and state some technical results; In Section 3, we prove Theorem 1.1 based on the results in Section 2; And Sections 4 and 5 give the proofs of these preliminary results stated in Section 2.

2. Preliminaries

We generally follow Stiebitz et al. [13] for notation and terminology. For a vertex $v \in V(G)$, denote $N_G(v)$ and $E_G(v)$ the set of vertices adjacent to v and the set of edges incident to v in G , respectively. Clearly, $d_G(v) = |E_G(v)|$ is the degree of v in G . For vertex sets $U, W \subseteq V(G)$, let $E_G(U, W)$ denote the set of all edges of G joining a vertex of U with a vertex of W . When $W = V(G) \setminus U$, we call $E_G(U, W)$ the **boundary** of U in G and denote by $\partial_G(U)$, that is, $\partial_G(U) = E_G(U, V(G) \setminus U)$. Let $d_G(U) = |\partial_G(U)|$. Let $E_G(u, w)$ and $E_G(u, W)$ denote $E_G(\{u\}, \{w\})$ and $E_G(\{u\}, W)$, respectively. For any two disjoint subgraphs F and H of G , we define $E_G(F, H) = E_G(V(F), V(H))$, $\partial_G(F) = \partial_G(V(F))$ and $d_G(F) = d_G(V(F))$. Given a graph property P , we say a subgraph $H \subseteq G$ is a **maximal subgraph with property P** if H is maximal among all subgraphs satisfying property P .

For a total-coloring φ of a graph G , a vertex set $W \subseteq V(G)$ is called **φ -distinct** if the colors assigned to the vertices in W and the edges incident to W are mutually distinct.

Theorem 2.1. Let G be a graph with maximum degree Δ sufficiently large, and let $V_1, \dots, V_m$ be m disjoint independent vertex sets of G . Suppose that G_0 is obtained from G by contracting each V_i to a single vertex. If $\chi'(G_0) \leq \Delta + \lceil 2\Delta^{35/36} \rceil$ and $|V_i| \leq \Delta^{1/36}$ for $i \in \{1, \dots, m\}$, then G has a total-coloring φ using at most $\Delta + \lfloor 10\Delta^{35/36} \rfloor$ colors such that V_i is φ -distinct for $i \in \{1, \dots, m\}$.

We will use the following three theorems to prove Theorem 2.1 in Section 4. The first one, due to Shannon in 1949 [12], gives an upper bound of chromatic index involving only the maximum degree.

Theorem 2.2 (Shannon's theorem). If G is a graph with maximum degree Δ , then $\chi'(G) \leq \lfloor 3\Delta/2 \rfloor$.

The second one, due to Hoeffding in 1963 [8], generalizes Chernoff bounds (the upper tail part).

Theorem 2.3 (Hoeffding's inequality). Let $X_1, \dots, X_n$ be random variables and $X = \sum_{i=1}^n X_i$ and $\mu = \mathbb{E}[X]$. If $a \leq X_i \leq b$ for all $i \in \{1, \dots, n\}$, then $\mathbb{P}[X \geq \mu + \gamma] \leq e^{-\frac{2\gamma^2}{n(b-a)^2}}$ for all $\gamma > 0$.

The third one is the symmetric case of the Lovász Local Lemma. (See Alon and Spencer [1], Corollary 5.1.2.)

Theorem 2.4 (The local lemma, symmetric case). Let $A_1, \dots, A_n$ be events in an arbitrary probability space. Suppose that each event A_i is mutually independent of all other events A_j but at most d , and $\mathbb{P}[A_i] \leq p$ for $i \in \{1, \dots, n\}$. If $4pd \leq 1$, then $\mathbb{P}[\bigwedge_{i=1}^n \overline{A_i}] > 0$.

A subgraph H of a graph G is an **induced subgraph** if H contains all edges of G that have both endpoints in $V(H)$. An induced subgraph H of graph G with $|H| \geq 3$ odd is said to be **k -dense** if $2|E(H)| > (k-1)(|H|-1)$, **k -near-dense** if H is k -dense and $\chi'(H) = k$, and **k -exact-dense** if $2|E(H)| = k(|H|-1)$ and $\chi'(H) = k$.

Lemma 2.5. If G is a graph with $\chi'(G) \geq \Delta(G) + 2$, then G contains a $\chi'(G)$ -near-dense subgraph.

Proof. Since $\chi'(G) \geq \Delta(G) + 2$, by Theorem 1.2 there exists a subgraph $H \subseteq G$ with $|H| \geq 3$ odd such that $\chi'(G) = \left\lceil \frac{2|E(H)|}{|H|-1} \right\rceil$. Then $\frac{2|E(H)|}{|H|-1} > \chi'(G) - 1$, which in turn gives that H is $\chi'(G)$ -dense and $\chi'(H) \geq \chi'(G)$. Combining this with the fact that $\chi'(H) \leq \chi'(G)$, we get that H is $\chi'(G)$ -near-dense subgraph of G . $\square$

Remark 2.6. The following properties follow directly from the definition.

- (1) A k -exact-dense graph is k -near-dense;

- (2) A k -near-dense graph is k_1 -dense for $k_1 \leq k$;
- (3) If G is a k -dense graph, then $\chi'(G) \geq k$;
- (4) If $\chi'(G) = k$, then $2|E(H)| \leq k(|H| - 1)$ for every subgraph $H \subseteq G$ with $|H|$ odd, moreover, if the equality holds, then H must be an induced subgraph; and
- (5) If H is a $\chi'(G)$ -dense subgraph of G , then H is $\chi'(G)$ -near-dense.

Theorem 2.7. *Let $\Delta \geq 2$ be a positive integer and $r \in (\frac{1}{2}, 1)$ be a real number such that $\Delta^r \geq \lceil \Delta^{1-r} \rceil$, and let G be a graph such that $\Delta(G) \leq \Delta$ and $\chi'(G) \geq \Delta + 2\Delta^r$. If G itself is a $\chi'(G)$ -near-dense graph, then for every $k \geq \chi'(G)$, there exists a k -exact-dense spanning supgraph G^* such that $d_{G^*}(v) \leq k - \Delta^{1-r} - (\Delta - d_G(v))$ for each $v \in V(G)$.*

The proof of Theorem 2.7 will be presented in Section 5. Let G be a graph and $W \subseteq V(G)$. Let G/W denote the graph obtained from G by contracting the set W , that is, we replace in G the set W with a new vertex w , each boundary edge $e \in E_G(x, y)$ with $x \notin W$ and $y \in W$ by an edge with endpoints x and w , and delete all edges in $G[W]$. We call G/W a **minor** of G . We notice that the term minor defined here is slightly broader than the commonly used term minor which requires that $G[W]$ is connected. Given an induced subgraph H of G , we define $G/H = G/V(H)$. Moreover, we say that G/H is obtained from G by contracting H and G is obtained from G/H by uncontracting H . The following simple observation is needed in our proof.

Lemma 2.8. *Let G be a graph and H be a k -dense subgraph of G . Let v_H be the resulting vertex from the contraction of H . If F is a k -dense subgraph of G/H containing v_H , then the graph F^* obtained from F by uncontracting v_H is also a k -dense subgraph of G .*

Proof. On the one hand, since both F and H are k -dense, $|F|$ and $|H|$ are both at least 3 and odd, and satisfy the following inequalities.

$$2|E(F)| > (k-1)(|F| - 1) \quad \text{and} \quad 2|E(H)| > (k-1)(|H| - 1).$$

On the other hand, considering that F^* is obtained from F by uncontracting v_H , we gain that $|F^*| = |F| + |H| - 1$ and $|E(F^*)| = |E(F)| + |E(H)|$. Hence, $|F^*|$ is odd and

$$2|E(F^*)| = 2|E(F)| + 2|E(H)| > (k-1)(|F| + |H| - 2) = (k-1)(|F^*| - 1),$$

which in turn gives that F^* is a k -dense subgraph of G . $\square$

The following result was obtained in Cao et al. [3] (Lemma 3.2 on page 4). For completeness, we include the proof here.

Lemma 2.9. *If a graph G satisfies $\chi'(G) \geq \Delta(G) + 2$, then two maximal $\chi'(G)$ -exact-dense subgraphs (if exists) are either the same or disjoint.*

Proof. Let $k = \chi'(G)$ and $\Delta = \Delta(G)$. Let H_1 and H_2 be two distinct maximal k -exact-dense subgraphs of G such that $V(H_1) \cap V(H_2) \neq \emptyset$. By Remark 2.6-(4), both H_1 and H_2 are induced subgraphs. Let $F_1 = H_1 - V(H_1 \cap H_2)$ and $F_2 = H_2 - V(H_1 \cap H_2)$. Following the maximality of H_1 and H_2 , we have $F_1 \neq \emptyset \neq F_2$.

We claim that $|H_1 \cap H_2|$ is odd. Otherwise, both $|F_1|$ and $|F_2|$ are odd. Since F_1 and F_2 are both subgraphs of G , $2|E(F_i)| \leq k(|F_i| - 1)$ for $i \in \{1, 2\}$. Since H_i is k -exact-dense for $i \in \{1, 2\}$, we have $2|E(H_i)| = k(|H_i| - 1)$. Note that $2|E(H_i)| = 2|E(F_i)| + 2|E(H_1 \cap H_2)| + 2|E(H_1 \cap H_2, F_i)|$. Thus, we have the following.

$$2|E(H_1 \cap H_2)| + 2|E(H_1 \cap H_2, F_i)| \geq k(|H_i| - 1) - k(|F_i| - 1) = k \cdot |H_1 \cap H_2| \quad (1)$$

Since $d_G(H_1 \cap H_2) \geq |E(H_1 \cap H_2, F_1)| + |E(H_1 \cap H_2, F_2)|$, we have

$$2|E(H_1 \cap H_2)| + d_G(H_1 \cap H_2) \geq \sum_{i=1}^2 (|E(H_1 \cap H_2)| + |E(H_1 \cap H_2, F_i)|) \geq k \cdot |H_1 \cap H_2|,$$

where we applied (1) for the last inequality.

On the other hand, $2|E(H_1 \cap H_2)| + d_G(H_1 \cap H_2) = \sum_{v \in V(H_1 \cap H_2)} d_G(v) \leq \Delta \cdot |H_1 \cap H_2|$. So, we get $\Delta \geq k$, giving a contradiction to $k \geq \Delta + 2$.

Since $|H_1 \cap H_2|$ is odd and $H_1 \cap H_2 \subseteq G$, we have $2|E(H_1 \cap H_2)| \leq k(|H_1 \cap H_2| - 1)$. Recall that $2|E(H_i)| = k(|H_i| - 1)$ for $i \in \{1, 2\}$. Thus,

$$\begin{aligned} 2|E(H_1 \cup H_2)| &= 2|E(H_1)| + 2|E(H_2)| - 2|E(H_1 \cap H_2)| \\ &\geq k(|H_1| - 1) + k(|H_2| - 1) - k(|H_1 \cap H_2| - 1) \\ &= k(|H_1 \cup H_2| - 1). \end{aligned}$$

In addition, we have $2|E(H_1 \cup H_2)| \leq k(|H_1 \cup H_2| - 1)$ because of $\chi'(H_1 \cup H_2) \leq \chi'(G) = k$. Hence, $H_1 \cup H_2$ is a k -exact-dense subgraph, giving a contradiction to the maximality of H_1 and H_2 . $\square$

3. Proof of Theorem 1.1

Let φ be an edge-coloring of a graph G with colors from a palette C . For each vertex $v \in V(G)$, we define the following two color sets

$$\varphi(v) = \{\varphi(e) : e \in E_G(v)\} \text{ and } \overline{\varphi}(v) = C \setminus \varphi(v),$$

and call $\varphi(v)$ the set of **colors present** at v and $\overline{\varphi}(v)$ the set of **colors missing** at v . We call φ a **k -edge-coloring** of G if $|C| \leq k$. Similarly, a total-coloring φ' of G is called a **k -total-coloring** if it uses no more than k colors.

We now present the proof of Theorem 1.1. Let G be a graph with maximum degree Δ sufficiently large and $\chi'(G) \geq \Delta + 10\Delta^{35/36}$. Let $k = \chi'(G)$, $k_1 = \Delta + \lceil 10\Delta^{35/36} \rceil$ and $k_0 = \Delta + \lceil 2\Delta^{35/36} \rceil$. Clearly, $k \geq k_1 \geq k_0 \geq \Delta + 2$.

Claim 3.1. *If a subgraph $H \subseteq G$ is $(k_0 + 1)$ -dense, then*

$$|H| < \Delta^{1/36} \quad \text{and} \quad d_G(H) < \Delta - (k_0 - \Delta)(|H| - 1).$$

Proof. By the Handshaking Lemma, $\Delta \cdot |H| \geq \sum_{v \in V(H)} d_G(v) = 2|E(H)| + d_G(H)$. Since H is $(k_0 + 1)$ -dense, we have $2|E(H)| > k_0(|H| - 1)$, and so $\Delta \cdot |H| > k_0(|H| - 1) + d_G(H)$. Hence, $d_G(H) < \Delta - (k_0 - \Delta)(|H| - 1)$. Applying $k_0 = \Delta + \lceil 2\Delta^{35/36} \rceil \geq \Delta + 2\Delta^{35/36}$ and $d_G(H) \geq 0$, we get $|H| < \Delta^{1/36}$. $\square$

In the following algorithm, we define a sequence $\mathcal{G}_t = (G_0, G_1, \dots, G_t)$ of minors of G and a companion sequence $\mathcal{H}_t = (H_0, \dots, H_{t-1})$ for $t \geq 0$.

Algorithm 1. Let $G_0 = G$. Initially, we set $\mathcal{G}_0 = (G_0)$ and $\mathcal{H}_0 = \emptyset$. Note that $\chi'(G_0) = k \geq \Delta(G_0) + 2$. Lemma 2.5 shows that G_0 has a k -near-dense subgraph. Let H_0 be a maximal k -near-dense subgraph of G_0 and $G_1 = G_0/H_0$. Set $\mathcal{G}_1 = (G_0, G_1)$ and $\mathcal{H}_1 = (H_0)$.

Suppose that we have defined a sequence $\mathcal{G}_t = (G_0, G_1, \dots, G_t)$ and its companion sequence $\mathcal{H}_t = (H_0, \dots, H_{t-1})$ for some $t \geq 1$. If $\chi'(G_t) \leq k_0$, then we stop and let $T = t$. Otherwise, we have $\chi'(G_t) > k_0 \geq \Delta + 2$. Using the result $\Delta \geq \Delta(G_t)$ shown in Claim 3.3 below, we get $\chi'(G_t) \geq \Delta(G_t) + 2$. By Lemma 2.5, G_t has a $\chi'(G_t)$ -near-dense subgraph. Let H_t be a maximal $\chi'(G_t)$ -near-dense subgraph in G_t and $G_{t+1} = G_t/H_t$. Set $\mathcal{G}_{t+1} = (G_0, G_1, \dots, G_t, G_{t+1})$ and $\mathcal{H}_{t+1} = (H_0, \dots, H_{t-1}, H_t)$.

In Algorithm 1, for each $t \in \{0, \dots, T-1\}$, since H_t is $\chi'(G_t)$ -near-dense subgraph of G_t , we have $|H_t| \geq 3$, and so $|G_{t+1}| \leq |G_t| - 2$. Hence, T is well defined, that is, Algorithm 1 terminates after a finite number of steps. We call $\mathcal{G}_T = (G_0, \dots, G_T)$ a **maximal dense-minor-sequence**, and H_t the **companion** of G_t for $t \in \{0, \dots, T-1\}$. Denote by v_{H_t} the resulting vertex from the contraction of H_t .

For each G_t , we call each vertex in $V(G_t) \setminus V(G)$ a **contracted vertex**. So, the vertices in $V(G_t)$ are divided into two classes: the original vertices of G and the contracted vertices. Algorithm 1 naturally generates an onto function $f_t : V(G) \rightarrow V(G_t)$ such that the pre-image $f_t^{-1}(v)$ is the vertex set of G whose contraction results in v if v is a contracted vertex, and $f_t^{-1}(v) = \{v\}$ otherwise. We call $f_t^{-1}(v)$ the **root** of v for each $v \in V(G_t)$.

Claim 3.2. *For each contracted vertex $v \in V(G_t)$, the subgraph $G[f_t^{-1}(v)]$ induced by the root of v is $(k_0 + 1)$ -dense.*

Proof. We first notice that H_s is $(k_0 + 1)$ -dense for every $s \in \{0, \dots, T-1\}$ because H_s is $\chi'(G_s)$ -near-dense and $\chi'(G_s) \geq k_0 + 1$.

Suppose that v is the resulting vertex from the contraction of H_s for some $s \leq t-1$. If H_s does not contain any contracted vertex, then $f_t^{-1}(v) = V(H_s)$. Since H_s is $(k_0 + 1)$ -dense, we are done. Suppose that H_s contains some contracted vertices. Let r be the

largest index such that the contraction of H_r results in a contracted vertex, say u , in H_s . Let S_r be the subgraph of G_r obtained from H_s by uncontracting u back to H_r . Applying Lemma 2.8 (with $G = G_r$, $H = H_r$, $F = H_s$ and $F^* = S_r$), we see that S_r is $(k_0 + 1)$ -dense. Repeating this process, we eventually obtain that $G[f_t^{-1}(v)]$ is $(k_0 + 1)$ -dense. $\square$

Claim 3.3. $\Delta(G_t) \leq \Delta$ for any $t \in \{0, \dots, T\}$.

Proof. For any vertex $v \in V(G_t)$, if v is a contracted vertex, by Claim 3.2 we have that $G[f_t^{-1}(v)]$ is $(k_0 + 1)$ -dense. Combining this with Claim 3.1, we have $d_{G_t}(v) = d_G(G[f_t^{-1}(v)]) < \Delta$. If v is not a contracted vertex, then $d_{G_t}(v) = d_G(v) \leq \Delta$. Hence, $\Delta(G_t) \leq \Delta$. $\square$

Claim 3.4. For each $t \in \{0, \dots, T - 1\}$, we have $\chi'(G_{t+1}) \leq \chi'(G_t)$.

Proof. Suppose on the contrary that $\chi'(G_{t+1}) > \chi'(G_t)$ for some $t \in \{0, \dots, T - 1\}$. Since $t \leq T - 1$, we have $\chi'(G_t) > k_0 \geq \Delta + 2$, and so $\chi'(G_{t+1}) > \Delta + 2$. Combining this with $\Delta(G_{t+1}) \leq \Delta$ (Claim 3.3), we have $\chi'(G_{t+1}) > \Delta(G_{t+1}) + 2$. Applying Lemma 2.5 to G_{t+1} , we get a $\chi'(G_{t+1})$ -near-dense subgraph S_{t+1} in G_{t+1} .

We claim that $v_{H_t} \in V(S_{t+1})$. Otherwise, S_{t+1} itself is a subgraph of G_t , and so $\chi'(G_{t+1}) = \chi'(S_{t+1}) \leq \chi'(G_t)$, giving a contradiction to the assumption $\chi'(G_{t+1}) > \chi'(G_t)$. Let S_t be the subgraph of G_t obtained from S_{t+1} by uncontracting v_{H_t} , i.e., replacing v_{H_t} with H_t in S_{t+1} . Since S_{t+1} is $\chi'(G_{t+1})$ -dense and $\chi'(G_{t+1}) > \chi'(G_t)$, it follows that S_{t+1} is $\chi'(G_t)$ -dense. Combining this with the fact that H_t is a $\chi'(G_t)$ -dense subgraph of G_t and applying Lemma 2.8, we see that S_t is also $\chi'(G_t)$ -dense subgraph of G_t . By Remark 2.6-(5), S_t is a $\chi'(G_t)$ -near-dense subgraph of G_t . However, S_t contains H_t as proper subgraph, which gives a contradiction to the maximality of H_t . $\square$

For each $t \in \{0, \dots, T\}$ and any subgraph $G'_t \subseteq G_t$, let $W(G'_t) = V(G'_t) \setminus V(G)$, the set of contracted vertices in G'_t . Let $F_t = G - \cup_{v \in V(G_t)} E(G[f_t^{-1}(v)])$. For each $v \in V(G_t) \setminus W(G_t)$, we have that $f_t^{-1}(v) = \{v\}$ is a singleton set, and so $E(G[f_t^{-1}(v)]) = \emptyset$. Hence, we have that

$$\begin{aligned} \cup_{v \in V(G_t)} E(G[f_t^{-1}(v)]) &= \cup_{w \in W(G_t)} E(G[f_t^{-1}(w)]) \\ F_t &= G - \cup_{w \in W(G_t)} E(G[f_t^{-1}(w)]). \end{aligned}$$

Claim 3.5. For every $t \in \{0, \dots, T\}$, there is a k -total-coloring φ_t of F_t such that the root $f_t^{-1}(w)$ is φ_t -distinct for every $w \in W(G_t)$.

Since $G_0 = G$, we have $W(G_0) = \emptyset$, which in turn gives $F_0 = G$. Hence, the coloring φ_0 in Claim 3.5 is a k -total-coloring of G , which gives Theorem 1.1. The remainder is dedicated to the proof of Claim 3.5.

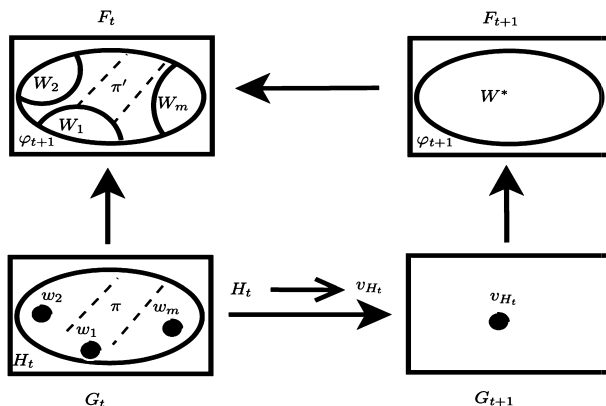


Fig. 1. $\varphi_{t+1} + \pi' \rightarrow \varphi_t$.

Proof of Claim 3.5. We first consider the case that $t = T$. In this case, we have $\chi'(G_t) \leq k_0 = \Delta + \lceil 2\Delta^{35/36} \rceil$. For any $w \in W(G_t)$, by Claim 3.2 $G[f_t^{-1}(w)]$ is a $(k_0 + 1)$ -dense subgraph of G , and so $|f_t^{-1}(w)| < \Delta^{1/36}$ (by Claim 3.1). Note that $f_t^{-1}(w)$ for $w \in W(G_t)$ are mutually disjoint independent vertex sets in graph F_t . By Theorem 2.1, there exists a k_1 -total-coloring φ_t of F_t such that $f_t^{-1}(w)$ is φ_t -distinct for all $w \in W(G_t)$. Since $k_1 \leq k$, Claim 3.5 holds for F_t .

Assume that $t \leq T - 1$ and Claim 3.5 holds for F_{t+1} , i.e., there is a k -total-coloring φ_{t+1} of F_{t+1} such that $f_{t+1}^{-1}(w)$ are φ_{t+1} -distinct for all $w \in W(G_{t+1})$. We will find a desired k -total-coloring of F_t based on φ_{t+1} so that Claim 3.5 holds for F_t . Recall that H_t is a maximal $\chi'(G_t)$ -near-dense subgraph of G_t and $G_{t+1} = G_t/H_t$.

Let $W^* = f_{t+1}^{-1}(v_{H_t}) = f_t^{-1}(V(H_t))$, $W(H_t) = \{w_1, \dots, w_m\}$, and $W_i = f_t^{-1}(w_i)$ for $i \in \{1, \dots, m\}$. By definition, we have

$$F_t = G - \cup_{w \in W(G_t) \setminus W(H_t)} E(G[f_t^{-1}(w)]) - \cup_{w_i \in W(H_t)} E(G[W_i])$$

$$F_{t+1} = G - \cup_{w \in W(G_{t+1}) \setminus \{v_{H_t}\}} E(G[f_{t+1}^{-1}(w)]) - E(G[W^*]).$$

Note that $W(G_t) \setminus W(H_t) = W(G_{t+1}) \setminus \{v_{H_t}\}$ and $f_{t+1}^{-1}(w) = f_t^{-1}(w)$ for any $w \in W(G_t) \setminus W(H_t)$, and $\cup_{i=1}^m W_i \subseteq W^*$. Hence, F_{t+1} is a spanning subgraph of F_t and $E(F_t - F_{t+1}) = E(G[W^*]) - \cup_{w_i \in W(H_t)} E(G[W_i]) = E(F_t[W^*])$. Moreover, $F_{t+1}[W^*]$ is an independent vertex set. Hence, the k -total-coloring φ_{t+1} of F_{t+1} is a partial coloring of F_t except that edges in $E(F_t[W^*])$ are yet colored. We will find an edge-coloring π' of $F_t[W^*]$ using colors $1, \dots, k$ such that its combination with φ_{t+1} gives a desired k -total-coloring of F_t .

Note that H_t is obtained from $F_t[W^*]$ by contracting $W_1, \dots, W_m$ into $w_1, \dots, w_m$, respectively. Hence, an edge-coloring π of H_t gives an edge-coloring π' of $F_t[W^*]$ with the property that for every W_i , all edges incident to it are assigned different colors. We will find a k -edge-coloring π of H_t to get an edge-coloring of $F_t[W^*]$ to fulfill our goal stated above. This strategy is depicted in Fig. 1.

Since H_t is a $\chi'(G_t)$ -near-dense subgraph of G_t , we have $\chi'(H_t) = \chi'(G_t)$. Applying it with $k = \chi'(G) \geq \chi'(G_t) > k_0$, we get $k \geq \chi'(H_t) > k_0 = \Delta + \lceil 2\Delta^{35/36} \rceil \geq \Delta + 2\Delta^{35/36}$. By Claim 3.3, we have $\Delta(H_t) \leq \Delta(G_t) \leq \Delta$. From the assumption of Δ being sufficiently large, we have $\Delta^{35/36} \geq \lceil \Delta^{1/36} \rceil$. Applying Theorem 2.7 with $r = \frac{35}{36}$, we see that H_t has a k -exact-dense spanning supgraph H_t^* such that $d_{H_t^*}(v) \leq k - \Delta^{1/36} - (\Delta - d_{H_t}(v))$ for every $v \in V(H_t)$. By the definition of k -exact-dense, $2|E(H_t^*)| = k(|H_t^*| - 1)$ and $\chi'(H_t^*) = k$.

Let π^* be a k -edge-coloring of H_t^* . Since a matching of H_t^* contains at most $(|H_t^*| - 1)/2$ edges, each color class of π^* must be a near perfect matching. So, each color is missing at exact one vertex. Hence, $\overline{\pi^*}(v)$ are pairwise disjoint for $v \in V(H_t)$ and the following holds.

$$|\overline{\pi^*}(v)| = k - d_{H_t^*}(v) \geq \Delta^{1/36} + (\Delta - d_{H_t}(v)). \quad (2)$$

Since $H_t \subseteq H_t^*$, by restricting π^* to H_t we get a k -edge-coloring π of H_t . For each vertex $v \in V(H_t)$, we have $\overline{\pi}(v) \supseteq \overline{\pi^*}(v)$.

On the other hand, for each $v \in V(H_t)$, we claim that

$$|f_t^{-1}(v)| < \Delta^{1/36} \quad \text{and} \quad d_{F_t - W^*}(f_t^{-1}(v)) \leq \Delta - d_{H_t}(v) \quad (3)$$

where $d_{F_t - W^*}(f_t^{-1}(v))$ denotes the number of the edges with one endpoint in $f_t^{-1}(v)$ and the other endpoint in $V(F_t) \setminus W^*$.

For the first inequality of (3), if v is a contracted vertex, i.e., $v \in W(H_t) = \{w_1, \dots, w_m\}$, then $G[f_t^{-1}(v)]$ is $(k_0 + 1)$ -dense. By Claim 3.1, we have $|f_t^{-1}(v)| < \Delta^{1/36}$. If v is not a contracted vertex, then $f_t^{-1}(v) = \{v\}$. Hence, $|f_t^{-1}(v)| = |\{v\}| = 1 < \Delta^{1/36}$.

For the second inequality of (3), we notice that $d_{F_t}(f_t^{-1}(v)) = d_{G_t}(v)$ and $d_{F_t[W^*]}(f_t^{-1}(v)) = d_{H_t}(v)$. Hence,

$$d_{F_t - W^*}(f_t^{-1}(v)) = d_{F_t}(f_t^{-1}(v)) - d_{F_t[W^*]}(f_t^{-1}(v)) = d_{G_t}(v) - d_{H_t}(v) \leq \Delta - d_{H_t}(v).$$

Combining (2) and (3), we get following inequality for every $v \in V(H_t)$.

$$|\overline{\pi^*}(v)| > |f_t^{-1}(v)| + d_{F_t - W^*}(f_t^{-1}(v)) \quad (4)$$

Under the k -total-coloring φ_{t+1} of F_{t+1} , for each vertex $v \in V(H_t)$, let C_v denote the set of colors assigned to the vertices in $f_t^{-1}(v)$ and edges incident to $f_t^{-1}(v)$. Note that $f_t^{-1}(v) \subseteq W^*$ for $v \in V(H_t)$. Since $W^* = f_{t+1}^{-1}(v_{H_t})$ is an independent vertex set in F_{t+1} and W^* is φ_{t+1} -distinct, the color sets C_v for $v \in V(H_t)$ are mutually disjoint and

$$|C_v| = |f_t^{-1}(v)| + d_{F_{t+1}}(f_t^{-1}(v)) = |f_t^{-1}(v)| + d_{F_t - W^*}(f_t^{-1}(v)) < |\overline{\pi^*}(v)|.$$

Recall that the color sets $\overline{\pi^*}(v)$ for $v \in V(H_t)$ are mutually disjoint. Hence, by permuting colors, we may assume that

$$\overline{\pi}(v) \supseteq \overline{\pi^*}(v) \supseteq C_v \quad \text{for every } v \in V(H_t) \quad (5)$$

Let π' be the edge-coloring of $F_t[W^*]$ generated from the coloring π of H_t , that is, $\pi'(e') = \pi(e)$ if $e' \in F_t[W^*]$ is the image of $e \in E(H_t)$ under the natural corresponding between $E(H_t)$ and $E(F_t[W^*])$. For every vertex $v \in V(H_t)$, by (5) we have $\overline{\pi'}(f_t^{-1}(v)) = \overline{\pi}(v) \supseteq C_v$. So, no colors in C_v are assigned to edges incident to $f_t^{-1}(v)$ in $F_t[W^*]$ under coloring π' . Hence, the combination of the coloring φ_{t+1} and π' gives a k -total-coloring φ_t of F_t . For each w_i , there are three disjoint sets of colors involving W_i : $\varphi_{t+1}(W_i)$, $\varphi_{t+1}(E(W_i, G - W^*))$, and $\pi'(E_G(W_i, W^* - W_i)) = \pi(E_{H_t}(w_i))$. Hence, W_i is φ_t -distinct. Additionally, we note that coloring φ_t agrees with φ_{t+1} on every edge and every vertex not in $F_t[W^*]$. Therefore, for every contracted vertex $w \in W(G_t)$, the vertex set $f_t^{-1}(w)$ is φ_t -distinct, and so φ_t is the desired k -total-coloring of F_t . $\square$

4. Proof of Theorem 2.1

In order to simplify the presentation of the proof, we omit floors and ceilings and treat large numbers as integers whenever this does not affect the argument. We reserved enough room in the calculation to absorb the differences.

Let G be a graph with maximum degree Δ sufficiently large. Let $V_1, \dots, V_m$ be m mutually disjoint independent vertex sets of G , and let G_0 be obtained from G by contracting each V_i to a single vertex. Suppose that $\chi'(G_0) \leq \Delta + 2\Delta^{35/36}$ and $|V_\ell| \leq \Delta^{1/36}$ for $\ell \in \{1, \dots, m\}$. We will show that G has a total-coloring η using at most $\Delta + 10\Delta^{35/36}$ colors such that all V_ℓ are η -distinct.

Set $k = \Delta + 10\Delta^{35/36}$ and $k_0 = \Delta + 2\Delta^{35/36}$. Let φ_0 be a k_0 -edge-coloring of G_0 using the colors $1, \dots, k_0$. Since $V_1, \dots, V_m$ are independent vertex sets, φ_0 naturally becomes a k_0 -edge-coloring φ of G with the additional property that for each $\ell \in \{1, \dots, m\}$, edges incident to V_ℓ are assigned different colors.

Let K be obtained from G by adding edges to V_ℓ such that $K[V_\ell]$, the subgraph induced by V_ℓ , is a complete (simple) graph for $\ell \in \{1, \dots, m\}$. Since $|V_\ell| \leq \Delta^{1/36}$ for $\ell \in \{1, \dots, m\}$, we have $\Delta(K) < \Delta + \Delta^{1/36}$.

Let $s = \lceil \Delta^{16/36} \rceil$ and $t = \lfloor k_0/s \rfloor$. By definition, we have

$$\frac{\Delta + 2\Delta^{35/36}}{\Delta^{16/36} + 1} - 1 < t \leq \Delta^{20/36} + 2\Delta^{19/36} \quad \text{and} \quad s \cdot t \leq k_0.$$

We specify s sets of colors: $C_1 = \{1, \dots, t\}, \dots, C_s = \{(s-1)t + 1, \dots, st\}$. Clearly, $C_1 \cup \dots \cup C_s = \{1, \dots, st\}$ is a subset of the set of colors used by the edge-coloring φ .

Definition 4.1. With respect to a partition $U_1, \dots, U_s$ of $V(G)$, we say that an edge $e \in E(G)$ **discords** a vertex $u \in V(G)$ if $\varphi(e) \in C_i$ and $u \in U_i$ for some $i \in \{1, \dots, s\}$.

Moreover, we say that e **conflicts** with u if either e discords a vertex in V_ℓ when $u \in V_\ell$ for some $\ell \in \{1, \dots, m\}$ or e discords u itself when $u \notin \cup_{\ell=1}^m V_\ell$.

Claim 4.2. *There is a partition $U_1, \dots, U_s$ of $V(G)$ such that*

- (i). *for each vertex v and U_i , $|N_K(v) \cap U_i| \leq t - 1$,*
- (ii). *for each vertex v , there are at most $4\Delta^{34/36}$ edges e incident with v that conflicts with the other endpoint of e in G .*

Proof. Assign each vertex to a uniformly random part with probability $\frac{1}{s}$ (where of course, these choices are made independently). For each pair (v, i) we let $A_{v,i}$ be the event that (i) fails to hold for (v, i) and B_v be the event that (ii) fails to hold for v . We will use Hoeffding's Inequality to prove the following inequalities by assuming that Δ is sufficiently large.

- A.** $\mathbb{P}[A_{v,i}] < e^{-\Delta^{1/18}} < \frac{1}{4\Delta^3}$ for every pair (v, i) , and
- B.** $\mathbb{P}[B_v] < e^{-\Delta^{1/18}} < \frac{1}{4\Delta^3}$ for every v .

We first complete the proof of Claim 4.2 based on (A) and (B) before giving their proofs. For each vertex $v \in V(G)$, let $D(v)$ denote the union of $N_G(v)$ and these V_ℓ that contain a neighbor of v . Since $|V_\ell| \leq \Delta^{1/36}$ for $\ell \in \{1, \dots, m\}$, it follows that $|D(v)| \leq \Delta \cdot \Delta^{1/36} = \Delta^{37/36}$. Note that events B_v and $A_{v,i}$ are determined by the partition assignments of the vertices in $D(v)$. Thus, by the Mutual Independence Principle, they are mutually independent of all events concerning vertices which are not in $D(v) \cup N_G(D(v))$. Notice that each vertex v involves with one event B_v and s events $A_{v,1}, \dots, A_{v,s}$. So, every event is mutually independent of all but at most $(s+1)|D(v) \cup N_G(D(v))| \leq (s+1)(|D(v)| \cdot \Delta) \leq (\lceil \Delta^{16/36} \rceil + 1) \cdot \Delta^{37/36} \cdot \Delta < \Delta^3$ other events. Since $4 \cdot \frac{1}{4\Delta^3} \cdot \Delta^3 = 1$, by Theorem 2.4 (the Local Lemma) there is a partition satisfying both (i) and (ii).

Proof of A. Given a pair (v, i) with $v \in V(G)$ and $1 \leq i \leq s$, we will show that $\mathbb{P}[A_{v,i}] < e^{-\Delta^{1/18}}$. For each $u \in N_K(v)$, let X_u be a 0–1 variable such that $X_u = 1$ if $u \in U_i$ and $X_u = 0$ otherwise. Let $X = \sum_{u \in N_K(v)} X_u$. Clearly, X is the number of neighbors of v assigned to U_i in K . Recall that $|N_K(v)| < \Delta + \Delta^{1/36}$. Since $\mathbb{E}[X_u] = \mathbb{P}[X_u = 1] = 1/s$ for each $u \in N_K(v)$, we have

$$\mu = \mathbb{E}[X] = \sum_{u \in N_K(v)} \mathbb{E}[X_u] = \frac{|N_K(v)|}{s} < (\Delta + \Delta^{1/36})/\Delta^{16/36} = \Delta^{20/36} + \Delta^{-15/36}.$$

Since $t > (\Delta + 2\Delta^{35/36})/(\Delta^{16/36} + 1) - 1$ and Δ is large, a simple calculation gives us $t > \Delta^{20/36} + \Delta^{-15/36} + \Delta^{19/36} > \mu + \Delta^{19/36}$. Applying Hoeffding's Inequality with $b = 1$ and $a = 0$, we get the following.

$$\mathbb{P}[A_{v,i}] = \mathbb{P}[X \geq t] \leq \mathbb{P}[X > \mu + \Delta^{19/36}] \leq e^{-\frac{2(\Delta^{19/36})^2}{|N_K(v)|}} < e^{-\frac{2\Delta^{38/36}}{2\Delta}} = e^{-\Delta^{1/18}}$$

In the above inequality, we used that $|N_k(v)| < \Delta + \Delta^{1/36} \leq 2\Delta$. $\square$

Proof of B. Given a vertex $v \in V(G)$, we wish to show that $\mathbb{P}[B_v] < e^{-\Delta^{1/18}}$. For each color set C_i , let E_{C_i} denote the set of edges in G assigned colors in C_i . Let $I = N_G(v) \setminus (\cup_{\ell=1}^m V_\ell)$ and $L = \{\ell \in \{1, \dots, m\} : N_G(v) \cap V_\ell \neq \emptyset\}$. For $u \in I$, let X_u be the number of edges in $E_G(v, u)$ that conflict with u , that is, the number of edges in $E_G(v, u)$ that discord u . For $i \in \{1, \dots, s\}$, if $u \in U_i$ then there are $|E_G(v, u) \cap E_{C_i}|$ edges in $E_G(v, u)$ that conflict with u . Hence, we have the following.

$$\mathbb{E}[X_u] = \sum_{i=1}^s (|E_G(v, u) \cap E_{C_i}| \cdot \mathbb{P}[u \in U_i]) = \frac{|E_G(v, u)|}{s} \quad (6)$$

For each $\ell \in L$, let X_ℓ be the number of pairs (e, w) with $e \in E_G(v, V_\ell)$ and $w \in V_\ell$ such that e discords w , and let X_ℓ^* be the number of edges incident with v that conflict with the other endpoint in V_ℓ . Since $V_\ell = \cup_{i=1}^s (V_\ell \cap U_i)$, we have

$$X_\ell^* = \sum_{1 \leq i \leq s, V_\ell \cap U_i \neq \emptyset} |E_G(v, V_\ell) \cap E_{C_i}| \leq \sum_{i=1}^s |E_G(v, V_\ell) \cap E_{C_i}| \cdot |V_\ell \cap U_i| = X_\ell.$$

For each $w \in V_\ell$, let X_w be the number of edges $e \in E_G(v, V_\ell)$ discording w . Clearly, $X_\ell = \sum_{w \in V_\ell} X_w$. Similar to (6), we have $\mathbb{E}[X_w] = |E_G(v, V_\ell)|/s$. Hence,

$$\mathbb{E}[X_\ell] = \sum_{w \in V_\ell} \mathbb{E}[X_w] = \frac{|E_G(v, V_\ell)| \cdot |V_\ell|}{s} \leq \frac{|E_G(v, V_\ell)| \cdot \Delta^{1/36}}{s}. \quad (7)$$

We divide each of I and L into two subsets as follows.

$$I_1 = \{u \in I : |E_G(v, u)| \leq \Delta^{14/36}\} \text{ \& \& } I_2 = \{u \in I : |E_G(v, u)| > \Delta^{14/36}\} \\ L_1 = \{\ell \in L : |E_G(v, V_\ell)| \leq \Delta^{14/36}\} \text{ \& \& } L_2 = \{\ell \in L : |E_G(v, V_\ell)| > \Delta^{14/36}\}$$

Let $Y = \sum_{u \in I_1} X_u + \sum_{\ell \in L_1} X_\ell$, $Z = \sum_{u \in I_2} X_u + \sum_{\ell \in L_2} X_\ell$, and $X = \sum_{u \in I} X_u + \sum_{\ell \in L} X_\ell$. Clearly, $X = Y + Z$. By (6) and (7), we have

$$\mathbb{E}[X] = \sum_{u \in I} \mathbb{E}[X_u] + \sum_{\ell \in L} \mathbb{E}[X_\ell] \leq \frac{1}{s} \sum_{u \in I} |E_G(v, u)| + \frac{1}{s} \sum_{\ell \in L} |E_G(v, V_\ell)| \cdot \Delta^{1/36} \\ \leq \frac{d_G(v) \cdot \Delta^{1/36}}{\Delta^{16/36}} \leq \Delta^{21/36}.$$

Consequently, we have $\mathbb{E}[Y] \leq \mathbb{E}[X] \leq \Delta^{21/36}$ and $\mathbb{E}[Z] \leq \mathbb{E}[X] \leq \Delta^{21/36}$.

For each $u \in I_1$, we have $X_u \leq |E_G(v, u)| \leq \Delta^{14/36}$. For each $\ell \in L_1$, we know $X_\ell \leq |E_G(v, V_\ell)| \cdot |V_\ell| \leq \Delta^{15/36}$. Notices that $|I_1| + |L_1| \leq |N_G(v)| \leq \Delta$. Applying Hoeffding's Inequality with $n = |I_1| + |L_1|$, $b = \Delta^{15/36}$ and $a = 0$, we get the following inequality.

$$\mathbb{P}[Y > 2\Delta^{34/36}] \leq \mathbb{P}[Y > \mathbb{E}[Y] + \Delta^{34/36}] \leq e^{-\frac{2(\Delta^{34/36})^2}{\Delta(\Delta^{15/36})^2}} = e^{-2\Delta^{\frac{1}{18}}} < \frac{e^{-\Delta^{\frac{1}{18}}}}{2}$$

For each color set C_i , since $|C_i| = t \leq \Delta^{20/36} + 2\Delta^{19/36}$, it follows that $|E_G(v, u) \cap E_{C_i}| \leq t$ for each $u \in N_G(v)$. So, for each $u \in I$, there are at most t edges in $E_G(v, u)$ discarding u . Hence $X_u \leq t$. For each $\ell \in L$, there are at most $t \cdot |V_\ell|$ pairs (e, w) with $e \in E_G(v, V_\ell)$ and $w \in V_\ell$ such that $w \in U_i$ and $\varphi(e) \in C_i$ for some i , and so $X_\ell \leq t \cdot |V_\ell| \leq t \cdot \Delta^{1/36}$. Note that $t \cdot \Delta^{1/36} \leq \Delta^{21/36} + 2\Delta^{20/36} \leq 1.4\Delta^{21/36}$ (because Δ is large) and $|I_2| + |L_2| < d_G(v)/\Delta^{14/36} \leq \Delta^{22/36}$. Applying Hoeffding's Inequality with $n = |I_2| + |L_2|$, $b = 1.4\Delta^{21/36}$ and $a = 0$, we get the following inequality (where we used $1.4^2 < 2$).

$$\mathbb{P}[Z > 2\Delta^{34/36}] \leq \mathbb{P}[Z > \mathbb{E}[Z] + \Delta^{34/36}] \leq e^{-\frac{2(\Delta^{34/36})^2}{\Delta^{22/36}(1.4\Delta^{21/36})^2}} < e^{-\Delta^{\frac{1}{9}}} < \frac{e^{-\Delta^{\frac{1}{18}}}}{2}$$

By definition, $\sum_{u \in I} X_u + \sum_{\ell \in L} X_\ell^*$ is exactly the number of edges e incident with v that conflicts with the other endpoint of e in G . Since $X_\ell^* \leq X_\ell$ for $\ell \in \{1, \dots, s\}$, we have $\sum_{u \in I} X_u + \sum_{\ell \in L} X_\ell^* \leq X$. Hence,

$$\mathbb{P}[B_v] \leq \mathbb{P}[X > 4\Delta^{34/36}] \leq \mathbb{P}[Y > 2\Delta^{34/36}] + \mathbb{P}[Z > 2\Delta^{34/36}] < e^{-\Delta^{1/18}},$$

which completes the proof of Claim 4.2. $\square$

Let $U_1, \dots, U_s$ be a partition of $V(G)$ satisfying both (i) and (ii). By (i), for each U_i , by using the simple greedy procedure, we get a vertex-coloring of $K[U_i]$ with colors in C_i , which in turn gives a vertex-coloring θ of K using at most k_0 colors. Since in K each V_ℓ is a clique for each $\ell \in \{1, \dots, m\}$, it follows that θ is also a vertex-coloring of G satisfying that each vertex in any V_ℓ is assigned a different color. It is noteworthy that the combination of φ and θ may not produce a total-coloring of G since some edges may have the same color as their endpoints.

Let $e \in E(G)$ and u be an endpoint of e . If $u \notin \cup_{\ell=1}^m V_\ell$, we say e **rejects** u if $\varphi(e) = \theta(u)$; if $u \in V_\ell$ for some $\ell \in \{1, \dots, m\}$, we say e **rejects** u if $\varphi(e) \in \theta(V_\ell)$. In either case, we call e a **reject edge**. From the construction of vertex-coloring θ , we see that if e rejects u then e must conflict with u , but the converse may not be true. For each vertex $v \in V(G)$, the **reject degree** of v , denoted by Rej_v , is the number of edges e incident with v that rejects the other endpoint of e . By (ii), we have $Rej_v \leq 4\Delta^{34/36}$ for each $v \in V(G)$.

Let R be a spanning subgraph of G induced by all reject edges. If $v \notin \cup_{\ell=1}^m V_\ell$, then there is at most one edge incident with v that rejects v . If $v \in V_\ell$ for some ℓ , there are

at most $|V_\ell| \leq \Delta^{1/36}$ edges incident with v sharing a common color with a vertex in V_ℓ , so there are at most $\Delta^{1/36}$ edges incident with v that rejects v . For each $v \in V(G)$, the degree $d_R(v)$ of v in R is bounded above by the sum of Rej_v and the number of edges that rejects v . Therefore, $d_R(v) \leq 4\Delta^{34/36} + \Delta^{1/36} \leq 5\Delta^{34/36}$ for every vertex $v \in V(G)$, and so $\Delta(R) \leq 5\Delta^{34/36}$.

Let R^* be obtained from graph R by contracting each V_ℓ to a single vertex. Since $|V_\ell| \leq \Delta^{1/36}$ for $\ell \in \{1, \dots, m\}$, it follows that $\Delta(R^*) \leq \Delta(R) \cdot \Delta^{1/36} \leq 5\Delta^{35/36}$. By Shannon's Theorem, $\chi'(R^*) \leq \frac{3}{2}(5\Delta^{35/36}) \leq 8\Delta^{35/36}$. Let π^* be an edge-coloring of R^* using at most $8\Delta^{35/36}$ new colors different from $\Delta + 2\Delta^{35/36}$ colors used in φ and θ . Clearly, π^* gives an edge-coloring π of R such that for each V_ℓ , all edges in R incident to V_ℓ have different colors. By combining vertex-coloring θ , edge-coloring φ on $E(G) \setminus E(R)$ and edge-coloring π on $E(R)$, we get a total-coloring η of G such that all V_ℓ are η -distinct. Note that we used at most $\Delta + 2\Delta^{35/36} + 8\Delta^{35/36} = \Delta + 10\Delta^{35/36}$ colors in total, which completes the proof of Theorem 2.1. $\square$

5. Proof of Theorem 2.7

Let $\Delta \geq 2$ be an integer and $r \in (\frac{1}{2}, 1)$ be a real number such that $\Delta^r \geq \lceil \Delta^{1-r} \rceil$, and let G be a graph such that $\Delta(G) \leq \Delta$ and $\chi'(G) \geq \Delta + 2\Delta^r$. Assume that G itself is a $\chi'(G)$ -near-dense graph. For any $k \geq \chi'(G)$, we will show that G has a k -exact-dense spanning supgraph G^* such that $d_{G^*}(v) \leq k - \Delta^{1-r} - (\Delta - d_G(v))$ for every $v \in V(G)$.

For each $v \in V(G)$, let $\lambda(v) = \Delta - d_G(v)$ and call it the **deficiency** of v in G with respect to Δ . For any vertex set $W \subseteq V(G)$, let $\lambda(W) = \sum_{w \in W} \lambda(w)$; and for a subgraph $F \subseteq G$, let $\lambda(F) = \lambda(V(F))$. Let $k_0 = \Delta + \lceil 2\Delta^r \rceil$ and $k_1 = \chi'(G)$. Clearly, $k \geq k_1 \geq k_0 \geq \Delta + 2\Delta^r$. We first prove the following two claims.

Claim 5.1. $\Delta > (2\Delta^r - 1)(|G| - 1) + \lambda(G)$. In particular, $|G| - 1 < \Delta^{1-r} \leq \Delta^r$.

Proof. Since G is k_1 -near-dense, we have that $2|E(G)| > (k_1 - 1)(|G| - 1) \geq (k_0 - 1)(|G| - 1)$. According to the definition of deficiency, we have that $\Delta \cdot |G| = \sum_{v \in V(G)} (d_G(v) + \lambda(v)) = \sum_{v \in V(G)} d_G(v) + \lambda(G) = 2|E(G)| + \lambda(G)$. Hence, $\Delta \cdot |G| > (k_0 - 1)(|G| - 1) + \lambda(G)$, which gives $\Delta > (k_0 - \Delta - 1)(|G| - 1) + \lambda(G)$. Since $k_0 \geq \Delta + 2\Delta^r$, we further have that $\Delta > (2\Delta^r - 1)(|G| - 1) + \lambda(G)$. Because of $\Delta^r \geq 1$ and $\lambda(G) \geq 0$, we have $\Delta > \Delta^r(|G| - 1)$, and so $|G| - 1 < \Delta^{1-r} \leq \Delta^r$. $\square$

Claim 5.2. Let $W \subseteq V(G)$ with $|W|$ odd. If $W \neq V(G)$, then $d_G(W) > \Delta^r(2|\overline{W}| - 1) + \lambda(\overline{W})$, where $\overline{W} = V(G) \setminus W$.

Proof. Let $D = G[W]$ and $F = G[\overline{W}]$ be the subgraphs of G induced by W and $\overline{W}$, respectively. We note that $2|E(D)| \leq k_1(|D| - 1)$ due to that $\chi'(D) \leq \chi'(G) = k_1$ and $|D|$ is odd. Since G is k_1 -near-dense and $|G| = |D| + |\overline{W}|$, we have

$$2|E(G)| > (k_1 - 1)(|G| - 1) = k_1(|D| - 1) + k_1 \cdot |\overline{W}| - (|G| - 1).$$

Consequently, we have the following.

$$2|E(G)| > 2|E(D)| + k_1 \cdot |F| - (|G| - 1) \quad (8)$$

Applying $\sum_{v \in V(G)} d_G(v) = \sum_{v \in V(D)} d_G(v) + \sum_{v \in V(F)} d_G(v)$, we get the following.

$$2|E(G)| = 2|E(D)| + d_G(V(D)) + \Delta \cdot |F| - \lambda(F). \quad (9)$$

Combining (8) and (9), we get

$$\begin{aligned} d_G(V(D)) &> (k_1 - \Delta)|F| + \lambda(F) - (|G| - 1) \geq 2\Delta^r \cdot |F| - (|G| - 1) + \lambda(F) \\ &> \Delta^r(2|F| - 1) + \lambda(F) \quad (\text{since } |G| - 1 < \Delta^r) \end{aligned}$$

Since $V(D) = W$, $|F| = |\overline{W}|$ and $\lambda(F) = \lambda(\overline{W})$, we are done. $\square$

Let G^* be a spanning supgraph of G such that

- i. $\chi'(G^*) \leq k$ and $d_{G^*}(v) \leq k - \lceil \Delta^{1-r} \rceil - \lambda(v)$ for each $v \in V(G)$, and
- ii. $|E(G^*)|$ is maximum subject to the two conditions above.

Since $k \geq \chi'(G) \geq \Delta + 2\Delta^r$ and $d_G(v) = \Delta - \lambda(v) \leq k - 2\Delta^r - \lambda(v) \leq k - \lceil \Delta^{1-r} \rceil - \lambda(v)$ for every $v \in V(G)$, we see that graph G itself satisfies two conditions in **i**. So, such a graph G^* exists. Since $\Delta \geq 2 > 1$, we have $\lceil \Delta^{1-r} \rceil \geq 2$. Condition **i** implies that $\Delta(G^*) \leq k - \lceil \Delta^{1-r} \rceil \leq k - 2$. By the assumption that $\chi'(G^*) \leq k$ and $|G^*| = |G|$ is odd, we have $2|E(G^*)| \leq k(|G| - 1)$. Thus, in order to prove that G^* is k -exact-dense, we only need to show that $2|E(G^*)| \geq k(|G| - 1)$; and get a contradiction if $2|E(G^*)| > k(|G| - 1)$. Let $U = \{u \in V(G) : d_{G^*}(u) < k - \lceil \Delta^{1-r} \rceil - \lambda(u)\}$.

Claim 5.3. *If $|U| \geq 2$, then U is contained in a k -exact-dense subgraph of G^* .*

Proof. For any two distinct vertices $u, v \in U$, we add a new edge e with endpoints u, v to G^* and let $G' = G^* \cup \{e\}$. According to the definition of U , we have $d_{G'}(u) = d_{G^*}(u) + 1 \leq k - \lceil \Delta^{1-r} \rceil - \lambda(u)$ and $d_{G'}(v) = d_{G^*}(v) + 1 \leq k - \lceil \Delta^{1-r} \rceil - \lambda(v)$, which in turn gives $d_{G'}(w) \leq k - \lceil \Delta^{1-r} \rceil - \lambda(w)$ for any $w \in V(G)$. Consequently, $\Delta(G') \leq k - \lceil \Delta^{1-r} \rceil \leq k - 2$. Since $|E(G')| > |E(G^*)|$, by the maximality of $|E(G^*)|$ we have $\chi'(G') > k \geq \Delta(G') + 2$. By Theorem 1.2, G' contains a subgraph F with $|F| \geq 3$ odd such that $2|E(F)| > k(|F| - 1)$, which implies that $\chi'(F) > k$. Since $F - e \subseteq G^*$ and $\chi'(G^*) \leq k < \chi'(F)$, we have $e \in E(F) \setminus E(G^*)$, $\chi'(F - e) = \chi'(G^*) = k$ and $2|E(F - e)| = k(|F| - 1)$. Hence, $F - e$ is a k -exact-dense subgraph of G^* that contains both u and v .

Fixing u , we have shown that for any vertex $v \in U$ with $v \neq u$, there is a maximal k -exact-dense subgraph in G^* containing both u and v . Since $\chi'(G^*) = k \geq \Delta(G^*) + 2$, by

Lemma 2.9 these maximal k -exact-dense subgraphs are the same graph, and so Claim 5.3 holds. $\square$

By Claim 5.3, we consider the following three cases: (1) $U = \emptyset$; (2) $|U| = 1$; and (3) $|U| \geq 2$.

Case 1: $U = \emptyset$. In this case, we have $d_{G^*}(v) = k - \lceil \Delta^{1-r} \rceil - \lambda(v)$ for all $v \in V(G)$. Hence,

$$\begin{aligned} 2|E(G^*)| &= \sum_{v \in V(G)} d_{G^*}(v) = (k - \lceil \Delta^{1-r} \rceil)|G| - \lambda(G) \\ &= k(|G| - 1) + k - \lceil \Delta^{1-r} \rceil \cdot |G| - \lambda(G). \end{aligned}$$

Applying $\Delta > (2\Delta^r - 1)(|G| - 1) + \lambda(G)$ (Claim 5.1) to $k \geq \Delta + 2\Delta^r$, we get

$$k > (2\Delta^r - 1)|G| + \lambda(G) + 1 > \lceil \Delta^{1-r} \rceil \cdot |G| + \lambda(G)$$

since $2\Delta^r - 1 > \Delta^r \geq \lceil \Delta^{1-r} \rceil$. Therefore, $2|E(G^*)| > k(|G| - 1)$, giving a contradiction.

Case 2: $|U| = 1$. Let $U = \{u\}$ and $F^* = G^* - u$. Applying Claim 5.2 with $W = \{u\}$, we get $d_{G^*}(u) \geq d_G(u) > \Delta^r(2(|G| - 1) - 1) + \lambda(G - u) \geq \Delta^r(|G| - 1) + \lambda(G - u)$. Hence,

$$\begin{aligned} 2|E(G^*)| &= \sum_{v \in V(G)} d_{G^*}(v) = \sum_{v \in V(G) \setminus \{u\}} (k - \lceil \Delta^{1-r} \rceil - \lambda(v)) + d_{G^*}(u) \\ &> (k - \lceil \Delta^{1-r} \rceil)(|G| - 1) - \lambda(G - u) + \Delta^r(|G| - 1) + \lambda(G - u) \\ &\geq (k - \lceil \Delta^{1-r} \rceil + \Delta^r)(|G| - 1) \geq k(|G| - 1), \quad (\text{since } \Delta^r \geq \lceil \Delta^{1-r} \rceil) \end{aligned}$$

which gives a contradiction.

Case 3: $|U| \geq 2$. By Claim 5.3, there is a maximal k -exact-dense subgraph D^* containing U . If $D^* = G^*$, then we are done. We now suppose that $D^* \neq G^*$, that is, G^* is not k -exact-dense. Since $k \geq \chi'(G^*) \geq \chi'(D^*) = k$, we have $\chi'(G^*) = k$. Not being k -exact-dense gives $2|E(G^*)| < k(|G| - 1)$. Let $F^* = G^* - V(D^*)$. Hence,

$$2|E(G^*)| < k(|D^*| + |F^*| - 1) = k(|D^*| - 1) + k \cdot |F^*|.$$

Since D^* is k -exact-dense, $k(|D^*| - 1) = 2|E(D^*)|$. Given $U \subseteq V(D^*)$, we have $V(F^*) \cap U = \emptyset$. Consequently, $d_{G^*}(v) = k - \lceil \Delta^{1-r} \rceil - \lambda(v)$ for every vertex $v \in V(F^*)$. Hence,

$$\begin{aligned} k|F^*| &= \sum_{v \in V(F^*)} d_{G^*}(v) + \lceil \Delta^{1-r} \rceil \cdot |F^*| + \lambda(F^*) \\ &= 2|E(F^*)| + |E_{G^*}(D^*, F^*)| + \lceil \Delta^{1-r} \rceil \cdot |F^*| + \lambda(F^*). \end{aligned}$$

Therefore, $2|E(G^*)| < 2|E(D^*)| + 2|E(F^*)| + |E_{G^*}(D^*, F^*)| + \lceil \Delta^{1-r} \rceil \cdot |F^*| + \lambda(F^*)$. Combining this with $2|E(G^*)| = 2|E(D^*)| + 2|E(F^*)| + 2|E_{G^*}(D^*, F^*)|$, we get the following.

$$|E_{G^*}(D^*, F^*)| < \lceil \Delta^{1-r} \rceil \cdot |F^*| + \lambda(F^*). \quad (10)$$

Note that $|E_{G^*}(D^*, F^*)| = d_{G^*}(V(D^*)) \geq d_G(V(D^*))$. Applying Claim 5.2 with $W = V(D^*)$ and $\overline{W} = V(F^*)$ and using $|F^*| \geq 2$, we get

$$d_G(V(D^*)) > \Delta^r(2|F^*| - 1) + \lambda(F^*) > \Delta^r|F^*| + \lambda(F^*) \geq \lceil \Delta^{1-r} \rceil \cdot |F^*| + \lambda(F^*).$$

Hence, $|E_{G^*}(D^*, F^*)| > \lceil \Delta^{1-r} \rceil \cdot |F^*| + \lambda(F^*)$, giving a contradiction to (10). $\square$

6. Conclusion and acknowledgment

A natural strength of Theorem 1.1 is whether there is a universal constant c such that for a graph with maximum degree Δ sufficiently large, if $\chi'(G) \geq \Delta + c$ then $\chi''(G) = \chi'(G)$. Unfortunately, there are a couple of barriers in our proof such as Theorem 2.7 which limits the best bounds we might get is $\Delta + \Delta^{1/2}$. We thank Molloy and Reed for their book [10] that gave us several inspirational ideas.

Data availability

No data was used for the research described in the article.

References

- [1] Noga Alon, Joel H. Spencer, *The Probabilistic Method*, fourth edition, Wiley Series in Discrete Mathematics and Optimization, John Wiley & Sons, Inc., Hoboken, NJ, 2016.
- [2] Mehdi Behzad, *Graphs and Their Chromatic Numbers*, ProQuest LLC, Ann Arbor, MI, 1965, Thesis (Ph.D.)—Michigan State University.
- [3] Yan Cao, Guantao Chen, Guangming Jing, A note on Goldberg's conjecture of total chromatic numbers, *J. Graph Theory* (2021), <https://doi.org/10.1002/jgt.22771>.
- [4] Guantao Chen, Guangming Jing, Wenan Zang, Proof of the Goldberg-Seymour Conjecture on edge-colorings of multigraphs, arXiv:1901.10316, 2019.
- [5] M.K. Goldberg, Multigraphs with a chromatic index that is nearly maximal, *Diskretn. Anal.* 23 (3–7) (1973) 72.
- [6] Mark K. Goldberg, Edge-coloring of multigraphs: recoloring technique, *J. Graph Theory* 8 (1) (1984) 123–137.
- [7] R.P. Gupta, *Studies in the Theory of Graphs*, PhD thesis, Tata Institute of Fundamental Research, Bombay, 1967.
- [8] Wassily Hoeffding, Probability inequalities for sums of bounded random variables, *J. Am. Stat. Assoc.* 58 (1963) 13–30.
- [9] Michael Molloy, Bruce Reed, A bound on the total chromatic number, *Combinatorica* 18 (2) (1998) 241–280.
- [10] Michael Molloy, Bruce Reed, *Graph Colouring and the Probabilistic Method*, Algorithms and Combinatorics, vol. 23, Springer-Verlag, Berlin, 2002.
- [11] P.D. Seymour, On multicolourings of cubic graphs, and conjectures of Fulkerson and Tutte, *Proc. Lond. Math. Soc.* (3) 38 (3) (1979) 423–460.

- [12] Claude E. Shannon, A theorem on coloring the lines of a network, *J. Math. Phys.* 28 (1949) 148–151.
- [13] Michael Stiebitz, Diego Scheide, Bjarne Toft, Lene M. Favrholdt, Vizing's theorem and Goldberg's conjecture, in: *Graph Edge Coloring*, in: *Wiley Series in Discrete Mathematics and Optimization*, John Wiley & Sons, Inc., Hoboken, NJ, 2012, With a preface by Stiebitz and Toft.
- [14] V.G. Vizing, Critical graphs with a given chromatic class (in Russian) *Diskretn. Anal.* 5 (1965) 9–17.
- [15] V.G. Vizing, Some unsolved problems in graph theory, *Usp. Mat. Nauk* 23 (6 (144)) (1968) 117–134.