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On Segre Products, *F*-regularity, and Finite Frobenius Representation Type

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Dedicated to Ngo Viet Trung on the occasion of his 70th birthday, in celebration of his many contributions to commutative algebra

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Abstract
We study the behavior of various properties of commutative Noetherian rings under Segre products, with a special focus on properties in positive prime characteristic defined using the Frobenius endomorphism. Specifically, we construct normal graded rings of finite Frobenius representation type that are not Cohen-Macaulay.

Keywords Noetherian rings · Segre products · Frobenius endomorphism

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1 Introduction

We study the behavior of various properties of commutative Noetherian rings under Segre products, with a special focus on properties in positive prime characteristic defined using the Frobenius endomorphism. Segre products of rings arise rather naturally in the context of projective varieties: while the product of affine spaces $\mathbb{A}^m$ and $\mathbb{A}^n$ is readily identified with

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$\mathbb{A}^{m+n}$, it is the *Segre embedding* that gives the product of projective spaces $\mathbb{P}^m$ and $\mathbb{P}^n$ the structure of a projective variety:

$$\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{m+n+mn}, \qquad ((a_0, \dots, a_m), (b_0, \dots, b_n)) \mapsto (a_0b_0, a_0b_1, \dots, a_mb_n).$$

At the level of homogeneous coordinate rings, this corresponds to

$$\mathbb{P}^m \times \mathbb{P}^n \;=\; \operatorname{Proj} \mathbb{F}[x_0y_0,\; x_0y_1,\; \dots,\; x_my_n],$$

where $\mathbb{P}^m := \operatorname{Proj} \mathbb{F}[x_0, \dots, x_m]$ and $\mathbb{P}^n := \operatorname{Proj} \mathbb{F}[y_0, \dots, y_n]$.

More generally, for $\mathbb{N}$-graded rings $R = \oplus_{n\geq 0} R_n$ and $S = \oplus_{n\geq 0} S_n$, finitely generated over a field $R_0 = \mathbb{F} = S_0$, the *Segre product* of R and S is the $\mathbb{N}$-graded ring

$$R \# S := \bigoplus_{n\geq 0} R_n \otimes_{\mathbb{F}} S_n.$$

It is readily seen that $R \# S$ is a subring of the tensor product $R \otimes_{\mathbb{F}} S$; moreover, $R \# S$ is a direct summand of $R \otimes_{\mathbb{F}} S$ as an $R \# S$-module, equivalently the inclusion of rings

$$R \# S \hookrightarrow R \otimes_{\mathbb{F}} S$$

is pure; it follows from this that if $\mathbb{F}$ is a field of positive characteristic, and R and S are F-pure or F-regular, then the same is also true for $R \# S$. What is perhaps surprising is that the converse also holds, provided that the $\mathbb{N}$-grading on each of the rings R and S is irredundant; this is proved here as Theorem 3.1, see also [10, Theorem 5.2]. The additional hypothesis on the grading is indeed required in view of Example 3.2.

While the properties F-purity and F-regularity are inherited by pure subrings, the property of being F-rational is not, as established by the second author in [32]. Nonetheless, we show that if R and S are F-rational rings of positive prime characteristic, then $R \# S$ is also F-rational, Theorem 4.1. The converse to this is false, see Example 4.2.

Lastly, we turn to the property of finite Frobenius representation type (FFRT); the notion is due to Smith and Van den Bergh [27], and it follows readily from their results that if R and S are $\mathbb{N}$-graded reduced rings, finitely generated over a perfect field $R_0 = \mathbb{F} = S_0$ of positive characteristic, then $R \# S$ has FFRT. We use this to construct normal graded rings that are not Cohen-Macaulay, but have the FFRT property.

The observation that Segre products readily yield large families of normal graded rings that are not Cohen-Macaulay goes back at least to Chow [2], who established necessary and sufficient conditions for the Segre product of Cohen-Macaulay rings to be Cohen-Macaulay; Hochster and Roberts [15, §14] observed that under mild hypotheses, Chow's results may be recovered via the Künneth formula for sheaf cohomology. Subsequently, Goto and Watanabe [6] established a more general Künneth formula for local cohomology that extends this circle of ideas; this and other ingredients are summarized next.

2 Preliminaries

We first record the Künneth formula for local cohomology [6, Theorem 4.1.5]:

Theorem 2.1 (Goto-Watanabe) *Let R and S be normal $\mathbb{N}$-graded rings, finitely generated over a field $R_0 = \mathbb{F} = S_0$. Set $\mathfrak{m}_R$, $\mathfrak{m}_S$, and $\mathfrak{m}$ to be the homogeneous maximal ideals of the rings R, S, and $R \# S$ respectively. Suppose M and N are finitely generated $\mathbb{Z}$-graded modules over R and S respectively, such that $H^k_{\mathfrak{m}_R}(M) = 0 = H^k_{\mathfrak{m}_S}(N)$ for $k = 0, 1$.*

Then, for each $k \geq 0$, the local cohomology of the $\mathbb{Z}$-graded $R \# S$-module

$$M \# N := \bigoplus_{n \in \mathbb{Z}} M_n \otimes_{\mathbb{F}} N_n$$

is given by

$$H^k_{\mathfrak{m}}(M \# N) \;=\; \Big(M \# H^k_{\mathfrak{m}_S}(N)\Big) \oplus \Big(H^k_{\mathfrak{m}_R}(M) \# N\Big) \oplus \bigoplus_{i+j=k+1} \Big(H^i_{\mathfrak{m}_R}(M) \# H^j_{\mathfrak{m}_S}(N)\Big).$$

Our proof of Theorem 3.1 uses the description of normal graded rings in terms of $\mathbb{Q}$-divisors, due to Dolgačev [4], Pinkham [20], and Demazure [3], that we review next. A $\mathbb{Q}$-*divisor* on a normal projective variety X is a $\mathbb{Q}$-linear combination of codimension one irreducible subvarieties of X. Let $D = \sum n_i V_i$ be a $\mathbb{Q}$-divisor, where $n_i \in \mathbb{Q}$, and the subvarieties V_i are distinct. Set

$$\lfloor D \rfloor := \sum \lfloor n_i \rfloor V_i,$$

where $\lfloor n \rfloor$ is the greatest integer less than or equal to n. We define

$$\mathcal{O}_X(D) := \mathcal{O}_X(\lfloor D \rfloor).$$

Let $K(X)$ denote the field of rational functions on X. Each $g \in K(X)$ defines a Weil divisor $\operatorname{div}(g)$ by considering the zeros and poles of g with appropriate multiplicity. As these multiplicities are integers, it follows that for a $\mathbb{Q}$-divisor D one has

$$\begin{aligned} H^0(X, \mathcal{O}_X(\lfloor D \rfloor)) &= \{g \in K(X) \mid \operatorname{div}(g) + \lfloor D \rfloor \geq 0\} \\ &= \{g \in K(X) \mid \operatorname{div}(g) + D \geq 0\} = H^0(X, \mathcal{O}_X(D)). \end{aligned}$$

A $\mathbb{Q}$-divisor D is *ample* if ND is an ample Cartier divisor for some $N \in \mathbb{N}$. In this case, the *generalized section ring* $\Gamma_*(X, D)$ is the $\mathbb{N}$-graded ring

$$\Gamma_*(X, D) := \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD))T^n,$$

where T is an element of degree 1, transcendental over $K(X)$. The following is [3, 3.5]:

Theorem 2.2 (Demazure) *Let R be an $\mathbb{N}$-graded normal domain that is finitely generated over a field R_0. Let T be a homogeneous element of degree 1 in the fraction field of R. Then there exists a unique ample $\mathbb{Q}$-divisor D on $X :=$* Proj *R such that*

$$R_n \;=\; H^0(X, \mathcal{O}_X(nD))T^n \quad \textit{for each } n \geq 0.$$

Let $D = \sum (s_i/t_i) V_i$ be a $\mathbb{Q}$-divisor where the V_i are distinct, s_i and t_i are relatively prime integers, and $t_i > 0$. Following [29, Theorem 2.8], the *fractional part* of D is

$$D' := \sum \frac{t_i - 1}{t_i} V_i.$$

This definition is motivated by the fact that one then has

$$-\lfloor -nD \rfloor \;=\; \lfloor D' + nD \rfloor$$

for each integer n, so that taking the graded dual of

$$[H^{\dim R}_{\mathfrak{m}}(R)]_{-n} = H^{\dim X}(X, \mathcal{O}_X(-nD))$$

yields

$$[\omega_R]_n \;=\; H^0(X, \mathcal{O}_X(K_X + D' + nD)),$$

where ω_R is the graded canonical module of $R := \Gamma_*(X, D)$, and K_X is the canonical divisor of X. The following is [31, Theorem 3.3]; note that

$$H^{\dim X}(X, \mathcal{O}_X(K_X + D')) \;=\; H^{\dim X}(X, \mathcal{O}_X(K_X))$$

is the rank one vector space $[H^{\dim R}_{\mathfrak{m}}(\omega_R)]_0$.

Theorem 2.3 (Watanabe) *Let X be a normal projective variety of characteristic $p > 0$, and K_X its canonical divisor. Let D be an ample $\mathbb{Q}$-divisor, and set $R := \Gamma_*(X, D)$. Then:*

(i) The ring R is F-pure if and only if the Frobenius map below is injective:

$$F\colon H^{\dim X}(X, \mathcal{O}_X(K_X + D')) \to H^{\dim X}(X, \mathcal{O}_X(pK_X + pD')).$$

(ii) Let η be a nonzero element of $H^{\dim X}(X, \mathcal{O}_X(K_X + D'))$. Then the ring R is F-regular if and only if for each integer $n > 0$ and each nonzero element c of $H^0(X, \mathcal{O}_X(nD))$, there exists an integer $e > 0$ such that $cF^e(\eta)$ is a nonzero element of

$$H^{\dim X}(X, \mathcal{O}_X(p^e(K_X + D') + nD)).$$

3 *F*-regularity and *F*-purity

The theory of tight closure was introduced by Hochster and Huneke in [12], and further developed in the graded context in [13]. A ring R of positive prime characteristic is *weakly F-regular* if each ideal of R equals its tight closure, while R is F-regular if each localization of R is weakly F-regular. Following [11, p. 166], a ring R of positive prime characteristic is *strongly F-regular* if $N^*_M = N$ for each pair of R-modules $N \subseteq M$. When R is an $\mathbb{N}$-graded ring that is finitely generated over a field R_0 of positive characteristic, as is the case in the present paper, the properties of weak F-regularity, F-regularity, and strong F-regularity coincide by [17, Corollary 3.3].

The following theorem may be viewed as an extension of [10, Theorem 5.2], where it was proved under the hypothesis that the rings contain homogeneous elements of degree 1:

Theorem 3.1 *Let R and S be normal $\mathbb{N}$-graded rings, finitely generated over a perfect field $R_0 = \mathbb{F} = S_0$ of positive characteristic. Suppose that the fraction fields of R as well as S contain homogeneous elements of degree* 1.

Then the Segre product $R \# S$ is F-regular (respectively, F-pure) if and only if R and S are F-regular (respectively, F-pure).

Proof If the rings R and S are F-regular or F-pure, then the same holds for their tensor product $R \otimes_{\mathbb{F}} S$, see for example the proof of $2 \implies 3$ in [10, Theorem 5.2]. The property, then, is inherited by the pure subring $R \# S$; it is only the converse that requires the additional hypothesis on the grading:

Let D_X and D_Y be ample $\mathbb{Q}$-divisors on $X := \operatorname{Proj} R$ and $Y := \operatorname{Proj} S$ respectively, such that $R = \Gamma_*(X, D_X)$ and $S = \Gamma_*(Y, D_Y)$. Set $Z := X \times Y$, and let $\pi_1\colon Z \to X$ and $\pi_2\colon Z \to Y$ be the respective projection morphisms. For each integer $n \geq 0$ one has

$$H^0(Z, \mathcal{O}_Z(\pi_1^*(nD_X) + \pi_2^*(nD_Y))) \;=\; H^0(X, \mathcal{O}_X(nD_X)) \otimes H^0(Y, \mathcal{O}_X(nD_Y)),$$

from which it follows that

$$R \# S = \Gamma_*(Z, \pi_1^*(D_X) + \pi_2^*(D_Y)).$$

Setting $D_Z := \pi_1^*(D_X) + \pi_2^*(D_Y)$, one has

$$D'_Z = \pi_1^*(D'_X) + \pi_2^*(D'_Y),$$

so the Frobenius map F as in Theorem 2.3 (i) takes the form

$$\begin{array}{ccc} H^{d_1+d_2}(Z, \mathcal{O}_Z(K_Z + D'_Z)) & \xrightarrow{\cong} & H^{d_1}(X, \mathcal{O}_X(K_X + D'_X)) \otimes H^{d_2}(Y, \mathcal{O}_Y(K_Y + D'_Y)) \\ {\scriptstyle F}\downarrow & & {\scriptstyle F}\downarrow \\ H^{d_1+d_2}(Z, \mathcal{O}_Z(pK_Z + pD'_Z)) & \xrightarrow{\cong} & H^{d_1}(X, \mathcal{O}_X(pK_X + pD'_X)) \otimes H^{d_2}(Y, \mathcal{O}_Y(pK_Y + pD'_Y)) \end{array}$$

where $d_1 := \dim X$ and $d_2 := \dim Y$. Let η_1 and η_2 be nonzero elements of the rank one vector spaces $H^{d_1}(X, \mathcal{O}_X(K_X + D'))$ and $H^{d_2}(Y, \mathcal{O}_Y(K_Y + D'_Y))$ respectively.

If $R \# S$ is F-pure, the injectivity of the vertical arrows in the diagram displayed above implies that $F(\eta_1 \otimes \eta_2) = F(\eta_1) \otimes F(\eta_2)$ is nonzero, and hence that the maps

$$H^{d_1}(X, \mathcal{O}_X(K_X + D'_X)) \xrightarrow{F} H^{d_1}(X, \mathcal{O}_X(pK_X + pD'_X))$$

and

$$H^{d_2}(Y, \mathcal{O}_Y(K_Y + D'_Y)) \xrightarrow{F} H^{d_2}(Y, \mathcal{O}_Y(pK_Y + pD'_Y))$$

are injective; it follows that the rings R and S are F-pure.

Next, assume that $R \# S$ is F-regular. Fix $n > 0$, and consider nonzero elements

$$c_1 \in H^0(X, \mathcal{O}_X(nD_X)) \qquad \text{and} \qquad c_2 \in H^0(Y, \mathcal{O}_Y(nD_Y)).$$

Then $c_1 \otimes c_2$ is a nonzero element of $H^0(Z, \mathcal{O}_X(nD_Z))$, so the F-regularity of $R \# S$ implies that there exists $e > 0$ such that

$$(c_1 \otimes c_2)F^e(\eta_1 \otimes \eta_2) = c_1 F^e(\eta_1) \otimes c_2 F^e(\eta_2)$$

is a nonzero element of

$$\begin{aligned} & H^{d_1+d_2}(Z, \mathcal{O}_Z(p^e(K_Z + D'_Z) + nD_Z)) \\ \cong\ & H^{d_1}(X, \mathcal{O}_X(p^e(K_X + D'_X) + nD_X)) \otimes H^{d_2}(Y, \mathcal{O}_Y(p^e(K_Y + D'_Y) + nD_Y)). \end{aligned}$$

But then the elements

$$c_1 F^e(\eta_1) \in H^{d_1}(X, \mathcal{O}_X(p^e(K_X + D'_X) + nD_X))$$

and

$$c_2 F^e(\eta_2) \in H^{d_2}(Y, \mathcal{O}_Y(p^e(K_Y + D'_Y) + nD_Y))$$

are nonzero, implying that the rings R and S are F-regular. □

The hypothesis that the $\mathbb{N}$-grading on R and S is irredundant is indeed required:

Example 3.2 Consider the hypersurface $R := \mathbb{F}_2[x, y, z]/(x^2 + y^3 + z^3)$ where x, y, z have degrees $3, 2, 2$, respectively, and $S := \mathbb{F}_2[u, v]$ where u and v have degree 2. The ring

R is not F-pure or F-regular since the element x belongs to the Frobenius closure of the ideal $(y, z)R$. However, since the ring S is supported only in even degrees, one has

$$R \# S \;=\; R^{(2)} \# S \;=\; \mathbb{F}_2[y, z] \# \mathbb{F}_2[u, v] \;=\; \mathbb{F}_2[uy, uz, vy, vz],$$

which is F-regular. Note that while the fraction field of R contains homogeneous elements of degree 1, the fraction field of S does not.

4 *F*-rationality

Following [11, p. 125], a local ring of positive prime characteristic is F-rational if it is a homomorphic image of a Cohen-Macaulay ring, and each ideal generated by a system of parameters is tightly closed; a Noetherian ring of positive prime characteristic is F-rational if its localization at each maximal ideal (equivalently, at each prime ideal) is F-rational. With this definition, an F-rational ring is normal and Cohen-Macaulay.

For the case of interest in this paper, let R be an $\mathbb{N}$-graded normal domain that is a finitely generated algebra over a field R_0 of positive characteristic. Then R is F-rational if and only if the ideal generated by some (equivalently, any) homogeneous system of parameters for R is tightly closed; see [13, Theorem 4.7] and the preceding remark.

Smith [25] proved that F-rational rings have rational singularities; the converse, more precisely the theorem that rings with rational singularities have F-rational type, is due independently to Hara [7] and to Mehta and Srinivas [19].

Let R be a finitely generated algebra over a field of characteristic zero; Boutot's theorem states that if R has rational singularities, then so does each pure subring of R [1]. The corresponding statement for F-rational rings turns out to be false: in [32] the second author constructed an example of an F-rational ring with a pure subring that is not F-rational. Nonetheless, we have:

Theorem 4.1 *Suppose R and S are F-rational $\mathbb{N}$-graded rings, finitely generated over a perfect field $R_0 = \mathbb{F} = S_0$ of positive characteristic. Then $R \# S$ is F-rational.*

Proof Note that R and S are Cohen-Macaulay; it suffices to assume that they have positive dimension, in which case $a(R) < 0$ and $a(S) < 0$ by [5, Satz 3.1] or [30, Theorem 2.2]. Using this, the Künneth formula shows that $R \# S$ is Cohen-Macaulay and that

$$H^d_{\mathfrak{m}}(R \# S) \;=\; H^{\dim R}_{\mathfrak{m}_R}(R) \# H^{\dim S}_{\mathfrak{m}_S}(S), \tag{4.1.1}$$

where $d := \dim(R \# S)$, and $\mathfrak{m}_R$, $\mathfrak{m}_S$, and $\mathfrak{m}$ are the homogeneous maximal ideals of the rings R, S, and $R \# S$ respectively. The hypothesis that $\mathbb{F}$ is perfect ensures that the ring $R \# S$ is normal. By [9, Corollary 6.8], the ring $R \otimes_{\mathbb{F}} S$ is F-rational.

It suffices to show that the zero submodule of (4.1.1) is tightly closed. Suppose, to the contrary, that c and η are nonzero homogenous elements of $R \# S$ and $H^d_{\mathfrak{m}}(R \# S)$ respectively, with $cF^e(\eta) = 0$ in $H^d_{\mathfrak{m}}(R \# S)$ for $e \gg 0$. It follows that $cF^e(\eta)$ is also zero for $e \gg 0$, when regarded as an element of

$$H^{\dim R}_{\mathfrak{m}_R}(R) \otimes_{\mathbb{F}} H^{\dim S}_{\mathfrak{m}_S}(S).$$

But then η, regarded as an element of the module above, is in the tight closure of zero; this contradicts the F-rationality of $R \otimes_{\mathbb{F}} S$. □

In contrast with Theorem 3.1, $R \# S$ may be F-rational even when R and S are not:

Example 4.2 Let $\mathbb{F}$ be a field of positive characteristic, and consider the hypersurfaces

$$R := \mathbb{F}[x, y, z]/(x^2 + y^3 + z^7) \qquad \text{and} \qquad S := \mathbb{F}[u, v, w]/(u^4 + v^5 + w^5),$$

with x, y, z having degrees 21, 14, 6 respectively, and u, v, w having degrees 5, 4, 4 respectively. Then $a(R) = 1$ and $a(S) = 7$, so R and S are not F-rational. Note that the gradings are irredundant, i.e., as in the hypotheses of Theorem 3.1, the fraction fields of R as well as S contain homogeneous elements of degree 1.

Since $[H^2_{\mathfrak{m}_R}(R)]_{\geq 0}$ is supported only in degree 1, and $[H^2_{\mathfrak{m}_S}(S)]_{\geq 0}$ in degrees 2, 3, and 7, the Künneth formula shows that $R \# S$ is Cohen-Macaulay, and also that $a(R \# S) = -5$. Suppose that the characteristic of $\mathbb{F}$ is at least 7. Then the Frobenius action on each of

$$[H^2_{\mathfrak{m}_R}(R)]_{\leq -5} \qquad \text{and} \qquad [H^2_{\mathfrak{m}_S}(S)]_{\leq -5}$$

and hence on $H^2_{\mathfrak{m}_R}(R) \# H^2_{\mathfrak{m}_S}(S)$ is injective. Moreover, we claim that $R \# S$ has an isolated non F-regular point: to see this, let $r \otimes s$ be a nonzero homogeneous element of $R \# S$ of positive degree; then the ring

$$(R \# S)_{r \otimes s} \;=\; R_r \# S_s$$

is a pure subring of the regular ring $R_r \otimes_{\mathbb{F}} S_s$, and is hence F-regular. It follows that $R \# S$ is F-rational by [13, Theorem 7.1].

5 Finite Frobenius Representation Type

The notion of rings of finite Frobenius representation type (FFRT) is due to Smith and Van den Bergh; it is an essential ingredient in their proof of the following remarkable theorem: If R is a graded direct summand of a polynomial ring over a perfect field $\mathbb{F}$ of positive characteristic, then the ring of $\mathbb{F}$-linear differential operators on R is a simple ring, see [27, Theorem 1.3]. This is striking in that the corresponding statement is not known for polynomial rings over fields of characteristic zero.

Subsequently, the FFRT property has found several other applications: Seibert [23] proved that over rings with FFRT, the Hilbert-Kunz multiplicity is rational; tight closure commutes with localization for rings with FFRT by Yao [33]; if R is a Gorenstein ring with FFRT, Takagi and Takahashi [28] proved that each local cohomology module of the form $H^k_{\mathfrak{a}}(R)$ has finitely many associated primes; the Gorenstein hypothesis may be removed, as proved subsequently by Hochster and Núñez-Betancourt [14].

A reduced ring R of positive prime characteristic p, satisfying the Krull-Schmidt theorem, is said to have *finite Frobenius representation type* if there exists a finite set $\mathcal{S}$ of R-modules such that for each $q = p^e$, each indecomposable summand of $R^{1/q}$ is isomorphic to an element of $\mathcal{S}$. When R is Cohen-Macaulay, each indecomposable summand of $R^{1/q}$ is a maximal Cohen-Macaulay R-module; thus, Cohen-Macaulay rings of finite representation type have FFRT, though the latter property is much weaker: e.g., in the graded setting, the FFRT property is inherited by direct summands [27, Proposition 3.1.6].

Key examples of rings with FFRT include those that are graded direct summands of polynomial rings; such rings are also F-regular, and hence Cohen-Macaulay. Recent work on the FFRT property includes that of Hara and Ohkawa [8], where they study the property for 2-dimensional normal graded rings in terms of $\mathbb{Q}$-divisors, and [21, 22] where Raedschelders, Špenko, and Van den Bergh prove that over an algebraically closed field of characteristic

$p \geq \max\{n-2, 3\}$, the Plücker homogeneous coordinate ring of the Grassmannian $G(2, n)$ has FFRT.

Our goal here is to construct normal rings with FFRT that are not Cohen-Macaulay. Note that a Stanley-Reisner ring over a perfect field has FFRT by [16, Example 2.36], though such a ring need not be Cohen-Macaulay. Our interest here, however, is primarily in normal domains. We first record:

Lemma 5.1 *Let $\mathbb{F}$ be a perfect field of positive characteristic, and let R and S be reduced rings that are finitely generated $\mathbb{F}$-algebras. Suppose, moreover, that R, S, and $R \otimes_{\mathbb{F}} S$ satisfy the Krull-Schmidt theorem. Then, if R and S have FFRT, so does $R \otimes_{\mathbb{F}} S$.*

Proof If R and S have FFRT, there exist indecomposable R-modules $M_1, \ldots, M_m$, and indecomposable S-modules $N_1, \ldots, N_n$ such that for each $q = p^e$, one has

$$R^{1/q} \cong \bigoplus M_i \quad \text{and} \quad S^{1/q} \cong \bigoplus N_j,$$

where, in each case, the index set depends on q, and modules may be repeated within the direct sum. Set $T := R \otimes_{\mathbb{F}} S$. Then

$$T^{1/q} \cong R^{1/q} \otimes_{\mathbb{F}} S^{1/q} \cong \left(\bigoplus M_i\right) \otimes_{\mathbb{F}} \left(\bigoplus N_j\right) \cong \bigoplus \left(M_i \otimes_{\mathbb{F}} N_j\right).$$

Each of the mn modules of the form $M_i \otimes_{\mathbb{F}} N_j$ is a direct sum of finitely many indecomposable T-modules. This provides a finite set of indecomposable T-modules that contains an isomorphic copy of each indecomposable summand of $T^{1/q}$ for each $q = p^e$. □

Proposition 5.2 *Let R and S be $\mathbb{N}$-graded reduced rings, finitely generated over a perfect field $R_0 = \mathbb{F} = S_0$ of positive characteristic. If R and S have FFRT, then the rings $R \otimes_{\mathbb{F}} S$ and $R \# S$ also have FFRT.*

Proof The statement regarding the tensor product follows immediately from the lemma, bearing in mind that the Krull-Schmidt theorem holds for $\mathbb{N}$-graded rings A with A_0 a field.

The assertion about the Segre product follows from [27, Proposition 3.1.6], since $R \# S$ is a graded direct summand of the tensor product $R \otimes_{\mathbb{F}} S$. □

Example 5.3 Let $\mathbb{F}$ be a perfect field of characteristic $p \geq 7$, and consider the hypersurface $R := \mathbb{F}[x, y, z]/(x^2 + y^3 - z^p)$, with x, y, z having degrees $3p, 2p, 6$ respectively. Note that the ring R is sandwiched between $A := \mathbb{F}[x, y]$ and $A^{1/p} = \mathbb{F}[x^{1/p}, y^{1/p}]$, since

$$z = x^{2/p} + y^{3/p}.$$

As A is a polynomial ring, and hence has finite representation type, it follows that R has FFRT by [24, Observation 3.7, Theorem 3.10]. Set $S := \mathbb{F}[u, v]$, where u and v are indeterminates with degree 1. Then the ring $R \# S$ has FFRT by Proposition 5.2. However, since $a(R) = p - 6 > 0$, the Künneth formula shows that $R \# S$ is not Cohen-Macaulay.

Remark 5.4 The examples above are characteristic-specific: to illustrate, let $p \geq 7$ be a prime integer, and let $\mathbb{F}$ now be an *arbitrary* field. Set $\mathbb{P}^1 := \operatorname{Proj} \mathbb{F}[u, v]$, with points of $\mathbb{P}^1$ parametrized by u/v. If $p = 6k + 1$, consider the $\mathbb{Q}$-divisor

$$D := \frac{1}{2}(0) - \frac{1}{3}(\infty) - \frac{k}{p}(-1). \tag{5.4.1}$$

Then $\Gamma_*(\mathbb{P}^1, D) := \bigoplus H^0(\mathbb{P}^1, nD)T^n$ is the $\mathbb{F}$-algebra generated by

$$z := \frac{v^2(u+v)}{u^3}T^6, \quad y := \frac{v^{4k+1}(u+v)^{2k}}{u^{6k+1}}T^{2p}, \quad x := \frac{v^{6k+1}(u+v)^{3k}}{u^{9k+1}}T^{3p},$$

where T is an indeterminate of degree one. It is readily seen that $\Gamma_*(\mathbb{P}^1, D)$ is a hypersurface with defining equation $z^p = x^2 + y^3$.

If $p = 6k - 1$, consider instead the $\mathbb{Q}$-divisor

$$D := \frac{1}{3}(\infty) + \frac{k}{p}(-1) - \frac{1}{2}(0). \tag{5.4.2}$$

In this case, $\Gamma_*(\mathbb{P}^1, D)$ is the $\mathbb{F}$-algebra generated by

$$z := \frac{u^3}{v^2(u+v)}T^6, \quad y := \frac{u^{6k-1}}{v^{4k-1}(u+v)^{2k}}T^{2p}, \quad x := \frac{u^{9k-1}}{v^{6k-1}(u+v)^{3k}}T^{3p}.$$

Once again, $\Gamma_*(\mathbb{P}^1, D)$ is a hypersurface with defining equation $z^p = x^2 + y^3$.

Note that the denominators occurring in the $\mathbb{Q}$-divisor D in (5.4.1) and (5.4.2) are 2, 3, and p. It follows from [8, Theorem 7.2] that if the characteristic of $\mathbb{F}$ is not 2, 3, or p, then the hypersurface $\mathbb{F}[x, y, z]/(x^2 + y^3 - z^p)$ does not have FFRT.

This raises the question:

Question 5.5 Let R be a normal graded domain, finitely generated over a field of characteristic zero. If R has dense FFRT type, i.e., there exists a dense set of prime integers p for which the mod p reductions R_p have FFRT, then is R a Cohen-Macaulay ring?

A related question is the following; see also [18, Question 9.1].

Question 5.6 Let R be a normal graded domain, finitely generated over a field of characteristic zero. If R has dense FFRT type, then is R an F-regular ring?

The converse is false: [26, Theorem 5.1] provides an example of an F-regular hypersurface R, over a field of characteristic zero, for which each mod p reduction R_p has a local cohomology module of the form $H^3_I(R_p)$ that has infinitely many associated prime ideals; it follows from [28, Theorem 3.9] or [14, Theorem 5.7] that, for each prime integer p, the mod p reduction R_p does not have FFRT.

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References

1. Boutot, J.-F.: Singularités rationnelles et quotients par les groupes réductifs. Invent. Math. **88**, 65–68 (1987)
2. Chow, W.-L.: On unmixedness theorem. Amer. J. Math. **86**, 799–822 (1964)
3. Demazure, M.: Anneaux gradués normaux. In: Introduction á la théorie des singularités II. Travaux en Cours, vol. 37, pp. 35–68. Hermann, Paris (1988)
4. Dolgačev, I.V.: Automorphic forms, and quasihomogeneous singularities. Funkcional. Anal. i Priložen. **9**, 67–68 (1975)

5. Flenner, H.: Rationale quasihomogene Singularitäten. Arch. Math. **36**, 35–44 (1981)
6. Goto, S., Watanabe, K.-i.: On graded rings I. J. Math. Soc. Japan. **30**, 179–213 (1978)
7. Hara, N.: A characterisation of rational singularities in terms of injectivity of Frobenius maps. Amer. J. Math. **120**, 981–996 (1998)
8. Hara, N., Ohkawa, R.: The FFRT property of two-dimensional graded rings and orbifold curves. Adv. Math. **370**, 107215 (2020)
9. Hashimoto, M.: Cohen-Macaulay F-injective homomorphisms. In: Geometric and combinatorial aspects of commutative algebra (Messina, 1999), Lecture Notes in Pure and Appl. Math., vol. 217, pp. 231–244. Dekker, New York (2001)
10. Hashimoto, M.: Surjectivity of multiplication and F-regularity of multigraded rings. In: Commutative algebra (Grenoble/Lyon, 2001), Contemp. Math., vol. 331, pp. 153–170. Amer. Math. Soc., Providence, RI (2003)
11. Hochster, M.: Foundations of tight closure theory. http://www.math.lsa.umich.edu/~hochster/. Accessed 1 Mar 2023
12. Hochster, M., Huneke, C.: Tight closure, invariant theory, and the Briançon-Skoda theorem. J. Amer. Math. Soc. **3**, 31–116 (1990)
13. Hochster, M., Huneke, C.: Tight closure of parameter ideals and splitting in module-finite extensions. J. Algebraic Geom. **3**, 599–670 (1994)
14. Hochster, M., Núñez-Betancourt, L.: Support of local cohomology modules over hypersurfaces and rings with FFRT. Math. Res. Lett. **24**, 401–420 (2017)
15. Hochster, M., Roberts, J.: Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay. Adv. Math. **13**, 115–175 (1974)
16. Kamoi, Y.: A study of Noetherian G-graded rings. Ph.D. thesis, Tokyo Metropolitan University (1995)
17. Lyubeznik, G., Smith, K.E.: Strong and weak F-regularity are equivalent for graded rings. Amer. J. Math. **121**, 1279–1290 (1999)
18. Mallory, D.: Finite F-representation type for homogeneous coordinate rings of non-Fano varieties. arXiv preprint arXiv:2207.08966
19. Mehta, V.B., Srinivas, V.: A characterization of rational singularities. Asian J. Math. **1**, 249–271 (1997)
20. Pinkham, H.: Normal surface singularities with C^* action. Math. Ann. **227**, 183–193 (1977)
21. Raedschelders, T., Špenko, Š, Van den Bergh, M.: The Frobenius morphism in invariant theory. Adv. Math. **348**, 183–254 (2019)
22. Raedschelders, T., Špenko, Š, Van den Bergh, M.: The Frobenius morphism in invariant theory II. Adv. Math. **410**, 108587 (2022)
23. Seibert, G.: The Hilbert-Kunz function of rings of finite Cohen-Macaulay type. Arch. Math. (Basel) **69**, 286–296 (1997)
24. Shibuta, T.: One-dimensional rings of finite F-representation type. J. Algebra **332**, 434–441 (2011)
25. Smith, K.E.: F-rational rings have rational singularities. Amer. J. Math. **119**, 159–180 (1997)
26. Singh, A.K., Swanson, I.: Associated primes of local cohomology modules and of Frobenius powers. Int. Math. Res. Not. **33**, 1703–1733 (2004)
27. Smith, K.E., Van den Bergh, M.: Simplicity of rings of differential operators in prime characteristic. Proc. London Math. Soc. **75**, 32–62 (1997)
28. Takagi, S., Takahashi, R.: D-modules over rings with finite F-representation type. Math. Res. Lett. **15**, 563–581 (2008)
29. Watanabe, K.-i.: Some remarks concerning Demazure's construction of normal graded rings. Nagoya Math. J. **83**, 203–211 (1981)
30. Watanabe, K.-i.: Rational singularities with k^*-action. In: Commutative algebra (Trento, 1981), Lecture Notes in Pure and Appl. Math., vol. 84, pp. 339–351. Dekker, New York (1983)
31. Watanabe, K.-i.: F-regular and F-pure normal graded rings. J. Pure Appl. Algebra **71**, 341–350 (1991)
32. Watanabe, K.-i.: F-rationality of certain Rees algebras and counterexamples to "Boutot's Theorem" for F-rational rings. J. Pure Appl. Algebra **122**, 323–328 (1997)
33. Yao, Y.: Modules with finite F-representation type. J. London Math. Soc. **2**(72), 53–72 (2005)