



Exponential Entropy Dissipation for Weakly Self-Consistent Vlasov–Fokker–Planck Equations

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Abstract

We study long-time dynamical behaviors of weakly self-consistent Vlasov–Fokker–Planck equations. We introduce Hessian matrix conditions on mean-field kernel functions, which characterizes the exponential convergence of solutions in L^1 distances. The matrix condition is derived from the dissipation of a selected Lyapunov functional, namely auxiliary Fisher information functional. We verify proposed matrix conditions in examples.

Keywords Auxiliary mean-field Fisher information functional · Information gamma calculus · Mean-field information Hessian matrix

1 Introduction

Weakly self-consistent Vlasov–Poisson–Fokker–Planck equations (Bolley et al. 2010; Guillin et al. 2021; Villani 2009) play essential roles in mathematical physics and

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probability with applications in modeling and machine learning sampling problems. The equation describes the probability density's evolution of particles, which interact with each other from interaction energies while under white noise perturbations.

Consider a mean-field underdamped Langevin diffusion process

$$\begin{cases} dx_t = v_t dt \\ dv_t = -v_t dt - \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} \nabla_x W(x_t, y) f(t, y, \tilde{v}) d\tilde{v} dy + \nabla_x U(x_t) \right) dt + \sqrt{2} dB_t, \end{cases}$$

where $(x_t, v_t) \in \mathbb{T}^d \times \mathbb{R}^d$ presents an identical particle's position and velocity, $\mathbb{T}^d$ is a d dimensional torus representing a position domain, and B_t is a standard Brownian motion in $\mathbb{R}^d$. Each identical particle interacts with each other through a mean-field interaction potential W and a confinement potential U . We assume that $W \in C^\infty(\mathbb{T}^d \times \mathbb{T}^d)$ is a symmetric kernel function, that is, $W(x, y) = W(y, x)$. Denote $f = f(t, x, v)$ as the probability density function of the stochastic process (x_t, v_t) . The density function f follows a nonlinear Fokker–Planck equation:

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f - \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} \nabla_x W(x, y) f(t, y, \tilde{v}) d\tilde{v} dy + \nabla_x U(x) \right) \cdot \nabla_v f \\ = \nabla_v \cdot (f v) + \nabla_v \cdot (\nabla_v f). \end{aligned} \quad (1.1)$$

An equilibrium of Eq. (1.1) satisfies

$$f_\infty(x, v) = \frac{1}{Z} e^{-\frac{\|v\|^2}{2} - \int_{\mathbb{T}^d \times \mathbb{R}^d} W(x, y) f_\infty(y, \tilde{v}) dy d\tilde{v} - U(x)},$$

where $Z = \int_{\mathbb{T}^d \times \mathbb{R}^d} e^{-\frac{\|v\|^2}{2} - \int_{\mathbb{T}^d \times \mathbb{R}^d} W(x, y) f_\infty(y, \tilde{v}) dy d\tilde{v} - U(x)} dx dv < +\infty$ is a normalization constant. This is known as a nonlinear Gibbs distribution. We are interested in studying long-time behaviors of density functions. How fast does function f converge to f_∞ ?

In this paper, we establish an exponential convergence result for the solution of (1.1) in both functional free energy and L^1 distances. Our method follows from a Lyapunov method, where the Lyapunov functional is selected as auxiliary Fisher information functionals. From the dissipation of Lyapunov functionals along with PDE (1.1), we derive a matrix condition, which guarantees the exponential convergence decay result in both functional free energy and L^1 distances. Explicit examples are studied.

In the literature, various properties of the Vlasov–Fokker–Planck equation have been studied, e.g., Bouchut (1993), Bouchut and Dolbeault (1995), Carrillo and Soler (1995), Degond (1986), Esposito et al. (2010). The original Vlasov equation involves the inverse of the Laplacian operator as the interaction kernel function. For simplicity, we only focus on the weakly self-consistent kernel functions, where W is a given smooth function. The existence of a smooth solution and the regularity property have recently been studied in Cesbron and Herda (2023), see also the general approach regarding the regularity properties in Villani (2009)[Appendix A.21]. The convergence for particle systems with mean-field interactions has been studied in

Guillin et al. (2019, 2021). And Guillin et al. (2021) proves the exponential convergence in H^1 norms; Bolley et al. (2010), Wang (2021) show exponential convergence results in Wasserstein-2 type distances. In addition, Guillin et al. (2021) studies the exponential contraction of the solution in the Wasserstein-1 distance for nonconvex confinement potential energies. We remark that there are comparison studies and discussions between H^1 and L^1 for the Fokker–Planck equations; see the detailed argument in Markowich and Villani (2000). One important fact of the L^1 norm is that this ensures the density has finite mass in physics. In particular, the L^1 distance is closely related to the Helmholtz free energy for physical systems through the Csiszàr–Kullback inequality or the Pinsker inequality. Our convergence analysis of the functional free energy is crucial for statistical physics-oriented equations, e.g., spatially homogeneous Fokker–Planck–Landau equation in plasma physics (Desvillettes and Villani 2000), Fokker–Planck equation for granular media (Benedetto et al. 1998), etc. Furthermore, our methods are closely related to but technically different from Villani’s hypocoercivity methods (Villani 2009). Villani’s methods estimate the first-order dissipation, which prove the $O(t^{-\infty})$ decay for $W(x, y) = W(x - y)$; see [Villani (2009), Theorem 56]. Meanwhile, we estimate the second-order dissipation and obtain a Hessian matrix condition of interaction kernel function W and potential function U to determine the exponential convergence result. As in Feng and Li (2023, 2021), we develop entropy dissipation methods (Arnold and Carlen 2000; Arnold and Erb 2014; Arnold et al. 2000, 2001; Bakry and Émery 1985; Baudoin and Garofalo 2016; Calogero 2012; Carrillo et al. 2003; Chow et al. 2018; Li 2021) for Eq. (1.1). It is worth mentioning that the convergence analysis of mean-field Langevin dynamics and nonlinear Fokker–Planck equations are important in artificial intelligence (AI) sampling problems (Carrillo et al. 2021; Garbuno-Inigo et al. 2020; Hu et al. 2019; Kazeykina et al. 2020; Li et al. 2021). This is to design mean-field Markov–Chain–Monte–Carlo (MCMC) sampling algorithms. The convergence analysis of f toward f_∞ plays a key role in AI theory. It helps in designing algorithmic reliable kinetic sampling methods (Feng and Li 2021; Li et al. 2021; Ma et al. 2019) in Bayesian inverse problems. In this direction, our result estimates the exponential decay rates in both functional free energy and L^1 distance for kinetic-type degenerate mean-field stochastic processes. Our explicit condition for the potential function V and the interaction kernel W are more straightforward to verify in the applications than those using the Logarithmic Sobolev-type inequalities as the assumption.

This paper is organized as follows. In Sect. 2, we present the main result of this paper. In particular, we give the Hessian matrix conditions for (1.1), under which we establish the global exponential convergence results in time in both the functional free energy and the norm L^1 . In Sect. 3, we verify the proposed conditions in examples. In Sects. 4 and 5, we provide the proofs of the main results.

2 Main Results

In this section, we present the main result of this paper. We first introduce the notions of free energy, Fisher information, and an auxiliary functional, which are the Lyapunov functionals in this paper. We next present the main theorem, which holds under a mean

field information matrix condition. This is based on a Lyapunov method, under which we derive an exponential Lyapunov constant for PDE (1.1). We last present examples.

2.1 Notations

We briefly introduce the analytical property for the solution of PDE (1.1). Assume that $W(x, y) = W(y, x)$ and $W \in C^2(\mathbb{T}^d \times \mathbb{T}^d)$. Denote $f_0 = f_0(x, v)$ as a probability density on $\Omega := \mathbb{T}^d \times \mathbb{R}^d$, such that all moments of density function f_0 are finite, that is, $\int_{\Omega} \|v\|^k f_0(x, v) dx dv < +\infty$, for all $k \in \mathbb{Z}_+$. We assume that there exists a unique smooth solution $f = f(t, x, v)$ of Eq. (1.1). See details in Villani (2009).

We consider a Lyapunov functional for (1.1), which is often named free energy. For convenience of presentation, denote a probability density space supported on $\Omega = \mathbb{T}^d \times \mathbb{R}^d$.

$$\mathcal{P} = \left\{ f \in L^1(\Omega) : \int_{\Omega} f(x, v) dx dv = 1, \quad f \geq 0 \right\}.$$

Definition 2.1 (*Free energy*) Define a functional $\mathcal{E} : \mathcal{P} \rightarrow \mathbb{R}$ as

$$\begin{aligned} \mathcal{E}(f) = & \int_{\Omega} f(x, v) \log f(x, v) dx dv + \int_{\Omega} \frac{1}{2} \|v\|^2 f(x, v) dx dv \\ & + \frac{1}{2} \int_{\Omega \times \Omega} W(x, y) f(x, v) f(y, \tilde{v}) dx dv dy d\tilde{v} + \int_{\Omega} U(x) f(x, v) dx dv. \end{aligned} \quad (2.1)$$

In this paper, we study the convergence behavior of (1.1) through functional $\mathcal{E}(f)$. We check that when f_{∞} is the minimizer of $\mathcal{E}$, then

$$\begin{aligned} \frac{\delta}{\delta f(x, v)} \mathcal{E}(f)|_{f=f_{\infty}} = & \log f_{\infty}(x, v) + 1 + \frac{1}{2} \|v\|^2 \\ & + \int_{\Omega} W(x, y) f_{\infty}(y, \tilde{v}) dy d\tilde{v} + U(x) = C, \end{aligned}$$

where $\frac{\delta}{\delta f}$ is the L^2 first variation operator w.r.t. f , and C is a constant. In other words,

$$f_{\infty}(x, v) = \frac{1}{Z} e^{-\frac{\|v\|^2}{2} - \int_{\Omega} W(x, y) f_{\infty}(y, \tilde{v}) dy d\tilde{v} - U(x)},$$

where $Z = \int_{\Omega} e^{-\frac{\|v\|^2}{2} - \int_{\Omega} W(x, y) f_{\infty}(y, \tilde{v}) dy d\tilde{v} - U(x)} dx dv < +\infty$ is a normalization constant. In the literature, we note that functional $\mathcal{E}(f)$ is often named the *free energy*, and f_{∞} is called a *nonlinear Gibbs distribution* (Carlen et al. 2003; Esposito et al. 2010).

In this paper, we mainly study the long-time dynamical behavior of $f(t, x, v)$ for general interaction kernel function W and potential function U . In particular, we shall investigate that how fast does the Lyapunov functional $\mathcal{E}(f(t, \cdot))$ converge to $\mathcal{E}(f_{\infty})$.

To study this convergence, we introduce some necessary notations and functionals. Denote $I_d \in \mathbb{R}^{d \times d}$ as the identity matrix. Let

$$a = \begin{pmatrix} 0 \\ I_d \end{pmatrix} \in \mathbb{R}^{2d \times d}, \quad z = \begin{pmatrix} z_1 I_d \\ z_2 I_d \end{pmatrix} \in \mathbb{R}^{2d \times d}, \quad (2.2)$$

where $z_1, z_2 \in \mathbb{R}$ are two given constants. Using above matrices, we define the following functionals to characterize the decay of Lyapunov functional $\mathcal{E}$. Denote

$$\frac{\delta}{\delta f(x, v)} \mathcal{E}(f) = \log f(x, v) + 1 + \frac{1}{2} \|v\|^2 + \int_{\Omega} W(x, y) f(y, \tilde{v}) dy d\tilde{v} + U(x).$$

Definition 2.2 (Fisher information functionals) Define a functional $\mathcal{DE}_a: \mathcal{P} \rightarrow \mathbb{R}_+$ as

$$\mathcal{DE}_a(f) := \int_{\Omega} \langle \nabla_{x,v} \frac{\delta}{\delta f(x, v)} \mathcal{E}(f), a a^T \nabla_{x,v} \frac{\delta}{\delta f(x, v)} \mathcal{E}(f) \rangle f(x, v) dx dv. \quad (2.3)$$

Define an auxiliary functional $\mathcal{DE}_z: \mathcal{P} \rightarrow \mathbb{R}_+$ as

$$\mathcal{DE}_z(f) := \int_{\Omega} \langle \nabla_{x,v} \frac{\delta}{\delta f(x, v)} \mathcal{E}(f), z z^T \nabla_{x,v} \frac{\delta}{\delta f(x, v)} \mathcal{E}(f) \rangle f(x, v) dx dv. \quad (2.4)$$

It is known that $\mathcal{DE}_a$, named “Fisher information functional,” equals to the decay of free energy $\mathcal{E}$ along with the solution of PDE (1.1). In other words,

$$\frac{d}{dt} \mathcal{E}(f(t, \cdot, \cdot)) = -\mathcal{DE}_a(f(t, \cdot, \cdot)) \leq 0. \quad (2.5)$$

This result is stated in Lemma 4.1. We note that functional $\mathcal{DE}_a$ itself cannot guarantee the L^1 decay of the solution, due to the degeneracy of the subelliptic operator in PDE (1.1). To overcome this degeneracy issue, we construct an additional functional $\mathcal{DE}_z$. We call $\mathcal{DE}_z$ the “auxiliary Fisher information functional.” Shortly, we demonstrate that the designed auxiliary functional $\mathcal{DE}_z$ is useful in establishing the decay rate of the degenerate subelliptic operator in (1.1).

2.2 Main Result

We are ready to present the main result. We first provide a matrix eigenvalue assumption.

Definition 2.3 (Mean-field information matrix) Define a symmetric matrix function $\mathfrak{R} \in \mathbb{R}^{4d \times 4d}$, such that

$$\mathfrak{R}(z, x, y) = \frac{1}{2} \begin{pmatrix} A(x, y) & B(x, y) \\ B(x, y) & A(y, x) \end{pmatrix},$$

where $A, B \in \mathbb{R}^{2d \times 2d}$ are defined below: Denote

$$\begin{aligned} V(x, y) &= U(x) + W(x, y), \\ A(x, y) &= \begin{pmatrix} z_1 z_2 \mathbb{I}_d & \frac{1}{2}[(1 + z_1 z_2 + z_2^2)\mathbb{I}_d - z_1^2 \nabla_{xx}^2 V(x, y)] \\ \frac{1}{2}[(1 + z_1 z_2 + z_2^2)\mathbb{I}_d - z_1^2 \nabla_{xx}^2 V(x, y)] & (1 + z_2^2)\mathbb{I}_d - z_1 z_2 \nabla_{xx}^2 V(x, y) \end{pmatrix}, \\ A(y, x) &= \begin{pmatrix} z_1 z_2 \mathbb{I}_d & \frac{1}{2}[(1 + z_1 z_2 + z_2^2)\mathbb{I}_d - z_1^2 \nabla_{yy}^2 V(y, x)] \\ \frac{1}{2}[(1 + z_1 z_2 + z_2^2)\mathbb{I}_d - z_1^2 \nabla_{yy}^2 V(y, x)] & (1 + z_2^2)\mathbb{I}_d - z_1 z_2 \nabla_{yy}^2 V(y, x) \end{pmatrix}, \end{aligned}$$

and

$$B(x, y) = \begin{pmatrix} 0 & -\frac{z_1^2}{2} \nabla_{xy}^2 W(x, y) \\ -\frac{z_1^2}{2} \nabla_{xy}^2 W(x, y) & -z_1 z_2 \nabla_{xy}^2 W(x, y) \end{pmatrix}.$$

Assumption 2.1 (*Mean-field information matrix condition*) Assume that there exists a constant $\lambda > 0$, such that

$$\mathfrak{R}(z, x, y) \geq \lambda \begin{pmatrix} aa^\top + zz^\top & 0 \\ 0 & aa^\top + zz^\top \end{pmatrix}, \quad (2.6)$$

where a, z are defined in (2.2), such that

$$aa^\top + zz^\top = \begin{pmatrix} z_1^2 \mathbb{I}_d & z_1 z_2 \mathbb{I}_d \\ z_1 z_2 \mathbb{I}_d & (1 + z_2^2) \mathbb{I}_d \end{pmatrix}.$$

Under the above matrix eigenvalue condition, we next prove the following main theorem.

Theorem 2.1 Suppose that Assumption 2.1 holds, and there exists a smooth solution $f(t, x, v)$ of (1.1). Then the following exponential convergence result is satisfied.

$$\mathcal{E}(f(t, \cdot, \cdot)) - \mathcal{E}(f_\infty) \leq \frac{1}{2\lambda} e^{-2\lambda t} [\mathcal{DE}_{a,z}(f_0) - \mathcal{DE}_{a,z}(f_\infty)], \quad (2.7)$$

where

$$\mathcal{DE}_{a,z}(f) := \mathcal{DE}_a(f) + \mathcal{DE}_z(f). \quad (2.8)$$

Corollary 2.2 Suppose Assumption 2.1 holds, and there exists a smooth solution $f(t, x, v)$ of (1.1). Assume that there exists a sufficient small constant $C_W > 0$, such that

$$\max_{(x,y) \in \mathbb{T}^d \times \mathbb{T}^d} |W(x, y)| \leq C_W. \quad (2.9)$$

Then the following L^1 distance convergence holds.

$$\int_{\Omega} |f(t, x, v) - f_{\infty}(x, v)| dx dv \leq C e^{-\lambda t} \sqrt{\mathcal{DE}_{a,z}(f_0) - \mathcal{DE}_{a,z}(f_{\infty})},$$

for some constant $C > 0$.

Remark 2.1 The functional $\mathcal{E}$ is also significant in physics. It is the Helmholtz free energy. Thus, the convergence analysis of the kinetic Fokker–Planck equation in terms of Helmholtz free energy is crucial for statistical physics-oriented equations. The second law of thermodynamics shows that free energy dissipation equals the negative Fisher information; see Lemma 4.1. The analysis in this paper further introduces the convergence rate of the free energy. Regarding Lyapunov’s functions, there are other choices, including H^{-1} distances. To our knowledge, they are not oriented for analyzing the Helmholtz free energy. We refer interesting readers to Markowich and Villani (2000) for the importance of L^1 distances and free energy estimations.

2.3 Examples

We last present two concrete examples of L^1 distance exponential convergence results for different kernels W and potentials U in (1.1). We leave their proofs with detailed conditions in Sect. 3.

Example 2.1 Assume $W(x, y) = W(y, x)$, $U(x) \neq 0$. For $z_1 = 1$ and $z_2 = 0.3$, assume

$$\underline{\lambda} l_d \leq \nabla_{xx}^2(U(x) + W(x, y)) \leq \bar{\lambda} l_d, \quad 2\underline{\lambda} - \bar{\lambda}^2 > 0.08,$$

and for sufficiently small $\varepsilon > 0$,

$$\|\nabla_{x,y}^2 W(x, y)\|_F = O(\varepsilon), \quad \max_{(x,y) \in \mathbb{T}^d \times \mathbb{T}^d} |W(x, y)| < 1,$$

where $\|\cdot\|_F$ is the matrix Frobenius norm. Then Assumption 2.1 is satisfied. Thus, the exponential convergence results in (2.7) and (2.8) hold.

Example 2.2 Consider $W(x, y) = W(y, x)$, $U(x) \neq 0$. Assume

$$\begin{cases} \nabla_{xy}^2 W(x, y) = Q_W^{-1} \text{Diag}(\lambda_1^W, \dots, \lambda_d^W) Q_W, \\ \nabla_{xx}^2 W(x, y) = Q_W^{-1} \text{Diag}(\tilde{\lambda}_1^W, \dots, \tilde{\lambda}_d^W) Q_W, \\ \underline{\lambda} l_d \leq \nabla_{xx}^2 U(x) \leq \bar{\lambda} l_d, \end{cases}$$

where Q_W denotes the orthogonal matrix for the eigenvalue decomposition of $\nabla^2 W$. Let $\underline{\lambda} = \bar{\lambda} = 0.9$, $z_1 = 1$, and $z_2 = 0.3$. If the following condition holds,

$$-0.538 < \tilde{\lambda}_i^W < 0.297, \quad i = 1, \dots, d, \quad \max_{(x,y) \in \mathbb{T}^d \times \mathbb{T}^d} |W(x, y)| < 1,$$

then for sufficiently small λ_i^W , Assumption 2.1 holds true. Thus, the exponential convergence results in (2.7) and (2.8) hold. In particular, for $d = 1$, if $\tilde{\lambda}_i^W = -0.12$, then $|\lambda_i^W| < 10^{-3}$ is enough to guarantee Assumption 2.1. If $W(x, y) = W(x - y)$ and $\tilde{\lambda}_i^W = -\lambda_i^W$ is small enough, Assumption 2.1 holds.

Remark 2.2 (*Comparisons with Carrillo et al. (2003)*) The mentioned paper studies a matrix eigenvalue condition for gradient-drift Fokker–Planck equation. In example 2.2, we work on a matrix eigenvalue condition for degenerate non-gradient-drift Fokker–Planck equation.

Remark 2.3 (*Comparisons with Guillin et al. (2021)*) The paper in this remark establishes the exponential convergence results in weighted Sobolev space. Meanwhile, we show the exponential convergence results in L^1 distance.

Remark 2.4 The work of Villani (2009) analyzes the case for $U \equiv 0$ and $W(x, y) = W(x - y)$. We shall show that Assumption 2.1 does not hold for constant matrices a and z . This implies that exponential decay does not hold in this example. In this sense, our result does not improve the $O(t^{-\infty})$ convergence result in Villani (2009)[Theorem 56].

3 Verification of Assumptions in Examples

In this section, we verify Assumption 2.1 in two examples.

3.1 Proof of Example 2.1

Lemma 3.1 *Assume that*

$$\begin{cases} \nabla_{xy}^2 W(x, y) = Q_W^{-1} \text{Diag}(\lambda_1^W, \dots, \lambda_d^W) Q_W, \\ \nabla_{xx}^2 W(x, y) + \nabla_{xx}^2 U(x) = Q_V^{-1} \text{Diag}(\tilde{\lambda}_1^W, \dots, \tilde{\lambda}_d^W) Q_V, \end{cases} \quad (3.1)$$

where Q_W and Q_V denote the orthogonal matrix for the eigenvalue decomposition of $\nabla_{xy}^2 W(x, y)$ and $\nabla_{xx}^2 (W(x, y) + U(x))$ with $Q_W^{-1} = Q_W^T$ and $Q_V^{-1} = Q_V^T$.

(1) *If there exist positive constants $z_1, z_2, \lambda_{W_{xx}} > 0$, such that*

$$A(x, y) \geq \lambda_{W_{xx}} I_{2d}, \quad \text{and} \quad C_1 < \lambda_{W_{xx}} < C_2, \quad (3.2)$$

where $A(x, y)$ is defined in Definition 2.3, and

$$\begin{aligned} C_1 &= \sqrt{\frac{z_1^2 z_2^2 + \frac{z_1^4}{2} + \sqrt{z_1^4 z_2^4 + z_1^6 z_2^2}}{2}} |\lambda_i^W|, \\ C_2 &= \min \left\{ z_1 z_2 - \left| \frac{1}{2} [(1 + z_1 z_2 + z_2^2) - z_1^2 \tilde{\lambda}_i^W] \right|, \right. \end{aligned}$$

$$(1 + z_2^2) - z_1 z_2 \tilde{\lambda}_i^W - \left| \frac{1}{2} [(1 + z_1 z_2 + z_2^2) - z_1^2 \tilde{\lambda}_i^W] \right|, \\ i = 1, \dots, d,$$

then Assumption (2.1) holds.

(2) Suppose there exists constants $\bar{\lambda} \geq \underline{\lambda} > 0$, such that for any $(x, y) \in \mathbb{T}^d \times \mathbb{T}^d$,

$$\underline{\lambda} l_d \leq \nabla_{xx}^2 W(x, y) + \nabla_{xx}^2 U(x) \leq \bar{\lambda} l_d,$$

Assume that $\nabla_{xy}^2 W(x, y) = 0$, $z_1 = 1$, and there exist constant $z_2 \in (0, \frac{1+\sqrt{5}}{2})$ and $\delta > 0$, such that $\underline{\lambda}$, $\bar{\lambda}$ satisfies the following conditions:

$$2\underline{\lambda} - \bar{\lambda}^2 > 1 - \delta, \quad [2(z_2 - z_2^2)]\underline{\lambda} + 2z_2 + 2z_2^3 - z_2^4 - 3z_2^2 > \delta, \quad (3.3)$$

then Assumption 2.1 holds

Proof Case 1: According to Definition 2.3, we have

$$\begin{aligned} \mathfrak{R}(z, x, y) &= \frac{1}{2} \begin{pmatrix} A(x, y) & B(x, y) \\ B(x, y) & A(y, x) \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 0 & B(x, y) \\ B(x, y) & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} A(x, y) & 0 \\ 0 & A(y, x) \end{pmatrix} \\ &= \frac{1}{2} (\tilde{\mathcal{J}}_1 + \tilde{\mathcal{J}}_2). \end{aligned}$$

We want to get a positive lower bound for the spectrum of $\tilde{\mathcal{J}}_2$. Applying the Gershgorin circle theorem. it is sufficient to require the following condition, for $V(x, y) = W(x, y) + U(x)$,

$$\begin{pmatrix} z_1 z_2 l_d & \frac{1}{2} [(1 + z_1 z_2 + z_2^2) l_d - z_1^2 \nabla_{xx}^2 V(x, y)] \\ \frac{1}{2} [(1 + z_1 z_2 + z_2^2) l_d - z_1^2 \nabla_{xx}^2 V(x, y)] & (1 + z_2^2) l_d - z_1 z_2 \nabla_{xx}^2 V(x, y) \end{pmatrix} \\ - \lambda_{W_{xx}} l_{2d} \geq 0.$$

such that there exists constant $\lambda_{W_{xx}} > 0$ satisfying

$$\tilde{\mathcal{J}}_2 \succeq \lambda_{W_{xx}} \begin{pmatrix} l_{2d} & 0 \\ 0 & l_{2d} \end{pmatrix}.$$

According to the eigenvalue decomposition of $\nabla_{xx}^2 W(x, y) + \nabla_{xx}^2 U(x)$, it is thus sufficient to prove the following inequalities, for $i = 1, \dots, d$,

$$\begin{aligned} z_1 z_2 - \left| \frac{1}{2} [(1 + z_1 z_2 + z_2^2) - z_1^2 \tilde{\lambda}_i^W] \right| &\geq \lambda_{W_{xx}}, \\ (1 + z_2^2) - z_1 z_2 \tilde{\lambda}_i^W - \left| \frac{1}{2} [(1 + z_1 z_2 + z_2^2) - z_1^2 \tilde{\lambda}_i^W] \right| &\geq \lambda_{W_{xx}}. \end{aligned} \quad (3.4)$$

Given such a positive $\lambda_{W_{xx}} > 0$, we analyze the first term $\tilde{\mathcal{J}}_1$, such that

$$\begin{pmatrix} 0 & \mathbf{B}(x, y) \\ \mathbf{B}(x, y) & 0 \end{pmatrix} + \lambda_{W_{xx}} \begin{pmatrix} \mathbf{I}_{2d} & 0 \\ 0 & \mathbf{I}_{2d} \end{pmatrix} \succeq 0. \quad (3.5)$$

According to Schur complement for symmetric matrix function (see (Vandenberghe and Boyd 2004)[Appendix A.5]), this is equivalent to the following condition:

$$\begin{cases} \lambda_{W_{xx}} > 0; \\ \lambda_{W_{xx}}^2 \mathbf{I}_{2d} - \mathbf{B}^2(x, y) \succeq 0. \end{cases} \quad (3.6)$$

Recall

$$\begin{aligned} \mathbf{B}(x, y) &= \begin{pmatrix} 0 & -\frac{z_1^2}{2} \nabla_{xy}^2 W(x, y) \\ -\frac{z_1^2}{2} \nabla_{xy}^2 W(x, y) & -z_1 z_2 \nabla_{xy}^2 W(x, y) \end{pmatrix} \\ &= \begin{pmatrix} 0 & \mathbf{Q}_W^{-1} \left[\mathbf{Diag} \left\{ \frac{-z_1^2}{2} \lambda_i^W \right\}_{i=1}^d \right] \mathbf{Q}_W \\ \mathbf{Q}_W^{-1} \left[\mathbf{Diag} \left\{ \frac{-z_1^2}{2} \lambda_i^W \right\}_{i=1}^d \right] \mathbf{Q}_W & \mathbf{Q}_W^{-1} \left[\mathbf{Diag} \left\{ -z_1 z_2 \lambda_i^W \right\}_{i=1}^d \right] \mathbf{Q}_W \end{pmatrix} \\ &= \begin{pmatrix} 0 & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{pmatrix}. \end{aligned}$$

We have

$$\lambda_{W_{xx}}^2 \mathbf{I}_{2d} - \mathbf{B}^2(x, y) = \begin{pmatrix} \lambda_{W_{xx}}^2 \mathbf{I}_d - \mathbf{B}_{12}^2 & -\mathbf{B}_{12} \mathbf{B}_{22} \\ -\mathbf{B}_{12} \mathbf{B}_{22} & \lambda_{W_{xx}}^2 \mathbf{I}_d - \mathbf{B}_{22}^2 - \mathbf{B}_{12}^2 \end{pmatrix} = \begin{pmatrix} \mathbf{C}_{11}^W & \mathbf{C}_{12}^W \\ \mathbf{C}_{12}^W & \mathbf{C}_{22}^W \end{pmatrix},$$

where we denote

$$\begin{aligned} \mathbf{C}_{11}^W &= \mathbf{Q}_W^{-1} \left[\mathbf{Diag} \left\{ \lambda_{W_{xx}}^2 - \frac{z_1^4}{4} (\lambda_i^W)^2 \right\}_{i=1}^d \right] \mathbf{Q}_W \\ \mathbf{C}_{12}^W &= \mathbf{Q}_W^{-1} \left[\mathbf{Diag} \left\{ -z_1 z_2 \frac{z_1^2}{2} (\lambda_i^W)^2 \right\}_{i=1}^d \right] \mathbf{Q}_W \\ \mathbf{C}_{22}^W &= \mathbf{Q}_W^{-1} \left[\mathbf{Diag} \left\{ \lambda_{W_{xx}}^2 - (\lambda_i^W)^2 (z_1^2 z_2^2 + \frac{z_1^4}{4}) \right\}_{i=1}^d \right] \mathbf{Q}_W. \end{aligned}$$

Applying Schur complement, it is equivalent to, for $i = 1, \dots, d$,

$$\begin{cases} \lambda_{W_{xx}}^2 - (\lambda_i^W)^2 \frac{z_1^4}{4} \geq 0; \\ [\lambda_{W_{xx}}^2 - (\lambda_i^W)^2 (z_1^2 z_2^2 + \frac{z_1^4}{4})][\lambda_{W_{xx}}^2 - \frac{z_1^4}{4} (\lambda_i^W)^2] - z_1^2 z_2^2 \frac{z_1^4}{4} (\lambda_i^W)^4 \geq 0. \end{cases} \quad (3.7)$$

This is equivalent to

$$\begin{cases} \frac{\lambda_{W_{xx}}^2}{(\lambda_i^W)^2} - (z_1^2 z_2^2 + \frac{z_1^4}{4}) \geq 0; \\ \frac{\lambda_{W_{xx}}^4}{(\lambda_i^W)^4} - (z_1^2 z_2^2 + \frac{z_1^4}{2}) \frac{\lambda_{W_{xx}}^2}{(\lambda_i^W)^2} + \frac{z_1^8}{16} \geq 0. \end{cases} \quad (3.8)$$

Solving the second inequality, we get

$$\frac{\lambda_{W_{xx}}^2}{(\lambda_i^W)^2} \geq \frac{z_1^2 z_2^2 + \frac{z_1^4}{2} + \sqrt{z_1^4 z_2^4 + z_1^6 z_2^2}}{2}.$$

Notice that

$$\frac{z_1^2 z_2^2 + \frac{z_1^4}{2} + \sqrt{z_1^4 z_2^4 + z_1^6 z_2^2}}{2} > (z_1^2 z_2^2 + \frac{z_1^4}{4}).$$

It is sufficient to prove the following inequality for (3.8):

$$\lambda_{W_{xx}}^2 \geq \frac{z_1^2 z_2^2 + \frac{z_1^4}{2} + \sqrt{z_1^4 z_2^4 + z_1^6 z_2^2}}{2} (\lambda_i^W)^2. \quad (3.9)$$

Thus, for $\lambda_{W_{xx}}, \lambda_i^W > 0, i = 1, \dots, d$, combining with (3.4), the matrix $\mathfrak{R}$ is positive definite, if the following condition holds:

$$\begin{cases} \lambda_{W_{xx}} \leq z_1 z_2 - |\frac{1}{2}[(1 + z_1 z_2 + z_2^2) - z_1^2 \tilde{\lambda}_i^W]|, \\ \lambda_{W_{xx}} \leq (1 + z_2^2) - z_1 z_2 \tilde{\lambda}_i^W - |\frac{1}{2}[(1 + z_1 z_2 + z_2^2) - z_1^2 \tilde{\lambda}_i^W]|, \\ \lambda_{W_{xx}}^2 \geq \frac{z_1^2 z_2^2 + \frac{z_1^4}{2} + \sqrt{z_1^4 z_2^4 + z_1^6 z_2^2}}{2} (\lambda_i^W)^2, \end{cases} \quad (3.10)$$

which is condition (3.2). The proof is completed.

Case 2: We apply the Schur complement for symmetric matrix function $\mathfrak{R}$. The following conditions are equivalent.

- (1) $\mathfrak{R} \succeq 0$ ($\mathfrak{R}$ is positive definite).
- (2) $A(x, y) \succeq 0, (I_{2d} - A(x, y)A^{-1}(x, y))B(x, y) = 0, A(y, x) - B(x, y)A^{-1}(x, y)sB(x, y) \succeq 0$.

According to our assumption $\nabla_{xy}^2 W = 0$, thus $B = 0$. We only need to show that $A(x, y)$ is positive definite. Similar arguments then apply to $A(y, x)$. Denote

$$A(x, y) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

with

$$A_{11} = z_1 z_2 l_d, \quad A_{12} = A_{21} = \frac{1}{2}[(1 + z_1 z_2 + z_2^2)l_d - z_1^2 \nabla_{xx}^2 V(x, y)], \quad (3.11)$$

$$A_{22} = (1 + z_2^2)l_d - z_1 z_2 \nabla_{xx}^2 V(x, y). \quad (3.12)$$

Applying the Schur complement for symmetric matrix A , it is equivalent to find $z_1, z_2 \in \mathbb{R}^1$, such that

$$z_1 z_2 > 0, \quad 1 + z_2^2 - \bar{\lambda} z_1 z_2 > 0,$$

and

$$A_{22} - A_{12} A_{11}^{-1} A_{21} \geq 0 \iff z_1 z_2 A_{22} - A_{12}^2 \geq 0.$$

By direct computation, it is equivalent to the following condition,

$$z_1 z_2 [(1 + z_2^2)l_d - z_1 z_2 \nabla_{xx}^2 V(x, y)] - \frac{1}{4}((1 + z_1 z_2 + z_2^2)l_d - z_1^2 \nabla_{xx}^2 V(x, y))^2 \geq 0.$$

By a direct computation, it is equivalent to the following inequality:

$$-z_1^4 (\nabla_{xx}^2 V)^2 + [2(1 + z_1 z_2 - z_2^2)z_1^2] \nabla_{xx}^2 V + [2z_1 z_2 + 2z_1 z_2^3 - 1 - (z_1^2 + 2)z_2^2 - z_2^4]l_d > 0.$$

Based on the assumption of $\nabla_{xx}^2 W(x, y) + \nabla_{xx}^2 U(x) = \nabla_{xx}^2 V(x, y)$, we have

$$\nabla_{xx}^2 V(x, y) = Q_V^{-1} \mathbf{Diag}(\tilde{\lambda}_1^W, \dots, \tilde{\lambda}_d^W) Q_V.$$

In particular, we assume $0 < \underline{\lambda} \leq \tilde{\lambda}_1^W \leq \tilde{\lambda}_2^W \leq \dots \leq \tilde{\lambda}_d^W \leq \bar{\lambda}$, where $\tilde{\lambda}_1^W, \dots, \tilde{\lambda}_d^W$ are eigenvalues of matrix $\nabla_{xx}^2 W(x, y) + \nabla_{xx}^2 U(x)$, and $\underline{\lambda}, \bar{\lambda}$ are lower bound and upper bound of these eigenvalues, respectively. Applying the lower and upper bound of the eigenvalues, it is sufficient to prove the following conditions:

$$\begin{cases} z_1 z_2 > 0, & (1 + z_1 z_2 - z_2^2) > 0; \\ -z_1^4 \bar{\lambda}^2 + [2(1 + z_1 z_2 - z_2^2)z_1^2] \underline{\lambda} + 2z_1 z_2 + 2z_1 z_2^3 - 1 - (z_1^2 + 2)z_2^2 - z_2^4 > 0. \end{cases} \quad (3.13)$$

Let $z_1 = 1$, then (3.3) implies (3.13). Let $1 + z_2 - z_2^2 > 0$, for a small constant $\delta > 0$, there exists $z_2 > 0$ such that $[2(z_2 - z_2^2)] \underline{\lambda} + 2z_2 + 2z_2^3 - z_2^4 - 3z_2^2 > \delta$, which completes the proof for condition (1). $\square$

Proof of Example 2.1 We provide a simple proof of Example 2.1 by applying Condition (2) in Lemma 3.1. If $\underline{\lambda} l_d \leq \nabla_{xx}^2 (U(x) + W(x, y)) \leq \bar{\lambda} l_d$, which satisfies the condition(2) in Lemma 3.1, then as long as $\|\nabla_{x,y}^2 W(x, y)\|_F = O(\varepsilon)$ is small enough for

$\varepsilon > 0$, matrix $\mathfrak{R}$ remains positive definite. In particular, if we pick $z_1 = 1$, $z_2 = 0.3$, $\delta = 0.02$, this implies

$$2\underline{\lambda} - \bar{\lambda}^2 > 1 - \delta, \quad [2(z_2 - z_2^2)]\underline{\lambda} + 2z_2 + 2z_2^3 - z_2^4 - 3z_2^2 > 0.02,$$

i.e., $2\underline{\lambda} - \bar{\lambda}^2 > 0.08$, and $0.42\underline{\lambda} + 0.3759 > 0.02$. Hence, we require that $2\underline{\lambda} - \bar{\lambda}^2 > 0.08$. $\square$

3.2 Proof of Example 2.2

Similar to the previous section, we apply Schur complement to derive the positive definite condition for $\mathfrak{R}$. In particular, we rewrite matrix $\mathfrak{R}$ in the following form, where we separate the potential function $U(\cdot)$ and the interacting potential function $W(x, y)$.

Definition 3.1 (*Reformulation of mean-field information matrix*) Define a symmetric matrix function $\mathfrak{R} \in \mathbb{R}^{4d \times 4d}$, such that

$$\mathfrak{R}(z, x, y) = \frac{1}{2} \begin{pmatrix} A_1(x, y) & B(x, y) \\ B(x, y) & A_1(y, x) \end{pmatrix} + \frac{1}{2} \begin{pmatrix} A_2(x, y) & 0 \\ 0 & A_2(y, x) \end{pmatrix},$$

where A_1, A_2 , and $B \in \mathbb{R}^{2d \times 2d}$ are defined below:

$$\begin{aligned} A_1(x, y) &= \begin{pmatrix} 0 & -\frac{z_1^2}{2} \nabla_{xx}^2 W(x, y) \\ -\frac{z_1^2}{2} \nabla_{xx}^2 W(x, y) & -z_1 z_2 \nabla_{xx}^2 W(x, y) \end{pmatrix}, \\ A_1(y, x) &= \begin{pmatrix} 0 & -\frac{z_1^2}{2} \nabla_{yy}^2 W(y, x) \\ -\frac{z_1^2}{2} \nabla_{yy}^2 W(y, x) & -z_1 z_2 \nabla_{yy}^2 W(y, x) \end{pmatrix}, \\ B(x, y) &= \begin{pmatrix} 0 & -\frac{z_1^2}{2} \nabla_{xy}^2 W(x, y) \\ -\frac{z_1^2}{2} \nabla_{xy}^2 W(x, y) & -z_1 z_2 \nabla_{xy}^2 W(x, y) \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} A_2(x, y) &= \begin{pmatrix} z_1 z_2 I_d & \frac{1}{2} [(1 + z_1 z_2 + z_2^2) I_d - z_1^2 \nabla_{xx}^2 U(x)] \\ \frac{1}{2} [(1 + z_1 z_2 + z_2^2) I_d - z_1^2 \nabla_{xx}^2 U(x)] & (1 + z_2^2) I_d - z_1 z_2 \nabla_{xx}^2 U(x) \end{pmatrix}, \\ A_2(y, x) &= \begin{pmatrix} z_1 z_2 I_d & \frac{1}{2} [(1 + z_1 z_2 + z_2^2) I_d - z_1^2 \nabla_{yy}^2 U(y)] \\ \frac{1}{2} [(1 + z_1 z_2 + z_2^2) I_d - z_1^2 \nabla_{yy}^2 U(y)] & (1 + z_2^2) I_d - z_1 z_2 \nabla_{yy}^2 U(y) \end{pmatrix}. \end{aligned}$$

Lemma 3.2 Assume that

$$\begin{cases} \nabla_{xy}^2 W(x, y) = Q_W^{-1} \text{Diag}(\lambda_1^W, \dots, \lambda_d^W) Q_W, \\ \nabla_{xx}^2 W(x, y) = Q_W^{-1} \text{Diag}(\tilde{\lambda}_1^W, \dots, \tilde{\lambda}_d^W) Q_W, \\ \underline{\lambda} I_d \leq \nabla_{xx}^2 U(x) \leq \bar{\lambda} I_d. \end{cases} \quad (3.14)$$

And there exists $z_1 > 0$ and $z_2 > 0$, such that

$$A_2 \succeq \lambda_U I_{2d},$$

where we denote λ_U as the spectrum lower bound for A_2 defined in Definition 3.1. Then if λ_i^W sufficiently small, and

$$\tilde{\lambda}_i^W \in \left(\frac{-2\lambda_U(z_2 + \sqrt{z_1^2 + z_2^2})}{z_1^3}, \frac{2\lambda_U(\sqrt{z_1^2 + z_2^2} - z_2)}{z_1^3} \right), \quad i = 1, \dots, d, \quad (3.15)$$

then Assumption (2.1) holds.

Proof According to Definition 3.1, we have

$$\begin{aligned} \mathfrak{R}(z, x, y) &= \frac{1}{2} \begin{pmatrix} A_1(x, y) & B(x, y) \\ B(x, y) & A_1(y, x) \end{pmatrix} + \frac{1}{2} \begin{pmatrix} A_2(x, y) & 0 \\ 0 & A_2(y, x) \end{pmatrix} \\ &= \frac{1}{2} (\mathcal{J}_1 + \mathcal{J}_2). \end{aligned}$$

According to condition (3.4) in the proof of Lemma 3.1, we find a sufficient condition such that $\mathcal{J}_2$ is positive definite, i.e., there exists constant $\lambda_U > 0$, such that

$$\mathcal{J}_2 \succeq \lambda_U \begin{pmatrix} I_{2d} & 0 \\ 0 & I_{2d} \end{pmatrix}.$$

Next, we analyze the first term $\mathcal{J}_1$, such that $\mathcal{J}_1 + \mathcal{J}_2$ is positive definite, which implies that $\mathcal{J}_1$ could be potentially non-positive definite. We shall show that

$$\begin{pmatrix} A_1(x, y) & B(x, y) \\ B(x, y) & A_1(y, x) \end{pmatrix} + \begin{pmatrix} \lambda_U I_{2d} & 0 \\ 0 & \lambda_U I_{2d} \end{pmatrix} \succeq 0.$$

According to Schur complement, this is equivalent to the following condition:

$$\begin{cases} A_1(x, y) + \lambda_U I_{2d} \succeq 0; \\ A_1(y, x) + \lambda_U I_{2d} - B(x, y)[A_1(x, y) + \lambda_U I_{2d}]^{-1}B(x, y) \succeq 0. \end{cases} \quad (3.16)$$

Based on Assumption 3.14, the first condition is represented below:

$$\lambda_U \begin{pmatrix} I_d & 0 \\ 0 & I_d \end{pmatrix} + A_1(y, x)$$

$$= \begin{pmatrix} \lambda_U I_d & Q_W^{-1} [\text{Diag} \{ -\frac{z_1^2}{2} \tilde{\lambda}_i^W \}_{i=1}^d] Q_W \\ Q_W^{-1} [\text{Diag} \{ -\frac{z_1^2}{2} \tilde{\lambda}_i^W \}_{i=1}^d] Q_W & Q_W^{-1} [\text{Diag} \{ \lambda_U - z_1 z_2 \tilde{\lambda}_i^W \}_{i=1}^d] Q_W \end{pmatrix} \succeq 0.$$

Thus, the first condition in (3.16) is equivalent to

$$\lambda_U (\lambda_U - z_1 z_2 \tilde{\lambda}_i^W) - \frac{z_1^4}{4} (\tilde{\lambda}_i^W)^2 > 0, \quad \text{for } i = 1, \dots, d, \quad (3.17)$$

which implies

$$\tilde{\lambda}_i^W \in \left(\frac{-2\lambda_U (z_2 + \sqrt{z_1^2 + z_2^2})}{z_1^3}, \frac{2\lambda_U (\sqrt{z_1^2 + z_2^2} - z_2)}{z_1^3} \right). \quad (3.18)$$

Similarly, we have

$$\begin{aligned} B(x, y) &= \begin{pmatrix} 0 & Q_W^{-1} [\text{Diag} \{ -\frac{z_1^2}{2} \lambda_i^W \}_{i=1}^d] Q_W \\ Q_W^{-1} [\text{Diag} \{ -\frac{z_1^2}{2} \lambda_i^W \}_{i=1}^d] Q_W & Q_W^{-1} [\text{Diag} \{ -z_1 z_2 \lambda_i^W \}_{i=1}^d] Q_W \end{pmatrix} \\ &= \begin{pmatrix} 0 & B_{12} \\ B_{21} & B_{22} \end{pmatrix}. \end{aligned}$$

Since each block of matrix $A_1(x, y) + \lambda_U I_{2d}$ is diagonal, we have

$$[A_1(x, y) + \lambda_U I_{2d}]^{-1} = \begin{pmatrix} A_{11}^W & A_{12}^W \\ A_{21}^W & A_{22}^W \end{pmatrix},$$

where we denote

$$D_i = \lambda_U (\lambda_U - z_1 z_2 \tilde{\lambda}_i^W) - \frac{z_1^4}{4} (\tilde{\lambda}_i^W)^2, \quad \text{for } i = 1, \dots, d,$$

and

$$\begin{aligned} A_{11}^W &= Q_W^{-1} [\text{Diag} \left(\left\{ \frac{1}{D_i} [\lambda_U - z_1 z_2 \tilde{\lambda}_i^W] \right\}_{i=1}^d \right)] Q_W \\ A_{12}^W &= A_{21}^W = Q_W^{-1} [\text{Diag} \left(\left\{ \frac{z_1^2 \tilde{\lambda}_i^W}{2D_i} \right\}_{i=1}^d \right)] Q_W \\ A_{22}^W &= Q_W^{-1} [\text{Diag} \left(\left\{ \frac{\lambda_U}{D_i} \right\}_{i=1}^d \right)] Q_W. \end{aligned}$$

Using the above explicit representation, we obtain

$$A_1(y, x) + \lambda_U I_{2d} - B(x, y)[A(x, y) + \lambda_U I_{2d}]^{-1}B(x, y)$$

$$\begin{aligned}
&= A_1(y, x) + \lambda_U l_{2d} \\
&\quad - \begin{pmatrix} B_{12} A_{22}^W B_{21} & B_{12} A_{21}^W B_{12} + B_{12} A_{22}^W B_{22} \\ B_{21} A_{12}^W B_{21} + B_{22} A_{22}^W B_{21} & B_{21} A_{11}^W B_{12} + B_{22} A_{21}^W B_{12} + B_{21} A_{12}^W B_{22} + B_{22} A_{22}^W B_{22} \end{pmatrix} \\
&= \begin{pmatrix} \tilde{C}_{11} & \tilde{C}_{12} \\ \tilde{C}_{21} & \tilde{C}_{22} \end{pmatrix},
\end{aligned}$$

where we denote

$$\begin{aligned}
\tilde{C}_{11} &= Q_W^{-1} \left[\text{Diag} \left(\left\{ \lambda_U - \left[\frac{z_1^4 \lambda_U (\lambda_i^W)^2}{4D_i} \right] \right\}_{i=1}^d \right) \right] Q_W, \\
\tilde{C}_{12} &= Q_W^{-1} \left[\text{Diag} \left(\left\{ -\frac{z_1^2}{2} \tilde{\lambda}_i^W - \left[\frac{z_1^3 z_2 (\lambda_i^W)^2 \lambda_U}{2D_i} + \frac{z_1^6 (\lambda_i^W)^2 \tilde{\lambda}_i^W}{8D_i} \right] \right\}_{i=1}^d \right) \right] Q_W, \\
\tilde{C}_{22} &= Q_W^{-1} \left[\text{Diag} \left(\left\{ \lambda_U - z_1 z_2 \tilde{\lambda}_i^W - \left[\frac{z_1^4 (\lambda_i^W)^2 (\lambda_U - z_1 z_2 \tilde{\lambda}_i^W)}{4D_i} \right. \right. \right. \\
&\quad \left. \left. \left. + \frac{z_1^5 z_2 (\lambda_i^W)^2 \tilde{\lambda}_i^W}{2D_i} + \frac{\lambda_U (z_1 z_2)^2 (\lambda_i^W)^2}{D_i} \right] \right\}_{i=1}^d \right) \right] Q_W.
\end{aligned}$$

Applying Schur complement and under the condition $D_i > 0$ for $i = 1, \dots, d$ (i.e., (3.18)), the second condition in (3.16) is equivalent to, for $i = 1, \dots, d$,

$$\begin{cases} 4\lambda_U D_i - z_1^4 \lambda_U (\lambda_i^W)^2 > 0, \\ 4[4\lambda_U D_i - z_1^4 \lambda_U (\lambda_i^W)^2] \times \\ [4D_i (\lambda_U - z_1 z_2 \tilde{\lambda}_i^W) - [z_1^4 (\lambda_i^W)^2 (\lambda_U - z_1 z_2 \tilde{\lambda}_i^W) + 2z_1^5 z_2 (\lambda_i^W)^2 \tilde{\lambda}_i^W \\ + 4\lambda_U (z_1 z_2)^2 (\lambda_i^W)^2]] - [4D_i z_1^2 \tilde{\lambda}_i^W + 4z_1^3 z_2 (\lambda_i^W)^2 \lambda_U + z_1^6 (\lambda_i^W)^2 \tilde{\lambda}_i^W]^2 > 0. \end{cases} \quad (3.19)$$

Recall that $D_i = \lambda_U (\lambda_U - z_1 z_2 \tilde{\lambda}_i^W) - \frac{z_1^4}{4} (\tilde{\lambda}_i^W)^2$, for $i = 1, \dots, d$. For the first inequality in (3.19), it is sufficient to require that, for $(\lambda_i^W)^2$ small enough,

$$D_i = \lambda_U (\lambda_U - z_1 z_2 \tilde{\lambda}_i^W) - \frac{z_1^4}{4} (\tilde{\lambda}_i^W)^2 > 0.$$

For the second inequality in (3.19), denote $\lambda_U - z_1 z_2 \tilde{\lambda}_i^W = \alpha$. We observe the following simplification:

$$\begin{aligned}
&4[4D_i \lambda_U - z_1^4 (\lambda_i^W)^2 \lambda_U] \\
&\quad \times [4D_i \alpha - (z_1^4 (\lambda_i^W)^2 \alpha + 2z_1^5 z_2 (\lambda_i^W)^2 \tilde{\lambda}_i^W + 4\lambda_U (z_1 z_2)^2 (\lambda_i^W)^2)] \\
&\quad - [4D_i z_1^2 \tilde{\lambda}_i^W + 4z_1^3 z_2 (\lambda_i^W)^2 \lambda_U + z_1^6 (\lambda_i^W)^2 \tilde{\lambda}_i^W]^2 > 0, \\
&\Rightarrow 4\lambda_U [4D_i - \varepsilon_1] \times [4D_i \alpha - \varepsilon_2] - [4D_i z_1^2 \tilde{\lambda}_i^W + \varepsilon_3]^2 > 0, \\
&\Rightarrow 64\alpha D_i^2 \lambda_U - 16D_i^2 z_1^4 (\tilde{\lambda}_i^W)^2 + o(\varepsilon) > 0, \\
&\Rightarrow 64D_i^2 (\lambda_U \alpha - \frac{1}{4} z_1^4 (\tilde{\lambda}_i^W)^2) + o(\varepsilon) > 0. \quad (3.20)
\end{aligned}$$

Here all ε terms depend on $(\lambda_i^W)^2$. And the leading term $(\lambda_U \alpha - \frac{1}{4} z_1^4 (\tilde{\lambda}_i^W)^2) > 0$ is the same as 3.17. Thus, we finish the proof as long as ε is small enough. $\square$

Proof of Example 2.2 According to the above Lemma, let $\underline{\lambda} = \bar{\lambda} = 0.9$, $z_1 = 1$, and $z_2 = 0.3$, we obtain $\lambda_U \approx 0.2$. By direct computations, condition (3.15) implies

$$-0.4(0.3 + \sqrt{1.09}) < \tilde{\lambda}_i^W < 0.4(\sqrt{1.09} - 0.3), \quad i = 1, \dots, d.$$

In particular, if we choose $\tilde{\lambda}_i^W = -0.12$, plugging into (3.20), it is sufficient to require $\lambda_i^W < 10^{-3}$, such that $o(\varepsilon)$ is small enough. $\square$

Proof of Remark 2.4 For $U \equiv 0$, and $W(x, y) = W(x - y)$, we have $\nabla_{xx}^2 W(x - y) = -\nabla_{xy}^2 W(x - y)$ and $\lambda_i^W = -\tilde{\lambda}_i^W$. According to the eigenvalue decomposition, we have

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix} = \text{Diag}(Q_W^{-1}, Q_W^{-1}, Q_W^{-1}, Q_W^{-1}) \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{B} & \hat{A} \end{pmatrix} \text{Diag}(Q_W, Q_W, Q_W, Q_W),$$

where

$$\begin{aligned} \hat{A} &= \begin{pmatrix} z_1 z_2 l_d & \frac{1}{2}[(1 + z_1 z_2 + z_2^2)l_d - z_1^2 \text{Diag}\{\tilde{\lambda}_i^W\}_{i=1}^d] \\ \frac{1}{2}[(1 + z_1 z_2 + z_2^2)l_d - z_1^2 \text{Diag}\{\tilde{\lambda}_i^W\}_{i=1}^d] & (1 + z_2^2)l_d - z_1 z_2 \text{Diag}\{\tilde{\lambda}_i^W\}_{i=1}^d \end{pmatrix}, \\ \hat{B} &= \begin{pmatrix} 0 & \text{Diag}\{-\frac{z_1^2}{2}\lambda_i^W\}_{i=1}^d \\ \text{Diag}\{-\frac{z_1^2}{2}\lambda_i^W\}_{i=1}^d & \text{Diag}\{-z_1 z_2 \lambda_i^W\}_{i=1}^d \end{pmatrix}. \end{aligned}$$

For $\mathfrak{R}$ positive definite, it is equivalent to the following condition:

$$\begin{pmatrix} \hat{A} & \hat{B} \\ \hat{B} & \hat{A} \end{pmatrix} \succeq \lambda l_{4d},$$

for some positive constant $\lambda > 0$. It is thus sufficient to prove the following inequalities, for $i = 1, \dots, d$, and $z_1 > 0$, $z_2 > 0$,

$$\begin{aligned} z_1 z_2 - \frac{1}{2}[(1 + z_1 z_2 + z_2^2) - z_1^2 \tilde{\lambda}_i^W] - \frac{z_1^2}{2} |\lambda_i^W| &\geq \lambda, \\ (1 + z_2^2) - z_1 z_2 \tilde{\lambda}_i^W - \frac{1}{2}[(1 + z_1 z_2 + z_2^2) - z_1^2 \tilde{\lambda}_i^W] - \frac{z_1^2}{2} |\lambda_i^W| - z_1 z_2 |\lambda_i^W| &\geq \lambda. \end{aligned} \quad (3.21)$$

Assume that $\tilde{\lambda}_i^W = -\lambda_i^W > 0$, and $\frac{1}{2}[(1 + z_1 z_2 + z_2^2) - z_1^2 \tilde{\lambda}_i^W] > 0$, then the above inequality is reduced to the following inequalities:

$$\frac{z_1 z_2 - 1 - z_2^2}{2} \geq \lambda, \quad -\frac{z_1 z_2 - 1 - z_2^2}{2} - 2z_1 z_2 \tilde{\lambda}_i^W \geq \lambda. \quad (3.22)$$

It is obvious that such a constant λ does not exist unless $z_1 z_2 - 1 - z_2^2 = 0$, $\lambda = 0$, and $\tilde{\lambda}_i^W = 0$, for $i = 1, \dots, d$. This implies that exponential decay does not hold for $U \equiv 0$ and $W(x, y) = W(x - y)$. In this case, the $O(t^{-\infty})$ convergence derived in Villani (2009)[Theorem 56] seems to be optimal. Meanwhile, from (3.22), we can provide an estimate of the negative lower bound for eigenvalues of $\mathfrak{R}$. $\square$

Remark 3.1 Following the above remark, for $U \equiv 0$, and $W(x, y) = W(x - y)$, we have $A_1(y, x) = A_1(x, y) = -B(x, y)$. Similar to the condition (3.22), matrix A_2 is at most semi-positive definite. Let $z_1 z_2 - 1 - z_2^2 = 0$, then $A_2 \geq 0$. We also note that

$$\begin{aligned} & (\phi_1(x, v) \ \phi_2(x, v) \ \phi_1(y, \tilde{v}) \ \phi_2(y, \tilde{v})) \mathfrak{R} (\phi_1(x, v) \ \phi_2(x, v) \ \phi_1(y, \tilde{v}) \ \phi_2(y, \tilde{v}))^\top \\ & \geq \frac{1}{2} (\phi_1(x, v) - \phi_1(y, \tilde{v}) \ \phi_2(x, v) - \phi_2(y, \tilde{v})) A_1(x, y) \begin{pmatrix} \phi_1(x, v) - \phi_1(y, \tilde{v}) \\ \phi_2(x, v) - \phi_2(y, \tilde{v}) \end{pmatrix}. \end{aligned} \quad (3.23)$$

The above condition (3.23) recovers the similar Hessian matrix for non-degenerate gradient flow equations in Carrillo et al. (2003). However, matrix A_1 is always negative definite even if we assume that $\nabla^2 W$ is positive definite. These facts show major differences between degenerate and non-degenerate gradient flow equations.

4 Proofs of Theorem 2.1 and Corollary 2.2

In this section, we present the main proof in this paper. For simplicity of presentation, we denote $f = f(t, x, v)$ as the solution of PDE (1.1).

We rewrite (1.1) in the following equivalent form:

$$\partial_t f = \nabla_{x,v} \cdot (f\gamma) + \nabla_{x,v} \cdot (faa^\top \nabla_{x,v} \frac{\delta}{\delta f} \mathcal{E}(f)), \quad (4.1)$$

where

$$\gamma = J \nabla_{x,v} [\frac{v^2}{2} + \int_{\mathbb{T}^d \times \mathbb{R}^d} W(x, y) f(t, y, \tilde{v}) d\tilde{v} dy + U(x)], \quad (4.2)$$

and

$$J = \begin{pmatrix} 0 & -I_d \\ I_d & 0 \end{pmatrix}_{2d \times 2d}. \quad (4.3)$$

Formulation (4.1) is known as the flux-gradient flow (Li et al. 2021) or Pre-Generic (Duong and Ottobre 2021). We summarize several lemmas below.

Proposition 4.1 For γ and J defined in (4.2) and (4.3), the PDE (1.1) is equivalent to the following form

$$\partial_t f = \nabla_{x,v} \cdot (f\gamma) + \nabla \cdot (faa^\top \nabla_{x,v} \frac{\delta}{\delta f} \mathcal{E}(f)). \quad (4.4)$$

Furthermore, the following identity holds:

$$\nabla_{x,v} \cdot (f\gamma) = \nabla_{x,v} \cdot (fJ\nabla \frac{\delta}{\delta f} \mathcal{E}(f)) = f \langle \nabla_{x,v} \frac{\delta}{\delta f} \mathcal{E}(f), \gamma \rangle. \quad (4.5)$$

In particular, we denote

$$W \circledast \rho = \int_{\mathbb{T}^d \times \mathbb{R}^d} W(x, y) f(t, y, \tilde{v}) dy d\tilde{v}, \quad \rho(t, y) = \int_{\mathbb{R}^d} f(t, y, \tilde{v}) d\tilde{v}. \quad (4.6)$$

Proof First, we observe that

$$\begin{aligned} & \left(- \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} \nabla_x W(x, y) f(t, y, \tilde{v}) d\tilde{v} dy + \nabla_x U(x) \right) \right. \\ & \quad \left. = -J\nabla_{x,v} \left[\frac{v^2}{2} + \int_{\mathbb{T}^d \times \mathbb{R}^d} W(x, y) f(t, y, \tilde{v}) d\tilde{v} dy + U(x) \right] \right). \end{aligned}$$

Furthermore, using the fact $\nabla \cdot (J\nabla \phi) = 0$ for any smooth function ϕ , (1.1) is equivalent to the following formulation:

$$\begin{aligned} & \partial_t f - \nabla_{x,v} \cdot \left(fJ\nabla_{x,v} \left[\frac{v^2}{2} + \int_{\mathbb{T}^d \times \mathbb{R}^d} W(x, y) f(t, y, \tilde{v}) d\tilde{v} dy + U(x) \right] \right) \\ & = \nabla_{x,v} \cdot \left(f a a^T \nabla_{x,v} \left[\log f + \frac{v^2}{2} + \int_{\mathbb{T}^d \times \mathbb{R}^d} W(x, y) f(t, y, \tilde{v}) d\tilde{v} dy + U(x) \right] \right). \end{aligned}$$

Applying the fact that $\frac{\delta}{\delta f} \mathcal{E} = [\log f + 1 + \frac{v^2}{2} + \int_{\mathbb{T}^d \times \mathbb{R}^d} W(x, y) f(t, y, \tilde{v}) d\tilde{v} dy + U(x)]$, we have

$$\partial_t f - \nabla_{x,v} \cdot (f\gamma) = \nabla \cdot (f a a^T \nabla_{x,v} \frac{\delta}{\delta f} \mathcal{E}(f)),$$

where $\gamma = J\nabla_{x,v} [\frac{v^2}{2} + \int_{\mathbb{T}^d \times \mathbb{R}^d} W(x, y) f(t, y, \tilde{v}) d\tilde{v} dy + U(x)]$. Furthermore, we observe that

$$\begin{aligned} \nabla_{x,v} \cdot (f\gamma) &= f \langle \nabla_{x,v} \log f, J\nabla_{x,v} [\frac{v^2}{2} + \int_{\mathbb{T}^d \times \mathbb{R}^d} W(x, y) f(t, y, \tilde{v}) d\tilde{v} dy + U(x)] \rangle \\ &= f \langle \nabla_{x,v} \frac{\delta}{\delta f} \mathcal{E}(f), \gamma \rangle, \end{aligned}$$

where we add $\langle \nabla_{x,v} [\frac{v^2}{2} + U(x) + W \circledast \rho(x)], J\nabla_{x,v} [\frac{v^2}{2} + U(x) + W \circledast \rho(x)] \rangle = 0$ in the last step. Notice that $\nabla \cdot (fJ\nabla \log f) = 0$, we obtain

$$\nabla_{x,v} \cdot (f\gamma) = \nabla \cdot (fJ\nabla_{x,v} \frac{\delta}{\delta f} \mathcal{E}(f)).$$

□

Lemma 4.1 Under the assumption $W(x, y) = W(y, x)$, we have

$$\frac{\partial}{\partial t} \mathcal{E}(f) = - \int_{\Omega} (\nabla \frac{\delta}{\delta f} \mathcal{E}(f), aa^T \nabla \frac{\delta}{\delta f} \mathcal{E}(f)) f dx dv = -\mathcal{D}\mathcal{E}_a(f),$$

where $\mathcal{E}(f)$ is the free energy defined in (2.1), and $\mathcal{D}\mathcal{E}_a(f)$ is defined in (2.3).

Proof Clearly, PDE (1.1) in formulation (4.1) can be written below. From equality (4.5), we have

$$\partial_t f = \nabla_{x,v} \cdot (f(aa^T + J)\nabla_{x,v} \frac{\delta}{\delta f} \mathcal{E}(f)).$$

Thus,

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(f) &= \int_{\Omega} \frac{\delta}{\delta f} \mathcal{E}(f) \cdot \partial_t f dx dv \\ &= \int_{\Omega} \frac{\delta}{\delta f} \mathcal{E}(f) \cdot \nabla_{x,v} \cdot (f(aa^T + J)\nabla_{x,v} \frac{\delta}{\delta f} \mathcal{E}(f)) dx dv \\ &= - \int_{\Omega} (\nabla_{x,v} \frac{\delta}{\delta f} \mathcal{E}, (aa^T + J)\nabla_{x,v} \frac{\delta}{\delta f} \mathcal{E}(f)) f dx dv \\ &= - \int_{\Omega} (\nabla_{x,v} \frac{\delta}{\delta f} \mathcal{E}, aa^T \nabla_{x,v} \frac{\delta}{\delta f} \mathcal{E}(f)) f dx dv, \end{aligned}$$

where we use the fact that

$$\left(\nabla_{x,v} \frac{\delta}{\delta f} \mathcal{E}, J \nabla_{x,v} \frac{\delta}{\delta f} \mathcal{E}(f) \right) = 0.$$

□

Lemma 4.2 (Technical Lemma) Suppose Assumption 2.1 holds. Then

$$\begin{aligned} \partial_t \mathcal{D}\mathcal{E}_{a,z}(f) &\leq -2 \int_{\Omega \times \Omega} \Re(\delta \mathcal{E}, \delta \mathcal{E}) f(t, x, v) f(t, y, \tilde{v}) dx dv dy d\tilde{v} \\ &\leq -2\lambda[\mathcal{D}\mathcal{E}_a(f) + \mathcal{D}\mathcal{E}_z(f)]. \end{aligned}$$

We leave the proof of Lemma 4.2 in Sect. 5.

Proof of Theorem 2.1 According to Lemma 4.2, we have

$$\partial_t \mathcal{D}\mathcal{E}_{a,z}(f) \leq -2\lambda \mathcal{D}\mathcal{E}_{a,z}(f).$$

Furthermore, using the fact that

$$-\frac{d}{dt} \mathcal{E}(f) = \mathcal{D}\mathcal{E}_a(f) \leq \mathcal{D}\mathcal{E}_{a,z}(f),$$

we have

$$\begin{aligned}
 -[\mathcal{DE}_{a,z}(f) - \mathcal{DE}_{a,z}(f_\infty)] &= \int_t^\infty \frac{d}{ds} \mathcal{DE}_{a,z}(f(s, \cdot, \cdot)) ds \\
 \text{Step A: } &\leq -2\lambda \int_t^\infty \mathcal{DE}_{a,z}(f(s, \cdot, \cdot)) ds \\
 &\leq -2\lambda \int_t^\infty \mathcal{DE}_a(f(s, \cdot, \cdot)) ds \\
 &= -2\lambda \int_t^\infty -\frac{d}{ds} \mathcal{E}(f(s, \cdot, \cdot)) ds \\
 \text{Step B: } &= -2\lambda [\mathcal{E}(f) - \mathcal{E}(f_\infty)].
 \end{aligned}$$

From Step B, we have

$$[\mathcal{E}(f) - \mathcal{E}(f_\infty)] \leq \frac{1}{2\lambda} [\mathcal{DE}_{a,z}(f) - \mathcal{DE}_{a,z}(f_\infty)]. \quad (4.7)$$

From Step A, we have

$$-[\mathcal{DE}_{a,z}(f) - \mathcal{DE}_{a,z}(f_\infty)] \leq -2\lambda \int_t^\infty \mathcal{DE}_{a,z}(f(s, \cdot, \cdot)) - \mathcal{DE}_{a,z}(f_\infty) ds.$$

Applying Gronwall's inequality, we have

$$\mathcal{DE}_{a,z}(f) - \mathcal{DE}_{a,z}(f_\infty) \leq e^{-2\lambda t} [\mathcal{DE}_{a,z}(f_0) - \mathcal{DE}_{a,z}(f_\infty)].$$

Using the inequality (4.7), we have

$$\mathcal{E}(f) - \mathcal{E}(f_\infty) \leq \frac{1}{2\lambda} [\mathcal{DE}_{a,z}(f) - \mathcal{DE}_{a,z}(f_\infty)] \leq \frac{1}{2\lambda} e^{-2\lambda t} [\mathcal{DE}_{a,z}(f_0) - \mathcal{DE}_{a,z}(f_\infty)],$$

which finishes the proof. $\square$

Proof of Corollary 2.2 Recall that

$$\begin{aligned}
 \mathcal{E}(f) &= \int_\Omega f \log f dx dv + \frac{1}{2} \int_\Omega \|v\|^2 f dx dv \\
 &\quad + \frac{1}{2} \int_{\Omega \times \Omega} W(x, y) f(x, v) f(y, \tilde{v}) dx dv dy d\tilde{v} + \int_\Omega U(x) f(x, v) dx dv.
 \end{aligned}$$

Similarly, we denote $\mathcal{E}(f_\infty)$ as the free energy associated with the equilibrium f_∞ . We know that f_∞ satisfies the equation,

$$f_\infty(x, v) = \frac{1}{Z} e^{-\frac{1}{2}\|v\|^2 - \int_\Omega W(x, y) f_\infty(y, v) dy dv - U(x)},$$

which implies that

$$\log f_{\infty} = -\frac{1}{2}\|v\|^2 - \int_{\Omega} W(x, y) f_{\infty}(y, v) dy dv - U(x) - \log Z,$$

where Z is the normalization constant. We thus obtain

$$\mathcal{E}(f_{\infty}) = -\frac{1}{2} \int_{\Omega \times \Omega} W(x, y) f_{\infty}(x, v) f_{\infty}(y, \tilde{v}) dx dv dy d\tilde{v} - \log Z.$$

Following the above representation of $\mathcal{E}(f_{\infty})$, and denoting $f = f(t, \cdot, \cdot)$, we derive

$$\begin{aligned} & \mathcal{E}(f) - \mathcal{E}(f_{\infty}) \\ &= \int_{\Omega} f(t, x, v) \log f(t, x, v) dx dv + \frac{1}{2} \int_{\Omega} \|v\|^2 f(t, x, v) dx dv \\ & \quad + \frac{1}{2} \int_{\Omega \times \Omega} W(x, y) f(t, x, v) f(t, y, \tilde{v}) dx dv dy d\tilde{v} + \int_{\Omega} U(x) f(t, x, v) dx dv \\ & \quad + \frac{1}{2} \int_{\Omega \times \Omega} W(x, y) f_{\infty}(x, v) f_{\infty}(y, \tilde{v}) dx dv dy d\tilde{v} + \log Z \\ &= \int_{\Omega} f \log \frac{f}{f_{\infty}} dx dv + \int_{\Omega} f \log f_{\infty} dx dv + \frac{1}{2} \int_{\Omega} \|v\|^2 f dx dv \\ & \quad + \int_{\Omega} U(x) f dx dv + \log Z \\ & \quad + \frac{1}{2} \int_{\Omega \times \Omega} W(x, y) f(t, x, v) f(t, y, \tilde{v}) dx dv dy d\tilde{v} \\ & \quad + \frac{1}{2} \int_{\Omega \times \Omega} W(x, y) f_{\infty}(x, v) f_{\infty}(y, \tilde{v}) dx dv dy d\tilde{v} \\ &= \int_{\Omega} f \log \frac{f}{f_{\infty}} dx dv - \int_{\Omega \times \Omega} f(t, x, \tilde{v}) W(x, y) f_{\infty}(y, v) dy dv dx d\tilde{v} \\ & \quad + \frac{1}{2} \int_{\Omega \times \Omega} W(x, y) f(t, x, v) f(t, y, \tilde{v}) dx dv dy d\tilde{v} \\ & \quad + \frac{1}{2} \int_{\Omega \times \Omega} W(x, y) f_{\infty}(x, v) f_{\infty}(y, \tilde{v}) dx dv dy d\tilde{v} \\ &= \int_{\Omega} f \log \frac{f}{f_{\infty}} dx dv + \frac{1}{2} \int_{\Omega \times \Omega} W(x, y) (f(t, x, v) \\ & \quad - f_{\infty}(x, v)) (f(t, y, \tilde{v}) - f_{\infty}(y, \tilde{v})) dx dy dv d\tilde{v}, \\ &\geq \int_{\Omega} f \log \frac{f}{f_{\infty}} dx dv - \frac{1}{2} \int_{\Omega \times \Omega} |W(x, y)| \cdot |(f(t, x, v) \\ & \quad - f_{\infty}(x, v))| \cdot |(f(t, y, \tilde{v}) - f_{\infty}(y, \tilde{v}))| dx dy dv d\tilde{v} \\ &\geq \int_{\Omega} f \log \frac{f}{f_{\infty}} dx dv - \frac{1}{2} C_W \|f - f_{\infty}\|_{L^1}^2, \end{aligned}$$

where the last inequality follows from our assumption in (2.9). Applying the Csiszár-Kullback-Pinsker inequality, we have

$$\begin{aligned}\mathcal{E}(f) - \mathcal{E}(f_\infty) &\geq \int_{\Omega} f \log \frac{f}{f_\infty} dx dv - \frac{1}{2} C_W \|f - f_\infty\|_{L^1}^2 \\ &\geq \frac{1}{2} \|f - f_\infty\|_{L^1}^2 - \frac{1}{2} C_W \|f - f_\infty\|_{L^1}^2.\end{aligned}$$

The proof is thus completed assuming that $C_W < 1$. $\square$

5 Proofs of Technical Lemmas

In this section, we provide all proofs of technical lemmas in the previous section. For simplicity of notations, we use the integration notation that $\int = \int_{\Omega}$. Besides, we denote $\Omega = \Omega_x \times \Omega_v$, $\Omega_x = \mathbb{T}^d$, $\Omega_v = \mathbb{R}^d$, and use the notation ρ to represent the marginal density function of f on the spatial domain, i.e.,

$$\rho(t, x) = \int_{\Omega_v} f(t, x, v) dv. \quad (5.1)$$

5.1 Notations and Information Gamma Operators

In this subsection, we prepare some notations for later on computations in technical lemmas.

Denote $\phi \in C^\infty(\Omega)$ as a smooth testing function. Denote the Kolmogorov operator $L := L(f)$ for PDE (1.1) as

$$L\phi = v \nabla_x \phi - \nabla_v \phi \cdot \nabla_x \left(\int_{\Omega} W(x, y) f(t, y, \tilde{v}) d\tilde{v} dy + U(x) \right) + \Delta_v \phi - v \cdot \nabla_v \phi.$$

We also denote the L^2 adjoint operator with respect to the Lebesgue measure of L as L^* . In other words, (1.1) can be written as

$$\partial_t f = L^* f.$$

Following Proposition 4.1, we decompose the operator L as

$$L\phi = \tilde{L}\phi - \langle \gamma, \nabla \phi \rangle,$$

where

$$\tilde{L}\phi = \nabla \cdot (aa^T \nabla \phi) + \langle aa^T \nabla (-\frac{v^2}{2} - U(x) - W \circledast \rho(x)), \nabla \phi \rangle.$$

Denote a z -direction generator:

$$\tilde{L}_z \phi = \nabla \cdot (zz^T \nabla \phi) + \left\langle zz^T \nabla \left(-\frac{v^2}{2} - U(x) - W \circledast \rho(x) \right), \nabla \phi \right\rangle.$$

We first define Gamma one bilinear forms $\Gamma_1, \Gamma_1^z: C^\infty(\Omega) \times C^\infty(\Omega) \rightarrow C^\infty(\Omega)$ by

$$\Gamma_1(\phi, \phi) = \langle a^T \nabla \phi, a^T \nabla \phi \rangle_{\mathbb{R}^d}, \quad \Gamma_1^z(\phi, \phi) = \langle z^T \nabla \phi, z^T \nabla \phi \rangle_{\mathbb{R}^d}. \quad (5.2)$$

Next, we recall the following *information gamma calculus* introduced in Feng and Li (2021), where a and z are chosen as constant matrices. Define the following three bilinear forms:

$$\tilde{\Gamma}_2, \tilde{\Gamma}_2^z, \Gamma_{\mathcal{I}_{a,z}}: C^\infty(\Omega) \times C^\infty(\Omega) \rightarrow C^\infty(\Omega).$$

We denote

$$\tilde{\Gamma}_2(\phi, \phi) = \frac{1}{2} \tilde{L} \Gamma_1(\phi, \phi) - \Gamma_1(\tilde{L} \phi, \phi),$$

and

$$\tilde{\Gamma}_2^z(\phi, \phi) = \frac{1}{2} \tilde{L} \Gamma_1^z(\phi, \phi) - \Gamma_1^z(\tilde{L} \phi, \phi).$$

We denote the irreversible Gamma operator:

$$\Gamma_{\mathcal{I}_{a,z}}(\phi, \phi) = (\tilde{L} \phi + \tilde{L}_z \phi) \langle \nabla \phi, \gamma \rangle - \frac{1}{2} \langle \nabla (\Gamma_1(\phi, \phi) + \Gamma_1^z(\phi, \phi)), \gamma \rangle. \quad (5.3)$$

We first present the following proposition for a and z defined in (2.2), which is a special case of Feng and Li (2021)[Proposition 9]. See other (generalized) Bakry–Emery formulation in Baudoin and Garofalo (2016) for degenerate operator L and Bakry and Émery (1985); Arnold and Carlen (2000) for non-degenerate operator L .

Proposition 5.1 *For any $\phi(x, v) \in C^\infty(\Omega)$, we have*

$$\begin{aligned} \tilde{\Gamma}_2(\phi, \phi) + \tilde{\Gamma}_2^z(\phi, \phi) &= \sum_{i=1}^d \left[z_1^2 |\partial_{x_i x_i}^2 \phi|^2 + (1 + z_2^2) |\partial_{x_i + d x_i + d}^2 \phi|^2 + 2z_1 z_2 |\partial_{x_i x_i + d}^2 \phi|^2 \right] \\ &\quad - \sum_{i=1}^d \sum_{\hat{k}=1}^{2d} a_i^T \nabla [a a^T \nabla (-\frac{v^2}{2} - U(x) - W \circledast \rho(x))]_{\hat{k}} \partial_{x_{\hat{k}}} \phi a_i^T \nabla \phi \\ &\quad - \sum_{k=1}^d \sum_{\hat{k}=1}^{2d} z_k^T \nabla [a a^T \nabla (-\frac{v^2}{2} - U(x) - W \circledast \rho(x))]_{\hat{k}} \partial_{x_{\hat{k}}} \phi z_k^T \nabla \phi, \end{aligned}$$

where we denote $(x_1, \dots, x_d, v_1, \dots, v_d) = (x_1, \dots, x_{2d})$, and $\nabla\phi = (\partial_{x_1}f, \dots, \partial_{x_{2d}}\phi)$. Furthermore, we denote $a^\top = (a_1^\top, \dots, a_d^\top)^\top \in \mathbb{R}^{d \times 2d}$ and $z^\top = (z_1^\top, \dots, z_d^\top)^\top \in \mathbb{R}^{d \times 2d}$.

We next prove the following equivalent formulation for the irreversible Gamma operator. For simplicity of presentation, we use the following notation in the rest of the paper:

$$\delta\mathcal{E} = \frac{\delta}{\delta f}\mathcal{E}(f). \quad (5.4)$$

Lemma 5.1 For irreversible Gamma operator $\Gamma_{\mathcal{I}_{a,z}} = \Gamma_{\mathcal{I}_a} + \Gamma_{\mathcal{I}_z}$ defined in (5.3), we have

$$\begin{aligned} \int \Gamma_{\mathcal{I}_a}(\delta\mathcal{E}, \delta\mathcal{E}) f dx dv &= - \int \langle aa^\top \nabla \delta\mathcal{E}, \nabla \gamma \nabla \delta\mathcal{E} \rangle f dx dv, \\ \int \Gamma_{\mathcal{I}_z}(\delta\mathcal{E}, \delta\mathcal{E}) f dx dv &= - \int \langle zz^\top \nabla \delta\mathcal{E}, \nabla \gamma \nabla \delta\mathcal{E} \rangle f dx dv. \end{aligned}$$

Proof We lay out the proof for the first identity with matrix a . The second identity for matrix z can be proved in a similar manner. Recall

$$\begin{aligned} \Gamma_{\mathcal{I}_{a,z}}(\delta\mathcal{E}, \delta\mathcal{E}) &= (\tilde{L}\delta\mathcal{E} + \tilde{L}_z\delta\mathcal{E}) \langle \nabla \delta\mathcal{E}, \gamma \rangle - \frac{1}{2} \langle \nabla (\Gamma_1(\delta\mathcal{E}, \delta\mathcal{E}) + \Gamma_1^z(\delta\mathcal{E}, \delta\mathcal{E})), \gamma \rangle \\ &= \Gamma_{\mathcal{I}_a}(\delta\mathcal{E}, \delta\mathcal{E}) + \Gamma_{\mathcal{I}_z}(\delta\mathcal{E}, \delta\mathcal{E}), \end{aligned}$$

and

$$\begin{aligned} \tilde{L}\delta\mathcal{E} &= \nabla \cdot (aa^\top \nabla \delta\mathcal{E}) + \langle aa^\top \nabla (-\frac{v^2}{2} - U(x) - W \circledast \rho(x)), \nabla \delta\mathcal{E} \rangle, \\ \tilde{L}_z\delta\mathcal{E} &= \nabla \cdot (zz^\top \nabla \delta\mathcal{E}) + \langle zz^\top \nabla (-\frac{v^2}{2} - U(x) - W \circledast \rho(x)), \nabla \delta\mathcal{E} \rangle. \end{aligned}$$

Then we have

$$\begin{aligned} &\int \Gamma_{\mathcal{I}_a}(\delta\mathcal{E}, \delta\mathcal{E}) f dx dv \\ &= \int \left[(\tilde{L}\delta\mathcal{E}) \langle \nabla \delta\mathcal{E}, \gamma \rangle - \frac{1}{2} \langle \nabla (\Gamma_1(\delta\mathcal{E}, \delta\mathcal{E})), \gamma \rangle \right] f dx dv \\ &= \int \left[\nabla \cdot ((aa^\top) \nabla \delta\mathcal{E}) \langle \nabla \delta\mathcal{E}, \gamma \rangle + \langle \nabla \delta\mathcal{E}, \gamma \rangle \langle \nabla \delta\mathcal{E}, (aa^\top) \nabla (-\frac{v^2}{2} - U(x) \right. \\ &\quad \left. - W \circledast \rho(x)) \rangle \right] f dx dv + \int \frac{1}{2} \nabla \cdot (f\gamma) \langle \nabla \delta\mathcal{E}, (aa^\top) \nabla \delta\mathcal{E} \rangle dx dv. \end{aligned}$$

Using the fact $\nabla \cdot (f\gamma) = f \langle \nabla \delta\mathcal{E}, \gamma \rangle = f \langle \nabla \delta\mathcal{E}, \gamma \rangle$, we have

$$\int \Gamma_{\mathcal{I}_a}(\delta\mathcal{E}, \delta\mathcal{E}) f dx dv$$

$$\begin{aligned}
&= \int \left[\nabla \cdot (aa^T \nabla \delta \mathcal{E}) \langle \nabla \delta \mathcal{E}, \gamma \rangle + \langle \nabla \delta \mathcal{E}, \gamma \rangle \langle \nabla \delta \mathcal{E}, aa^T \nabla \delta \mathcal{E} \rangle \right. \\
&\quad \left. \delta \mathcal{E}, aa^T \nabla \left(-\frac{v^2}{2} - U(x) - W \circledast \rho(x) \right) \right] f dx dv \\
&\quad + \int \frac{1}{2} \langle \nabla \delta \mathcal{E}, \gamma \rangle \langle \nabla \delta \mathcal{E}, aa^T \nabla \delta \mathcal{E} \rangle f dx dv. \tag{5.5}
\end{aligned}$$

Applying integration by parts for the first term, we have

$$\begin{aligned}
&\int \nabla \cdot (aa^T \nabla \delta \mathcal{E}) \langle \nabla \delta \mathcal{E}, \gamma \rangle f dx dv \\
&= - \int \left[\langle aa^T \nabla \delta \mathcal{E}, \nabla \log f \rangle \langle \nabla \delta \mathcal{E}, \gamma \rangle + \langle aa^T \nabla \delta \mathcal{E}, \nabla^2 \delta \mathcal{E} \gamma \rangle \right] f dx dv \\
&\quad - \int \langle aa^T \nabla \delta \mathcal{E}, \nabla \gamma \nabla \delta \mathcal{E} \rangle f dx dv.
\end{aligned}$$

Plugging the above equality in (5.5) and using the fact $\nabla \delta \mathcal{E} = \nabla \log f - \nabla \left(-\frac{v^2}{2} - U(x) - W \circledast \rho(x) \right)$, we have

$$\begin{aligned}
&\int \Gamma_{\mathcal{I}_a}(\delta \mathcal{E}, \delta \mathcal{E}) f dx \\
&= \int -\frac{1}{2} (\nabla \delta \mathcal{E}, \gamma) (\nabla \delta \mathcal{E}, aa^T \nabla \delta \mathcal{E}) f dx dv \\
&\quad - \int \left[\langle aa^T \nabla \delta \mathcal{E}, \nabla \gamma \nabla \delta \mathcal{E} \rangle + \langle aa^T \nabla \delta \mathcal{E}, \nabla^2 \delta \mathcal{E} \gamma \rangle \right] f dx dv \\
&= \int -\frac{1}{2} \langle \nabla f, \gamma \rangle \langle \nabla \delta \mathcal{E}, aa^T \nabla \delta \mathcal{E} \rangle dx dv \\
&\quad + \int \frac{1}{2} \langle \nabla \left(-\frac{v^2}{2} - U(x) - W \circledast \rho(x) \right), \gamma \rangle \langle \nabla \delta \mathcal{E}, aa^T \nabla \delta \mathcal{E} \rangle f dx dv \\
&\quad - \int \left[\langle aa^T \nabla \delta \mathcal{E}, \nabla \gamma \nabla \delta \mathcal{E} \rangle + \langle aa^T \nabla \delta \mathcal{E}, \nabla^2 \delta \mathcal{E} \gamma \rangle \right] f dx dv.
\end{aligned}$$

Applying integration by parts for the first term and using the identity $\nabla \cdot (\gamma) = 0$, we have

$$\begin{aligned}
&\int -\frac{1}{2} (\nabla f, \gamma) (\nabla \delta \mathcal{E}, aa^T \nabla \delta \mathcal{E}) dx dv \\
&= \int \frac{1}{2} \nabla \cdot (\gamma \langle \nabla \delta \mathcal{E}, aa^T \nabla \delta \mathcal{E} \rangle) f dx dv \\
&= \int \frac{1}{2} \langle \gamma, \langle \nabla \delta \mathcal{E}, \nabla (aa^T \nabla \delta \mathcal{E}) \rangle \rangle + \int \langle aa^T \nabla \delta \mathcal{E}, \nabla^2 \delta \mathcal{E} \gamma \rangle f dx dv.
\end{aligned}$$

By summing over all above terms, we obtain

$$\int \Gamma_{\mathcal{I}_a}(\delta \mathcal{E}, \delta \mathcal{E}) f dx dv = - \int \langle \nabla \gamma \nabla \delta \mathcal{E}, aa^T \nabla \delta \mathcal{E} \rangle f dx dv$$

In the above equality, we use the fact that $\langle \nabla(-\frac{v^2}{2} - U(x) - W \circledast \rho(x)), \gamma \rangle = 0$ and $\nabla(aa^T) = 0$ since a is constant matrix. This completes the proof. $\square$

5.2 Proof of Lemma 4.2

The proof of Lemma 4.2 is divided into the following lemmas.

Lemma 5.2 For $\mathcal{DE}_a(f)$ defined in (2.3), we have the following equality:

$$\begin{aligned} \partial_t \mathcal{DE}_a(f) &= -2 \int [\tilde{\Gamma}_2(\delta \mathcal{E}, \delta \mathcal{E}) - \langle aa^T \nabla \delta \mathcal{E}, \nabla \gamma \nabla \delta \mathcal{E} \rangle] f dx dv \\ &\quad - 2 \int \langle \nabla_{xy}^2 W(x, y)(J + aa^T) \nabla_{y, \tilde{v}} \delta \mathcal{E}(y, \tilde{v}), aa^T \nabla_{x, v} \delta \mathcal{E}(x, v) \rangle \\ &\quad f(x, v) f(y, \tilde{v}) dx dv dy d\tilde{v}. \end{aligned}$$

Proof For $\mathcal{DE}_a(f)$ defined in (2.3), and the structure of matrix a , we have

$$\mathcal{DE}_a(f) = \int \langle \nabla \delta \mathcal{E}, aa^T \nabla \delta \mathcal{E} \rangle f dx dv.$$

We derive the dissipation of $\mathcal{DE}_a(f)$ as below, where we denote $\nabla = \nabla_{x, v}$,

$$\begin{aligned} \partial_t \mathcal{DE}_a(f) &= 2 \int \langle \nabla \partial_t \delta \mathcal{E}, aa^T \nabla \delta \mathcal{E} \rangle f dx dv + \int \langle \nabla \delta \mathcal{E}, aa^T \nabla \delta \mathcal{E} \rangle \partial_t f dx dv \\ &= -2 \int \langle (\partial_t \delta \mathcal{E})(x) \nabla \cdot (f aa^T \nabla \delta \mathcal{E}) dx dv + \int \langle \nabla \delta \mathcal{E}, aa^T \nabla \delta \mathcal{E} \rangle \partial_t f dx dv \\ &= -2 \int \langle \delta^2 \mathcal{E}(x, v, y, \tilde{v}) \partial_t f(y, \tilde{v}) \nabla \cdot (f aa^T \nabla \delta \mathcal{E})(x, v) dx dv \\ &\quad + \int \langle \nabla \delta \mathcal{E}, aa^T \nabla \delta \mathcal{E} \rangle \partial_t f dx dv \\ &= -2 \int \langle W(x, y) \partial_t f(y, \tilde{v}) \nabla \cdot (f aa^T \nabla \delta \mathcal{E})(x, v) dx dv dy d\tilde{v} \\ &\quad - 2 \int \langle \frac{1}{f} \partial_t f \rangle \nabla \cdot (f aa^T \nabla \delta \mathcal{E}) dx dv \\ &\quad + \int \langle \nabla \delta \mathcal{E}, aa^T \nabla \delta \mathcal{E} \rangle \partial_t f dx dv. \end{aligned}$$

Hence,

$$\begin{aligned} \partial_t \mathcal{DE}_a(f) &= -2 \int W(x, y) [\nabla_{y, \tilde{v}} \cdot (f aa^T \nabla_{y, \tilde{v}} \delta \mathcal{E}(y, \tilde{v})) \\ &\quad + \nabla_{y, \tilde{v}} \cdot (f \gamma)] \nabla_{x, v} \cdot (f aa^T \nabla_{x, v} \delta \mathcal{E}(x, v)) dx dv dy d\tilde{v} \end{aligned}$$

$$\begin{aligned}
& -2 \int \frac{1}{f} [\nabla_{x,v} \cdot (faa^T \nabla_{x,v} \delta \mathcal{E}) + \nabla_{x,v} \cdot (f\gamma)] \nabla_{x,v} \cdot (faa^T \nabla_{x,v} \delta \mathcal{E}) dx dv \\
& + \int \langle \nabla \delta \mathcal{E}, aa^T \nabla \delta \mathcal{E} \rangle [\nabla \cdot (faa^T \nabla_{x,v} \delta \mathcal{E}) + \nabla_{x,v} \cdot (f\gamma)] dx dv \\
& = -2 \int W(x, y) \nabla_{y, \tilde{v}} \cdot (faa^T \nabla_{y, \tilde{v}} \delta \mathcal{E}(y, \tilde{v})) \nabla_{x,v} \cdot (faa^T \nabla_{x,v} \delta \mathcal{E}(x, v)) dx dv dy d\tilde{v} \cdots \mathcal{J}_{11} \\
& -2 \int W(x, y) \nabla_{y, \tilde{v}} \cdot (f\gamma) \nabla_{x,v} \cdot (faa^T \nabla_{x,v} \delta \mathcal{E}(x, v)) dx dv dy d\tilde{v} \cdots \mathcal{J}_{21} \\
& -2 \int \frac{1}{f} \nabla_{x,v} \cdot (faa^T \nabla_{x,v} \delta \mathcal{E}) \nabla_{x,v} \cdot (faa^T \nabla_{x,v} \delta \mathcal{E}) dx dv \cdots \mathcal{J}_{12} \\
& -2 \int \frac{1}{f} \nabla_{x,v} \cdot (f\gamma) \nabla_{x,v} \cdot (faa^T \nabla_{x,v} \delta \mathcal{E}) dx dv \cdots \mathcal{J}_{22} \\
& + \int \langle \nabla \delta \mathcal{E}, aa^T \nabla \delta \mathcal{E} \rangle \nabla \cdot (faa^T \nabla_{x,v} \delta \mathcal{E}) dx dv \cdots \mathcal{J}_{13} \\
& + \int \langle \nabla \delta \mathcal{E}, aa^T \nabla \delta \mathcal{E} \rangle \nabla_{x,v} \cdot (f\gamma) dx dv \cdots \mathcal{J}_{23} \\
& = \mathcal{J}_{11} + \mathcal{J}_{12} + \mathcal{J}_{13} + \mathcal{J}_{21} + \mathcal{J}_{22} + \mathcal{J}_{23}.
\end{aligned}$$

We first have

$$\mathcal{J}_{11} = -2 \int \langle \nabla_{xy}^2 W(x, y) aa^T \nabla_{y, \tilde{v}} \delta \mathcal{E}(y, \tilde{v}), aa^T \nabla_{x,v} \delta \mathcal{E}(x, v) \rangle f(x, v) f(y, \tilde{v}) dx dv dy d\tilde{v}.$$

Next, using the identity $\nabla_{x,v} \cdot (f\gamma) = \nabla \cdot (fJ\nabla \delta \mathcal{E})$, we have

$$\begin{aligned}
\mathcal{J}_{21} & = -2 \int W(x, y) \nabla_{y, \tilde{v}} \cdot (f\gamma) \nabla_{x,v} \cdot (faa^T \nabla_{x,v} \delta \mathcal{E}) dx dv dy d\tilde{v} \\
& = -2 \int \langle \nabla_{xy}^2 W(x, y) J \nabla_{y, \tilde{v}} \delta \mathcal{E}(y, \tilde{v}), aa^T \nabla_{x,v} \delta \mathcal{E}(x, v) \rangle f(x, v) f(y, \tilde{v}) dx dv dy d\tilde{v}.
\end{aligned}$$

Furthermore, applying the identity $\nabla_{x,v} \cdot (f\gamma) = f \langle \nabla \delta \mathcal{E}, \gamma \rangle$, and we have

$$\begin{aligned}
& \mathcal{J}_{22} + \mathcal{J}_{23} \\
& = -2 \int \nabla_{x,v} \cdot (f\gamma) \frac{\nabla_{x,v} \cdot (faa^T \nabla_{x,v} \delta \mathcal{E})}{f} f dx dv \\
& \quad + \int \langle \nabla \delta \mathcal{E}, aa^T \nabla \delta \mathcal{E} \rangle \nabla_{x,v} \cdot (f\gamma) dx dv \\
& = -2 \int \nabla_{x,v} \cdot (f\gamma) \left[\langle \nabla_{x,v} \log f, aa^T \nabla_{x,v} \delta \mathcal{E} \rangle + \nabla_{x,v} \cdot (aa^T \nabla_{x,v} \delta \mathcal{E}) \right] dx dv \\
& \quad + \int \langle \nabla \delta \mathcal{E}, aa^T \nabla \delta \mathcal{E} \rangle \nabla_{x,v} \cdot (f\gamma) dx dv \\
& = -2 \int \nabla_{x,v} \cdot (f\gamma) \left[\langle \nabla_{x,v} \delta \mathcal{E}, aa^T \nabla_{x,v} \delta \mathcal{E} \rangle + \tilde{L} \delta \mathcal{E} \right] dx dv \\
& \quad + \int \langle \nabla \delta \mathcal{E}, aa^T \nabla \delta \mathcal{E} \rangle \nabla_{x,v} \cdot (f\gamma) dx dv
\end{aligned}$$

$$\begin{aligned}
&= -2 \int \langle \nabla \delta \mathcal{E}, \gamma \rangle \tilde{L} \delta \mathcal{E} f \, dx dv - \int \langle \nabla \delta \mathcal{E}, aa^T \nabla \delta \mathcal{E} \rangle \nabla_{x,v} \cdot (f \gamma) \, dx dv \\
&= -2 \left[\int \langle \nabla \delta \mathcal{E}, \gamma \rangle \tilde{L} \delta \mathcal{E} f \, dx dv - \frac{1}{2} \int \langle \nabla \Gamma_1(\delta \mathcal{E}, \delta \mathcal{E}), \gamma \rangle f \, dx dv \right] \\
&= -2 \int \tilde{\Gamma}_{\mathcal{I}_a}(\delta \mathcal{E}, \delta \mathcal{E}) f \, dx dv.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mathcal{J}_{12} + \mathcal{J}_{13} &= -2 \int \frac{1}{f} \nabla \cdot (f aa^T \nabla_{x,v} \delta \mathcal{E}) \nabla_{x,v} \cdot (f aa^T \nabla_{x,v} \delta \mathcal{E}) \, dx dv \\
&\quad + \int \langle \nabla \delta \mathcal{E}, aa^T \nabla \delta \mathcal{E} \rangle \nabla \cdot (f aa^T \nabla_{x,v} \delta \mathcal{E}) \, dx dv \\
&= -2 \int \left[\langle \nabla_{x,v} \delta \mathcal{E}, aa^T \nabla_{x,v} \delta \mathcal{E} \rangle + \tilde{L} \delta \mathcal{E} \right] \tilde{L}^* f \, dx dv \\
&\quad + \int \langle \nabla \delta \mathcal{E}, aa^T \nabla \delta \mathcal{E} \rangle \tilde{L}^* f \, dx dv \\
&= -2 \int \left[\frac{1}{2} \tilde{L} \Gamma_1(\delta \mathcal{E}, \delta \mathcal{E}) - \Gamma_1(\tilde{L} \delta \mathcal{E}, \delta \mathcal{E}) \right] f \, dx dv \\
&= -2 \int \tilde{\Gamma}_2(\delta \mathcal{E}, \delta \mathcal{E}) f \, dx dv.
\end{aligned}$$

Applying Lemma 5.1 and combining the above terms, we finish the proof. $\square$

Lemma 5.3 For $\mathcal{DE}_z(f)$ defined in (2.4), we have the following equality

$$\begin{aligned}
\partial_t \mathcal{DE}_z(f) &= -2 \int [\tilde{\Gamma}_2^z(\delta \mathcal{E}, \delta \mathcal{E}) - \langle zz^T \nabla \delta \mathcal{E}, \nabla \gamma \nabla \delta \mathcal{E} \rangle] f \, dx dv \\
&\quad - 2 \int \langle \nabla_{xy}^2 W(x, y) (J + aa^T) \nabla_{y,\tilde{v}} \delta \mathcal{E}(y, \tilde{v}), zz^T \nabla_{x,v} \delta \mathcal{E}(x, v) \rangle f(x, v) f(y, \tilde{v}) \, dx dv dy d\tilde{v}.
\end{aligned}$$

Proof For $\mathcal{DE}_z(f)$ defined in (2.3), and the structure of matrix a , we have

$$\mathcal{DE}_z(f) = \int \langle \nabla \delta \mathcal{E}, zz^T \nabla \delta \mathcal{E} \rangle f \, dx dv.$$

We derive the dissipation of $\mathcal{DE}_z(f)$ as below:

$$\begin{aligned}
&\partial_t \mathcal{DE}_z(f) \\
&= 2 \int \langle \nabla \partial_t \delta \mathcal{E}, zz^T \nabla \delta \mathcal{E} \rangle f \, dx dv + \int \langle \nabla \delta \mathcal{E}, zz^T \nabla \delta \mathcal{E} \rangle \partial_t f \, dx dv \\
&= -2 \int \langle (\partial_t \delta \mathcal{E})(x) \nabla \cdot (f zz^T \nabla \delta \mathcal{E}) \, dx dv + \int \langle \nabla \delta \mathcal{E}, zz^T \nabla \delta \mathcal{E} \rangle \partial_t f \, dx dv \\
&= -2 \int (\delta^2 \mathcal{E}(x, v, y, \tilde{v}) \partial_t f(y, \tilde{v}) \nabla \cdot (f zz^T \nabla \delta \mathcal{E})(x, v) \, dx dv \\
&\quad + \int \langle \nabla \delta \mathcal{E}, zz^T \nabla \delta \mathcal{E} \rangle \partial_t f \, dx dv
\end{aligned}$$

$$\begin{aligned}
&= -2 \int (W(x, y) \partial_t f(y, \tilde{v})) \nabla \cdot (f z z^T \nabla \delta \mathcal{E})(x, v) dx dv dy d\tilde{v} \\
&\quad - 2 \int \left(\frac{1}{f} \partial_t f \right) \nabla \cdot (f z z^T \nabla \delta \mathcal{E}) dx dv \\
&\quad + \int \langle \nabla \delta \mathcal{E}, z z^T \nabla \delta \mathcal{E} \rangle \partial_t f dx dv.
\end{aligned}$$

Plugging in the equation for $\partial_t f$, we have

$$\begin{aligned}
&\partial_t \mathcal{D}\mathcal{E}_z(f) \\
&= -2 \int W(x, y) [\nabla_{y, \tilde{v}} \cdot (f a a^T \nabla_{y, \tilde{v}} \delta \mathcal{E}(y, \tilde{v})) \\
&\quad + \nabla_{y, \tilde{v}} \cdot (f \gamma)] \nabla_{x, v} \cdot (f z z^T \nabla_{x, v} \delta \mathcal{E}(x, v)) dx dv dy d\tilde{v} \\
&\quad - 2 \int \frac{1}{f} [\nabla_{x, v} \cdot (f a a^T \nabla_{x, v} \delta \mathcal{E}) + \nabla_{x, v} \cdot (f \gamma)] \nabla_{x, v} \cdot (f z z^T \nabla_{x, v} \delta \mathcal{E}) dx dv \\
&\quad + \int \langle \nabla \delta \mathcal{E}, z z^T \nabla_{x, v} \delta \mathcal{E} \rangle [\nabla \cdot (f a a^T \nabla_{x, v} \delta \mathcal{E}) + \nabla_{x, v} \cdot (f \gamma)] dx dv \\
&= -2 \int W(x, y) \nabla_{y, \tilde{v}} \cdot (f a a^T \nabla_{y, \tilde{v}} \delta \mathcal{E}(y, \tilde{v})) \nabla_{x, v} \cdot (f z z^T \nabla_{x, v} \delta \mathcal{E}(x, v)) dx dv dy d\tilde{v} \cdots \mathcal{J}_{11}^z \\
&\quad - 2 \int W(x, y) \nabla_{y, \tilde{v}} \cdot (f \gamma) \nabla_{x, v} \cdot (f z z^T \nabla_{x, v} \delta \mathcal{E}(x, v)) dx dv dy d\tilde{v} \cdots \mathcal{J}_{21}^z \\
&\quad - 2 \int \frac{1}{f} \nabla_{x, v} \cdot (f a a^T \nabla_{x, v} \delta \mathcal{E}) \nabla_{x, v} \cdot (f z z^T \nabla_{x, v} \delta \mathcal{E}) dx dv \cdots \mathcal{J}_{12}^z \\
&\quad - 2 \int \frac{1}{f} \nabla_{x, v} \cdot (f \gamma) \nabla_{x, v} \cdot (f z z^T \nabla_{x, v} \delta \mathcal{E}) dx dv \cdots \mathcal{J}_{22}^z \\
&\quad + \int \langle \nabla \delta \mathcal{E}, z z^T \nabla \delta \mathcal{E} \rangle \nabla_{x, v} \cdot (f a a^T \nabla_{x, v} \delta \mathcal{E}) dx dv \cdots \mathcal{J}_{13}^z \\
&\quad + \int \langle \nabla \delta \mathcal{E}, z z^T \nabla \delta \mathcal{E} \rangle \nabla_{x, v} \cdot (f \gamma) dx dv \cdots \mathcal{J}_{23}^z \\
&= \mathcal{J}_{11}^z + \mathcal{J}_{12}^z + \mathcal{J}_{13}^z + \mathcal{J}_{21}^z + \mathcal{J}_{22}^z + \mathcal{J}_{23}^z.
\end{aligned}$$

We first have

$$\mathcal{J}_{11}^z = -2 \int \langle \nabla_{xy}^2 W(x, y) a a^T \nabla_{y, \tilde{v}} \delta \mathcal{E}(y, \tilde{v}), z z^T \nabla_{x, v} \delta \mathcal{E}(x, v) \rangle f(x, v) f(y, \tilde{v}) dx dv dy d\tilde{v}.$$

Next, using the identity $\nabla_{x, v} \cdot (f \gamma) = \nabla_{x, v} \cdot (f J \nabla_{x, v} \delta \mathcal{E})$, we have

$$\begin{aligned}
\mathcal{J}_{21}^z &= -2 \int W(x, y) \nabla_{y, \tilde{v}} \cdot (f \gamma) \nabla_{x, v} \cdot (f z z^T \nabla_{x, v} \delta \mathcal{E}) dx dv dy d\tilde{v} \\
&= -2 \int \langle \nabla_{xy}^2 W(x, y) J \nabla_{y, \tilde{v}} \delta \mathcal{E}(y, \tilde{v}), z z^T \nabla_{x, v} \delta \mathcal{E}(x, v) \rangle f(x, v) f(y, \tilde{v}) dx dv dy d\tilde{v}.
\end{aligned}$$

Furthermore, applying the identity $\nabla_{x, v} \cdot (f \gamma) = f \langle \nabla_{x, v} \delta \mathcal{E}, \gamma \rangle$, we have

$$\mathcal{J}_{22} + \mathcal{J}_{23}$$

$$\begin{aligned}
&= -2 \int \nabla_{x,v} \cdot (f\gamma) \frac{\nabla_{x,v} \cdot (fzz^T \nabla_{x,v} \delta \mathcal{E})}{f} f dx dv \\
&+ \int \langle \nabla \delta \mathcal{E}, zz^T \nabla \delta \mathcal{E} \rangle \nabla_{x,v} \cdot (f\gamma) dx dv \\
&= -2 \int \nabla_{x,v} \cdot (f\gamma) \left[\langle \nabla_{x,v} \log f, zz^T \nabla_{x,v} \delta \mathcal{E} \rangle + \nabla_{x,v} \cdot (zz^T \nabla_{x,v} \delta \mathcal{E}) \right] dx dv \\
&+ \int \langle \nabla \delta \mathcal{E}, zz^T \nabla \delta \mathcal{E} \rangle \nabla_{x,v} \cdot (f\gamma) dx dv \\
&= -2 \int \nabla_{x,v} \cdot (f\gamma) \left[\langle \nabla_{x,v} \delta \mathcal{E}, zz^T \nabla_{x,v} \delta \mathcal{E} \rangle + \tilde{L}_z \delta \mathcal{E} \right] dx dv \\
&+ \int \langle \nabla \delta \mathcal{E}, zz^T \nabla \delta \mathcal{E} \rangle \nabla_{x,v} \cdot (f\gamma) dx dv \\
&= -2 \int \langle \nabla \delta \mathcal{E}, \gamma \rangle \tilde{L}_z \delta \mathcal{E} f dx dv - \int \langle \nabla \delta \mathcal{E}, zz^T \nabla \delta \mathcal{E} \rangle \nabla_{x,v} \cdot (f\gamma) dx dv \\
&= -2 \left[\int \langle \nabla \delta \mathcal{E}, \gamma \rangle \tilde{L}_z \delta \mathcal{E} f dx dv - \frac{1}{2} \int \langle \nabla \Gamma_1^z(\delta \mathcal{E}, \delta \mathcal{E}), \gamma \rangle f dx dv \right] \\
&= -2 \int \tilde{\Gamma}_{\mathcal{I}_z}(\delta \mathcal{E}, \delta \mathcal{E}) f dx dv.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mathcal{J}_{12}^z + \mathcal{J}_{13}^z &= -2 \int \frac{1}{f} \nabla \cdot (faa^T \nabla_{x,v} \delta \mathcal{E}) \nabla_{x,v} \cdot (fzz^T \nabla_{x,v} \delta \mathcal{E}) dx dv \\
&+ \int \langle \nabla \delta \mathcal{E}, zz^T \nabla \delta \mathcal{E} \rangle \nabla \cdot (faa^T \nabla_{x,v} \delta \mathcal{E}) dx dv \\
&= -2 \int \left[\langle \nabla_{x,v} \delta \mathcal{E}, zz^T \nabla_{x,v} \delta \mathcal{E} \rangle + \tilde{L}_z \delta \mathcal{E} \right] \tilde{L}^* f dx dv \\
&+ \int \langle \nabla \delta \mathcal{E}, zz^T \nabla \delta \mathcal{E} \rangle \tilde{L}^* f dx dv \\
&= -2 \int \left[\frac{1}{2} \tilde{L} \Gamma_1^z(\delta \mathcal{E}, \delta \mathcal{E}) - \Gamma_1(\tilde{L}_z \delta \mathcal{E}, \delta \mathcal{E}) \right] f dx dv \\
&= -2 \int \left[\frac{1}{2} \tilde{L} \Gamma_1^z(\delta \mathcal{E}, \delta \mathcal{E}) - \Gamma_1^z(\tilde{L} \delta \mathcal{E}, \delta \mathcal{E}) + \Gamma_1^z(\tilde{L} \delta \mathcal{E}, \delta \mathcal{E}) - \Gamma_1(\tilde{L}_z \delta \mathcal{E}, \delta \mathcal{E}) \right] \\
&f dx dv = -2 \int \tilde{\Gamma}_2(\delta \mathcal{E}, \delta \mathcal{E}) f dx dv,
\end{aligned}$$

where the last equality follows from the following observation

$$\int [\Gamma_1^z(\tilde{L} \delta \mathcal{E}, \delta \mathcal{E}) - \Gamma_1(\tilde{L}_z \delta \mathcal{E}, \delta \mathcal{E})] f dx dv = 0, \quad (5.6)$$

for constant matrices a and z . We first observe that

$$\begin{aligned}
 & \int \Gamma_1^z(\tilde{L}\delta\mathcal{E}, \delta\mathcal{E}) f dx dv \\
 &= \int \Gamma_1^z \left(\langle \nabla(-\frac{v^2}{2} - U(x) - W \circledast \rho(x)), aa^T \nabla \delta\mathcal{E} \rangle \right. \\
 &\quad \left. + \nabla \cdot (aa^T \nabla \delta\mathcal{E}), \delta\mathcal{E} \right) f dx dv \\
 &= \int \Gamma_1^z \left(\Gamma_1(-\frac{v^2}{2} - U(x) - W \circledast \rho(x), \delta\mathcal{E}), \delta\mathcal{E} \right) f \\
 &\quad + \Gamma_1^z \left(\nabla \cdot (aa^T \nabla \delta\mathcal{E}), \delta\mathcal{E} \right) f dx dv \\
 &= \int \Gamma_1^z \left(\Gamma_1(-\frac{v^2}{2} - U(x) - W \circledast \rho(x), \delta\mathcal{E}), \delta\mathcal{E} \right) f dx dv \\
 &\quad - \int \nabla \cdot (aa^T \nabla \delta\mathcal{E}) \nabla \cdot (fzz^T \nabla \delta\mathcal{E}) dx dv \\
 &= \int \Gamma_1^z \left(\Gamma_1(-\frac{v^2}{2} - U(x) - W \circledast \rho(x), \delta\mathcal{E}), \delta\mathcal{E} \right) f dx dv \\
 &\quad - \int \nabla \cdot (aa^T \nabla \delta\mathcal{E}) \nabla \cdot (fzz^T \nabla \delta\mathcal{E}) dx dv \\
 &= \int \Gamma_1^z \left(\Gamma_1(-\frac{v^2}{2} - U(x) - W \circledast \rho(x), \delta\mathcal{E}), \delta\mathcal{E} \right) f dx dv \\
 &\quad - \int \nabla \cdot (aa^T \nabla \delta\mathcal{E}) \nabla \cdot (zz^T \nabla \delta\mathcal{E}) f dx dv \\
 &\quad - \int \nabla \cdot (aa^T \nabla \delta\mathcal{E}) \langle \nabla \log f, zz^T \nabla \delta\mathcal{E} \rangle f dx dv.
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 & \int \Gamma_1(\tilde{L}_z \delta\mathcal{E}, \delta\mathcal{E}) f dx dv \\
 &= \int \Gamma_1 \left(\Gamma_1^z(-\frac{v^2}{2} - U(x) - W \circledast \rho(x), \delta\mathcal{E}), \delta\mathcal{E} \right) f dx dv \\
 &\quad - \int \nabla \cdot (aa^T \nabla \delta\mathcal{E}) \nabla \cdot (zz^T \nabla \delta\mathcal{E}) f dx dv \\
 &\quad - \int \nabla \cdot (zz^T \nabla \delta\mathcal{E}) \langle \nabla \log f, aa^T \nabla \delta\mathcal{E} \rangle f dx dv.
 \end{aligned}$$

Furthermore, by direct expansion of the gradient, we have

$$\begin{aligned}
 & \int \nabla \cdot (aa^T \nabla \delta\mathcal{E}) \langle \nabla \log f, zz^T \nabla \delta\mathcal{E} \rangle f dx dv \\
 &= - \int \langle a^T \nabla \delta\mathcal{E}, a^T \nabla \langle z^T \nabla \log f, z^T \nabla \delta\mathcal{E} \rangle \rangle f dx dv
 \end{aligned}$$

$$\begin{aligned}
&= - \int \sum_{i=1}^d \sum_{j=1}^d \sum_{\hat{i}, \hat{j}, \tilde{i}, \tilde{j}=1}^{2d} \left(a_{i\hat{i}}^T \partial_{x_i} \delta \mathcal{E} a_{i\tilde{i}}^T \partial_{x_{\tilde{i}}} [z_{j\hat{j}}^T \partial_{x_{\hat{j}}} \log f z_{j\tilde{j}}^T \partial_{x_{\tilde{j}}} \delta \mathcal{E}] \right) f dx dv \\
&= - \int \sum_{i=1}^d \sum_{j=1}^d \sum_{\hat{i}, \hat{j}, \tilde{i}, \tilde{j}=1}^{2d} \left(a_{i\hat{i}}^T a_{i\tilde{i}}^T z_{j\hat{j}}^T z_{j\tilde{j}}^T \partial_{x_i} \delta \mathcal{E} \partial_{x_{\tilde{i}}}^2 \log f \partial_{x_{\tilde{j}}} \delta \mathcal{E} \right) f dx dv \\
&\quad - \int \sum_{i=1}^d \sum_{j=1}^d \sum_{\hat{i}, \hat{j}, \tilde{i}, \tilde{j}=1}^{2d} \left(a_{i\hat{i}}^T a_{i\tilde{i}}^T z_{j\hat{j}}^T z_{j\tilde{j}}^T \partial_{x_i} \delta \mathcal{E} \partial_{x_{\tilde{j}}} \log f \partial_{x_{\tilde{i}}}^2 \delta \mathcal{E} \right) f dx dv \\
&= \int \nabla \cdot (z z^T \nabla \delta \mathcal{E}) \langle \nabla \log f, a a^T \nabla \delta \mathcal{E} \rangle f dx dv,
\end{aligned}$$

where we denote $(x_1, \dots, x_d, v_1, \dots, v_d) = (x_1, \dots, x_{2d})$ and use the fact that a and z are constant matrices. Similarly, we have

$$\begin{aligned}
&\int \Gamma_1^z \left(\Gamma_1 \left(-\frac{v^2}{2} - U(x) - W \circledast \rho(x), \delta \mathcal{E} \right), \delta \mathcal{E} \right) f dx dv \\
&= \int \Gamma_1 \left(\Gamma_1^z \left(-\frac{v^2}{2} - U(x) - W \circledast \rho(x), \delta \mathcal{E} \right), \delta \mathcal{E} \right) f dx dv,
\end{aligned}$$

which proves equation (5.6). Applying Lemma 5.1 and combining the above terms, we complete the proof. $\square$

We are now ready to prove Lemma 4.2. Recall the mean-field Langevin dynamics

$$\begin{cases} dx_t = v_t dt \\ dv_t = (-v_t - \nabla_x \tilde{V}(x_t, f)) dt + \sqrt{2} dB_t, \end{cases} \quad (5.7)$$

where f is the solution of PDE (1.1), and we denote

$$\tilde{V}(x, f) = \int_{\Omega} W(x, y) f(t, y, \tilde{v}) dy d\tilde{v} + U(x).$$

Proof of Lemma 4.2 We first present the explicit formulation of matrix $\mathfrak{R}$. Following from Lemma 5.2 and Lemma 5.3, we have

$$\begin{aligned}
&\partial_t \mathcal{D} \mathcal{E}_{a,z}(f) \\
&= -2 \int [\tilde{\Gamma}_2(\delta \mathcal{E}, \delta \mathcal{E}) + \tilde{\Gamma}_2^z(\delta \mathcal{E}, \delta \mathcal{E}) - \langle (a a^T + z z^T) \nabla \delta \mathcal{E}, \nabla \gamma \nabla \delta \mathcal{E} \rangle] f dx dv \\
&\quad - 2 \int \nabla_{xy}^2 W(x, y) \langle (J + a a^T) \nabla_{y, \tilde{v}} \delta \mathcal{E}(y, \tilde{v}), \\
&\quad (a a^T + z z^T) \nabla_{x, v} \delta \mathcal{E}(x, v) \rangle f(x, v) f(y, \tilde{v}) dx dv dy d\tilde{v}.
\end{aligned}$$

Following from the definition of $\tilde{\Gamma}_2$ and $\tilde{\Gamma}_2^z$ from Section 4 and Proposition 5.1 we have

$$\begin{aligned}
& \tilde{\Gamma}_2(\delta\mathcal{E}, \delta\mathcal{E}) + \tilde{\Gamma}_2^z(\delta\mathcal{E}, \delta\mathcal{E}) \\
&= \sum_{i=1}^d \left[z_1^2 |\partial_{x_i x_i}^2 \delta\mathcal{E}|^2 + (1 + z_2^2) |\partial_{x_i+d x_i+d}^2 \delta\mathcal{E}|^2 + 2z_1 z_2 |\partial_{x_i x_i+d}^2 \delta\mathcal{E}|^2 \right] \\
&\quad - \sum_{i=1}^d \sum_{\hat{k}=1}^{2d} a_i^T \nabla [aa^T \nabla (-\frac{v^2}{2} - U(x) - W \circledast \rho(x))]_{\hat{k}} \partial_{x_{\hat{k}}} \delta\mathcal{E} a_i^T \nabla \delta\mathcal{E} \\
&\quad - \sum_{k=1}^d \sum_{\hat{k}=1}^{2d} z_k^T \nabla [aa^T \nabla (-\frac{v^2}{2} - U(x) - W \circledast \rho(x))]_{\hat{k}} \partial_{x_{\hat{k}}} \delta\mathcal{E} z_k^T \nabla \delta\mathcal{E}, \\
&= \|\mathfrak{H}\text{ess}\delta\mathcal{E}\|_F^2 + \mathfrak{R}_{a,z}(\delta\mathcal{E}, \delta\mathcal{E}),
\end{aligned}$$

where we define

$$\begin{aligned}
\|\mathfrak{H}\text{ess}\delta\mathcal{E}\|_F^2 &= \sum_{i=1}^d \left[z_1^2 |\partial_{x_i x_i}^2 \delta\mathcal{E}|^2 + (1 + z_2^2) |\partial_{x_i+d x_i+d}^2 \delta\mathcal{E}|^2 + 2z_1 z_2 |\partial_{x_i x_i+d}^2 \delta\mathcal{E}|^2 \right] \\
\mathfrak{R}_{a,z}(\delta\mathcal{E}, \delta\mathcal{E}) &= - \sum_{i=1}^d \sum_{\hat{k}=1}^{2d} a_i^T \nabla [aa^T \nabla (-\frac{v^2}{2} - U(x) - W \circledast \rho(x))]_{\hat{k}} \partial_{x_{\hat{k}}} \delta\mathcal{E} a_i^T \nabla \delta\mathcal{E} \\
&\quad - \sum_{k=1}^d \sum_{\hat{k}=1}^{2d} z_k^T \nabla [aa^T \nabla (-\frac{v^2}{2} - U(x) - W \circledast \rho(x))]_{\hat{k}} \partial_{x_{\hat{k}}} \delta\mathcal{E} z_k^T \nabla \delta\mathcal{E}.
\end{aligned}$$

According to the above computation, we have

$$\begin{aligned}
& \int_{\Omega} [(\tilde{\Gamma}_2 + \tilde{\Gamma}_2^z)(\delta\mathcal{E}, \delta\mathcal{E}) - \langle (aa^T + zz^T) \nabla \delta\mathcal{E}, \nabla \gamma \nabla \delta\mathcal{E} \rangle] f dx dv \\
&\quad + \int_{\Omega \times \Omega} \langle \nabla_{xy}^2 W(x, y) (J + aa^T) \nabla_{y, \tilde{v}} \delta\mathcal{E}(y, \tilde{v}), \\
&\quad (aa^T + zz^T) \nabla_{x, v} \delta\mathcal{E}(x, v) \rangle f(x, v) f(y, \tilde{v}) dx dv dy d\tilde{v} \\
&\geq \mathfrak{R}_1 + \mathfrak{R}_2,
\end{aligned}$$

where we denote

$$\begin{aligned}
\mathfrak{R}_1 &= \int_{\Omega \times \Omega} \langle \nabla_{xy}^2 W(x, y) (J + aa^T) \nabla_{y, \tilde{v}} \delta\mathcal{E}(y, \tilde{v}), \\
&\quad (aa^T + zz^T) \nabla_{x, v} \delta\mathcal{E}(x, v) \rangle f(x, v) f(y, \tilde{v}) dx dv dy d\tilde{v}. \\
\mathfrak{R}_2 &= \int_{\Omega} [\mathfrak{R}_{a,z}(\delta\mathcal{E}, \delta\mathcal{E}) - \langle (aa^T + zz^T) \nabla \delta\mathcal{E}, \nabla \gamma \nabla \delta\mathcal{E} \rangle] f(x, v) dx dv.
\end{aligned}$$

If Assumption 2.1 holds, we have

$$\begin{aligned}
& \int_{\Omega} [(\tilde{\Gamma}_2 + \tilde{\Gamma}_2^z)(\delta\mathcal{E}, \delta\mathcal{E}) - \langle (aa^T + zz^T)\nabla\delta\mathcal{E}, \nabla\gamma\nabla\delta\mathcal{E} \rangle] f dx dv \\
& + \int_{\Omega \times \Omega} \langle \nabla_{xy}^2 W(x, y)(J + aa^T)\nabla_{y, \tilde{v}}\delta\mathcal{E}(y, \tilde{v}), \\
& (aa^T + zz^T)\nabla_{x, v}\delta\mathcal{E}(x, v) \rangle f(x, v)f(y, \tilde{v}) dx dv dy d\tilde{v} \\
& \geq \mathfrak{R}_1 + \mathfrak{R}_2 = \int_{\Omega \times \Omega} \mathfrak{R}(\nabla\delta\mathcal{E}, \nabla\delta\mathcal{E}) f(t, x, v)f(t, y, \tilde{v}) dx dv dy d\tilde{v} \\
& \geq \lambda \int_{\Omega} (\Gamma_1(\delta\mathcal{E}, \delta\mathcal{E}) + \Gamma_1^z(\delta\mathcal{E}, \delta\mathcal{E})) f dx dv = \lambda[\mathcal{D}\mathcal{E}_a(f) + \mathcal{D}\mathcal{E}_z(f)].
\end{aligned}$$

Thus, we only need to show that $\mathfrak{R}$ is indeed the matrix defined in Definition 2.3. With some abuse of notations, we denote $\mathfrak{R} = \mathfrak{R}_1 + \mathfrak{R}_2$. In the following, we derive the explicit formulation of $\mathfrak{R}$ as defined in Definition 2.3 and Definition 3.1.

Case 1: For $d=1$, the $\mathfrak{R}$ has the following two parts. For constant matrices $a = (0 \ 1)^T$ and $z = (z_1 \ z_2)^T$,

$$\begin{aligned}
\mathfrak{R}_1 &= \int \langle \nabla_{xy}^2 W(x, y)(J + aa^T)\nabla_{y, \tilde{v}}\delta\mathcal{E}(y, \tilde{v}), \\
& (aa^T + zz^T)\nabla_{x, v}\delta\mathcal{E}(x, v) \rangle f(x, v)f(y, \tilde{v}) dx dv dy d\tilde{v}. \\
&= \int (\nabla\delta\mathcal{E}(y, \tilde{v}))^T \text{sym} \left((aa^T + zz^T)^T \nabla_{xy}^2 W(x, y)(aa^T + J) \right) \\
& (\nabla\delta\mathcal{E}(x, v)) f(x, v)f(y, \tilde{v}) dx dv dy d\tilde{v}.
\end{aligned}$$

To be precise, we denote $\mathcal{J} = \int_{\Omega \times \Omega}$ in $\mathfrak{R}_1$. Plugging in the matrices a , z , and J , we have the following symmetrization of the matrix,

$$\begin{aligned}
& \text{sym} \left((aa^T + zz^T)^T \nabla_{xy}^2 W(x, y)(aa^T + J) \right) \\
&= \text{sym} \left((aa^T + zz^T) \nabla_{xy}^2 W(x, y)(aa^T + J) \right) \\
&= \text{sym} \left(\begin{pmatrix} z_1^2 & z_1 z_2 \\ z_1 z_2 & (1 + z_2^2) \end{pmatrix}_{2 \times 2} \nabla_{xy}^2 W(x, y) \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}_{2 \times 2} \right) \\
&= \text{sym} \left(\begin{pmatrix} z_1^2 & z_1 z_2 \\ z_1 z_2 & (1 + z_2^2) \end{pmatrix}_{2 \times 2} \begin{pmatrix} 0 & -\nabla_{xy}^2 W(x, y) \\ 0 & 0 \end{pmatrix}_{2d \times 2d} \right) \\
&= \text{sym} \begin{pmatrix} 0 & -z_1^2 \nabla_{xy}^2 W(x, y) \\ 0 & -z_1 z_2 \nabla_{xy}^2 W(x, y) \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\frac{z_1^2}{2} \nabla_{xy}^2 W(x, y) \\ -\frac{z_1^2}{2} \nabla_{xy}^2 W(x, y) & -z_1 z_2 \nabla_{xy}^2 W(x, y) \end{pmatrix}.
\end{aligned}$$

This implies that

$$\mathfrak{R}_1 = \int (\nabla \delta \mathcal{E}(y, \tilde{v}))^\top \begin{pmatrix} 0 & -\frac{z_1^2}{2} \nabla_{xy}^2 W(x, y) \\ -\frac{z_1^2}{2} \nabla_{xy}^2 W(x, y) & -z_1 z_2 \nabla_{xy}^2 W(x, y) \end{pmatrix} (\nabla \delta \mathcal{E}(x, v)) f(x, v) f(y, \tilde{v}) dx dv dy d\tilde{v}.$$

As for the second term, we have

$$\begin{aligned} \mathfrak{R}_2 &= \int_{\Omega} [\mathfrak{R}_{a,z}(\delta \mathcal{E}, \delta \mathcal{E}) - \langle (aa^\top + zz^\top) \nabla \delta \mathcal{E}, \nabla \gamma \nabla \delta \mathcal{E} \rangle] f(x, v) dx dv \\ &= \int_{\Omega} (\nabla \delta \mathcal{E})^\top \mathfrak{R}_2 \nabla \delta \mathcal{E} f(x, v) dx dv. \end{aligned}$$

By direct computation and matrix symmetrization, the matrix $\mathfrak{R}_2$ has the following representation,

$$\begin{aligned} \mathfrak{R}_2 &= \begin{pmatrix} 0 & 0 \\ 0 & -\frac{\partial^2(-\frac{v^2}{2} - \tilde{V}(x, f))}{\partial v^2} \end{pmatrix} + \frac{1}{2} \left[\begin{pmatrix} 0 & -z_1^\top \nabla \left(\frac{\partial(-\frac{v^2}{2} - \tilde{V}(x, f))}{\partial v} \right) \\ -z_1^\top \nabla \left(\frac{\partial(-\frac{v^2}{2} - \tilde{V}(x, f))}{\partial v} \right) \end{pmatrix} z^\top \right. \\ &\quad \left. + z \left(0 - z_1^\top \nabla \left(\frac{\partial(-\frac{v^2}{2} - \tilde{V}(x, f))}{\partial v} \right) \right) \right] \\ &\quad - \frac{1}{2} [(\nabla \gamma)^\top aa^\top + aa^\top \nabla \gamma] - \frac{1}{2} [(\nabla \gamma)^\top zz^\top + zz^\top \nabla \gamma], \end{aligned}$$

with $(\nabla \gamma)_{ij} = \nabla_i \gamma_j$, such that

$$\nabla \gamma = \begin{pmatrix} 0 & \int \nabla_{xx}^2 W(x, y) f(t, y, v) dy dv + \nabla_{xx} U(x) \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \nabla_{xx}^2 \tilde{V}(x, f) \\ -1 & 0 \end{pmatrix},$$

and

$$zz^\top = \begin{pmatrix} z_1^2 & z_1 z_2 \\ z_1 z_2 & z_2^2 \end{pmatrix}, \quad aa^\top = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

By direct computations, we have

$$\begin{aligned} \mathfrak{R}_2 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2} \left[\begin{pmatrix} 0 \\ z_2 \end{pmatrix} (z_1, z_2) + \begin{pmatrix} z_1 \\ 0 \end{pmatrix} (0, z_2) \right] - \frac{1}{2} [(\nabla \gamma)^\top aa^\top + aa^\top \nabla \gamma] \\ &\quad - \frac{1}{2} [(\nabla \gamma)^\top zz^\top + zz^\top \nabla \gamma] \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{2} z_1 z_2 \\ \frac{1}{2} z_1 z_2 & z_2^2 \end{pmatrix} - \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} - \begin{pmatrix} -z_1 z_2 & \frac{1}{2} [\nabla^2 \tilde{V} z_1^2 - z_2^2] \\ \frac{1}{2} [\nabla^2 \tilde{V} z_1^2 - z_2^2] & \nabla^2 \tilde{V} z_1 z_2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} [z_1 z_2 + z_2^2 - \nabla^2 \tilde{V} z_1^2 + 1] & \frac{1}{2} [z_1 z_2 + z_2^2 - \nabla^2 \tilde{V} z_1^2 + 1] \\ \frac{1}{2} [z_1 z_2 + z_2^2 - \nabla^2 \tilde{V} z_1^2 + 1] & 1 + z_2^2 - \nabla^2 \tilde{V} z_1 z_2 \end{pmatrix}. \end{aligned}$$

For notation simplicity, we denote $U(x, v) = \nabla \delta \mathcal{E}(x, v)$ and $U(y, \tilde{v}) = \nabla \delta \mathcal{E}(y, \tilde{v})$. Recall that $\tilde{V}(x, f) = \int_{\Omega} W(x, y) f(t, y, \tilde{v}) dy d\tilde{v} + U(x)$, we have

$$\begin{aligned}
 & \mathfrak{R}_1 + \mathfrak{R}_2 \\
 &= \int_{\Omega \times \Omega} U^T(y, \tilde{v}) \begin{pmatrix} 0 & -\frac{z_1^2}{2} \nabla_{xy}^2 W(x, y) \\ -\frac{z_1^2}{2} \nabla_{xy}^2 W(x, y) & -z_1 z_2 \nabla_{xy}^2 W(x, y) \end{pmatrix} \\
 & \quad U(x, v) f(x, v) f(y, \tilde{v}) dx dv dy d\tilde{v} \\
 &+ \int_{\Omega} U^T(x, v) \begin{pmatrix} z_1 z_2 & \frac{1}{2} [z_1 z_2 + z_2^2 - \nabla_{xx}^2 \tilde{V} z_1^2 + 1] \\ \frac{1}{2} [z_1 z_2 + z_2^2 - \nabla_{xx}^2 \tilde{V} z_1^2 + 1] & 1 + z_2^2 - \nabla_{xx}^2 \tilde{V} z_1 z_2 \end{pmatrix} \\
 & \quad U(x, v) f(x, v) dx dv \\
 &= \int_{\Omega \times \Omega} U^T(y, \tilde{v}) \begin{pmatrix} 0 & -\frac{z_1^2}{2} \nabla_{xy}^2 W(x, y) \\ -\frac{z_1^2}{2} \nabla_{xy}^2 W(x, y) & -z_1 z_2 \nabla_{xy}^2 W(x, y) \end{pmatrix} \\
 & \quad U(x, v) f(x, v) f(y, \tilde{v}) dx dv dy d\tilde{v} \\
 &+ \int_{\Omega} U^T(x, v) \begin{pmatrix} z_1 z_2 & \frac{1}{2} [z_1 z_2 + z_2^2 - \nabla_{xx}^2 U z_1^2 + 1] \\ \frac{1}{2} [z_1 z_2 + z_2^2 - \nabla_{xx}^2 U z_1^2 + 1] & 1 + z_2^2 - \nabla_{xx}^2 U z_1 z_2 \end{pmatrix} \\
 & \quad U(x, v) f(x, v) dx dv \\
 &+ \int_{\Omega \times \Omega} U^T(x, v) \begin{pmatrix} 0 & -\frac{z_1^2}{2} \nabla_{xx}^2 W(x, y) \\ -\frac{z_1^2}{2} \nabla_{xx}^2 W(x, y) & -z_1 z_2 \nabla_{xx}^2 W(x, y) \end{pmatrix} \\
 & \quad U(x, v) f(y, \tilde{v}) f(x, v) dy d\tilde{v} dx dv \\
 &= \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3.
 \end{aligned}$$

Using the fact that $W(x, y) = W(y, x)$, and $\nabla_{xx}^2 W(x, y) = \nabla_{yy}^2 W(y, x)$, we have

$$\begin{aligned}
 & \mathcal{T}_1 + \mathcal{T}_3 \\
 &= \frac{1}{2} \int_{\Omega \times \Omega} U^T(x, v) \begin{pmatrix} 0 & -\frac{z_1^2}{2} \nabla_{xx}^2 W(x, y) \\ -\frac{z_1^2}{2} \nabla_{xx}^2 W(x, y) & -z_1 z_2 \nabla_{xx}^2 W(x, y) \end{pmatrix} \\
 & \quad U(x, v) f(y, \tilde{v}) f(x, v) dy d\tilde{v} dx dv \\
 &+ \frac{1}{2} \int_{\Omega \times \Omega} U^T(y, \tilde{v}) \begin{pmatrix} 0 & -\frac{z_1^2}{2} \nabla_{yy}^2 W(y, x) \\ -\frac{z_1^2}{2} \nabla_{yy}^2 W(y, x) & -z_1 z_2 \nabla_{yy}^2 W(y, x) \end{pmatrix} \\
 & \quad U(y, \tilde{v}) f(y, \tilde{v}) f(x, v) dy d\tilde{v} dx dv \\
 &+ \int_{\Omega \times \Omega} U^T(y, \tilde{v}) \begin{pmatrix} 0 & -\frac{z_1^2}{2} \nabla_{xy}^2 W(x, y) \\ -\frac{z_1^2}{2} \nabla_{xy}^2 W(x, y) & -z_1 z_2 \nabla_{xy}^2 W(x, y) \end{pmatrix} \\
 & \quad U(x, v) f(x, v) f(y, \tilde{v}) dx dv dy d\tilde{v} \\
 &= \int_{\Omega \times \Omega} \left[\frac{1}{2} U^T(x, v) A_1(x, y) U(x, v) + \frac{1}{2} U^T(y, \tilde{v}) A_1(y, x) U(y, \tilde{v}) \right. \\
 & \quad \left. + U^T(y, \tilde{v}) B(x, y) U(x, v) \right] f(x, v) f(y, \tilde{v}) dx dv dy d\tilde{v}
 \end{aligned}$$

$$= \frac{1}{2} \int_{\Omega \times \Omega} (\mathbf{U}^\top(x, v) \mathbf{U}^\top(y, \tilde{v})) \begin{pmatrix} \mathbf{A}_1(x, y) & \mathbf{B}(x, y) \\ \mathbf{B}(x, y) & \mathbf{A}_1(y, x) \end{pmatrix} \begin{pmatrix} \mathbf{U}(x, v) \\ \mathbf{U}(y, \tilde{v}) \end{pmatrix} f(x, v) f(y, \tilde{v}) dx dv dy d\tilde{v},$$

where we denote

$$\begin{aligned} \mathbf{A}_1(x, y) &= \begin{pmatrix} 0 & -\frac{z_1^2}{2} \nabla_{xx}^2 W(x, y) \\ -\frac{z_1^2}{2} \nabla_{xx}^2 W(x, y) & -z_1 z_2 \nabla_{xx}^2 W(x, y) \end{pmatrix}, \\ \mathbf{A}_1(y, x) &= \begin{pmatrix} 0 & -\frac{z_1^2}{2} \nabla_{yy}^2 W(y, x) \\ -\frac{z_1^2}{2} \nabla_{yy}^2 W(y, x) & -z_1 z_2 \nabla_{yy}^2 W(y, x) \end{pmatrix}, \\ \mathbf{B}(x, y) &= \begin{pmatrix} 0 & -\frac{z_1^2}{2} \nabla_{xy}^2 W(x, y) \\ -\frac{z_1^2}{2} \nabla_{xy}^2 W(x, y) & -z_1 z_2 \nabla_{xy}^2 W(x, y) \end{pmatrix}. \end{aligned}$$

Combining $\mathcal{T}_2$ with the above term, we have

$$\begin{aligned} &\mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 \\ &= \frac{1}{2} \int_{\Omega \times \Omega} (\mathbf{U}^\top(x, v) \mathbf{U}^\top(y, \tilde{v})) \begin{pmatrix} \mathbf{A}_1(x, y) + 2\mathbf{A}_2(x, y) & \mathbf{B}(x, y) \\ \mathbf{B}(x, y) & \mathbf{A}_1(y, x) \end{pmatrix} \begin{pmatrix} \mathbf{U}(x, v) \\ \mathbf{U}(y, \tilde{v}) \end{pmatrix} f(x, v) f(y, \tilde{v}) dx dv dy d\tilde{v} \\ &= \frac{1}{2} \int_{\Omega \times \Omega} (\mathbf{U}^\top(x, v) \mathbf{U}^\top(y, \tilde{v})) \begin{pmatrix} \mathbf{A}_1(x, y) + \mathbf{A}_2(x, y) & \mathbf{B}(x, y) \\ \mathbf{B}(x, y) & \mathbf{A}_1(y, x) + \mathbf{A}_2(y, x) \end{pmatrix} \begin{pmatrix} \mathbf{U}(x, v) \\ \mathbf{U}(y, \tilde{v}) \end{pmatrix} f(x, v) f(y, \tilde{v}) dx dv dy d\tilde{v}, \end{aligned}$$

where we denote

$$\begin{aligned} \mathbf{A}_2(x, y) &= \begin{pmatrix} \frac{1}{2}[z_1 z_2 + z_2^2 - \nabla_{xx}^2 U(x) z_1^2 + 1] & \frac{1}{2}[z_1 z_2 + z_2^2 - \nabla_{xx}^2 U(x) z_1^2 + 1] \\ \frac{1}{2}[z_1 z_2 + z_2^2 - \nabla_{xx}^2 U(x) z_1^2 + 1] & 1 + z_2^2 - \nabla_{xx}^2 U(x) z_1 z_2 \end{pmatrix} \\ \mathbf{A}_2(y, x) &= \begin{pmatrix} \frac{1}{2}[z_1 z_2 + z_2^2 - \nabla_{yy}^2 U(y) z_1^2 + 1] & \frac{1}{2}[z_1 z_2 + z_2^2 - \nabla_{yy}^2 U(y) z_1^2 + 1] \\ \frac{1}{2}[z_1 z_2 + z_2^2 - \nabla_{yy}^2 U(y) z_1^2 + 1] & 1 + z_2^2 - \nabla_{yy}^2 U(y) z_1 z_2 \end{pmatrix}. \end{aligned} \quad (5.8)$$

This produced the matrix tensor for Definition 3.1. Combining the above matrix terms, we have

$$\mathfrak{R} = \frac{1}{2} \begin{pmatrix} \mathbf{A}(x, y) & \mathbf{B}(x, y) \\ \mathbf{B}(x, y) & \mathbf{A}(y, x) \end{pmatrix},$$

where we denote

$$\begin{aligned} A(x, y) &= A_1(x, y) + A_2(x, y) \\ &= \begin{pmatrix} 0 & -\frac{z_1^2}{2} \nabla_{xx}^2 W(x, y) \\ -\frac{z_1^2}{2} \nabla_{xx}^2 W(x, y) & -z_1 z_2 \nabla_{xx}^2 W(x, y) \end{pmatrix} \\ &\quad + \begin{pmatrix} z_1 z_2 & \frac{1}{2} [(1 + z_1 z_2 + z_2^2) - z_1^2 \nabla_{xx}^2 U(x)] \\ \frac{1}{2} [(1 + z_1 z_2 + z_2^2) - z_1^2 \nabla_{xx}^2 U(x)] & (1 + z_2^2) - z_1 z_2 \nabla_{xx}^2 U(x) \end{pmatrix} \\ &= \begin{pmatrix} z_1 z_2 & \frac{1}{2} [(1 + z_1 z_2 + z_2^2) - z_1^2 \nabla_{xx}^2 V(x, y)] \\ \frac{1}{2} [(1 + z_1 z_2 + z_2^2) - z_1^2 \nabla_{xx}^2 V(x, y)] & (1 + z_2^2) - z_1 z_2 \nabla_{xx}^2 V(x, y) \end{pmatrix}, \end{aligned}$$

and

$$A(y, x) = \begin{pmatrix} z_1 z_2 & \frac{1}{2} [(1 + z_1 z_2 + z_2^2) - z_1^2 \nabla_{yy}^2 V(y, x)] \\ \frac{1}{2} [(1 + z_1 z_2 + z_2^2) - z_1^2 \nabla_{yy}^2 V(y, x)] & (1 + z_2^2) - z_1 z_2 \nabla_{yy}^2 V(y, x) \end{pmatrix},$$

with $V(x, y) = W(x, y) + U(x)$, and

$$B(x, y) = \begin{pmatrix} 0 & -\frac{z_1^2}{2} \nabla_{xy}^2 W(x, y) \\ -\frac{z_1^2}{2} \nabla_{xy}^2 W(x, y) & -z_1 z_2 \nabla_{xy}^2 W(x, y) \end{pmatrix}.$$

This produces the matrix tensor for Definition 2.3.

Case 2: $d \geq 2$. We first demonstrate the derivation for $d = 2$. By direction computations, we have

$$a = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^T, \quad z = \begin{pmatrix} z_1 & 0 & z_2 & 0 \\ 0 & z_1 & 0 & z_2 \end{pmatrix}^T,$$

and

$$\begin{aligned} &\text{sym}((aa^T + zz^T)^T \nabla_{xy}^2 W(x, y) (aa^T + J)) \\ &= \text{sym}((aa^T + zz^T) \nabla_{xy}^2 W(x, y) (aa^T + J)) \\ &= \text{sym} \left(\begin{pmatrix} z_1^2 l_2 & z_1 z_2 l_2 \\ z_1 z_2 l_2 & (1 + z_2^2) l_2 \end{pmatrix}_{4 \times 4} \begin{pmatrix} \nabla_{xy}^2 W(x, y) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -l_2 \\ l_2 & l_2 \end{pmatrix}_{2d \times 2d} \right) \\ &= \begin{pmatrix} 0 & -\frac{z_1^2}{2} \nabla_{xy}^2 W(x, y) \\ -\frac{z_1^2}{2} \nabla_{xy}^2 W(x, y) & -z_1 z_2 \nabla_{xy}^2 W(x, y) \end{pmatrix}. \end{aligned}$$

Similar to $d = 1$, by routine computation, we have

$$\mathfrak{R}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \frac{1}{2}z_1z_2 & 0 \\ 0 & 0 & 0 & \frac{1}{2}z_1z_2 \\ \frac{1}{2}z_1z_2 & 0 & z_2^2 & 0 \\ 0 & \frac{1}{2}z_1z_2 & 0 & z_2^2 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \\ - \begin{pmatrix} -z_1z_2 & 0 & \frac{1}{2}(z_1^2\nabla_{x_1x_1}\tilde{V} - z_2^2) & \frac{1}{2}z_1^2\nabla_{x_1x_2}\tilde{V} \\ 0 & -z_1z_2 & \frac{1}{2}z_1^2\nabla_{x_1x_2}\tilde{V} & \frac{1}{2}(z_1^2\nabla_{x_2x_2}\tilde{V} - z_2^2) \\ \frac{1}{2}(z_1^2\nabla_{x_1x_1}\tilde{V} - z_2^2) & \frac{1}{2}z_1^2\nabla_{x_1x_2}\tilde{V} & z_1z_2\nabla_{x_1x_1}\tilde{V} & z_1z_2\nabla_{x_1x_2}\tilde{V} \\ \frac{1}{2}z_1^2\nabla_{x_1x_2}\tilde{V} & \frac{1}{2}(z_1^2\nabla_{x_1x_1}\tilde{V} - z_2^2) & z_1z_2\nabla_{x_1x_2}\tilde{V} & z_1z_2\nabla_{x_2x_2}\tilde{V} \end{pmatrix},$$

where $\tilde{V}(x, f) = \int_{\Omega} W(x, y) f(t, y, v) dv dy + U(x)$. We then combine the two matrices following the proof for $d = 1$. Similarly, for any $d \geq 2$, we have

$$\mathfrak{R}(z, x, y) = \frac{1}{2} \begin{pmatrix} A(x, y) & B(x, y) \\ B(x, y) & A(y, x) \end{pmatrix} \in \mathbb{R}^{4d \times 4d},$$

where

$$A(x, y) = \begin{pmatrix} 0 & -\frac{z_1^2}{2}\nabla_{xx}^2 W(x, y) \\ -\frac{z_1^2}{2}\nabla_{xx}^2 W(x, y) & -z_1z_2\nabla_{xx}^2 W(x, y) \end{pmatrix} \\ + \begin{pmatrix} z_1z_2l_d & \frac{1}{2}[(1 + z_1z_2 + z_2^2)l_d - z_1^2\nabla_{xx}^2 U(x)] \\ \frac{1}{2}[(1 + z_1z_2 + z_2^2)l_d - z_1^2\nabla_{xx}^2 U(x)] & (1 + z_2^2)l_d - z_1z_2\nabla_{xx}^2 U(x) \end{pmatrix} \\ = \begin{pmatrix} z_1z_2l_d & \frac{1}{2}[(1 + z_1z_2 + z_2^2)l_d - z_1^2\nabla_{xx}^2 V(x, y)] \\ \frac{1}{2}[(1 + z_1z_2 + z_2^2)l_d - z_1^2\nabla_{xx}^2 V(x, y)] & (1 + z_2^2)l_d - z_1z_2\nabla_{xx}^2 V(x, y) \end{pmatrix},$$

for $V(x, y) = W(x, y) + U(x)$, and

$$B(x, y) = \begin{pmatrix} 0 & -\frac{z_1^2}{2}\nabla_{xy}^2 W(x, y) \\ -\frac{z_1^2}{2}\nabla_{xy}^2 W(x, y) & -z_1z_2\nabla_{xy}^2 W(x, y) \end{pmatrix}.$$

Separating the matrix $A(x, y)$ into $A_1(x, y)$ and $A_2(x, y)$, we derive the matrix defined in Definition 3.1. $\square$

References

- Arnold, A., Carlen, E.: A generalized Bakry-Emery condition for non-symmetric diffusions. In: Equadiff 99: (In 2 Volumes), pp. 732–734. World Scientific (2000)
- Arnold, A., Erb, J.: Sharp entropy decay for hypocoercive and non-symmetric Fokker–Planck equations with linear drift. arXiv preprint [arXiv:1409.5425](https://arxiv.org/abs/1409.5425) (2014)
- Arnold, A., Markowich, P., Toscani, G.: On large time asymptotics for drift-diffusion-Poisson systems. Transp. Theory Stat. Phys. **29**(3–5), 571–581 (2000)

- Arnold, A., Markowich, P., Toscani, G., Unterreiter, A.: On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker–Planck type equations. *Commun. Partial Differ. Equ.* **26**, 43 (2001)
- Bakry, D., Émery, M.: Diffusions hypercontractives. In: *Seminaire de Probabilités XIX 1983/84*, pp. 177–206. Springer (1985)
- Baudoin, F., Garofalo, N.: Curvature-dimension inequalities and Ricci lower bounds for sub-Riemannian manifolds with transverse symmetries. *J. Eur. Math. Soc.* **19**(1), 151–219 (2016)
- Benedetto, D., Caglioti, E., Carrillo, J.A., Pulvirenti, M.: A non-Maxwellian steady distribution for one-dimensional granular media. *J. Stat. Phys.* **91**, 979–990 (1998)
- Bolley, F., Guillain, A., Malrieu, F.: Trend to equilibrium and particle approximation for a weakly self-consistent Vlasov–Fokker–Planck equation. *ESAIM: Math. Model. Numer. Anal.* **44**(5), 867–884 (2010)
- Bouchut, F.: Existence and uniqueness of a global smooth solution for the Vlasov–Poisson–Fokker–Planck system in three dimensions. *J. Funct. Anal.* **111**(1), 239–258 (1993)
- Bouchut, F., Dolbeault, J.: On long time asymptotics of the Vlasov–Fokker–Planck equation and of the Vlasov–Poisson–Fokker–Planck system with Coulombic and Newtonian potentials. *Differ. Integral Equ.* **8**(3), 487–514 (1995)
- Calogero, S.: Exponential convergence to equilibrium for kinetic Fokker–Planck equations. *Commun. Partial Differ. Equ.* **37**(8), 1357–1390 (2012)
- Carlen, E., Carvalho, M., Esposito, R., Lebowitz, J., Marra, R.: Free energy minimizers for a two-species model with segregation and liquid–vapour transition. *Nonlinearity* **16**(3), 1075 (2003)
- Carrillo, J., Hoffmann, F., Stuart, A., Vaes, U.: Consensus based sampling. *arXiv preprint [arXiv:2106.02519](https://arxiv.org/abs/2106.02519)* (2021)
- Carrillo, J.A., McCann, R.J., Villani, C.: Kinetic equilibration rates for granular media and related equations: entropy dissipation and mass transportation estimates. *Rev. Mat. Iberoam.* **19**(3), 971–1018 (2003)
- Carrillo, J., Soler, J.: On the initial value problem for the Vlasov–Poisson–Fokker–Planck system with initial data in L^p spaces. *Math. Methods Appl. Sci.* **18**(10), 825–839 (1995)
- Cesbron, L., Herda, M.: On a Vlasov–Fokker–Planck equation for stored electron beams. *arXiv preprint [arXiv:2307.15964](https://arxiv.org/abs/2307.15964)* (2023)
- Chow, S.-N., Li, W., Zhou, H.: Entropy dissipation of Fokker–Planck equations on graphs. *Discrete Contin. Dyn. Syst.* **38**(10), 4929–4950 (2018)
- Desvillettes, L., Villani, C.: On the spatially homogeneous landau equation for hard potentials part ii: h-theorem and applications: H-theorem and applications. *Commun. Partial Differ. Equ.* **25**, 261–298 (2000)
- Degond, P.: Global existence of smooth solutions for the Vlasov–Fokker–Planck equation in 1 and 2 space dimensions. *Ann. Sci. École Norm. Sup.* **19**(4), 519 (1986)
- Duong, M.H., Ottobre, M.: Non-reversible processes: GENERIC, Hypocoercivity and fluctuations (2021)
- Esposito, R., Guo, Y., Marra, R.: Stability of the front under a Vlasov–Fokker–Planck dynamics. *Arch. Ration. Mech. Anal.* **195**(1), 75–116 (2010)
- Feng, Q., Li, W.: Entropy dissipation for degenerate stochastic differential equations via sub-Riemannian density manifold. *Entropy* **25**, 786 (2023)
- Feng, Q., Li, W.: Hypocoercive entropy dissipation for stochastic differential equations. *arXiv preprint [arXiv:2102.00544](https://arxiv.org/abs/2102.00544)* (2021)
- Garbuno-Inigo, A., Hoffmann, F., Li, W., Stuart, A.M.: Interacting Langevin diffusions: gradient structure and ensemble Kalman sampler. *SIAM J. Appl. Dyn. Syst.* **19**(1), 412–441 (2020)
- Guillin, A., Bris, P.L., Monmarche, P.: Convergence rates for the Vlasov–Fokker–Planck equation and uniform in time propagation of chaos in non convex cases. *arXiv preprint [arXiv:2105.09070](https://arxiv.org/abs/2105.09070)* (2021)
- Guillin, A., Liu, W., Wu, L., Zhang, C.: Uniform Poincaré and logarithmic Sobolev inequalities for mean field particles systems. *arXiv preprint [arXiv:1909.07051](https://arxiv.org/abs/1909.07051)* (2019)
- Guillin, A., Liu, W., Wu, L., Zhang, C.: The kinetic Fokker–Planck equation with mean field interaction. *J. Math. Pures Appl.* **150**, 1–23 (2021)
- Hu, K., Ren, Z., Siska, D., Szpruch, L.: Mean-field Langevin dynamics and energy landscape of neural networks. *arXiv preprint [arXiv:1905.07769](https://arxiv.org/abs/1905.07769)* (2019)
- Kazeykina, A., Ren, Z., Tan, X., Yang, J.: Ergodicity of the underdamped mean-field Langevin dynamics. *arXiv preprint [arXiv:2007.14660](https://arxiv.org/abs/2007.14660)* (2020)
- Li, W.: Transport information geometry: Riemannian calculus on probability simplex. *Inf. Geom.* **5**, 161 (2021)

- Li, W., Liu, S., Osher, S.: Controlling conservation laws i: entropy–entropy flux. arXiv preprint [arXiv:2111.05473](https://arxiv.org/abs/2111.05473) (2021)
- Ma, Y.-A., Chatterji, N., Cheng, X., Flammarion, N., Bartlett, P., Jordan, M.I.: Is there an analog of Nesterov acceleration for MCMC? arXiv preprint [arXiv:1902.00996](https://arxiv.org/abs/1902.00996) (2019)
- Markowich, P.A., Villani, C.: On the trend to equilibrium for the Fokker–Planck equation: an interplay between physics and functional analysis. *Mat. Contemp.* **19**, 1–29 (2000)
- Vandenbergh, L., Boyd, S.: *Convex Optimization*, vol. 1. Cambridge University Press, Cambridge (2004)
- Villani, C.: Hypocoercivity, vol. 202, No. 950. *Memoirs of the American Mathematical Society* (2009)
- Wang, F.-Y.: Exponential ergodicity for fully non-dissipative McKean–Vlasov SDEs. arXiv preprint [arXiv:2101.12562](https://arxiv.org/abs/2101.12562) (2021)

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