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Ozone Groups and Centers of Skew Polynomial Rings

Kenneth Chan[1], Jason Gaddis[2,*], Robert Won[3], and James J. Zhang[1]
[1]Department of Mathematics, University of Washington, Seattle, Box 354350, WA 98195, USA, [2]Department of Mathematics, Miami University, Oxford, OH 45056, USA, and [3]Department of Mathematics, George Washington University, Washington, DC 20052, USA

**Correspondence to be sent to: e-mail: gaddisj@miamioh.edu*

We introduce the ozone group of a noncommutative algebra A, defined as the group of automorphisms of A, which fix every element of its center. In order to initiate the study of ozone groups, we study polynomial identity (PI) skew polynomial rings, which have long proved to be a fertile testing ground in noncommutative algebra. Using the ozone group and other invariants defined herein, we give explicit conditions for the center of a PI skew polynomial ring to be Gorenstein (resp. regular) in low dimension.

Introduction

The centers of polynomial identity (PI) Artin–Schelter regular algebras of global dimension three, including the skew polynomial rings of three variables, have been studied by Artin [1] and Mori [27] from the point of view of noncommutative projective geometry. A motivation of this paper is to study the center of a PI skew polynomial ring of any number of variables from an algebraic point of view.

One of our aims is to search for new invariants that control the structure of a PI skew polynomial ring and its center, as well as the interplay between them. For example, are there combinatorial invariants that are closely related to the Calabi–Yau property of a PI skew polynomial ring or the Gorenstein property of its center? In our results below,

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we provide several invariants that provide an affirmative answer to this question (e.g., $\mathfrak{pg}_S$ in Theorem 0.10 or the element $\mathfrak{oj}_S\,\mathfrak{pg}_S$ in Theorem 0.11).

We start with a more general class of algebras, as we hope that some of the ideas in this paper will work more generally. Throughout, let $\Bbbk$ denote a base field with $\operatorname{char}\Bbbk = 0$. Let A be a noetherian PI Artin–Schelter regular algebra with center Z. The first combinatorial/algebraic invariant we introduce is the following, which will be studied in more detail in [9].

Definition 0.1 ([9]). The *ozone group* of A is

$$\operatorname{Oz}(A) := \operatorname{Aut}_{Z\text{-alg}}(A).$$

The observant reader will notice a similarity between the ozone group and the Galois group of a field extension $E \subset F$. The classical Galois group $\operatorname{Gal}(F/E)$ is not an ozone group, *per se*, as the ozone group of any commutative ring is trivial. However, both Galois groups and ozone groups involve automorphisms of an overstructure, which fix a central substructure. This relationship will be considered further in [9].

By [9], the order of the ozone group satisfies the following condition:

$$1 \le |\operatorname{Oz}(A)| \le \operatorname{rk}(A_Z).$$

It is quite surprising that the skew polynomial rings play a role at one of the extreme cases of the above inequalities and that they can, in fact, be characterized in terms of properties of the ozone group [Lemma 0.2].

Let $\mathbf{p} := (p_{ij}) \in M_n(\Bbbk)$ be a multiplicatively antisymmetric matrix. The *skew polynomial ring*

$$S_{\mathbf{p}} := \Bbbk_{\mathbf{p}}[x_1, \ldots, x_n]$$

is the $\Bbbk$-algebra generated by $\{x_1, \ldots, x_n\}$ and subject to relations

$$x_j x_i = p_{ij} x_i x_j \text{ for all } 1 \le i, j \le n.$$

By [13, Theorem 2], the parameter $\mathbf{p}$, up to a permutation of $\{1, \ldots, n\}$, is an algebra invariant of $S_{\mathbf{p}}$. It is easy to see that $S_{\mathbf{p}}$ is PI (i.e., $S_{\mathbf{p}}$ satisfies a polynomial identity) if and only if each p_{ij} is a root of unity and this is the setting we consider in this paper.

Skew polynomial rings have been studied extensively and their ring theoretic properties are well-understood. In particular, $S_{\mathbf{p}}$ is an Artin–Schelter regular algebra of global and Gelfand–Kirillov dimension n. Skew polynomial rings provide a good set of examples to test theories related to Artin–Schelter regular algebras in general.

However, there are still unsolved questions concerning skew polynomial rings. For example, when $p \neq 1$ is a root of unity, it is unknown if $\Bbbk_p[x_1, x_2, x_3]$ is cancellative [28, Question 0.8] and it is not clear how to describe its full automorphism group. In the current paper, we pay more attention to some homological questions.

Lemma 0.2 ([9]). Suppose $\Bbbk$ is algebraically closed. Let A be a noetherian PI Artin–Schelter regular algebra generated in degree 1. Then A is isomorphic to a skew polynomial ring if and only if $\mathrm{Oz}(A)$ is abelian and $|\,\mathrm{Oz}(A)| = \mathrm{rk}(A_Z)$.

We note that the hypothesis on $\Bbbk$ is not necessary for the forward direction in Lemma 0.2.

For most of the results in this paper, we will assume the following hypothesis.

Hypothesis 0.3. Let $S = S_{\mathbf{p}} = \Bbbk_{\mathbf{p}}[x_1, \ldots, x_n]$ be a PI skew polynomial ring with $\deg(x_i) = 1$ for all i, and let $Z = ZS_{\mathbf{p}}$ denote the center of S. Let ξ be a primitive ℓth root of unity such that $p_{ij} = \xi^{b_{ij}}$ for some integer b_{ij}. By convention, we choose the b_{ij} such that $b_{ii} = 0$ and $b_{ji} = -b_{ij}$ for all i and j. We assume that ℓ is minimal and that $\ell > 1$ (which is equivalent to the noncommutativity of S). Hence, $\gcd(b_{ij}, \ell)_{1 \le i,j \le n} = 1$. Implicitly, we fix a set of generators $\{x_1, \ldots, x_n\}$.

We let $\overline{\mathbb{Z}}$ denote $\mathbb{Z}/\ell\mathbb{Z}$ and let $\overline{a}$ (where $a \in \mathbb{Z}, \mathbb{Z}^n$, or $M_n(\mathbb{Z})$) denote the image of a modulo ℓ. Let $B = (b_{ij})$, so B and $\overline{B}$ are $n \times n$ skew-symmetric matrices.

Let ϕ_i denote the automorphism of S given by conjugation by x_i, that is,

$$\phi_i(f) = x_i^{-1} f x_i \text{ for all } f \in S,$$

and let O be the subgroup of $\mathrm{Aut}_{\mathrm{gr}}(S)$ generated by $\{\phi_1, \ldots, \phi_n\}$.

It turns out that O is the ozone group $\mathrm{Oz}(S)$ [9]. For each fixed i, we have $\phi_i(x_s) = \xi^{b_{is}} x_s = p_{is} x_s$ for all s and the order of ϕ_i is

$$o(\phi_i) = \ell / \gcd\{b_{1i}, \ldots, b_{ni}, \ell\}.$$

It is easy to see that the center of $S_{\mathbf{p}}$ is

$$ZS_{\mathbf{p}} = S_{\mathbf{p}}^{O}.$$

Hence, we are able to use tools of noncommutative invariant theory to reveal properties of the center $ZS_{\mathbf{p}}$.

In general, it is difficult to discern properties of $S_{\mathbf{p}}$ simply by examining its parameters $\mathbf{p} := (p_{ij})$. However, in low-dimension ($n \leq 4$), many such properties can be worked out explicitly. In particular, we are interested in studying the center $ZS_{\mathbf{p}}$ in terms of $\mathbf{p}$.

Note that most of the questions we consider are easily addressed in the case $n = 2$. For a large part of this paper, we focus on the cases $n = 3$ and $n = 4$. We notice that there are some similarities and differences between the cases of n being odd or even. In some cases, we are able to say something about the case of higher n, and we hope that some of the ideas and results presented here can be extended to general n.

Several of our results make use of the *Pfaffian* of the matrix B associated to $S_{\mathbf{p}}$, which we denote by $\mathrm{pf}(B)$. The Pfaffian was introduced by Cayley [6], based on earlier work of Jacobi [15]. In particular, for even n, Cayley showed that the Pfaffian of a skew-symmetric matrix is $\sqrt{\det(B)}$. When n is odd, every $n \times n$ matrix has Pfaffian 0. Now suppose n is even and let $A = (a_{ij})$ be a skew symmetric $n \times n$ matrix. Denote by $A_{\hat{i}\hat{j}}$ the submatrix of A obtained by deleting both the ith row and column and the jth row and column. The Pfaffian of A may be computed as

$$\mathrm{pf}(A) = \sum_{k=2}^{n} (-1)^k a_{1k}\, \mathrm{pf}(A_{\hat{1}\hat{k}}).$$

Next we define a few more combinatorial invariants. For the remainder of this introduction, we assume Hypothesis 0.3. In particular, we assume $S = S_{\mathbf{p}}$ is a PI skew polynomial ring with center $Z = ZS_{\mathbf{p}}$. Define

$$\mathfrak{f}_i = \gcd\{d_i \mid \exists\, d_1, \dots, \widehat{d_i}, \dots, d_n \text{ with } x_1^{d_1} \cdots x_i^{d_i} \cdots x_n^{d_n} \in Z\}. \tag{E0.3.1}$$

By Proposition 1.8(2), the set of integers $\{\mathfrak{f}_1, \dots, \mathfrak{f}_n\}$ with multiplicities is an algebra invariant of S.

In this paper, we would like to show that $\{\mathfrak{f}_1, \dots, \mathfrak{f}_n\}$ also play an essential role in several properties of the algebra $S_{\mathbf{p}}$. They have been used to control the automorphism group of S. For example, if $\mathfrak{f}_i \geq 2$ for all i, then every automorphism of S is affine and every locally nilpotent derivation of S is zero [7, Theorem 3]. So, these numbers deserve further attention. It would be nice to have a simple formula for $\mathfrak{f}_i$ in terms of $\{p_{ij}\}$.

Now we define the "ozone" version of three invariants studied in [21]. The *ozone Jacobian* of S is defined to be

$$\mathfrak{oj}_S := \prod_{i=1}^{n} x_i^{\mathfrak{f}_i - 1}, \tag{E0.3.2}$$

the *ozone arrangement* of S is defined to be

$$\mathfrak{oa}_S := \prod_{\mathfrak{f}_i > 1} x_i, \tag{E0.3.3}$$

and the *ozone discriminant* of S is defined to be

$$\mathfrak{od}_S := \prod_{\mathfrak{f}_i > 1} x_i^{\mathfrak{f}_i} = \mathfrak{oj}_S \, \mathfrak{oa}_S. \tag{E0.3.4}$$

All three of these definitions depend on the chosen generating set $\{x_1, \ldots, x_n\}$. However, by Proposition 1.8(1), up to an element of $\Bbbk^\times$, $\mathfrak{oj}_S$, $\mathfrak{oa}_S$, and $\mathfrak{od}_S$ are algebra invariants of $S_{\mathbf{p}}$.

We also consider the *product of generators* of S, which is defined to be

$$\mathfrak{pg}_S := \prod_{i=1}^{n} x_i.$$

Note that $\mathfrak{pg}_S$ is not an algebra invariant.

We now summarize our main results and the structure of this paper. In Section 1, we recall some basic definitions. In Section 2, we work out some basic facts concerning the reflections in O. In Sections 3, 4, and 5, the main results are proven. More details on those sections are given below. We think the results are inspiring, though the proofs are not difficult. At the end, in Section 6, we list some questions and give some examples.

0.1 Auslander's Theorem and smallness

In [2], Auslander proved that for V a finite-dimensional $\Bbbk$-space, $A = \Bbbk[V]$, and G a finite subgroup of $\mathrm{GL}(V)$, the map

$$A\#G \to \mathrm{End}_{A^G}(A) \tag{E0.3.5}$$

$$a\#g \mapsto \begin{pmatrix} A & \to & A \\ b & \mapsto & ag(b) \end{pmatrix}$$

is an isomorphism if and only if G is small (contains no pseudo-reflections). This map may be defined for any algebra A and any finite subgroup G of $\mathrm{Aut}(A)$, though in general it may not be injective or surjective. We say Auslander's Theorem holds for the pair (A, G) (or (A, G) satisfies Auslander's Theorem) if (E0.3.5) is an isomorphism.

Auslander's Theorem is a critical component in the study of the McKay correspondence, and has been studied in the noncommutative setting [10, 11, 14, 31]. Bao, He, and the fourth author introduced the notion of pertinency in [3, 4]. As above, let A be an algebra and G a finite subgroup of $\mathrm{Aut}(A)$. Then the *pertinency* of the G action on A is defined to be

$$\mathsf{p}(A,G) = \mathrm{GKdim}(A) - \mathrm{GKdim}\left(\frac{A\#G}{(f_G)}\right), \tag{E0.3.6}$$

where $f_G = \sum_{g\in G} 1\#g \in A\#G$.

In Section 3, we study Auslander's Theorem for the pair (S, O). There is a notion of a reflection in the noncommutative setting related to the trace series of a graded algebra [Definition 2.1(1)]. However, for the skew polynomial rings S, a diagonal automorphism g is a reflection if and only if it is a classical pseudo-reflection when restricted to $\bigoplus_{i=1}^n \Bbbk x_i$ [Example 2.2].

Theorem 0.4. Assume Hypothesis 0.3. The following are equivalent.

(1) Auslander's Theorem holds for (S, O).
(2) The pertinency satisfies $\mathsf{p}(S, O) \geq 2$.
(3) The O-action is small in the sense of Definition 2.1(3).
(4) The O-action is small in the classical sense, that is, it contains no pseudo-reflection when restricted to $\bigoplus_{i=1}^n \Bbbk x_i$.
(5) The only Artin–Schelter regular algebra T satisfying $Z \subseteq T \subseteq S$ is $T = S$.
(6) The ozone Jacobian $\mathfrak{oj}_S = 1$, namely, $\mathfrak{f}_i = 1$ for all i.
(7) For each i, there is an element of the form $x_1^{a_1} \cdots x_i \cdots x_n^{a_n}$ in Z.

Part (6) of the above theorem shows that $\mathfrak{oj}_S$ (or the set $\{\mathfrak{f}_1, \ldots, \mathfrak{f}_n\}$) serves as an invariant that can be used to determine whether Auslander's Theorem holds for (S, O). For small n, we have the following result in terms of the matrix $B = (b_{ij})$.

Theorem 0.5. Assume Hypothesis 0.3.

(1) Let $n = 2$. Then (S, O) does not satisfy Auslander's Theorem.
(2) Let $n = 3$. Then (S, O) satisfies Auslander's Theorem if and only if $\gcd(b_{ij}, \ell) = 1$ for each $i \neq j$.
(3) Let $n = 4$. Then (S, O) satisfies Auslander's Theorem if and only if
   - $\mathrm{pf}(B) \equiv 0 \pmod{\ell}$, and
   - there does not exist an index j and an integer k such that $kb_{ij} \equiv 0 \pmod{\ell}$ for all but one i.

0.2 Regular center

In Section 4, we consider the question of when $Z := ZS_{\mathbf{p}}$ is regular, that is, under what conditions is Z a polynomial ring. Again, in the case $n = 2$, this is clear (see Theorem 0.7(1)). For the next theorem, we refer to Definition 1.6 for the notation $\mathfrak{j}_{S,O}$ and $\mathfrak{a}_{S,O}$.

Theorem 0.6. Assume Hypothesis 0.3. The following are equivalent.

(1) Z is regular.
(2) $Z = \Bbbk[x_1^{w_1}, \cdots, x_n^{w_n}]$ for some $w_i \geq 1$.
(3) $Z = \Bbbk[x_1^{\mathfrak{f}_1}, \cdots, x_n^{\mathfrak{f}_n}]$.
(4) S is a free module over Z.
(5) O is a reflection group.
(6) $|O| = \frac{\ell^n}{\prod_{s=1}^n \gcd\{b_{1s}, b_{2s}, \cdots, b_{ns}, \ell\}}$.
(7) $|O| = \prod_{i=1}^n \mathfrak{f}_i$.
(8) $O \cong \prod_{s=1}^n \langle \phi_s \rangle$.
(9) $x_i^{\mathfrak{f}_i} \in Z$ for all i.

When Z is regular, then $\mathfrak{o}\mathfrak{j}_S$ is equal to $\mathfrak{j}_{S,O}$ and $\mathfrak{o}\mathfrak{a}_S$ is equal to $\mathfrak{a}_{S,O}$.

For small n, we have the following theorem in terms of the parameters $\mathbf{p}$ or the matrix B. This result makes use of the Smith normal form of the matrix B.

Theorem 0.7. Assume Hypothesis 0.3.

(1) Let $n = 2$. Then $Z = \Bbbk[x_1^{\ell}, x_2^{\ell}]$ so Z is regular.
(2) Let $n = 3$. Then Z is regular if and only if the orders of $\{p_{ij}\}_{i<j}$ are pairwise coprime.
(3) Let $n = 4$. Then Z is regular if and only if the orders of $\{q_{ij}\}_{i<j}$ are pairwise coprime, where the q_{ij} are defined in Corollary 4.8.

Remark 0.8. In the case where $\ell \mid \mathrm{pf}(B)$, we have $q_{ij} = p_{ij}$, so the conditions for Z to be regular for $n = 3$ and 4 look similar.

For arbitrary n, we have the following interesting partial result. Let $o(p_{ij})$ denote the order of the root of unity p_{ij}.

Theorem 0.9. Assume Hypothesis 0.3. Suppose the orders $\{o(p_{ij})\}_{i<j}$ are pairwise coprime. Then Z is regular. As a consequence,

(1) $\mathfrak{f}_i = \prod_j o(p_{ij}) = \prod_{j \neq i} o(p_{ij})$ for all i.
(2) $|O| = \ell^2$.

Using the above theorem, one can easily construct many examples of skew polynomial rings with regular centers. We are not aware of any analogous result for other families of Artin–Schelter regular algebras.

0.3 Gorenstein center

In Section 5, we study the question of when $Z := ZS_{\mathbf{p}}$ is Gorenstein. By a theorem of Watanabe [29, Theorem 1], the invariant ring of $\Bbbk[x_1, \ldots, x_n]$ by a finite linear group K is Gorenstein when $K \subseteq \mathrm{SL}_n(\Bbbk)$. There is a suitable replacement for the group $\mathrm{SL}_n(\Bbbk)$ in the noncommutative setting using the *homological determinant* hdet : $K \to \Bbbk^\times$, as introduced by Jorgensen and the fourth author [16]. We refer to that reference for the full generalization, as computing hdet in the case of the elements of $\mathrm{Oz}(S)$ acting on S is relatively trivial [Example 1.4].

Theorem 0.10. Assume Hypothesis 0.3. The following are equivalent.

(1) S is Calabi–Yau (see Definition 1.2).
(2) The O-action on S has trivial homological determinant.
(3) $\prod_{s=1}^{n} p_{is} = 1$ for all $i = 1, \ldots, n$.
(4) $\mathfrak{pg}_S \in Z$.

As a consequence, if one of the above holds, then Z is Gorenstein and all statements in Theorems 0.4 and 0.11 hold.

When the O-action has trivial homological determinant, then Z is Gorenstein [16, Theorem 3.3]. More generally, we let H denote the subgroup of O generated by reflections. Then O/H acts on S^H. By [19, Theorem 0.2], $Z = S^O$ is Gorenstein if and only if the O/H-action has trivial homological determinant. By a result of Kirkman and the fourth author [21, Theorem 2.4], we can show that Z is Gorenstein if and only if the ozone Jacobian $\mathfrak{oj}_S$ is equal to the Jacobian $\mathfrak{j}_{S,O}$ defined in [21, Definition 0.3] (also see Definition 1.6). We summarize these results as follows.

Theorem 0.11. Assume Hypothesis 0.3 and let H denote the subgroup of O generated by reflections. The following are equivalent.

(1) Z is Gorenstein.
(2) The O/H-action on S^H has trivial homological determinant.

(3) $\phi_i(\mathfrak{oj}_S\,\mathfrak{pg}_S) = \mathfrak{oj}_S\,\mathfrak{pg}_S$, or $\prod_{s=1}^{n} p_{is}^{\mathfrak{f}_s} = 1$ for all i.

(4) $\mathfrak{oj}_S\,\mathfrak{pg}_S \in Z$.

By the above theorem, the centrality of $\mathfrak{oj}_S\mathfrak{pg}_S$ serves an indicator for Z being Gorenstein. Similarly, by Theorem 0.10, the centrality of $\mathfrak{pg}_S$ serves as an indicator for S being Calabi–Yau. Using Theorems 0.4, 0.6, 0.10, and 0.11, one can easily show the following.

Corollary 0.12. Assume Hypothesis 0.3. Then the following hold.

(1) S is Calabi–Yau if and only if Z is Gorenstein and Auslander's Theorem holds for (S, O).

(2) If S is Calabi–Yau, then Z is not regular.

It is natural to ask if there are similar results for other classes of Artin–Schelter regular algebras. For example, if A is a noetherian graded down-up algebra with parameters (α, β), then there is a canonical element $\mathfrak{b} \in A$ such that A is Calabi–Yau if and only if $\mathfrak{b} \in Z(A)$ [Proposition 6.3]. It would be interesting to know if $\mathfrak{b}$ can be defined homologically.

For small n, we are able to give conditions that are equivalent to the Gorensteinness of Z in terms of parameters $\mathbf{p}$ or the matrix B.

Theorem 0.13. Assume Hypothesis 0.3.

(1) Let $n = 2$. Then $Z = \Bbbk[x_1^{\ell}, x_2^{\ell}]$ so Z is Gorenstein.

(2) Let $n = 3$. For each $1 \le i, j \le 3$, let $b'_{ij} = \gcd(b_{ij}, \ell)$. Then Z is Gorenstein if and only if

$$\overline{B}(b'_{23}, b'_{13}, b'_{12})^T = 0$$

in $\overline{\mathbb{Z}}^3$.

(3) Let $n = 4$. Then Z is Gorenstein if and only if

$$\frac{\ell}{\gcd(\mathrm{pf}(B), \ell)}\overline{B}(v_1, v_2, v_3, v_4)^T = 0$$

in $\overline{\mathbb{Z}}^4$ where $v_i = \gcd(\ell, \{b_{jk} \mid j, k \neq i\})$ for $i = 1, \ldots, 4$.

0.4 Graded isolated singularities

By Remark 3.7, under Hypothesis 0.3, Z is regular or does not have isolated singularities.

1 Preliminaries

In this section, we review some concepts that will be used throughout the paper. Some of the definitions can be found in the survey paper of Kirkman [22].

Definition 1.1. A connected graded algebra A is called *Artin–Schelter Gorenstein* (or *AS Gorenstein*, for short) if A has injective dimension $d < \infty$ on the left and on the right, and

$$\mathrm{Ext}^i_A({}_A\Bbbk, {}_AA) \cong \mathrm{Ext}^i_A(\Bbbk_A, A_A) \cong \delta_{id}\Bbbk(\mathfrak{l}),$$

where δ_{id} is the Kronecker-delta. If in addition A has finite global dimension and finite Gelfand–Krillov dimension (GKdim), then A is called *Artin–Schelter regular* (or *AS regular*, for short) of dimension d.

For an algebra A, the *enveloping algebra* of A is $A^e = A \otimes A^{\mathrm{op}}$. If σ is an automorphism of A, then ${}^\sigma A$ is the A^e-module that, as a $\Bbbk$-vector space, is just A, but where the natural action is twisted by σ on the left: that is,

$$(a \otimes b) \cdot c = \sigma(a)cb$$

for all $a \otimes b \in A^e$ and $c \in A$.

Definition 1.2. Suppose A is AS regular of dimension d. Then there is a graded algebra automorphism μ of A, called the *Nakayama automorphism*, such that

$$\mathrm{Ext}^d_{A^e}(A, A^e) \cong {}^\mu A.$$

When $\mu = \mathrm{Id}$, then we say that A is *Calabi–Yau*.

Next we recall the trace series and homological determinant.

Definition 1.3. Let A be a connected graded algebra with A_j denoting the the homogeneous part of A of degree j. The *trace series* of a graded algebra automorphism σ of A is defined to be

$$\operatorname{Tr}_A(\sigma, t) := \sum_{j=0}^{\infty} (\operatorname{tr}(\sigma|_{A_j}))t^j,$$

where $\operatorname{tr}(\sigma|_{A_j})$ is the usual trace of the $\Bbbk$-linear map σ restricted to A_j.

When σ is the identity, we recover the Hilbert series of A, namely, $\operatorname{Tr}_A(\operatorname{Id}, t) = h_A(t)$. If A is AS regular, then $h_A(t) = 1/r(t)$ where $r(t)$ is an integral polynomial of degree $\mathfrak{l} \geq d := \operatorname{gldim} A$ and $r(t) = 1 + r_1 t + \cdots + r_{\mathfrak{l}-1} t^{\mathfrak{l}-1} + r_{\mathfrak{l}} t^{\mathfrak{l}}$ where $r_{\mathfrak{l}} = (-1)^d$. For every graded algebra automorphism σ of A, $\operatorname{Tr}_A(\sigma, t)$ must be equal to $1/q(t)$ where $q(t) = 1 + q_1 t + \cdots + q_{\mathfrak{l}-1} t^{\mathfrak{l}-1} + q_{\mathfrak{l}} t^{\mathfrak{l}}$. By [16, Lemma 2.6], the homological determinant of σ is defined to be

$$\operatorname{hdet}(\sigma) = (-1)^d q_{\mathfrak{l}} = r_{\mathfrak{l}} q_{\mathfrak{l}}. \tag{E1.3.1}$$

One nice property of homological determinant is that the map

$$\operatorname{hdet} : \operatorname{Aut}_{\operatorname{gr}}(A) \to \Bbbk^{\times}$$

is a group homomorphism [16, Proposition 2.5]. When we consider a finite group G acting on A, we say that the G-action has trivial homological determinant if $\operatorname{hdet}(G) = 1$.

Example 1.4. Let A be the weighted skew polynomial ring $S_{\mathbf{p}}$ with $\deg(x_i) > 0$ for all i. Let σ be a diagonal automorphism of A determined by

$$\sigma(x_i) = a_i x_i$$

for all i, where $a_i \in \Bbbk^{\times}$. Such an automorphism is also denoted by $\operatorname{diag}(a_i)$. It is easy to see that the trace series of σ is

$$\operatorname{Tr}_A(\sigma, t) = \prod_{i=1}^{n} (1 - a_i t^{\deg(x_i)})^{-1}.$$

By (E1.3.1), the homological determinant of σ is

$$\operatorname{hdet}(\sigma) = \prod_{i=1}^{n} a_i. \tag{E1.4.1}$$

Another way of expressing the hdet is

$$\sigma\left(\prod_{i=1}^{n} x_i\right) = \text{hdet}(\sigma)\left(\prod_{i=1}^{n} x_i\right). \qquad \text{(E1.4.2)}$$

Note that in this example, $\text{hdet}(\sigma)$ is the determinant of the matrix $\text{diag}(a_i)$. In general this is not true, see [22, Example 1.8(2)].

Lemma 1.5. Assume Hypothesis 0.3.

(1) For all i, $\text{hdet}\,\phi_i = \prod_{s=1}^{n} p_{is}$.
(2) O has trivial homological determinant if and only if $\prod_{s=1}^{n} p_{is} = 1$ for all i.
(3) If $n = 3$, then O has trivial homological determinant if and only if there exists a root of unity p such that

$$(p_{ij}) = \begin{pmatrix} 1 & p & p^{-1} \\ p^{-1} & 1 & p \\ p & p^{-1} & 1 \end{pmatrix}.$$

(4) If $n = 4$, then O has trivial homological determinant if and only if there exist roots of unity p, q, r such that

$$(p_{ij}) = \begin{pmatrix} 1 & q & p & p^{-1}q^{-1} \\ q^{-1} & 1 & r & qr^{-1} \\ p^{-1} & r^{-1} & 1 & pr \\ pq & q^{-1}r & p^{-1}r^{-1} & 1 \end{pmatrix}.$$

Proof. Part (1) follows directly from (E1.4.1). Parts (2), (3), and (4) now follow from (1). ■

The notions of the Jacobian and reflection arrangement of a finite group action on an AS regular algebra were introduced in [21, Definition 0.3]. We will soon consider diagonal actions on $S_{\mathbf{p}}$ after the definition (e.g., G is a subgroup of O).

Definition 1.6 ([21, Definition 0.3]). Let A be a noetherian AS regular algebra and G is a finite subgroup of $\text{Aut}_{\text{gr}}(A)$.

(1) Let $A_{\text{hdet}^{-1}} := \{x \in A \mid \sigma(x) = (\text{hdet}(\sigma))^{-1}x, \ \forall\, \sigma \in G\}$. If $A_{\text{hdet}^{-1}}$ is a free A^G-module of rank one on both sides generated by an element, denoted by $\mathsf{j}_{A,G}$, then $\mathsf{j}_{A,G}$ is called the *Jacobian* of the G-action. By [21, Theorem 2.4], the Jacobian $\mathsf{j}_{A,G}$ exists if and only if A^G is AS Gorenstein.

(2) Let $A_{\mathrm{hdet}} := \{x \in A \mid \sigma(x) = \mathrm{hdet}(\sigma)x,\ \forall\, \sigma \in G\}$. If A_{hdet} is a free A^G-module of rank one on both sides generated by an element, denoted by $\mathfrak{a}_{A,G}$, then $\mathfrak{a}_{A,G}$ is called the *reflection arrangement* of the G-action. By [21, Theorem 0.2], the reflection arrangement $\mathfrak{a}_{A,G}$ exists if A^G is AS regular.

Lemma 1.7. Assume Hypothesis 0.3 and let G be a finite subgroup of diagonal automorphisms of S. Then $\mathfrak{pg}_S := \prod_{i=1}^n x_i$ is an element in S_{hdet}. As a consequence, if $\mathfrak{a}_{S,G}$ exists, then it is a factor of $\mathfrak{pg}_S$.

Proof. The main assertion follows from (E1.4.2). The consequence is clear. ■

Recall that $\mathfrak{oj}_S$, $\mathfrak{oa}_S$, and $\mathfrak{od}_S$ are defined (E0.3.2)–(E0.3.4).

Proposition 1.8. Assume Hypothesis 0.3.

(1) Up to nonzero scalars, $\mathfrak{oj}_S$, $\mathfrak{oa}_S$, and $\mathfrak{od}_S$ are algebra invariants of S. Namely, they are independent of the chosen generating set $\{x_1, \ldots, x_n\}$.

(2) The set $\{\mathfrak{f}_1, \ldots, \mathfrak{f}_n\}$ with multiplicities is an algebra invariant of S.

Proof. (1) We use the notation introduced in [7] without giving detailed definitions. For each $s \in \{1, \ldots, n\}$, let T_s be the set defined as before [7, Lemma 2.9] and let $d_w(S/Z)$ be the discriminant of S over its center Z introduced in [7, Definition 1.2(3)] where $w = \mathrm{rk}(S_Z)$. It follows from [7, Lemma 2.9(2)] that $\mathfrak{f}_s > 1$ if and only if T_s is empty. Combined with [7, Theorem 2.11(2)], we obtain that $\mathfrak{f}_s > 1$ if and only if x_s divides $d_w(S/Z)$, In other words, $d_w(S/Z) =_{\Bbbk^\times} \prod_{i, \mathfrak{f}_i > 1} x_i^{\lambda_i}$ for some $\lambda_i \geq 1$. By definition, $d_w(S/Z)$ is an algebra invariant of A up to a nonzero scalar [7].

Since S is $\mathbb{N}^n$-graded, the prime decomposition of $x_i^{\lambda_i}$ is unique and $\{x_i \mid \mathfrak{f}_i > 1\}$ is a complete list of prime factors of $d_w(S/Z)$. Therefore, the list $I := \{\Bbbk^\times x_i \mid \mathfrak{f}_i > 1\}$ (after adding nonzero scalars) is an algebra invariant of S. As a consequence, $\mathfrak{oa}_S$ is an algebra invariant.

Note that, for each $\Bbbk^\times x_i \in I$,

$$\mathfrak{f}_i = \gcd\{d \mid x_i^d a \in Z \text{ for some } a \in S \text{ with } x_i \nmid a\}.$$

This implies that the set $\{\Bbbk^\times x_i^{\mathfrak{f}_i} \mid \mathfrak{f}_i > 1\}$ is also an algebra invariant of S. As a consequence, $\mathfrak{od}_S$ is an algebra invariant of S.

Since $\mathfrak{oj}_S = \mathfrak{od}_S(\mathfrak{oa}_S)^{-1}$, $\mathfrak{oj}_S$ is an algebra invariant.

(2) By part (1), the set $\{\mathfrak{f}_i \mid \mathfrak{f}_i > 1\}$ is an algebra invariant of S. Since $\{\mathfrak{f}_1, \ldots, \mathfrak{f}_n\} = \{\mathfrak{f}_i \mid \mathfrak{f}_i > 1\} \cup \{1, \ldots, 1\}$ considered as a set with multiplicities, the assertion follows. ■

2 Reflections and Reflection Groups

Throughout this section, we assume Hypothesis 0.3. Recall that a linear automorphism g of a finite-dimensional vector space V is a *pseudo-reflection* if g fixes a codimension one subspace. In the noncommutative case, we need to use the trace series [Definition 1.3] to define a reflection.

Definition 2.1 ([18, Definition 1.4]). Let A be an AS regular algebra of Gelfand–Kirillov dimension n.

(1) We call a graded algebra automorphism σ of A a *reflection* of A if $\sigma \neq \mathrm{Id}$ has finite order and the trace series of σ has the form

$$\mathrm{Tr}_A(\sigma, t) = \frac{1}{(1-t)^{n-1}q(t)} \quad \text{where} \quad q(1) \neq 1.$$

(2) A finite subgroup $G \subseteq \mathrm{Aut}_{\mathrm{gr}}(A)$ is called a *reflection group* if it is generated by reflections.

(3) A finite subgroup $G \subseteq \mathrm{Aut}_{\mathrm{gr}}(A)$ is called *small* if it does not contain any reflections.

Example 2.2. Let $S = S_{\mathbf{p}}$ with $\deg(x_i) = 1$ for all i and let $\sigma = \mathrm{diag}(a_i)$, $a_i \in \Bbbk^\times$, as in Example 1.4. By that example, the trace series of σ is

$$\mathrm{Tr}_S(\sigma, t) = \prod_{i=1}^{n} (1 - a_i t)^{-1}.$$

Hence, σ is a reflection if and only if $a_i = 1$ for all i but one. Therefore, σ is a reflection if and only if $\sigma|_{S_1}$ is a pseudo-reflection. As a consequence, O is small in the sense of Definition 2.1(3) if and only if O is small in the classical sense.

One noncommutative version of the Shephard–Todd–Chevalley theorem is the following.

Theorem 2.3 ([20, Theorem 5.5]). Let $S = S_{\mathbf{p}}$ and let G be a finite group of graded algebra automorphisms of S. Then S^G has finite global dimension if and only if G is generated by reflections of S (in this case, S^G is again a skew polynomial ring with weighted grading).

In this paper, we will only apply this theorem when every element of G is a diagonal automorphism of S. In fact, we will only consider the case when G is a subgroup of $\mathrm{Oz}(S)$. We now introduce the notion of an ozone subring.

Definition 2.4. Let A be a noetherian PI AS regular algebra with center Z.

(1) A subring R of A is called *ozone* if R is AS regular and $Z \subseteq R \subseteq A$.
(2) The set of all ozone subrings of A is denoted by $\Phi_Z(A)$.
(3) If R is a minimal element in $\Phi_Z(A)$ via inclusion, then R is called a *mozone* subring of A.

Note that A itself is an ozone subring of A, so $\Phi_Z(A)$ is not empty. If Z is AS regular (i.e., a commutative polynomial ring) or if $A = Z$ is itself commutative, then Z is the unique mozone subring of A. Assume Hypothesis 0.3 and let H be the subgroup of $O := \mathrm{Oz}(S)$ that is generated by reflections. Let $B := S^H$. By Theorem 2.3, B is a skew polynomial ring (and hence an ozone subring of S). We will show that B is a mozone subring of S. (See also Question 6.2(6).)

Proposition 2.5. Assume Hypothesis 0.3. Let H be the subgroup of O generated by reflections in O. Then S^H is a mozone subring of S.

Proof. Since H fixes S^H, there is an induced action of O/H on S^H. By the proof of [19, Proposition 4.12], this action is small in the sense of Definition 2.1(3). Hence, by [3, Theorem 5.5], we therefore have that $S^H\#(O/H) \cong \mathrm{End}_{S^O}(S^H)$ (which is analogous to a weighted version of Theorem 0.4).

Now suppose for contradiction that there is an AS regular algebra R such that $Z \subseteq R \subsetneq S^H$. By [18, Lemma 1.10(b)], S^H is a graded free module over R. Hence, we can write $S^H = R \oplus R(-d_1) \oplus \cdots \oplus R(-d_n)$ for some integers d_i, at least one of which is positive. Then $\mathrm{End}_{S^O}(S^H)$ contains elements of negative degree, which contradicts that it is isomorphic to $S^H\#(O/H)$. ■

Recall from (E0.3.1) that

$$\mathfrak{f}_i = \gcd\{d_i \mid \exists\, d_1, \ldots, \widehat{d_i}, \ldots, d_n \text{ with } x_1^{d_1} \cdots x_i^{d_i} \cdots x_n^{d_n} \in Z\}.$$

Let M be the subring of S generated by $\{x_1^{\mathfrak{f}_1}, \ldots, x_n^{\mathfrak{f}_n}\}$. It is straightforward to verify that $M \in \Phi_Z(S)$.

Proposition 2.6. Assume Hypothesis 0.3. Let H be the subgroup of O generated by reflections in O.

(1) The subgroup H is isomorphic to $\prod_{i=1}^{n}\langle r_i\rangle$ where r_i is a diagonal automorphism of S determined by

$$r_i : x_j \mapsto \begin{cases} x_j & j \neq i \\ c_i x_i & j = i \end{cases}$$

for some root of unity c_i.

(2) For each i, let w_i be the order of c_i given in part (1). For each i, j, define $q_{ij} = p_{ij}^{w_i w_j}$ and let $\mathbf{q} = (q_{ij})$. Then $S^H = \Bbbk_{\mathbf{q}}[x_1^{w_1}, \dots, x_n^{w_n}]$.

(3) For each i, $w_i = \mathfrak{f}_i$.

(4) $\mathfrak{f}_i = \min\{d_i > 0 \mid \exists\, d_1, \dots, \widehat{d_i}, \dots, d_n \text{ with } x_1^{d_1} \cdots x_i^{d_i} \cdots x_n^{d_n} \in Z\}$.

Proof. (1) Let H_i be the subgroup of H that is generated by reflections r of the form

$$r(x_j) = \begin{cases} x_j & j \neq i \\ \alpha_i x_i & j = i \end{cases}$$

for some $\alpha_i \in \Bbbk$. Since every element in O is a diagonal automorphism, by Example 2.2, every reflection is of the form of r given above. Since H is generated by reflections in O, H is generated by the union of the H_i. It is clear that the product $\prod_{i=1}^{n} H_i$ is a subgroup of H. Thus $H = \prod_{i=1}^{n} H_i$. Note that each H_i is a finite subgroup of $\Bbbk^\times$, whence it is cyclic. Therefore, each H_i is of the form $\langle r_i\rangle$ where r_i is a diagonal reflection automorphism of S. The assertion follows.

(2) It is clear that $S^H \supseteq \Bbbk_{\mathbf{q}}[x_1^{w_1}, \dots, x_n^{w_n}]$. Since S is $\mathbb{Z}^n$-graded and H consists of $\mathbb{Z}^n$-graded algebra automorphisms of S, S^H is $\mathbb{Z}^n$-graded. So to prove $S^H \subseteq \Bbbk_{\mathbf{q}}[x_1^{w_1}, \dots, x_n^{w_n}]$, we only need to consider monomials in S^H. Suppose that $f := x_1^{v_1} \cdots x_n^{v_n}$ is an element of S^H. Since $f = r_i(f) = c_i^{v_i} f$ for all i, we see that $w_i \mid v_i$. The assertion follows.

(3) By part (2) and the fact that $H \subseteq O$, we have

$$Z = S^O \subseteq S^H = \Bbbk_{\mathbf{q}}[x_1^{w_1}, \dots, x_n^{w_n}].$$

Therefore, every monomial in Z is a product of $x_1^{w_1}, \dots, x_n^{w_n}$. By definition, $w_i \mid \mathfrak{f}_i$ for all i. Next we show that $\mathfrak{f}_i \mid w_i$. Let c_i' be a primitive $\mathfrak{f}_i$th root of unity. By the definition of the $\mathfrak{f}_i$, we obtain that Z is a subring of $M' := \Bbbk_{(q'_{ij})}[x_1^{\mathfrak{f}_1}, \dots, x_n^{\mathfrak{f}_n}]$ where $q'_{ij} = p_{ij}^{\mathfrak{f}_i \mathfrak{f}_j}$. Let r_i' be the diagonal automorphism of S determined by

$$x_j \mapsto \begin{cases} x_j & j \neq i \\ c_i' x_i & j = i. \end{cases}$$

It is clear that r_i' preserves M' and hence Z so $r_i' \in O$. Since r_i' is a reflection, therefore $r_i' \in H$ and consequently, $x_i^{w_i} = r_i'(x_i^{w_i}) = (c_i')^{w_i} x_i^{w_i}$. This implies that $(c_i')^{w_i} = 1$ or $\mathfrak{f}_i \mid w_i$ as required.

(4) By definition, $\mathfrak{f}_i$ is given by (E0.3.1). For each i, there are finitely many central elements, say $z(1), \ldots, z(s)$, defined by

$$z(j) := x_1^{d_1^j} \cdots x_i^{d_i^j} \cdots x_n^{d_n^j} \in Z$$

such that $\mathfrak{f}_i = \gcd\{d_i^1, \cdots, d_i^s\}$. Write $\mathfrak{f}_i = \sum_{j=1}^s c_j d_i^j$. Choose an integer α such that $c_j' := c_j + \alpha\ell > 0$ for all j. Then $\mathfrak{f}_i = \left(\sum_{j=1}^s c_j' d_i^j\right) - \beta\ell$ for some β and, up to a scalar, we have

$$\prod_j z(j)^{c_j'} = x_1^{d_1'} \cdots x_i^{\mathfrak{f}_i} \cdots x_n^{d_n'} x_i^{\beta\ell} \in Z.$$

This implies that $x_1^{d_1'} \cdots x_i^{\mathfrak{f}_i} \cdots x_n^{d_n'} \in Z$. The claim follows. ■

Theorem 2.7. Assume Hypothesis 0.3. If Z is Gorenstein, then $\mathfrak{oj}_S$ is equal to the Jacobian $\mathrm{j}_{S,O}$.

Proof. Let H be the subgroup of O generated by reflections in O. By Proposition 2.6(2,3), $S^H = \Bbbk[x_1^{\mathfrak{f}_1}, \cdots, x_n^{\mathfrak{f}_n}]$. Then

$$S = \bigoplus_{0 \le s_i \le \mathfrak{f}_i - 1} x_1^{s_1} \cdots x_n^{s_n} S^H. \tag{E2.7.1}$$

Since $\mathfrak{pg}_S \in S_{\mathrm{hdet}_O}$ (see (E1.4.2)) and since the H-action on $\chi := \prod_{i=1}^n x_i^{\mathfrak{f}_i}$ is trivial, $\mathfrak{oj}_S = \chi(\mathfrak{pg}_S)^{-1} \in S_{\mathrm{hdet}_H^{-1}}$. The decomposition (E2.7.1) shows that $S_{\mathrm{hdet}_H^{-1}} = \mathfrak{oj}_S S^H$ or equivalently, $\mathfrak{oj}_S$ is $\mathrm{j}_{S,H}$. Therefore, $S_{\mathrm{hdet}_O^{-1}} \subseteq S_{\mathrm{hdet}_H^{-1}} = \mathfrak{oj}_S S^H$.

We claim that $\mathfrak{oj}_S \in S_{\mathrm{hdet}_O^{-1}}$. To see that we note that $\mathfrak{oj}_S = \chi(\mathfrak{pg}_S)^{-1}$. For every $g \in O$, let $\overline{g}$ be the induced algebra automorphism of S^H. Since $Z = (S^H)^{O/H}$ is Gorenstein, then the O/H-action on S^H has trivial homological determinant by [19, Theorem 0.2]. Since S^H is a skew polynomial ring and $\overline{g}$ is a diagonal action, O/H-action having trivial homological determinant implies that

$$g(\chi) = \overline{g}\left(\prod_{i=1}^n x_i^{\mathfrak{f}_i}\right) = \prod_{i=1}^n x_i^{\mathfrak{f}_i} = \chi,$$

see (E1.4.2). Hence,

$$g(\mathfrak{oj}_S) = g(\chi \mathfrak{pg}_S^{-1}) = g(\chi)(g(\mathfrak{pg}_S))^{-1} = \chi\ \mathrm{hdet}^{-1}(g)(\mathfrak{pg}_S)^{-1} = \mathrm{hdet}^{-1}(g)\mathfrak{oj}_S.$$

This implies that $\mathfrak{oj}_S \in S_{\mathrm{hdet}_O^{-1}}$. Then it is easy to check that $S_{\mathrm{hdet}_O^{-1}} = \mathfrak{oj}_S S^O$. In other words, $\mathfrak{oj}_S = \mathfrak{j}_{S,O}$. ■

Proposition 2.6, and part (4) in particular, make computing the $\mathfrak{f}_i$ straightforward in small dimension, as demonstrated by the following result.

Lemma 2.8. Assume Hypothesis 0.3 and let $n = 3$. Then=10

$$\mathfrak{f}_1 = \gcd(b_{23}, \ell), \quad \mathfrak{f}_2 = \gcd(b_{13}, \ell), \quad \mathfrak{f}_3 = \gcd(b_{12}, \ell).$$

Proof. We prove the first equality, as the others are similar. First note that $x_1^{b_{23}} x_2^{\ell - b_{13}} x_3^{b_{12}} \in Z$, and so $\mathfrak{f}_1 \mid b_{23}$.

Since $x^\ell \in Z$, then $\mathfrak{f}_1 \mid \ell$, so $\mathfrak{f}_1 \mid \gcd(b_{23}, \ell)$. On the other hand, there exists $a = x_1^{u_1} x_2^{u_2} x_3^{u_3} \in Z$ with $u_1 = \mathfrak{f}_1$ by Proposition 2.6(4). By considering $[x_2, a] = [x_3, a] = 0$, we have

$$b_{23}u_3 \equiv b_{12}u_1 \mod \ell$$
$$b_{23}u_2 \equiv -b_{13}u_1 \mod \ell.$$

Hence, if $d \mid b_{23}$ and $d \mid \ell$, then $d \mid b_{12}u_1$ and $d \mid b_{13}u_1$. But since $d \mid b_{23}u_1$ and $\gcd(b_{23}, b_{12}, b_{13}) = 1$, then $d \mid u_1$. Thus, $d \mid \mathfrak{f}_1$, so $\gcd(b_{23}, \ell) \mid \mathfrak{f}_1$. ■

For $i = 1, 2, 3$, set

$$g_i = o(\phi_i) = \frac{\ell}{\gcd(b_{ij}, b_{ik}, \ell)} \text{ with } \{i, j, k\} = \{1, 2, 3\}.$$

Then it follows from Lemma 2.8 that

$$g_i = \frac{\ell}{\gcd(\mathfrak{f}_j, \mathfrak{f}_k, \ell)} \text{ with } \{i, j, k\} = \{1, 2, 3\}.$$

Write $g = (g_1, g_2, g_3)$ and $\mathfrak{f} = (\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3)$. By Theorem 0.6, Z is regular if and only if $g = \mathfrak{f}$.

3 Auslander's Theorem

Throughout this section, we assume Hypothesis 0.3. In this section, we seek to understand when the Auslander map

$$S\#O \to \mathrm{End}_{S^O}(S)$$

is an isomorphism. Recall, from Definition 2.1, that a finite subgroup $G \subseteq \mathrm{Aut}_{\mathrm{gr}}(A)$ is called small if it does not contain any reflections. By Example 2.2, when $A = S_{\mathbf{p}}$, G is small if and only if G, considered as a subgroup of $\mathrm{GL}(\bigoplus_{i=1}^n \Bbbk x_i)$, is small in the classical sense.

Proof of Theorem 0.4. (1) $\Leftrightarrow$ (2): By [4, Theorem 0.3], the Auslander map is an isomorphism for the pair (S, O) if and only if $\mathsf{p}(S, O) \geq 2$.

(1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4): Since each ϕ_i is a diagonal automorphism, by [3, Theorem 5.5], the Auslander map is an isomorphism for the pair (S, O) if and only if O is small in the classical sense, that is, if and only if O, when restricted to $\bigoplus_{i=1}^n \Bbbk x_i$, contains no pseudo-reflections of $\bigoplus_{i=1}^n \Bbbk x_i$.

(3) $\Leftrightarrow$ (5): If O is small, then, by Proposition 2.5, we have that S is the unique mozone subring of S. Conversely, if O is not small, then let $H \leq O$ be the non-trivial subgroup of reflections. By Proposition 2.6, $S^H \subsetneq S$ is a mozone subring of S, and hence S is not a mozone subring.

(3) $\Leftrightarrow$ (6): By Proposition 2.6(2,3), $S^H = \Bbbk[x_1^{\mathfrak{f}_1}, \cdots, x_n^{\mathfrak{f}_n}]$. Hence, $H = \{1\}$ if and only if $\mathfrak{f}_i = 1$ for all i.

(6) $\Leftrightarrow$ (7): This follows from Proposition 2.6(4). ■

For the rest of the section, we consider the cases of small n. In the $n = 2$ case, with $S = \Bbbk_p[x_1, x_2]$ for $p \neq 1$, we have $S^O = \Bbbk[x_1^\ell, x_2^\ell]$ where ℓ is the order of the root of unity p. Consequently, S is free over S^O. It follows that $\mathrm{End}_{S^O}(S)$ has negative degree maps and so the Auslander map $S\#O \to \mathrm{End}_{S^O}(S)$ is not an isomorphism. Next we consider $n = 3$ and 4.

Let $g = \phi_1^{u_1} \cdots \phi_n^{u_n} \in O$ and write $\mathbf{u} = (u_i)$ for the (column) vector with components u_i. For a vector $\mathbf{x}$, we use $\mathbf{x}^T$ to denote its transpose. The restriction of g to S_1 is given by the diagonal matrix $\mathrm{diag}(\xi^{v_i})$ where $(v_i)^T = \mathbf{v}^T = \mathbf{u}^T\overline{B}$, viewing $\mathbf{v}$ and $\mathbf{u}$ as elements of $\overline{\mathbb{Z}}^n$. It is clear that g is a pseudo-reflection if $\mathbf{v} = \lambda\mathbf{e}_i$ for some nonzero $\lambda \in \overline{\mathbb{Z}}$ and some $1 \leq i \leq n$. We can therefore express the condition of O being small in terms of nonexistence of solutions of some linear equations over $\overline{\mathbb{Z}}$. Namely, O having no pseudo-reflections is equivalent to the equation $\mathbf{u}^T\overline{B} = \lambda\mathbf{e}_i^T$ having no solution for any nonzero $\lambda \in \overline{\mathbb{Z}}$. Taking transposes, this is equivalent to the equation

$$\overline{B}\mathbf{y} = \lambda\mathbf{e}_i, \tag{E3.0.1}$$

having no solutions $\mathbf{y} \in \overline{\mathbb{Z}}^n$ for any nonzero $\lambda \in \overline{\mathbb{Z}}$ and any $1 \leq i \leq n$.

The next lemma allows us to reduce to the case that no x_i is central.

Lemma 3.1. Suppose x_1 is central (equivalently, $\phi_1 = \mathrm{id}$). Let R be the subalgebra of S generated by $x_2, \ldots, x_n$, and let $O' = \langle \phi_i\big|_R \mid i = 2, \ldots, n\rangle$. Then $\mathsf{p}(R, O') = \mathsf{p}(S, O)$ so (S, O) satisfies Auslander's Theorem if and only if (R, O') does.

Proof. Since O acts trivially on x_1, by (E0.3.6), we have

$$\begin{aligned}\mathsf{p}(S, O) &= \mathrm{GKdim}(S) - \mathrm{GKdim}(S\#O/(f_O)) \\ &= \mathrm{GKdim}(R[x_1]) - \mathrm{GKdim}(R[x_1]\#O/(f_O)) \\ &= (\mathrm{GKdim}(R) + 1) - \mathrm{GKdim}((R\#O'/(f_{O'}))[x_1]) \\ &= (\mathrm{GKdim}(R) + 1) - (\mathrm{GKdim}(R\#O'/(f_{O'})) + 1) \\ &= \mathrm{GKdim}(R) - \mathrm{GKdim}(R\#O'/(f_{O'})) \\ &= \mathsf{p}(R, O').\end{aligned}$$

■

We begin by considering the case $n = 3$. For a root of unity p, let $o(p)$ denote its order.

Proposition 3.2. Assume Hypothesis 0.3 with $n = 3$. Then (S, O) satisfies Auslander's Theorem if and only if each p_{ij} is a primitive ℓth root of unity for all $i \neq j$.

Proof. By Lemma 3.1, we may assume that no $\phi_i = \mathrm{Id}$.

Assume O is small so that no power of the ϕ_i are reflections. Note that $\phi_1^{o(p_{12})} = \mathrm{diag}(1, 1, p_{13}^{o(p_{12})})$. By hypothesis, this implies that $o(p_{13})$ divides $o(p_{12})$. Using the same logic on the other ϕ_i gives $o(p_{12}) = o(p_{13}) = o(p_{23})$. That is, each of the p_{ij} is a primitive ℓth root of unity.

Conversely, assume that each of the p_{ij} is a primitive ℓth root of unity and so $\gcd(b_{ij}, \ell) = 1$ for each b_{ij}. We wish to show that equation (E3.0.1) has no solutions. Now we compute

$$\overline{B}\mathbf{y} = \overline{B}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_2 b_{12} + y_3 b_{13} \\ -y_1 b_{12} + y_3 b_{23} \\ -y_1 b_{13} - y_2 b_{23} \end{bmatrix}.$$

Without loss of generality, suppose that the first two entries of $\overline{B}\mathbf{y}$ are equal to 0. Then

$$\begin{aligned} b_{12}(y_1 b_{13} + y_2 b_{23}) &= b_{13}(y_1 b_{12}) + b_{23}(y_2 b_{12}) \\ &= b_{13}(y_3 b_{23}) - b_{23}(y_3 b_{13}) = 0 \end{aligned}$$

and hence $\overline{B}\mathbf{y} = 0$. Therefore, equation (E3.0.1) has no solutions and so O is small. ■

Example 3.3. As a consequence of the proof of Proposition 3.2, when $n = 3$, if no power of any ϕ_i is a reflection, then O is small. By contrast, in the $n = 4$ case, it is possible for O to contain a reflection even if no power of any ϕ_i is a reflection.

Let ξ be a primitive sixth root of unity and set

$$p_{12} = p_{13} = p_{23} = \xi$$
$$p_{14} = p_{24} = p_{34} = -1.$$

Then

$$\phi_1 = \mathrm{diag}(1, \xi, \xi, -1) \qquad \phi_2 = \mathrm{diag}(\xi^{-1}, 1, \xi, -1)$$
$$\phi_3 = \mathrm{diag}(\xi^{-1}, \xi^{-1}, 1, -1) \qquad \phi_4 = \mathrm{diag}(-1, -1, -1, 1).$$

(Here we consider the map ϕ_i as a matrix form when restricted to the degree 1 part of S.) So no power of any ϕ_i is a reflection. However,

$$\psi = \phi_1\phi_2\phi_3 = \mathrm{diag}(\xi^{-2}, 1, \xi^2, -1)$$

and so $\psi^3 = \mathrm{diag}(1, 1, 1, -1)$. That is, ψ^3 is a reflection. By an easy computation, $\mathfrak{f}_i = 2$ for all $i = 1, 2, 3, 4$. As a consequence,

$$x_1^2 x_2^2 x_3^2 x_4^2 = (\mathfrak{oj}_S)^2 = (\mathfrak{oa}_S)^2 = (\mathfrak{pg}_S)^2 = \mathfrak{od}_S$$

up to nonzero scalars. Using the results stated in the introduction, we have

1. (1) (S, O) does not satisfy Auslander's Theorem as $\mathfrak{f}_1 \neq 1$ [Theorem 0.4(6)],
2. (2) S is not Calabi–Yau as $\mathfrak{pg}_S \notin Z$ [Theorem 0.10(4)], and
3. (3) Z is not Gorenstein as $\mathfrak{oj}_S\, \mathfrak{pg}_S \notin Z$ [Theorem 0.11(4)].

We will shortly give necessary and sufficient conditions on the entries b_{ij} of B for O to be small when $n = 4$. The following result gives a necessary condition for any n, and is of independent interest.

Proposition 3.4. Assume Hypothesis 0.3. If O is small, then $\ell \mid \mathrm{pf}(B)$.

Proof. When n is odd, $\mathrm{pf}(B) = 0$, and so the statement is vacuously true. Now suppose n is even. We will prove the contrapositive: if $\mathrm{pf}(B) \neq 0$ in $\overline{\mathbb{Z}}$, then O contains a reflection.

Recall the adjugate matrix $\mathrm{adj}(B)$ is defined by the property

$$\mathrm{adj}(B)B = B\,\mathrm{adj}(B) = \det(B)I.$$

There is a Pfaffian version of this for skew symmetric B (with n even), defined by

$$\mathrm{Padj}(B)B = B\,\mathrm{Padj}(B) = \mathrm{pf}(B)I$$

[5, Corollary 1, page 46]. If $\mathrm{pf}(B) \not\equiv 0 \pmod{\ell}$, we can use the above to solve the equation $B\mathbf{y} = \mathrm{pf}(B)\mathbf{e}_i$. For any i,

$$\mathbf{y} = \mathrm{Padj}(B)\mathbf{e}_i.$$

By the discussion preceding equation (E3.0.1), this produces a reflection in O. ■

Proposition 3.5. Assume Hypothesis 0.3 and let $n = 4$. Then O is small if and only if the following two conditions hold:

(1) $\mathrm{pf}(B) \equiv 0 \pmod{\ell}$ and

(2) there does not exist an index j and an integer k such that $kb_{ij} \equiv 0 \pmod{\ell}$ for all but one i.

Proof. As discussed at the beginning of this section, the smallness of O is equivalent to equation (E3.0.1) having no solutions.

The previous proposition shows that if O is small, then $\mathrm{pf}(B) \equiv 0 \pmod{\ell}$, so condition (1) holds. Further, if there did exist an index i and an integer k such that $kb_{ij} \equiv 0 \pmod{\ell}$ for all but one j, then $\overline{B}(k\mathbf{e}_j) = kb_{ij}\mathbf{e}_i$, and so we have a solution to (E3.0.1). Hence, if O is small then conditions (1) and (2) hold.

Conversely, suppose that conditions (1) and (2) hold. We wish to show that (E3.0.1) does not have any solutions. Without loss of generality, suppose to the contrary that there is a solution, say $\mathbf{u}$, when $i = 4$, so that $\lambda\mathbf{e}_4 = \overline{B\mathbf{u}}$. Multiplying both sides by $\mathrm{Padj}(B)$ gives

$$\lambda\overline{\mathrm{Padj}(B)}\mathbf{e}_4 = \overline{\mathrm{Padj}(B)Bu} = \overline{\mathrm{pf}(B)I\mathbf{u}} = 0.$$

Using [5, Definition 1, page 46], one easily computes that

$$\lambda\overline{\mathrm{Padj}(B)}\mathbf{e}_4 = \lambda\overline{(-b_{23}, b_{13}, -b_{12}, 0)^t}.$$

Now since $b_{12}\lambda \equiv b_{13}\lambda \equiv 0 \pmod{\ell}$, condition (2) implies that $b_{14}\lambda \equiv 0 \pmod{\ell}$. Similarly, since $b_{12}\lambda \equiv b_{23}\lambda \equiv 0 \pmod{\ell}$, then $b_{24}\lambda \equiv 0 \pmod{\ell}$. And since $b_{14}\lambda \equiv b_{24}\lambda \equiv 0 \pmod{\ell}$, then $b_{34}\lambda \equiv 0 \pmod{\ell}$. But since $\lambda \neq 0$, this implies that $\gcd(b_{ij}, \ell) \neq 1$, which is a contradiction. ■

Now we are ready to prove Theorem 0.5.

Proof of Theorem 0.5. Note that the case $n = 2$ is trivial. The case $n = 3$ follows from Proposition 3.2 while the $n = 4$ case follows from Proposition 3.5. ■

Remark 3.6. We remark that condition (2) in Proposition 3.5 is equivalent to the condition that for all $1 \le i \le 4$, no ϕ_i^k is a reflection. Hence, in both the cases $n = 3$ and $n = 4$, the smallness of O is equivalent to the Pfaffian being 0 (which is automatic when $n = 3$) and no power of a generator ϕ_i being a reflection. In Example 3.3, the Pfaffian of B was nonzero.

Remark 3.7. Let R be an AS regular algebra satisfying $\mathrm{gldim}(R) \ge 2$ and G a finite subgroup of $\mathrm{Aut}_{\mathrm{gr}}(R)$. We say R^G has *graded isolated singularities* if $\mathsf{p}(R, G) = \mathrm{GKdim}(R)$.

Consider the case that $R = S_{\mathbf{p}}$ on n variables and $G = O$ as above. By [3, Lemma 5.4], it suffices to compute $\mathsf{p}(A, G)$ where $A = \Bbbk[x_1, \dots, x_n]$. But in this setting, $\mathsf{p}(A, G) = n$ is equivalent to G acting freely on $A_1 \backslash \{0\}$ (see the references given in [11, page 4320]). It is clear that this fails since, for example, $\phi_i(x_i) = x_i$, Hence, choosing any nontrivial ϕ_i (one of which must exist if R is noncommutative), shows that G *does not* act freely on A_1. Thus $ZS_{\mathbf{p}}$ does not have isolated singularities.

On the other hand, there are other AS regular algebras A such that $Z(A)$ has an isolated singularity [9].

4 Regular Center

Throughout this section, we assume Hypothesis 0.3. We consider the question of determining when the center of S is regular (equivalently, Z is a polynomial ring).

Recall that $\overline{B}$ is the matrix obtained from B by reduction mod ℓ. Let $\overline{K}$ denote the kernel of $\overline{B}$ and K be its inverse image in $\mathbb{Z}^n$. For $i = 1, \dots, n$, denote by $K_i \subseteq \mathbb{Z}$ the projection of K onto its ith component. If p is a prime number, then let $\mathbb{Z}_{(p)}$ denote the localization of $\mathbb{Z}$ at the prime ideal (p). If M is a $\mathbb{Z}$-module and $m \in M$, we use the notation $m \otimes 1$ to denote the image of m in $M \otimes \mathbb{Z}_{(p)}$.

Lemma 4.1. Assume Hypothesis 0.3.

(1) Let $x_1^{u_1} \cdots x_n^{u_n}$ be a monomial in S. Write $\mathbf{u} = (u_1, \dots, u_n)^T$ and $x^{\mathbf{u}} := x_1^{u_1} \cdots x_n^{u_n}$. Then $x^{\mathbf{u}}$ is central if and only if $\mathbf{u} \in K$.

(2) We have $Z = \Bbbk[x_1^{\mathfrak{f}_1}, \dots, x_n^{\mathfrak{f}_n}]$ if and only if $\mathfrak{f}_i \mathbf{e}_i \in K$ for each $i = 1, \dots, n$. Equivalently, $\mathfrak{f}_i \mathbf{e}_i \otimes 1 \in K \otimes \mathbb{Z}_{(p)}$ for every prime $p \mid \ell$ and $i = 1, \dots, n$.

Proof. (1) For any $1 \le i \le n$ we have

$$x_i \left(x_1^{u_1} \cdots x_n^{u_n}\right) = \xi^{b_{i1}u_1+\cdots+b_{in}u_n} \left(x_1^{u_1} \cdots x_n^{u_n}\right) x_i$$
$$= \xi^{(B\mathbf{u})_i} \left(x_1^{u_1} \cdots x_n^{u_n}\right) x_i.$$

Hence, $x^{\mathbf{u}}$ is central if and only if $\xi^{(B\mathbf{u})_i} = 1$ for all i if and only if $\mathbf{u} \in K$.

(2) By part (1) and definition, $\mathfrak{f}_i = \gcd(K_i)$. If $\mathfrak{f}_i \mathbf{e}_i \in K$, then by part (1), we have $x_i^{\mathfrak{f}_i} \in Z$. Suppose that $x^{\mathbf{u}} \in Z$, or equivalently, $\mathbf{u} \in K$. Then by the definition of $\mathfrak{f}_i$ (see (E0.3.1)), there exist integers a_i such that $\mathfrak{f}_i a_i = u_i$, so that $(x_1^{\mathfrak{f}_1})^{a_1} \cdots (x_n^{\mathfrak{f}_n})^{a_n} = x^{\mathbf{u}}$. Hence, $Z \subseteq \Bbbk[x_1^{\mathfrak{f}_1}, \ldots, x_n^{\mathfrak{f}_n}]$. The converse is clear. ■

Proof of Theorem 0.6. (1) ⇔ (5): This is [20, Theorem 5.5].

(1) ⇔ (4): This follows from [18, Lemma 1.10].

(5) ⇒ (3): By the proof of Proposition 2.6, when $O = H, Z = S^O = S^H$ is the subring of the form $\Bbbk[x_1^{\mathfrak{f}_1}, \cdots, x_n^{\mathfrak{f}_n}]$.

(3) ⇒ (2) ⇒ (1): These implications are clear.

(3) ⇔ (7): Since O is generated by $\{\phi_i\}_{i=1}^n$, it is preserved under base field extension. Further both assertions in (7) and (3) are preserved under base field extension. So we can assume that $\Bbbk$ is algebraically closed, whence we can use Lemma 0.2. Given (3), it is easy to see that $\operatorname{rk}(S_Z) = \prod_{i=1}^n \mathfrak{f}_i$, and this is the order of O by Lemma 0.2. Conversely, $Z \subseteq \Bbbk[x_1^{\mathfrak{f}_1}, \cdots, x_n^{\mathfrak{f}_n}] := B \subseteq S$ by definition. Then $\operatorname{rk}(S_B)\operatorname{rk}(B_Z) = \operatorname{rk}(S_Z) = |O|$ where the last equation is Lemma 0.2. Further, O acts on B with $Z = B^O$. Hence, Z is a direct summand of B (see [18, Lemma 1.11] and [26, Corollary 1.12]). Since $\operatorname{rk}(S_B) = \prod_{i=1}^n \mathfrak{f}_i$, then $|O| = \prod_{i=1}^n \mathfrak{f}_i$ implies that $\operatorname{rk}(B_Z) = 1$ or equivalently, $Z = B = \Bbbk[x_1^{\mathfrak{f}_1}, \cdots, x_n^{\mathfrak{f}_n}]$.

(3) ⇒ (6): Since O is generated by $\{\phi_1, \cdots, \phi_n\}$, there is a surjective map $\prod_{s=1}^n \langle \phi_s \rangle \to O$ and consequently,

$$|O| \le \frac{\ell^n}{\prod_{s=1}^n \gcd\{b_{1s}, b_{2s}, \cdots, b_{ns}, \ell\}}.$$

Similar to the argument in (3) ⇔ (7), we can assume that $\Bbbk$ is algebraically closed. Again by Lemma 0.2, $|O| = \operatorname{rk}(S_Z) = \prod_{i=1}^n \mathfrak{f}_i$. Since $x_i^{\mathfrak{f}_i} \in Z$, $p_{is}^{\mathfrak{f}_i} = 1$ after applying ϕ_s to $x_i^{\mathfrak{f}_i}$. Equivalently, $b_{is}\mathfrak{f}_i$ is divisible by ℓ. Then

$$\gcd\{b_{i1}, \cdots, b_{in}, \ell\}\mathfrak{f}_i = \gcd\{b_{i1}\mathfrak{f}_i, \cdots, b_{in}\mathfrak{f}_i, \ell\mathfrak{f}_i\} = \ell a$$

for some integer a. Therefore, $\mathfrak{f}_i \ge \frac{\ell}{\gcd\{b_{i1}, \cdots, b_{in}, \ell\}}$. Thus,

$$|O| \ge \frac{\ell^n}{\prod_{i=1}^n \gcd\{b_{i1}, \cdots, b_{in}, \ell\}}.$$

This proves equality. In fact, we also obtain that

$$\mathfrak{f}_i = \frac{\ell}{\gcd\{b_{i1}, \cdots, b_{in}, \ell\}}. \tag{E4.1.1}$$

(6) ⇔ (8): Since the order of ϕ_i is $\frac{\ell}{\gcd\{b_{i1},\cdots,b_{in},\ell\}}$, (6) implies that the surjective map $O \to \prod_{i=1}^{n} \langle \phi_i \rangle$ is bijective. The converse is similar.

(6) ⇒ (2): Let $w_i = \frac{\ell}{\gcd\{b_{i1},\cdots,b_{in},\ell\}}$. Then $w_i b_{is}$ is divisible by ℓ. This implies that $x_i^{w_i} \in Z$ for all i. Let $C = \Bbbk[x_1^{w_1}, \cdots, x_n^{w_n}]$. So $\operatorname{rk}(S_C) = \frac{\ell^n}{\prod_{i=1}^{n} \gcd\{b_{i1},\cdots,b_{in},\ell\}}$, which is equal to $|O|$. Since $\operatorname{rk}(S_Z) = |O|$, we obtain that $\operatorname{rk}(Z_C) = 1$. Since Z is Cohen–Macaulay [16, Lemma 3.1], then $C = Z$.

(9) ⇔ (3): This is Lemma 4.1(2).

When Z is regular, it is Gorenstein. By Theorem 2.7, $\mathfrak{o}\mathfrak{j}_S$ is equal to $\mathfrak{j}_{S,O}$. By a decomposition like (E2.7.1), one sees that $\mathfrak{o}\mathfrak{a}_S$ is equal to $\mathfrak{a}_{S,O}$. ∎

Remark 4.2. We now describe an idea (or rather an algorithm) for testing regularity and Gorensteinness of Z that works for any n. We divide it into several steps.

(1) Recall that if M is any matrix over a PID, then there exists a diagonal matrix D and invertible matrices L, R such that $D = LMR$. The matrix D is the *Smith normal form* of M. We will apply this to the matrix B.

(2) By definition, K is the preimage of $\overline{K}$ where $\overline{K}$ is the kernel of the map $L_{\overline{B}} : \overline{\mathbb{Z}}^n \to \overline{\mathbb{Z}}^n$ (as a left multiplication by the matrix $\overline{B}$). We also use $\overline{B}$ to denote this (right) $\mathbb{Z}$-module endomorphism $L_{\overline{B}}$ if no confusion occurs. By definition there is a short exact sequence

$$0 \to \ell(\mathbb{Z}^n) \to K \to \overline{K} \to 0.$$

As a consequence, K contains $\ell \mathbf{e}_i$ for each i. We may consider $\{\ell \mathbf{e}_i\}_{i=1}^n$ as a subset of a generating set of K. If necessary, we also view $\overline{B}$ as the composition map $\mathbb{Z}^n \to \overline{\mathbb{Z}}^n \xrightarrow{L_{\overline{B}}} \overline{\mathbb{Z}}^n$. In this setting, K is the kernel of $\overline{B}$.

(3) Since $\mathbb{Z}$ is integrally closed and K is finitely generated, the conditions in Lemma 4.1(2) can be checked locally (at prime p for all $p \mid \ell$), and to do this it is convenient to have a generating set for $K_{(p)} := K \otimes \mathbb{Z}_{(p)}$. To this end, we compute the kernel of the map $\overline{B}_{(p)} := \overline{B} \otimes \mathbb{Z}_{(p)}$ for each prime $p \mid \ell$, in other words, we produce a generating set for $K_{(p)}$, which can be glued together to get a generating set for K.

(4) Recall that for an integer m and a prime p, $\nu_p(m)$ denotes the maximal integer a such that $p^a \mid m$. Recall from Hypothesis 0.3 that $B = (b_{ij})_{i,j}$ is an $n \times n$

skew symmetric matrix over $\mathbb{Z}$. We further introduce some notations: fix a prime p dividing ℓ and let

$$N = \nu_p(\ell), \quad \alpha_{ij} = \min\{N, \nu_p(b_{ij})\}, \quad \text{and} \quad \alpha = \min\{N, \nu_p(\mathrm{pf}(B))\}. \qquad \text{(E4.2.1)}$$

(5) For each $p \mid \ell, \overline{\mathbb{Z}} \otimes \mathbb{Z}_{(p)} \cong \mathbb{Z}/(p^N)$ and $K_{(p)}$ is the kernel of the left multiplication map

$$\overline{B}_{(p)} : \mathbb{Z}^n_{(p)} \to (\mathbb{Z}/(p^N))^n.$$

(6) By part (2), for each $p \mid \ell, p^N \mathbf{e}_i \in K_{(p)}$. So we are interested in other generators in $K_{(p)}$. In other words, we are interested in generators of $\overline{K}_{(p)}$.

(7) Given the Smith normal form $D = LBR$ for B, the equation $\overline{B}_{(p)}\mathbf{u} = 0$ is equivalent to $B\mathbf{u} \equiv 0 \pmod{p^N}$ and is equivalent to $D\mathbf{v} \equiv 0 \pmod{p^N}$ where $\mathbf{v} = R^{-1}\mathbf{u}$. Hence, to compute the kernel of $\overline{B}_{(p)}$, we compute the kernel of $D \bmod p^N$ (or equivalently, the kernel of the map $\overline{D}_{(p)}$) and apply R to the generators to obtain a set of generators $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ for the kernel of $\overline{B}_{(p)}$. If we let $\mathfrak{f}_{p,j} = \gcd_{1 \le i \le r}((\mathbf{u}_i)_j)$ (computed in $\mathbb{Z}_{(p)}$), then $\mathfrak{f}_{p,j}$ is of the form $p^a u$ where a is a nonnegative integer and u is a unit in $\mathbb{Z}_{(p)}$. So we can write $\mathfrak{f}_{p,j} = p^a$, which is called the *standard form* of $\mathfrak{f}_{p,j}$. Using the standard forms, one sees that $\mathfrak{f}_j$ is the lcm of the $\mathfrak{f}_{p,j}$ as p runs over the prime divisors of ℓ. Since $\mathfrak{f}_{p,j} = \mathfrak{f}_j$ in $\mathbb{Z}_{(p)}$, we will also use $\mathfrak{f}_j$ for $\mathfrak{f}_{p,j}$ in the middle of the proofs.

(8) By Lemma 4.1 (resp. Lemma 5.1 in the next section), to determine if the center Z is regular (resp. Gorenstein), it is enough check whether $\mathfrak{f}_i\mathbf{e}_i \in K$ (resp. $(\mathfrak{f}_1, \ldots, \mathfrak{f}_n)^T \in K$).

(9) For the argument below, we will fix a prime divisor p of ℓ. Throughout the rest of this section, we will use the convention introduced in this remark.

Using the "algorithm" discussed above, we give explicit conditions equivalent to the regularity of Z in terms of the parameters b_{ij} in the cases $n = 3$ and $n = 4$.

Proposition 4.3. Assume Hypothesis 0.3 and retain the convention introduced in (E4.2.1). Assume, without loss of generality, that $\alpha_{12} = \min\{\alpha_{ij}\} = 0$.

(1) If $n = 3$, then $K_{(p)}$ is generated as a $\mathbb{Z}_{(p)}$-module by $p^N\mathbf{e}_i$ for $i = 1, 2$ and

$$\frac{1}{b_{12}} \begin{bmatrix} b_{23} \\ -b_{13} \\ b_{12} \end{bmatrix}.$$

(2) If $n = 4$, then $K_{(p)}$ is generated as a $\mathbb{Z}_{(p)}$-module by $p^N \mathbf{e}_i$ for $i = 1, 2$ and

$$p^{N-\alpha}\begin{bmatrix} b_{23}/b_{12} \\ -b_{13}/b_{12} \\ 1 \\ 0 \end{bmatrix}, \quad p^{N-\alpha}\begin{bmatrix} b_{24}/b_{12} \\ -b_{14}/b_{12} \\ 0 \\ 1 \end{bmatrix}.$$

Proof. (1) Let $n = 3$. The Smith normal form $D = LBR$ of B over the ring $\mathbb{Z}_{(p)}$ is

$$D = \begin{bmatrix} b_{12} & 0 & 0 \\ 0 & -b_{12} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b_{23}/b_{12} & -b_{13}/b_{12} & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 1 & b_{23}/b_{12} \\ 1 & 0 & -b_{13}/b_{12} \\ 0 & 0 & 1 \end{bmatrix}.$$

Applying the argument in Remark 4.2(7), the kernel of $D_{(p)}$ is generated, as a $\mathbb{Z}_{(p)}$-module, by $p^N\mathbf{e}_1, p^N\mathbf{e}_2, \mathbf{e}_3$. Applying R to these gives the stated generators.

(2) Let $n = 4$. The Smith normal form of B is given by

$$D = \begin{bmatrix} b_{12} & 0 & 0 & 0 \\ 0 & -b_{12} & 0 & 0 \\ 0 & 0 & \frac{1}{b_{12}}\operatorname{pf}(B) & 0 \\ 0 & 0 & 0 & -\frac{1}{b_{12}}\operatorname{pf}(B) \end{bmatrix},$$

$$L = \frac{1}{b_{12}}\begin{bmatrix} b_{12} & 0 & 0 & 0 \\ 0 & b_{12} & 0 & 0 \\ b_{23} & -b_{13} & b_{12} & 0 \\ b_{24} & -b_{14} & 0 & b_{12} \end{bmatrix},$$

$$R = \frac{1}{b_{12}}\begin{bmatrix} 0 & b_{12} & b_{24} & b_{23} \\ b_{12} & 0 & -b_{14} & -b_{13} \\ 0 & 0 & 0 & b_{12} \\ 0 & 0 & b_{12} & 0 \end{bmatrix}.$$

Using similar reasoning as above, the kernel of $D_{(p)}$ is generated by $p^N\mathbf{e}_1, p^N\mathbf{e}_2, p^{N-\alpha}\mathbf{e}_3$, $p^{N-\alpha}\mathbf{e}_4$. Again, applying R to these gives the stated generators. ■

We will write $\mathfrak{f}_i\mathbf{e}_i$ for $\mathfrak{f}_i\mathbf{e}_i \otimes 1 \in K_{(p)} := K \otimes \mathbb{Z}_{(p)}$ if no confusion occurs.

Proposition 4.4. Keep the assumptions of Proposition 4.3. If $n = 3$, then $\mathfrak{f}_i\mathbf{e}_i \in K_{(p)}$ for all $i = 1, 2, 3$ if and only if $\alpha_{23} = \alpha_{13} = N$.

Proof. Using the generators of $K_{(p)}$ from Proposition 4.3 and the definition of $\mathfrak{f}_i$ (cf. (E0.3.1)), we have, up to units in $\mathbb{Z}_{(p)}$,

$$\mathfrak{f}_1 = b_{23}/b_{12}, \qquad \mathfrak{f}_2 = b_{13}/b_{12}, \qquad \mathfrak{f}_3 = 1. \tag{E4.4.1}$$

(Strictly speaking these should be $\mathfrak{f}_{p,1}, \mathfrak{f}_{p,2}$, and $\mathfrak{f}_{p,3}$, respectively.) Suppose $\mathfrak{f}_i \mathbf{e}_i \in K_{(p)}$ for all i. In particular, for $i = 3$, there exists $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}_{(p)}$ such that

$$\begin{bmatrix} \lambda_1 p^N \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \lambda_2 p^N \\ 0 \end{bmatrix} + \begin{bmatrix} \lambda_3 b_{23}/b_{12} \\ -\lambda_3 b_{13}/b_{12} \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence, $\lambda_3 = 1$. The first component gives $\lambda_1 = -b_{23} p^{-N}/b_{12}$, so $\nu_p(\lambda_1) = \alpha_{23} - N$ and a necessary and sufficient condition for $\lambda_1 \in \mathbb{Z}_{(p)}$ is $\alpha_{23} = N$. Similarly, we get $\alpha_{13} = N$, and this proves the forward implication. The converse is clear. ■

Proposition 4.5. Keep the assumptions of Proposition 4.3. If $n = 4$, then $\mathfrak{f}_i \mathbf{e}_i \in K_{(p)}$ for all $i = 1, \ldots, 4$ if and only if

$$\alpha_{ij} \geq \alpha \tag{E4.5.1}$$

for $(i,j) = (1,3), (1,4), (2,3), (2,4)$.

Proof. Using the generators of $K \otimes \mathbb{Z}_{(p)}$ from Proposition 4.3 and the definition of $\mathfrak{f}_i$, we have

$$\begin{aligned} \mathfrak{f}_1 &= \gcd\left(p^N, p^{N-\alpha}\frac{b_{23}}{b_{12}}, p^{N-\alpha}\frac{b_{24}}{b_{12}}\right) = \gcd\left(p^{N-\alpha}\frac{b_{23}}{b_{12}}, p^{N-\alpha}\frac{b_{24}}{b_{12}}\right) \\ \mathfrak{f}_2 &= \gcd\left(p^N, p^{N-\alpha}\frac{b_{13}}{b_{12}}, p^{N-\alpha}\frac{b_{14}}{b_{12}}\right) = \gcd\left(p^{N-\alpha}\frac{b_{13}}{b_{12}}, p^{N-\alpha}\frac{b_{14}}{b_{12}}\right) \\ \mathfrak{f}_3 &= \mathfrak{f}_4 = p^{N-\alpha}. \end{aligned} \tag{E4.5.2}$$

Suppose $\mathfrak{f}_i \mathbf{e}_i \in K_{(p)}$ for all i. In particular, for $i = 3$, there exists $\lambda_1, \ldots, \lambda_4 \in \mathbb{Z}_{(p)}$ such that

$$\begin{bmatrix} \lambda_1 p^N \\ \lambda_2 p^N \\ 0 \\ 0 \end{bmatrix} + p^{N-\alpha} \begin{bmatrix} \lambda_3 b_{23}/b_{12} \\ -\lambda_3 b_{13}/b_{12} \\ \lambda_3 \\ 0 \end{bmatrix} + p^{N-\alpha} \begin{bmatrix} \lambda_4 b_{24}/b_{12} \\ -\lambda_4 b_{14}/b_{12} \\ 0 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ p^{N-\alpha} \\ 0 \end{bmatrix}.$$

Hence, $\lambda_4 = 0$ and $\lambda_3 = 1$. The first component gives $\lambda_1 = -p^{-\alpha} b_{23}/b_{12}$, so $\nu_p(\lambda_1) = -\alpha + \alpha_{23}$. So $\lambda_1 \in \mathbb{Z}_{(p)}$ if and only if $\alpha_{23} \geq \alpha$. The same argument on the second component gives $\alpha_{13} \geq \alpha$. We can perform the same calculations for $i = 4$ to obtain $\alpha_{14} \geq \alpha$ and $\alpha_{24} \geq \alpha$.

Conversely, assume the inequalities (E4.5.1) hold. Then $N \leq N - \alpha + \alpha_{ij}$ so $\mathfrak{f}_1 = \mathfrak{f}_2 = p^N$. Hence, $\mathfrak{f}_i \mathbf{e}_i \in K_{(p)}$ for $i = 1, 2$. Next, we take the third generator of $K_{(p)}$ (cf. Proposition 4.3) and subtract from it multiples of the first two

$$\mathfrak{f}_3 \mathbf{e}_3 = p^{N-\alpha} \begin{bmatrix} b_{23}/b_{12} \\ -b_{13}/b_{12} \\ 1 \\ 0 \end{bmatrix} - \beta_1 p^N \mathbf{e}_1 + \beta_2 p^N \mathbf{e}_2,$$

where $\beta_1 = p^{-\alpha} b_{23}/b_{12}$ and $\beta_2 = p^{-\alpha} b_{13}/b_{12}$. The inequalities (E4.5.1) can be rearranged so that $\alpha_{ij} - \alpha \geq 0$. The left-hand side of this inequality is $\nu_p(\beta_k)$ for appropriate i, j, k. Hence $\beta_1, \beta_2 \in \mathbb{Z}_{(p)}$, so $\mathfrak{f}_3 \mathbf{e}_3 \in K_{(p)}$. Similar computations with the fourth generator of K yield $\mathfrak{f}_4 \mathbf{e}_4 \in K_{(p)}$. This completes the proof. ■

Next, we globalize the above results that are local at p. First, we need a technical lemma.

Lemma 4.6. Let $r_1, \ldots, r_k$ be roots of unity with orders $o_1, \ldots, o_k$. Let $s = o_1 \cdots o_k$ and ζ be a primitive s-th root of unity. Then the orders are pairwise coprime if and only if there exist integers $n_1, \ldots, n_k$ such that $\gcd(n_i, s) = 1$ and

$$r_i = \zeta^{n_i o_1 \cdots \hat{o}_i \cdots o_k}$$

Proof. We prove the forward direction only, since the reverse implication is immediate. Since r_i has order o_i, it is a primitive o_i-th root of unity. Now $\zeta^{o_1 \cdots \hat{o}_i \cdots o_k}$ is also a primitive o_i-th root of unity, so there exists an integer m_i, such that m_i and o_i are coprime, and

$$r_i = \zeta^{m_i s_i},$$

where $s_i = o_1 \cdots \hat{o}_i \cdots o_k$. The integers m_i and s may not be coprime, but we can choose an integer j such that $n_i := m_i + j o_i$ and s are coprime. Firstly, since o_i and s_i are coprime, we have $\{m_i + j o_i\}_{j=0}^{s_i} = \mathbb{Z}/s_i\mathbb{Z}$ as sets. Hence, there exists an integer j such that n_i is coprime to s_i. But m_i and o_i are assumed to be coprime, so n_i and o_i are coprime, which means n_i and s are coprime. Finally,

$$\zeta^{n_i s_i} = \zeta^{(m_i + j o_i) s_i} = \zeta^{m_i s_i + j s} = \zeta^{m_i s_i} = r_i,$$

and we are done. ■

Corollary 4.7. Assume Hypothesis 0.3 and let $n = 3$. Then, Z is regular if and only if the orders of p_{12}, p_{13} and p_{23} are pairwise coprime. Equivalently, there exist pairwise coprime integers $a, b, c \geq 1$ such that $p_{12} = \xi^{ab}$, $p_{13} = \xi^{acn_0}$ and $p_{23} = \xi^{bcm_0}$ where ξ is a primitive ℓth root of unity with $\ell = abc$ and n_0, m_0 are coprime to ℓ.

Proof. For each prime p dividing ℓ, we have $\nu_p(o(p_{ij})) = N - \alpha_{ij}$. The orders of $\{p_{ij}\}_{i<j}$ being pairwise coprime is equivalent to at least two of $\{\alpha_{ij}\}_{i<j}$ being equal to N. By Proposition 4.4 and Lemma 4.1, this occurs if and only if Z regular. The last statement follows from Lemma 4.6. ■

Corollary 4.8. Assume Hypothesis 0.3 and let $n = 4$. Let $\rho = \gcd(\ell, \mathrm{pf}(B))$, $c_{ij} = \gcd(b_{ij}, \rho)$, ω be a primitive ρth root of unity, and $q_{ij} = \omega^{c_{ij}}$. Then Z is regular if and only if the orders of $\{q_{ij}\}_{i<j}$ are pairwise coprime. Equivalently, there exist pairwise coprime integers $o_{ij} \geq 1$ such that

$$q_{ij} = \omega^{n_{ij}\hat{o}_{ij}},$$

where $(n_{ij}, \rho) = 1$ and $\prod_{i<j} o_{ij} = \rho$ and $\hat{o}_{ij} = \rho / o_{ij}$.

Proof. We first show that the conditions on α_{ij} in Proposition 4.5 imply that $\alpha_{34} \geq \alpha$ as well. Rearranging the equation for the Pfaffian gives

$$b_{12}b_{34} = \mathrm{pf}(B) + b_{13}b_{24} - b_{14}b_{23}.$$

Taking p-valuations and noting that $\alpha_{12} = 0$ gives the inequality.

Let p be a prime factor of ρ. Then $\nu_p(o(q_{ij})) = \nu_p(\rho) - \nu_p(c_{ij})$. We also have $\nu_p(\rho) = \alpha$ and $\nu_p(c_{ij}) = \min\{\alpha_{ij}, \alpha\}$. Proposition 4.5 and Lemma 4.1 say that Z is regular if and only if $\alpha_{ij} \geq \alpha$ for $(i,j) \neq (1,2)$. These conditions imply $\nu_p(o(q_{ij})) = 0$ except $(i,j) = (1,2)$. Hence, $o(q_{ij})$ are pairwise coprime.

Conversely, we get that $\nu_p(o(q_{ij})) = 0$ for all $(i,j) \neq (1,2)$. This means $\min\{\alpha_{ij}, \alpha\} = \alpha$ implying $\alpha_{ij} \geq \alpha$ for such (i,j).

The last statement follows from Lemma 4.6. ■

We finish this section with the proofs of Theorems 0.7 and 0.9.

Proof of Theorem 0.7. In the $n = 2$ case, $Z = \Bbbk[x_1^\ell, x_2^\ell]$ so the result is clear. The result in the $n = 3$ case follows from Corollary 4.7, while the result in the $n = 4$ case is due to Corollary 4.8. ■

Proof of Theorem 0.9. If the orders of $\{p_{ij}\}_{i<j}$ are pairwise coprime, then for any prime $p \mid \ell$, the matrix $\overline{B}_{(p)}$ has rank 2. In particular, $\overline{B}_{(p)}$ is similar to the block diagonal matrix formed from

$$\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$$

and a zero matrix of the appropriate dimension. Hence, $K_{(p)}$ is generated by $p^{N-\nu_p(b)}\mathbf{e}_1$, $p^{N-\nu_p(b)}\mathbf{e}_2$ and $\mathbf{e}_3, \dots, \mathbf{e}_n$. This shows that $\nu_p(\mathfrak{f}_i) = N - \nu_p(b)$ for $i = 1, 2$ and $\nu_p(\mathfrak{f}_j) = 0$ for $j = 3, \dots, n$. The result follows from Lemma 4.1(2).

Part (1) of the consequence follows from (E4.1.1). By definition, $\ell = \prod_{i<j} o(p_{ij})$. Then part (2) of the consequence follows from part (1) and Theorem 0.6(7). ■

5 Gorenstein Center

Throughout this section, we assume Hypothesis 0.3. Here we study the question of when the center of S is Gorenstein.

Proof of Theorem 0.11. (1) ⇔ (2): This is [19, Theorem 0.2].

(2) ⇔ (3): Let $\overline{\phi_i}$ denote the image of ϕ_i in O/H. Since $S^H = \Bbbk[x_1^{\mathfrak{f}_1}, \cdots, x_n^{\mathfrak{f}_n}]$ (see the proof of Proposition 2.6) and $\overline{\phi_i} \in O/H$ acts on S^H as a diagonal map for each i, (E1.4.2) implies that

$$\operatorname{hdet}_{O/H}(\overline{\phi_i}) = \overline{\phi_i}(x_1^{\mathfrak{f}_1} \cdots x_n^{\mathfrak{f}_n})(x_1^{\mathfrak{f}_1} \cdots x_n^{\mathfrak{f}_n})^{-1} = \prod_{s=1}^{n} p_{is}^{\mathfrak{f}_s}.$$

Since $\mathfrak{oj}_S\mathfrak{pg}_S = x_1^{\mathfrak{f}_1} \cdots x_n^{\mathfrak{f}_n}$, we see that (2) ⇔ (3).

(3) ⇔ (4): This is clear as an element $f \in Z$ if and only if $\phi_i(f) = f$ for all i. ■

Next we prove Theorem 0.10.

Proof of Theorem 0.10. (2) ⇔ (3): This is Lemma 1.5(2).

(1) ⇔ (3): The Nakayama automorphism μ of S follows from [24, Proposition 4.1], and it is easy to show that (3) holds if and only if $\mu = \mathrm{id}$. Equivalently, S is Calabi–Yau.

(3) ⇔ (4): This follows by an easy computation.

Suppose that O-action has trivial homological determinant. By [31, Theorem 1.21], Auslander's Theorem holds for (S, O). Hence, all statements in Theorem 0.4 hold. Since Auslander's Theorem holds for (S, O), H is trivial. Then Theorem 0.11(2) holds. Hence, all statements in Theorem 0.11 hold. ■

Theorem 0.11(3) gives us necessary and sufficient conditions for $Z = S^O$ to be Gorenstein. It is more convenient for us to express the condition in terms of the matrix B.

Lemma 5.1 (≡Theorem 0.11(3)). Assume Hypothesis 0.3. The center Z of S is Gorenstein if and only if the following equation holds in $\overline{\mathbb{Z}}^n$:

$$\overline{B}(\mathfrak{f}_1, \ldots, \mathfrak{f}_n)^T = 0.$$

We already computed the $\{\mathfrak{f}_i\}$ for $n = 3, 4$ in the previous section, so we are ready to prove Theorem 0.13.

Proof of Theorem 0.13. (1) The result for $n = 2$ is well-known.

(2) The result for $n = 3$ is immediate from Lemma 2.8.

(3) Let $\mathbf{v} = \ell / \gcd(\mathrm{pf}(B), \ell)(v_1, \ldots, v_4)^T$. The image of this vector in $(\mathbb{Z}_{(p)})^4$ is a unit multiple of

$$p^{N-\alpha}\begin{bmatrix} \gcd(b_{23}, b_{24}, b_{34}) \\ \gcd(b_{13}, b_{14}, b_{34}) \\ \gcd(b_{12}, b_{14}, b_{24}) \\ \gcd(b_{12}, b_{13}, b_{23}) \end{bmatrix}.$$

After relabelling if necessary, we can assume b_{12} is a unit. The equation

$$b_{12}b_{34} = \mathrm{pf}(B) + b_{13}b_{24} - b_{14}b_{23}$$

shows that $\alpha_{34} \geq \alpha$, which means we can drop b_{34} from the arguments in the first two gcds above. Comparing with (E4.5.2) shows that the ith component of $\mathbf{v}$ has the same p-valuation as $\mathfrak{f}_i$. This holds for all prime factors p of ℓ so we are done. ■

We end this section by discussing some examples and showcase the subtleties in the results above. We remark that for a commutative ring, the following set of implications hold:

regular $\Rightarrow$ hypersurface ring $\Rightarrow$ complete intersection $\Rightarrow$ Gorenstein.

In this paper, we have focused on the regular and Gorenstein properties of the center of $S_{\mathbf{p}}$, but it would be interesting to determine conditions equivalent to the center of $S_{\mathbf{p}}$ being a hypersurface ring or a complete intersection. Recall that Z is a (commutative) hypersurface ring if $Z \cong \Bbbk[x_1, \ldots, x_n]/(f)$ for some homogeneous polynomial f. In this case, the Hilbert series satisfies $h_Z(t) = p(t)/q(t)$ where $p(t)$ is a cyclotomic polynomial.

Example 5.2. Set $n = 3$.

(1) By Theorem 0.10, if S is Calabi-Yau, then

$$Z = \Bbbk[x_1^\ell, x_2^\ell, x_3^\ell, x_1x_2x_3]/(x_1^\ell x_2^\ell x_3^\ell - (x_1x_2x_3)^\ell)$$

is a hypersurface ring.

(2) Let $\ell > 1$ and consider the following B matrix:

$$B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

By Lemma 2.8 and the subsequent discussion, $\mathfrak{f} = (1, 1, \ell)$ and $g = (\ell, \ell, \ell)$. It follows that Z is generated by $\{x^\ell, y^\ell, z^\ell, xy\}$ and so

$$Z \cong \Bbbk[y_1, y_2, y_3, y_4]/(y_4^\ell - y_1y_2).$$

Thus, Z is a hypersurface ring that is not Calabi–Yau.

(3) Let $\ell = 24$ and consider the following matrix:

$$B_k = \begin{pmatrix} 0 & 4 & 6 \\ -4 & 0 & k \\ -6 & -k & 0 \end{pmatrix},$$

where $k = 3$ or 9. In either case, Lemma 2.8 gives $\mathfrak{f} = (3, 6, 4)$ and $g = (12, 24, 8)$. By Lemma 5.1, S is Gorenstein when $k = 9$ and non-Gorenstein when $k = 3$. Suppose $k = 9$, then Z is generated by x_1^{12}, x_2^{24}, x_3^8, and $x_1^3x_2^6x_3^4$. One checks that the Hilbert series of Z is

$$h_Z(t) = \frac{p(t)}{q(t)} = \frac{1 + t^{13} + t^{18} + t^{31}}{(1 - t^{12})(1 - t^{24})(1 - t^8)}.$$

It is clear that $p(t)$ is not cyclotomic, and thus Z is not a hypersurface ring.

6 Questions and Comments

In this section, we list some questions and comments related to the theorems given in this paper. Much of this paper has been devoted to the study of the algebras $S_{\mathbf{p}}$. In addition to classifying those $S_{\mathbf{p}}$ such that $ZS_{\mathbf{p}}$ is regular or Gorenstein for $n \geq 5$, one could consider the following problems.

Question 6.1.

(1) What conditions on the PI skew polynomial ring $S_{\mathbf{p}}$ are equivalent to the center $ZS_{\mathbf{p}}$ being a hypersurface ring or a complete intersection ring?

(2) Let $S_{\mathbf{p}}$ be the skew polynomial ring in n variables. By Remark 3.7, $0 < \mathsf{p}(S_{\mathbf{p}}, O) < n$. For each $0 < i < n$, can we find some skew polynomial ring S such that $\mathsf{p}(S, O) = i$?

(3) (Ken Goodearl) The matrix B controls the PI degree of S [12]. Is there a direct computation of PI degree from the parameters b_{ij}? Is the PI degree related to the other invariants in this paper?

(4) (Colin Ingalls) In cases that the center $ZS_{\mathbf{p}}$ is not Gorenstein, under what conditions on $S_{\mathbf{p}}$ is it $\mathbb{Q}$-Gorenstein (see [8])?

(5) The classification of skew polynomial rings up to isomorphism is known (see [13]). Two skew polynomial rings are *birationally equivalent* if their associated quotient division rings are isomorphic. What is the classification of skew polynomial rings up to birational equivalence?

As we have mentioned, the algebras $S_{\mathbf{p}}$ are viewed as a good testing ground for many problems in noncommutative invariant theory. It would be interesting to study the problems in this paper more generally.

Question 6.2. Let A be a noetherian PI AS regular algebra.

(1) Is there is a version of Corollary 0.12(1) or (2) for A?

(2) Does (1) $\Leftrightarrow$ (3) in Theorem 0.10 hold for A? Namely, is it true that the homological determinant of $\mathrm{Oz}(A)$ is trivial if and only if A is Calabi–Yau?

(3) Can we define invariants $\mathfrak{oj}_A$, $\mathfrak{oa}_A$, and $\mathfrak{od}_A$ such that they generalize $\mathfrak{oj}_S$, $\mathfrak{oa}_S$, and $\mathfrak{od}_S$ and control properties of A and its center?

(4) For the skew polynomial ring $S_{\mathbf{p}}$, the ozone group O acts on $S_{\mathbf{p}}$ such that $ZS_{\mathbf{p}} = S_{\mathbf{p}}^O$. Is there a semisimple Hopf algebra H acting on A such that $Z(A) = A^H$?

(5) Suppose A is generated in degree 1 and suppose that the center of A is $\Bbbk[c_1, \cdots, c_n]$ where $\deg c_i > 1$ for every i. Does it hold that $\mathrm{Aut}(A)$ is affine?

(5) Related to this is the notion of LND-*rigidity*. Let $\mathrm{LND}(A)$ denote the intersection of all kernels of locally nilpotent derivations of A. Under the above hypotheses, does it hold that $\mathrm{LND}(A) = \{0\}$. We note that both of these hold when A is a skew polynomial ring.

(6) Is there a unique mozone subring of A? That is, does $\Phi_Z(A)$ have a minimum element?

Question 6.2(4) would be especially interesting in the case when A is a (PI) Sklyanin algebra.

One source of interesting examples that may be useful for studying some of the above questions are graded noetherian down-up algebras.

Let $\alpha, \beta \in \Bbbk$ with $\beta \neq 0$. A noetherian graded down-up algebra $A = A(\alpha, \beta)$ is generated as an algebra by x and y and subject to the relations

$$x^2y - \alpha xyx - \beta yx^2 = 0 = xy^2 - \alpha yxy - \beta y^2x.$$

It is well-known that A is AS regular, but is not isomorphic to any $S_{\mathbf{p}}$. The group of graded algebra automorphisms of A was computed by Kirkman and Kuzmanovich [17, Proposition 1.1]. If $\beta \neq \pm 1$, then every graded algebra automorphism of A is diagonal. By [25, (1.5.6)], the Nakayama automorphism of A is determined by

$$\mu : x \mapsto -\beta x, \quad y \mapsto -\beta^{-1} y.$$

Let ω_1 and ω_2 be the roots of the characteristic equation

$$w^2 - \alpha w - \beta = 0$$

and let $\Omega_i = xy - \omega_i yx$ for $i = 1, 2$. It is easy to see that for $\{i, j\} = \{1, 2\}$ we have

$$x\Omega_i = \omega_j \Omega_i x, \quad \omega_j y \Omega_i = \Omega_i y.$$

Note that A is PI if and only if both ω_1 and ω_2 are roots of unity.

Let

$$\mathfrak{b} := \Omega_1 \Omega_2.$$

If $\sigma \in \mathrm{Aut}_{\mathrm{gr}}(A)$ is diagonal, then $\mathrm{hdet}(\sigma) = \left(\det \sigma|_{A_1}\right)^2$ [17, Theorem 1.5]. Using this fact, one can easily check that

$$\sigma(\mathfrak{b}) = \mathrm{hdet}(\sigma)\mathfrak{b}. \tag{E6.2.1}$$

This equation should be compared with (E1.4.2).

Proposition 6.3. Let $A = A(\alpha, \beta)$ be a PI noetherian graded down-up algebra. Then A is Calabi–Yau if and only if $\mathfrak{b} \in Z(A)$.

Proof. Recall that A is Calabi–Yau if and only if the Nakayama automorphism μ of A is the identity, if and only if $\beta = -1$, if and only if $\omega_1\omega_2 = 1$, if and only if $\Omega_1\Omega_2 \in Z(A)$ by computation. ■

It is easy to show, using these generating sets for $Z(A)$ given in [23, 30], that $Z(A)$ is regular when $(\alpha,\beta) = (0,1)$. We conjecture that this is the only case that $Z(A)$ is regular.

Question 6.4. Suppose $A = A(\alpha,\beta)$ be a PI noetherian graded down up algebra. For what parameters (α,β) is $Z(A)$ regular (resp. Gorenstein)?

The following should be compared with Theorem 0.11.

Proposition 6.5. Let H be a finite subgroup of $\mathrm{Aut}_{\mathrm{gr}}(A)$ consisting of diagonal automorphisms. The following are equivalent.

(1) The H-action on A has trivial homological determinant.
(2) A^H is Gorenstein.
(3) $\mathfrak{b} \in A^H$.

Proof. (1) ⇔ (2): By [18, Proposition 6.4], H does not contain any reflections. By [19, Corollary 4.11], A^H is Gorenstein if and only if hdet_H is trivial.

(1) ⇔ (3): By (E6.2.1), hdet_H is trivial if and only if $g(\mathfrak{b}) = \mathfrak{b}$ for all $g \in H$, if and only if $\mathfrak{b} \in A^H$. ■

The following should be compared with Theorem 0.11.

Corollary 6.6. The following are equivalent.

(1) A^O is Gorenstein where $O = \mathrm{Oz}(A)$.
(2) $\mathfrak{b} \in A^O$.

Proof. By [9], the ozone group O consists of diagonal automorphisms. The assertion follows from Proposition 6.5 by setting $H = O$. ■

Question 6.7. Since $Z \subseteq A^O$, then clearly $\mathfrak{b} \in Z$ implies that $\mathfrak{b} \in A^O$. Does the opposite implication hold?

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