WEIGHTED HOMOLOGICAL REGULARITIES

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ABSTRACT. Let A be a noetherian connected graded algebra. We introduce and study homological invariants that are weighted sums of the homological and internal degrees of cochain complexes of graded A-modules, providing weighted versions of Castelnuovo–Mumford regularity, Tor-regularity, Artin–Schelter regularity, and concavity. In some cases an invariant (such as Tor-regularity) that is infinite can be replaced with a weighted invariant that is finite, and several homological invariants of complexes can be expressed as weighted homological regularities. We prove a few weighted homological identities some of which unify different classical homological identities and produce interesting new ones.

INTRODUCTION

Let $\Bbbk$ be a base field and let A be a connected graded $\Bbbk$-algebra. If X is a complex of graded left A-modules, then there are two natural gradings on X, namely, the gradings by homological and internal degrees. Properties of A can be reflected in the relationships between these degrees. For example, A is Koszul if the trivial graded A-module $\Bbbk$ has a minimal free resolution of the form

(E0.0.1) $\qquad \cdots \to A(-i)^{\beta_i} \to A(-i+1)^{\beta_{i-1}} \to \cdots \to A(-1)^{\beta_1} \to A \to \Bbbk \to 0,$

or equivalently, $\operatorname{Tor}_i^A(\Bbbk, \Bbbk)_j = 0$ for all $j \neq i$. In this case we say that the trivial A-module $\Bbbk$ has a *linear resolution*, or that the *Tor-regularity of* $\Bbbk$ is 0. The Tor-regularity of a complex X is defined to be

(E0.0.2) $\qquad \operatorname{Torreg}(X) = \sup_{i,j \in \mathbb{Z}} \{j - i \mid \operatorname{Tor}_i^A(\Bbbk, X)_j \neq 0\}.$

This supremum of a particular linear combination of internal and homological degrees provides a measure of the growth of the degrees of generators of the free modules in a minimal free resolution of X.

When A is a noetherian commutative graded algebra generated in degree one, the Tor-regularity of the trivial module is either zero or infinity [AP01], so the Tor-regularity measures only whether A is Koszul or not. Jørgensen and Dong–Wu [Jør99, Jør04, DW09] studied the Tor- and Ext-regularities for noncommutative algebras, and further results on these regularities were the subject of [KWZ21]. As shown in [KWZ21, Example 2.4(4)], for any non-negative integer n, there is a noncommutative algebra whose trivial graded module $\Bbbk$ has Tor-regularity n, and

so the Tor-regularity of the trivial graded module of a noncommutative algebra provides more information than whether or not the algebra is Koszul.

Castelnuovo–Mumford regularity (CM regularity for short) was introduced for commutative graded algebras as the supremum of another linear combination of homological and internal degrees (in this case the homological degree involves local cohomology). Over a noetherian commutative graded algebra, every finitely generated graded module has finite CM regularity. In the noncommutative case, CM regularity was studied by Jørgensen and Dong–Wu [Jør99, Jør04, DW09] and explored further in [KWZ21]; for a noncommutative noetherian algebra, some finitely generated modules may have infinite CM regularity [KWZ21, Example 5.1]. In [KWZ21, Definition 0.6] a new notion of regularity involving both internal and homological degrees, the Artin–Schelter regularity, was introduced; it measures how close an algebra is to being an Artin-Schelter regular algebra [Definition 0.1].

In this paper we introduce *weighted* versions of classical homological invariants such as the Tor-, Ext- and CM regularities, as well as a weighted version of the Artin–Schelter regularity. These weighted invariants are defined as extrema of general weighted sums of homological and internal degrees of certain complexes, and hence they extend the original version of these invariants. Moreover, we will see that other useful invariants, such as the sup, inf, projective dimension and depth of a complex, can be viewed in the context of weighted regularities of complexes. These weighted invariants can provide new finite invariants, even for commutative algebras. For example, Proposition 5.8 gives a condition which guarantees the existence of some weight such that a *weighted* Tor-regularity [Definition 0.2] of the trivial module is finite. In particular, for a noetherian commutative algebra it implies that such a weight always exists.

We now define the weighted regularities that will be the focus of this paper. An $\mathbb{N}$-graded algebra A is called *connected graded* if $A_0 = \Bbbk$. For a connected graded algebra A, let $\mathfrak{m} = A_{\geq 1}$ and $\Bbbk = A/\mathfrak{m}$. An important class of connected graded algebras in this paper are the *Artin–Schelter regular* algebras [AS87] which play a central role in noncommutative algebraic geometry and representation theory.

Definition 0.1 ([AS87, p. 171]). A connected graded algebra T is called *Artin–Schelter Gorenstein* (or *AS Gorenstein*, for short) if the following conditions hold:

(a) T has injective dimension $d < \infty$ on the left and on the right,
(b) $\operatorname{Ext}_T^i({}_T\Bbbk, {}_TT) = \operatorname{Ext}_T^i(\Bbbk_T, T_T) = 0$ for all $i \neq d$, and
(c) $\operatorname{Ext}_T^d({}_T\Bbbk, {}_TT) \cong \operatorname{Ext}_T^d(\Bbbk_T, T_T) \cong \Bbbk(\mathfrak{l})$ for some integer $\mathfrak{l}$. Here $\mathfrak{l}$ is called the *AS index* of T.

In this case, we say T is of type $(d, \mathfrak{l})$. If in addition,

(d) T has finite global dimension, and
(e) T has finite Gelfand–Kirillov dimension,

then T is called *Artin–Schelter regular* (or *AS regular*, for short) of dimension d.

In this paper we generally reserve the letters S and T for AS regular algebras. Note that the only commutative AS regular algebras are polynomial rings and so AS regular algebras are regarded as noncommutative versions of commutative polynomial rings. Recall that the ith local cohomology of a graded left A-module M is defined to be

$$H_{\mathfrak{m}}^i(M) = \lim_{n \to \infty} \operatorname{Ext}_A^i(A/\mathfrak{m}^n, M).$$

If X is a complex of graded left A-modules, one can define the ith local cohomology of X, denoted by $H_{\mathfrak{m}}^i(X)$, similarly, as in [Jør97, Jør04].

Definition 0.2. Fix a real number ξ. Let A be a noetherian connected graded algebra and let X be a nonzero complex of graded left A-modules.

(1) The ξ-*Tor-regularity* of X is defined to be

$$\mathrm{Torreg}_\xi(X) = \sup_{i,j\in\mathbb{Z}} \{j - \xi i \mid \mathrm{Tor}_i^A(\Bbbk, X)_j \neq 0\}.$$

If $\xi = 1$, then $\mathrm{Torreg}_\xi(X)$ agrees with the usual Tor-regularity $\mathrm{Torreg}(X)$ defined in (E0.0.2) [Jør99, Jør04, DW09].

(2) The ξ-*Castelnuovo–Mumford regularity* (or ξ-*CM regularity*, for short) of X is defined to be

$$\mathrm{CMreg}_\xi(X) = \sup_{i,j\in\mathbb{Z}} \{j + \xi i \mid H_{\mathfrak{m}}^i(X)_j \neq 0\}.$$

If $\xi = 1$, then $\mathrm{CMreg}_\xi(X)$ agrees with the usual Castelnuovo–Mumford regularity $\mathrm{CMreg}(X)$ defined in [Jør99, Jør04, DW09].

(3) The ξ-*Artin–Schelter regularity* (or ξ-*AS regularity*) of A is defined to be

$$\mathrm{ASreg}_\xi(A) = \mathrm{Torreg}_\xi(\Bbbk) + \mathrm{CMreg}_\xi(A).$$

If $\xi = 1$, then $\mathrm{ASreg}_\xi(A)$ agrees with the AS regularity introduced in [KWZ21].

The notions of regularity given in parts (1) and (2) above are natural generalizations of the classical Tor- and Castelnuovo–Mumford regularities [MB66, Eis95, EG84]. These weighted homological invariants provide useful information about the graded algebra A and graded modules (or complexes of modules) over A. In Section 2 we will extend these definitions of weighted regularities to consider weights of the form $\xi = (\xi_0, \xi_1)$ and more general linear combinations of homological and internal degrees [Definitions 2.1, 2.3, and 2.5]. For simplicity in this introduction we consider only the case where $\xi_0 = 1$ and $\xi_1 = \xi$.

For a finitely generated graded A-module M, the relations between the regularities $\mathrm{Torreg}(M)$ and $\mathrm{CMreg}(M)$ have been studied in the literature. When A is a polynomial ring generated in degree 1, $\mathrm{Torreg}(M) = \mathrm{CMreg}(M)$ [EG84], but this is not the case for all AS regular algebras [DW09, Theorem 5.4], see also Theorem 4.3. Other relations between these invariants were established in the commutative case [Röm08] and were extended to the noncommutative case in [Jør99, Jør04, DW09]. In this paper we provide further relations between the weighted versions of these invariants in the noncommutative case. The following two theorems extend [Jør04, Theorems 2.5 and 2.6], [Röm08, Theorem 4.2], and [DW09, Proposition 5.6].

Theorem 0.3. *Let A be a noetherian connected graded algebra with balanced dualizing complex. Let X be a nonzero object in $\mathrm{D}_{\mathrm{fg}}^{\mathrm{b}}(A\text{-}\mathrm{Gr})$ and let $\xi \in \mathbb{R}$.*

(1) (Theorem 3.3)

$$\mathrm{Torreg}_\xi(X) \leq \mathrm{CMreg}_\xi(X) + \mathrm{Torreg}_\xi(\Bbbk).$$

(2) (Theorem 3.5)

$$\mathrm{CMreg}_\xi(X) \leq \mathrm{Torreg}_\xi(X) + \mathrm{CMreg}_\xi(A).$$

(3)
$$\mathrm{ASreg}_\xi(A) \geq 0.$$

Theorem 0.4 (Theorem 3.10). *Let A be a noetherian connected graded algebra with balanced dualizing complex. Let X be a nonzero object in $\mathrm{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-}\mathrm{Gr})$ of finite projective dimension.*

(1) *Suppose $0 \leq \xi \leq 1$. Then*
$$\mathrm{CMreg}_\xi(X) = \mathrm{Torreg}_\xi(X) + \mathrm{CMreg}_\xi(A).$$

(2) *For all $\xi \ll 0$,*
$$\mathrm{CMreg}_\xi(X) = \mathrm{Torreg}_\xi(X) + \mathrm{CMreg}_\xi(A).$$

If M is a graded vector space, let $\deg(M)$ denote the maximal degree of the nonzero homogeneous elements in M, as in equation (E1.1.1). (A more general two-parameter definition of the weighted degree of a complex is given in equation (E1.1.3).)

Remark 0.5. Retain the hypotheses of Theorem 0.4.

(1) If $\xi > 1$, then by Remarks 2.6(1) and 3.11(1), the conclusion of Theorem 0.4 may fail to hold, even when A is AS regular and $X = \Bbbk$.

(2) It is unknown if Theorem 0.4(2) holds for all $\xi < 0$, see Remark 3.11(2).

(3) The famous Auslander–Buchsbaum formula in the graded setting can be recovered from by taking $\lim_{\xi \to -\infty} \frac{1}{\xi}(-)$ in Theorem 0.4(2) (see Corollary 3.12(1)). Hence Theorem 0.4 unifies the Auslander–Buchsbaum formula [Jør98, Theorem 3.2] with [Röm08, Theorem 4.2] (in the commutative case) and [DW09, Proposition 5.6] and [KWZ21, Theorem 0.7] (in the noncommutative case).

(4) In addition to part (3), when taking $\lim_{\xi \to -\infty}(-)$ of Theorem 0.4(2), in Corollary 3.12(2), we obtain a new homological identity

(E0.5.1) $$\deg H^{d(X)}_{\mathfrak{m}}(X) = \deg \mathrm{Tor}^A_{p(X)}(\Bbbk, X) + \deg H^{d(A)}_{\mathfrak{m}}(A),$$

where $p(X) := \mathrm{pdim}(X)$ and $d(X) := \mathrm{depth}(X)$. We call (E0.5.1) a *refined Auslander–Buchsbaum formula*.

In [KWZ21, Theorem 0.8], we generalized a result of Dong and Wu [DW09, Theorems 4.10 and 5.4] to the not-necessarily Koszul setting to show that a noetherian connected graded algebra A with balanced dualizing complex is AS regular if and only if $\mathrm{ASreg}(A) = 0$. Here, we extend this result to the weighted setting. The Cohen–Macaulay property will be defined in Definition 1.2.

Theorem 0.6. *Let A be a noetherian connected graded algebra with balanced dualizing complex. Then the following are equivalent:*

(1) *A is AS regular.*

(2) *There exists a $\xi \leq 1$ such that $\mathrm{ASreg}_\xi(A) = 0$.*

(3) *A is Cohen–Macaulay and there exists a $\xi \in \mathbb{R}$ such that $\mathrm{ASreg}_\xi(A) = 0$.*

It is an open question if the hypothesis that $\xi \leq 1$ can be removed from part (2) or the hypothesis that A is Cohen–Macaulay can be removed from part (3).

By Example 2.2(3), if T is AS regular (or AS Gorenstein) of type $(d, \mathfrak{l})$, then

(E0.6.1) $$\mathrm{CMreg}_\xi(T) = \xi d - \mathfrak{l}$$

which will appear in several places in this paper. If T is AS regular and $\xi \leq 1$, then we also have

$$\mathrm{Torreg}_\xi(\Bbbk) = -\xi d + \mathfrak{l} = -\mathrm{CMreg}_\xi(T),$$

which follows from a direct computation, or from (E0.6.1) and the fact $\mathrm{ASreg}_\xi(A) = 0$. Some further computations of weighted regularities in the non-Koszul case are provided in Remark 2.6.

As an immediate consequence of Theorem 0.6, we have the following corollary.

Corollary 0.7. *Let A be a noetherian AS Gorenstein algebra of type $(d, \mathfrak{l})$. Suppose there is a $\xi \in \mathbb{R}$ such that*

$$\deg \mathrm{Tor}_i^A(\Bbbk, \Bbbk) \leq \xi i + \mathfrak{l} - \xi d$$

for all $i \geq 0$. Then A is AS regular.

Note that Corollary 0.7 recovers [DW09, Theorem 4.10] by setting $\mathfrak{l} = d$ and $\xi = 1$. We also prove a weighted version of [Jør99, Corollary 5.2], which can be used to compute the weighted CM regularity of a finitely generated graded module over an AS Gorenstein algebra.

Theorem 0.8 (Theorem 4.6)**.** *Suppose A is a noetherian AS Gorenstein algebra of type $(d, \mathfrak{l})$. Let $\xi \leq 1$ be a real number and let $M \neq 0$ be a finitely generated graded left A-module with finite projective dimension.*

(1) *Let w be an integer with $0 \leq w \leq d$. Then*

$$\max_{0 \leq j \leq w} \{\deg H_{\mathfrak{m}}^j(M) + \xi j\} = -\mathfrak{l} + \xi d + \max_{d-w \leq j \leq d} \{\deg \mathrm{Tor}_j^A(\Bbbk, M) - \xi j\}.$$

(2) *In particular, if w is chosen to be maximal with the property that $H_{\mathfrak{m}}^w(M) \neq 0$, we have*

$$\mathrm{CMreg}_\xi(M) = -\mathfrak{l} + \xi d + \max_{d-w \leq j \leq d} \{\deg \mathrm{Tor}_j^A(\Bbbk, M) - \xi j\}.$$

(3) *If, further, M is s-Cohen–Macaulay, then $p := d - s$ is the projective dimension of M and*

$$\mathrm{CMreg}_\xi(M) = -\mathfrak{l} + \xi s + \deg(\mathrm{Tor}_p^A(\Bbbk, M)).$$

The above results are generalizations of classical results in various directions, for example, from commutative algebras to noncommutative algebras, from modules to complexes, and from unweighted regularities to weighted regularities.

Homological invariants, as well as homological identities have many applications. In [KWZ22, Theorems 0.8 and 0.10], the authors used regularities to bound the maximal degree of generators of the invariant rings under Hopf actions. In [KWZ21, Corollary 0.11], the authors demonstrated how to control the Koszul property of an AS regular algebra A by using a finite map $T \to A$. Following the ideas of Backelin [Bac86], we can also use finite maps to control the Koszul property of higher Veronese subrings.

In Section 5 we consider the case when there is a finite graded map from a noetherian AS regular algebra T into a connected graded algebra A. In Proposition 5.8 we show there exists a weight ξ with $\mathrm{Torreg}_\xi(X) < \infty$ for all $X \in \mathsf{D}_{\mathrm{fg}}^{\mathrm{b}}(A\text{-Gr})$. This result can be applied to all noetherian commutative graded algebras, where the Tor-regularity is not always finite. As the value of $\mathrm{Torreg}(_A\Bbbk)$ is related to the Koszul property of A, the values of $\mathrm{Torreg}_\xi(_A\Bbbk)$ are related to the Koszul property

of the Veronese subrings of A [Bac86, Corollary, p. 81]. See a related result in Proposition 5.8(1).

Corollary 0.9 (Corollary 5.9). *Let A be a noetherian algebra generated in degree 1 and suppose there is a finite map $T \to A$ where T is a noetherian connected graded algebra of finite global dimension. Then $A^{(d)}$ is Koszul for $d \gg 0$.*

When A is commutative, Corollary 5.9 recovers a very nice result of Mumford [Mum10, Theorem 1]. If we remove the hypothesis that there is a finite map from a noetherian connected graded algebra T of finite global dimension to A, it is an open question if the conclusion of Corollary 0.9 holds, see Question 5.10.

The paper is organized as follows. Section 1 recalls some basic definitions and properties of homological algebra (including local cohomology). In Section 2 we provide the full definitions of the weighted regularities defined in Definition 0.2, and explicitly compute these invariants in several examples. Section 3 proves generalizations of some equalities and inequalities from [Jør04, DW09] that comprise Theorems 0.3 and 0.4. Theorems 0.6 and 0.8 on weighted AS regularities and the computation of weighted CM regularities of graded modules over AS Gorenstein algebras are proved in Section 4. In Section 5 we provide some comments and remarks about related homological invariants such as concavity, rate, and slope.

1. PRELIMINARIES

For an $\mathbb{N}$-graded $\Bbbk$-algebra A, we let $A\text{-}\mathrm{Gr}$ denote the category of $\mathbb{Z}$-graded left A-modules. When convenient, we identify the category of graded right A-modules with $A^{\mathrm{op}}\text{-}\mathrm{Gr}$ where A^{op} is the opposite ring of A. The derived category of graded A-modules is denoted by $\mathsf{D}(A\text{-}\mathrm{Gr})$. We use the standard notation $\mathsf{D}^+(A\text{-}\mathrm{Gr})$, $\mathsf{D}^-(A\text{-}\mathrm{Gr})$, and $\mathsf{D}^{\mathrm{b}}(A\text{-}\mathrm{Gr})$ for the full subcategories of (cochain) complexes $X = (X^n)$ of $\mathbb{Z}$-graded left A-modules which are bounded below (i.e., $X^n = 0$ for all $n \ll 0$), bounded above (i.e., $X^n = 0$ for all $n \gg 0$), and bounded (i.e., $X^n = 0$ for all $|n| \gg 0$), respectively. We use the subscript fg to denote the full subcategories consisting of complexes with finitely generated cohomology, e.g., $\mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-}\mathrm{Gr})$. We adopt the standard convention that a left A-module M can be viewed as a complex concentrated in position 0.

Let ℓ be an integer. For a graded A-module M, the shifted A-module $M(\ell)$ is defined by

$$M(\ell)_m = M_{m+\ell}$$

for all $m \in \mathbb{Z}$. For a cochain complex $X = (X^n, d_X^n : X^n \to X^{n+1})$, we define two notions of shifting: $X(\ell)$ shifts the degrees of each graded vector space $X(\ell)_m^i = X_{m+\ell}^i$ (together with differential $d_{X(\ell)}^i = d_X^i$) for all $i, m \in \mathbb{Z}$ and $X[\ell]$ shifts the complex $X[\ell]^i = X^{i+\ell}$ (together with differential $d_{X[\ell]}^i = (-1)^\ell d_X^i$) for all $i \in \mathbb{Z}$.

Let $M = \bigoplus_{d \in \mathbb{Z}} M_d$ be a $\mathbb{Z}$-graded $\Bbbk$-vector space. We say that M is *locally finite* if $\dim_\Bbbk M_d < \infty$ for all $d \in \mathbb{Z}$.

Definition 1.1. Let $A := \bigoplus_{i \geq 0} A_i$ be a locally finite $\mathbb{N}$-graded algebra. The *Hilbert series* of A is defined to be

$$h_A(t) = \sum_{i \in \mathbb{N}} (\dim_\Bbbk A_i) t^i.$$

Similarly, if $M = \bigoplus_{i \in \mathbb{Z}} M_i$ is a locally finite $\mathbb{Z}$-graded A-module (or $\mathbb{Z}$-graded vector space), the *Hilbert series of M* is defined to be

$$h_M(t) = \sum_{i \in \mathbb{Z}} (\dim_{\Bbbk} M_i) t^i.$$

Define the *degree* of M to be the maximal degree of the nonzero homogeneous elements in M, namely,

(E1.1.1) $\deg(M) = \inf\{d \mid (M)_{\geq d} = 0\} - 1 = \sup\{d \mid (M)_d \neq 0\} \quad \in \quad \mathbb{Z} \cup \{\pm\infty\}.$

By convention, we define $\deg(0) = -\infty$. Similarly, we define

(E1.1.2) $\mathrm{ged}(M) = \sup\{d \mid (M)_{\leq d} = 0\} + 1 = \inf\{d \mid (M)_d \neq 0\} \quad \in \quad \mathbb{Z} \cup \{\pm\infty\}.$

By convention, we define $\mathrm{ged}(0) = \infty$. Now we fix $\xi := (\xi_0, \xi_1)$ a pair of real numbers not both zero and move from the case $(1, \xi)$ considered in the introduction to the general case $\xi = (\xi_0, \xi_1)$. We will see in Lemma 3.1(1) that if $\xi_0 > 0$, then we can often rescale so that $\xi_0 = 1$. For a cochain complex X, the *ξ-weighted degree* of X is defined to be

(E1.1.3) $$\deg_\xi(X) = \sup_{m,n \in \mathbb{Z}} \{\xi_0 m + \xi_1 n \mid H^n(X)_m \neq 0\}.$$

Similarly, the *ξ-weighted* ged of X is defined to be

(E1.1.4) $$\mathrm{ged}_\xi(X) = \inf_{m,n \in \mathbb{Z}} \{\xi_0 m + \xi_1 n \mid H^n(X)_m \neq 0\}.$$

Recall that the *supremum* and *infimum* of X are defined to be

$$\sup(X) = \sup_{n \in \mathbb{Z}} \{n \mid H^n(X) \neq 0\} \quad \text{and} \quad \inf(X) = \inf_{n \in \mathbb{Z}} \{n \mid H^n(X) \neq 0\}.$$

It is clear that, if X is nonzero in $\mathsf{D}(A\text{-Gr})$, then

$$\sup(X) = \deg_{(0,1)}(X) \quad \text{and} \quad \inf(X) = \mathrm{ged}_{(0,1)}(X).$$

Further,

$$-\inf(X) = \deg_{(0,-1)}(X) \quad \text{and} \quad -\sup(X) = \mathrm{ged}_{(0,-1)}(X).$$

If

$$X = \quad \cdots \to X^{s-1} \to X^s \to X^{s+1} \to \cdots,$$

then the brutal truncations of X are denoted by

$$X^{\geq s} := \quad \cdots \to 0 \to \cdots \to 0 \to X^s \to X^{s+1} \to \cdots$$

and

$$X^{\leq s} := \quad \cdots \to X^{s-1} \to X^s \to 0 \to \cdots \to 0 \to \cdots.$$

Let A be a connected graded algebra with graded Jacobson radical $\mathfrak{m} := A_{\geq 1}$. Let $\Bbbk$ also denote the graded A-bimodule $A/\mathfrak{m}$. For a graded left A-module M, let

(E1.1.5) $$t_i^A(_AM) = \deg \mathrm{Tor}_i^A(\Bbbk, M).$$

If M is a graded right A-module, let

(E1.1.6) $$t_i^A(M_A) = \deg \mathrm{Tor}_i^A(M, \Bbbk).$$

It is clear that $t_i^A(_A\Bbbk) = t_i^A(\Bbbk_A)$. If the context is clear, we will use $t_i^A(M)$ instead of $t_i^A(_AM)$ (or $t_i^A(M_A)$).

For each graded left A-module M, we define

$$\Gamma_{\mathfrak{m}}(M) = \{x \in M \mid A_{\geq n}x = 0 \text{ for some } n \geq 1\} = \lim_{n \to \infty} \mathrm{Hom}_A(A/A_{\geq n}, M)$$

and call this the $\mathfrak{m}$-*torsion submodule* of M. It is standard that the functor $\Gamma_{\mathfrak{m}}(-)$ is a left exact functor A-Gr $\to A$-Gr. Since the category A-Gr has enough injectives, the ith right derived functors, denoted by $H_{\mathfrak{m}}^i$ or $\mathrm{R}^i\,\Gamma_{\mathfrak{m}}$, are defined and called the *local cohomology functors*. Explicitly, one has

$$H_{\mathfrak{m}}^i(M) = \mathrm{R}^i\,\Gamma_{\mathfrak{m}}(M) := \lim_{n\to\infty} \mathrm{Ext}_A^i(A/A_{\geq n}, M).$$

See [AZ94, VDB97] for more details. For a complex X of graded left A-modules, the local cohomology functors are defined by

$$H_{\mathfrak{m}}^i(X) = \mathrm{R}^i\,\Gamma_{\mathfrak{m}}(X) := \lim_{n\to\infty} \mathrm{Ext}_A^i(A/A_{\geq n}, X).$$

Definition 1.2. Let A be a connected graded noetherian algebra. Let M be a finitely generated graded left A-module. We call M *s-Cohen–Macaulay* or simply *Cohen–Macaulay* if $H_{\mathfrak{m}}^i(M) = 0$ for all $i \neq s$ and $H_{\mathfrak{m}}^s(M) \neq 0$. We say A is *Cohen–Macaulay* if ${}_A A$ is Cohen–Macaulay.

Throughout the rest of this paper, we assume the following hypothesis; we refer the reader to [Yek92] for the definitions of a dualizing complex and a balanced dualizing complex.

Hypothesis 1.3. *Let A be a noetherian connected graded algebra with balanced dualizing complex. In this case by [VDB97, Theorem 6.3] the balanced dualizing complex will be given by $\mathrm{R}\,\Gamma_{\mathfrak{m}}(A)'$, where $'$ represents the graded vector space dual.*

The local cohomological dimension of a graded A-module M is defined to be

$$\mathrm{lcd}(M) := \sup\{i \in \mathbb{Z} \mid H_{\mathfrak{m}}^i(M) \neq 0\}$$

and the cohomological dimension of $\Gamma_{\mathfrak{m}}$ is defined to be

$$\mathrm{cd}(\Gamma_{\mathfrak{m}}) = \sup_{M \in A\text{-Gr}} \{\mathrm{lcd}(M)\}.$$

We will use the following *Local Duality Theorem* of Van den Bergh several times.

Theorem 1.4. *Let A be a noetherian connected graded $\Bbbk$-algebra with $\mathrm{cd}(\Gamma_{\mathfrak{m}}) < \infty$ and let C be a connected graded algebra.*

(1) [VDB97, Theorem 5.1] *For any $X \in \mathsf{D}((A \otimes C^{\mathrm{op}})\text{-Gr})$ there is an isomorphism*

$$\mathrm{R}\,\Gamma_{\mathfrak{m}}(X)' \cong \mathrm{RHom}_A(X, \mathrm{R}\,\Gamma_{\mathfrak{m}}(A)')$$

in $\mathsf{D}((C \otimes A^{\mathrm{op}})\text{-Gr})$.

(2) [Jør04, Observation 2.3] *Assume Hypothesis 1.3. If $X \in \mathsf{D}_{\mathrm{fg}}^{\mathrm{b}}(A\text{-Gr})$, then $\mathrm{R}\,\Gamma_{\mathfrak{m}}(X)' \in \mathsf{D}_{\mathrm{fg}}^{\mathrm{b}}(A^{\mathrm{op}}\text{-Gr})$.*

Proof. (2) By [VDB97, Theorem 6.3], $R := \mathrm{R}\,\Gamma_{\mathfrak{m}}(A)'$ is the balanced dualizing complex over A. By a basic property of a balanced dualizing complex [Yek92, Proposition 3.4], $\mathrm{RHom}_A(X, R) \in \mathsf{D}_{\mathrm{fg}}^{\mathrm{b}}(A^{\mathrm{op}}\text{-Gr})$. By part (1), $\mathrm{R}\,\Gamma_{\mathfrak{m}}(X)' \cong \mathrm{RHom}_A(X, R)$, and so the assertion follows. $\square$

2. Weighted versions of homological regularities

In this section we introduce weighted versions of several homological invariants in the noncommutative setting and provide some sample computations of these regularities. Castelnuovo–Mumford regularity in the noncommutative setting was first studied by Jørgensen in [Jør99, Jør04] and later by Dong and Wu [DW09].

Throughout, we fix an ordered pair $\xi = (\xi_0, \xi_1)$ of real numbers. In the introduction, we identified ξ with $(1, \xi)$.

Definition 2.1. Let X be a nonzero object in $\mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-Gr})$.

(1) The ξ-*Castelnuovo–Mumford regularity* of X is defined to be

$$\mathrm{CMreg}_\xi(X) = \deg_\xi(\mathrm{R}\,\Gamma_{\mathfrak{m}}(X)).$$

If $\xi_0 \neq 0$, then it is clear that

$$\mathrm{CMreg}_\xi(X) = \sup_{i \in \mathbb{Z}}\{\xi_0 \deg(H^i_{\mathfrak{m}}(X)) + \xi_1 i\}.$$

(2) If $\xi = (1,1)$, then $\mathrm{CMreg}_\xi(X)$ becomes the ordinary $\mathrm{CMreg}(X)$ as defined in [Jør04, Definition 2.1].

By Theorem 1.4(2), if $0 \neq X \in \mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-Gr})$ then $\mathrm{R}\,\Gamma_{\mathfrak{m}}(X) \not\cong 0$ and $\mathrm{R}\,\Gamma_{\mathfrak{m}}(X)' \in \mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A^{\mathrm{op}}\text{-Gr})$. It follows that, if $\xi_0 \geq 0$, then $\mathrm{CMreg}_\xi(X)$ is finite. However, if A does not have a balanced dualizing complex then $\mathrm{CMreg}_\xi(A)$ and $\mathrm{CMreg}_\xi(X)$ can be infinite (e.g. [KWZ21, Example 5.1]).

Example 2.2. Suppose that $\xi_0 \geq 0$.

(1) If M is a finite-dimensional nonzero graded left A-module, then

$$\mathrm{CMreg}_\xi(M) = \xi_0 \deg(M).$$

A more general case is considered in part (4).

(2) Let A be an AS Gorenstein algebra of type $(d, \mathfrak{l})$. Then $\mathrm{CMreg}_\xi(A) = \xi_1 d - \xi_0 \mathfrak{l}$.

(3) Let A be an AS regular algebra of type $(d, \mathfrak{l})$. By [SZ97, Proposition 3.1], when regarded as a rational function, $\deg_t h_A(t) = -\mathfrak{l}$. Hence,

$$\mathrm{CMreg}_\xi(A) = \xi_1 d - \xi_0 \mathfrak{l} = \xi_1 \operatorname{gldim} A + \xi_0 \deg_t h_A(t).$$

(4) If M is s-Cohen–Macaulay, then, by definition,

$$\mathrm{CMreg}_\xi(M) = \xi_1 s + \xi_0 \deg(H^s_{\mathfrak{m}}(M)).$$

(5) If $\xi = (0,1)$ and $X \in \mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-Gr})$, then

$$\mathrm{CMreg}_{(0,1)}(X) = \sup(\mathrm{R}\,\Gamma_{\mathfrak{m}}(X)).$$

(6) Recall from [Jør97] that the depth of a complex X is defined to be

$$\mathrm{depth}(X) := \inf(\mathrm{RHom}_A(\Bbbk, X)).$$

By [DW09, Lemma 2.6], $\mathrm{depth}(X) = \inf(\mathrm{R}\,\Gamma_{\mathfrak{m}}(X))$. If $\xi = (0, -1)$ and $X \in \mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-Gr})$, then

$$\mathrm{CMreg}_{(0,-1)}(X) = -\inf(\mathrm{R}\,\Gamma_{\mathfrak{m}}(X)) = -\mathrm{depth}(X).$$

Definition 2.3. Let X be a nonzero object in $\mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-Gr})$. The ξ-*Ext-regularity* of X is defined to be

$$\begin{aligned}
\mathrm{Extreg}_\xi(X) &= -\mathrm{ged}_\xi(\mathrm{R}\,\mathrm{Hom}_A(X, \Bbbk)) \\
&= -\inf_{i \in \mathbb{Z}}\{\xi_0 \operatorname{ged}(\mathrm{Ext}^i_A(X, \Bbbk)) + \xi_1 i\},
\end{aligned}$$

where the second equality holds when $\xi_0 \neq 0$. The *Tor-regularity* of X is defined to be

$$\begin{aligned}
\text{Torreg}_\xi(X) &= \deg_\xi(\Bbbk \otimes_A^L X) \\
&= \sup_{i \in \mathbb{Z}}\{\xi_0 \deg(\text{Tor}_{-i}^A(\Bbbk, X)) + \xi_1 i\} \\
&= \sup_{i \in \mathbb{Z}}\{\xi_0 \deg(\text{Tor}_i^A(\Bbbk, X)) - \xi_1 i\}
\end{aligned}$$

where the second equality holds when $\xi_0 \neq 0$.

By [DW09, Remark 4.5], if X has a finitely generated minimal free resolution over A, then $\text{Extreg}_\xi(X) = \text{Torreg}_\xi(X)$.

Example 2.4. The following examples are clear.

(1) If $M \in A\text{-Gr}$ and $r = \text{Torreg}_\xi(M)$ and $\xi_0 > 0$, then

$$t_i^A({}_A M) := \deg(\text{Tor}_i^A(\Bbbk, M)) \leq \xi_0^{-1}(r + \xi_1 i)$$

for all i.

(2) $\text{Torreg}_\xi(A) = \text{Extreg}_\xi(A) = 0$.

(3) Suppose $X \in \mathsf{D}_{\text{fg}}^{\text{b}}(A\text{-Gr})$. If $\xi = (0, -1)$, then

$$\text{Torreg}_{(0,-1)}(X) = \text{pdim}(X)$$

where pdim denotes the projective dimension.

(4) Suppose $\xi_0 > 0$. Let A be a Koszul algebra as defined in the introduction (E0.0.1), and let $g = \text{gldim}\, A$. By definition, $\deg \text{Tor}_A^i(\Bbbk, \Bbbk) = i$ for all $0 \leq i \leq g$ and $-\infty$ otherwise. This implies that

(E2.4.1)
$$\text{Torreg}_\xi({}_A\Bbbk) = \begin{cases} 0 & \xi_1 \geq \xi_0, \\ +\infty & \xi_1 < \xi_0 \text{ and } g = \infty, \\ g(\xi_0 - \xi_1) & \xi_1 < \xi_0 \text{ and } g < \infty. \end{cases}$$

(5) Suppose $\xi_0 > 0$ and $\xi_1 < \xi_0$. Let $g = \text{gldim}\, A$. Since $\deg \text{Tor}_A^i(\Bbbk, \Bbbk) \geq i$ for all $0 \leq i \leq g$, we obtain

$$\text{Torreg}_\xi({}_A\Bbbk) = \begin{cases} +\infty & g = \infty, \\ \deg \text{Tor}_A^g(\Bbbk, \Bbbk)\xi_0 - g\xi_1 & g < \infty. \end{cases}$$

Using the weighted versions of the Tor (or Ext) and CM regularities, we define a weighted version of the Artin-Schelter regularity of a connected graded algebra A, extending the unweighted ($\xi = (1, 1)$) version, which was introduced in [KWZ21].

Definition 2.5. The *ξ-Artin–Schelter regularity* (or *ξ-AS regularity*) of a connected graded algebra A is

$$\text{ASreg}_\xi(A) := \text{Torreg}_\xi(\Bbbk) + \text{CMreg}_\xi(A).$$

If there exists a noetherian AS regular algebra T and a finite map $T \to A$, then it follows from Proposition 5.8 that there is a ξ such that $\text{ASreg}_\xi(A) < \infty$.

Remark 2.6.

(1) When $\xi_0 > 0$, it follows from Theorem 3.3 and Example 2.4(2) that $\text{Torreg}_\xi({}_A\Bbbk) \geq -\text{CMreg}_\xi(A)$, or equivalently $\text{ASreg}_\xi(A) \geq 0$. Let T be

a non-Koszul AS regular algebra of global dimension 3 that is generated in degree 1. Then T is of type $(3, 4)$ and

$$t_i^T(\Bbbk) = \begin{cases} 0, & i = 0, \\ 1, & i = 1, \\ 3, & i = 2, \\ 4, & i = 3. \end{cases}$$

By Example 2.2(3), $\mathrm{CMreg}_\xi(T) = -4\xi_0 + 3\xi_1$ and it is easy to check that, if $\xi_0 = 1$, then

$$\mathrm{Torreg}_\xi(\Bbbk) = \max\{0, 1 - \xi_1, 3 - 2\xi_1, 4 - 3\xi_1\} = \begin{cases} 4 - 3\xi_1 & \xi_1 \leq 1, \\ 3 - 2\xi_1, & 1 \leq \xi_1 \leq 1.5, \\ 0, & 1.5 \leq \xi_1. \end{cases}$$

As a consequence, $\mathrm{Torreg}_\xi({}_T\Bbbk) = -\mathrm{CMreg}_\xi(T)$ if $\xi_1 \leq 1$ and $\mathrm{Torreg}_\xi({}_T\Bbbk) > -\mathrm{CMreg}_\xi(T)$ if $\xi_1 > 1$. Note that for any ξ, by Example 2.2(1),

$$\mathrm{CMreg}_\xi(\Bbbk) = 0.$$

So, if $\xi_1 > 1$, Theorem 0.4 (or Theorem 3.10) fails even when $X = \Bbbk$.

(2) Let T be as in part (1) and let $B = T[x_1, \cdots, x_n]$ for some integer $n \geq 1$, where $\deg x_s = 1$ for all $1 \leq s \leq n$. Then B is of type $(3 + n, 4 + n)$ and

$$t_i^B({}_B\Bbbk) = \begin{cases} 0, & i = 0, \\ 1, & i = 1, \\ i + 1, & 2 \leq i \leq n + 3. \end{cases}$$

By Example 2.2(4), $\mathrm{CMreg}_\xi(B) = -(n+4)\xi_0 + (n+3)\xi_1$ and it is easy to check that, if $\xi_0 = 1$, then

$$\mathrm{Torreg}_\xi(\Bbbk_B) = \begin{cases} (n+4) - (n+3)\xi_1 & \xi_1 \leq 1, \\ 3 - 2\xi_1, & 1 \leq \xi_1 \leq 1.5, \\ 0, & 1.5 \leq \xi_1. \end{cases}$$

Similarly, $\mathrm{Torreg}_\xi(\Bbbk_B) > -\mathrm{CMreg}_\xi(B)$ if $\xi_1 > 1$.

(3) Suppose $\xi_0 = 1$ and $\xi_1 \leq 1$. Let T be a noetherian AS regular algebra. By [SZ97, (3–4), p. 1600], $t_{i-1}^T(\Bbbk) < t_i^T(\Bbbk)$ for all $1 \leq i \leq d := \mathrm{gldim}\, T$. Since each $t_i^T(\Bbbk)$ is an integer, we have $t_{i-1}^T(\Bbbk) - (i - 1) \leq t_i^T(\Bbbk) - i$. Since $\xi_1 \leq 1$, we obtain that

$$t_{i-1}^T(\Bbbk) - \xi_1(i - 1) \leq t_i^T(\Bbbk) - \xi_1 i$$

for all $1 \leq i \leq d$. This implies that

$$\mathrm{Torreg}_\xi(\Bbbk) = t_d^T(\Bbbk) - \xi_1 d = \mathfrak{l} - \xi_1 d$$

by [SZ97, Proposition 3.1(4)]. By Example 2.2(3),

$$\mathrm{Torreg}_\xi({}_T\Bbbk) = -\mathrm{CMreg}_\xi(T),$$

or equivalently

$$\mathrm{ASreg}_\xi(T) = 0$$

(compare to Theorem 0.6).

(4) Let $\xi = (1, \xi_1)$. Let $A = \Bbbk[x]$ with $\deg(x) = 2$. Then A is of type $(1, 2)$ and $\deg \operatorname{Tor}_A^1(\Bbbk, \Bbbk) = 2$. We have

$$\operatorname{CMreg}_\xi(A) = -2 + \xi_1$$

and

$$\operatorname{Torreg}_\xi(\Bbbk) = \begin{cases} 2 - \xi_1 & \xi_1 \le 2, \\ 0 & \xi_1 > 2. \end{cases}$$

As a consequence,

$$\operatorname{ASreg}_\xi(\Bbbk) = \begin{cases} 0 & \xi_1 \le 2, \\ -2 + \xi_1 & \xi_1 > 2. \end{cases}$$

The following is a generalization of [KWZ21, Lemma 2.3].

Lemma 2.7. *Assume that $\xi_0 > 0$. Suppose that T and A are connected graded algebras. Then*

$$\operatorname{Torreg}_\xi({}_{T \otimes A}\Bbbk) = \operatorname{Torreg}_\xi({}_T\Bbbk) + \operatorname{Torreg}_\xi({}_A\Bbbk).$$

Proof. Let P be a projective resolution of $\Bbbk$ as a right T-module and let Q be a projective resolution of $\Bbbk$ as a right A-module. Let X and Y denote the complexes given by tensoring P and Q with ${}_T\Bbbk$ and ${}_A\Bbbk$ respectively. Then $\operatorname{Tor}_i^T(\Bbbk, \Bbbk)$ and $\operatorname{Tor}_i^A(\Bbbk, \Bbbk)$ can be computed by taking homology of X and Y, respectively. Further, $\operatorname{Tor}_i^{T \otimes A}(\Bbbk, \Bbbk)$ can be computed by taking homology of the complex $X \otimes Y$. By the Künneth formula (see, e.g. [Rot09, Theorem 10.8.1]), we have

$$\bigoplus_{p+q=n} \operatorname{Tor}_p^T(\Bbbk, \Bbbk) \otimes \operatorname{Tor}_q^A(\Bbbk, \Bbbk) \cong \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y)$$

$$\cong H_n(X \otimes Y) \cong \operatorname{Tor}_n^{T \otimes A}(\Bbbk, \Bbbk)$$

where $\otimes$ stands for $\otimes_\Bbbk$. Therefore,

$$\operatorname{Torreg}_\xi({}_{T \otimes A}\Bbbk) = \sup_{n \in \mathbb{Z}} \{\xi_0 \deg(\operatorname{Tor}_n^{T \otimes A}(\Bbbk, \Bbbk)) - \xi_1 n\}$$

$$= \sup_{p,q \in \mathbb{Z}} \{\xi_0 \deg(\operatorname{Tor}_p^T(\Bbbk, \Bbbk)) + \xi_0 \deg(\operatorname{Tor}_q^A(\Bbbk, \Bbbk)) - \xi_1(p+q)\}$$

$$= \sup_{p \in \mathbb{Z}} \{\xi_0 \deg(\operatorname{Tor}_p^T(\Bbbk, \Bbbk)) - \xi_1 p\} + \sup_{q \in \mathbb{Z}} \{\xi_0 \deg(\operatorname{Tor}_q^A(\Bbbk, \Bbbk)) - \xi_1 q\}$$

$$= \operatorname{Torreg}_\xi({}_T\Bbbk) + \operatorname{Torreg}_\xi({}_A\Bbbk),$$

as desired. $\square$

Example 2.8. Let $\xi_0 = 1$.

Let T be the noetherian AS regular algebra given in Remark 2.6(1). Then

$$\operatorname{Torreg}_\xi({}_T\Bbbk) = \max\{0, 1 - \xi_1, 3 - 2\xi_1, 4 - 3\xi_1\} = \begin{cases} 4 - 3\xi_1 & \xi_1 \le 1, \\ 3 - 2\xi_1 & 1 \le \xi_1 \le 1.5, \\ 0 & 1.5 \le \xi_1. \end{cases}$$

Let A be an affine (commutative) noetherian Koszul algebra that has infinite global dimension. Then

$$\operatorname{Torreg}_\xi({}_A\Bbbk) = \max_{n \in \mathbb{N}}\{n - n\xi\} = \begin{cases} \infty & \xi_1 > 1, \\ 0 & \xi_1 \le 1. \end{cases}$$

By Lemma 2.7, we obtain that

$$\mathrm{Torreg}_\xi(_{A\otimes T}\Bbbk) = \begin{cases} \infty & \xi_1 > 1, \\ 4 - 3\xi_1 & \xi_1 \leq 1. \end{cases}$$

3. Equalities and inequalities

In this section we study the relationships between the weighted regularities defined in the previous section, generalizing results of Jørgensen, Dong, and Wu [Jør99, Jør04, DW09] and proving Theorem 0.3. In this section we fix a pair of real numbers $\xi = (\xi_0, \xi_1)$.

Recall that for a cochain complex X and $\ell \in \mathbb{Z}$, the complex $X(\ell)$ shifts the degrees of each graded vector space $X(\ell)_m^i = X_{m+\ell}^i$ while the complex $X[\ell]$ shifts the complex $X[\ell]^i = X^{i+\ell}$.

Lemma 3.1. *Let A be a noetherian connected graded algebra and X be a nonzero complex of graded left A-modules. The following statements hold.*

(1) *Let $\lambda > 0$ and $\xi' = \lambda\xi$. Then*

$$\deg_{\xi'}(X) = \lambda \deg_\xi(X).$$

The above equation also holds if $\deg$ is replaced with ged, CMreg, Extreg, or Torreg.

(2) *Suppose $\deg_\xi(X)$ is finite. Then*

$$\deg_\xi(X[1]) = \deg_\xi(X) - \xi_1 \quad \text{and} \quad \deg_\xi(X(1)) = \deg_\xi(X) - \xi_0.$$

Similar equations hold if $\deg$ is replaced with ged, CMreg, Extreg, or Torreg.

(3) *Suppose $\deg_\xi(X)$ is finite. Assume that ξ_0 and ξ_1 are nonzero such that $\xi_0^{-1}\xi_1$ is irrational. Then the numbers in the collection*

$$\{\deg_\xi(X[m](n))\}_{m,n\in\mathbb{Z}}$$

are distinct. This assertion holds if $\deg$ is replaced with ged, CMreg, Extreg, or Torreg.

In the following parts, we assume that $\{\xi_n := (\xi_{0,n}, \xi_{1,n})\}_{n\geq 1}$ is a sequence of pairs such that $\lim_{n\to\infty} \xi_n = \xi$. In parts (4) and (5), we further assume that $\xi_0 > 0$.

(1) *Suppose $X \in \mathsf{D}^\mathrm{b}(A\text{-}\mathrm{Gr})$. If $\deg H^j(X) < \infty$ for all j, then*

$$\lim_{n\to\infty} \deg_{\xi_n}(X) = \deg_\xi(X).$$

Similarly, if $\mathrm{ged}\, H^j(X) > -\infty$ for all j, then

$$\lim_{n\to\infty} \mathrm{ged}_{\xi_n}(X) = \mathrm{ged}_\xi(X).$$

(2) *If Y is a nonzero object in $\mathsf{D}^\mathrm{b}_{\mathrm{fg}}(A\text{-}\mathrm{Gr})$, then $\lim_{n\to\infty} \mathrm{CMreg}_{\xi_n}(Y) = \mathrm{CMreg}_\xi(Y)$.*

(3) *If Y is a nonzero object in $\mathsf{D}^\mathrm{b}_{\mathrm{fg}}(A\text{-}\mathrm{Gr})$ of finite projective dimension, then $\lim_{n\to\infty} \mathrm{Torreg}_{\xi_n}(Y) = \mathrm{Torreg}_\xi(Y)$ and $\lim_{n\to\infty} \mathrm{Extreg}_{\xi_n}(Y) = \mathrm{Extreg}_\xi(Y)$.*

Proof. (1) Let $\lambda > 0$ and $\xi' = \lambda\xi$. If $\deg_\xi(X)$ is finite, then

$$\deg_{\xi'}(X) = \sup_{m,n\in\mathbb{Z}} \{\lambda\xi_0 m + \lambda\xi_1 n \mid H^n(X)_m \neq 0\}$$

$$= \lambda \sup_{m,n\in\mathbb{Z}} \{\xi_0 m + \xi_1 n \mid H^n(X)_m \neq 0\} = \lambda \deg_\xi(X).$$

If $\deg_\xi(X)$ is infinite, then so is $\deg_{\xi'}(X)$. A similar proof works for $\mathrm{ged}_\xi(X)$, and the result holds for $\mathrm{CMreg}_\xi(X)$, $\mathrm{Extreg}_\xi(X)$, and $\mathrm{Torreg}_\xi(X)$, since they can each be expressed as $\deg_\xi$ or ged_ξ of certain complexes of A-modules.

(2) Suppose that $\deg_\xi(X)$ is finite. Then

$$\deg_\xi(X[1]) = \sup_{m,n\in\mathbb{Z}} \{\xi_0 m + \xi_1 n \mid H^n(X[1])_m \neq 0\}$$

$$= \sup_{m,n\in\mathbb{Z}} \{\xi_0 m + \xi_1 n \mid H^{n+1}(X)_m \neq 0\}$$

$$= \sup_{m,n\in\mathbb{Z}} \{\xi_0 m + \xi_1(n-1) \mid H^n(X)_m \neq 0\} = \deg_\xi(X) - \xi_1.$$

Further,

$$\deg_\xi(X(1)) = \sup_{m,n\in\mathbb{Z}} \{\xi_0 m + \xi_1 n \mid H^n(X(1))_m \neq 0\}$$

$$= \sup_{m,n\in\mathbb{Z}} \{\xi_0 m + \xi_1 n \mid H^n(X)_{m+1} \neq 0\}$$

$$= \sup_{m,n\in\mathbb{Z}} \{\xi_0(m-1) + \xi_1 n \mid H^n(X)_m \neq 0\} = \deg_\xi(X) - \xi_0.$$

The proofs for $\mathrm{ged}_\xi(X)$, $\mathrm{CMreg}_\xi(X)$, $\mathrm{Torreg}_\xi(X)$, and $\mathrm{Extreg}_\xi(X)$ are similar.

(3) Suppose $\deg_\xi(X) = d$ is finite and that ξ_0, ξ_1 are nonzero such that $\xi_0^{-1}\xi_1$ is irrational. By the above result, for $m,n \in \mathbb{Z}$, we have $\deg_\xi(X[m](n)) = d - m\xi_1 - n\xi_0$. If $\deg_\xi(X[m](n)) = \deg_\xi(X[m'](n'))$ then, $(m-m')\xi_1 = (n'-n)\xi_0$. Since $\xi_0^{-1}\xi_1$ is irrational and $\xi_0, \xi_1 \neq 0$, therefore $m = m'$ and $n = n'$. The same proof shows that the result holds for $\mathrm{ged}_\xi(X)$, $\mathrm{CMreg}_\xi(X)$, $\mathrm{Torreg}_\xi(X)$, and $\mathrm{Extreg}_\xi(X)$.

(4) Let $\{\xi_n = (\xi_{0,n}, \xi_{1,n})\}_{n\geq 1}$ be a sequence such that $\lim_{n\to\infty} \xi_n = \xi$. Then

$$\lim_{n\to\infty} \deg_{\xi_n}(X) = \lim_{n\to\infty} \sup_{i,j\in\mathbb{Z}} \{\xi_{0,n}i + \xi_{1,n}j \mid H^j(X)_i \neq 0\}.$$

Since $X \in \mathsf{D}^\mathrm{b}(A\text{-Gr})$, only finitely many $H^j(X)$ are nonzero and so the above supremum can be taken over finitely many j.

If $\deg H^j(X) < \infty$ for each j, by hypothesis $\xi_0 > 0$, then the supremum is taken over a finite set, and so the convergence holds. If some $\deg H^j(X) = \infty$ (and $\xi_0 > 0$), then $\deg_\xi(X)$ is infinite, and $\deg_{\xi_n}(X)$ is also infinite when $n \gg 0$. A similar proof shows that the assertion for $\mathrm{ged}_\xi(X)$ holds.

(5) If $Y \in \mathsf{D}^\mathrm{b}_\mathrm{fg}(A\text{-Gr})$ is nonzero, then $X := \mathrm{R}\Gamma_\mathfrak{m}(Y)$ is a nonzero object in $\mathsf{D}^\mathrm{b}(A\text{-Gr})$. By Theorem 1.4(2), $\deg H^i(X) < \infty$ for all i. Hence, by part (4), the assertion holds for $\mathrm{CMreg}_\xi(Y) = \deg_\xi(\mathrm{R}\Gamma_\mathfrak{m}(Y)) = \deg_\xi(X)$.

(6) If $Y \in \mathsf{D}^\mathrm{b}_\mathrm{fg}(A\text{-Gr})$ is nonzero with finite projective dimension, then $\Bbbk \otimes_A^L Y$ and $\mathrm{R}\mathrm{Hom}_A(Y, \Bbbk)$ are nonzero in $\mathsf{D}^\mathrm{b}(\Bbbk\text{-Gr})$. Further, in each degree of the complex, homology is finite-dimensional. Hence, by part (4), the assertion holds for $\mathrm{Torreg}_\xi(Y) = \deg_\xi(\Bbbk \otimes_A^L Y)$ and $\mathrm{Extreg}_\xi(Y) = \mathrm{ged}_\xi \mathrm{R}\mathrm{Hom}_A(Y, \Bbbk)$. $\qquad\square$

Proposition 3.2. *Suppose that A is a noetherian connected graded algebra with balanced dualizing complex. Assume that $\xi_0 > 0$. Let Z be a nonzero object in $\mathsf{D}^-(A\text{-Gr})$ with $\deg_\xi(Z)$ finite. Then*

$$\mathrm{Extreg}_\xi(Z) \leq \deg_\xi(Z) + \mathrm{Torreg}_\xi(\Bbbk).$$

Proof. By Lemma 3.1(1), we may assume that $\xi_0 = 1$. Let $p := \deg_\xi(Z)$, which is finite by hypothesis. If $\mathrm{Torreg}_\xi(\Bbbk) = \infty$, then the assertion holds trivially, so we may assume that $r := \mathrm{Torreg}_\xi(\Bbbk)$ is finite.

By Example 2.4(1) and our assumption that $\xi_0 = 1$,

(E3.2.1) $$\deg(\mathrm{Tor}_i^A(\Bbbk, \Bbbk)) \leq r + \xi_1 i$$

for all i.

Let

(E3.2.2) $$F: \qquad \cdots \to F_s \to F_{s-1} \to \cdots \to F_1 \to A \to \Bbbk \to 0$$

be a minimal projective resolution of the right A-module $\Bbbk$. By (E3.2.1), the generators of F_m are placed in degrees less than or equal to $r + \xi_1 m$ for every m and so F_m can be written as a finite coproduct

$$F_m = \coprod_j A(-\sigma_{m,j})$$

where $\sigma_{m,j}$ are integers $\leq r + \xi_1 m$. Taking Matlis duals in (E3.2.2), $I := F'$ is a minimal injective resolution of the left A-module $\Bbbk$ which has

$$I^m = \coprod_j A'(\sigma_{m,j})$$

for each $m \geq 0$.

Since $\xi_0 = 1$ and $p = \deg_\xi(Z)$ (or equivalently $\deg H^n(Z) \leq p - \xi_1 n$ for all n), we have

$$H^{-n}(Z)_{>p+\xi_1 n} = 0$$

for all n. For each m, $\mathrm{Ext}_A^m(H^{-n}(Z), \Bbbk)$ is a subquotient of $\mathrm{Hom}_A(H^{-n}(Z), I^m)$, which is

$$\mathrm{Hom}_A\left(H^{-n}(Z), \coprod_j A'(\sigma_{m,j})\right) \cong \coprod_j (H^{-n}(Z))'(\sigma_{m,j}),$$

and this vanishes in degree less than $-p - \xi_1 n - r - \xi_1 m$. Hence

(E3.2.3) $$\mathrm{Ext}_A^m(H^{-n}(Z), \Bbbk)_{<-p-\xi_1 n - r - \xi_1 m} = 0$$

for all m, n. By [Jør04, Lemma 1.2], there is a convergent spectral sequence

$$E_2^{m,n} := \mathrm{Ext}_A^m(H^{-n}(Z), \Bbbk) \Longrightarrow \mathrm{Ext}_A^{m+n}(Z, \Bbbk),$$

and since (E3.2.3) shows that $(E_2^{m,n})_{<-p-r-\xi_1(m+n)} = 0$, it follows that

$$\mathrm{Ext}_A^q(Z, \Bbbk)_{<-p-r-\xi_1 q} = 0.$$

This condition is equivalent to $\mathrm{ged}(\mathrm{Ext}_A^q(Z, \Bbbk)) \geq -p - r - \xi_1 q$ for all q. By Definition 2.3, $\mathrm{Extreg}_\xi(Z) \leq p + r$ as desired. $\qquad\square$

The following is a generalization of [Jør04, Theorem 2.5], and establishes Theorem 0.3(1). Combining this result with Example 2.4(2) also yields Theorem 0.3(3).

Theorem 3.3. *Assume that $\xi_0 > 0$. Let A be a noetherian connected graded algebra with a balanced dualizing complex and let X be a nonzero object in $\mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-}\mathrm{Gr})$. Then*

$$\mathrm{Torreg}_\xi(X) = \mathrm{Extreg}_\xi(X) \leq \mathrm{CMreg}_\xi(X) + \mathrm{Torreg}_\xi(\Bbbk).$$

Proof. Let $Z = \mathrm{R}\,\Gamma_{\mathfrak{m}}(X)$. By Theorem 1.4(2), Z' is nonzero in $\mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A^{\mathrm{op}}\text{-}\mathrm{Gr})$. Hence $\deg_\xi(Z)$ is a finite number and $Z \in \mathsf{D}^{\mathrm{b}}(A\text{-}\mathrm{Gr})$. By Proposition 3.2,

$$\mathrm{Extreg}_\xi(Z) \leq \deg_\xi(Z) + \mathrm{Torreg}_\xi(\Bbbk).$$

By definition,

$$\mathrm{CMreg}_\xi(X) = \deg_\xi(\mathrm{R}\,\Gamma_{\mathfrak{m}}(X)) = \deg_\xi(Z).$$

By [Jør04, Proposition 1.1],

$$\mathrm{RHom}_A(X, \Bbbk) \cong \mathrm{RHom}_A(\mathrm{R}\,\Gamma_{\mathfrak{m}}(X), \Bbbk) = \mathrm{RHom}_A(Z, \Bbbk).$$

Hence $\mathrm{Extreg}_\xi(X) = \mathrm{Extreg}_\xi(Z)$ and so the assertion follows. $\square$

Note that [Jør04, Theorem 2.5] is a special case of Theorem 3.3 where $\xi = (1,1)$. By considering some specific weights, we obtain the following corollary.

Corollary 3.4. *Assume that $\xi_0 > 0$. Let A be a noetherian connected graded algebra with a balanced dualizing complex and let X be a nonzero object in $\mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-}\mathrm{Gr})$. If $\mathrm{Torreg}_\xi(\Bbbk)$ is finite, then so is $\mathrm{Torreg}_\xi(X)$.*

Proof. Since X has a finitely generated minimal free resolution, $\mathrm{Extreg}_\xi(X) = \mathrm{Torreg}_\xi(X)$. Hence, by Theorem 3.3,

$$\mathrm{Torreg}_\xi(X) \leq \mathrm{CMreg}_\xi(X) + \mathrm{Torreg}_\xi(\Bbbk),$$

and since $\mathrm{CMreg}_\xi(X)$ is finite (see comments after Definition 2.1), the result follows. $\square$

Proof of Theorem 0.3(3). Let $X = A$ in Theorem 3.3, we obtain that

$$\mathrm{ASreg}_\xi(A) := \mathrm{CMreg}_\xi(A) + \mathrm{Torreg}_\xi(\Bbbk) \geq \mathrm{Extreg}_\xi(A) = 0.$$

$\square$

Theorem 3.3 has other consequences by setting ξ to be special values. For example, if A is AS regular, when we set $\xi_1 = 1$ and take the limit as $\xi_0 \to 0^+$, we obtain that

(E3.4.1) $$\sup(X) \leq \mathrm{lcd}(X).$$

Theorem 3.5 is a generalization of [Jør04, Theorem 2.6], and it provides a proof of Theorem 0.3(2).

Theorem 3.5. *Assume that $\xi_0 > 0$. Let A be a noetherian connected graded algebra with a balanced dualizing complex and let X be a nonzero object in $\mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-}\mathrm{Gr})$. Then*

$$\mathrm{CMreg}_\xi(X) \leq \mathrm{Extreg}_\xi(X) + \mathrm{CMreg}_\xi(A).$$

Proof. By Lemma 3.1(1), we may assume that $\xi_0 = 1$. As noted after Definition 2.1, $\mathrm{CMreg}_\xi(A)$ is finite. If $\mathrm{Extreg}_\xi(X)$ is infinite, then the assertion holds trivially. So we may assume that $r := \mathrm{Extreg}_\xi(X)$ is finite, and hence $\mathrm{Torreg}_\xi(X) = r$ is finite. Let F be a minimal free resolution of X. By definition,

$$\deg(\mathrm{Tor}_i^A(\Bbbk, X)) \leq r + \xi_1 i$$

for all integers i. This implies that the generators of F_m are in degrees less than or equal to $r + \xi_1 m$ for each integer m. Therefore F_m can be written as a finite coproduct

$$F_m = \coprod_j A(-\sigma_{m,j})$$

where $\sigma_{m,j}$ are integers $\leq r + \xi_1 m$.

Let $p := \mathrm{CMreg}_\xi(A)$. By [Jør04, Observation 2.3], $\mathrm{CMreg}_\xi(A_A) = \mathrm{CMreg}_\xi({}_A A)$, so we have

(E3.5.1) $$H^n_{\mathfrak{m}^{op}}(A)_{>p-\xi_1 n} = 0$$

for each integer n.

Now $\mathrm{Tor}^A_{-m}(H^n_{\mathfrak{m}^{op}}(A), X)$ is a subquotient of $H^n_{\mathfrak{m}^{op}}(A) \otimes_A F_{-m}$ which is

$$H^n_{\mathfrak{m}^{op}}(A) \otimes_A \coprod_j A(-\sigma_{-m,j}) \cong \coprod_j H^n_{\mathfrak{m}^{op}}(A)(-\sigma_{-m,j}),$$

and the latter vanishes in degree larger than $p - \xi_1 n + r - \xi_1 m$ by (E3.5.1). Hence

(E3.5.2) $$\mathrm{Tor}^A_{-m}(H^n_{\mathfrak{m}^{op}}(A), X)_{>p+r-\xi_1 n - \xi_1 m} = 0$$

for all m, n.

By [Jør04, Lemma 1.3], there is a convergent spectral sequence

$$E_2^{m,n} := \mathrm{Tor}^A_{-m}(H^n_{\mathfrak{m}^{op}}(A), X) \Longrightarrow H^{m+n}_{\mathfrak{m}}(X).$$

Since (E3.5.2) shows that $(E_2^{m,n})_{>p+r-\xi_1(n+m)} = 0$ for all m, n, the spectral sequence implies that

$$H^q_{\mathfrak{m}}(X)_{>p+r-\xi_1 q} = 0$$

for all q. By definition, this is equivalent to

$$\mathrm{CMreg}_\xi(X) \leq r + p = \mathrm{Extreg}_\xi(X) + \mathrm{CMreg}_\xi(A).$$

□

If $\xi = (1,1)$, then Theorem 3.5 recovers [Jør04, Theorem 2.6]. The following is an immediate corollary of Theorems 3.3 and 3.5.

Corollary 3.6. *Assume that $\xi_0 > 0$. Let A be a noetherian connected graded algebra with a balanced dualizing complex. Suppose that $\mathrm{Torreg}_\xi(\Bbbk) = -\mathrm{CMreg}_\xi(A)$ (which is a finite number, denoted by c). Then for all nonzero X in $\mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-}\mathrm{Gr})$, $\mathrm{Extreg}_\xi(X) = \mathrm{CMreg}_\xi(X) + c$.*

In Lemma 3.7, reg_ξ can be CMreg_ξ, or Extreg_ξ or Torreg_ξ.

Lemma 3.7. *Suppose that $\xi = (\xi_0, \xi_1)$ with $\xi_0 > 0$. Let $X \to Y \to Z \to X[1]$ be a distinguished triangle in $\mathsf{D}(A\text{-}\mathrm{Gr})$. Then*

(1) $\mathrm{reg}_\xi(Y) \leq \max\{\mathrm{reg}_\xi(X), \mathrm{reg}_\xi(Z)\}$.
(2) $\mathrm{reg}_\xi(X) \leq \max\{\mathrm{reg}_\xi(Y), \mathrm{reg}_\xi(Z) + \xi_1\}$.
(3) $\mathrm{reg}_\xi(Z) \leq \max\{\mathrm{reg}_\xi(X) - \xi_1, \mathrm{reg}_\xi(Y)\}$.

Proof. We will prove the assertions only for $\mathrm{reg}_\xi = \mathrm{CMreg}_\xi$. The other proofs are similar.

(1) Starting from the distinguished triangle $X \to Y \to Z \to X[1]$, we obtain a distinguished triangle

$$\mathrm{R}\,\Gamma_{\mathfrak{m}}(X) \to \mathrm{R}\,\Gamma_{\mathfrak{m}}(Y) \to \mathrm{R}\,\Gamma_{\mathfrak{m}}(Z) \to \mathrm{R}\,\Gamma_{\mathfrak{m}}(X)[1]$$

which implies that there is a long exact sequence

$$\cdots \to H_{\mathfrak{m}}^{n-1}(Z) \to H_{\mathfrak{m}}^n(X) \to H_{\mathfrak{m}}^n(Y) \to H_{\mathfrak{m}}^n(Z) \to H_{\mathfrak{m}}^{n+1}(X) \to \cdots .$$

Hence, for all integers n, we have $\deg H_{\mathfrak{m}}^n(Y) \leq \max\{\deg H_{\mathfrak{m}}^n(X), \deg H_{\mathfrak{m}}^n(Z)\}$. Since $\xi_0 > 0$, we obtain that

$$\xi_0 \deg H_{\mathfrak{m}}^n(Y) + \xi_1 n \leq \max\{\xi_0 \deg H_{\mathfrak{m}}^n(X) + \xi_1 n, \xi_0 \deg H_{\mathfrak{m}}^n(Z) + \xi_1 n\}$$
$$\leq \max\{\mathrm{CMreg}_\xi(X), \mathrm{CMreg}_\xi(Z)\}$$

for all n. This implies that

$$\mathrm{CMreg}_\xi(Y) \leq \max\{\mathrm{CMreg}_\xi(X), \mathrm{CMreg}_\xi(Z)\}$$

as desired.

(2) By rotation, we have a distinguished triangle $Z[-1] \to X \to Y \to Z$. By part (1), we have

$$\mathrm{CMreg}_\xi(X) \leq \max\{\mathrm{CMreg}_\xi(Y), \mathrm{CMreg}_\xi(Z[-1])\}.$$

The assertion now follows from Lemma 3.1(2).

(3) The proof is similar to the proof of part (2). $\qquad \square$

Our next result is a generalization [DW09, Proposition 5.6]. We will use the following notation. Let

$$(\text{E3.7.1}) \qquad Y := \cdots \to 0 \to F^{-w} \to F^{-(w-1)} \to \cdots \to F^{-1} \to F^0 \to 0 \to \cdots$$
$$= \cdots \to 0 \to F_w \to F_{w-1} \to \cdots \to F_1 \to F_0 \to 0 \to \cdots$$

be a minimal free resolution of a complex in $\mathrm{D}_{\mathrm{fg}}^{\mathrm{b}}(A\text{-Gr})$ of finite projective dimension that is bounded below at position 0. Let

$$Z = F_w[w], \quad \text{and} \quad X = \cdots \to 0 \to 0 \to F_{w-1} \to \cdots \to F_1 \to F_0 \to 0 \to \cdots .$$

Observe that we have a distinguished triangle

$$(\text{E3.7.2}) \qquad\qquad\qquad X \to Y \to Z \to X[1].$$

For each $0 \leq s \leq w$, write

$$(\text{E3.7.3}) \qquad\qquad\qquad F^{-s} = F_s := \coprod_{j=1}^{n_s} A(-\sigma_{s,j})$$

for some integers $\sigma_{s,j}$ and write

$$(\text{E3.7.4}) \qquad\qquad\qquad \sigma_s := \max_{1 \leq j \leq n_s}\{\sigma_{s,j}\}.$$

Lemma 3.8. *Retain the above notation* ((E3.7.1)–(E3.7.4)). *Let* $\xi_0 > 0$.

(1) *Then*

$$\mathrm{Torreg}_\xi(Y) = \max_{0 \leq s \leq w}\{\mathrm{Torreg}_\xi(F_s[s])\}.$$

As a consequence,

$$\mathrm{Torreg}_\xi(Y) = \max\{\mathrm{Torreg}_\xi(X), \mathrm{Torreg}_\xi(Z)\}.$$

(2) *If* $Z \neq 0$ *and* $\xi_1 \ll 0$, *then* $\mathrm{Torreg}_\xi(Y) = \mathrm{Torreg}_\xi(Z)$.

Proof. (1) For each $0 \leq s \leq w$, write $F_s = \coprod_{j=1}^{n_s} A(-\sigma_{s,j})$ as in (E3.7.3). By definition, $\mathrm{Tor}_s^A(\Bbbk, Y) = \coprod_{j=1}^{n_s} \Bbbk(-\sigma_{s,j})$ and consequently,

$$(\text{E3.8.1}) \quad \mathrm{Torreg}_\xi(Y) = \max_{0 \leq s \leq w} \left\{ \xi_0 \max_{1 \leq j \leq n_s} \{\sigma_{s,j}\} - \xi_1 s \right\} = \max_{0 \leq s \leq w} \{\xi_0 \sigma_s - \xi_1 s\}.$$

Since $\mathrm{Torreg}_\xi(F_s[s]) = \xi_0 \sigma_s - \xi_1 s$, the main assertion follows, and the consequence follows from the main assertion.

(2) Since $Z \neq 0$, $\sigma_w := \max_{1 \leq j \leq n_w} \{\sigma_{w,j}\}$ is finite. It follows from (E3.8.1) that, when $\xi_1 \ll 0$, $\mathrm{Torreg}_\xi(Y) = \sigma_w - \xi_1 w$. Since $\mathrm{Torreg}_\xi(Z) = \sigma_w - \xi_1 w$ by definition, the assertion follows. $\square$

We continue to use the notation introduced before Lemma 3.8.

Lemma 3.9. *Retain the above notation. Suppose that $Z \neq 0$ and that $\xi_0 > 0$.*

(1)
$$\mathrm{CMreg}_\xi(Y) \leq \max_{0 \leq s \leq w} \{\mathrm{CMreg}_\xi(F_s[s])\}.$$

(2)
$$\mathrm{depth}(Y) \geq \min_{1 \leq s \leq w} \{\mathrm{depth}(F_s[s])\} = -w + \mathrm{depth}(A) = \mathrm{depth}(Z).$$

(3) *Let $d = \mathrm{depth}(A)$ and $f = -w + \mathrm{depth}(A)$. Then*
$$\deg H_{\mathfrak{m}}^f(Y) \leq \deg H_{\mathfrak{m}}^d(A) + \sigma_w = \deg H_{\mathfrak{m}}^f(F_w[w]) = \deg H_{\mathfrak{m}}^f(Z).$$

(4) *Let $\alpha = \deg H_{\mathfrak{m}}^f(Z) (= \deg H_{\mathfrak{m}}^d(A) + \sigma_w)$. Then the natural map*
$$H_{\mathfrak{m}}^f(Z)_\alpha \to H_{\mathfrak{m}}^{f+1}(X)_\alpha = H_{\mathfrak{m}}^f(X[1])_\alpha$$
is zero.

(5) *If $\xi_1 \ll 0$, then*
$$\mathrm{CMreg}_\xi(Y) = \max_{0 \leq s \leq w} \{\mathrm{CMreg}_\xi(F_s[s])\} = \mathrm{CMreg}_\xi(F_w[w]) = \mathrm{CMreg}_\xi(Z).$$

Proof. (1) This follows by induction on w, (E3.7.2), and Lemma 3.7(1).

(2) First of all it is clear that
$$\mathrm{depth}(Z) = \mathrm{depth}(F_w[w]) = \mathrm{depth}\left(\coprod_{j=1}^{n_w} A(-\sigma_{w,j})[w]\right) = -w + \mathrm{depth}(A).$$

It is obvious that $\min_{1 \leq s \leq w}\{\mathrm{depth}(F_s[s])\} = -w + \mathrm{depth}(A)$.

Next, we prove $\mathrm{depth}(Y) \geq \min_{1 \leq s \leq w}\{\mathrm{depth}(F_s[s])\}$ by induction on w. Let $\overline{Y} = \mathrm{R}\,\Gamma_{\mathfrak{m}}(Y)$, $\overline{X} = \mathrm{R}\,\Gamma_{\mathfrak{m}}(X)$, and $\overline{Z} = \mathrm{R}\,\Gamma_{\mathfrak{m}}(Z)$. By Example 2.2(6), $\mathrm{depth}(Y) = \inf(\overline{Y})$. A similar assertion holds for X and Z. It follows from (E3.7.2) that there is a distinguished triangle

$$(\text{E3.9.1}) \qquad \overline{X} \to \overline{Y} \to \overline{Z} \to \overline{X}[1].$$

Hence $\inf(\overline{Y}) \geq \min\{\inf(\overline{X}), \inf(\overline{Z})\}$, consequently,
$$\mathrm{depth}(Y) = \inf(\overline{Y}) \geq \min\{\inf(\overline{X}), \inf(\overline{Z})\} = \min\{\mathrm{depth}(X), \mathrm{depth}(Z)\}.$$

By definition, X is a version of Y with w being replaced by $w - 1$. Hence the first inequality follows by the above inequality and induction on w.

(3) It is clear that
$$\deg H_{\mathfrak{m}}^f(Z) = \deg H_{\mathfrak{m}}^f(F_w[w]) = \deg H_{\mathfrak{m}}^d(A) + \sigma_w.$$

It remains to show that $\deg H_{\mathfrak{m}}^f(Y) \leq \deg H_{\mathfrak{m}}^f(Z)$.

By part (2),

$$\operatorname{depth}(X) \geq \min_{0 \leq s \leq w-1} \{\operatorname{depth}(F_s[s])\} \geq -(w-1) + \operatorname{depth}(A) = f + 1.$$

As a consequence, $H_{\mathfrak{m}}^i(X) = 0$ for all $i \leq f$. Taking the cohomology group of (E3.9.1) gives rise to a long exact sequence

$$(\text{E3.9.2}) \qquad \cdots \to 0 \to 0(= H_{\mathfrak{m}}^f(X)) \to H_{\mathfrak{m}}^f(Y) \to H_{\mathfrak{m}}^f(Z) \to H_{\mathfrak{m}}^{f+1}(X) \to \cdots.$$

Therefore $H_{\mathfrak{m}}^f(Y)$ is a graded subspace of $H_{\mathfrak{m}}^f(Z)$, consequently, $\deg H_{\mathfrak{m}}^f(Y) \leq \deg H_{\mathfrak{m}}^f(Z)$ as desired.

(4) Since Y is a minimal free complex, the image of the map $Z \to X[1]$ is in $\mathfrak{m}X$. Define a complex U where $U^s = X^s$ for all $1 \leq s \leq w-2$ and $U^{w-1} = \coprod_{\sigma_{w-1,j} < \sigma_w} A(-\sigma_{w-1,j})$. Then U is a subcomplex of X and the image of the map $Z \to X[1]$ is in $U[1]$. Hence the composition $Z \to U[1] \to X[1]$ is the map $Z \to X[1]$. It suffices to show that the map $H_{\mathfrak{m}}^f(Z)_\alpha \to H_{\mathfrak{m}}^f(U[1])_\alpha$ is zero. By part (3),

$$\deg H_{\mathfrak{m}}^f(U[1]) \leq \deg(H_{\mathfrak{m}}^f(U^{w-1}[w])) \leq \deg H_{\mathfrak{m}}^d(A) + (\sigma_w - 1) = \alpha - 1.$$

This means that $H_{\mathfrak{m}}^f(U[1])_\alpha = 0$, and consequently, the map $H_{\mathfrak{m}}^f(Z)_\alpha \to H_{\mathfrak{m}}^f(U[1])_\alpha$ is zero.

(5) By definition, $\operatorname{CMreg}_\xi(Z) = \operatorname{CMreg}_\xi(F_w[w])$. When $\xi_1 \ll 0$,

$$\operatorname{CMreg}_\xi(F_w[w]) = \xi_0 \deg(H_{\mathfrak{m}}^d(A) + \sigma_w) + \xi_1(-w + \operatorname{depth}(A)) = \xi_0 \alpha + \xi_1 f.$$

It is clear that, when $\xi_1 \ll 0$,

$$\max_{0 \leq s \leq w} \{\operatorname{CMreg}_\xi(F_s[s])\} = \operatorname{CMreg}_\xi(F_w[w]) = \operatorname{CMreg}_\xi(Z).$$

By part (1), it remains to show that $\operatorname{CMreg}_\xi(Y) \geq \operatorname{CMreg}_\xi(Z)$. By part (4) and (E3.9.2), $H_{\mathfrak{m}}^f(Y)_\alpha \cong H_{\mathfrak{m}}^f(Z)_\alpha$. By the definition of α and f, $H_{\mathfrak{m}}^f(Z)_\alpha \neq 0$. So $\deg H_{\mathfrak{m}}^f(Y) \geq \alpha$ (in fact =). Therefore $\operatorname{CMreg}_\xi(Y) \geq \xi_0 \alpha + \xi_1 f = \operatorname{CMreg}_\xi(Z)$ as desired. $\square$

We are now ready to prove Theorem 0.4.

Theorem 3.10. *Retain the above notation. Let W be nonzero in $\mathrm{D}_{\mathrm{fg}}^{\mathrm{b}}(A\text{-}\mathrm{Gr})$ with finite projective dimension. Suppose $\xi_0 > 0$.*

(1) *If $0 \leq \xi_1 \leq \xi_0$, then*

$$\operatorname{CMreg}_\xi(W) = \operatorname{Torreg}_\xi(W) + \operatorname{CMreg}_\xi(A).$$

(2) *If $\xi_1 \ll 0$, then*

$$\operatorname{CMreg}_\xi(W) = \operatorname{Torreg}_\xi(W) + \operatorname{CMreg}_\xi(A).$$

Note that [KWZ21, Theorem 2.8] is a special case of Theorem 3.10 by taking $\xi = (1,1)$. Part of the proof of Theorem 3.10 is similar to the proof of [KWZ21, Theorem 2.8].

Proof of Theorem 3.10. (1) By Lemma 3.1(1), we may assume that $\xi_0 = 1$. So the condition $\xi_1 \leq \xi_0$ becomes $\xi := \xi_1 \leq 1$. By Lemma 3.1(5, 6), we can assume that $\xi \neq 0, 1$.

By Lemma 3.1(2), after a complex shift, we may assume that $W^n = 0$ for all $n \geq 1$. Let F be a minimal free resolution of W, so it can be written as

$$F : \qquad \cdots \to 0 \to F^{-s} \xrightarrow{d^{-s}} \cdots \to F^{-1} \xrightarrow{d^{-1}} F^0 \to 0 \to \cdots$$

for some $s \geq 0$. (Note that this F is different from the complex introduced in (E3.7.1).) We will prove the assertion by induction on s, which is the projective dimension of W.

For the initial step, we assume that $s = 0$, or $W = F^0 = \bigoplus_i A(-a_i)$ for some integers a_i. In this case, it is clear that

$$\text{Torreg}_\xi(W) = \text{Torreg}_\xi(F^0) = \max_i\{a_i\} =: a.$$

By Lemma 3.1(2),

$$\text{CMreg}_\xi(W) = \text{CMreg}_\xi\left(\bigoplus_i A(-a_i)\right) = \max_i\{\text{CMreg}_\xi(A(-a_i))\}$$
$$= \text{CMreg}_\xi(A) + \max_i\{a_i\} = \text{CMreg}_\xi(A) + \text{Torreg}_\xi(W).$$

So the assertion holds for $W = F^0$ as required.

For the inductive step, assume that $s > 0$. Let $F^{\leq -1}$ be the brutal truncation of the complex F

$$F^{\leq -1} : \qquad \cdots \to 0 \to F^{-s} \to \cdots \to F^{-1} \to 0 \to 0 \to \cdots,$$

which is obtained by replacing F^0 with 0. We have a distinguished triangle in $\mathsf{D}^b_{\text{fg}}(A\text{-Gr})$

(E3.10.1) $$F^0 \xrightarrow{f} F \to F^{\leq -1} \to F^0[1]$$

where F^0 is viewed as a complex concentrated at position 0 and f is the inclusion. Let G be the complex $F^{\leq -1}[-1]$, which is a minimal free complex concentrated in position $\{-(s-1), \cdots, 0\}$. Then we have a distinguished triangle in $\mathsf{D}^b_{\text{fg}}(A\text{-Gr})$

(E3.10.2) $$G \xrightarrow{\phi_2} F^0 \xrightarrow{f} F \to G[1]$$

obtained by rotating (E3.10.1). By the induction hypothesis, the assertion holds for both G and F^0. We need to show that the assertion holds for W, or equivalently, for F, as $F \cong W$ in $\mathsf{D}^b_{\text{fg}}(A\text{-Gr})$. By Theorem 0.3(2) (= Theorem 3.5), it suffices to show

(E3.10.3) $$\text{CMreg}_\xi(F) \geq \text{Torreg}_\xi(F) + \text{CMreg}_\xi(A).$$

We fix the following temporary notation:

$$a = \text{Torreg}_\xi(F^0), \quad b = \text{Torreg}_\xi(G), \quad c = \text{Torreg}_\xi(F) = \text{Torreg}_\xi(W),$$

and

$$\alpha = \text{CMreg}_\xi(F^0), \quad \beta = \text{CMreg}_\xi(G), \quad \gamma = \text{CMreg}_\xi(F) = \text{CMreg}_\xi(W).$$

Note that a is an integer. By definition, the minimality of F, and Lemma 3.8(1), we have

$$c = \max\left\{\text{Torreg}_\xi(F^0), \text{Torreg}_\xi(F^{\leq -1})\right\} = \max\{a, b - \xi\}.$$

Therefore, we have

(E3.10.4) $$a \leq c \quad \text{and} \quad b - \xi \leq c.$$

There are only three cases to consider:

> **Case 1.** $c = a$ and $a \geq b$,
> **Case 2.** $c = b - \xi$ (where $0 < \xi < 1$ by the first paragraph), and
> **Case 3.** $c = a$ and $a < b$.

Case 1. Suppose that $c = a$ and $a \geq b$. By the definition of a, we have $F^0 = A(-a) \oplus C^0$ where C^0 is a graded left free A-module. Let $\phi_1 : F^0 \to A(-a)$ be the corresponding projection. By the definition of $\alpha := \mathrm{CMreg}_\xi(F^0)$, there is an integer $j \in \mathbb{Z}$ such that $H_\mathfrak{m}^j(F^0)_{\alpha - \xi j} \neq 0$ and the induced projection

$$\tau_1 := H_\mathfrak{m}^j(\phi_1)_{\alpha - \xi j} : \quad H_\mathfrak{m}^j(F^0)_{\alpha - \xi j} \to H_\mathfrak{m}^j(A(-a))_{\alpha - \xi j}$$

is nonzero. The triangle (E3.10.2) gives rise to a long exact sequence
(E3.10.5)
$$\cdots \to H_\mathfrak{m}^j(G)_{\alpha - \xi j} \xrightarrow{\tau_2} H_\mathfrak{m}^j(F^0)_{\alpha - \xi j} \to H_\mathfrak{m}^j(F)_{\alpha - \xi j} \to H_\mathfrak{m}^{j+1}(G)_{\alpha - \xi j} \to \cdots .$$

Let $\phi_2 : G \to F^0$ as in (E3.10.2). If

$$\tau_2 := H_\mathfrak{m}^j(\phi_2)_{\alpha - \xi j} : \quad H_\mathfrak{m}^j(G)_{\alpha - \xi j} \to H_\mathfrak{m}^j(F^0)_{\alpha - \xi j}$$

is not surjective, then (E3.10.5) implies that $H_\mathfrak{m}^j(F)_{\alpha - \xi j} \neq 0$. By definition, the assumption that $a = c$, and the induction hypothesis, we have

$$\begin{aligned}
\mathrm{CMreg}_\xi(F) &\geq \alpha - \xi j + \xi j = \alpha \\
&= a + \mathrm{CMreg}_\xi(A) = c + \mathrm{CMreg}_\xi(A) \\
&= \mathrm{Torreg}_\xi(F) + \mathrm{CMreg}_\xi(A)
\end{aligned}$$

as desired. It remains to show that τ_2 is not surjective. This follows from the following claim.

Claim. If $b < a + 1$ (which covers Case 1), then $\tau_2 := H_\mathfrak{m}^j(\phi_2)_{\alpha - \xi j}$ is not surjective.

Proof of Claim. Assume to the contrary that τ_2 is surjective. Then so is the composition

$$\tau_3 := \tau_1 \circ \tau_2 : \quad H_\mathfrak{m}^j(G)_{\alpha - \xi j} \to H_\mathfrak{m}^j(A(-a))_{\alpha - \xi j}.$$

In particular, τ_3 is not the zero map. Note that

$$\tau_3 = \tau_1 \circ \tau_2 = H_\mathfrak{m}^j(\phi_1)_{\alpha - \xi j} \circ H_\mathfrak{m}^j(\phi_2)_{\alpha - \xi j} = H_\mathfrak{m}^j(\phi_1 \circ \phi_2)_{\alpha - \xi j},$$

which implies that $\phi_3 := \phi_1 \circ \phi_2$ is nonzero in $\mathsf{D}_{\mathrm{fg}}^{\mathrm{b}}(A\text{-Gr})$. Consider F as the cone of the map $\phi_2 : G \to F^0$; it is clear that ϕ_2 is the map from the top row G to the middle row F^0 in the following diagram

$$
\begin{array}{ccccccccc}
F^{-s} & \longrightarrow & \cdots & \longrightarrow & F^{-2} & \longrightarrow & F^{-1} & \longrightarrow & 0 \\
\downarrow {\scriptstyle 0} & & & & \downarrow {\scriptstyle 0} & & \downarrow {\scriptstyle d^{-1} = \phi_2} & & \\
0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & F^0 & \longrightarrow & 0 \\
\downarrow {\scriptstyle 0} & & & & \downarrow {\scriptstyle 0} & & \downarrow {\scriptstyle \phi_1} & & \\
0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & A(-a) & \longrightarrow & 0.
\end{array}
$$

Note that F^{-1} is the zeroth term in the minimal free resolution of G. Since $b < a+1$, F^{-1} is generated in degree $< a+1$. Since a is an integer, F^{-1} is generated in degree $\leq a$. Since F is a minimal free resolution of W, $\mathrm{im}\, \phi_2 \subseteq \mathfrak{m} F^0$, and consequently,

im $\phi_3 \subseteq \mathfrak{m}A(-a)$. For every generator x in F^{-1} which has degree $\leq a$, the image $\phi_3(x)$ lies in $\mathfrak{m}A(-a)$, which has degree at least $a+1$. Therefore $\phi_3(x) = 0$. This implies that $\phi_3(F^{-1}) = 0$, yielding a contradiction. So we have proved the claim. $\square$

Case 2. Suppose $c = b - \xi$. Since $\xi > 0$ by hypothesis, $c < b$. By the definition of $\beta := \mathrm{CMreg}_\xi(G)$, there is an integer $j \in \mathbb{Z}$ such that $H_\mathfrak{m}^j(G)_{\beta - \xi j} \neq 0$. The triangle (E3.10.2) gives rise to a long exact sequence

(E3.10.6) $\qquad \cdots \to H_\mathfrak{m}^{j-1}(F)_{\beta - \xi j} \to H_\mathfrak{m}^j(G)_{\beta - \xi j} \to H_\mathfrak{m}^j(F^0)_{\beta - \xi j} \to \cdots .$

By the induction hypothesis, the assumption that $c < b$, (E3.10.4), and the definitions of α, β, a and b, we have

$$\beta = \mathrm{CMreg}_\xi(G) = \mathrm{Torreg}_\xi(G) + \mathrm{CMreg}_\xi(A) = b + \mathrm{CMreg}_\xi(A)$$
$$> c + \mathrm{CMreg}_\xi(A) \geq a + \mathrm{CMreg}_\xi(A) = \mathrm{CMreg}_\xi(F^0)$$
$$= \alpha,$$

which implies that $H_\mathfrak{m}^j(F^0)_{\beta - \xi j} = 0$. Since $H_\mathfrak{m}^j(G)_{\beta - \xi j} \neq 0$ by definition, (E3.10.6) implies that $H_\mathfrak{m}^{j-1}(F)_{\beta - \xi j} \neq 0$. By definition, $\mathrm{CMreg}_\xi(F) \geq \beta - \xi j + \xi(j-1) = \beta - \xi$. This inequality implies that

$$\mathrm{CMreg}_\xi(F) \geq \beta - \xi = \mathrm{CMreg}_\xi(G) - \xi$$
$$= \mathrm{Torreg}_\xi(G) + \mathrm{CMreg}_\xi(A) - \xi = b + \mathrm{CMreg}_\xi(A) - \xi$$
$$= c + \mathrm{CMreg}_\xi(A) = \mathrm{Torreg}_\xi(F) + \mathrm{CMreg}_\xi(A),$$

as desired, see (E3.10.3).

Case 3. Finally, suppose that $c = a$ and $a < b$. Since $c = \max\{a, b - \xi\}$, we must have $b - \xi \leq c = a < b$. If $b - \xi = c$, the assertion follows from Case 2. It remains to show the assertion when $b - \xi < c = a < b$. Recall that $0 < \xi < 1$.

By the definition of $\beta := \mathrm{CMreg}_\xi(G)$, there is an integer $j \in \mathbb{Z}$ such that $H_\mathfrak{m}^j(G)_{\beta - \xi j} \neq 0$. Since $a < b$, by the induction hypothesis, $\alpha < \beta$, consequently, $H_\mathfrak{m}^j(F^0)_{\beta - \xi j} = 0$. By (E3.10.6), we obtain that $H_\mathfrak{m}^{j-1}(F)_{\beta - \xi j} \neq 0$. Therefore

$$\mathrm{CMreg}_\xi(F) \geq \beta - \xi j + \xi(j-1) = \beta - \xi.$$

By the definition of $\alpha := \mathrm{CMreg}_\xi(F^0)$, there is another integer, still denoted by j, such that $H_\mathfrak{m}^j(F^0)_{\alpha - \xi j} \neq 0$. Since $b - \xi < a$ or $b < a + \xi$, by the induction hypothesis, $\beta < \alpha + \xi$, consequently, $H_\mathfrak{m}^{j+1}(G)_{\alpha - \xi j} = 0$. Since $b < a + \xi < a + 1$, by Claim, τ_2 is not surjective. By (E3.10.5), we obtain that $H_\mathfrak{m}^j(F)_{\alpha - \xi j} \neq 0$. Therefore

$$\mathrm{CMreg}_\xi(F) \geq \alpha - \xi j + \xi j = \alpha.$$

Combining the two previous inequalities, using the induction hypothesis and the fact that $c = \max\{a, b - \xi\}$, we obtain that

$$\mathrm{CMreg}_\xi(F) \geq \max\{\alpha, \beta - \xi\}$$
$$= \max\{a + \mathrm{CMreg}_\xi(A), b - \xi + \mathrm{CMreg}_\xi(A)\}$$
$$= \max\{a, b - \xi\} + \mathrm{CMreg}_\xi(A)$$
$$= c + \mathrm{CMreg}_\xi(A) = \mathrm{Torreg}_\xi(F) + \mathrm{CMreg}_\xi(A)$$

as desired.

Combining these three cases completes the proof.

(2) We will use the notation introduced before Lemma 3.8.

First of all, when $W = A$, we have

$$\mathrm{CMreg}_\xi(W) = \mathrm{CMreg}_\xi(A) = 0 + \mathrm{CMreg}_\xi(A) = \mathrm{Torreg}_\xi(A) + \mathrm{CMreg}_\xi(A)$$
$$= \mathrm{Torreg}_\xi(W) + \mathrm{CMreg}_\xi(A),$$

so the assertion holds for $W = A$. By Lemma 3.1(1), the assertion holds for $W = Z := [\coprod_{j=1}^{n_w} A(-\sigma_{w,j})][w]$. When $\xi_1 \ll 0$, we have, by setting $W = Y$,

$$\mathrm{CMreg}_\xi(W) = \mathrm{CMreg}_\xi(Y) = \mathrm{CMreg}_\xi(Z) \quad \text{by Lemma 3.9(5)}$$
$$= \mathrm{Torreg}_\xi(Z) + \mathrm{CMreg}_\xi(A)$$
$$= \mathrm{Torreg}_\xi(Y) + \mathrm{CMreg}_\xi(A) \quad \text{by Lemma 3.8(2)}$$
$$= \mathrm{Torreg}_\xi(W) + \mathrm{CMreg}_\xi(A)$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Remark 3.11. Retain the hypotheses of Theorem 3.10.

(1) If $\xi_1 > \xi_0 > 0$, the conclusion of Theorem 3.10(1),

(E3.11.1) $$\mathrm{CMreg}_\xi(W) = \mathrm{Torreg}_\xi(W) + \mathrm{CMreg}_\xi(A)$$

may fail to hold for some W. For example, let A be a noetherian Koszul AS regular algebra of type $(d, \mathfrak{l})$ with $d = \mathfrak{l} \geq 1$. By Example 2.2(2), $\mathrm{CMreg}_\xi(A) = \xi_1 d - \xi_0 \mathfrak{l} = (\xi_1 - \xi_0)d$. It is easy to check that $\mathrm{Torreg}_\xi(\Bbbk) = 0$ when $\xi_1 > \xi_0 > 0$. By Example 2.2(1), $\mathrm{CMreg}_\xi(\Bbbk) = 0$. Then

$$\mathrm{CMreg}_\xi(\Bbbk) = 0 < (\xi_1 - \xi_0)d = \mathrm{Torreg}_\xi(\Bbbk) + \mathrm{CMreg}_\xi(A).$$

(2) By Theorem 3.10(2) (E3.11.1) holds for all $\xi_1 \ll 0$. It is unknown if equation (E3.11.1) holds for all $\xi_1 < 0$.

Next we recover the theorem of Auslander and Buchsbaum [Jør98, Theorem 3.2] as a special case of Theorem 3.10.

Corollary 3.12. *Let A be a noetherian connected graded algebra with balanced dualizing complex. Let X be a nonzero object in $\mathrm{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-Gr})$ with finite projective dimension.*

(1) [*The Auslander–Buchsbaum Formula*]

$$\mathrm{pdim}(X) + \mathrm{depth}(X) = \mathrm{depth}(A).$$

(2) *Let $p(X) = \mathrm{pdim}(X)$ and $d(X) = \mathrm{depth}(X)$. Then*

$$\deg H_{\mathfrak{m}}^{d(X)}(X) = \deg \mathrm{Tor}_{p(X)}^{A}(\Bbbk, X) + \deg H_{\mathfrak{m}}^{d(A)}(A).$$

Proof. Let $\xi_0 > 0$ be fixed. When $\xi_1 \ll 0$, by definition,

$$\mathrm{CMreg}_\xi(X) = \sup_{i \in \mathbb{Z}}\{\xi_0 \deg(H_{\mathfrak{m}}^{i}(X)) + \xi_1 i\}$$
$$= \xi_0 \deg(H_{\mathfrak{m}}^{d(X)}(X)) + \xi_1 d(X),$$
$$\mathrm{Torreg}_\xi(X) = \sup_{i \in \mathbb{Z}}\{\xi_0 \deg(\mathrm{Tor}_{i}^{A}(\Bbbk, X)) - \xi_1 i\}$$
$$= \xi_0 \deg(\mathrm{Tor}_{p(X)}^{A}(\Bbbk, X)) - \xi_1 p(X).$$

As a special case of the first equation, when $\xi_1 \ll 0$,

$$\mathrm{CMreg}_\xi(A) = \xi_0 \deg(H_\mathfrak{m}^{d(A)}(A)) + \xi_1 d(A).$$

By Theorem 3.10, we have, for all $\xi_1 \ll 0$,

$$\mathrm{CMreg}_\xi(X) = \mathrm{Torreg}_\xi(X) + \mathrm{CMreg}_\xi(A),$$

or equivalently,

$$\xi_0 \deg(H_\mathfrak{m}^{d(X)}(X)) + \xi_1 d(X) = \xi_0 \deg(\mathrm{Tor}_{p(X)}^A(\Bbbk, X)) - \xi_1 p(X) + \xi_0 \deg(H_\mathfrak{m}^{d(A)}(X)) + \xi_1 d(A).$$

Therefore both parts (1) and (2) follow from the above equation by comparing the coefficients on ξ_0 and ξ_1 as ξ_1 varies. $\qquad\square$

To conclude this section, we generalize [Jør04, Theorem 3.1] by removing the Koszul assumption. As noted after Definition 2.1, if M is a finitely generated left A-module then $\mathrm{CMreg}_\xi(M)$ is finite.

Theorem 3.13. *Let* $\xi := (1, \xi_1)$, $\epsilon := \max\{0, \xi_1 - 1\}$, *and* $c := \mathrm{Torreg}_\xi(\Bbbk)$. *Let* M *be a nonzero finitely generated graded left A-module. Then for every integer* $s \geq \mathrm{CMreg}_\xi(M)$, *we have*

$$i \leq \mathrm{ged}\, \mathrm{Tor}_i^A(\Bbbk, M_{\geq s}(s)) \leq \deg \mathrm{Tor}_i^A(\Bbbk, M_{\geq s}(s)) \leq c + \epsilon + i\xi_1$$

whenever $\mathrm{Tor}_i^A(\Bbbk, M_{\geq s}(s)) \neq 0$.

Proof. Since $M_{\geq s}(s)$ is a module generated by elements of non-negative degrees, it is clear that $i \leq \mathrm{ged}\, \mathrm{Tor}_i^A(\Bbbk, M_{\geq s}(s))$ when $\mathrm{Tor}_i^A(\Bbbk, M_{\geq s}(s)) \neq 0$. It remains to show that if $s \geq \mathrm{CMreg}_\xi(M)$, then $\deg \mathrm{Tor}_i^A(\Bbbk, M_{\geq s}(s)) \leq c + \epsilon + i\xi_1$ for all i, or equivalently that $\mathrm{Torreg}_\xi(M_{\geq s}(s)) \leq c + \epsilon$.

Let $s \geq \mathrm{CMreg}_\xi(M)$. It is clear that $\deg M/M_{\geq s} \leq s - 1$. By Example 2.2(1),

$$\mathrm{CMreg}_\xi(M/M_{\geq s}) = \deg M/M_{\geq s} \leq s - 1.$$

Applying Lemma 3.7(2) to the short exact sequence

$$0 \to M_{\geq s} \to M \to M/M_{\geq s} \to 0,$$

we obtain that $\mathrm{CMreg}_\xi(M_{\geq s}) \leq \max\{s, s - 1 + \xi_1\} = s + \epsilon$. By Theorem 3.3,

$$\mathrm{Torreg}_\xi(M_{\geq s}) \leq \mathrm{CMreg}_\xi(M_{\geq s}) + \mathrm{Torreg}_\xi(\Bbbk) \leq s + \epsilon + c.$$

By Lemma 3.1(2) we obtain that $\mathrm{Torreg}_\xi(M_{\geq s}(s)) \leq c + \epsilon$ as desired. $\qquad\square$

If A is Koszul then $c = \mathrm{Torreg}_\xi(\Bbbk) = 0$, and if, in addition, $\xi_1 = 1$ then Theorem 3.13 says that $M_{\geq s}(s)$ has a linear resolution, which recovers [Jør04, Theorem 3.1].

4. WEIGHTED ARTIN–SCHELTER REGULARITY

Extending Definition 0.2(3) to any $\xi = (\xi_0, \xi_1)$ the weighted AS-regularity (or ξ-AS regularity) of A is defined to be

$$\mathrm{ASreg}_\xi(A) = \mathrm{Torreg}_\xi(\Bbbk) + \mathrm{CMreg}_\xi(A).$$

In this section we prove results that are related to AS Gorenstein and AS regular algebras. We begin with a generalization of a nice result of Dong and Wu [DW09, Theorem 4.10] that provides a proof of part of Theorem 0.6, showing that (1) and (3) are equivalent; parts (1) and (2) are equivalent by [KWZ21, Theorem 0.8]. Note that by Remark 2.6(1,3), the existence of ξ such that $\mathrm{ASreg}_\xi(A) = 0$ in part (ii) of Theorem 4.1 does not imply that $\mathrm{ASreg}_\zeta(A) = 0$ for all ζ.

Theorem 4.1. *Let A be a noetherian connected graded algebra with balanced dualizing complex. Then the following are equivalent:*

(i) *A is AS regular.*

(ii) *A is Cohen–Macaulay and there exists a $\xi = (\xi_0, \xi_1)$ with $\xi_0 > 0$ such that $\mathrm{ASreg}_\xi(A) = 0$.*

An AS Gorenstein algebra is called *standard* if $\mathfrak{l} = d$. When A is Koszul, then [DW09, Theorem 4.10] can be recovered from Theorem 4.1 since, by definition, standard AS Gorenstein algebras satisfy (ii) in the above theorem. Note that [KWZ21, Theorem 3.2] is a special case of Theorem 4.1 by taking $\xi = (1, 1)$. Our proof of Theorem 4.1 is very close to the proof of [KWZ21, Theorem 3.2].

Proof of Theorem 4.1. We first prove that (i) implies (ii). Suppose that A is AS regular of type $(d, \mathfrak{l})$. It is well-known that A is Cohen–Macaulay. If we let $\xi = (1, 1)$, then
$$\mathrm{CMreg}_\xi(A) = d - \mathfrak{l} = -\mathrm{Torreg}_\xi(\Bbbk)$$
by Remark 2.6(3), and so part (ii) holds.

We now show that (ii) implies (i). Let A be noetherian connected graded with balanced dualizing complex. If $\mathrm{pdim}\,\Bbbk < \infty$, then A has finite global dimension. Since A is noetherian, if it has finite global dimension, then it has finite GK dimension. By [Zha97, Theorem 0.3], A is AS Gorenstein and so A is AS regular by definition. Hence, it suffices to show that $\mathrm{pdim}\,\Bbbk < \infty$.

Let
$$(E4.1.1) \qquad F: \quad \cdots \to F_i \to \cdots \to F_0 \to {}_A\Bbbk \to 0$$
be a minimal free resolution of the trivial left A-module ${}_A\Bbbk$. Since A is Cohen–Macaulay, by [VDB97, Theorem 6.3], the balanced dualizing complex over A is
$$R \cong \mathrm{R}\,\Gamma_{\mathfrak{m}}(A)' \cong \omega[d]$$
where ω is a dualizing A-bimodule and $d = \mathrm{lcd}(A)$. By Theorem 1.4(1), for every complex X of left graded A-modules,
$$(E4.1.2) \qquad \mathrm{R}\,\Gamma_{\mathfrak{m}}(X)' \cong \mathrm{R}\,\mathrm{Hom}_A(X, R) \cong \mathrm{R}\,\mathrm{Hom}_A(X, \omega[d]).$$
Since the dualizing complex has finite injective dimension, (E4.1.2), taking X to be an A-module M, implies that
$$d = \mathrm{injdim}(\omega) < \infty.$$
As a consequence of (E4.1.2), $\Gamma_{\mathfrak{m}}$ has cohomological dimension d.

For each $j \geq 0$, let $Z_j(F)$ denote the jth syzygy of the complex F (E4.1.1). We will show that $Z_j(F) = 0$ for $j \gg 0$, which implies that $\mathrm{pdim}\,\Bbbk < \infty$ as desired. Assume to the contrary that there is an increasing sequence $j_1 < j_2 < \cdots$ such that $Z_{j_s}(F) \neq 0$ for all $s \geq 1$. Then $Z_j(F) \neq 0$ for all $j \geq 0$. Note that
$$\cdots \to F_{j+2} \to F_{j+1} \to Z_j(F) \to 0$$
is a minimal free resolution of $Z_j(F)$.

Claim. For all $j \geq d$, $t_{j+1}^A(\Bbbk) \leq t_j^A(\Bbbk)$.

Proof of the claim. By the balanced dualizing complex condition, $\mathrm{Ext}_A^i(\Bbbk, \omega) = 0$ for all $i \neq d = \mathrm{injdim}\,\omega$. By induction on syzygies, we have $\mathrm{Ext}_A^i(Z_{d-1}(F), \omega) = 0$ for all $i \neq 0$. Further, by induction, one sees that $\mathrm{Ext}_A^i(Z_{j-1}(F), \omega) = 0$ for all

$i \neq 0$ and all $j \geq d$. From now on, we fix $j \geq d$. By (E4.1.2), we obtain that $H_{\mathfrak{m}}^i(Z_{j-1}(F)) = 0$ for all $i \neq d$. Since A is Cohen–Macaulay, $H_{\mathfrak{m}}^i(F_j) = 0$ for all $i \neq d$. Applying $\operatorname{R}\Gamma_{\mathfrak{m}}(-)$ to the short exact sequence

$$0 \to Z_j(F) \to F_j \to Z_{j-1}(F) \to 0,$$

we obtain a long exact sequence, which has only three nonzero terms yielding a short exact sequence

$$0 \to H_{\mathfrak{m}}^d(Z_j(F)) \to H_{\mathfrak{m}}^d(F_j) \to H_{\mathfrak{m}}^d(Z_{j-1}(F)) \to 0.$$

The above short exact sequence implies that $\deg H_{\mathfrak{m}}^d(Z_j(F)) \leq \deg H_{\mathfrak{m}}^d(F_j)$. By definition,

(E4.1.3) $$\operatorname{CMreg}_\xi(Z_j(F)) \leq \operatorname{CMreg}_\xi(F_j).$$

By Corollary 3.6, for any $X \in \mathsf{D}_{\mathrm{fg}}^{\mathrm{b}}(A\text{-Gr})$

(E4.1.4) $$\operatorname{Torreg}_\xi(X) = \operatorname{CMreg}_\xi(X) + c,$$

where $c = -\operatorname{CMreg}_\xi(A)$. Then

$$
\begin{aligned}
t_{j+1}^A(\Bbbk) = t_0^A(F_{j+1}) &= t_0^A(Z_j(F)) \\
&= \frac{1}{\xi_0}(\xi_0 t_0^A(Z_j(F)) - \xi_1 0) \\
&\leq \frac{1}{\xi_0} \sup\{\xi_0 t_i^A(Z_j(F)) - \xi_1 i \mid i \in \mathbb{Z}\} \\
&= \frac{1}{\xi_0} \operatorname{Torreg}_\xi(Z_j(F)) \\
&= \frac{1}{\xi_0}(\operatorname{CMreg}_\xi(Z_j(F)) + c) \qquad \text{by (E4.1.4)} \\
&\leq \frac{1}{\xi_0}(\operatorname{CMreg}_\xi(F_j) + c) \qquad \text{by (E4.1.3)} \\
&= \frac{1}{\xi_0} \operatorname{Torreg}_\xi(F_j) \\
&= \frac{1}{\xi_0} \sup\{\xi_0 t_i^A(F_j) - \xi_1 i \mid i \in \mathbb{Z}\} \\
&= \frac{1}{\xi_0}(\xi_0 t_0^A(F_j) - \xi_1 0) = t_0^A(F_j) \\
&= t_j^A(\Bbbk)
\end{aligned}
$$

as desired. This finishes the proof of the claim. □

Since A is connected graded and since F is the minimal free resolution of $_A\Bbbk$, we have

(E4.1.5) $$t_j^A(\Bbbk) \geq j$$

whenever $F_j \neq 0$. Then for $j \gg 0$, the claim contradicts (E4.1.5). Therefore we obtain a contradiction, and hence $\operatorname{pdim}\Bbbk < \infty$ as required. □

Remark 4.2. Suppose [KWZ22, Hypothesis 2.7] holds. Then, by Example 2.2(3), $\operatorname{CMreg}_\xi(A) = \xi_0 \deg_t h_A(t) + \xi_1 d$. Then (ii) is equivalent to

$$\operatorname{Torreg}_\xi(\Bbbk) = -(\xi_0 \deg_t h_A(t) + \xi_1 d)$$

which could be easier to compute than computing these regularities from their definitions.

Proof of Theorem 0.6. (1) $\Leftrightarrow$ (3) This is Theorem 4.1.

(1) $\Rightarrow$ (2) See the first paragraph of the proof of Theorem 4.1.

(2) $\Rightarrow$ (1) By the second paragraph of the proof of Theorem 4.1, it suffices to show that $\operatorname{pdim} \Bbbk < \infty$. If $\xi = 1$, the assertion follows from [KWZ21, Theorem 0.8]. Note that the hypothesis $\operatorname{ASreg}_\xi(A) = 0$ implies that $\operatorname{Torreg}_\xi(\Bbbk) < \infty$. If $\xi < 1$, it follows from (E2.4.1) that $\operatorname{pdim} \Bbbk < \infty$ as desired. $\qquad\square$

Proof of Corollary 0.7. By definition, the hypothesis implies that $\operatorname{Torreg}_\xi(\Bbbk) \leq \mathfrak{l} - \xi d = -\operatorname{CMreg}_\xi(A)$. Thus $\operatorname{ASreg}_\xi(A) \leq 0$. By Theorem 0.3(3), $\operatorname{ASreg}_\xi(A) = 0$. Now the assertion follows from Theorem 0.6(3$\Rightarrow$1). $\qquad\square$

The following result is a weighted and noncommutative (and non-generated in degree 1) version of [Röm08, Theorem 1.3(i$\Leftrightarrow$ii$\Leftrightarrow$v)].

Theorem 4.3. *Let A be a noetherian connected graded algebra with balanced dualizing complex. Then the following are equivalent:*

(1) *A is AS regular.*

(2) *There exists a pair $\xi = (\xi_0, \xi_1)$ with $\xi_0 > 0$ and $\xi_1 \leq \xi_0$ such that*
$$\operatorname{CMreg}_\xi(X) = \operatorname{Torreg}_\xi(X) + \operatorname{CMreg}_\xi(A)$$
for all $0 \neq X \in \mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-}\mathrm{Gr})$.

(3) *There exists a pair $\xi = (\xi_0, \xi_1)$ with $\xi_0 > 0$ and $\xi_1 \leq \xi_0$ such that*
$$\operatorname{CMreg}_\xi(M) = \operatorname{Torreg}_\xi(M) + \operatorname{CMreg}_\xi(A)$$
for all noetherian modules $0 \neq M \in A\text{-}\mathrm{Gr}$.

(4) *There exist c and a pair $\xi = (\xi_0, \xi_1)$ with $\xi_0 > 0$ and $\xi_1 \leq \xi_0$ such that*
$$\operatorname{CMreg}_\xi(X) = \operatorname{Torreg}_\xi(X) - c$$
for all $0 \neq X \in \mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-}\mathrm{Gr})$.

(5) *There exist c and a pair $\xi = (\xi_0, \xi_1)$ with $\xi_0 > 0$ and $\xi_1 \leq \xi_0$ such that*
$$\operatorname{CMreg}_\xi(M) = \operatorname{Torreg}_\xi(M) - c$$
for all noetherian modules $0 \neq M \in A\text{-}\mathrm{Gr}$.

Proof. (1) $\Rightarrow$ (2) This follows from Theorem 4.1 and Corollary 3.6.

(2) $\Rightarrow$ (3), (2) $\Rightarrow$ (4), (2) $\Rightarrow$ (5), (3) $\Rightarrow$ (5), (4) $\Rightarrow$ (5) Trivial.

(5) $\Rightarrow$ (1) Let $M = A$, then $\operatorname{CMreg}_\xi(A) = \operatorname{Torreg}_\xi(A) - c = -c$. Therefore $c = -\operatorname{CMreg}_\xi(A)$. Let $M = \Bbbk$. Then $0 = \operatorname{CMreg}_\xi(\Bbbk) = \operatorname{Torreg}_\xi(\Bbbk) - c$ which implies that $c = \operatorname{Torreg}_\xi(\Bbbk)$. Therefore $\operatorname{ASreg}_\xi(A) = \operatorname{CMreg}_\xi(A) + \operatorname{Torreg}_\xi(\Bbbk) = -c + c = 0$. Now the assertion follows from Theorem 0.6. $\qquad\square$

Remark 4.4. By Remark 2.6(1), if A is AS regular, $\operatorname{Torreg}_\xi(\Bbbk)$ may not be zero. So [Röm08, Theorem 1.3(i$\Leftrightarrow$iv)] cannot be generalized to the noncommutative case.

The next two theorems are weighted versions of [Jør99, Theorem 5.1] and [Jør99, Corollary 5.2]. In the following theorems and their proofs, we use both
$$0 \to F^{-s} \to \cdots \to F^0 \to 0$$
and
$$0 \to F_s \to \cdots \to F_0 \to 0$$

to denote the same complex F after identifying F^{-i} with F_i. Similarly, we use both $H^{-i}(F)$ and $H_i(F)$ for the same (co)homology.

Suppose that $F := 0 \to F^{-s} \to \cdots \to F^0 \to 0$ is a minimal complex of finitely generated free A-modules bounded left at position zero. Application of the functor $(-)^\vee := \operatorname{Hom}_A(-, A)$ on F yields the minimal free complex $F^\vee$ that is bounded above at position zero.

Theorem 4.5. *Retain the above notation. Let $\xi \le 1$ be a real number. Then, for all integers $0 \le c \le s$, we have*

$$\max_{0 \le j \le c} \left\{ -\operatorname{ged}(H^{s-j}(F^\vee)) + \xi j \right\} = \max_{0 \le j \le c} \left\{ \deg(H_{s-j}(\Bbbk \otimes_A F)) + \xi j \right\}.$$

Proof. We proceed by induction on $c \ge 0$.

For the initial step, let $c = 0$. If $F^{-s} = 0$, then both sides of the equation are $-\infty$, so we may assume that $F^{-s} \ne 0$. The complex $F^\vee$ is minimal, and the top term in this complex is $(F^\vee)^s = (F^{-s})^\vee = (F_s)^\vee$. Therefore

$$-\operatorname{ged} H^s(F^\vee) = -\operatorname{ged}(F^\vee)^s = -\operatorname{ged}(F_s^\vee),$$

which is equal to the maximal degree of the generators of F_s. But by the minimality of the complex this is again equal to $\deg H_s(\Bbbk \otimes_A F)$, so

$$-\operatorname{ged} H^s(F^\vee) + \xi 0 = \deg H_s(\Bbbk \otimes_A F) + \xi 0,$$

and the equation holds for $c = 0$.

For the inductive step, we employ the following notation: for an integer $e \ge 0$, let

$$x_e = \max_{0 \le j \le e} \left\{ -\operatorname{ged}(H^{s-j}(F^\vee)) + \xi j \right\}$$

$$y_e = \max_{0 \le j \le e} \left\{ \deg(H_{s-j}(\Bbbk \otimes_A F)) + \xi j \right\},$$

and suppose that $x_e = y_e$ for every $e \le c$. Set

$$\alpha = \text{maximal degree of a generator in } F_{s-c} = F^{-s+c},$$

$$\beta = \text{maximal degree of a generator in } F_{s-c-1} = F^{-s+c+1}.$$

There are two cases to consider.

Case 1. Suppose $\beta \le \alpha - 1$. In this case, using the fact that $\xi \le 1$, we obtain that

$$y_{c+1} = \max\{y_c, \beta + \xi(c+1)\} \le \max\{y_c, \beta + 1 + \xi c\} \le \max\{y_c, \alpha + \xi c\} = y_c.$$

By the induction hypothesis and the definition of y_c, we have $x_c = y_c \ge \alpha + \xi c$. Since $\operatorname{ged} H^{s-c-1}(F^\vee) \ge \operatorname{ged}(F^\vee)^{s-c-1}$, we obtain that

$$-\operatorname{ged} H^{s-c-1}(F^\vee) + \xi(c+1) \le -\operatorname{ged}(F^\vee)^{s-c-1} + \xi(c+1)$$
$$= \beta + \xi(c+1) \le \alpha + \xi c \le x_c,$$

whence $x_{c+1} = x_c$. Therefore, $x_{c+1} = x_c = y_c = y_{c+1}$.

Case 2. Suppose $\beta \ge \alpha$. In this case we have $y_{c+1} = \max\{y_c, \beta + \xi(c+1)\}$ as before. On the other hand, $F^\vee$ is minimal, so since $-\beta$ is the minimal degree of the generators of $(F^\vee)^{s-c-1}$ and $-\alpha$ is the minimal degree of the generators of $(F^\vee)^{s-c}$, the inequality $-\beta \le -\alpha$ implies that any element in the minimal degree of $(F^\vee)^{s-c-1}$ is mapped to zero in $(F^\vee)^{s-c}$ (since it has to have image inside $\mathfrak{m}(F^\vee)^{s-c}$, and this module begins in degree $-\alpha + 1$). By minimality of the

complex, elements in the minimal degree of $(F^\vee)^{s-c-1}$ are not in the image of the differential, hence $\operatorname{ged} H^{s-c-1}(F^\vee) = -\beta$. Now we obtain that

$$x_{c+1} = \max\{x_c, -\operatorname{ged} H^{s-c-1}(F^\vee) + \xi(c+1)\} = \max\{x_c, \beta + \xi(c+1)\},$$

which implies that $x_{c+1} = y_{c+1}$. By induction, we have proved the claim.

$\square$

Theorem 4.6 can be used to compute the weighted CM regularity of a finitely generated nonzero graded module over an AS Gorenstein algebra.

Theorem 4.6 (Theorem 0.8). *Suppose A is a noetherian, connected graded AS Gorenstein algebra of type $(d, \mathfrak{l})$. Let $\xi \leq 1$ be a real number (and also by abuse the notation let $\xi = (1, \xi)$). Let $M \neq 0$ be a finitely generated left graded A-module with finite projective dimension.*

(1) *Let w be an integer with $0 \leq w \leq d$. Then*

$$\max_{0 \leq j \leq w} \left\{ \deg H_{\mathfrak{m}}^j(M) + \xi j \right\} = -\mathfrak{l} + \xi d + \max_{d-w \leq j \leq d} \left\{ \deg \operatorname{Tor}_j^A(\Bbbk, M) - \xi j \right\}.$$

(2) *In particular, if $0 \leq w \leq d$ is the maximum integer such that $H_{\mathfrak{m}}^w(M) \neq 0$, we have*

$$\operatorname{CMreg}_\xi(M) = -\mathfrak{l} + \xi d + \max_{d-w \leq j \leq d} \left\{ \deg \operatorname{Tor}_j^A(\Bbbk, M) - \xi j \right\}.$$

(3) *If, further, M is s-Cohen–Macaulay, then*

$$\operatorname{CMreg}_\xi(M) = -\mathfrak{l} + \xi s + \deg(\operatorname{Tor}_{d-s}^A(\Bbbk, M)).$$

Proof. (1) Let p be the projective dimension of M. By the Auslander–Buchsbaum Formula, $p \leq \operatorname{depth}(A) = d$. Suppose $0 \to F_p \to \cdots \to F_0 \to M \to 0$ is a minimal free resolution of M and write

$$F := \qquad \cdots \to 0 \to F_p \to \cdots \to F_0 \to 0 \cdot$$

(where we have removed the M term). By Theorem 1.4(1) and the hypothesis that A is AS Gorenstein of type $(d, \mathfrak{l})$, for all integers i, we have

$$H_{\mathfrak{m}}^i(M) \cong \operatorname{Ext}_A^{d-i}(M, A)'(\mathfrak{l}).$$

Note that $\operatorname{Ext}_A^j(M, A)$ can be computed by using the complex $F^\vee := \operatorname{Hom}_A(F, A)$. Then we have

$$\max_{0 \leq j \leq w} \{\deg(H_{\mathfrak{m}}^j(M)) + \xi j\} = \max_{0 \leq j \leq w} \left\{ \deg\left(\operatorname{Ext}_A^{d-j}(M, A)'\right) - \mathfrak{l} + \xi j \right\}$$

$$= \max_{0 \leq j \leq w} \left\{ -\operatorname{ged}(H^{d-j}(F^\vee)) - \mathfrak{l} + \xi j \right\}$$

$$= -\mathfrak{l} + \xi d + \max_{0 \leq j \leq w} \left\{ -\operatorname{ged}(H^{d-j}(F^\vee)) - \xi d + \xi j \right\}$$

$$= -\mathfrak{l} + \xi d + \max_{d-w \leq k \leq d} \left\{ -\operatorname{ged}(H^k(F^\vee)) - \xi k \right\}$$

$$= -\mathfrak{l} + \xi d + \max_{d-w \leq k \leq p} \left\{ -\operatorname{ged}(H^k(F^\vee)) - \xi k \right\} \quad \text{since } \operatorname{pdim} M = p$$

$$= -\mathfrak{l} + \xi d + \max_{0 \leq i \leq p-d+w} \left\{ -\operatorname{ged}(H^{p-i}(F^\vee)) - \xi p + \xi i \right\}$$

$$= -\mathfrak{l} + \xi d - \xi p + \max_{0 \leq i \leq p-d+w} \left\{ -\operatorname{ged}(H^{p-i}(F^\vee)) + \xi i \right\}.$$

It is clear that

$$\max_{d-w\leq j\leq d}\left\{\deg(\mathrm{Tor}_j^A(\Bbbk,M))-\xi j\right\}=\max_{p-d\leq i\leq p-d+w}\left\{\deg(\mathrm{Tor}_{p-i}^A(\Bbbk,M))-\xi p+\xi i\right\}$$

$$=\max_{0\leq i\leq p-d+w}\left\{\deg(\mathrm{Tor}_{p-i}^A(\Bbbk,M))-\xi p+\xi i\right\}$$

$$=-\xi p+\max_{0\leq i\leq p-d+w}\left\{\deg(\mathrm{Tor}_{p-i}^A(\Bbbk,M))+\xi i\right\}.$$

Now by Theorem 4.5, we have

$$\max_{0\leq i\leq p-d+w}\left\{-\mathrm{ged}(H^{p-i}(F^\vee))+\xi i\right\}=\max_{0\leq i\leq p-d+w}\left\{\deg(\mathrm{Tor}_{p-i}^A(\Bbbk,M))+\xi i\right\}.$$

Therefore the assertion follows.

Part (2) is a special case of part (1).

For part (3), if M is s-Cohen–Macaulay, then $s=\mathrm{depth}(M)$, and by the Auslander–Buchsbaum formula, $\mathrm{pdim}(M)=\mathrm{depth}(A)-\mathrm{depth}(M)=d-s$ is the projective dimension of M. Taking $w=s=d-p$, it follows from part (2) that

$$\mathrm{CMreg}_\xi(M)=-\mathfrak{l}+\xi d+\max_{d-s\leq j\leq d}\{\deg\mathrm{Tor}_j^A(\Bbbk,M)-\xi j\}$$

$$=-\mathfrak{l}+\xi d+\max_{d-s\leq j\leq d-s}\left\{\deg\mathrm{Tor}_j^A(\Bbbk,M)-\xi j\right\}$$

$$\text{since }\mathrm{pdim}(M)=d-s$$

$$=-\mathfrak{l}+\xi d+\deg\mathrm{Tor}_{d-s}^A(\Bbbk,M)-\xi(d-s)$$

$$=-\mathfrak{l}+\xi s+\deg\mathrm{Tor}_{d-s}^A(\Bbbk,M)$$

as desired. □

Remark 4.7. Retain the hypothesis as Theorem 4.6(3). Let p be the projective dimension of M.

(1) Recall that $t^j(M)=\deg\mathrm{Tor}_j^A(\Bbbk,M)$ for all i. Note that $\mathrm{CMreg}_\xi(M)=\deg H_{\mathfrak{m}}^s(M)+\xi s$. It follows from Theorem 4.6(3) that

$$t^p(M)=\deg H_{\mathfrak{m}}^s(M)+\mathfrak{l}$$

which is a special case of Corollary 3.12(2).

(2) By Theorem 4.6(3, 2) (taking $w=d$), we obtain that

$$t^p(M)-\xi p=\mathrm{CMreg}_\xi(M)+\mathfrak{l}-\xi s-\xi p$$

$$=\max_{0\leq j\leq d}\left\{t^j(M)-\xi j\right\}$$

for all $\xi\leq 1$. Therefore, for each j, $t^j(M)-\xi j\leq t^p(M)-\xi p$ for all $\xi\leq 1$. By taking $\xi=1$, we have $t^p(M)-t^j(M)\geq p-j$ for all $0\leq j\leq p$.

5. Related invariants

In this section we consider concavity, rate, and slope, homological invariants that are related to our weighted regularities and homological invariants that have been studied in the literature.

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5.1. **Concavity.** In this subsection we use the letters A or B for connected graded noetherian algebras, S or T for connected graded noetherian AS regular algebras, and G for a general locally finite graded noetherian algebra.

In [KWZ21, Definition 0.9], we introduced the notion of the *concavity* of a numerical invariant $\mathcal{P}$. We recall the definition here. A graded algebra map $\phi : A \to G$ is called *finite* if $_A G$ and G_A are finitely generated. For a locally finite graded noetherian algebra G, let

$$\Phi(G) = \{T \mid T \text{ is AS regular and there is a finite map } \phi : T \to G\}.$$

Definition 5.1 ([KWZ21, Definition 0.9]). Let G be a locally finite graded noetherian algebra. Let $\mathcal{P}$ be any numerical invariant that is defined for locally finite $\mathbb{N}$-graded noetherian rings (or connected graded noetherian AS regular algebras). The $\mathcal{P}$-*concavity* of G is defined to be

$$c_{\mathcal{P}}(G) := \inf_{T \in \Phi(G)} \{\mathcal{P}(T)\}.$$

If no such T exists, we define $c_{\mathcal{P}}(G) = \infty$.

When $\mathcal{P} = -\mathrm{CMreg}$ we call

$$c(G) := c_{-\mathrm{CMreg}}(G)$$

simply the *concavity* of G.

In this subsection we introduce the weighted version of concavity, that is, when $\mathcal{P} = -\mathrm{CMreg}_{\xi}$ in the above definition.

Definition 5.2. Let G be a locally finite graded noetherian algebra and let $\xi \in \mathbb{R}$. The ξ-*concavity* of G is defined to be

$$c_{\xi}(G) := c_{-\mathrm{CMreg}_{\xi}}(G).$$

Now we prove the following weighted analogues of [KWZ21, Theorem 0.10(1) and Proposition 4.1(4)], which follow from weighted versions of the proofs in [KWZ21].

Proposition 5.3.
(1) *Let t be a commutative indeterminate and assume that $\xi \geq \deg t$. Then $c_{\xi}(G[t]) = c_{\xi}(G)$.*
(2) *Let T be a noetherian AS regular algebra. Suppose $0 \leq \xi \leq 1$. Then*

$$c_{\xi}(T) = -\mathrm{CMreg}_{\xi}(T).$$

Proof. (1) There is a finite map $G[t] \to G$ given by sending t to 0. Hence, taking $\mathcal{P} = -\mathrm{CMreg}_{\xi}$ in [KWZ21, Proposition 4.1(1)], we have $c_{\xi}(G[t]) \geq c_{\xi}(G)$.

Fix a real number $\epsilon > 0$. By definition of $c_{\xi}(G)$, there is a noetherian AS regular algebra T of type $(d, \mathfrak{l})$ and a finite map $\phi : T \to G$ such that $-\mathrm{CMreg}_{\xi}(T) \leq c_{\xi}(G) + \epsilon$. Then $T[t] \to G[t]$ is a finite map. Hence,

$$c_{\xi}(G[t]) \leq -\mathrm{CMreg}_{\xi}(T[t]) = -(\xi(d+1) - (\mathfrak{l} + \deg t))$$
$$= -(\xi d - \mathfrak{l}) - (\xi - \deg t) \leq -(\xi d - \mathfrak{l})$$
$$= -\mathrm{CMreg}_{\xi}(T) \leq c_{\xi}(G) + \epsilon.$$

Since ϵ was arbitrary, we obtain that $c_{\xi}(G[t]) \leq c_{\xi}(G)$. Combined with the previous paragraph, we conclude that $c_{\xi}(G[t]) = c_{\xi}(G)$.

(2) Fix a noetherian AS regular algebra T. Recall that $0 \leq \xi \leq 1$. Suppose S is any noetherian AS regular algebra and suppose $S \to T$ is a finite map. By Theorem 0.4 and the fact that $\operatorname{Torreg}_\xi({}_S T) \geq 0$, we obtain that

$$\operatorname{CMreg}_\xi(S) = \operatorname{CMreg}_\xi({}_S T) - \operatorname{Torreg}_\xi({}_S T) \leq \operatorname{CMreg}_\xi(T)$$

and hence, $-\operatorname{CMreg}_\xi(S) \geq -\operatorname{CMreg}_\xi(T)$. Therefore, $c_\xi(T) \geq -\operatorname{CMreg}_\xi(T)$. By definition, it is clear that $c_\xi(T) \leq -\operatorname{CMreg}_\xi(T)$, and so we have equality, as desired. $\square$

5.2. **Rate (or rate of growth of homology).** We first recall the notion of the rate of growth of homology that was introduced by Backelin [Bac86, p.81].

Definition 5.4. Let A be a connected graded algebra. The *rate of the homology* of A is defined to be

$$\operatorname{rate}(A) := \max \left\{ 1, \sup_{i \geq 2} \{ (t^i({}_A \Bbbk) - 1)/(i - 1) \} \right\}.$$

It is clear that $\operatorname{rate}(A) \geq 1$. If A is a commutative finitely generated connected graded algebra, then by Corollary 5.9, $\operatorname{rate}(A)$ is finite. We show that $\operatorname{rate}(A)$ is finite if and only if $\operatorname{Torreg}_\xi(\Bbbk)$ is finite for some ξ, and in Proposition 5.8 we provide a sufficient condition, for the finiteness of $\operatorname{Torreg}_\xi(\Bbbk)$ that includes the case when A is commutative.

Question 5.5. Suppose that A is connected graded and noetherian (not necessarily generated in degree 1).

(1) Is $\operatorname{rate}(A)$ always finite?
(2) If the answer to part (1) is no, is there a natural condition on A that guarantees the finiteness of $\operatorname{rate}(A)$?
(3) If A has a balanced dualizing complex, is $\operatorname{rate}(A)$ finite?

The finiteness of $\operatorname{rate}(A)$ is particularly interesting due to a result of Backelin, which we now describe. Suppose that A is generated in degree 1. Then A is Koszul if and only if $\operatorname{rate}(A) = 1$. Backelin proved that the finiteness of $\operatorname{rate}(A)$ is related to the Koszul property of the Veronese subrings of A. Let $d \geq 2$ be an integer and define the *dth Veronese subring of A*:

$$A^{(d)} := \bigoplus_{i \geq 0} A_{di}.$$

In this setting, we regrade so that elements in A_{di} have degree i.

Theorem 5.6 ([Bac86, Corollary, p. 81]). *Let A be a connected graded algebra generated in degree 1. If $d \geq \operatorname{rate}(A)$, then $A^{(d)}$ is Koszul.*

Earlier, in the context of commutative algebra (and algebraic geometry) Mumford proved that if A is a connected graded finitely generated commutative algebra, then the Veronese subring $A^{(d)}$ is Koszul for $d \gg 0$, see [Mum88, lemma, p. 282] and [Mum10, Theorem 1]. Hence, Backelin's result extends Mumford's result to the noncommutative setting, with the additional assumptions that A is generated in degree 1 and $\operatorname{rate}(A)$ is finite. In general, Mumford's result fails in the noncommutative setting—see [SZ94, Corollary 3.2], an example that is not generated in degree 1. In Example 5.11 we show for this algebra A and for $\xi \geq 3$, $\operatorname{Torreg}_\xi(X) < \infty$ for all $X \in \mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-Gr})$.

Lemma 5.7. *Suppose A is connected graded and noetherian.*

(1) *Suppose $r := \operatorname{rate}(A)$ is finite Then*
$$\operatorname{Torreg}_{(1,r)}(\Bbbk) \leq \max\{0, 1 - r, t_1^A(\Bbbk) - r\}.$$

(2) *Suppose that $a := \operatorname{Torreg}_\xi(\Bbbk)$ is finite for some ξ. Let $a' = \max\{a, 1 - \xi\}$. Then $\operatorname{rate}(A) \leq \max\{1, a' + 2\xi - 1\}$.*

(3) *Therefore $\operatorname{rate}(A)$ is finite if and only if $\operatorname{Torreg}_\xi(\Bbbk)$ is finite for some ξ.*

Proof. (1) By definition, for all $i \geq 2$, $t_i^A(\Bbbk) - 1 \leq r(i-1)$. Then $t_i^A(\Bbbk) - ri \leq 1 - r$. The assertion follows.

(2) By definition, $t_i^A(\Bbbk) \leq a + i\xi \leq a' + i\xi$ for all $i \geq 2$. Then
$$\sup_{i \geq 2}\{(t_i^A(\Bbbk) - 1)/(i-1)\} \leq \sup_{i \geq 2}\{(a' + i\xi - 1)/(i-1)\}$$
$$= \sup_{n \geq 1}\{[(a' + \xi - 1) + n\xi]/n\} = a' + 2\xi - 1.$$

(3) This is an immediate consequence of parts (1) and (2). The assertion follows. $\square$

Proposition 5.8 provides a criterion for the finiteness of $\operatorname{Torreg}_\xi(\Bbbk)$ for some ξ.

Proposition 5.8. *Let A be a noetherian connected graded algebra and suppose that there is a finite map $\phi : T \to A$ where T is a noetherian connected graded algebra of finite global dimension. Let*
$$c := \max\left\{t_0^T(_TA), \max_{1 \leq s \leq \operatorname{pdim}(_TA)}\{t_s^T(_TA)/s\}\right\} < \infty.$$

(1) *Write $\xi = (1, \xi)$. If $\xi \geq c$, then $\operatorname{Torreg}_\xi(_A\Bbbk) < \infty$.*

(2) *If further T is AS regular and $\xi \geq c$, then $\operatorname{Torreg}_\xi(X) < \infty$ for all $X \in \mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(A\text{-}\mathrm{Gr})$.*

(3) *Suppose T is AS regular. Then $\operatorname{ASreg}_\xi(A) < \infty$ for all $\xi \gg 0$.*

Proof. (1) It is enough to show the assertion for $\xi = (1, c)$. By definition of c, we have
$$t_0^T(_TA) \leq c \text{ and } t_s^T(_TA) \leq cs \qquad \text{for all } s \geq 1.$$
Let
$$d := \max_{0 \leq s \leq \operatorname{gldim} T}\{t_s^A(\Bbbk) - cs\},$$
which is clearly finite. We claim that $t_j^A(\Bbbk) - cj \leq d$ for all $j \geq 0$, which is equivalent to the main assertion. We prove this by induction. By the definition of d, the claim holds for all $0 \leq j \leq \operatorname{gldim} T$.

Now assume that $j > \operatorname{gldim} T$. We will use the change of rings spectral sequence given in [Rot09, Theorem 10.60], namely:
$$E_{p,q}^2 := \operatorname{Tor}_p^A\left(\operatorname{Tor}_q^T(\Bbbk_T, A), {}_AM\right) \implies \operatorname{Tor}_{p+q}^T(\Bbbk_T, {}_TM).$$

Letting $M = \Bbbk$ in this spectral sequence and assuming the induction hypothesis that
$$t_s^A(\Bbbk) - cs \leq d$$

for all $s \leq j - 1$, we see that for $r \geq 2$,

$$
\begin{aligned}
\deg E^r_{j-r,r-1} \leq \deg E^2_{j-r,r-1} &\leq \max_{p \leq j-1} \{t^A_p(\Bbbk) + t^T_{j-1-p}(_T A)\} \\
&\leq \max\{t^A_{j-1}(\Bbbk) + t^T_0(_T A), \max_{0 \leq p \leq j-1}\{t^A_p(\Bbbk) + t^T_{j-1-p}(_T A)\}\} \\
&\leq \max\{c(j-1) + d + c, \max_{0 \leq p \leq j-1}\{(cp + d) + (j - 1 - p)c\}\} \\
&= d + cj.
\end{aligned}
$$

Note that the incoming differentials to $E^r_{j,0}$, for $r \geq 2$, are all zero, and the outgoing differentials from $E^r_{j,0}$ land at $E^r_{j-r,r-1}$ with $\deg E^r_{j-r,r-1} \leq d + cj$. When $j > \operatorname{gldim} T$, $E^\infty_{j,0} = \operatorname{Tor}^T_j(\Bbbk, \Bbbk) = 0$. Hence $E^2_{j,0}$ has a filtration such that each subfactor is a submodule of some $E^r_{j-r,r-1}$ where $2 \leq r \leq j$. Therefore

$$
\begin{aligned}
t^A_j(\Bbbk) = \deg \operatorname{Tor}^A_j(\Bbbk, \Bbbk) &= \deg \operatorname{Tor}^A_j(\Bbbk \otimes_T A, \Bbbk) \\
&= \deg E^2_{j,0} \leq \max_{r \geq 2}\{\deg E^r_{j-r,r-1}\} \\
&\leq d + cj.
\end{aligned}
$$

This finishes the inductive step and the proof of the main assertion.

(2) Since T has a balanced dualizing complex, so does A, via the finite map $\phi : T \to A$. The assertion follows from part (1) and Theorem 0.3(1).

(3) The assertion follows from part (2) and the definition of $\operatorname{ASreg}_\xi(A)$. $\square$

The above proposition shows that if A is finitely generated and commutative then there exists a weight ξ such that $\operatorname{Torreg}_\xi(_A\Bbbk)$ is finite; hence by Lemma 5.7, $\operatorname{rate}(A)$ is finite, as noted earlier.

Corollary 5.9. *Let A be a noetherian connected graded algebra generated in degree 1 and suppose there is a finite map $T \to A$ where T is a noetherian connected graded algebra of finite global dimension. Then $\operatorname{rate}(A)$ is finite and hence $A^{(d)}$ is Koszul for $d \gg 0$.*

Proof. The assertion follows from Proposition 5.8, Lemma 5.7, and Theorem 5.6. $\square$

When A is commutative and finitely generated, then there is a surjective map from a polynomial ring to A. So Corollary 5.9 recovers Mumford's result [Mum10, Theorem 1]. This motivates the following questions that are related to Question 5.5.

Question 5.10. Let A be a noetherian connected graded algebra.

(1) Suppose A is generated in degree 1. Is then $A^{(d)}$ Koszul for $d \gg 0$?

If the answer to part (1) is no, we further ask

(2) Suppose A is generated in degree 1. Is there a natural homological condition such that $A^{(d)}$ is Koszul for $d \gg 0$? For example, if A has a balanced dualizing complex is $A^{(d)}$ Koszul for $d \gg 0$?

(3) Suppose further that A is PI. Is then $A^{(d)}$ Koszul for $d \gg 0$?

If A is not generated in degree 1, then it is not necessary that $A^{(d)}$ is Koszul for $d \gg 0$. We can therefore ask part (2) in this setting, namely:

(4) If A is not necessarily generated in degree 1, is there a natural homological condition which guarantees that $A^{(d)}$ is Koszul for $d \gg 0$?

Example 5.11 shows that the hypothesis in Proposition 5.8(2) is sufficient, but not necessary.

Example 5.11. Assume that $\Bbbk = \mathbb{C}$. Let U be the algebra $\Bbbk\langle x, y\rangle/(yx - xy - x^2)$ and let $R = \Bbbk + Uy$. The algebra U is noetherian AS regular of global dimension two and R is noetherian and generated by y and xy. It follows from [SZ94, Theorem 2.3] that there is no finite map from a noetherian AS regular algebra T to R.

We claim that if $\xi_1 \geq 3$ (and write $\xi = (1, \xi_1)$), then $\mathrm{Torreg}_\xi(X) < \infty$ for all $X \in \mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(R\text{-Gr})$. We give a sketch of the proof below.

Claim 1. Consider U as a left graded R-module. Then $_RU$ is finitely generated and $\mathrm{Torreg}_\xi(U) < \infty$.

Proof of Claim 1. By [SZ94, (2.3.1)], we have a short exact sequence

$$0 \to Uhx \to R \oplus Rx \to U \to 0$$

where $h = (y^2 - 2xy)$. Using this we obtain the following minimal free resolution of the R-module $_RU$:

$$\cdots \to R(-9) \oplus R(-10) \to R(-6) \oplus R(-7) \to R(-3) \oplus R(-4) \to R \oplus R(-1) \to U \to 0$$

which implies that $t^i(_RU) = 3i + 1$ for all $i \geq 0$ and

$$\mathrm{Torreg}_\xi(_RU) = \begin{cases} \infty & \xi_1 < 3, \\ 1 & \xi_1 \geq 3. \end{cases}$$

In particular, when $\xi_1 \geq 3$, $\mathrm{Torreg}_\xi(_RU) < \infty$. $\qquad\square$

Claim 2. Suppose $\xi_1 \geq 3$. If M is a finitely generated graded left U-module, then $\mathrm{Torreg}_\xi(_RM) < \infty$.

Proof of Claim 2. Since U is AS regular, there is a minimal free resolution

$$0 \to P_2 \to P_1 \to P_0 \to M \to 0.$$

By Claim 1, $\mathrm{Torreg}_\xi(_RP_i) < \infty$ for $i = 0, 1, 2$ (as we assume $\xi_1 \geq 3$). By Lemma 3.7, $\mathrm{Torreg}_\xi(_RM) < \infty$. $\qquad\square$

Claim 3. Suppose $\xi_1 \geq 3$. If M is a finitely generated graded left R-module, then $\mathrm{Torreg}_\xi(_RM) < \infty$.

Proof of Claim 3. We use induction on the Krull dimension of M.

Case 1. Suppose M has Krull dimension 0. Then M is finite dimensional. If M is 1-dimensional, then it is of the form $\Bbbk(n)$, which is an U-module, and the assertion follows from Claim 2. If $\dim M > 1$, the assertion follows from Lemma 3.7 and the base case $\Bbbk(n)$. The minimal free resolution of the trivial module $_R\Bbbk$ (which was computed by Frank Moore):

$$\cdots \to R(-(3n-1)) \oplus R(-(3n-2)) \to \cdots \to R(-2) \oplus R(-1) \to R \to \Bbbk \to 0.$$

As a consequence

$$\mathrm{Torreg}_\xi(_R\Bbbk) = \begin{cases} \infty & \xi < 3, \\ 0 & \xi_1 \geq 3. \end{cases}$$

Case 2. Assume M has Krull dimension 1. By noetherian induction, we need to consider only the case when M is 1-critical. First we recall that 1-critical U-modules are graded shifts of $U/U(ax + by)$ (these are called point modules). Since Proj R = Proj U, every 1-critical R-module M is a submodule of a 1-critical U-module N such that N/M is finite dimensional. By Claim 2, Case 1, and Lemma 3.7, the assertion follows.

Case 3. Suppose M has Krull dimension 2. By noetherian induction, we need to consider only the case when M is 2-critical. Then M contains a submodule isomorphic to $R(n)$, and $M/R(n)$ has Krull dimension 1. So the assertion follows from Case 2 and Lemma 3.7. Combining these cases we finish the proof of Claim 3.

$\square$

Claim 4. Suppose $\xi_1 \geq 3$. If X is in $\mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(R\text{-Gr})$, then $\mathrm{Torreg}_\xi(X) < \infty$.

Proof of Claim 4. Define the *amplitude* of a complex X to be

$$\mathrm{amp}(X) := \sup(X) - \inf(X).$$

If $\mathrm{amp}(X) = 0$, then X is isomorphic to a complex shift of a module. The claim follows from Lemma 3.1(2) and Claim 3. If $\mathrm{amp}(X) > 0$, then by truncation, there are complexes Y and Z with smaller amplitude than $\mathrm{amp}(X)$ such that

$$Y \to X \to Z \to Y[1]$$

is a distinguished triangle. The claim follows from induction and Lemma 3.7(1). $\square$

The following corollary follows from the above example.

Corollary 5.12. *There is a noetherian connected graded algebra A with finite* $\mathrm{Torreg}_\xi(\Bbbk)$ *for some ξ, but not generated in degree 1, such that the Veronese subring* $A^{(d)}$ *is not Koszul for every $d \gg 0$.*

Proof. Let A be the algebra R in Example 5.11. By [SZ94, Corollary 3.2], the Veronese subring $A^{(d)}$ is not Koszul for all $d \gg 0$. By Example 5.11, $\mathrm{Torreg}_\xi(\Bbbk)$ is finite for $\xi = (1, 3)$. $\square$

5.3. **Slope.** Another homological invariant that is related to the ξ-Tor-regularity is the *slope* of a graded A-module M, which was introduced in [ACI10] for finitely generated commutative algebras.

Definition 5.13 ([ACI10, p. 197])**.** Let M be a graded left A-module. The *slope* of M is defined to be

$$\mathrm{slope}\ M := \sup_{i \geq 1} \frac{\deg(\mathrm{Tor}_i^A(\Bbbk, M)) - \deg(\mathrm{Tor}_0^A(\Bbbk, M))}{i}.$$

Following [Bac86], $\mathrm{rate}(A) = \mathrm{slope}_A(A_{\geq 1})$.

When A is a finitely generated commutative connected graded algebra, then in [ACI10, Corollary 1.3] it is proved that for every finitely generated graded A-module M, the slope of M is finite. The relationship between slope and CMreg and Torreg was discussed in [ACI10]. Also see [ACI15] (and the references therein) for the study of $t_i^A(M)$ in the commutative setting.

Remark 5.14. Let A be a connected graded noetherian algebra and M be a finitely generated graded left A-module.

(1) It is clear that the definition of the slope(M) makes sense in the noncommutative setting, too.
(2) If $s = s(M) := \mathrm{slope}(M)$ is finite, then $\mathrm{Torreg}_s(M) \leq t_0^A(M)$.
(3) If $\mathrm{Torreg}_\xi(M)$ is finite for some ξ, then $\mathrm{slope}(M) \leq \xi + |\,\mathrm{Torreg}_\xi(M) - t_0^A(M)|$.
(4) Combining parts (2) and (3), slope(M) is finite if and only if $\mathrm{Torreg}_\xi(M)$ is finite for some ξ.
(5) In the setting of Proposition 5.8(2), by part (4), slope(M) is always finite.
(6) c_ξ (in Definition 5.2), rate, slope, and Torreg_ξ are useful in understanding various homological properties, which was demonstrated in [ACI10, ACI15] in the commutative case.

We conclude the paper by asking a final question. If A is commutative, then by [AP01] if $\mathrm{Torreg}(\Bbbk) < \infty$, then $\mathrm{Torreg}(\Bbbk) = 0$. Although this is not true in the noncommutative setting, we remark that if A is commutative then $c(A) = 0$.

Question 5.15. If $c(A) = 0$ and $\mathrm{Torreg}(\Bbbk) < \infty$, then is $\mathrm{Torreg}(\Bbbk) = 0$ (or equivalently, A is Koszul)?

Acknowledgments

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