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IMPROVED CONCENTRATION OF LAGUERRE AND JACOBI ENSEMBLES*

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Abstract. We consider the asymptotic limits where certain parameters in the definitions of the Laguerre and Jacobi ensembles diverge. In these limits, Dette, Imhof, and Nagel proved that, up to a linear transformation, the joint probability distributions of the ensembles become more and more concentrated around the zeros of the Laguerre and Jacobi polynomials, respectively. In this paper, we improve the concentration bounds. Our proofs are similar to those in the original references, but the error analysis is improved and arguably simpler. For the first and second moments of the Jacobi ensemble, we further improve the concentration bounds implied by our aforementioned results.

Key words. Laguerre ensemble, Laguerre polynomial, Jacobi ensemble, Jacobi polynomial, Wishart ensemble

MSC codes. 60B10, 60B20

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1. Introduction.

The Gaussian, Wishart, and Jacobi ensembles are three classical ensembles in random matrix theory. They find numerous applications in physics, statistics, and other branches of applied science. The Gaussian (Wishart) ensemble is also known as the Hermite (Laguerre) ensemble due to its relationship with the Hermite (Laguerre) polynomial.

Of particular interest are the asymptotic limits where certain parameters in the definitions of the ensembles diverge. In these limits, Dette, Imhof, and Nagel [2, 3] proved that up to a linear transformation, the joint probability distributions of the Hermite, Laguerre, and Jacobi ensembles become more and more concentrated around the zeros of the Hermite, Laguerre, and Jacobi polynomials, respectively. These results allow us to transfer knowledge on the zeros of orthogonal polynomials to the corresponding ensembles.

In this paper, we improve the concentration bounds for the Laguerre and Jacobi probability distributions around the zeros of the Laguerre and Jacobi polynomials, respectively. Our proofs are similar to those in the original references [2, 3], but the error analysis is improved and arguably simpler. We also prove the concentration of the first and second moments of the Jacobi ensemble. The last result has found applications in quantum statistical mechanics [5].

The rest of this paper is organized as follows. Section 2 presents our main results, which are compared with previous results in the literature. Proofs are given in section 3.

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2. Results. In the literature, there is more than one definition of the Laguerre probability distribution. These definitions differ only by a linear transformation and are thus essentially equivalent. In this paper, we stick to one definition. When citing a result from the literature, we perform a linear transformation such that the result is presented for the definition we stick to. The same applies to the Jacobi case.

Let n be the number of random variables in an ensemble. Let β be the Dyson index, which can be an arbitrary positive number.

2.1. Laguerre ensemble. We draw $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ from the Laguerre ensemble.

DEFINITION 2.1 (Laguerre ensemble). *The probability density function of the β-Laguerre ensemble with parameters*

$$\alpha > (n-1)\frac{\beta}{2} \tag{2.1}$$

is

$$f_{\mathrm{Lag}}(\lambda_1, \lambda_2, \dots, \lambda_n) \propto \prod_{1\le i<j\le n} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^n \lambda_i^{\alpha - \frac{(n-1)\beta}{2} - 1} e^{-\lambda_i/2}, \quad \lambda_i > 0. \tag{2.2}$$

For certain values of β, the Laguerre ensemble arises as the probability density function of the eigenvalues of a Wishart matrix VV^*, where V is an $n \times \frac{2\alpha}{\beta}$ matrix with real ($\beta = 1$), complex ($\beta = 2$), or quaternionic ($\beta = 4$) entries. In each case, the entries of V are independent standard Gaussian random variables, and V^* denotes the conjugate transpose of V.

Let

$$L_n^{(p)}(x) := \sum_{i=0}^n \binom{n+p}{n-i} \frac{(-x)^i}{i!}, \quad p > -1 \tag{2.3}$$

be the Laguerre polynomial, whose zeros are all in the interval with endpoints [7]

$$2n + p - 2 \pm \sqrt{1 + 4(n-1)(n+p-1)\cos^2 \frac{\pi}{n+1}}. \tag{2.4}$$

Let $x_1 < x_2 < \cdots < x_n$ be the zeros of the Laguerre polynomial $L_n^{(2\alpha/\beta - n)}(x/\beta)$.

We are interested in the limit $\alpha \to \infty$ but do not assume that $n \to \infty$. Note that, if β is a constant, then $n \to \infty$ implies that $\alpha \to \infty$; see (2.1).

THEOREM 2.2 (Theorem 2.1 in [2]). *For any $0 < \epsilon < 1$,*

$$\Pr\left(\frac{1}{2\alpha} \max_{1\le i\le n} |\lambda_i - x_i| > \epsilon\right) \le 4n(1 + \epsilon^2/25)^\alpha e^{-\alpha\epsilon^2/25}. \tag{2.5}$$

This theorem can be restated as the following corollary.

COROLLARY 2.3. *There exist positive constants C_1, C_2 such that, for any $0 < \epsilon < 1$,*

$$\Pr\left(\frac{1}{2\alpha} \max_{1\le i\le n} |\lambda_i - x_i| > \epsilon\right) \le C_1 n e^{-C_2\alpha\epsilon^4}. \tag{2.6}$$

THEOREM 2.4 (Theorem 2.4 in [2]). *Let $\kappa \ge 1$ be a parameter. If*

$$n-1+1/\beta \le 2\alpha/\beta \le n-1+\kappa \quad \text{and} \quad 2\kappa\beta/\alpha < \epsilon < 1, \tag{2.7}$$

then there exist positive constants C_1, C_2, C_3 such that

$$\Pr\left(\frac{1}{2\alpha}\max_{1\le i\le n}|\lambda_i - x_i| > \epsilon\right) \le C_1 n(e^{-C_2\alpha\epsilon^2/\kappa} + e^{C_3\kappa^2\beta - C_2\alpha\epsilon^2}). \tag{2.8}$$

The original upper bound on $\Pr(\frac{1}{2\alpha}\max_{1\le i\le n}|\lambda_i - x_i| > \epsilon)$ in Theorem 2.4 of [2] is a complicated expression without implicit constants. The right-hand side of (2.8) is its simplification using implicit constants.

If condition (2.7) is satisfied, (2.8) may be an improvement of (2.6). In particular, for a constant β, the right-hand side of (2.8) becomes $C_1' n e^{-C_2'\alpha\epsilon^2}$ (C_1', C_2' are positive constants) if and only if κ is upper bounded by a constant.

As the main result of this subsection, Theorem 2.5 is an improvement of Corollary 2.3 and Theorem 2.4.

THEOREM 2.5. *There exist positive constants C_1, C_2 such that, for any $\epsilon > 0$,*

$$\Pr\left(\frac{1}{2\alpha}\max_{1\le i\le n}|\lambda_i - x_i| > \epsilon\right) \le C_1 n e^{-C_2\alpha\epsilon\min\{\epsilon,1\}}. \tag{2.9}$$

Let $n \le s$ be two positive integers, and let V be an $n\times s$ matrix whose elements are independent standard real Gaussian random variables. Then, VV^T is a real Wishart matrix, whose joint eigenvalue distribution is given by (2.2) with $\beta = 1$ and $\alpha = s/2$. Theorem 2.5 implies the following.

COROLLARY 2.6. *Let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ be the eigenvalues of VV^T and $x_1 < x_2 < \cdots < x_n$ be the zeros of the Laguerre polynomial $L_n^{(s-n)}(x)$. There exist positive constants C_1, C_2 such that, for any $\epsilon > 0$,*

$$\Pr\left(\frac{1}{s}\max_{1\le i\le n}|\lambda_i - x_i| > \epsilon\right) \le C_1 n e^{-C_2 s\epsilon\min\{\epsilon,1\}}. \tag{2.10}$$

Analogues of Corollary 2.6 for complex ($\beta = 2$) and quaternionic ($\beta = 4$) Wishart matrices also follow directly from Theorem 2.5.

Let

$$M_1^{\mathrm{L}} := \frac{1}{n}\sum_{i=1}^{n}\lambda_i \tag{2.11}$$

be the first moment of the Laguerre ensemble. The distribution of M_1^{L} has a particularly simple form.

FACT 1. *M_1^{L} is distributed as $\frac{1}{n}\chi^2_{2\alpha n}$, where χ^2_k denotes the chi-square distribution with k degrees of freedom.*

Thus, the concentration of M_1^{L} follows directly from the tail bound [9, 6] for the chi-square distribution.

The distribution of the second moment of the Laguerre ensemble does not have a simple form. Furthermore, it is complicated to obtain concentration bounds for the distribution, so we omit this analysis here.

2.2. Jacobi ensemble. We draw $\mu_1 \le \mu_2 \le \cdots \le \mu_n$ from the Jacobi ensemble.

DEFINITION 2.7 (Jacobi ensemble). *The probability density function of the β-Jacobi ensemble with parameters $a, b > 0$ is*

(2.12)
$$f_{\mathrm{Jac}}(\mu_1, \mu_2, \ldots, \mu_n) \propto \prod_{1\le i<j\le n} |\mu_i - \mu_j|^\beta \prod_{i=1}^n (1-\mu_i)^{a-1}(1+\mu_i)^{b-1}, \quad -1 \le \mu_i \le 1.$$

The Jacobi ensemble can be interpreted as the probability density function of the eigenvalues of a random matrix ensemble. In the complex ($\beta = 2$) case, let Q_1 and Q_2 be uniformly random projectors in $\mathbb{C}^{2n+a+b-2}$ with ranks n and $n+b-1$, respectively. Then, $\frac{1+\mu_1}{2}, \frac{1+\mu_2}{2}, \ldots, \frac{1+\mu_n}{2}$ are the nonzero eigenvalues of $Q_1Q_2Q_1$ [1]. Equivalently, they are the squared singular values of an $n\times(n+b-1)$ rectangular block within a Haar-random unitary matrix of dimension $2n+a+b-2$. A random matrix interpretation for general β is given in [8], but it has less of a natural connection to applications.

The Jacobi polynomial is defined as

$$P_n^{p,q}(y) := \frac{\Gamma(n+p+1)}{\Gamma(n+p+q+1)} \sum_{i=0}^n \frac{\Gamma(n+p+q+i+1)}{i!(n-i)!\Gamma(p+i+1)} \left(\frac{y-1}{2}\right)^i, \tag{2.13}$$

where Γ is the gamma function. It is well known that all zeros of the Jacobi polynomial are in the interval $(-1, 1)$. Let $y_1 < y_2 < \cdots < y_n$ be the zeros of the Jacobi polynomial $P_n^{2a/\beta-1, 2b/\beta-1}(y)$.

2.2.1. Pointwise approximation. In this subsubsection, we are interested in the limit $a + b \to \infty$ but do not assume that $\min\{a, b\} \to \infty$.

THEOREM 2.8 (Theorem 2.1 in [3]). *For any $0 < \epsilon \le 1/2$,*

$$\Pr\left(\max_{1\le i\le n} |\mu_i - y_i| > \epsilon\right) \le 4(2n-1)\left(1 + \frac{\epsilon^2}{162+2\epsilon^2}\right)^{a+b} e^{-\frac{(a+b)\epsilon^2}{162+2\epsilon^2}}. \tag{2.14}$$

This theorem can be restated as follows.

COROLLARY 2.9. *There exist positive constants C_1, C_2 such that, for any $0 < \epsilon \le 1/2$,*

$$\Pr\left(\max_{1\le i\le n} |\mu_i - y_i| > \epsilon\right) \le C_1 n e^{-C_2(a+b)\epsilon^4}. \tag{2.15}$$

As the main result of this subsubsection, Theorem 2.10 is an improvement of Corollary 2.9.

THEOREM 2.10. *There exist positive constants C_1, C_2 such that, for any $\epsilon > 0$,*

$$\Pr\left(\max_{1\le i\le n} |\mu_i - y_i| > \epsilon\right) \le C_1 n e^{-C_2(a+b)\epsilon^2}. \tag{2.16}$$

Section 3 of [3] presents several applications of Theorem 2.8. Most of them can be improved by using Theorem 2.10. We discuss one of them in detail.

Let β be a positive constant. Consider the limit $n \to \infty$ with

$$a = \omega(n), \quad a = \Theta(b). \tag{2.17}$$

Let $\delta(\cdot)$ be the Dirac delta. The semicircle law with radius r is a probability distribution on the interval $[-r, r]$ with density function

$$f_{\mathrm{SC}}(\mu) \propto \sqrt{r^2 - \mu^2}. \tag{2.18}$$

COROLLARY 2.11. *The empirical distribution*

$$f(\mu) := \frac{1}{n}\sum_{i=1}^{n}\delta\left(\mu - \sqrt{\frac{a+b}{2abn\beta}}\big((a+b)\mu_i + a - b\big)\right) \tag{2.19}$$

of linearly transformed μ_i converges weakly to the semicircle law with radius 2 almost surely.

For $\omega(n) = a = o(n^2/\ln n)$, Corollary 2.11 was proved in Example 3.4 of [3] using Theorem 2.8. Using Theorem 2.10 instead, the same proof becomes valid for any $a = \omega(n)$.

Corollary 2.11 is very similar to Theorem 2.1 in [11].

2.2.2. Moments. Theorem 2.10 implies the concentration of any smooth multivariate function of $\mu_1, \mu_2, \dots, \mu_n$. The main result of this subsubsection is tighter concentration bounds (than those implied by Theorem 2.10) for the first and second moments of the Jacobi ensemble.

Let

$$N := a + b + \beta(n-1). \tag{2.20}$$

Suppose that $\beta = \Theta(1)$ is a positive constant and that $a + b = \Omega(1)$. In this subsubsection, we are interested in the limit $N \to \infty$. This means that $a + b \to \infty$ or $n \to \infty$ or both.

Let

$$M_1^{\mathrm{J}} := \frac{1}{n}\sum_{i=1}^{n}\mu_i, \quad M_2^{\mathrm{J}} := \frac{1}{n}\sum_{i=1}^{n}(\mu_i - \mathbb{E}M_1^{\mathrm{J}})^2 \tag{2.21}$$

be the first and shifted second moments of the Jacobi ensemble. Equation (B.7) of [10] implies that

$$\mathbb{E}M_1^{\mathrm{J}} = \frac{b-a}{N}, \quad \mathbb{E}M_2^{\mathrm{J}} = \frac{\beta n(2a+\beta n)(2b+\beta n)}{2N^3} + O(1/N). \tag{2.22}$$

Indeed, $\mathbb{E}M_2^{\mathrm{J}}$ can be calculated exactly in closed form. The expression is lengthy and simplifies to the above using the Big-O notation.

THEOREM 2.12 (concentration of moments). *For any $\epsilon > 0$,*

$$\Pr(|M_1^J - \mathbb{E}M_1^J| > \epsilon) = O(e^{-\Omega(Nn\epsilon^2)}), \tag{2.23}$$

$$\Pr(|M_2^J - \mathbb{E}M_2^J| > \epsilon) = O(e^{-\Omega(N\epsilon)\min\{N\epsilon, n\}}). \tag{2.24}$$

Let

$$Y_1 := \frac{1}{n}\sum_{i=1}^{n}y_i, \quad Y_2 := \frac{1}{n}\sum_{i=1}^{n}(y_i - Y_1)^2 \tag{2.25}$$

be the mean and variance of the zeros of the Jacobi polynomial. From direct calculation (Appendix A), we find that

$$Y_1 = \frac{b-a}{N}, \quad Y_2 = \frac{\beta(n-1)\big(2a+\beta(n-1)\big)\big(2b+\beta(n-1)\big)}{N^2(2N-\beta)}. \tag{2.26}$$

Hence,

$$\mathbb{E}M_1^{\mathrm{J}} = Y_1, \quad \mathbb{E}M_2^{\mathrm{J}} = Y_2 + O(1/N). \tag{2.27}$$

COROLLARY 2.13. *For any* $\epsilon > 0$,

$$\Pr(|M_1^J - Y_1| > \epsilon) = O(e^{-\Omega(Nn\epsilon^2)}), \tag{2.28}$$

$$\Pr(|M_2^J - Y_2| > \epsilon) = O(e^{-\Omega(N\epsilon)\min\{N\epsilon, n\}}). \tag{2.29}$$

3. Proofs. The proofs of Theorems 2.5 and 2.10 are similar to those of Theorems 2.2 and 2.8 in [2, 3], respectively, but the error analysis is improved and arguably simpler.

The following lemma will be used multiple times.

LEMMA 3.1. *Let* m *be an integer, and let* p_i, q_i *be numbers such that* $|p_i - q_i| \le \delta$ *for* $i = 1, 2, \ldots, m$. *Then,*

$$\left|\prod_{i=1}^m p_i - \prod_{i=1}^m q_i\right| \le \delta \sum_{k=0}^{m-1} \prod_{i=1}^{m-k-1} |p_i| \prod_{j=m+1-k}^{m} |q_j|. \tag{3.1}$$

Proof.

$$\begin{aligned}\left|\prod_{i=1}^m p_i - \prod_{i=1}^m q_i\right| &\le \sum_{k=0}^{m-1}\left|\prod_{i=1}^{m-k} p_i \prod_{j=m+1-k}^{m} q_j - \prod_{i=1}^{m-k-1} p_i \prod_{j=m-k}^{m} q_j\right| \\ &\le \delta \sum_{k=0}^{m-1} \prod_{i=1}^{m-k-1} |p_i| \prod_{j=m+1-k}^{m} |q_j|. \end{aligned} \tag{3.2}$$

∎

Let C be a positive constant. For notational simplicity, we will reuse C in that its value may be different in different expressions or equations.

3.1. Laguerre ensemble: Proofs of Theorem 2.5 and Fact 1. For Theorem 2.5, it suffices to prove the following.

THEOREM 3.2. *For any* $\epsilon > 0$,

$$\Pr\left(\frac{1}{2\alpha}\max_{1\le i\le n}|\lambda_i - x_i| > 4\epsilon\right) \le 4ne^{-\alpha(\sqrt{1+\epsilon}-1)^2}. \tag{3.3}$$

Proof of Theorem 3.2. Let $X_{2\alpha}, X_{2\alpha-\beta}, X_{2\alpha-2\beta}, \ldots, X_{2\alpha-(n-1)\beta}, Y_\beta, Y_{2\beta}, \ldots, Y_{(n-1)\beta}$ be independent nonnegative random variables with $X_k^2 \sim \chi_k^2$ and $Y_l^2 \sim \chi_l^2$. Note that

$$\mathbb{E}(X_k^2) = k, \quad \mathrm{Var}(X_k^2) = 2k. \tag{3.4}$$

Lemma A.1 in [2] gives the tail bound (δ here and in all probability bounds below is positive)

$$\Pr(|X_k - \sqrt{k}| > \delta) \le 2(1 + \delta/\sqrt{k})^k e^{-\delta\sqrt{k} - \delta^2/2} \le 2e^{-\delta^2/2}. \tag{3.5}$$

Let $\mathbf{L}_{i,j}$ be the element in the ith row and jth column of a real symmetric $n \times n$ tridiagonal random matrix $\mathbf{L}$. "Tridiagonal" means that $\mathbf{L}_{i,j} = 0$ if $|i - j| > 1$. The diagonal and subdiagonal matrix elements are, respectively,

$$\mathbf{L}_{1,1} = X_{2\alpha}^2, \tag{3.6}$$

$$\mathbf{L}_{i,i} = X_{2\alpha-(i-1)\beta}^2 + Y_{(n+1-i)\beta}^2, \quad i = 2, 3, \ldots, n, \tag{3.7}$$

$$\mathbf{L}_{i+1,i} = X_{2\alpha-(i-1)\beta} Y_{(n-i)\beta}, \quad i = 1, 2, \ldots, n-1. \tag{3.8}$$

The joint eigenvalue distribution of $\mathbf{L}$ is given by [4] the Laguerre ensemble (Definition 2.1).

Let $\mathbf{L}'$ be a real symmetric $n\times n$ tridiagonal deterministic matrix, whose matrix elements are obtained by replacing X_k^2, Y_l^2 in (3.6) and (3.7) by their expectation values and replacing X_kY_l in (3.8) by $\sqrt{\mathbb{E}(X_k^2)\mathbb{E}(Y_l^2)}$; i.e.,

$$\mathbf{L}'_{1,1}=2\alpha, \tag{3.9}$$

$$\mathbf{L}'_{i,i}=2\alpha+(n+2-2i)\beta,\quad i=2,3,\ldots,n, \tag{3.10}$$

$$\mathbf{L}'_{i+1,i}=\sqrt{\big(2\alpha-(i-1)\beta\big)(n-i)\beta},\quad i=1,2,\ldots,n-1. \tag{3.11}$$

The eigenvalues of $\mathbf{L}'$ are the zeros of the Laguerre polynomial $L_n^{(2a/\beta-n)}(x/\beta)$ [2].

Let $\|\cdot\|$ denote the operator norm. Let $\mathbf{L}_{1,0}={\mathbf{L}'}_{1,0}=\mathbf{L}_{n+1,n}={\mathbf{L}'}_{n+1,n}:=0$. Let $\delta=\sqrt{2\alpha}(\sqrt{1+\epsilon}-1)$. Since

$$\max_{1\le i\le n}|\lambda_i-x_i|\le\|\mathbf{L}-\mathbf{L}'\|\le\max_{1\le i\le n}\{|\mathbf{L}_{i,i-1}-\mathbf{L}'_{i,i-1}|+|\mathbf{L}_{i,i}-\mathbf{L}'_{i,i}|+|\mathbf{L}_{i+1,i}-\mathbf{L}'_{i+1,i}|\}, \tag{3.12}$$

it suffices to show that

$$|\mathbf{L}_{i,i}-\mathbf{L}'_{i,i}|\le 4\alpha\epsilon,\quad |\mathbf{L}_{i+1,i}-\mathbf{L}'_{i+1,i}|\le 2\alpha\epsilon\quad\forall i \tag{3.13}$$

under the assumptions that

$$|X_k-\sqrt{k}|\le\delta,\quad |Y_l-\sqrt{l}|\le\delta\quad\forall k,l. \tag{3.14}$$

Indeed, (3.14) and Lemma 3.1 with $m=2$ imply that, for any $k,l\le 2\alpha$,

$$|X_k^2-k|\le\delta(2\sqrt{k}+\delta)\le\delta(2\sqrt{2\alpha}+\delta), \tag{3.15}$$

$$|X_kY_l-\sqrt{kl}|\le\delta(\sqrt{k}+\sqrt{l}+\delta)\le\delta(2\sqrt{2\alpha}+\delta)=2\alpha\epsilon. \tag{3.16}$$ □

Proof of Fact 1. Using the matrix model from [4] with entries given by (3.6), (3.7) and (3.8), we find that

$$M_1^{\mathrm{L}}\sim\frac{1}{n}\sum_{i=1}^n\mathbf{L}_{i,i}=\frac{1}{n}\sum_{i=1}^n X^2_{2\alpha-(i-1)\beta}+\frac{1}{n}\sum_{i=2}^n Y^2_{(n+1-i)\beta}\sim\frac{1}{n}\chi^2_{2\alpha n}. \tag{3.17}$$ □

3.2. Jacobi ensemble. For $k,l>0$, let $Z\sim B(k,l)$ denote a beta-distributed random variable on the interval $[-1,1]$ with probability density function

$$f_{\mathrm{beta}}(z)\propto(1-z)^{k-1}(1+z)^{l-1} \tag{3.18}$$

so that

$$\mathbb{E}Z=\frac{l-k}{k+l}. \tag{3.19}$$

Assume, without loss of generality, that $k\ge l$. Theorem 8 in [12] gives the tail bound

$$\Pr(Z>\mathbb{E}Z+\delta)\le 2e^{-C\min\left\{\frac{k^2\delta^2}{l},k\delta\right\}},\quad \Pr(Z<\mathbb{E}Z-\delta)\le 2e^{-\frac{Ck^2\delta^2}{l}}. \tag{3.20}$$

Note that $\Pr(Z > \mathbb{E}Z + \delta) = 0$ for $\delta \ge 1 - \mathbb{E}Z$. In this case, the first inequality above holds trivially. The tail bound (3.20) implies that

$$\Pr(|Z - \mathbb{E}Z| > \delta) \le 4e^{-Ck\delta^2}, \tag{3.21}$$

$$\Pr(Z > \mathbb{E}Z + 2\delta\sqrt{1+\mathbb{E}Z} + \delta^2) \le 2e^{-Ck\delta^2} \quad \forall \delta > 0. \tag{3.22}$$

Furthermore, for $0 < \delta < \sqrt{1+\mathbb{E}Z}$,

$$\Pr(Z < \mathbb{E}Z - 2\delta\sqrt{1+\mathbb{E}Z} + \delta^2) \le 2e^{-Ck\delta^2}. \tag{3.23}$$

Equations (3.22) and (3.23) imply that

$$\Pr(|\sqrt{1+Z} - \sqrt{1+\mathbb{E}Z}| > \delta) \le 4e^{-Ck\delta^2}. \tag{3.24}$$

Similarly,

$$\Pr(|\sqrt{1-Z} - \sqrt{1-\mathbb{E}Z}| > \delta) \le 4e^{-Ck\delta^2}. \tag{3.25}$$

3.2.1. Pointwise approximation: Proof of Theorem 2.10. Let $Z_2, Z_3, Z_4, \dots, Z_{2n}$ be independent random variables with distribution

$$Z_i \sim \begin{cases} B\big(a + (2n-i)\beta/4, b + (2n-i)\beta/4\big), & \text{even } i, \\ B\big(a + b + (2n-1-i)\beta/4, (2n+1-i)\beta/4\big), & \text{odd } i, \end{cases} \tag{3.26}$$

so that

$$\mathbb{E}Z_i = \frac{1}{a+b+(n-i/2)\beta} \times \begin{cases} b-a, & \text{even } i, \\ \beta/2 - a - b, & \text{odd } i. \end{cases} \tag{3.27}$$

Let $Z_1 := -1$.

Let $\mathbf{J}_{i,j}$ be the element in the ith row and jth column of a real symmetric $n \times n$ tridiagonal random matrix $\mathbf{J}$. The diagonal and subdiagonal matrix elements are, respectively,

$$\mathbf{J}_{i,i} = (1 - Z_{2i-1})Z_{2i} - (1 + Z_{2i-1})Z_{2i-2}, \tag{3.28}$$

$$\mathbf{J}_{i+1,i} = \sqrt{(1 - Z_{2i-1})(1 - Z_{2i}^2)(1 + Z_{2i+1})}. \tag{3.29}$$

The joint eigenvalue distribution of $\mathbf{J}/2$ is given by [8] the Jacobi ensemble (Definition 2.7).

Let $\mathbf{J}'$ be a real symmetric $n \times n$ tridiagonal deterministic matrix, whose matrix elements are obtained by replacing every random variable Z_i in (3.28) and (3.29) by $\mathbb{E}Z_i$; i.e.,

$$\mathbf{J}'_{i,i} = (1 - \mathbb{E}Z_{2i-1})\mathbb{E}Z_{2i} - (1 + \mathbb{E}Z_{2i-1})\mathbb{E}Z_{2i-2}, \tag{3.30}$$

$$\mathbf{J}'_{i+1,i} = \sqrt{(1 - \mathbb{E}Z_{2i-1})\big(1 - (\mathbb{E}Z_{2i})^2\big)(1 + \mathbb{E}Z_{2i+1})}. \tag{3.31}$$

The eigenvalues of $\mathbf{J}'/2$ are the zeros of the Jacobi polynomial $P_n^{2a/\beta-1,2b/\beta-1}(x)$ [3].

Let $\mathbf{J}_{1,0} = \mathbf{J}'_{1,0} = \mathbf{J}_{n+1,n} = \mathbf{J}'_{n+1,n} := 0$. Using (3.21), (3.24), and (3.25) and since

$$\max_{1\le i\le n} |\mu_i - y_i| \le \|\mathbf{J} - \mathbf{J}'\|/2 \le \max_{1\le i\le n} \{|\mathbf{J}_{i,i-1} - \mathbf{J}'_{i,i-1}| + |\mathbf{J}_{i,i} - \mathbf{J}'_{i,i}| + |\mathbf{J}_{i+1,i} - \mathbf{J}'_{i+1,i}|\}/2, \tag{3.32}$$

it suffices to show that

$$|\mathbf{J}_{i,i} - \mathbf{J}'_{i,i}| \le C\epsilon \quad \forall i, \tag{3.33}$$

$$|\mathbf{J}_{i+1,i} - \mathbf{J}'_{i+1,i}| \le C\epsilon \quad \forall i \tag{3.34}$$

under the assumptions that

$$|Z_i - \mathbb{E}Z_i| \le \epsilon \quad \forall i, \tag{3.35}$$

$$|\sqrt{1+Z_i} - \sqrt{1+\mathbb{E}Z_i}| \le \epsilon, \quad |\sqrt{1-Z_i} - \sqrt{1-\mathbb{E}Z_i}| \le \epsilon \quad \forall i. \tag{3.36}$$

Equation (3.33) follows from (3.35) and Lemma 3.1 with $m = 2$. Equation (3.34) follows from (3.36) and Lemma 3.1 with $m = 4$.

3.2.2. Moments: Proof of Theorem 2.12. Since $N = O(\max\{a+b, n\})$, it suffices to prove that

$$\Pr(|M_1^{\mathrm{J}} - \mathbb{E}M_1^{\mathrm{J}}| > \epsilon) = O(e^{-\Omega(a+b)n\epsilon^2}), \tag{3.37}$$

$$\Pr(|M_1^{\mathrm{J}} - \mathbb{E}M_1^{\mathrm{J}}| > \epsilon) = O(e^{-\Omega(n^2\epsilon^2)}), \tag{3.38}$$

$$\Pr(|M_2^{\mathrm{J}} - \mathbb{E}M_2^{\mathrm{J}}| > \epsilon) = O(e^{-\Omega(a+b)\epsilon\min\{N\epsilon, n\}}), \tag{3.39}$$

$$\Pr(|M_2^{\mathrm{J}} - \mathbb{E}M_2^{\mathrm{J}}| > \epsilon) = O(e^{-\Omega(n^2\epsilon^2)}). \tag{3.40}$$

We follow the proof of Theorem 2.10 and use the same notation. We have proved that

$$\Pr(|\mathbf{J}_{i,i} - \mathbf{J}'_{i,i}| > \delta) = O(e^{-\Omega(a+b)\delta^2}) \quad \forall i, \tag{3.41}$$

$$\Pr(|\mathbf{J}_{i+1,i} - \mathbf{J}'_{i+1,i}| > \delta) = O(e^{-\Omega(a+b)\delta^2}) \quad \forall i. \tag{3.42}$$

Let I_n be the identity matrix of order n. A straightforward calculation using (3.28) and (3.29) yields

$$M_1^{\mathrm{J}} = \frac{1}{n}\mathrm{tr}\frac{\mathbf{J}}{2} = \frac{1}{2n}\sum_{i=1}^{n}\mathbf{J}_{i,i} = \frac{1}{2n}\left(Z_{2n} - \sum_{i=2}^{2n} Z_{i-1}Z_i\right), \tag{3.43}$$

$$M_2^{\mathrm{J}} = \frac{1}{n}\mathrm{tr}((\mathbf{J}/2 - Y_1 I_n)^2) = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{J}_{i,i}/2 - Y_1)^2 + \frac{1}{2n}\sum_{i=1}^{n-1}\mathbf{J}_{i+1,i}^2 \tag{3.44}$$

$$= Y_1^2 - 2Y_1 M_1^{\mathrm{J}} + \frac{1}{2} + \frac{2Z_{2n-1}(1-Z_{2n}^2) + Z_2^2 + Z_{2n}^2}{4n} + M', \tag{3.45}$$

where

$$M' := \frac{1}{4n}\sum_{i=3}^{2n}\left(2Z_{i-2}(Z_{i-1}^2 - 1)Z_i + Z_{i-1}^2 Z_i^2\right). \tag{3.46}$$

We will use the Chernoff bound multiple times.

LEMMA 3.3. *Let $W_1, W_2, \ldots, W_n$ be independent real-valued random variables such that*

$$\mathbb{E}W_i = 0, \quad \Pr(|W_i| > x) = O\left(e^{-\min\left\{\frac{x}{r}, \frac{x^2}{s^2}\right\}}\right) \quad \forall i \tag{3.47}$$

for some $r, s > 0$*. Then,*

$$\Pr\left(\left|\frac{1}{n}\sum_{i=1}^{n} W_i\right| > \delta\right) = O\left(e^{-\Omega(n)\min\left\{\frac{\delta}{r}, \frac{\delta^2}{r^2+s^2}\right\}}\right). \tag{3.48}$$

Each W_i is a subexponential random variable in that its probability distribution satisfies (3.47). Thus, Lemma 3.3 is the Chernoff bound for subexponential random variables. For $r = 0^+$, W_i becomes a sub-Gaussian random variable, and Lemma 3.3 reduces to the Chernoff bound for sub-Gaussian random variables.

Proof of Lemma 3.3. The tail bound (3.47) implies that, for any $j > 0$,

$$\begin{aligned}\mathbb{E}(|W_i|^j) &= \int_0^\infty \Pr(|W_i|^j > x)\,\mathrm{d}x = \int_0^\infty jx^{j-1}\Pr(|W_i| > x)\,\mathrm{d}x \\ &= \int_0^\infty jx^{j-1} O(e^{-x/r} + e^{-x^2/s^2})\,\mathrm{d}x = O\big(r^j\Gamma(j+1) + s^j\Gamma(j/2+1)\big).\end{aligned} \tag{3.49}$$

Let t be such that $0 < t \le 1/(2r)$. Since $\mathbb{E}W_i = 0$,

$$\mathbb{E}e^{tW_i} = 1 + \sum_{j=2}^{\infty}\frac{t^j\mathbb{E}(W_i^j)}{j!} = 1 + \sum_{j=2}^{\infty} O\left((rt)^j + \frac{(st)^j\Gamma(j/2+1)}{j!}\right). \tag{3.50}$$

Using $(st)^j \le (st)^{j-1} + (st)^{j+1}$ for odd j,

$$\begin{aligned}\mathbb{E}e^{tW_i} &= 1 + \frac{O(rt)^2}{1-rt} + \sum_{j=1}^{\infty}(st)^{2j} O\left(\frac{\Gamma(j+1/2)}{(2j-1)!} + \frac{j!}{(2j)!} + \frac{\Gamma(j+3/2)}{(2j+1)!}\right) \\ &= 1 + O(rt)^2 + O(1)\sum_{j=1}^{\infty}\frac{(st)^{2j}}{j!} \le e^{c(r^2+s^2)t^2},\end{aligned} \tag{3.51}$$

where $c > 0$ is a constant. Recall the standard Chernoff argument:

$$\Pr\left(\frac{1}{n}\sum_{i=1}^{n} W_i > \delta\right) = \Pr(e^{t\sum_{i=1}^n W_i} > e^{nt\delta}) \le e^{-nt\delta}\mathbb{E}e^{t\sum_{i=1}^n W_i} = \prod_{i=1}^{n}\mathbb{E}e^{tW_i - t\delta}. \tag{3.52}$$

If $\delta \le c(r^2+s^2)/r$, we choose $t = \frac{\delta}{2c(r^2+s^2)}$ so that

$$\mathbb{E}e^{tW_i - t\delta} \le e^{-\frac{\delta^2}{4c(r^2+s^2)}}. \tag{3.53}$$

If $\delta > c(r^2+s^2)/r$, we choose $t = 1/(2r)$ so that

$$\mathbb{E}e^{tW_i - t\delta} \le e^{\frac{c(r^2+s^2)}{4r^2} - \frac{\delta}{2r}} \le e^{-\frac{\delta}{4r}}. \tag{3.54}$$

We complete the proof by combining these two cases. ∎

Lemma 3.4. *Let* $W_1, W_2, \ldots, W_n$ *be independent random variables on the interval* $[-1, 1]$ *such that*

$$\mathbb{E}W_i = 0, \quad \Pr(|W_i| > x) = O\left(e^{-\min\left\{(r+is)x, \frac{(r+is)^3x^2}{r^2}\right\}}\right) \quad \forall i \tag{3.55}$$

for some $r, s = \Omega(1)$*. Then,*

$$\Pr\left(\left|\frac{1}{n}\sum_{i=1}^{n} W_i\right| > \delta\right) = O(e^{-\Omega(n^2\delta^2)}). \tag{3.56}$$

Proof. For $i \geq (2t-r)/s$, by replacing (3.47) with (3.55), (3.51) implies that

$$\mathbb{E}e^{tW_i} = e^{\frac{O(t^2)}{(r+is)^2} + \frac{O(r^2t^2)}{(r+is)^3}}. \tag{3.57}$$

Since $|W_i| \leq 1$, we trivially have

$$\mathbb{E}e^{tW_i} \leq e^t. \tag{3.58}$$

The Chernoff argument (3.52) implies that

$$\begin{aligned}\Pr\left(\frac{1}{n}\sum_{i=1}^n W_i > \delta\right) &\leq e^{-nt\delta}\prod_{i=1}^n \mathbb{E}e^{tW_i} \\ &\leq e^{-nt\delta}\prod_{1\leq i<\frac{2t-r}{s}} e^t \times \prod_{\max\left\{\frac{2t-r}{s},1\right\}\leq i\leq n} e^{\frac{O(t^2)}{(r+is)^2}+\frac{O(r^2t^2)}{(r+is)^3}} \\ &\leq e^{-nt\delta+O(t^2)+\sum_{i=1}^\infty \frac{O(t^2)}{(r+is)^2}+\frac{O(r^2t^2)}{(r+is)^3}} = e^{O(t^2)-nt\delta}.\end{aligned} \tag{3.59}$$

We complete the proof by choosing $t = c'n\delta$ for a sufficiently small constant $c' > 0$. ∎

Proof of (3.37). Using (3.41) and Lemma 3.3 with $r = 0^+$,

$$\Pr\left(\left|\frac{1}{n}\sum_{\text{even } i}(\mathbf{J}_{i,i} - \mathbf{J}'_{i,i})\right| > \epsilon\right) = O(e^{-\Omega(a+b)n\epsilon^2}), \tag{3.60}$$

$$\Pr\left(\left|\frac{1}{n}\sum_{\text{odd } i}(\mathbf{J}_{i,i} - \mathbf{J}'_{i,i})\right| > \epsilon\right) = O(e^{-\Omega(a+b)n\epsilon^2}). \tag{3.61}$$

Then, (3.37) follows from (3.43) and the union bound. ∎

Proof of (3.38). The tail bound (3.21) implies that

$$\Pr(|Z_i - \mathbb{E}Z_i| > \delta) = O(e^{-\Omega(a+b+(n-i/2)\beta)\delta^2}) \tag{3.62}$$

so that

$$\Pr(|Z_{i-1}Z_i - \mathbb{E}Z_{i-1}\cdot\mathbb{E}Z_i| > \delta) = O\left(e^{-\Omega(a+b+(n-i/2)\beta)\delta\min\left\{\frac{(a+b+(n-i/2)\beta)^2\delta}{(a+b)^2},1\right\}}\right). \tag{3.63}$$

Using Lemma 3.4,

$$\Pr\left(\left|\frac{1}{n}\sum_{\text{even } i}(Z_{i-1}Z_i - \mathbb{E}Z_{i-1}\cdot\mathbb{E}Z_i)\right| > \epsilon\right) = O(e^{-\Omega(n^2\epsilon^2)}), \tag{3.64}$$

$$\Pr\left(\left|\frac{1}{n}\sum_{\text{odd } i}(Z_{i-1}Z_i - \mathbb{E}Z_{i-1}\cdot\mathbb{E}Z_i)\right| > \epsilon\right) = O(e^{-\Omega(n^2\epsilon^2)}). \tag{3.65}$$

Then, (3.38) follows from (3.43) and the union bound. ∎

Proof of (3.39). Equations (3.27), (3.28), (3.29), (3.30), and (3.31) imply that

$$|\mathbf{J}'_{i,i}/2 - Y_1| = O(n/N), \quad |\mathbf{J}'_{i+1,i}| = O(\sqrt{n/N}) \quad \forall i, \tag{3.66}$$

$$|\mathbb{E}((\mathbf{J}_{i,i}/2 - Y_1)^2) - (\mathbf{J}'_{i,i}/2 - Y_1)^2| = \frac{O(1)}{a+b} \quad \forall i, \tag{3.67}$$

$$|\mathbb{E}(\mathbf{J}^2_{i+1,i}) - \mathbf{J}'^2_{i+1,i}| = \frac{O(n)}{(a+b)N} \quad \forall i. \tag{3.68}$$

Equations (3.41), (3.42), and (3.66) imply that

$$\Pr\left(|(\mathbf{J}_{i,i}/2 - Y_1)^2 - (\mathbf{J}'_{i,i}/2 - Y_1)^2| > \delta\right) = O\left(e^{-\Omega(a+b)\delta\min\{N^2\delta/n^2,1\}}\right) \quad \forall i, \tag{3.69}$$

$$\Pr(|\mathbf{J}^2_{i+1,i} - \mathbf{J}'^2_{i+1,i}| > \delta) = O\left(e^{-\Omega(a+b)\delta\min\{N\delta/n,1\}}\right) \quad \forall i. \tag{3.70}$$

Using (3.67) and (3.68),

$$\Pr\left(|(\mathbf{J}_{i,i}/2 - Y_1)^2 - \mathbb{E}((\mathbf{J}_{i,i}/2 - Y_1)^2)| > \delta\right) = O\left(e^{-\Omega(a+b)\delta\min\{N^2\delta/n^2,1\}}\right) \quad \forall i, \tag{3.71}$$

$$\Pr\left(|\mathbf{J}^2_{i+1,i} - \mathbb{E}(\mathbf{J}^2_{i+1,i})| > \delta\right) = O\left(e^{-\Omega(a+b)\delta\min\{N\delta/n,1\}}\right) \quad \forall i. \tag{3.72}$$

Using Lemma 3.3,

$$\Pr\left(\left|\frac{1}{n}\sum_{\text{even } i}\left((\mathbf{J}_{i,i}/2 - Y_1)^2 - \mathbb{E}((\mathbf{J}_{i,i}/2 - Y_1)^2)\right)\right| > \epsilon\right) = O\left(e^{-\Omega(a+b)\epsilon\min\{N\epsilon,n\}}\right), \tag{3.73}$$

$$\Pr\left(\left|\frac{1}{n}\sum_{\text{odd } i}\left((\mathbf{J}_{i,i}/2 - Y_1)^2 - \mathbb{E}((\mathbf{J}_{i,i}/2 - Y_1)^2)\right)\right| > \epsilon\right) = O\left(e^{-\Omega(a+b)\epsilon\min\{N\epsilon,n\}}\right), \tag{3.74}$$

$$\Pr\left(\left|\frac{1}{n}\sum_{\text{even } i}\left(\mathbf{J}^2_{i+1,i} - \mathbb{E}(\mathbf{J}^2_{i+1,i})\right)\right| > \epsilon\right) = O\left(e^{-\Omega(a+b)\epsilon\min\{N\epsilon,n\}}\right), \tag{3.75}$$

$$\Pr\left(\left|\frac{1}{n}\sum_{\text{odd } i}\left(\mathbf{J}^2_{i+1,i} - \mathbb{E}(\mathbf{J}^2_{i+1,i})\right)\right| > \epsilon\right) = O\left(e^{-\Omega(a+b)\epsilon\min\{N\epsilon,n\}}\right). \tag{3.76}$$

Then, (3.39) follows from (3.44) and the union bound. ∎

Proof of (3.40). The tail bound (3.62) implies that

$$\begin{aligned}&\Pr\left(|2Z_{i-2}(Z^2_{i-1} - 1)Z_i + Z^2_{i-1}Z^2_i - \mathbb{E}(2Z_{i-2}(Z^2_{i-1} - 1)Z_i + Z^2_{i-1}Z^2_i)| > \delta\right)\\ &\quad = O\left(e^{-\Omega(a+b+(n-i/2)\beta)\delta\min\left\{\frac{(a+b+(n-i/2)\beta)^2\delta}{(a+b)^2},1\right\}}\right).\end{aligned} \tag{3.77}$$

Recall the definition (3.46) of M'. It can be proved in the same way as (3.38) that

$$\Pr(|M' - \mathbb{E}M'| > \epsilon) = O(e^{-\Omega(n^2\epsilon^2)}). \tag{3.78}$$

Equation (3.40) follows from (3.38), (3.45), and (3.78) and the union bound. ∎

Appendix A. Proof of (2.26). We write the Jacobi polynomial (2.13) as

$$P_n^{p,q}(y) = \frac{\Gamma(p+q+2n+1)}{2^n n!\Gamma(p+q+n+1)} \left(y^n + \sum_{j=0}^{n-1} c_j y^j \right). \tag{A.1}$$

Let $p = 2a/\beta - 1$ and $q = 2b/\beta - 1$. From direct calculation, we find that

$$c_{n-1} = \frac{2n(p+n)}{p+q+2n} - n = \frac{n(a-b)}{N}, \tag{A.2}$$

$$\begin{aligned} c_{n-2} &= n(n-1)\left(\frac{1}{2} - \frac{2(p+n)}{p+q+2n} + \frac{2(p+n)(p+n-1)}{(p+q+2n)(p+q+2n-1)}\right) \\ &= \frac{n(n-1)\big(2(a-b)^2 - \beta N\big)}{2N(2N-\beta)}. \end{aligned} \tag{A.3}$$

Hence,

$$Y_1 = -c_{n-1}/n = (b-a)/N, \tag{A.4}$$

$$\begin{aligned} Y_2 &= -Y_1^2 + \frac{1}{n}\sum_{j=1}^n y_j^2 = \frac{1}{n}\left(\sum_{j=1}^n y_j\right)^2 - Y_1^2 - \frac{1}{n}\sum_{j\neq k} y_j y_k \\ &= (n-1)Y_1^2 - 2c_{L-2}/n = \beta(n-1)(1-Y_1^2)/(2N-\beta). \end{aligned} \tag{A.5}$$

Appendix B. Moments of the Hermite ensemble. Fact 1 and Theorem 2.12 concern the moments of the Laguerre and Jacobi ensembles, respectively. For the Hermite ensemble, it is simple to calculate the distributions of the first and second moments exactly. The results are presented here for completeness.

DEFINITION B.1 (Hermite ensemble). *The probability density function of the β-Hermite ensemble is*

$$f_{\mathrm{Herm}}(\nu_1, \nu_2, \ldots, \nu_n) \propto \prod_{1\le i<j\le n} |\nu_i - \nu_j|^\beta \prod_{i=1}^n e^{-\nu_i^2/2}. \tag{B.1}$$

For $\beta = 1, 2, 4$, the Hermite ensemble gives the probability density function of the eigenvalues of an $n \times n$ self-adjoint matrix whose entries are real, complex, or quaternionic Gaussian random variables.

Let

$$M_1^{\mathrm{H}} := \frac{1}{n}\sum_{i=1}^n \nu_i, \quad M_2^{\mathrm{H}} := \frac{1}{n}\sum_{i=1}^n (\nu_i - \mathbb{E}M_1^{\mathrm{H}})^2 = \frac{1}{n}\sum_{i=1}^n \nu_i^2 \tag{B.2}$$

be the first and second moments of the Hermite ensemble, where we used the fact that $\mathbb{E}M_1^{\mathrm{H}} = 0$.

FACT 2. *M_1^H is distributed as $\mathcal{N}(0, 1/n)$, where $\mathcal{N}(0, \sigma^2)$ denotes the normal distribution with mean* 0 *and variance σ^2. M_2^H is distributed as $\frac{1}{n}\chi^2_{n+\beta n(n-1)/2}$.*

Proof. Let $g_1, g_2, \ldots, g_n, X_\beta, X_{2\beta}, \ldots, X_{(n-1)\beta}$ be independent random variables with

$$g_i \sim \mathcal{N}(0,1), \quad X_k^2 \sim \chi_k^2, \quad X_k \ge 0. \tag{B.3}$$

The eigenvalues of the real symmetric $n \times n$ tridiagonal random matrix

$$\mathbf{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}g_1 & X_\beta & & & & \\ X_\beta & \sqrt{2}g_2 & X_{2\beta} & & & \\ & X_{2\beta} & \sqrt{2}g_3 & X_{4\beta} & & \\ & & \ddots & \ddots & \ddots & \\ & & & X_{(n-2)\beta} & \sqrt{2}g_{n-1} & X_{(n-1)\beta} \\ & & & & X_{(n-1)\beta} & \sqrt{2}g_n \end{pmatrix} \tag{B.4}$$

are distributed according to f_{Herm} [4] so that

$$M_1^{\mathrm{H}} \sim \frac{1}{n}\mathrm{tr}\mathbf{H} = \frac{1}{n}\sum_{i=1}^{n} g_i \sim \mathcal{N}(0, 1/n), \tag{B.5}$$

$$M_2^{\mathrm{H}} \sim \frac{1}{n}\mathrm{tr}(\mathbf{H}^2) = \frac{1}{n}\sum_{i=1}^{n} g_i^2 + \frac{1}{n}\sum_{i=1}^{n-1} X_{i\beta}^2 \sim \frac{1}{n}\chi^2_{n+\beta n(n-1)/2}. \tag{B.6}$$

∎

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