An Eulerian finite element method for the linearized Navier–Stokes problem in an evolving domain

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The paper addresses an error analysis of an Eulerian finite element method used for solving a linearized Navier–Stokes problem in a time-dependent domain. In this study, the domain's evolution is assumed to be known and independent of the solution to the problem at hand. The numerical method employed in the study combines a standard backward differentiation formula-type time-stepping procedure with a geometrically unfitted finite element discretization technique. Additionally, Nitsche's method is utilized to enforce the boundary conditions. The paper presents a convergence estimate for several velocity–pressure elements that are inf-sup stable. The estimate demonstrates optimal order convergence in the energy norm for the velocity component and a scaled $L^2(H^1)$-type norm for the pressure component.

Keywords: interface Stokes problem; evolving interface; cutFEM.

1. Introduction

Fluid equations formulated in time-dependent domains are essential components of mathematical models used in a wide range of applications, including cardiovascular research and aerospace engineering (Formaggia *et al.*, 2010; Bazilevs *et al.*, 2013). The analysis of such equations is a classical topic in mathematical fluid mechanics (Solonnikov, 1977; Miyakawa & Teramoto, 1982; Solonnikov, 1987; Neustupa, 2009). Moreover, a significant body of literature addresses the development of computational methods aimed at numerically solving these problems. Well-established numerical techniques include immersed boundary methods, fictitious domain methods, methods based on Lagrangian and arbitrary Lagrangian-Eulerian formulations, space-time finite element formulations, level-set methods and extended finite element methods; see, e.g., Peskin (1977); Tezduyar *et al.* (1992); Hirt *et al.* (1997); Masud & Hughes (1997); Formaggia & Nobile (1999); Glowinski *et al.* (1999); Duarte *et al.* (2004); Gross & Reusken (2011).

In this paper, we focus on an Eulerian finite element method that utilizes a time-independent triangulation of $\mathbb{R}^3$ to solve a system of governing equations within a volume $\Omega(t)$ that smoothly evolves through the background mesh, a typical configuration for unfitted finite element methods. Specifically, we consider the CutFEM unfitted finite element method (Burman *et al.*, 2015) that incorporates Nitsche's method for boundary condition imposition and employs a ghost-penalty stabilization (Burman, 2010) to handle instabilities arising from arbitrary small 'cuts' made by $\Omega(t)$ within the background simplices.

For time stepping, we adopt an Eulerian procedure suggested in Lehrenfeld & Olshanskii (2019) that relies on the implicit extension of the solution from $\Omega(t)$ to its neighborhood of $\mathcal{O}(\Delta t)$. This combination of the CutFEM method and implicit extension-based time stepping was initially applied to two-phase flow problems in Claus & Kerfriden (2019), demonstrating its efficacy when used in conjunction with the level-set method for interface capturing. Recent studies in Burman *et al.* (2022) and von Wahl *et al.* (2022) have addressed the analysis of this method, considering equal-order stabilized and Taylor–Hood elements, respectively. Both of these analyses identified a major challenge: the lack of a weak divergence-free property of the time difference of the finite element solutions $(\mathbf{u}_h^n - \mathbf{u}_h^{n-1})/\Delta t$ with respect to the discrete pressure space at time t^n. The absence of this property makes it challenging to bound this term in a suitable norm and precluding optimal-order estimates for the pressure. This observation has also been made in the literature on adaptive-in-time finite element methods, where the pressure space varies in time due to mesh adaptation (Besier & Wollner, 2012; Brenner *et al.*, 2014). The use of equal-order finite elements and pressure stabilization in Burman *et al.* (2022) allows the authors to establish the optimal error estimate for velocity. However, for inf-sup stable Taylor–Hood elements, the coupling between pressure and velocity appears stronger, and the sub-optimality in pressure also hindered the authors of von Wahl *et al.* (2022) from obtaining the optimal order estimation for the velocity error. It is worth noting that von Wahl *et al.* (2022) also quantified the error resulting from an approximate reconstruction of the evolving 'exact' domain, $\Omega(t)$.

Despite the aforementioned theoretical challenges, numerical experiments have demonstrated optimal order convergence rates (von Wahl *et al.*, 2022). This raises the question of whether the analysis can be enhanced to provide support for the observed numerical evidence. This is the question addressed in the present paper. The setup of the problem and the methods here is similar to von Wahl *et al.* (2022), but we consider general inf-sup stable unfitted finite element pairs, essentially those covered in the analysis by Guzmán *et al.* (Guzmán & Olshanskii, 2018).

The main result established in this paper can be summarized as follows: optimal convergence rates are proven for the energy norm of velocity and a scaled $L^2(H^1)$-norm of the pressure under the constraint $h^2 \lesssim \Delta t \lesssim h$, where h represents the mesh size and Δt denotes the time step. This bridges the gap in the analysis up to the selection of the pressure norm. Notably, the use of a nonstandard pressure norm is vital in mitigating the lack of divergence-free property in the discrete time derivative. This argument aligns with the analysis in a recent study (Olshanskii *et al.*, 2023), which analyzed a finite element method for the Navier–Stokes equations posed on time-dependent surfaces.

In general, there is a scarcity of literature addressing error bounds for fully discrete solutions of fluid equations in evolving domains. However, under the simplifying assumption that the motion of the domain is given and decoupled from the flow solution, error bounds for the arbitrary Lagrangian–Eulerian and quasi-Lagrangian finite element methods for Stokes, Navier–Stokes and coupled Stokes–parabolic equations in moving domains can be found in Martín *et al.* (2009); Lozovskiy *et al.* (2018); Kesler *et al.* (2021). Similarly, error bounds for the unfitted characteristic finite element method within the same setup are provided in Ma *et al.* (2023).

The remainder of the paper is organized in five sections and an appendix. Section 2 formulates the linearized Navier–Stokes problem in evolving domains and introduces suitable extension operators utilized in the analysis. In particular, the numerical analysis relies on existence of a sufficiently regular divergence-free extension of the fluid velocity field in a neighborhood of $\Omega(t)$. The fully discrete numerical method based on a Nitsche-based CutFEM formulation is given in Section 3. Here, we present the scheme for general finite element Stokes pairs satisfying certain assumptions. Stability and convergence analysis is the subject of Section 4. In Section 5, we list three standard finite element pairs satisfying the assumptions. Finally, a proof of a 'discrete' trace estimate is found in Appendix A.

2. Problem formulation

We consider a time-dependent domain $\Omega(t) \subset \mathbb{R}^3$ with boundary $\Gamma(t) := \partial\Omega(t)$ whose motion is assumed to be known *a priori*. In particular, we assume a smooth solenoidal vector field $\mathbf{w} : \mathbb{R}^3 \times [0, T] \to \mathbb{R}^3$, for some final time $T > 0$ such that the normal velocity of the boundary is specified via

$$V_\Gamma = \mathbf{w} \cdot \mathbf{n}_\Gamma \quad \text{on } \Gamma(t), \tag{2.1}$$

where $\mathbf{n}_\Gamma$ denotes the outward unit normal of $\Gamma(t)$. We then consider the Oseen problem in the moving volume $\Omega(t)$:

$$
\begin{aligned}
\mathbf{u}_t + (\mathbf{w} \cdot \nabla)\mathbf{u} - \Delta\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega(t), \\
\operatorname{div}\mathbf{u} &= 0 \quad \text{in } \Omega(t), \\
\mathbf{u} &= \mathbf{w} \quad \text{on } \Gamma(t),
\end{aligned}
\tag{2.2}
$$

with initial condition $\mathbf{u}|_{t=0} = \mathbf{u}_0$ in $\Omega_0 := \Omega(0)$. As mentioned in the introduction, unfitted finite element methods for (2.2) were recently addressed in Burman *et al.* (2022); von Wahl *et al.* (2022) with suboptimal error bounds. We note that the previous studies (Burman *et al.*, 2022; von Wahl *et al.*, 2022) ignore the advection term $(\mathbf{w} \cdot \nabla)\mathbf{u}$ in (2.2). While this term does not lead to any additional difficulties in the analysis, we believe it is mechanically relevant to include it in this simplified model. By a standard argument, we can re-write the above problem for

$$\mathbf{u} = 0 \quad \text{on } \Gamma(t). \tag{2.3}$$

We assume the smooth velocity field $\mathbf{w} : \mathbb{R}^3 \times [0, T] \to \mathbb{R}^3$ is such that it defines the flow map $\Phi_t : \Omega(0) \to \Omega(t)$ as the material evolution of the fluid volume: for $\mathbf{z} \in \Omega_0$, the trajectory $\mathbf{x}(t, \mathbf{z}) = \Phi_t(\mathbf{z})$ solves

$$
\begin{cases}
\mathbf{x}(0, \mathbf{z}) = \mathbf{z}, \\
\frac{d}{dt}\mathbf{x}(t, \mathbf{z}) = \mathbf{w}(t, \mathbf{x}(\mathbf{z}, t)) \qquad t \in (0, T]
\end{cases}
\tag{2.4}
$$

for some final time $T > 0$. Equation (2.4) defines a smooth bijection between Ω_0 and $\Omega(t)$ for every $t \in [0, T]$. If $\partial\Omega_0 \in C^p$ and $\mathbf{w} \in \mathbf{C}^p(\mathbb{R}^3)$, then $\Gamma(t) \in C^p$; the flow map Φ_t also preserves the connectivity of $\Omega(t)$.

Summarizing, we are interested in the analysis of a finite element method for solving (2.2) with $\Omega(t) = \Phi_t(\Omega(0))$ and homogeneous Dirichlet boundary conditions (2.3).

2.1 *Extensions*

Let $\Omega(t) \subset \widehat{\Omega}$ for all $t \in [0, T]$, for a bounded polyhedral domain $\widehat{\Omega} \subset \mathbb{R}^3$. We define the two space-time domains $\mathcal{Q}$ and $\hat{\mathcal{Q}}$ as follows:

$$\mathcal{Q} := \bigcup_{t \in [0,T]} \Omega(t) \times \{t\} \subset \widehat{\mathcal{Q}} := \widehat{\Omega} \times [0, T] \subset \mathbb{R}^4.$$

For a domain $D \subset \mathbb{R}^3$ and some $\delta > 0$, we use the notation $\mathcal{O}_\delta(D)$ for the δ-neighborhood of D:

$$\mathcal{O}_\delta(D) = \{x \in \mathbb{R}^3 : \text{dist}(x, D) \leq \delta\}.$$

Denoting by $\mathbf{V}(t) = \{\mathbf{v} \in \mathbf{H}_0^1(\Omega(t)) : \text{div}\,\mathbf{v} = 0\}$, the subspace of divergence-free functions in $\mathbf{H}_0^1(\Omega(t))$, our goal now is to define an extension operator $\mathcal{E} : \mathbf{V}(t) \to \mathbf{H}^1(\widehat{\Omega})$ that preserves the divergence-free condition. To this end, we note that since $\text{div}\mathbf{u} = 0$, we can write $\mathbf{u} = \nabla \times \boldsymbol{\psi}$ in $\Omega(t)$ with a stream function that satisfies $\boldsymbol{\psi} \in \mathbf{W}^{k+1,p}(\Omega(t))$ and

$$\|\boldsymbol{\psi}\|_{W^{k+1,p}(\Omega(t))} \lesssim \|\mathbf{u}\|_{W^{k,p}(\Omega(t))} \quad \text{for } \mathbf{u} \in W^{k,p}(\Omega(t)), \tag{2.5}$$

$k \geq 0$, $1 < p < \infty$; see Girault & Raviart (1986); Costabel & McIntosh (2010).

REMARK 2.1 Here, the statement $A \lesssim B$ (resp., $A \gtrsim B$) to mean $A \leq cB$ (resp., $A \geq cB$) for some constant $c > 0$ independent of the spatial and temporal discretization parameters h and Δt introduced below and time t. The statement $A \simeq B$ means $A \lesssim B$ and $A \gtrsim B$.

For $\boldsymbol{\psi}_0 = \boldsymbol{\psi} \circ \Phi_t$ we consider Stein's extension: since the boundary of Ω_0 is smooth, there is a continuous linear extension operator $\mathcal{E}_0 : L^2(\Omega_0) \to L^2(\mathbb{R}^3)$, ($\mathcal{E}_0 \boldsymbol{\psi}_0 = \boldsymbol{\psi}_0$ in Ω_0), with the following properties (Stein, 2016, Section VI.3.1):

$$\|\mathcal{E}_0 \boldsymbol{\psi}_0\|_{W^{k,p}(\mathbb{R}^3)} \leq C_{\Omega_0} \|\boldsymbol{\psi}_0\|_{W^{k,p}(\Omega_0)}, \quad \text{for } \boldsymbol{\psi}_0 \in W^{k,p}(\Omega_0), \ k = 0, \ldots, m+1, \ 1 \leq p \leq \infty, \tag{2.6}$$

with any fixed $m \geq 0$. Here, the extension operator is performed component-wise, i.e., $(\mathcal{E}_0 \boldsymbol{\psi}_0)_i = \mathcal{E}_0(\boldsymbol{\psi}_0)_i$ for $i = 1, 2, 3$. For the extension $\mathcal{E}_\psi \boldsymbol{\psi} := (\mathcal{E}_0 \boldsymbol{\psi}_0) \circ \Phi_t^{-1}$ of $\boldsymbol{\psi}$, the following estimates follow from the analysis in Lehrenfeld & Olshanskii (2019):

$$\|\mathcal{E}_\psi \boldsymbol{\psi}\|_{H^k(\widehat{\Omega})} \lesssim \|\boldsymbol{\psi}\|_{H^k(\Omega(t))}, \quad k = 0, \ldots, m+1, \quad \|\mathcal{E}_\psi \boldsymbol{\psi}\|_{W^{4,5}(\widehat{\mathcal{Q}})} \lesssim \|\boldsymbol{\psi}\|_{W^{4,5}(\mathcal{Q})},$$

$$\|(\mathcal{E}_\psi \boldsymbol{\psi})_t\|_{H^m(\widehat{\Omega})} \lesssim (\|\boldsymbol{\psi}\|_{H^{m+1}(\Omega(t))} + \|\boldsymbol{\psi}_t\|_{H^m(\Omega(t))}). \tag{2.7}$$

We now define the velocity extension as follows:

$$\mathcal{E}\mathbf{u}(t) := \nabla \times (\mathcal{E}_\psi \boldsymbol{\psi}), \quad \text{for each } t \in [0, T]. \tag{2.8}$$

By construction, there holds

$$\text{div}\mathcal{E}\mathbf{u} = 0 \quad \text{in } \widehat{\Omega}.$$

For $\mathbf{u} \in L^\infty(0, T; \mathbf{H}^m(\Omega(t))) \cap \mathbf{W}^{3,5}(\mathcal{Q})$ such that $\text{div}\mathbf{u} = 0$ in $\Omega(t)$ for all $t \in (0, T)$ and any fixed integer $m \geq 0$, the following estimates follow from (2.5), (2.7), Poincare–Friedrich's inequality and the embedding $W^{3,5}(\widehat{\mathcal{Q}}) \subset W^{2,\infty}(\widehat{\mathcal{Q}})$:

$$\|\mathcal{E}\mathbf{u}\|_{H^k(\widehat{\Omega})} \lesssim \|\mathbf{u}\|_{H^k(\Omega(t))}, \quad k = 0, \ldots, m, \tag{2.9a}$$

$$\|\nabla(\mathcal{E}\mathbf{u})\|_{\widehat{\Omega}} \lesssim \|\nabla\mathbf{u}\|_{\Omega(t)}, \tag{2.9b}$$

$$\|\mathcal{E}\mathbf{u}\|_{W^{2,\infty}(\widehat{\mathcal{Q}})} \lesssim \|\mathbf{u}\|_{W^{3,5}(\mathcal{Q})}. \tag{2.9c}$$

Here, we use the standard notation for the L^2-norm $\|\cdot\|_D = \|\cdot\|_{L^2(D)}$ for some domain D. Furthermore, for $\mathbf{u} \in L^\infty(0, T; \mathbf{H}^m(\Omega(t)))$ such that $\mathbf{u}_t \in L^\infty(0, T; \mathbf{H}^{m-1}(\Omega(t)))$ it holds

$$\|(\mathcal{E}\mathbf{u})_t\|_{H^{m-1}(\widehat{\Omega})} \lesssim (\|\mathbf{u}\|_{H^m(\Omega(t))} + \|\mathbf{u}_t\|_{H^{m-1}(\Omega(t))}). \tag{2.10}$$

With an abuse of notation, we define the extension of the pressure as

$$\mathcal{E}p(t) = (\mathcal{E}_0(p \circ \Phi_t)) \circ \Phi_t^{-1}, \quad \text{for each } t \in [0, T]. \tag{2.11}$$

Then estimates (2.9a), (2.9c), with $\mathcal{E}\mathbf{u}$ and $\mathbf{u}$ replaced by $\mathcal{E}p$ and p, respectively, are satisfied (cf. Lehrenfeld & Olshanskii, 2019, Lemma 3.3). For the analysis, we need $\mathcal{E}\mathbf{u}$ and $\mathcal{E}p$ defined in $\mathcal{O}_\delta(\Omega(t)) \subset \widehat{\Omega}$, a δ-neighborhood of $\Omega(t)$ with $\delta \simeq \Delta t$.

3. The fully discrete finite element method

We adopt the basic framework in Lehrenfeld & Olshanskii (2019); Burman *et al.* (2022); von Wahl *et al.* (2022) to build a Nitsche-based CutFEM spatial discretization of the Stokes problem on an evolving domain.

3.1 *Approximate geometries*

Recall that $\widehat{\Omega} \subset \mathbb{R}^3$ is a polyhedral domain with $\Omega(t) \subset \widehat{\Omega}$ for all $t \in [0, T]$. For simplicity, we consider a time discretization with a uniform time-step $\Delta t = T/N$ for some $N \in \mathbb{N}$. We set $t_n = n\Delta t$, $\Omega^n = \Omega(t_n)$, $\Gamma^n = \Gamma(t_n)$ and $(\mathbf{u}^n, p^n) = (\mathbf{u}(t_n), p(t_n))$. We further set $\mathbf{w}^n_\infty = \|\mathbf{w}(t_n) \cdot \mathbf{n}_\Gamma\|_{L^\infty(\Gamma^n)}$. For practical purposes such as numerical integration, and similar to Lehrenfeld & Olshanskii (2019); Burman *et al.* (2022); von Wahl *et al.* (2022), we assume that the domains Ω^n are given by their approximations Ω^n_h (cf. (3.1)–(3.2) below). The boundary of Ω^n_h is denoted by Γ^n_h. To ensure that discrete solutions are well defined at subsequent time-steps, we extend the computational domain by a layer of thickness δ_h with $c_{\delta_h}\mathbf{w}^n_\infty \Delta t \le \delta_h$ with constant $1 \le c_{\delta_h} = O(1)$ such that $\text{dist}(\Omega^n_h, \Omega^{n+1}_h) \le \delta_h$ for all n.

We assume there is a bijective, Lipschitz continuous map $\Psi_n : \mathcal{O}_{\delta_h}(\Omega^n_h) \to \mathcal{O}_{\delta_h}(\Omega^n)$ that connects the approximate and exact domains at each time step. In particular, we assume Ψ_n satisfies $\mathcal{O}_{\delta_h}(\Omega^n) = \Psi_n(\mathcal{O}_{\delta_h}(\Omega^n_h))$, $\Omega^n = \Psi_n(\Omega^n_h)$, $\Gamma^n = \Psi_n(\Gamma^n_h)$ and the existence of a positive integer q such that

$$\|\Psi_n - \text{id}\|_{W^{j,\infty}(\mathcal{O}_{\delta_h}(\Omega^n_h))} \lesssim h^{q+1-j} \qquad j = 0, 1. \tag{3.1}$$
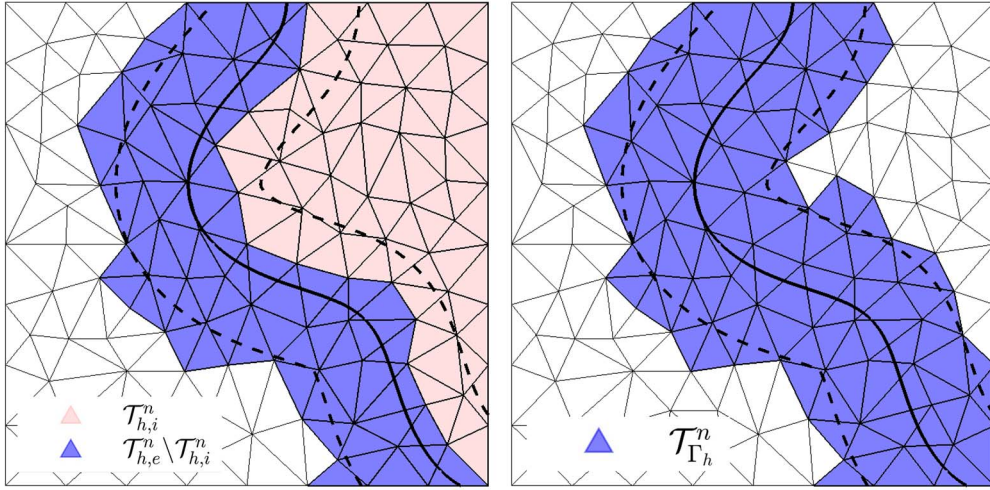
Fig. 1. Left: a depiction of the triangulation $\mathcal{T}_{h,i}^n$ and $\mathcal{T}_{h,e}^n \backslash \mathcal{T}_{h,i}^n$. Right: the triangulation $\mathcal{T}_{\Gamma_h}^n$ around a δ_h-neighborhood tubular region of Γ_h.

We refer to q as the geometric order of approximation. Such a mapping has been constructed in Gross *et al.* (2015) based on isoparametric mappings of geometries defined via level sets. Note that (3.1) implies

$$\text{dist}\left(\Omega^n, \Omega_h^n\right) \lesssim h^{q+1}. \tag{3.2}$$

3.2 *Triangulations*

We let $\mathcal{T}_h$ denote a shape-regular and quasi-uniform simplicial triangulation of the background domain $\widehat{\Omega}$ with $h = \max_{T \in \mathcal{T}_h} \text{diam}(T)$. Note the quasi-uniformity implies a constant $c > 0$ such that $h \leq c\,\text{diam}(T) =: h_T$ for all $T \in \mathcal{T}_h$.

We then define, for each time step *n*, the active triangulation and corresponding domain induced by the background triangulation (cf. Fig. 1):

$$\mathcal{T}_{h,e}^n = \left\{T \in \mathcal{T}_h : \text{dist}\left(\mathbf{x}, \Omega_h^n\right) \leq \delta_h \, \exists\, \mathbf{x} \in \bar{T}\right\}, \qquad \Omega_{h,e}^n = \text{int}\left(\bigcup_{T \in \mathcal{T}_{h,e}^n} \bar{T}\right).$$

We further definite the set of interior elements for Ω_h^n and associated domain at time step *n*:

$$\mathcal{T}_{h,i}^n = \left\{T \in \mathcal{T}_{h,e}^n : \text{int}(T) \subset \Omega_h^n\right\}, \qquad \Omega_{h,i}^n = \text{int}\left(\bigcup_{T \in \mathcal{T}_{h,i}^n} \bar{T}\right),$$

and denote by $\mathcal{F}_{h,i}^n$ (resp., $\mathcal{F}_{h,e}^n$) the set of interior faces of $\mathcal{T}_{h,i}^n$ (resp., $\mathcal{T}_{h,e}^n$), i.e.,

$$\mathcal{F}_{h,*}^n = \left\{F = \partial T_1 \cap \partial T_2 : T_1, T_2 \in \mathcal{T}_{h,*}^n, \, T_1 \neq T_2\right\} \qquad * \in \{i, e\}.$$

We further set $h_F = \text{diam}(F)$ for all $F \in \mathcal{F}_{h,e}$. Following Lehrenfeld & Olshanskii (2019); von Wahl *et al.* (2022), we define the elements in a strip around Γ_h^n:

$$\mathcal{T}_{\Gamma_h}^n := \left\{ T \in \mathcal{T}_{h,e}^n : \text{ dist}\left(x, \Gamma_h^n\right) \leq \delta_h \; \exists x \in \bar{T} \right\},$$

and define the set of faces in this strip:

$$\mathcal{F}_{\Gamma_h}^n := \left\{ F = \partial T_1 \cap \partial T_2 : T_1 \in \mathcal{T}_{h,e}^n, \; T_2 \in \mathcal{T}_{\Gamma_h}^n, \; T_1 \neq T_2, \; |\partial T_1 \cap \partial T_2| > 0 \right\}.$$

For any sub-triangulation $\mathcal{S}_h \subset \mathcal{T}_h$ and $m \in \mathbb{N}$, we set $H^m(\mathcal{S}_h)$ to be the piecewise Sobolev space with respect to $\mathcal{S}_h$, i.e., $q \in H^m(\mathcal{S}_h)$ implies q is an L^2 function on the domain induced by $\mathcal{S}_h$ and $q|_T \in H^m(T)$ for all $T \in \mathcal{S}_h$. Analogous vector-valued spaces are denoted in boldface.

3.3 *Finite element spaces and assumptions*

We denote by $P_m(\mathcal{T}_h)$ the space of piecewise polynomials of degree m with respect to $\mathcal{T}_h$, and set $P_m^c(\mathcal{T}_h) = P_m(\mathcal{T}_h) \cap H^1(\widehat{\Omega})$ to be its subspace of continuous, piecewise polynomials. Analogous vector-valued spaces are denoted in boldface. We consider a Stokes finite element pair $\mathbf{V}_h \times Q_h \subset \mathbf{H}^1(\widehat{\Omega}) \times L^2(\widehat{\Omega})$, consisting of piecewise polynomial spaces with respect to $\mathcal{T}_h$, and assume the following inclusions

$$\mathbf{P}_{\underline{m}_v}^c(\mathcal{T}_h) \subset \mathbf{V}_h \subset \mathbf{P}_{\overline{m}_v}^c(\mathcal{T}_h), \tag{3.3}$$

for some integers $1 \leq \underline{m}_v \leq \overline{m}_v$. We further assume there exists $m_q \in \mathbb{N}_0$ such that

$$Q_h = P_{m_q}(\mathcal{T}_h) \quad \text{or} \quad Q_h = P_{m_q}^c(\mathcal{T}_h). \tag{3.4}$$

We set $\mathbf{V}_h^n \subset \mathbf{H}^1(\Omega_{h,e}^n)$ to be the restriction of $\mathbf{V}_h$ to $\Omega_{h,e}^n$, and let Q_h^n be the restriction of Q_h to $\Omega_{h,e}^n$ with a zero-mean constraint on $\Omega_{h,i}^n$, i.e.,

$$Q_h^n = \left\{ q|_{\Omega_{h,e}^n} : \exists q \in Q_h \text{ such that } \int_{\Omega_{h,i}^n} q \, dx = 0 \right\}.$$

Note that, by construction, $\Omega_h^n \subset \Omega_{h,e}^{n-1}$, and therefore functions in $\mathbf{V}_h^{n-1} \times Q_h^{n-1}$ are well defined on Ω_h^n.

We define the Nitsche-type norms on $\mathbf{H}^1(\Omega_h^n) \cap \mathbf{H}^2(\mathcal{T}_{h,e}^n)\big|_{\Omega_h^n}$:

$$\|\mathbf{v}\|_n^2 := \|\nabla \mathbf{v}\|_{\Omega_h^n}^2 + h^{-1} \|\mathbf{v}\|_{\Gamma_h^n}^2 + h \|\nabla \mathbf{v}\|_{\Gamma_h^n}^2,$$

and further define the norm for piecewise smooth functions on the extended domains:

$$\|\mathbf{v}\|_{n,e}^2 := \|\nabla \mathbf{v}\|_{\Omega_{h,e}^n}^2 + \|\mathbf{v}\|_n^2.$$

Likewise, we define the weighted H^1-seminorm with respect to the interior mesh $\mathcal{T}_{h,i}^n$:

$$\|q\|_{n,i}^2 := h^2 \sum_{T \in \mathcal{T}_{h,i}^n} \|\nabla q\|_T^2 + h \sum_{F \in \mathcal{F}_{h,i}^n} \|[\![q]\!]\|_F^2,$$

where $[\![\cdot]\!]$ denotes the jump operator across an interior face. Note that $\|\cdot\|_{n,i}$ is a norm on $Q_h^n|_{\Omega_{h,i}^n}$. Similarly, we define weighted seminorm over the extended domain $\Omega_{h,e}^n$:

$$\|q\|_{n,e}^2 := h^2 \sum_{T \in \mathcal{T}_{h,e}^n} \|\nabla q\|_T^2 + h \sum_{F \in \mathcal{F}_{h,e}^n} \|[\![q]\!]\|_F^2.$$

Note that $\|q\|_{n,e}$ is a norm on Q_h^n, and *it will be our main pressure norm for stability and error analysis.*

In addition to the inclusions (3.3)–(3.4), we make the following assumptions to ensure stability of the discretization presented below.

ASSUMPTION 3.1　Assume that, given $q \in Q_h^n$, there exists $\mathbf{v}_h \in \mathbf{V}_h^n$ that satisfies

$$\|\mathbf{v}\|_{n,e} \lesssim \|q\|_{n,i}, \tag{3.5a}$$

$$\|q\|_{n,i}^2 \leq b_h^n(\mathbf{v}, q) := \int_{\Omega_h^n} (\operatorname{div} \mathbf{v}) q \, dx - \int_{\Gamma_h^n} (\mathbf{v} \cdot \mathbf{n}) q \, ds, \tag{3.5b}$$

$$\|\mathbf{v}\|_{\Omega_h^n} \lesssim h \|q\|_{n,i}. \tag{3.5c}$$

REMARK 3.2　The first two statements (3.5a)–(3.5b) are assumptions related to discrete inf-sup stability, but where the L^2 norm of the pressure function is replaced with the weighted H^1-norm. A variation of these conditions is shown to hold in the context of CutFEM for many standard stable Stokes pairs in Guzmán & Olshanskii (2018). Using a Verfürth-type trick, it is shown in this reference that, if (3.5a)–(3.5b) is satisfied then the discrete inf-sup condition with L^2 pressure norm holds:

$$\theta \|q\|_{\Omega_h^n} \leq \sup_{\mathbf{v} \in \mathbf{V}_h^n} \frac{b_h^n(\mathbf{v}, q)}{\|\mathbf{v}\|_{n,e}} + |q|_{J_h^n} \qquad \forall q \in Q_h^n,$$

where $\theta > 0$ is independent of h and how Γ_h^n cuts through the triangulation $\mathcal{T}_h$, and $|\cdot|_{J_h^n}$ is given by (4.1) below. We show below in Section 5 that the third condition (3.5c) is satisfied for several canonical pairs as well.

REMARK 3.3　Assumption 3.1 can be modified and slightly weakened by replacing $\Omega_{h,i}^n$ and $\mathcal{T}_{h,i}^n$ by a smaller domain and mesh, respectively, provided the pressure ghost-penalty compensates for the smaller domain. In particular, let $\tilde{\mathcal{T}}_{h,i}^n \subset \mathcal{T}_{h,i}^n$ be a sub-mesh with corresponding domain $\tilde{\Omega}_{h,i}^n = \operatorname{int}\left(\bigcup_{T \in \tilde{\mathcal{T}}_{h,i}^n} \bar{T}\right)$.

Then if

$$\|q\|_{\Omega_{h,e}^n} \lesssim \|q\|_{\tilde{\Omega}_{h,i}^n} + |q|_{J_h^n} \qquad \forall q \in Q_h^n, \tag{3.6}$$

then we can replace $\mathcal{T}_{h,i}^n$ by $\tilde{\mathcal{T}}_{h,i}^n$ and $\Omega_{h,i}^n$ by $\tilde{\Omega}_{h,i}^n$ in Assumption 3.1. This modified assumption is used in the case $\mathbf{V}_h \times Q_h$ is the Taylor–Hood pair.

3.4 *The CutFEM discretization*

The finite element method based on the backward Euler temporal discretization seeks, at each time step, the pair $(\mathbf{u}_h^n, p_h^n) \in \mathbf{V}_h^n \times Q_h^n$ such that

$$\int_{\Omega_h^n} \left(\frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}}{\Delta t} \right) \cdot \mathbf{v} \, dx + a_h^n \left(\mathbf{u}_h^n, \mathbf{v} \right) - b_h^n \left(\mathbf{v}, p_h^n \right) + b_h^n \left(\mathbf{u}_h^n, q \right) + \gamma_J J_h^n \left(p_h^n, q \right) = F^n(\mathbf{v}, q), \tag{3.7}$$

for all $\mathbf{v} \in \mathbf{V}_h^n$, $q \in Q_h^n$. Here, $b_h^n(\cdot, \cdot)$ is given by (3.5b), and the bilinear form $a_h^n(\cdot, \cdot)$ is defined as

$$\widehat{a}_h^n(\mathbf{u}, \mathbf{v}) = \int_{\Omega_h^n} \nabla \mathbf{u} : \nabla \mathbf{v} \, dx + \int_{\Omega_h^n} (\mathbf{w} \cdot \nabla \mathbf{u}) \cdot \mathbf{v} \, dx - \int_{\Gamma_h^n} \left([(\nabla \mathbf{u})\mathbf{n}] \cdot \mathbf{v} + [(\nabla \mathbf{v})\mathbf{n}] \cdot \mathbf{u} - \frac{\eta}{h} \mathbf{u} \cdot \mathbf{v} \right) ds,$$

$$a_h^n(\mathbf{u}, \mathbf{v}) = \widehat{a}_h^n(\mathbf{u}, \mathbf{v}) + \gamma_s s_h^n(\mathbf{u}, \mathbf{v}),$$

where $\gamma_s, \gamma_J, \eta \geq 1$ are user-defined constants. The bilinear forms $s_h^n(\cdot, \cdot)$ and $J_h^n(\cdot, \cdot)$ consist of ghost-penalty terms acting on $\mathbf{V}_h^n \times \mathbf{V}_h^n$ and $Q_h^n \times Q_h^n$, respectively, defined on an $O(\delta_h)$ neighborhood of Γ_h^n:

$$s_h^n(\mathbf{u}, \mathbf{v}) = \sum_{F \in \mathcal{F}_{\Gamma_h}^n} \sum_{k=1}^{\overline{m}_v} h^{2k-1} \int_F \left[\!\left[\partial_{\mathbf{n}_F}^k \mathbf{u} \right]\!\right] \left[\!\left[\partial_{\mathbf{n}_F}^k \mathbf{v} \right]\!\right] ds,$$

$$J_h^n(p, q) = \sum_{F \in \mathcal{F}_{\Gamma_h}^n} \sum_{k=0}^{m_q} h^{2k+1} \int_F \left[\!\left[\partial_{\mathbf{n}_F}^k p \right]\!\right] \left[\!\left[\partial_{\mathbf{n}_F}^k q \right]\!\right] ds, \tag{3.8}$$

and $\partial_{\mathbf{n}_F}^k$ denotes the kth-order directional derivative with respect to the normal of the face F. Here, $\overline{m}_v$ and m_q are the integers in (3.3)–(3.4). Finally, $F^n(\mathbf{v}, q)$ is a bounded linear functional on $\mathbf{V}_h^n \times Q_h^n$ with

$$\|F^n\|_* := \sup_{(\mathbf{v}, q) \in \mathbf{V}_h^n \times Q_h^n} \frac{F^n(\mathbf{v}, q)}{\left(\|\mathbf{v}\|_{n,e}^2 + \|q\|_{n,e}^2 \right)^{\frac{1}{2}}} < \infty. \tag{3.9}$$

In (3.7), it is given by

$$F^n(\mathbf{v}, q) = \int_{\Omega_h^n} \mathbf{f}^n \cdot \mathbf{v} \, dx,$$

but later we will consider a more general F^n for the purpose of analysis.

REMARK 3.4 We have assumed that $\mathbf{w}$ is a (given) smooth solenoidal vector field defined on $\mathbb{R}^3 \times [0, T]$. If this vector field is instead defined on $\cup_{t \in [0,T]} \Omega(t) \times \{t\}$, then a suitable extension to $\mathbf{w}$ would be used in the bilinear form $a_h^n(\cdot, \cdot)$. Such an approximation may not be solenoidal, in which case a standard skew-symmetry of the convective term would be required in the finite element method. The stability and convergence analysis results presented below still hold in this more general setting, albeit with slightly more technical arguments. We refer to Gross *et al.* (2015) for details.

REMARK 3.5 The ghost-penalty bilinear forms (3.8) both stabilize the solution of problem (3.7) due to irregular cuts as well as yield implicit extensions to $\Omega_{h,e}^n$. These terms also aid in algebraic stabilization, as the resulting condition number of the system is insensitive to how Γ_h^n intersects $\mathcal{T}_h$. The pressure ghost-stabilization form $J_h^n(\cdot, \cdot)$ ensures numerical stability as it provides an inf-sup-type stability condition of the pair $\mathbf{V}_h^n \times Q_h^n$ (cf. Remark 3.2).

There are now several types of ghost-penalty stabilization besides the 'derivative jump version' used in (3.8). These include the 'direct version' (Preuss, 2018) as well as the 'local projection stabilization version' (Burman, 2010). In principle, we can replace (3.8) with any choice of these types of ghost penalty versions, and the stability and convergence analysis presented below carries through with only superficial modifications. However, for clarity of presentation, we only focus on the derivative jump version in detail below.

Finally, we remark that the extension of the discrete pressure approximation to all of $\Omega_{h,e}^n$ is not required; in particular, the pressure ghost-penalty stabilization $J_h^n(\cdot, \cdot)$ only needs to be defined on a single layer of elements around Γ_h^n to ensure stability. However, we use the set of faces $\mathcal{F}_{\Gamma_h}^n$ for both terms in (3.8) to simplify the presentation.

4. Stability and convergence analysis

We denote by

$$|\mathbf{v}|_{s_h^n} = \sqrt{s_h^n(\mathbf{v}, \mathbf{v})} \quad \text{and} \quad |q|_{J_h^n} = \sqrt{J_h^n(q, q)} \tag{4.1}$$

the seminorms induced by the bilinear forms $s_h^n(\cdot, \cdot)$ and $J_h^n(\cdot, \cdot)$, respectively. We assume that the Nitsche penalty parameter η is chosen sufficiently large (but independent of h and the mesh-interface cut) such that $a_h^n(\cdot, \cdot)$ is coercive on $\mathbf{V}_h^n$ (cf. Burman & Hansbo, 2012). In particular, we assume $\eta > 0$ is chosen such that

$$a_h^n(\mathbf{v}, \mathbf{v}) \geq \frac{1}{2} \|\mathbf{v}\|_n^2 + \gamma_s |\mathbf{v}|_{s_h^n}^2 \qquad \forall \mathbf{v} \in \mathbf{V}_h^n. \tag{4.2}$$

Similar to Lehrenfeld & Olshanskii (2019); von Wahl *et al.* (2022), we assume that elements in the strip $\mathcal{T}_{h,e}^n \backslash \mathcal{T}_{h,i}^n$ can be reached from an uncut element in $\mathcal{T}_{h,i}^n$ by a path that crosses at most L faces with $L \lesssim (1 + \frac{\delta_h}{h})$; we refer to Lehrenfeld & Olshanskii (2019); von Wahl *et al.* (2022) to see why this is a reasonable assumption and how it relates to the shape-regularity of the triangulation $\mathcal{T}_h$. We consider the setting where L is uniformly bounded with respect to the discretization parameters, i.e., when $\delta_h \lesssim h$.

Recalling that $c_{\delta_h} \mathbf{w}_\infty^n \Delta t \leq \delta_h$ with $1 \leq c_{\delta_h} = O(1)$, this brings us to the time-step restriction:

$$\Delta t \lesssim h. \tag{4.3}$$

The condition (4.3) and $\|\mathbf{w}\|_{L^\infty(\mathcal{Q})} \lesssim 1$ implies

$$L \lesssim 1. \tag{4.4}$$

Thanks to (4.4) and standard properties of the stabilization terms (see, e.g., Massing *et al.*, 2014, Lemma 5.1), we have the following norm equivalences for all $\mathbf{v} \in \mathbf{V}_h^n$ and $q \in Q_h^n$:

$$
\begin{aligned}
\|\mathbf{v}\|_{\Omega_{h,e}^n}^2 &\simeq \|\mathbf{v}\|_{\Omega_h^n}^2 + h^2 |\mathbf{v}|_{s_h^n}^2, \\
\|\|\mathbf{v}\|\|_{n,e}^2 &\simeq \|\|\mathbf{v}\|\|_n^2 + |\mathbf{v}|_{s_h^n}^2, \\
\|\|q\|\|_{n,e}^2 &\simeq \|\|q\|\|_{n,i}^2 + |q|_{J_h^n}^2, \\
\|q\|_{\Omega_{h,e}^n}^2 &\simeq \|q\|_{\Omega_{h,i}^n}^2 + |q|_{J_h^n}^2.
\end{aligned}
\tag{4.5}
$$

4.1 *Preliminary results*

In this section, we collect some preliminary results used in the stability and the convergence analysis of the finite element method (3.7).

LEMMA 4.1 For h sufficiently small, there holds for all $\mathbf{v} \in \mathbf{V}_h^{n-1}$,

$$\|\mathbf{v}\|_{\Omega_h^n}^2 \leq \|\mathbf{v}\|_{\Omega_{h,e}^{n-1}}^2 \leq (1 + c_1 \Delta t) \|\mathbf{v}\|_{\Omega_h^{n-1}}^2 + \frac{\Delta t}{4} \|\|\mathbf{v}\|\|_{n-1}^2 + \Delta t L |\mathbf{v}|_{s_h^{n-1}}^2 \tag{4.6}$$

for a constant $c_1 > 0$ independent of h, Δt and how the boundary cuts through the triangulation.

Proof. From Lehrenfeld & Olshanskii (2019, Lemma 5.7), we have

$$\|\mathbf{v}\|_{\Omega_{h,e}^{n-1}}^2 \leq (1 + c_1(\epsilon)\Delta t) \|\mathbf{v}\|_{\Omega_h^{n-1}}^2 + c_2(\epsilon)\Delta t \|\nabla \mathbf{v}\|_{\Omega_h^{n-1}}^2 + c_3(\epsilon, h)\Delta t L |\mathbf{v}|_{s_h^{n-1}}^2,$$

with

$$
\begin{aligned}
c_1(\epsilon) &= c' c_{\delta_h} \mathbf{w}_\infty^n (1 + \epsilon^{-1}), & c_2(\epsilon) &= c' c_{\delta_h} \mathbf{w}_\infty^n \epsilon, \\
c_3(\epsilon, h) &= c_2(\epsilon) + c_4(\epsilon, h), & c_4(\epsilon, h) &= h^2 c' c_{\delta_h} \mathbf{w}_\infty^n (1 + \epsilon^{-1}),
\end{aligned}
$$

$c' > 0$ is a generic constant, and $\epsilon > 0$ is arbitrary. The result (4.6) follows from the inequality $\|\nabla \mathbf{v}\|_{\Omega_h^{n-1}} \leq \|\|\mathbf{v}\|\|_{n-1}$ and by taking ϵ such that $c_2(\epsilon) = \frac{1}{4}$ and h sufficiently small such that $c_4(\epsilon, h) \leq 1$. $\qquad \square$

LEMMA 4.2 There holds the following discrete Poincare inequality:

$$\|\mathbf{v}\|_{\Omega_h^n} \leq c_P \|\mathbf{v}\|_n \qquad \forall \mathbf{v} \in \mathbf{V}_h^n. \tag{4.7}$$

Proof. See Massing *et al.* (2014, Lemma 7.2).   □

The following continuity estimate for the bilinear form $a_h^n(\cdot, \cdot)$ is essentially given in Burman & Hansbo (2012) (also see Massing *et al.* (2014); von Wahl *et al.* (2022)) and follows from the Cauchy–Schwarz inequality, so the proof is omitted.

LEMMA 4.3 There holds

$$a_h^n(\mathbf{u}, \mathbf{v}) \lesssim \|\mathbf{u}\|_n \|\mathbf{v}\|_n + \gamma_s |\mathbf{u}|_{s_h^n} |\mathbf{v}|_{s_h^n} \qquad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}^{\overline{m}_v+1}\left(\mathcal{T}_{h,e}^n\right) \cap \mathbf{H}^1\left(\Omega_{h,e}^n\right). \tag{4.8}$$

The next result states a discrete trace inequality for discontinuous piecewise polynomial functions. Its proof is given in Appendix A (also see Buffa & Ortner, 2009, Theorem 4.4).

LEMMA 4.4 There holds

$$\|q\|_{\Gamma_h^n} \lesssim h^{-1} \|q\|_{n,e} \qquad \forall q \in Q_h^n. \tag{4.9}$$

4.2 *Stability analysis*

In this section, we derive stability results for the finite element method (3.7). First, we state the energy estimate for the finite element velocity in the following lemma. This result is essentially given in von Wahl *et al.* (2022, Theorem 5.9), but we provide a proof for completeness.

LEMMA 4.5 There holds for h sufficiently small, any $\varepsilon > 0$ and $k = 1, 2, \ldots$

$$\left\|\mathbf{u}_h^k\right\|_{\Omega_h^k}^2 + \sum_{n=1}^k \left\|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\right\|_{\Omega_h^n}^2 + \Delta t \sum_{n=1}^k \left(\frac{1}{4}\|\mathbf{u}_h^n\|_n^2 + \left(2\gamma_s - L - \frac{1}{2}\right)|\mathbf{u}_h^n|_{s_h^n}^2 + 2\gamma_J |p_h^n|_{J_h^n}^2\right)$$

$$\leq \exp(ct_k)\left(\left\|\mathbf{u}_h^0\right\|_{\Omega_h^0}^2 + \frac{\Delta t}{4}\left\|\mathbf{u}_h^0\right\|_0^2 + \Delta t L \left|\mathbf{u}_h^0\right|_{s_h^0}^2\right.$$

$$\left.+ \Delta t\left(c_e + \varepsilon^{-1}\right)\sum_{n=0}^k \|F^n\|_*^2 + \Delta t\varepsilon \sum_{n=0}^k \left\|p_h^n\right\|_{n,e}^2\right), \tag{4.10}$$

with constants c and c_e independent of the discretization parameters.

Proof. Taking $\mathbf{v} = \mathbf{u}_h^n$ and $q = p_h^n$ in (3.7), adding the two statements, applying (4.2) and using the algebraic identity $(a-b)a = \frac{1}{2}(a^2 - b^2) + \frac{1}{2}(a-b)^2$ yields

$$\frac{1}{2}\|\mathbf{u}_h^n\|_{\Omega_h^n}^2 - \frac{1}{2}\left\|\mathbf{u}_h^{n-1}\right\|_{\Omega_h^n}^2 + \frac{1}{2}\left\|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\right\|_{\Omega_h^n}^2 + \Delta t\left(\frac{1}{2}\|\mathbf{u}_h^n\|_n^2 + \gamma_s |\mathbf{u}_h^n|_{s_h^n}^2 + \gamma_J |p_h^n|_{J_h^n}^2\right) \leq \Delta t F^n\left(\mathbf{u}_h^n, p_h^n\right).$$

Using (3.9) and the Cauchy–Schwarz inequality, we estimate the right-hand side as follows:

$$F^n \left(\mathbf{u}_h^n, p_h^n \right) \leq \|F^n\|_* \left(\|\|\mathbf{u}_h^n\|\|_{n,e}^2 + \|\|p_h^n\|\|_{n,e}^2 \right)^{\frac{1}{2}} \leq \|F^n\|_* \left(\|\|\mathbf{u}_h^n\|\|_{n,e} + \|\|p_h^n\|\|_{n,e} \right)$$

$$\leq \sqrt{c_e/2} \|F^n\|_* \left(\|\|\mathbf{u}_h^n\|\|_n^2 + |\mathbf{u}_h^n|_{s_h^n}^2 \right)^{\frac{1}{2}} + \|F^n\|_* \|\|p_h^n\|\|_{n,e}$$

$$\leq \frac{1}{2} \left(c_e + \epsilon^{-1} \right) \|F^n\|_*^2 + \frac{1}{4} \left(\|\|\mathbf{u}_h^n\|\|_n^2 + |\mathbf{u}_h^n|_{s_h^n}^2 \right) + \frac{\varepsilon}{2} \|\|p_h^n\|\|_{n,e}^2 ,$$

where $c_e \geq 1$ satisfies $\|\|\mathbf{u}_h^n\|\|_{n,e}^2 \leq \frac{c_e}{2} (\|\|\mathbf{u}_h^n\|\|_n^2 + |\mathbf{u}_h^n|_{s_h^n}^2)$ (cf. (4.5)). This yields

$$\|\mathbf{u}_h^n\|_{\Omega_h^n}^2 - \|\mathbf{u}_h^{n-1}\|_{\Omega_h^n}^2 + \|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|_{\Omega_h^n}^2 + \Delta t \left(\frac{1}{2} \|\|\mathbf{u}_h^n\|\|_n^2 + \left(2\gamma_s - \frac{1}{2} \right) |\mathbf{u}_h^n|_{s_h^n}^2 + 2\gamma_J |p_h^n|_{J_h^n}^2 \right)$$

$$\leq \Delta t \left(\left(c_e + \varepsilon^{-1} \right) \|F^n\|_*^2 + \varepsilon \|\|p_h^n\|\|_{n,e}^2 \right). \quad (4.11)$$

Applying the estimate (4.6) (with $\mathbf{v} = \mathbf{u}_h^{n-1}$) into (11) and summing the result over $n = 1, \ldots, k$ yields

$$\|\mathbf{u}_h^k\|_{\Omega_h^k}^2 + \sum_{n=1}^k \|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|_{\Omega_h^n}^2 + \Delta t \sum_{n=1}^k \left(\frac{1}{4} \|\|\mathbf{u}_h^n\|\|_n^2 + \left(2\gamma_s - L - \frac{1}{2} \right) |\mathbf{u}_h^n|_{s_h^n}^2 + 2\gamma_J |p_h^n|_{J_h^n}^2 \right)$$

$$\leq \|\mathbf{u}_h^0\|_{\Omega_h^0} + \frac{\Delta t}{4} \|\|\mathbf{u}_h^0\|\|_0^2 + \Delta t L |\mathbf{u}_h^0|_{s_h^0}^2 + c_1 \Delta t \sum_{n=0}^{k-1} \|\mathbf{u}_h^n\|_{\Omega_h^n}^2$$

$$+ \Delta t \sum_{n=1}^k \left(\left(c_e + \varepsilon^{-1} \right) \|F^n\|_*^2 + \varepsilon \|\|p_h^n\|\|_{n,e}^2 \right).$$

The estimate (4.10) now follows from a discrete Gronwall inequality. □

For the complete stability result, we need to estimate the pressure term on the right-hand side of (4.10). The estimate is given in the next lemma.

LEMMA 4.6 Assume $h^2 \lesssim \Delta t$. Then

$$\Delta t \sum_{n=1}^k \|\|p_h^n\|\|_{n,e}^2 \lesssim \exp(ct_k) \left(\|\mathbf{u}_h^0\|_{\Omega_h^0}^2 + \Delta t \left(\|\|\mathbf{u}_h^0\|\|_0^2 + |\mathbf{u}_h^0|_{s_h^0}^2 \right) + \Delta t \sum_{n=0}^k \|F^n\|_*^2 \right).$$

Proof. Let $\mathbf{v} \in \mathbf{V}_h^n$ satisfy (3.5) with $q = p_h^n$. Then using the identity (3.7) and bounds in (3.5), (3.9), (4.8) and (4.5), we have

$$
\begin{aligned}
\left\|\left|p_h^n\right|\right\|_{n,i}^2 &\leq b_h^n\left(\mathbf{v}, p_h^n\right) \\
&= \int_{\Omega_h^n} \frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}}{\Delta t} \cdot \mathbf{v}\, \mathrm{d}x + a_h^n\left(\mathbf{u}_h^n, \mathbf{v}\right) - F^n(\mathbf{v}, 0) \\
&\lesssim \left(\frac{1}{\Delta t}\left\|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\right\|_{\Omega_h^n} \|\mathbf{v}\|_{\Omega_h^n} + \left\|\left|\mathbf{u}_h^n\right|\right\|_n \|\mathbf{v}\|_n + \gamma_s \left|\mathbf{u}_h^n\right|_{s_h^n} |\mathbf{v}|_{s_h^n} + \|F^n\|_* \left\|\left|\mathbf{v}\right|\right\|_{n,e}\right) \\
&\lesssim \left(\frac{h}{\Delta t}\left\|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\right\|_{\Omega_h^n} + \left\|\left|\mathbf{u}_h^n\right|\right\|_n + \gamma_s \left|\mathbf{u}_h^n\right|_{s_h^n} + \|F^n\|_*\right) \left\|\left|p_h^n\right|\right\|_{n,i}.
\end{aligned}
$$

Thus, we have

$$
\Delta t \left\|\left|p_h^n\right|\right\|_{n,i}^2 \lesssim \frac{h^2}{\Delta t}\left\|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\right\|_{\Omega_h^n}^2 + \Delta t \left(\left\|\left|\mathbf{u}_h^n\right|\right\|_n^2 + \left|\mathbf{u}_h^n\right|_{s_h^n}^2 + \|F^n\|_*^2\right). \tag{4.12}
$$

Combining this with (4.5) leads to the estimate of the pressure norm in the extended domain:

$$
\Delta t \left\|\left|p_h^n\right|\right\|_{n,e}^2 \lesssim \frac{h^2}{\Delta t}\left\|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\right\|_{\Omega_h^n}^2 + \Delta t \left(\left|p_h^n\right|_{J_h^n}^2 + \left\|\left|\mathbf{u}_h^n\right|\right\|_n^2 + \left|\mathbf{u}_h^n\right|_{s_h^n}^2 + \|F^n\|_*^2\right).
$$

Summing inequality over $n = 1, \ldots, k$, and using $h^2 \lesssim \Delta t$ and (4.10) gets

$$
\Delta t \sum_{n=1}^k \left\|\left|p_h^n\right|\right\|_{n,e}^2 \lesssim \sum_{n=1}^k \left\|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\right\|_{\Omega_h^n}^2 + \Delta t \sum_{n=1}^k \left(\left|p_h^n\right|_{J_h^n}^2 + \left\|\left|\mathbf{u}_h^n\right|\right\|_n^2 + \left|\mathbf{u}_h^n\right|_{s_h^n}^2\right) + \Delta t \sum_{n=1}^k \|F^n\|_*^2.
$$

All terms on the right-hand side of the last inequality are estimated in (4.10). Thus, by applying (4.10) with ε sufficiently small, but independent of the discretization parameters proves the lemma. $\qquad\square$

REMARK 4.7 The corresponding 2-step backward differentiation formula (BDF2) scheme is analogous to (3.7), but where the discrete time derivative is replaced by $\frac{3\mathbf{u}_h^{n+1} - 4\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}}{2\Delta t}$, and the computational mesh is enlarged. In particular, δ_h is replaced by $2\delta_h$ in the definition of $\mathcal{T}_{h,e}^n$ so that functions in $\mathbf{V}_h^{n-2}$ are well defined in Ω_h^n. In this setting, a stability result holds for the discrete velocity similar to (4.10), but where $\|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|_{\Omega_h^n}^2$ is replaced by $\|\mathbf{u}_h^n - 2\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}\|_{\Omega_h^n}^2$. The proof of this result is similar to that of Lemma 4.5, but using a different polarization identity, and so we omit the details.

The stability of the discrete pressure solution in the BDF2 scheme is more subtle and requires a different argument than Lemma 4.6. Analogous to (4.12), there holds

$$
\Delta t \left\|\left|p_h^n\right|\right\|_{n,i}^2 \lesssim \frac{h^2}{\Delta t}\left\|3\mathbf{u}_h^n - 4\mathbf{u}_h^{n-1} + \mathbf{u}_h^{n-2}\right\|_{\Omega_h^n}^2 + \Delta t \left(\left\|\left|\mathbf{u}_h^n\right|\right\|_n^2 + \left|\mathbf{u}_h^n\right|_{s_h^n}^2 + \|F^n\|_*^2\right),
$$

and therefore

$$
\sum_{n=1}^{k} \Delta t \left\|\!\left\|p_h^n\right\|\!\right\|_{n,e}^2 \lesssim \frac{h^2}{\Delta t} \sum_{n=0}^{k} \left\|\mathbf{u}_h^n\right\|_{\Omega_h^n}^2 + \Delta t \sum_{k=1}^{n} \left(\left\|\!\left\|\mathbf{u}_h^n\right\|\!\right\|_n^2 + \left|\mathbf{u}_h^n\right|_{s_h^n}^2 + \Delta t \left|p_h^n\right|_{J_h^n}^2 \right) + \Delta t \sum_{n=1}^{k} \|F^n\|_*^2
$$

$$
\leq \frac{Th^2}{\Delta t^2} \max_{0 \leq n \leq N} \left\|\mathbf{u}_h^n\right\|_{\Omega_h^n}^2 + \Delta t \sum_{k=1}^{n} \left(\left\|\!\left\|\mathbf{u}_h^n\right\|\!\right\|_n^2 + \left|\mathbf{u}_h^n\right|_{s_h^n}^2 + \Delta t \left|p_h^n\right|_{J_h^n}^2 \right) + \Delta t \sum_{n=1}^{k} \|F^n\|_*^2.
$$

Thus, for $h \lesssim \Delta t$, the terms in the right-hand side of this expression are uniformly bounded, hence obtaining a stability estimate for the discrete pressure solution. Note that when combined with (4.3), we have the relation $\Delta t \simeq h$ in the case of BDF2.

4.3 *Consistency*

The consistency of the scheme (3.7) largely follows the arguments in von Wahl *et al.* (2022, Lemma 5.14). First, we identify the extensions of the smooth exact solution $\mathcal{E}\mathbf{u}$ and $\mathcal{E}p$ with $\mathbf{u}$ and p, respectively, both of which satisfy (2.9). Recall that for $\mathbf{u}$, we consider the divergence-free extension from (2.8). We then set $\mathbb{U}^n = \mathbf{u}^n - \mathbf{u}_h^n$ and $\mathbb{P}^n = p^n - p_h^n$ to denote the errors at t_n.

LEMMA 4.8 There holds for all $(\mathbf{v}, q) \in \mathbf{V}_h^n \times Q_h^n$,

$$
\int_{\Omega_h^n} \frac{\mathbb{U}^n - \mathbb{U}^{n-1}}{\Delta t} \cdot \mathbf{v} \, dx + a_h^n(\mathbb{U}^n, \mathbf{v}) - b_h^n(\mathbf{v}, \mathbb{P}^n) + b_h^n(\mathbb{U}^n, q) + \gamma_J J_h^n(\mathbb{P}^n, q) = \mathfrak{C}_c^n(\mathbf{v}, q),
$$

where the consistency error $\mathfrak{C}_c^n(\mathbf{v}, q)$ satisfies

$$
\left|\mathfrak{C}_c^n(\mathbf{v}, q)\right| \lesssim h^q \|\mathbf{u}^n\|_{H^2(\Omega^n)} \|\!\|q\|\!\|_{n,e}
$$
$$
+ (\Delta t + h^q + h^{m_1} + h^{m_2}) \left(\|\mathbf{f}^n\|_{H^1(\Omega^n)} + \|\mathbf{u}\|_{W^{3,5}(\mathcal{Q})} + \|p^n\|_{H^{m_2}(\Omega^n)} + \|\mathbf{u}^n\|_{H^{m_1+1}(\Omega^n)} \right) \|\!\|\mathbf{v}\|\!\|_{n,e},
$$
$$
\tag{4.13}
$$

for any integers m_1, m_2 satisfying $m_1 \geq \overline{m}_v$ and $m_2 \geq m_q + 1$.

Proof. Recall that $\Psi_n : \mathcal{O}_{\delta_h}(\Omega_h^n) \to \mathcal{O}_{\delta_h}(\Omega_h)$ is the mapping that connects the approximate and exact domains and satisfies (3.1). Testing (2.2) with $\mathbf{v}^\ell := \mathbf{v} \circ \Psi_n^{-1}$, $\mathbf{v} \in \mathbf{V}_h^n$ and $q^\ell := q \circ \Psi_n^{-1}$, $q \in Q_h^n$ and integrating by parts we arrive at the identity

$$
\int_{\Omega^n} \frac{\partial \mathbf{u}^n}{\partial t} \mathbf{v}^\ell \, dx + \int_{\Omega^n} \nabla \mathbf{u}^n : \nabla \mathbf{v}^\ell \, dx - \int_{\Gamma^n} [(\nabla \mathbf{u}^n)\mathbf{n}] \cdot \mathbf{v}^\ell \, ds + \int_{\Omega^n} (\mathbf{w} \cdot \nabla \mathbf{u}^n) \cdot \mathbf{v}^\ell) \, dx
$$
$$
- \int_{\Omega^n} p^n \mathrm{div} \mathbf{v}^\ell \, dx + \int_{\Gamma^n} p^n (\mathbf{v}^\ell \cdot \mathbf{n}) \, ds - \int_{\Omega^n} q^\ell \mathrm{div} \mathbf{u}^n \, dx = \int_{\Omega^n} \mathbf{f}^n \cdot \mathbf{v}^\ell \, dx.
$$

Subtracting this identity from (3.7) gives the consistency term:

$$
\mathfrak{C}_c^n(\mathbf{v}, q) = \underbrace{\int_{\Omega^n} \mathbf{f}^n \cdot \mathbf{v}^\ell \, dx - \int_{\Omega_h^n} \mathbf{f}^n \cdot \mathbf{v} \, dx}_{=:\mathfrak{T}_1} + \underbrace{\int_{\Omega_h^n} \frac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\Delta t} \cdot \mathbf{v} \, dx - \int_{\Omega^n} \frac{\partial \mathbf{u}^n}{\partial t} \mathbf{v}^\ell \, dx}_{=:\mathfrak{T}_2}
$$

$$
+ \underbrace{\widehat{a}_h^n(\mathbf{u}^n, \mathbf{v}) - \int_{\Omega^n} \nabla \mathbf{u}^n : \nabla \mathbf{v}^\ell \, dx + \int_{\Gamma^n} [(\nabla \mathbf{u}^n)\mathbf{n}] \cdot \mathbf{v}^\ell \, ds - \int_{\Omega^n} (\mathbf{w} \cdot \nabla \mathbf{u}^n) \cdot \mathbf{v}^\ell \, dx}_{=:\mathfrak{T}_3}
$$

$$
+ \underbrace{\int_{\Omega^n} p^n \mathrm{div}\mathbf{v}^\ell \, dx - \int_{\Gamma^n} p^n (\mathbf{v}^\ell \cdot \mathbf{n}) \, ds - b_h^n(\mathbf{v}, p^n)}_{=:\mathfrak{T}_4}
$$

$$
- \underbrace{b_h^n(\mathbf{u}^n, q)}_{=:\mathfrak{T}_5} + \underbrace{\gamma_J J_h^n(p^n, q) + \gamma_s s_h^n(\mathbf{u}^n, \mathbf{v})}_{=:\mathfrak{T}_6}.
$$

Estimates for $\mathfrak{T}_1$ and $\mathfrak{T}_4$ are exactly the same as in von Wahl *et al.* (2022, Lemma 5.14):

$$
|\mathfrak{T}_1| \lesssim h^q \|\mathbf{f}^n\|_{H^1(\Omega^n)} \|\mathbf{v}\|_{\Omega_h^n}, \qquad |\mathfrak{T}_4| \lesssim \left(h^q \|p^n\|_{H^1(\Omega^n)} + h^{m_2} \|p^n\|_{H^{m_2}(\Omega^n)} \right) \|\mathbf{v}\|_{n,e} \tag{4.14}
$$

for any $m_2 \geq 1$. Likewise, the arguments in Lehrenfeld & Olshanskii (2019, Lemma 5.6) and von Wahl *et al.* (2022, Lemma 5.14) show

$$
|\mathfrak{T}_2| \lesssim (\Delta t + h^q) \|\mathbf{u}\|_{W^{2,\infty}(\widehat{\mathcal{Q}})} \|\mathbf{v}\|_{\Omega_h^n} \lesssim (\Delta t + h^q) \|\mathbf{u}\|_{W^{3,5}(\mathcal{Q})} \|\mathbf{v}\|_{\Omega_h^n}, \tag{4.15}
$$

where we used (2.9c) in the last inequality.

Unlike the problem considered in von Wahl *et al.* (2022), the bilinear form $\widehat{a}_h^n(\cdot, \cdot)$ includes convective terms. Nonetheless, the same arguments in (von Wahl *et al.*, 2022, Lemma 5.14) are valid, yielding the following estimate:

$$
|\mathfrak{T}_3| \lesssim \left(h^q \|\mathbf{u}\|_{W^{2,\infty}(\widehat{\mathcal{Q}})} + h^{m_1} \|\mathbf{u}^n\|_{H^{m_1+1}(\Omega^n)} \right) \|\mathbf{v}\|_{n,e}
$$
$$
\lesssim \left(h^q \|\mathbf{u}\|_{W^{3,5}(\mathcal{Q})} + h^{m_1} \|\mathbf{u}^n\|_{H^{m_1+1}(\Omega^n)} \right) \|\mathbf{v}\|_{n,e}, \tag{4.16}
$$

where $m_1 \geq 1$ is only dictated by the regularity of $\mathbf{u}^n$, and we have again used (2.9c).

On the other hand, the estimate of $\mathfrak{T}_5 = b_h^n(\mathbf{u}^n, q)$ should involve the elementwise scaled H^1-norm for the pressure (which is nonstandard and not provided in von Wahl *et al.*, 2022). Since the extension of $\mathbf{u}$ is divergence-free, the estimate of $\mathfrak{T}_5$ reduces to estimating the boundary term:

$$
\mathfrak{T}_5 = - \int_{\Gamma_h^n} (\mathbf{u}^n \cdot \mathbf{n}) q \, ds.
$$

Since $\Psi_n(\Gamma_h^n) = \Gamma^n$, there holds

$$\mathbf{u}^n \circ \Psi_n = 0 \qquad \text{on } \Gamma_h^n.$$

Using the estimate $\|\mathbf{u}^n - \mathbf{u}^n \circ \Psi_n\|_{\Gamma_h^n} \lesssim h^{q+1} \|\mathbf{u}^n\|_{H^2(\Omega^n)}$ (cf. Gross *et al.*, 2015, Lemma 7.3) and the discrete trace inequality in Lemma 4.4, we have

$$|\mathfrak{T}_5| = \left| \int_{\Gamma_h^n} \left(\mathbf{u}^n - \mathbf{u}^n \circ \Psi_n \right) \cdot \mathbf{n} \, q \, \mathrm{d}s \right| \le \left\| \mathbf{u}^n - \mathbf{u}^n \circ \Psi_n \right\|_{\Gamma_h^n} \|q\|_{\Gamma_h^n} \lesssim h^{q+1} \|\mathbf{u}^n\|_{H^2(\Omega^n)} \|q\|_{\Gamma_h^n}$$

$$\lesssim h^q \|\mathbf{u}^n\|_{H^2(\Omega^n)} \, \|q\|_{n,e} \,. \tag{4.17}$$

Finally, the consistency term involving ghost stabilization $\mathfrak{T}_6$ vanishes provided $\mathbf{u}^n \in \mathbf{H}^{\overline{m}_v+1}(\Omega_{h,e}^n)$ and $p^n \in H^{m_q+1}(\Omega_{h,e}^n)$. The estimate (4.13) then follows from (4.14)–(4.17) and the discrete Poincare inequality (4.7). □

4.4 *Error estimates*

In this section, we combine the stability and consistency estimates to obtain error estimates for the finite element method (3.7). As a first step, let $(\mathbf{u}_I^n, p_I^n) \in \mathbf{V}_h^n \times Q_h^n$ be approximations to the exact solution satisfying

$$\left\| \mathbf{u}^n - \mathbf{u}_I^n \right\|_{n,e} + \left| \mathbf{u}^n - \mathbf{u}_I^n \right|_{s_h^n} \lesssim h^{\underline{m}_v} \|\mathbf{u}^n\|_{H^{\overline{m}_v+1}\left(\Omega_{h,e}^n\right)} \lesssim h^{\underline{m}_v} \|\mathbf{u}^n\|_{H^{\overline{m}_v+1}(\Omega^n)},$$

$$\left\| p^n - p_I^n \right\|_{n,e} + \left| p^n - p_I^n \right|_{J_h^n} \lesssim h^{m_q+1} \|p^n\|_{H^{m_q+1}\left(\Omega_{h,e}^n\right)} \lesssim h^{m_q+1} \|p^n\|_{H^{m_q+1}(\Omega^n)}, \tag{4.18a}$$

and

$$h^{-1} \left\| \mathbf{u}^n - \mathbf{u}_I^n \right\|_{\Omega_h^n} \lesssim h^{\underline{m}_v} \|\mathbf{u}\|_{H^{\overline{m}_v+1}\left(\Omega_{h,e}^n\right)} \lesssim h^{\underline{m}_v} \|\mathbf{u}\|_{H^{\overline{m}_v+1}(\Omega^n)},$$

$$\left\| p^n - p_I^n \right\|_{\Omega_h^n} + h^{1/2} \left\| p^n - p_I^n \right\|_{\Gamma_h^n} \lesssim h^{m_q+1} \|p^n\|_{H^{m_q+1}\left(\Omega_{h,e}^n\right)} \lesssim h^{m_q+1} \|p^n\|_{H^{m_q+1}(\Omega^n)}. \tag{4.18b}$$

The existence of such $\mathbf{u}_I^n$ and p_I^n satisfying (4.18) follows from the inclusions (3.3)–(3.4) and standard scaling and interpolation arguments. We also assume the initial condition of the finite element method (3.7) is $\mathbf{u}_h^0 = \mathbf{u}_I^0$.

We then split the error into its interpolation and discretization errors:

$$\mathbb{U}^n = \underbrace{\left(\mathbf{u}^n - \mathbf{u}_I^n \right)}_{=:\boldsymbol{\eta}^n} + \underbrace{\left(\mathbf{u}_I^n - \mathbf{u}_h^n \right)}_{=:\mathbf{e}_h^n \in \mathbf{V}_h^n}, \qquad \mathbb{P}^n = \underbrace{\left(p^n - p_I^n \right)}_{=:\zeta^n} + \underbrace{\left(p_I^n - p_h^n \right)}_{=:\mathrm{d}_h^n \in Q_h^n}.$$

Then the pair $(\mathbf{e}_h^n, \mathrm{d}_h^n) \in \mathbf{V}_h^n \times Q_h^n$ satisfies

$$\int_{\Omega_h^n} \frac{\mathbf{e}_h^n - \mathbf{e}_h^{n-1}}{\Delta t} \cdot \mathbf{v} \, \mathrm{d}x + a_h^n\left(\mathbf{e}_h^n, \mathbf{v}\right) - b_h^n\left(\mathbf{v}, \mathrm{d}_h^n\right) + b_h^n\left(\mathbf{e}_h^n, q\right) + \gamma_J J_h\left(\mathrm{d}_h^n, q\right) = \mathfrak{C}_c^n(\mathbf{v}, q) + \mathfrak{C}_I^n(\mathbf{v}, q),$$
(4.19)

for all $(\mathbf{v}, q) \in \mathbf{V}_h^n \times Q_h^n$, where $\mathfrak{C}_c^n(\mathbf{v}, q)$ is given in Lemma 4.8 and

$$\mathfrak{C}_I^n(\mathbf{v}, q) = \underbrace{-\int_{\Omega_h^n} \frac{\boldsymbol{\eta}^n - \boldsymbol{\eta}^{n-1}}{\Delta t} \cdot \mathbf{v} \, \mathrm{d}x}_{=:\mathfrak{T}_7} \underbrace{- a_h^n(\boldsymbol{\eta}^n, \mathbf{v})}_{=:\mathfrak{T}_8} + \underbrace{b_h^n(\mathbf{v}, \zeta^n)}_{=:\mathfrak{T}_9} - \underbrace{b_h^n(\boldsymbol{\eta}^n, q)}_{=:\mathfrak{T}_{10}} - \underbrace{\gamma_J J_h^n(\zeta^n, q)}_{=:\mathfrak{T}_{11}}.$$

We now bound the terms in $\mathfrak{C}_I^n(\mathbf{v}, q)$ individually.

First, by continuity estimates and the approximation properties (4.18), we have

$$|\mathfrak{T}_i| \lesssim \left(h^{m_v} \|\mathbf{u}^n\|_{H^{\overline{m}_v+1}(\Omega^n)} + h^{m_q+1} \|p^n\|_{H^{m_q+1}(\Omega^n)} \right) \|\mathbf{v}\|_{n,e} \quad i = 8, 9, 11.$$
(4.20)

For the temporal interpolation error, there holds by Lehrenfeld & Olshanskii (2019, Lemma 5.7) and the discrete Poincare inequality (4.7),

$$\begin{aligned} |\mathfrak{T}_7| &\lesssim h^{m_v} \sup_{t \in [0,T]} \left(\|\mathbf{u}\|_{H^{\overline{m}_v+1}(\Omega(t))} + \|\mathbf{u}_t\|_{H^{\overline{m}_v}(\Omega(t))} \right) \|\mathbf{v}\|_{\Omega_h^n} \\ &\lesssim h^{m_v} \sup_{t \in [0,T]} \left(\|\mathbf{u}\|_{H^{\overline{m}_v+1}(\Omega(t))} + \|\mathbf{u}_t\|_{H^{\overline{m}_v}(\Omega(t))} \right) \|\mathbf{v}\|_{n,e}. \end{aligned}$$
(4.21)

For $\mathfrak{T}_{10}$, we integrate by parts to obtain

$$\mathfrak{T}_{10} = \int_{\Omega_h^n} (\mathrm{div}\,\boldsymbol{\eta}^n) q \, \mathrm{d}x - \int_{\Gamma_h^n} (\boldsymbol{\eta}^n \cdot \mathbf{n}) q \, \mathrm{d}s = \int_{\Omega_h^n} \boldsymbol{\eta}^n \cdot \nabla q \, \mathrm{d}x + \sum_{F \in \mathcal{F}_{h,e}^n} \int_{F \cap \Omega_h^n} \boldsymbol{\eta}^n \cdot \mathbf{n} \, [\![q]\!] \, \mathrm{d}s.$$

Consequently by an elementwise trace inequality and (4.18), there holds

$$
\begin{aligned}
|\mathfrak{T}_{10}| &\leq \left(\sum_{T \in \mathcal{T}_{h,e}^n} h_T^{-2} \|\boldsymbol{\eta}^n\|_T^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_{h,e}^n} h_T^2 \|\nabla q\|_T^2 \right)^{\frac{1}{2}} \\
&\quad + \left(\sum_{F \in \mathcal{F}_{h,e}^n} h_F^{-1} \|\boldsymbol{\eta}^n\|_F^2 \right)^{\frac{1}{2}} \left(\sum_{F \in \mathcal{F}_{h,e}^n} h_F \|[\![q]\!]\|_F^2 \right)^{\frac{1}{2}} \\
&\lesssim \left(\sum_{T \in \mathcal{T}_{h,e}^n} \left(h_T^{-2} \|\boldsymbol{\eta}^n\|_T^2 + \|\nabla \boldsymbol{\eta}^n\|_T^2 \right) \right)^{\frac{1}{2}} \left(h^2 \sum_{T \in \mathcal{T}_{h,e}^n} \|\nabla q\|_T^2 + h \sum_{F \in \mathcal{F}_{h,e}^n} \|[\![q]\!]\|_F^2 \right)^{\frac{1}{2}} \\
&\lesssim h^{\underline{m}_v} \|\mathbf{u}^n\|_{H^{\overline{m}_v+1}(\Omega^n)} \|q\|_{n,e} \,.
\end{aligned}
\tag{4.22}
$$

Summarizing (4.20)–(4.22), we proved the bound

$$
\left| \mathfrak{C}_I^n(\mathbf{v}, q) \right| \lesssim \left(h^{\underline{m}_v} \sup_{t \in [0,T]} \left(\|\mathbf{u}\|_{H^{\overline{m}_v+1}(\Omega(t))} + \|\mathbf{u}_t\|_{H^{\overline{m}_v}(\Omega(t))} \right) + h^{m_q+1} \|p^n\|_{H^{m_q+1}(\Omega^n)} \right) \\
\times \left(\|\mathbf{v}\|_{n,e} + \|q\|_{n,e} \right).
\tag{4.23}
$$

From (4.13) and (4.23), it follows that the functionals $\mathfrak{C}_c^n$ and $\mathfrak{C}_I^n$ are bounded as

$$
\left| \mathfrak{C}_c^n(\mathbf{v}, q) \right| + \left| \mathfrak{C}_I^n(\mathbf{v}, q) \right| \leq C(\Delta t + h^q + h^{\underline{m}_v} + h^{m_q})(\|\mathbf{v}\|_{n,e} + \|q\|_{n,e}),
$$

where $C > 0$ depends on Sobolev norms of the exact solution and the source function.

Note that $\mathbf{e}_h^n$ and d_h^n satisfy the same FE formulation (3.7) as $\mathbf{u}_h^n$ and p_h^n, but with the zero initial condition and the right-hand side functional given by $F^n(\mathbf{v}, q) = \mathfrak{C}_c^n(\mathbf{v}, q) + \mathfrak{C}_I^n(\mathbf{v}, q)$. Therefore, we can apply the stability results from Lemmas 4.5 and 4.6 to estimate $\mathbf{e}_h^n$ and d_h^n:

$$
\left\| \mathbf{e}_h^k \right\|_{\Omega_h^k}^2 + \Delta t \sum_{n=1}^{k} \left(\|\mathbf{e}_h^n\|_n^2 + \|d_h^n\|_{n,e}^2 \right) \leq C(\Delta t + h^q + h^{\underline{m}_v} + h^{m_q+1})^2.
$$

Applying the triangle inequality and the estimates (4.18) one more time leads to our final result.

THEOREM 4.9 Assume the solution $(\mathbf{u}, p)$ to (2.2) is sufficiently smooth and let $\mathbf{u}_h^n, p_h^n$ be the solution to (3.7). Assume that the discretization parameters satisfy $h^2 \lesssim \Delta t \lesssim h$. The following error estimate holds

$$\max_{1 \leq n \leq N} \left\| \mathbf{u}(t^n) - \mathbf{u}_h^n \right\|_{\Omega_h^n}^2 + \Delta t \sum_{n=1}^N \left(\left\|\| \mathbf{u}(t^n) - \mathbf{u}_h^n \right\|\|_n^2 + \left\|\| p(t^n) - p_h^n \right\|\|_{n,e}^2 \right)$$

$$\leq C(\Delta t + h^q + h^{\underline{m}_v} + h^{m_q+1})^2, \quad (4.24)$$

with a constant C independent of discretization parameters, but dependent on the solution $(\mathbf{u}, p)$ and final time T.

REMARK 4.10 Compared to the error estimates in von Wahl *et al.* (2022), Theorem 4.9 provide optimal-order error estimates for the velocity and pressure approximations. The key tool that differentiates our result is the application of a scaled $L^2(H^1)$ norm for the pressure approximation. This strategy provides the flexibility to effectively handle the nondivergence-free property of the discrete time derivative $(\mathbf{u}^n - \mathbf{u}_h^{n-1})/\Delta t$ in the stability analysis under the mesh constraint $h^2 \lesssim \Delta t \lesssim h$ (cf. von Wahl *et al.*, 2022, Lemma 5.10 and Lemma 4.6).

REMARK 4.11 In the Oseen problem 2.2, we have implicitly taken the viscosity $\nu = 1$ to simplify the presentation. If $\Delta \mathbf{u}$ is replaced by $\nu \Delta \mathbf{u}$, then the velocity ghost penalty term in CutFEM discretization needs to scale like ν to perform the convergence and stability analysis. Also, $\gamma_s s_h^n(\cdot, \cdot)$ would be replaced by $\nu^{-1} \gamma_s s_h^n(\cdot, \cdot)$. In this general setting, a version of Theorem 4.9 holds, but the constant $C > 0$ scales like $\exp(\nu^{-1}T)$; see von Wahl *et al.* (2022) for details.

5. Examples of finite element pairs satisfying Assumption 3.1

In this section, we show that several canonical finite element pairs for the Stokes problem satisfy the three inequalities (3.5) in Assumption 3.1.

5.1 *The Mini element*

For a tetrahedron $T \in \mathcal{T}_h$, let $b_T \in P_4(T)$ denote the standard quartic bubble function, i.e., the product of the barycentric coordinates of T. The lowest-order Mini pair with respect to $\mathcal{T}_h$ is given by Arnold *et al.* (1985)

$$\mathbf{V}_h = \left\{ \mathbf{v} \in \mathbf{H}^1(\widehat{\Omega}) : \ \mathbf{v}|_T \in \mathbf{P}_1(T) + b_T \mathbf{P}_0(T) \ \forall T \in \mathcal{T}_h \right\},$$

$$Q_h = \left\{ q \in H^1(\widehat{\Omega}) : \ q|_T \in P_1(T) \ \forall T \in \mathcal{T}_h \right\}.$$

In this setting, we can take $\underline{m}_v = 1, \overline{m}_v = 4$ and $m_q = 1$.

We now verify conditions (3.5). Given $q \in Q_h^n$, we set $\mathbf{v} \in \mathbf{V}_h^n$ so that $\mathbf{v}|_T = h_T^2 b_T \nabla q|_T$ for all $T \in \mathcal{T}_h^{n,i}$. The function $\mathbf{v}$ is extended to Ω_e^n by zero. The results in Guzmán & Olshanskii (2018, Section 6.5) show that (3.5a)–(3.5b) is satisfied. We also have by a simple scaling argument

$$\|\mathbf{v}\|_{\Omega^n}^2 = \sum_{T \in \mathcal{T}_h^{n,i}} \|\mathbf{v}\|_T^2 = \sum_{T \in \mathcal{T}_h^{n,i}} h_T^4 \|b_T \nabla q\|_T^2 \simeq \sum_{T \in \mathcal{T}_h^{n,i}} h_T^4 \|\nabla q\|_T^2 \lesssim h^2 \|\| q \|\|_{n,i}^2.$$

Thus, (3.5c) is satisfied as well.

5.2 *The Taylor–Hood pair*

The (generalized) Taylor–Hood finite element pair is given by

$$\mathbf{V}_h = \left\{ \mathbf{v} \in \mathbf{H}^1(\widehat{\Omega}) : \ \mathbf{v}|_T \in \mathbf{P}_m(T) \ \forall T \in \mathcal{T}_h \right\},$$

$$Q_h = \left\{ q \in H^1(\widehat{\Omega}) : \ q|_T \in \mathrm{P}_{m-1}(T) \ \forall T \in \mathcal{T}_h \right\},$$

where $m \geq 2$. Thus, in this case $\overline{m}_v = \underline{m}_v = m$ and $m_q = m - 1$ in (3.3)–(3.4). Denote by $\mathcal{E}_h^{n,i}$ the set of interior one-dimensional edges of the the interior triangulation $\mathcal{T}_h^{n,i}$. We then denote by $\tilde{\mathcal{T}}_h^{n,i}$ the members in $\mathcal{T}_h^{n,i}$ that have at least three edges in $\mathcal{E}_h^{n,i}$ (cf. Remark 3.3). We assume that the domain of pressure ghost-stabilization is chosen such that (3.6) is satisfied. This is the case provided c_{δ_h} is sufficiently large (but still $O(1)$).

We denote the set of interior edges of $\tilde{\mathcal{T}}_h^{n,i}$ by $\tilde{\mathcal{E}}_h^{n,i}$. Then for $e \in \tilde{\mathcal{E}}_h^{n,i}$, we let ϕ_e denote the quadratic bubble function associated with e, and let $\mathbf{t}_e$ be a unit tangent vector of e. Note that ϕ_e has support on the tetrahedra that have e as an edge, and the number of such tetrahedra is uniformly bounded due to the shape-regularity of $\tilde{\mathcal{T}}_{h,i}^n$.

For a given $q \in Q_h^n$, we define

$$\mathbf{v} = \sum_{e \in \tilde{\mathcal{E}}_h^{n,i}} h_e^2 \phi_e (\nabla q \cdot \mathbf{t}_e) \mathbf{t}_e.$$

Because q is continuous, we see that $\nabla q \cdot \mathbf{t}_e$ is single-valued on e, and thus $\mathbf{v}$ is continuous and a piecewise polynomial of degree m; hence, $\mathbf{v} \in \mathbf{V}_h^n$.

It is shown in Guzmán & Olshanskii (2018, Section 6.1) that (3.5a)–(3.5b) is satisfied, thus it remains to show (3.5c). This follows from the identity $\|\phi_e\|_\infty = 1$ and the shape-regularity and quasi-uniformity of the triangulation:

$$\|\mathbf{v}\|_{\tilde{\Omega}_h^n} \lesssim h^2 \sum_{T \in \tilde{\mathcal{T}}_{h,i}^n} \|\nabla q\|_T^2.$$

5.3 *The $P_3 - P_0$ pair*

As our final example, we consider the $\mathbf{P}_3 - \mathrm{P}_0$ pair. In particular, the discrete velocity space is the cubic Lagrange space, and the discrete pressure space consists of piecewise constants:

$$\mathbf{V}_h = \mathbf{P}_3^c(\mathcal{T}_h) = \left\{ \mathbf{v} \in \mathbf{H}^1(\widehat{\Omega}) : \ \mathbf{v}|_T \in \mathbf{P}_3(T) \ \forall T \in \mathcal{T}_h \right\},$$

$$Q_h = \mathrm{P}_0(\mathcal{T}_h) = \left\{ q \in L^2(\widehat{\Omega}) : \ q|_T \in \mathrm{P}_0(T) \ \forall T \in \mathcal{T}_h \right\}.$$

For each interior face $F \in \mathcal{F}_{h,i}^n$ with $F = \partial T_1 \cap \partial T_2$, we denote by $\mathbf{n}_j$ the outward unit normal of ∂T_j restricted to F. Then for given $q \in Q_h^n$, we define $\mathbf{v} \in \mathbf{V}_h^n$ such that for all $F \in \mathcal{F}_{h,i}^n$,

$$\int_F \mathbf{v} \cdot \mathbf{n}_1 \, \mathrm{d}s = -h_F \int_F (q_1 \mathbf{n}_1 + q_2 \mathbf{n}_2) \cdot \mathbf{n}_1 \, \mathrm{d}s = h_F \int_F [\![q]\!] \cdot \mathbf{n}_1 \, \mathrm{d}s,$$

where $q_j = q|_{T_j}$. Note that this condition implies $\int_F \mathbf{v} \cdot \mathbf{n}_F \, \mathrm{d}s = -h_F \int_F [\![q]\!] \cdot \mathbf{n}_F \, \mathrm{d}s$ for any unit normal of $F \in \mathcal{F}_{h,i}^n$. We further specify that $\mathbf{v} = 0$ on all vertices and edges in $\mathcal{T}_{h,i}^n$, $\mathbf{v} \times \mathbf{n}_F = 0$ on all faces $F \in \mathcal{F}_{h,i}^n$, and $\mathbf{v} = 0$ on the boundary of $\Omega_{h,i}^n$. We extend $\mathbf{v}$ to $\Omega_{h,e}^n$ by zero.

By the divergence theorem, and using that q is piecewise constant, we have

$$b_h^n(\mathbf{v}, q) = \int_{\Omega_{h,i}^n} (\mathrm{div}\,\mathbf{v}) q \, \mathrm{d}x = -\sum_{T \in \mathcal{T}_{h,i}^n} \int_{\partial T} q(\mathbf{v} \cdot \mathbf{n}_{\partial T}) \, \mathrm{d}s \gtrsim h \sum_{F \in \mathcal{F}_{h,i}^n} \| [\![q]\!] \|_F^2 = \|q\|_{n,i}^2.$$

Thus, (3.5b) is satisfied. A scaling argument also yields on each $T \in \mathcal{T}_{h,i}^n$,

$$|\mathbf{v}|_{H^m(T)} \lesssim h_T^{2-2m} \sum_{\mathcal{F}_{h,i}^n \ni F \subset \partial T} h_F \| [\![q]\!] \|_F^2.$$

Consequently, by another scaling argument,

$$\|\mathbf{v}\|_{n,e}^2 \lesssim \|\nabla \mathbf{v}\|_{\Omega_{h,i}^n}^2 + h^{-2} \|\mathbf{v}\|_{\Omega_{h,i}^n}^2 \lesssim h \sum_{F \in \mathcal{F}_{h,i}^n} \| [\![q]\!] \|_F^2 = \|q\|_{n,i}^2,$$

$$\|\mathbf{v}\|_{\Omega_h^n}^2 = \|\mathbf{v}\|_{\Omega_{h,i}^n}^2 \lesssim h^3 \sum_{F \in \mathcal{F}_{h,i}^{n,i}} \| [\![q]\!] \|_F^2 = h^2 \|q\|_{n,i}^2,$$

and therefore (3.5a) and (3.5c) are satisfied as well.

Funding

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Appendix A. Proof of Lemma 4.4

We first note that if $Q_h^n \subset H^1(\Omega_{h,e}^n)$, then a standard trace inequality and the definition of $\|\!|\cdot|\!\|_{n,e}$ yields

$$\|q\|_{\Gamma_h^n} \lesssim \|q\|_{H^1(\Omega_h^n)} \lesssim h^{-1} \|\!|q|\!\|_{n,e} + \|q\|_{\Omega_h^n}. \tag{A.1}$$

To establish (4.9) in this case, we first apply a standard Poincare–Friedrich inequality

$$\|q\|_{\Omega_{h,i}^n} \lesssim \|\nabla q\|_{\Omega_{h,i}^n} \qquad \forall q \in \mathring{L}^2\left(\Omega_{h,i}^n\right) \cap H^1\left(\Omega_{h,i}^n\right),$$

and (4.5) to conclude

$$\|q\|_{\Omega_h^n} \lesssim \|q\|_{\Omega_{h,i}^n} + |q|_{J_h^n} \lesssim \|\nabla q\|_{\Omega_{h,i}^n} + |q|_{J_h^n} \lesssim h^{-1}\left(\|\!|q|\!\|_{n,i} + |q|_{J_h^n}\right) \lesssim h^{-1} \|\!|q|\!\|_{n,e} \quad \forall q \in Q_h^n.$$

The estimate (4.9) then follows from this inequality and (A.1).

Thus, it suffices to prove (4.9) in the case Q_h^n consists of discontinuous polynomials. To this end, we introduce an enriching operator $E_h : Q_h^n \to Q_h^n \cap H^1(\Omega_{h,e}^n)$ constructed by averaging (Brenner & Ridgway Scott, 2008). Let

$$\mathcal{T}_T^n = \left\{ T' \in \mathcal{T}_{h,e}^n : \bar{T} \cap \bar{T}' \neq \emptyset \right\},$$

and let $\mathcal{F}_T^{n,I}$ denote the set of *interior* faces of $\mathcal{T}_T^n$. Then there holds

$$|q - E_h q|_{H^\ell(T)}^2 \lesssim h_T^{1-2\ell} \sum_{F \in \mathcal{F}_T^{n,I}} \|[\![q]\!]\|_{L^2(F)}^2 \qquad \ell = 0, 1. \tag{A.2}$$

It then follows from (A.2) and the trace inequality

$$\|q\|_{T \cap \Gamma_h^n} \lesssim h_T^{-1/2} \|q\|_T + h_T^{1/2} \|\nabla q\|_T \qquad \forall q \in H^1(T)$$

that

$$\|q - E_h q\|_{T_h}^2 = \sum_{T \in \mathcal{T}_{h,e}^n} \|q - E_h q\|_{T \cap \Gamma_h^n}^2$$

$$\lesssim \sum_{T \in \mathcal{T}_{h,e}^n} \left(h_T^{-1} \|q - E_h q\|_T^2 + h_T \|\nabla(q - E_h q)\|_T^2 \right)$$

$$\lesssim \sum_{F \in \mathcal{F}_{h,e}^n} \|[\![q]\!]\|_F^2 \lesssim h^{-1} \|q\|_{n,e}^2. \tag{A.3}$$

Furthermore, by a standard trace inequality and (A.2), we have

$$\|E_h q\|_{\Gamma_h^n}^2 \lesssim \|E_h q\|_{H^1(\Omega_h^n)}^2 \le \|E_h q\|_{H^1(\Omega_{h,e}^n)}^2$$

$$\lesssim \sum_{T \in \mathcal{T}_{h,e}^n} \|q\|_{H^1(T)}^2 + h^{-1} \sum_{F \in \mathcal{F}_{h,e}^n} \|[\![q]\!]\|_F^2$$

$$\lesssim h^{-2} \|q\|_{n,e}^2 + \|q\|_{\Omega_{h,e}^n}^2. \tag{A.4}$$

Combining (A.3)–(A.4) yields

$$\|q\|_{\Gamma_h^n} \lesssim h^{-1} \|q\|_{n,e} + \|q\|_{\Omega_{h,e}^n}. \tag{A.5}$$

Finally, since $q|_{\Omega_{h,i}^n} \in \mathring{L}^2(\Omega_{h,i}^n)$, we apply the discrete Poincare–Friedrich inequality (Brenner & Ridgway Scott, 2008, Theorem 10.6.12)

$$\|q\|_{\Omega_{h,i}^n}^2 \lesssim \sum_{T \in \mathcal{T}_{h,i}^n} \|\nabla q\|_T^2 + h^{-1} \sum_{F \in \mathcal{F}_{h,i}^n} \|[\![q]\!]\|_F^2 \lesssim h^{-2} \|q\|_{n,i}^2,$$

and (4.5) to conclude

$$\|q\|_{\Omega_{h,e}^n} \lesssim \|q\|_{\Omega_{h,i}^n} + |q|_{J_h^n} \lesssim h^{-1} \left(\|q\|_{n,i} + |q|_{J_h^n} \right) \lesssim h^{-1} \|q\|_{n,e}.$$

Combined with (A.5), we obtain (4.9).