

Small gaps and small spacings between zeta zeros

by

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*Dedicated to Andrzej Schinzel, with appreciation for his contributions
and gratitude for his many years of service to number theory*

1. Introduction. We assume the Riemann hypothesis (RH) throughout this paper. Let $1/2 + i\gamma$ be a nontrivial zero of the Riemann zeta-function $\zeta(s)$, and let $m(\gamma)$ denote its multiplicity. While we expect that these zeros are all simple with $m(\gamma) = 1$, currently we cannot exclude the existence of multiple zeros. In this paper, we address how the possible existence of multiple zeros affects the results we can prove on close pairs of zeros. We consider both the nondecreasing sequence $\{\gamma\}$ of positive ordinates $\gamma > 0$ which counts multiplicity, and also the increasing sequence $\{\gamma_d\}$ of distinct zeros with ordinates $\gamma_d > 0$ which does not count multiplicity. If all the zeros are simple, then these sequences are identical. As usual, we let $N(T)$ denote the number of zeros with $0 < \gamma \leq T$, counting multiplicity. Then

$$(1) \quad N(T) := \sum_{0 < \gamma \leq T} 1 = \sum_{0 < \gamma_d \leq T} m(\gamma_d) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right)$$

for $T \geq 2$, where the remainder term $S(T)$ equals $\frac{1}{\pi} \arg \zeta\left(\frac{1}{2} + iT\right)$ for $T \neq \gamma$, with the argument obtained by a continuous variation along the line segments joining the points 2 , $2 + iT$ and $\frac{1}{2} + iT$ starting with the value $\arg \zeta(2) = 0$. It is known that $S(T) = O(\log T)$ as $T \rightarrow \infty$.

There are two possible types of close pairs of zeros: pairs arising from a multiple zero and close pairs from two distinct zeros which have a positive

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distance between them. Of course, two distinct zeros can both be multiple zeros generating many pairs of zeros of each type. We use the word *spacing* between zeros for both types of close pairs of zeros, and reserve the word *gap* between pairs of zeros to mean a spacing with a strictly positive length. Denote by γ^+ the next term $\gamma \leq \gamma^+$ after γ in the sequence of ordinates of zeros. Similarly, we denote by γ_d^+ the next term $\gamma_d < \gamma_d^+$ in the sequence of distinct ordinates. By (1), the average of the consecutive spacings $\gamma^+ - \gamma$ is $2\pi/\log \gamma$, and to measure how close these zeros can get, we define

$$(2) \quad \mu := \liminf_{\gamma \rightarrow \infty} (\gamma^+ - \gamma) \frac{\log \gamma}{2\pi}.$$

Similarly, to measure small gaps between consecutive distinct zeros, we define

$$(3) \quad \mu_d := \liminf_{\gamma_d \rightarrow \infty} (\gamma_d^+ - \gamma_d) \frac{\log \gamma_d}{2\pi}.$$

More generally, we consider the distribution functions

$$(4) \quad D(\lambda, T) := \frac{1}{N(T)} \sum_{\substack{0 < \gamma \leq T \\ \gamma^+ - \gamma \leq \frac{2\pi\lambda}{\log T}}} 1 \quad \text{and} \quad D_d(\lambda, T) := \frac{1}{N(T)} \sum_{\substack{0 < \gamma_d \leq T \\ \gamma_d^+ - \gamma_d \leq \frac{2\pi\lambda}{\log T}}} 1.$$

Corresponding to (2) and (3), we define

$$\begin{aligned} \mu_D &:= \inf_{\lambda} \left\{ \lambda : \liminf_{T \rightarrow \infty} D(\lambda, T) > 0 \right\} \\ \mu_{D_d} &:= \inf_{\lambda} \left\{ \lambda : \liminf_{T \rightarrow \infty} D_d(\lambda, T) > 0 \right\}. \end{aligned}$$

Thus, if $\lambda > \mu_D$, then there are a positive proportion of spacings of consecutive zeros of length at most λ times the average spacing. Likewise, if $\lambda > \mu_{D_d}$, then there are a positive proportion of gaps between distinct consecutive zeros with length at most λ times the average spacing. Note trivially that

$$\mu \leq \mu_d \leq \mu_{D_d} \quad \text{and} \quad \mu \leq \mu_D \leq \mu_{D_d}.$$

There are three known methods for proving the existence of close pairs of zeros. The first method is due to Selberg, who proved unconditionally that, for a small positive constant δ ,

$$(5) \quad \mu_D \leq 1 - \delta.$$

Selberg's proof is based on moments of the remainder term $S(T)$ in (1). Though his proof was never published, Fujii [15, 16] gave an abbreviated proof of (5), and a more detailed argument was given by Heath-Brown in the appendix to Titchmarsh [34, Chapter 9]. For references to these results and corrections to some misprints that have occurred, see [13]. Selberg's proof is in two steps. First, it is proved that a positive proportion of consecutive zeros have gaps larger than the average spacing, and then this result is used

to infer a positive proportion of consecutive zeros with spacing less than the average spacing. In [31], the value $\delta = \frac{1}{2} \times 10^{-3 \cdot 10^{13}}$ is obtained.

Another method, which depends on RH, was introduced by Montgomery and Odlyzko [25], who used it to obtain the estimate $\mu \leq 0.5179$. There have been many refinements made (e.g. [3, 9, 14]), and the current best result is due to Preobrazhenskiĭ [28], who proved that

$$\mu \leq 0.515396.$$

Two different proofs of this method are given in [25] and [9] and the limitations of the method are discussed in [20]. This method has also been modified by Conrey, Ghosh, Goldston, Gonek, Heath-Brown [8] in 1985 to produce the positive proportion result $\mu_D \leq 0.77$. Soundararajan [32] refined it to obtain $\mu_D \leq 0.6876$, and Wu [35] later obtained

$$\mu_D \leq 0.6653.$$

A third method for finding close pairs of zeros is due to Montgomery [24] in his paper on pair correlation of zeros, and also assumes RH. Originally, Montgomery obtained the estimate $\mu \leq 0.68$. This method has been refined by other authors (see e.g. [5, 18]), with the current best result

$$(6) \quad \mu \leq 0.6039$$

due to Chirre, Gonçalves, and de Laat [7]. In the present paper, we prove that Montgomery's pair correlation method actually gives a result for a positive proportion of spacings, improving the best known bounds from the previous method used in [8, 32, 35].

THEOREM 1. *Assuming RH, we have*

$$\mu_D \leq 0.6039.$$

The results mentioned above, for all three methods, fail to exclude the possibility that the small spacings that are detected are composed entirely from zero spacings between multiple zeros (and not from actual gaps). We suspect that none of the three methods can be modified to prove the existence of gaps between zeros smaller than the average spacing, in other words that these methods are incapable of proving that $\mu_d < 1$. In this paper, we introduce a new method specifically designed to find small gaps between distinct zeros.

THEOREM 2. *Assuming RH, we have*

$$\mu_d \leq 0.991.$$

The method we introduce produces gaps between a simple zero and a distinct zero of odd multiplicity. Both configurations of γ_d and γ_d^+ , where γ_d is a simple zero or γ_d^+ is a simple zero, occur. However, we cannot guarantee that both of these distinct zeros are simple. Furthermore, our method does

not produce a positive proportion of gaps and so we do not obtain a result for μ_{D_d} . Nevertheless, in an interval $[T, 2T]$, we can show that there are $\gg_\varepsilon T^{1-\varepsilon}$ such small gaps between distinct zeros for any $\varepsilon > 0$.

COROLLARY 1. *Assume RH and let T be large. Then, for any constant $C > \log 4$, there are*

$$\gg T \exp\left(-C \frac{\log T}{\log \log T}\right)$$

consecutive ordinates $\gamma_d, \gamma_d^+ \in [T, 2T]$ of distinct zeros with $\gamma_d < \gamma_d^+$ and $\gamma_d^+ - \gamma_d \leq 0.991 \frac{2\pi}{\log T}$.

We conclude the introduction by mentioning that this paper was inspired, in part, by the recent work of Rodgers and Tao [30], who proved that the de Bruijn–Newman constant is nonnegative. The last step of their proof relied on knowing that $\mu_D < 1$. There are also other reasons for studying small gaps and small spacings between zeta zeros. For instance, there is a well-known connection between the existence of small spacings between the zeros of the zeta function and the class number problem for imaginary quadratic fields. See the works of Conrey and Iwaniec [12] and Montgomery and Weinberger [27] for more on this connection. For an overview of results on the complementary problem of proving the existence of large gaps between zeros of the zeta function, see [2] and the references therein.

2. Pair correlation of zeta zeros and the proof of Theorem 1. In this section, we investigate small spacings and small gaps between zeta zeros using Montgomery’s pair correlation method [24]. As usual, we define

$$F(\alpha) = F(\alpha, T) = \frac{1}{N(T)} \sum_{0 < \gamma, \gamma' \leq T} T^{i\alpha(\gamma - \gamma')} w(\gamma - \gamma'),$$

where α and $T \geq 2$ are real, $w(u) = 4/(4 + u^2)$. By Fourier inversion, for any function $r \in L^1(\mathbb{R})$ such that $\widehat{r} \in L^1(\mathbb{R})$, we have

$$(7) \quad \sum_{0 < \gamma, \gamma' \leq T} r\left((\gamma - \gamma') \frac{\log T}{2\pi}\right) w(\gamma - \gamma') = N(T) \int_{\mathbb{R}} \widehat{r}(\alpha) F(\alpha) d\alpha,$$

where the Fourier transform \widehat{r} is defined by

$$\widehat{r}(\alpha) = \int_{\mathbb{R}} r(u) e(-\alpha u) du, \quad e(x) := e^{2\pi i x}.$$

Assuming RH, it is known that F is real-valued, even, nonnegative, and that

$$(8) \quad F(\alpha) = (1 + o(1)) T^{-2\alpha} \log T + \alpha + o(1)$$

uniformly for $0 \leq |\alpha| \leq 1$ (see [19, 24]). Therefore, we can asymptotically evaluate the right-hand side of (7) when $\text{supp}(\widehat{r}) \subseteq [-1, 1]$. By exploiting

the fact that F is nonnegative, we can further specialize our conditions on r to prove the existence of close pairs of zeros.

For $\lambda > 0$, let $\mathcal{A}(\lambda)$ denote the class of even, continuous, and real-valued functions $r \in L^1(\mathbb{R})$ satisfying the following three conditions:

- (i) $r(0) = 1$;
- (ii) $r(u) \leq 0$ if $|u| > \lambda$;
- (iii) $\widehat{r}(\alpha) \geq 0$ for all $\alpha \in \mathbb{R}$.

It can be shown that if $r \in \mathcal{A}(\lambda)$, then $\widehat{r} \in L^1(\mathbb{R})$. We also let

$$n^* := \limsup_{T \rightarrow \infty} \frac{1}{N(T)} \sum_{0 < \gamma_d \leq T} m(\gamma_d)^2$$

and, by (1), we note that $n^* \geq 1$. Then the following theorem holds.

THEOREM 3. *Assume RH. Let $r \in \mathcal{A}(\lambda)$ and define*

$$(9) \quad c(\lambda; r) := \widehat{r}(0) - 1 + 2 \int_0^1 \alpha \widehat{r}(\alpha) d\alpha.$$

If there exists a $\lambda_0 > 0$ and an $r \in \mathcal{A}(\lambda_0)$ such that $c(\lambda_0; r) > 0$ for sufficiently large T , then we have

$$(10) \quad D(\lambda_0, T) \gg 1 \quad \text{and} \quad \mu_D \leq \lambda_0.$$

If there exists a $\lambda^ > 0$ and an $r \in \mathcal{A}(\lambda^*)$ such that $c(\lambda^*; r) > n^* - 1$ for sufficiently large T , then we have*

$$(11) \quad D_d(\lambda^*, T) \gg c(\lambda^*; r)^4 \quad \text{and} \quad \mu_{D_d} \leq \lambda^*.$$

We now show how Theorem 3 implies Theorem 1.

Proof of Theorem 1. The claim that $\mu_D \leq 0.6039$ follows from Theorem 3 and the example of Chirre, Gonçalves, and de Laat [7] used to prove (6). Observe that if r is contained in the class \mathcal{A}_{LP} of functions described in [7], then $r \in \mathcal{A}(\lambda)$ for any $\lambda \geq \inf \{a > 0 : r(x) \leq 0 \text{ for } |x| \geq a\}$ and therefore r satisfies the conditions of Theorem 3. ■

We also deduce the following corollary from Theorem 3.

COROLLARY 2. *Assuming RH, we have*

$$\mu_{D_d} \leq 1.0522.$$

This result for μ_{D_d} is derived in §4. It may seem paradoxical that the bound we obtain for μ_{D_d} is larger than the average spacing between zeros, but presently (on RH) we only know that at least 84.77% of the nontrivial zeros of $\zeta(s)$ are distinct, see [7, Theorem 4]. In other words, the average spacing between distinct zeros could be as large as 1.17966 times the average spacing between all zeros, counting multiplicity. Therefore our result does find small

gaps between distinct zeros with respect to the worst case scenario for the average spacing of distinct zeros.

Our proof of Theorem 3 is based upon the following two propositions.

PROPOSITION 1. *Assume RH. If $r \in \mathcal{A}(\lambda)$, then*

$$(12) \quad \sum_{0 < \gamma_d \leq T} m(\gamma_d)(m(\gamma_d) - 1) + \sum_{\substack{0 < \gamma, \gamma' \leq T \\ 0 < |\gamma - \gamma'| \leq \frac{2\pi\lambda}{\log T}}} 1 \geq (c(\lambda; r) - o(1))N(T),$$

where $c(\lambda; r)$ is defined in (9).

If we can find an r and a $\lambda_0 > 0$ such that $c(\lambda_0; r) > 0$, then from (12) we immediately conclude that $\mu \leq \lambda_0$. In fact, the previous results on small gaps established using Montgomery's pair correlation method in [5, 7, 18, 24] all use Proposition 1 to prove the existence of close pairs of zeros assuming RH. To deduce that one can obtain positive proportions for gap results from Proposition 1, we use the following result whose proof relies on techniques introduced by Selberg in the 1940s.

PROPOSITION 2. *Assume RH. Suppose that $T \geq 2$ and that k is a positive integer. Let*

$$(13) \quad n(t, \lambda) = N\left(t + \frac{2\pi\lambda}{\log T}\right) - N(t).$$

Then there is a positive absolute constant C such that

$$(14) \quad \sum_{0 < \gamma \leq T} n(\gamma, k)^{2k} \leq (Ck)^{2k} T \log T$$

and

$$(15) \quad \sum_{0 < \gamma \leq T} m(\gamma)^{2k-1} = \sum_{0 < \gamma_d \leq T} m(\gamma_d)^{2k} < (Ck)^{2k-1} T \log T.$$

As described in §3, our proof of Proposition 2 overlaps earlier results of a number of authors. We conclude this section with the proof of Proposition 1.

Proof of Proposition 1. For functions $r \in \mathcal{A}(\lambda)$, it follows that

$$r(u) \leq |r(u)| = \left| \int_{\mathbb{R}} \widehat{r}(\alpha) e(\alpha u) d\alpha \right| \leq \int_{\mathbb{R}} |\widehat{r}(\alpha)| d\alpha = \int_{\mathbb{R}} \widehat{r}(\alpha) d\alpha = r(0) = 1.$$

Hence

$$\sum_{\substack{0 < \gamma, \gamma' \leq T \\ 0 < |\gamma - \gamma'| \leq \frac{2\pi\lambda}{\log T}}} 1 \geq \sum_{0 < \gamma, \gamma' \leq T} r\left((\gamma - \gamma') \frac{\log T}{2\pi}\right) w(\gamma - \gamma') = N(T) \int_{\mathbb{R}} \widehat{r}(\alpha) F(\alpha) d\alpha.$$

Since F and \widehat{r} are nonnegative, the inequality is still valid if we restrict the integral on the right-hand side to any finite interval. Since F and \widehat{r} are even,

by (8) we see that

$$\begin{aligned} \sum_{\substack{0 < \gamma, \gamma' \leq T \\ |\gamma - \gamma'| \leq \frac{2\pi\lambda}{\log T}}} 1 &\geq N(T) \int_{-1}^1 \widehat{r}(\alpha) F(\alpha) d\alpha = N(T) \left(\widehat{r}(0) + 2 \int_0^1 \alpha \widehat{r}(\alpha) d\alpha + o(1) \right) \\ &= N(T)(1 + c(\lambda; r) + o(1)). \end{aligned}$$

On the other hand, note that

$$\sum_{\substack{0 < \gamma, \gamma' \leq T \\ |\gamma - \gamma'| \leq \frac{2\pi\lambda}{\log T}}} 1 = \sum_{0 < \gamma_d \leq T} m(\gamma_d)^2 + \sum_{\substack{0 < \gamma, \gamma' \leq T \\ 0 < |\gamma - \gamma'| \leq \frac{2\pi\lambda}{\log T}}} 1.$$

Therefore

$$\sum_{0 < \gamma_d \leq T} m(\gamma_d)^2 + \sum_{\substack{0 < \gamma, \gamma' \leq T \\ 0 < |\gamma - \gamma'| \leq \frac{2\pi\lambda}{\log T}}} 1 \geq N(T)(1 + c(\lambda; r) + o(1)).$$

Making use of the identity

$$N(T) = \sum_{0 < \gamma_d \leq T} m(\gamma_d)$$

and then rearranging terms, the desired result in (12) follows. ■

3. Proof of Proposition 2. In this section, after establishing two preliminary results, we prove Proposition 2. First, we obtain a useful explicit formula relating zeros in short intervals to a Dirichlet polynomial over primes. A version of this explicit formula first appeared in a paper of Montgomery and Odlyzko [25], and later in the works of Gonek [21] and Radziwiłł [29]. We require a version with a slightly more precise error term for our application.

LEMMA 1. *Assume RH. Then, for $\tau \geq 2$, we have*

$$\begin{aligned} (16) \quad \sum_{\gamma} \left(\frac{\sin \frac{1}{2}(\gamma - \tau) \log x}{\frac{1}{2}(\gamma - \tau) \log x} \right)^2 \\ = \frac{\log \frac{\tau}{2\pi}}{\log x} - \frac{2}{\log x} \sum_{n \leq x} \frac{\Lambda(n)}{n^{1/2}} \left(1 - \frac{\log n}{\log x} \right) \cos(\tau \log n) \\ + O\left(\frac{1}{\tau \log x}\right) + O\left(\frac{x^{1/2}}{(\tau \log x)^2}\right). \end{aligned}$$

Proof. Let $\alpha(s) = \sum_{n \geq 1} a_n n^{-s}$ be a Dirichlet series with abscissa of convergence σ_c . If $a > \max(0, \sigma_c)$, then

$$\sum_{n \leq x} a_n \log \frac{x}{n} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \alpha(s) \frac{x^s}{s^2} ds \quad \text{for } x > 0.$$

Note that the sum on the left-hand side is either empty or equals zero if

$x \leq 1$. Therefore, by three applications of this identity, we deduce that

$$\sum_{n \leq x} a_n \log \frac{x}{n} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \alpha(s) \frac{(x^{s/2} - x^{-s/2})^2}{s^2} ds$$

for $x > 1$. We take $\alpha(s) = -\frac{\zeta'}{\zeta}(s + \frac{1}{2} + i\tau)$ and see that

$$(17) \quad \sum_{n \leq x} \frac{\Lambda(n)}{n^{1/2+i\tau}} \log \frac{x}{n} = -\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{\zeta'}{\zeta} \left(s + \frac{1}{2} + i\tau \right) \frac{(x^{s/2} - x^{-s/2})^2}{s^2} ds.$$

We move the contour of integration to the imaginary axis. In doing so, we encounter a pole at $s = \frac{1}{2} - i\tau$. On the imaginary axis we form semicircular paths around the poles at the points $s = i(\gamma - \tau)$. We shrink the radii of these semicircles to zero, which gives half-residues, and the integral on the imaginary axis is defined by taking the Cauchy Principal Value at the point $s = i(\gamma - \tau)$. Thus, (17) becomes

$$(18) \quad \sum_{n \leq x} \frac{\Lambda(n)}{n^{\frac{1}{2}+i\tau}} \log \frac{x}{n} = -\frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\zeta'}{\zeta} \left(\frac{1}{2} + it + i\tau \right) \left(\frac{\sin \frac{1}{2}t \log x}{t} \right)^2 dt - \frac{1}{2} \sum_{\gamma} \left(\frac{\sin \frac{1}{2}(\gamma - \tau) \log x}{\frac{1}{2}(\gamma - \tau)} \right)^2 + \left(\frac{x^{\frac{1}{2}(\frac{1}{2}-i\tau)} - x^{-\frac{1}{2}(\frac{1}{2}-i\tau)}}{\frac{1}{2} - i\tau} \right)^2.$$

Recall from [17] that, for $\frac{1}{2} \leq \sigma \leq 2$, $t \geq 2$, and $s \neq \frac{1}{2} + i\gamma$, we have

$$\operatorname{Re} \frac{\zeta'}{\zeta}(\sigma + it) = -\frac{1}{2} \log \frac{t}{2\pi} + \sum_{\gamma} \frac{\sigma - \frac{1}{2}}{(\sigma - \frac{1}{2})^2 + (t - \gamma)^2} + O(1/t)$$

and therefore

$$\operatorname{Re} \frac{\zeta'}{\zeta} \left(\frac{1}{2} + it \right) = -\frac{1}{2} \log \frac{t}{2\pi} + O(1/t)$$

for $t \geq 2$ and $t \neq \gamma$. We take the real parts of both sides of (18). An easy calculation shows that the real part of the integral in (18) is equal to $\frac{1}{2}(\log x) \log \frac{\tau}{2\pi} + O((\log x)/\tau)$ for $\tau \geq 2$. Multiplying both sides of (18) by $2/(\log x)^2$ and then rearranging, we see that

$$(19) \quad \begin{aligned} \sum_{\rho} \left(\frac{\sin \frac{1}{2}(\gamma - \tau) \log x}{\frac{1}{2}(\gamma - \tau) \log x} \right)^2 \\ = \frac{\log \frac{\tau}{2\pi}}{\log x} - \frac{2}{\log x} \sum_{n \leq x} \frac{\Lambda(n)}{n^{1/2}} \left(1 - \frac{\log n}{\log x} \right) \cos(\tau \log n) \\ + O\left(\frac{1}{\tau \log x}\right) + O\left(\frac{x^{1/2}}{(\tau \log x)^2}\right). \end{aligned}$$

This completes the proof of Lemma 1. ■

LEMMA 2. *Assume RH. Let $T \geq 2$ be given. If $k \in \mathbb{N}$, then there is a positive absolute constant C such that*

$$(20) \quad \int_0^T n(t, k)^{2k} dt < (Ck)^{2k} T.$$

Proof. Since $(\sin x)/x$ is decreasing for $0 \leq x \leq \pi$, it follows that the summand on the left-hand side of (19) is greater than or equal to $4/\pi^2$ if $|\gamma - \tau| \leq \pi/\log x$. Let k be a positive integer, and set $x = T^{1/k} \geq 2$. Then, for $0 \leq \tau \leq T$, from Lemma 1 we see that

$$(21) \quad N\left(\tau + \frac{\pi k}{\log T}\right) - N\left(\tau - \frac{\pi k}{\log T}\right) \\ \ll k + \frac{k}{\log T} \left| \sum_{n \leq x} \frac{\Lambda(n)}{n^{1/2}} \left(1 - \frac{\log n}{\log x}\right) \cos(\tau \log n) \right| + \frac{T^{1/2k}}{\tau + 2}.$$

By (13) we see the left-hand side of (21) is $n(t, k)$ with $t = \tau - \pi k/\log T$. We raise both sides of (21) to the power $2k$, and integrate. Since $|\operatorname{Re} z| \leq |z|$ for any complex number z , it follows that

$$(22) \quad \int_0^T n(t, k)^{2k} dt < (Ck)^{2k} T + \left(\frac{Ck}{\log T} \right)^{2k} \int_0^T \left| \sum_{n \leq x} \frac{\Lambda(n)}{n^{1/2+it}} \left(1 - \frac{\log n}{\log x}\right) \right|^{2k} dt.$$

Here C is a suitable absolute constant which may be increased appropriately between steps in the argument.

For any real number $b > 0$, we have $|f(t) + g(t)|^b \leq 2^b (|f(t)|^b + |g(t)|^b)$, and therefore we can divide the sum on the right-hand side into the sum over primes plus the sum over prime powers, and estimate them separately. For the sum over prime powers, the series over the prime powers p^ℓ with $\ell > 2$ converges absolutely and makes the contribution $< (Ck/\log T)^{2k} T$. The sum over squares of primes is trivially $\ll \log x$, which makes a contribution $\ll C^{2k} T$. Thus both of these errors are covered by the first error term on the right-hand side of (22).

For the contribution coming from primes, put $R = \pi(x)$, and let p_1, \dots, p_R denote the first R primes. By the multinomial theorem,

$$\left(\sum_{p \leq x} \frac{\log p}{p^{1/2+it}} \left(1 - \frac{\log p}{\log x}\right) \right)^k \\ = \sum_{\substack{\mu_1, \dots, \mu_R \\ \sum_r \mu_r = k}} \binom{k}{\mu_1 \dots \mu_R} \prod_{r=1}^R \left(\frac{\log p_r}{p_r^{1/2+it}} \left(1 - \frac{\log p_r}{\log x}\right) \right)^{\mu_r} = \sum_n c(n) n^{-it}.$$

Here $n = \prod_r p_r^{\mu_r}$, and $c(n) = 0$ if $n > x^k = T$. By the usual approximate

Parseval identity for Dirichlet polynomials [26, Corollary 3], we see that

$$\begin{aligned} \int_0^T \left| \sum_{p \leq x} \frac{\log p}{p^{1/2+it}} \left(1 - \frac{\log p}{\log x} \right) \right|^{2k} dt &\ll T \sum_n |c(n)|^2 \\ &= T \sum_{\substack{\mu_1, \dots, \mu_R \\ \sum_r \mu_r = k}} \binom{k}{\mu_1 \ \dots \ \mu_R}^2 \prod_{r=1}^R \left(\frac{\log p_r}{p_r^{1/2}} \left(1 - \frac{\log p_r}{\log x} \right) \right)^{2\mu_r}. \end{aligned}$$

The multinomial coefficient above is less than or equal to $k!$ for any collection of μ_r , so the above is

$$\leq k!T \sum_{\substack{\mu_1, \dots, \mu_R \\ \sum_r \mu_r = k}} \binom{k}{\mu_1 \ \dots \ \mu_R} \prod_{r=1}^R \left(\frac{\log p_r}{p_r^{1/2}} \left(1 - \frac{\log p_r}{\log x} \right) \right)^{2\mu_r},$$

which by the multinomial theorem is

$$= k!T \left(\sum_{p \leq x} \frac{\log^2 p}{p} \left(1 - \frac{\log p}{\log x} \right)^2 \right)^k \sim k!T \left(\frac{\log^2 x}{12} \right)^k < \left(\frac{C}{k} \right)^k T (\log T)^{2k}.$$

Hence the contribution from primes in the second term on the right-hand side of (22) is less than $(Ck)^k T$, which for large k is smaller than the contribution of the first term. This completes the proof of Lemma 2. ■

REMARKS. (1) It is trivial that

$$\int_0^T n(t, k) dt \sim kT$$

as $T \rightarrow \infty$ with k fixed.

(2) A form of Lemma 2 has been proved unconditionally by Fujii [16], and he also proved there unconditionally a form of the multiplicity bound (15) in Proposition 2.

(3) The second term on the right-hand side of (22) contributes less than the first term, which suggests the possibility that

$$\int_0^T (n(t, k) - k)^{2k} dt < (Ck)^k T.$$

It might be interesting, and perhaps even valuable in some contexts, to establish such a result.

(4) Our estimate for the $2k$ th moment for a Dirichlet polynomial over primes in the above proof is essentially the same as the proof of Lemma 3 of Soundararajan [33]. The use of the approximate Parseval identity for Dirichlet polynomials in our argument allows us to replace the condition $x^k \leq T/\log T$ appearing in [33] by $x^k \ll T$.

Proof of Proposition 2. For $t \in \mathbb{R}$ such that $0 < t \leq T$, we put $I(t) = [t, t + 2\pi/\log T]$. We observe that if $\gamma \in I(t)$, then $(\gamma, \gamma + 2\pi k/\log T] \subset (t, t + 2\pi(k+1)/\log T]$, and hence for such γ we have $n(\gamma, k) \leq n(t, k+1)$. For ordinates γ with $0 < \gamma \leq T$, we put $J(\gamma) = [\gamma - 2\pi/\log T, \gamma]$ and we note that $\gamma \in I(t)$ if and only if $t \in J(\gamma)$. Thus $n(\gamma, k)^{2k} \leq n(t, k+1)^{2k}$ for $t \in J(\gamma)$. We average over these t to see that

$$n(\gamma, k)^{2k} \leq \frac{\log T}{2\pi} \int_{J(\gamma)} n(t, k+1)^{2k} dt.$$

Summing over γ , we see that

$$\begin{aligned} \sum_{0 < \gamma \leq T} n(\gamma, k)^{2k} &\leq \frac{\log T}{2\pi} \sum_{\gamma \leq T} \int_{J(\gamma)} n(t, k+1)^{2k} dt \\ &= \frac{\log T}{2\pi} \int_0^T n(t, k+1)^{2k} \left(\sum_{\gamma \in I(t)} 1 \right) dt. \end{aligned}$$

The number of $\gamma \in I(t)$ is $n(t, 1)$ unless t is the ordinate of a zero, which occurs at only finitely many points in $(0, T]$. Thus, the right-hand side above equals

$$\frac{\log T}{2\pi} \int_0^T n(t, k+1)^{2k} n(t, 1) dt \leq \frac{\log T}{2\pi} \int_0^T n(t, k+1)^{2k+1} dt.$$

Now $n(t, k+1)$ is a nonnegative integer, so $n(t, k+1)^{2k+1} \leq n(t, k+1)^{2k+2}$ for all t . Hence, by Lemma 2, the right-hand side above is

$$\leq \frac{\log T}{2\pi} \int_0^T n(t, k+1)^{2k+2} dt \ll (Ck)^{2k} T \log T.$$

This establishes the desired bound in (14).

To prove (15), observe that

$$n(t, k)^{2k} = \left(\sum_{t \leq \gamma_d \leq t+2\pi k/\log T} m(\gamma_d) \right)^{2k} \geq \sum_{t \leq \gamma_d \leq t+2\pi k/\log T} m(\gamma_d)^{2k}.$$

Integrating both sides of this inequality and applying Lemma 2, it follows that

$$\int_0^T \sum_{t \leq \gamma_d \leq t+2\pi k/\log T} m(\gamma_d)^{2k} dt \leq \int_0^T n(t, k)^{2k} dt < (Ck)^{2k} T.$$

Moreover, the left-hand side above is

$$\geq \sum_{0 < \gamma_d \leq T} m(\gamma_d)^{2k} \int_{\gamma_d - 2\pi k/\log T}^{\gamma_d} 1 dt = \frac{2\pi k}{\log T} \sum_{0 < \gamma_d \leq T} m(\gamma_d)^{2k}.$$

Combining these estimates gives the bound in (15). This completes the proof of Proposition 2. ■

There is considerable overlap between (14) in Proposition 2 and Lemma 10 of Radziwiłł [29]. Radziwiłł proves a relationship between small gaps between zeros of $\zeta(s)$ and zeros of $\zeta'(s)$ close to the half-line. Assuming RH, he proves a form of our Lemma 1 and then uses a moment argument based on the Landau–Gonek formula [21]. In contrast, as mentioned above, our proof of Lemma 2 is similar to the argument used by Soundararajan [33, Lemma 3].

4. Proofs of Theorem 3 and Corollary 2. We now deduce Theorem 3 from Propositions 1 and 2.

Proof of Theorem 3. Assume RH. For $r \in \mathcal{A}(\lambda)$, suppose that $\lambda > 0$ is a number for which $c(\lambda; r) > 0$. Then, by (12), we have

$$\sum_{0 < \gamma_d \leq T} m(\gamma_d)(m(\gamma_d) - 1) + \sum_{\substack{0 < \gamma, \gamma' \leq T \\ 0 < |\gamma - \gamma'| \leq \frac{2\pi\lambda}{\log T}}} 1 \geq \frac{1}{2}c(\lambda; r)N(T)$$

for sufficiently large T . Consequently, at least one of the following is true:

$$(23) \quad \sum_{0 < \gamma_d \leq T} m(\gamma_d)(m(\gamma_d) - 1) \geq \frac{1}{4}c(\lambda; r)N(T)$$

or

$$(24) \quad \sum_{\substack{0 < \gamma, \gamma' \leq T \\ 0 < |\gamma - \gamma'| \leq \frac{2\pi\lambda}{\log T}}} 1 \geq \frac{1}{4}c(\lambda; r)N(T).$$

If (23) holds, then by Cauchy's inequality we have

$$\left(\frac{1}{4}c(\lambda; r)N(T)\right)^2 \leq \left(\sum_{\substack{0 < \gamma_d \leq T \\ m(\gamma_d) > 1}} 1\right) \left(\sum_{0 < \gamma_d \leq T} m(\gamma_d)^4\right).$$

By taking $k = 2$ in Proposition 2, we see that the second factor on the right-hand side above is $\ll T \log T$. Therefore

$$c(\lambda; r)^2 N(T) \ll \sum_{\substack{0 < \gamma_d \leq T \\ m(\gamma_d) > 1}} 1 \leq N(T)D(0, T) \leq N(T)D(\lambda, T),$$

which proves (10) in this case.

We now need to consider the case where (24) holds. For this, we first need to find an upper bound for the sum on the left-hand side of (24). By definition, $n(t, \lambda)$ counts ordinates (with multiplicity) in the interval $(t, t + 2\pi\lambda/\log T]$. Hence $n(\gamma, \lambda)$ counts ordinates γ' with multiplicity $m(\gamma')$ such that $\gamma < \gamma' \leq \gamma + 2\pi\lambda/\log T$. Thus

$$\sum_{\substack{0 < \gamma, \gamma' \leq T \\ 0 < |\gamma - \gamma'| \leq \frac{2\pi\lambda}{\log T}}} 1 = 2 \sum_{0 < \gamma \leq T} n(\gamma, \lambda) = 2 \sum_{0 < \gamma_d \leq T} m(\gamma_d) n(\gamma_d, \lambda).$$

Two applications of Cauchy's inequality give

$$\sum_{\substack{0 < \gamma, \gamma' \leq T \\ 0 < |\gamma - \gamma'| \leq \frac{2\pi\lambda}{\log T}}} 1 \leq 2 \left(\sum_{0 < \gamma_d \leq T} m(\gamma_d)^2 \right)^{1/2} \left(\sum_{\substack{0 < \gamma_d \leq T \\ n(\gamma_d, \lambda) \geq 1}} 1 \right)^{1/4} \left(\sum_{0 < \gamma \leq T} n(\gamma, \lambda)^4 \right)^{1/4}.$$

Since $n(\gamma_d, \lambda) \geq 1$ if and only if $\gamma_d^+ - \gamma_d \leq 2\pi\lambda/\log T$, we conclude from both (14) and (15) of Proposition 2 and the definitions in (4) that

$$(25) \quad \sum_{\substack{0 < \gamma, \gamma' \leq T \\ 0 < |\gamma - \gamma'| \leq \frac{2\pi\lambda}{\log T}}} 1 \ll (T \log T) D_d(\lambda, T)^{1/4} \leq (T \log T) D(\lambda, T)^{1/4}.$$

Assuming that (24) holds, we see from (25) that $D(\lambda, T) \gg c(\lambda; r)^4$ and (10) is established.

To prove (11), we have

$$\sum_{0 < \gamma_d \leq T} m(\gamma_d)(m(\gamma_d) - 1) \leq (n^* - 1 + o(1))N(T),$$

and the result follows immediately from (12) and (25). This completes the proof of Theorem 3. ■

Proof of Corollary 2. To make our calculations easy to verify, we will use the Selberg minorant for the characteristic function of the interval $[-1, 1]$. Slightly better results can be obtained by using the more elaborate methods of [7]. Let

$$(26) \quad R(x) = \left(\frac{\sin \pi x}{\pi x} \right)^2 \frac{1}{1 - x^2}$$

so that $R(0) = 1$ and $R(x) \leq 0$ for $|x| \geq 1$. We note that

$$\widehat{R}(t) = \begin{cases} 1 - |t| + \frac{\sin 2\pi|t|}{2\pi} & \text{if } |t| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $R \in \mathcal{A}(1)$. For any $\lambda > 0$, set $r(u) = R(u/\lambda)$ and note that $r \in \mathcal{A}(\lambda)$ with $\widehat{r}(\alpha) = \lambda \widehat{R}(\lambda\alpha)$. Thus

$$\begin{aligned} c(\lambda; r) &= \widehat{r}(0) - 1 + 2 \int_0^1 \alpha \widehat{r}(\alpha) d\alpha \\ &= \lambda - 1 + 2\lambda \int_0^{\min(1, 1/\lambda)} \alpha \left(1 - \lambda\alpha + \frac{\sin 2\pi\lambda\alpha}{2\pi} \right) d\alpha, \end{aligned}$$

and a straightforward numerical calculation shows that

$$c(0.60729; r) > 0.$$

Therefore, by Theorem 3, we conclude that $\mu_D \leq 0.60729$. By comparison, the result $\mu_D \leq 0.6039$ in (6) is obtained from a much more complicated minorant.

To find distinct gaps, we need $c(\lambda; r) > n^* - 1$. Using the best known bound $n^* \leq 1.3208$ from [7], we find that $c(1.05214; r) > 0.3208$, which establishes the bound $\mu_{D_d} \leq 1.05214$. ■

REMARK. We mention in passing that since

$$c(1; r) = \frac{1}{3} - \frac{1}{2\pi^2} = 0.282673 \dots,$$

in order to prove $\mu_{D_d} < 1$, we would need a bound of $n^* \leq 1.2826$.

5. Proof of Theorem 2. We now introduce a new method in order to prove that $\mu_d < 1$. Write the functional equation for the zeta function as $\zeta(s) = \chi(s)\zeta(1-s)$, where $\chi(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi}{2}s\right)\Gamma(1-s)$. It is not hard to see that $\chi(s) = \chi(1-s)^{-1}$ and $|\chi(\frac{1}{2}+it)| = 1$ for real t . The Hardy Z -function is defined by

$$Z(t) = \chi\left(\frac{1}{2}-it\right)^{1/2}\zeta\left(\frac{1}{2}+it\right) = \chi\left(\frac{1}{2}+it\right)^{1/2}\zeta\left(\frac{1}{2}-it\right).$$

It follows from the functional equation that $Z(t)$ is real for $t \in \mathbb{R}$, that $|Z(t)| = |\zeta(\frac{1}{2}+it)|$, and that $Z(t)$ changes sign when t corresponds to an ordinate of a zero of odd multiplicity of $\zeta(s)$ on the critical line.

Assume RH. If $\gamma \in [T, 2T]$ is an ordinate of a simple zero of $\zeta(s)$, and $0 < a < \mu_d$, then

$$Z'(\gamma)Z\left(\gamma + \frac{2\pi a}{\log T}\right) > 0$$

when T is sufficiently large. This inequality is easily established by considering cases. If $Z'(\gamma) > 0$, then $Z\left(\gamma + \frac{2\pi a}{\log T}\right) > 0$ for $0 < a < \mu_d$. If $Z'(\gamma) < 0$, then $Z\left(\gamma + \frac{2\pi a}{\log T}\right) < 0$ for $0 < a < \mu_d$. Finally, if $Z'(\gamma) = 0$, then γ corresponds to a multiple zero of $\zeta(s)$. Likewise, under the same assumptions,

$$Z'(\gamma)Z\left(\gamma + \frac{2\pi a}{\log T}\right)|f(\gamma)|^2 \geq 0$$

for any function f . Consequently, if there exist choices of κ and f such that

$$\sum_{T < \gamma \leq 2T} Z'(\gamma)Z\left(\gamma + \frac{2\pi\kappa}{\log T}\right)|f(\gamma)|^2 < 0$$

when T is sufficiently large, then it follows that $\mu_d \leq \kappa$. Note that this method is detecting spacings between simple zeros of $Z(t)$ and sign changes of $Z(t)$ at another location. In other words, it is detecting (nonzero) gaps

between the ordinates of simple zeros of $\zeta(s)$ and ordinates of zeros of $\zeta(s)$ of odd multiplicity. We could alternatively study mean-values of the form

$$\sum_{T < \gamma \leq 2T} Z'(\gamma) Z\left(\gamma - \frac{2\pi\kappa}{\log T}\right) |f(\gamma)|^2$$

in order to detect gaps between zeros of odd multiplicity followed by simple zeros.

As in Selberg's proof that a positive proportion of the zeros of $\zeta(s)$ are on the critical line, we are studying sign changes of the Hardy Z -function on the scale of average spacing between zeros. This intuition suggests that we should choose the test function f to mollify the product $Z'(\gamma)Z\left(\gamma + \frac{2\pi\kappa}{\log T}\right)$. It is convenient to choose a mollifier of the form

$$M(s, P) = \sum_{n \leq y} \frac{\mu(n)P\left(\frac{\log y/n}{\log y}\right)}{n^s},$$

where $\mu(n)$ is the Möbius function, $y = T^\vartheta$, and $P(x)$ is a polynomial satisfying $P(0) = 0$. Thus, if there exist choices of $\kappa, \eta, \vartheta, P$ such that

$$\Sigma(\kappa; \eta, P) := \sum_{T < \gamma \leq 2T} Z'(\gamma) Z\left(\gamma + \frac{2\pi\kappa}{\log T}\right) \left| M\left(\frac{1}{2} + i\gamma + \frac{2\pi i\eta}{\log T}, P\right) \right|^2 < 0$$

when T is sufficiently large, then $\mu_d \leq \kappa$. We choose $0 < \eta < \kappa$, so that we are simultaneously mollifying both $Z'(\gamma)$ and $Z\left(\gamma + \frac{2\pi\kappa}{\log T}\right)$. This mean-value can be analyzed using techniques similar to those introduced by Conrey, Ghosh, and Gonek [11] and Bui and Heath-Brown [1], who were interested in estimating the proportion of simple zeros of the zeta function.

In particular, let

$$M(s, g) = \sum_{n \leq y} \frac{\mu(n)g\left(\frac{\log y/n}{\log y}\right)}{n^s},$$

where $y = T^\vartheta$ and g is entire with $g(0) = 0$. Let Q_1, Q_2 be polynomials and let

$$\begin{aligned} I(a, b, g_1, g_2, Q_1, Q_2) &= \sum_{T < \gamma \leq 2T} Q_1\left(-\frac{d}{da}\right) \zeta\left(\rho + \frac{a}{\log T}\right) \\ &\quad \times Q_2\left(-\frac{d}{db}\right) \zeta\left(1 - \rho + \frac{b}{\log T}\right) M(\rho, g_1) M(1 - \rho, g_2). \end{aligned}$$

Then the following estimate holds.

THEOREM 4 (Conrey–Ghosh–Gonek). *If $0 < \vartheta < 1/2$ and $a, b \ll 1$, then as $T \rightarrow \infty$ we have*

$$\begin{aligned}
I(a, b, g_1, g_2, Q_1, Q_2) \sim & \frac{T \log T}{2\pi} \frac{\partial^2}{\partial u \partial v} \left\{ \left(\frac{1}{\vartheta} \int_0^1 g_1 g_2 \, dx + \int_0^1 g_1 \, dx \int_0^1 g_2 \, dx \right) \right. \\
& \times \left(\int_0^1 T_a Q_1 T_b Q_2 \, dx - \int_0^1 T_a Q_1 \, dx \int_0^1 T_b Q_2 \, dx \right) \\
& \left. + \int_0^1 g_1 \, dx \int_0^1 g_2 \, dx \left(Q_1(0) - \int_0^1 T_a Q_1 \, dx \right) \left(Q_2(0) - \int_0^1 T_b Q_2 \, dx \right) \right\} \Big|_{u=v=0},
\end{aligned}$$

where $g_1 = g_1(x + u)$, $g_2 = g_2(x + v)$,

$$T_a Q_1 = e^{-a(x+\vartheta u)} Q_1(x + \vartheta u),$$

and

$$T_b Q_2 = e^{-b(x+\vartheta v)} Q_2(x + \vartheta v).$$

This result was stated without a proof by Conrey, Ghosh, and Gonek [10, Theorem 2]. It can be deduced in a straightforward manner from the more general result of Heap, Li, and Zhao [22, Theorem 2], which builds on the previous work in [1, 11].

Proof of Theorem 2. It is not difficult to convert the mean-value $\Sigma(\kappa; \eta, P)$ into a mean-value of the form that is estimated in Theorem 4. We start by observing that if $\rho = \frac{1}{2} + i\gamma$ is a zero of $\zeta(s)$, then it follows from the definition of $Z(t)$ that

$$Z'(\gamma) = -i\chi(\rho)^{1/2} \zeta'(1 - \rho).$$

Hence

$$\begin{aligned}
Z'(\gamma) Z\left(\gamma + \frac{2\pi\kappa}{\log T}\right) \\
= -i\chi(\rho)^{1/2} \chi\left(1 - \rho - \frac{2\pi i\kappa}{\log T}\right)^{1/2} \zeta'(1 - \rho) \zeta\left(\rho + \frac{2\pi i\kappa}{\log T}\right).
\end{aligned}$$

Using the Stirling's formula approximation

$$\chi(s + a)\chi(1 - s + b) = \left(\frac{t}{2\pi}\right)^{-a-b} \left(1 + O\left(\frac{1}{1 + |t|}\right)\right)$$

for t large and a, b uniformly bounded, we have

$$\begin{aligned}
\chi(\rho)^{1/2} \chi\left(1 - \rho - \frac{2\pi i\kappa}{\log T}\right)^{1/2} &= \left(\frac{\gamma}{2\pi}\right)^{\pi i\kappa/\log T} \left(1 + O\left(\frac{1}{T}\right)\right) \\
&= e^{\pi i\kappa} (1 + O((\log T)^{-1}))
\end{aligned}$$

for $\gamma \in (T, 2T]$. Therefore

$$(27) \quad \begin{aligned} \Sigma(\kappa; \eta, P) &= -ie^{\pi i \kappa} \sum_{T < \gamma \leq 2T} \zeta'(1 - \rho) \zeta\left(\rho + \frac{2\pi i \kappa}{\log T}\right) \\ &\quad \times M\left(1 - \rho - \frac{2\pi i \eta}{\log T}, P\right) M\left(\rho + \frac{2\pi i \eta}{\log T}, P\right) \\ &+ O\left((\log T)^{-1} \sum_{T < \gamma \leq 2T} \left| \zeta'(1 - \rho) \zeta\left(\rho + \frac{2\pi i \kappa}{\log T}\right) \right| \left| M\left(\rho + \frac{2\pi i \eta}{\log T}, P\right) \right|^2\right). \end{aligned}$$

Using Theorem 4 and Cauchy's integral formula, we can deduce the bounds

$$\begin{aligned} \sum_{T < \gamma \leq 2T} \left| \zeta\left(\rho + \frac{2\pi i \kappa}{\log T}\right) M\left(\rho + \frac{2\pi i \eta}{\log T}, P\right) \right|^2 &\ll T \log T, \\ \sum_{T < \gamma \leq 2T} \left| \zeta'(\rho) M\left(\rho + \frac{2\pi i \eta}{\log T}, P\right) \right|^2 &\ll T \log^3 T, \end{aligned}$$

for any fixed κ , η , and P . Hence, using Cauchy's inequality and these bounds to estimate the error term in (27), it follows from Theorem 4 that

$$\begin{aligned} \Sigma(\kappa; \eta, P) &= -ie^{\pi i \kappa} \sum_{T < \gamma \leq 2T} \zeta'(1 - \rho) \zeta\left(\rho + \frac{2\pi i \kappa}{\log T}\right) \\ &\quad \times M\left(1 - \rho - \frac{2\pi i \eta}{\log T}, P\right) M\left(\rho + \frac{2\pi i \eta}{\log T}, P\right) \\ &+ O(T \log T) \\ &= -ie^{\pi i \kappa} (\log T) I(2\pi i \kappa, 0, g_1, g_2, 1, -x) + O(T \log T) \end{aligned}$$

with

$$g_1(x) = P(x) \exp(-2\pi i \vartheta \eta(1 - x))$$

and

$$g_2(x) = P(x) \exp(2\pi i \vartheta \eta(1 - x)).$$

The choice $\vartheta = 0.4999$, $\kappa = 0.991$, $\eta = 0.6$, and $P(x) = x$ gives

$$\Sigma(0.991; 0.6, x) = (-0.00155 \dots + o(1)) \frac{T \log^2 T}{2\pi},$$

and hence Theorem 2 follows. ■

REMARK. We have not tried to find the smallest admissible value of κ in the above argument, and instead focussed on finding simple choices of η and P that yield a value of $\kappa < 1$.

6. Proof of Corollary 1. Let T be large and let $\vartheta, \kappa, \eta, P$ be chosen as in the proof of Theorem 2. Then

$$\sum_{T < \gamma \leq 2T} Z'(\gamma) Z\left(\gamma + \frac{2\pi\kappa}{\log T}\right) \left| M\left(\frac{1}{2} + i\gamma + \frac{2\pi i\eta}{\log T}, P\right) \right|^2 < 0.$$

Observe that every negative term in this sum corresponds to a zero with $Z'(\gamma) Z\left(\gamma + \frac{2\pi\kappa}{\log T}\right) < 0$, however not necessarily every term in this sum is negative. Moreover, Theorem 4 shows that this sum is $\gg T \log^2 T$ in magnitude. Therefore,

$$\begin{aligned} T \log^2 T &\ll \left| \sum_{T < \gamma \leq 2T} Z'(\gamma) Z\left(\gamma + \frac{2\pi\kappa}{\log T}\right) \left| M\left(\frac{1}{2} + i\gamma + \frac{2\pi i\eta}{\log T}, P\right) \right|^2 \right| \\ &\leq \left| \sum_{\substack{T < \gamma \leq 2T \\ Z'(\gamma) Z\left(\gamma + \frac{2\pi\kappa}{\log T}\right) < 0}} Z'(\gamma) Z\left(\gamma + \frac{2\pi\kappa}{\log T}\right) \left| M\left(\frac{1}{2} + i\gamma + \frac{2\pi i\eta}{\log T}, P\right) \right|^2 \right|. \end{aligned}$$

Applying Cauchy's inequality to the latter sum, we deduce that

$$\begin{aligned} T^2 \log^4 T &\leq \left(\sum_{\substack{T < \gamma \leq 2T \\ Z'(\gamma) Z\left(\gamma + \frac{2\pi\kappa}{\log T}\right) < 0}} 1 \right) \\ &\quad \times \left(\sum_{T < \gamma \leq 2T} Z'(\gamma)^2 Z\left(\gamma + \frac{2\pi\kappa}{\log T}\right)^2 \left| M\left(\frac{1}{2} + i\gamma + \frac{2\pi i\eta}{\log T}, P\right) \right|^4 \right), \end{aligned}$$

where (by positivity) we have extended the second sum on the right-hand side to all zeros with $T < \gamma \leq 2T$. By [4] and [6], for $T < \gamma \leq 2T$, we have

$$Z'(\gamma)^2 Z\left(\gamma + \frac{2\pi\kappa}{\log T}\right)^2 \ll \exp\left((\log 4 + o(1)) \frac{\log T}{\log \log T}\right).$$

An upper bound for $|\zeta(\frac{1}{2} + it)|$ is given in [6], whereas the results in [4] combined with a standard argument using the functional equation, Stirling's formula, and Cauchy's integral formula can be used to show that a bound of similar strength holds for $|\zeta'(\frac{1}{2} + it)|$ and $|Z'(t)|$.

Now for $\{a(n)\} \subseteq \mathbb{C}$ with $|a(n)| \ll n^\varepsilon$ and $x \leq T^{1-\varepsilon}$ for $\varepsilon > 0$, it is known (e.g. using the Landau–Gonek explicit formula [21, Theorem 1] or contour integration [23, Theorem 5.1]) that

$$\sum_{0 < \gamma \leq T} \left| \sum_{n \leq x} \frac{a(n)}{n^{1/2+i\gamma}} \right|^2 \sim N(T) \sum_{n \leq x} \frac{|a(n)|^2}{n} - \frac{T}{\pi} \operatorname{Re} \sum_{n \leq x} \frac{(\Lambda * a)(n) \overline{a(n)}}{n}$$

as $T \rightarrow \infty$. From this, letting $d(n)$ denote the number of divisors of n , it

follows that

$$\begin{aligned} \sum_{T < \gamma \leq 2T} \left| M\left(\frac{1}{2} + i\gamma + \frac{2\pi i\kappa}{\log T}, P\right) \right|^4 \\ \ll T \log T \sum_{n \leq y^2} \frac{d(n)^2}{n} + T \sum_{n \leq y^2} \frac{(\Lambda * d)(n)d(n)}{n} \ll T \log^5 T \end{aligned}$$

for our choice of $M(s, P)$ and y in the proof of Theorem 2. Hence

$$\begin{aligned} \sum_{T < \gamma \leq 2T} Z'(\gamma)^2 Z\left(\gamma + \frac{2\pi\kappa}{\log T}\right)^2 \left| M\left(\frac{1}{2} + i\gamma + \frac{2\pi i\eta}{\log T}, P\right) \right|^4 \\ \ll T \exp\left((\log 4 + o(1)) \frac{\log T}{\log \log T}\right). \end{aligned}$$

Combining these estimates, we conclude that

$$\sum_{\substack{T < \gamma \leq 2T \\ Z'(\gamma)Z\left(\gamma + \frac{2\pi\kappa}{\log T}\right) < 0}} 1 \gg T \exp\left(-(\log 4 + o(1)) \frac{\log T}{\log \log T}\right).$$

In this way, we deduce Corollary 1 from the proof of Theorem 2.

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