Abstract

Despite the importance of graphical reasoning, graph construction and interpretation has been shown to be non-trivial. Paoletti et al. (2023) presented a framework that allows for a fine-grained analysis of students' graphical reasoning as they conceive of graphs as representing two covarying quantities. In this paper, we show how the framework can be used to characterize a student's meanings and reasoning related to such graphical reasoning, to diagnose complexities in a student's developing such reasoning *and* to design tasks that provide opportunities to resolve such complexities. We draw on data from a teaching experiment with a sixth-grade student in the U.S. to highlight how the framework allowed us to identify indications and contraindications of the student's engaging in reasoning compatible with the framework. Further, we describe how this analysis supported us in designing a task that was aligned with the framework and proved productive in supporting the students' learning. We conclude with a discussion of our findings and their implications for task design and future research.

Keywords: Graphs and Graphing; Covariational Reasoning; Quantitative Reasoning; Teaching Experiment; Middle School

1. Introduction

Constructing and interpreting graphs are critical skills for students to develop both as learners in STEM fields and as informed citizens (e.g., Gantt et al., 2023b; Glazer, 2011; Potgieter et al., 2008). However, the difficulties that students and teachers experience with graph construction and interpretation are well-documented (e.g., Glazer, 2011; Thompson et al., 2017). In response, a small but growing body of literature has begun to explore ways to support students' development of graphing meanings across various grade levels (Ellis et al., 2015; Hattikudur et al., 2012; Johnson, 2015; Liang & Moore, 2021; Nathan & Kim, 2007; Paoletti & Moore, 2017; Rolfes et al., 2020).

One form of graphing reasoning, referred to as *emergent graphical shape thinking* (EGST; Moore, 2020; Moore & Thompson, 2015), is important across STEM disciplines (Paoletti et al., 2020). Paoletti et al. (2023) presented a framework (described in Section 2) that allows for fine-grained analysis of students' meanings and reasoning as they develop EGST. They presented the activity of two advanced, 8^{th}-grade students (who had taken Algebra I and high school Geometry) to highlight the ways students' meanings might progress as they develop EGST. However, they did not describe how the framework could be used to provide indications or contraindications of student thinking progressing towards EGST or to support the design of novel tasks to support such reasoning.

The goal of this paper is to show how the framework can be used to identify indications and contraindications of meanings and reasoning related to EGST *and* to design tasks that support students' making progress toward EGST. By focusing both on analyzing students' reasoning and on designing tasks, we provide important implications for researchers and teachers aiming to support students' developing graphing meanings and for curriculum developers designing tasks to support such meanings.

2. Literature review and theoretical framework

Moore and Thompson (Moore, 2020; Moore & Thompson, 2015) described emergent graphical shape thinking (EGST) as conceiving of:

> a graph simultaneously as what is made (a trace) and how it is made (covariation)… [EGST] entails assimilating a graph as a trace in progress (or envisioning an already produced graph in terms of replaying its emergence), with the trace being a record of the relationship between covarying quantities. (Moore & Thompson, 2015, p. 785)

EGST is important for interpreting graphs across STEM fields. For example, interpreting graphs in many STEM textbooks and practitioner journals requires such reasoning (Paoletti et al., 2020). Further, there is evidence that students in middle school (e.g., Ellis et al., 2015; Paoletti, 2019) and high school (e.g., Johnson, 2015) can develop meanings for graphs that include elements of EGST.

Extending the work of Moore and Thompson, Paoletti et al. (2023) provided a framework that described components necessary for students to engage in EGST. In this section, we describe Paoletti et al.'s (2023) framework, including a synthesis of relevant literature around components of the framework where appropriate. Figure 1 provides an overview of the original framework. Visually, we note that students' situational quantitative and covariational reasoning (abbreviated M.S. to capture *meanings* and *situational*) is adjacent to their reasoning with graphical representations (abbreviated M.R. to capture *meanings* and *representation*). This organization conveys that students must engage in reasoning that bridges these meanings (Gantt et al., 2023b) as a prerequisite to engaging in EGST (abbreviated M.E. to capture *meanings* and *emergent*). In the sections that follow, we elaborate on each of these constructs.

Figure 1

A re-creation of the framework from Paoletti et al. (2023)

<table>
<tr><th>(M.S) Situational quantitative and covariational reasoning</th><th>(M.R) Reasoning with graphical representations of covarying quantities</th></tr>
<tr><td>M.S.1. Construct quantities in a contextualized or decontextualized situation.</td><td>M.R.1. Consider a varying segment length as representing a quantity's magnitude.</td></tr>
<tr><td>M.S.2. Coordinate how two quantities change in relation to each other.</td><td>M.R.2. Consider variations in two orthogonal segment lengths on axes in a coordinate system[1] in relation to two covarying quantities.</td></tr>
<tr><td>M.S.3. Develop an operative image of covariation that entails a multiplicative object.</td><td>M.R.3. Conceive of or anticipate a point as a multiplicative object in the coordinate system simultaneously representing the two segments' magnitudes.</td></tr>
<tr><th colspan="2">(M.E) Emergent graphical shape thinking</th></tr>
<tr><td colspan="2">M.E.0. Conceive a point as a multiplicative object in a coordinate system (M.R.3) whose motion is constrained by the covarying quantities conceived of in the situational multiplicative object (M.S.3).
M.E.1. Conceive of or imagine a graph being produced by the trace of the point (M.E.0) as quantities covary.
M.E.2. Consider various situations that produce the same final graph via different traces.</td></tr>
</table>

2.1 Meanings

Paoletti et al. (2023) emphasized the importance of students' developing *meanings* for various situational and representational objects. Consistent with Thompson's (2016) use, we characterize a student's meanings as the collection of actions, objects, or schemes they can call on when they assimilate a situation. For example, when a student assimilates a drawn picture as a linear graph, their meanings could entail a combination of interrelated elements including a constant rate of change, tricks involved in determining the slope of the drawn graph, and/or an analytic rule that represents the relationship.

All meanings are expressed in-the-moment in relation to a particular context or task. Some in-the-moment meanings are stable whereas others are only experienced in-the-moment; when a student first makes sense of a new idea, their in-the-moment meanings are often fleeting. Students typically need repeated opportunities to (re)construct in-the-moment meanings before the meanings become stable (Thompson, 2011). Once a meaning is stable, the student can spontaneously use the meaning in novel situations. Hence, Paoletti et al.'s (2023) framework emphasized providing students repeated opportunities to develop stable meanings for the components necessary for EGST.

2.2. Quantitative and covariational reasoning (M.S.1 and M.S.2)

Students need to construct and reason about quantities as a pre-requisite to engaging in EGST (Paoletti et al., 2023). Leveraging Steffe, Thompson, and colleagues' (e.g., Smith & Thompson, 2008; Steffe, 1991) stance on quantitative reasoning, we contend that a quantity is a conceptual entity an individual constructs as they conceive of a measurable attribute of some object or phenomena[1]. Because quantities are individual constructions, researchers (and teachers) cannot presume that a student's meanings for a particular quantity are compatible with their intentions. Researchers (e.g., Paoletti, 2015; Paoletti et al., 2019; Moore & Carlson, 2012) have provided numerous examples in which students develop meanings for a quantity that work for the particular student but are inconsistent with the quantity researchers intended for them to explore.

Although a student's quantitative reasoning can involve both numeric and non-numeric reasoning (Johnson, 2012; Moore et al., 2019), Smith and Thompson (2008) emphasize quantitative reasoning as "thinking about and representing… the relationships inherent in the quantities themselves, not the specific numeric values they take on" (p. 111). Hence, the essence of quantitative reasoning is non-numeric, having more to do with an individual's meanings for attributes of objects or phenomena than with particular numeric values. However, we have often observed students using numeric values as they develop meanings for a quantity or for how

[1] To construct a quantity, an individual needs to conceive of a reference point, direction, and unit to measure the quantity (see Joshua et al., 2015; Thompson, 2011 for more on this).

quantities change together (e.g., Paoletti & Vishnubhotla, 2022). Hence, numeric quantitative reasoning can support students as they develop meanings for quantities (Johnson, 2012). Our goal, however, is to support students in developing conceptions for quantities that are non-numeric (i.e., based in magnitudes, or 'amount-ness', rather than constrained to specific numeric values). One way researchers can infer that students conceive of non-numeric quantities as they describe the intensity of change of a quantity (e.g., a student describing a quantity as growing quickly or growing slowly; see Johnson, 2012).

As students construct two (or more) quantities, they can engage in covariational reasoning (Carlson et al., 2002; Saldanha & Thompson, 1998; Thompson & Carlson, 2017). Saldanha and Thompson (1998) defined an individual to be reasoning covariationally when they are "holding in mind a sustained image of two quantities' values (magnitudes) simultaneously" (p. 298). Carlson et al. (2002) highlighted that two stages of covariational reasoning involve conceiving of the directional covariation and the amounts of change (hereafter AoC) relationship between two quantities.[2] Researchers (e.g., Moore, 2014; Paoletti & Vishnubhotla, 2022) have shown how coordinating directional and AoC relationships can form a foundation for students' graphing activity.

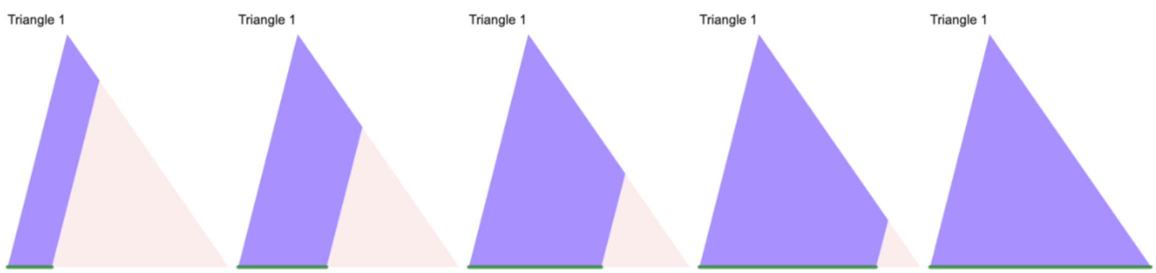
We use the growing shape depicted in Figure 2 to describe these two stages. After observing the shape grow through an animated series of screenshots, a student may first construct the shape's base length and area as individual quantities (M.S.1). The student may then conceive that as the shape's green base length increases, the purple area also increases (M.S.2 at the stage of directional covariation). The student may further explore *how* the two quantities

[2] We note there are numerous frameworks used to analyze students' covariational reasoning (see Thompson & Carlson, 2017 for a review). Particular to this report, Carlson et al.'s (2002) description of amounts of change is a particular form of the chunky continuous reasoning in Thompson and Carlson's (2017) framework. We use Carlson et al.'s (2002) language given its greater specificity than newer frameworks.

increase together via an AoC relationship. For instance, from the animation represented in Figure 2, the student may construct an AoC of the purple area as a quantity of "chunks" of purple area added between the screenshots of the growing triangle (Gantt et al., 2022); the student may construct an AoC of green base length similarly via the added distance between screenshots. Relating to the original quantities, the student may conclude that as the green base length increases, the purple area increases and that for equal AoC in the green base length, the AoC of the shape's purple area decreases (M.S.2 at the AoC stage).

Figure 2

Several screenshots of a growing shape that results in a triangle.



2.3. Reasoning with Graphical Representations (M.R.1 and M.R.2)

Students can construct quantities and reason about covarying quantities without representing these quantities via typical mathematical representations (e.g., diSessa et al., 1991; Paoletti & Moore, 2017). In order to represent a covariational relationship graphically in a Cartesian coordinate system, Paoletti et al.'s (2023) framework emphasizes the importance of students conceiving of a magnitude bar's length as representing the magnitude (or amount) of a quantity (M.R.1) prior to their using two orthogonal magnitude bars to represent covarying quantities (M.R.2). Lee and colleagues (Lee & Hardison, 2016; Lee et al., 2020; Paoletti et al., 2023) describe that the process of representing a quantity via a magnitude bar requires an individual to:

> establish quantities within the given space/phenomenon, disembed (Steffe & Olive, 2010) these quantities (i.e., extract them from the situation while maintaining an awareness of the quantities within the situation), and project them onto some new space, which is different from the space in which the quantities were originally conceived. (Lee et al., 2020, p. 34)

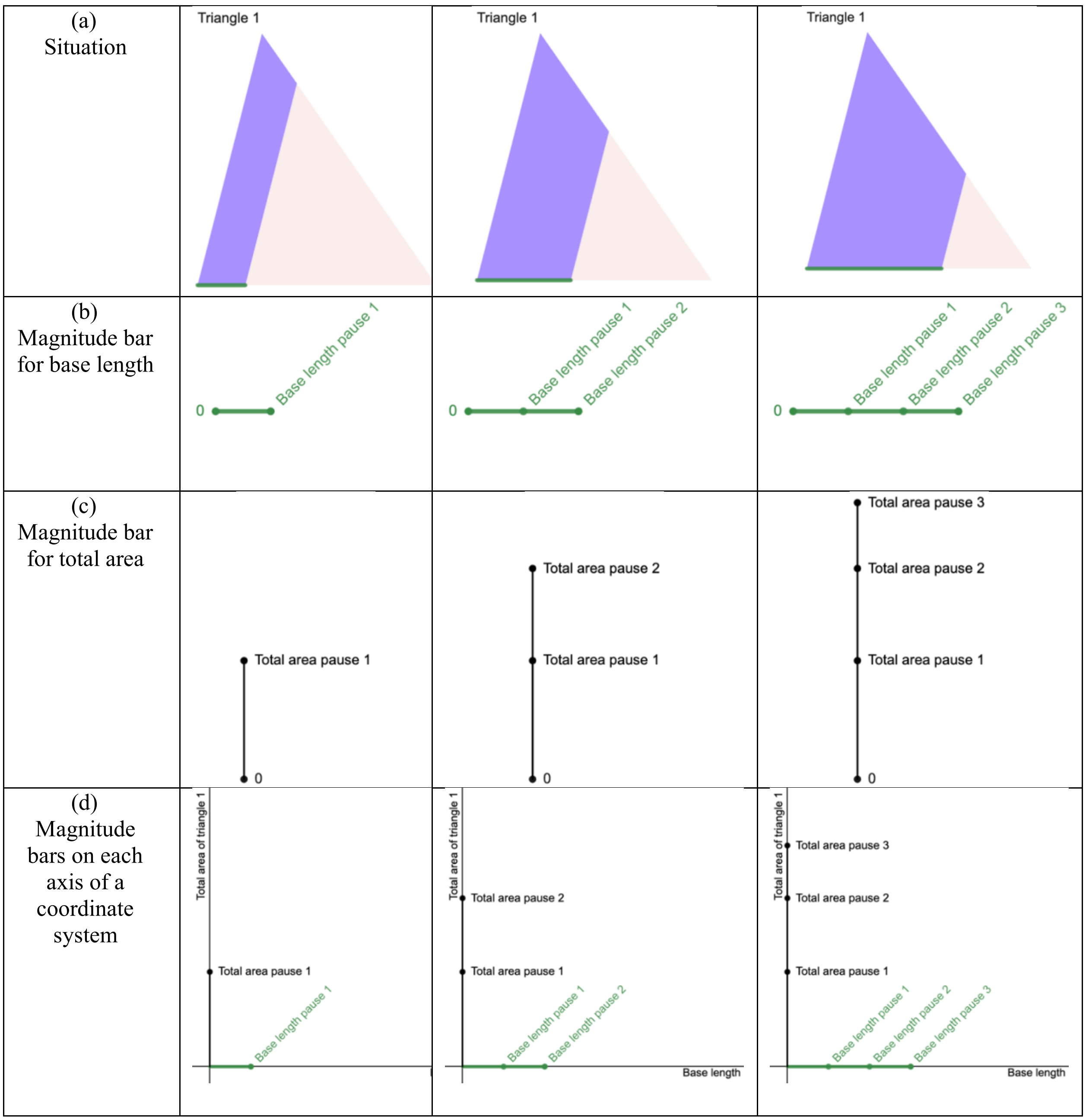
Magnitude bars serve as the 'new space' students can use to represent disembedded quantities. Representing quantities via magnitude bars is fundamental to students' bridging their meanings for situational quantities and graphical representations of such quantities.

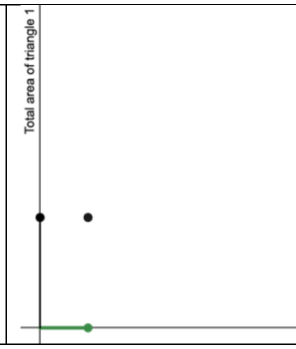
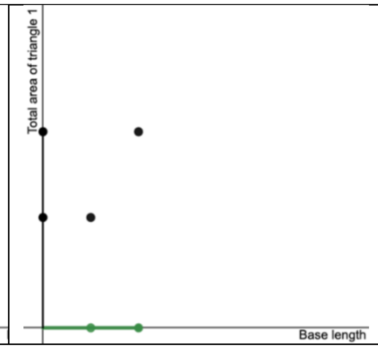
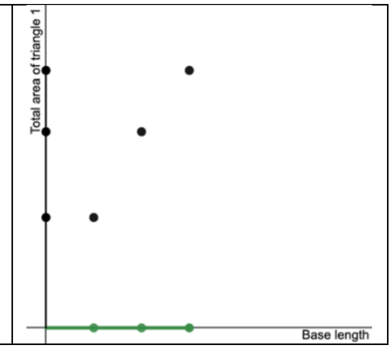
We note several mathematics education researchers have used magnitude bars (sometimes referred to as dynagraphs) for various purposes. Stevens (2023) synthesized the use of magnitude bars in the extant literature. She indicated that researchers have described ways magnitude bars can support students in developing meanings for functional relationships (Antonini et al., 2020; Goldenberg et al., 1992; Hollebrands et al., 2021), for linear measurement (Curry et al., 2006; Kamii, 2006), and fractions (Beckmann & Izsák, 2020; Yeo 2021). Particular to students' graphing activity, several researchers have described using magnitude bars (Liang & Moore, 2021; Paoletti et al., 2021; Stevens et al., 2017; Stevens 2023) or finger dragging activity that mimics magnitude bars (Castillo-Garsow, 2012; Thompson, 2002) to support students in representing quantities via orthogonal magnitude bars along axes.

We use Figure 3 to exemplify how magnitude bars can support students' graphing reasoning. After conceiving of the situation depicted in Figure 2, a student can consider how they can use a bar to represent the magnitude of each of the changing quantities. For example, considering the first three screenshots (Figure 3a), a student can conceive of magnitude bars whose changes reflect a base length increasing with equal AoC (Figure 3b) and a total area increasing with decreasing AoC (Figure 3c). Such reasoning bridges the students' meanings for each quantity in the situation with a graphical representation of the quantity (M.S.1↔M.R.1). The student can then consider overlaying these magnitude representations along axes (Figure 3d) to represent two conceived covarying quantities graphically (M.S.2↔M.R.2).

Figure 3

Several screenshots of a growing shape that results in a triangle.



(e) A point representing a multiplicative object			

The above example highlights how using magnitude bars can be helpful for students' developing meanings for graphs. However, researchers have also noted conflations that can occur when such bars are used (Liang & Moore, 2020; Stevens, 2023). For example, Stevens (2023) described how one student compared the relative lengths of two magnitude bars despite the two situational quantities being unlike (i.e., length and area). The student was given magnitude bar corresponding to a varying length and potential magnitude bars corresponding to the area of a growing shape. To narrow down the potential magnitude bars representing a shapes area, the student argued that the magnitude bar representing the area *must* be longer than the magnitude bar representing only a length because area involves the multiplication of two lengths. The student used this (non-normative) reasoning to decide which bar represented area rather than relying on the rate at which length and area were changing together (e.g., constant rate vs. increasing rate). From these studies, we identify that students can experience conflations with magnitude bars when they rely on non-quantitative meanings. Hence, it is important to support students in explicitly connecting their use of magnitude bars to their reasoning about quantities (M.S.2↔M.R.2).

2.4. Multiplicative objects (M.S.3, M.R.3, and M.E.0)

According to Paoletti et al.'s (2023) framework, constructing multiplicative objects[3] is the third meaning necessary for EGST in both situational (M.S.) and representational (M.R.) settings. Describing constructing multiplicative objects situationally, Saldanha and Thompson (1998) stated:

> [Covariational reasoning] entails coupling the two quantities, so that, in one's understanding, a multiplicative object is formed of the two. As a multiplicative object, one tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value. (p. 299)

Hence, constructing covarying quantities as a multiplicative object entails anticipating the simultaneity of the two quantities' values in relation to each other. Because covariational reasoning is developmental (Carlson et al., 2002; Paoletti et al., 2022; Saldanha & Thompson, 1998), students typically think of one quantity first, then the second, then back to the first, and so on. Engaging in this process can eventually lead to their maintaining an explicit and persistent awareness that both quantities simultaneously have a value. For example, using the context from Figure 2, a student may first consider a particular base length, then a corresponding area, then a new base length and a new corresponding area (M.S.2). After repeatedly engaging in such activity, the student can couple the two quantities to conceive of a multiplicative object (M.S.3) if they anticipate that, for all values of the green base length, there are simultaneous corresponding values for the shape's purple area.

Graphically, a dynamic coordinate point can be conceived of as a multiplicative object representing the simultaneity of two varying bars' magnitudes represented on each axis. Figure

[3] Thompson and colleagues' (Thompson et al., 2017; Saldanha & Thompson, 1998) use of multiplicative object is akin to the logical multiplication necessary for multiple classification, seriation, and simultaneity (e.g., a young child conceiving of an apple as both red and sweet).

3e depicts three such instantiations of a point corresponding to the endpoints of the two magnitude bars on the axes (M.R.3). If the student understands that this point simultaneously represents the two quantities they have conceived as varying together, they have bridged their meanings for situational and graphical multiplicative objects (M.S.3↔M.R.3). According to Paoletti et al. (2023), such reasoning is a prerequisite for EGST (M.E.0).

3. Methods

We conducted a 10-session teaching experiment (Steffe & Thompson, 2000) to further explore students' activity and the design of tasks using the EGST framework (Paoletti et al., 2023). In what follows, we describe participants, contexts, data analysis, and the initial tasks. Table 1 provides an overview of the teaching experiment sessions, including student attendance, the primary task students engaged with, and a short description of the goals of each session.

Table 1

An overview of the teaching experiment

Session	Task	Students present	Goal
1	Faucet Task	Kennedy and Zee	Establishing quantitative relationships and developing understandings of coordinate systems (see Paoletti et al., 2023)
2	Faucet Task	Kennedy and Zee	
3	Faucet Task	Kennedy and Zee	
4	Growing Triangle	Kennedy	Constructing an increasing by more relationship and representing the relationship graphically (see Paoletti et al., 2023)
5	Growing Triangle	Kennedy and Zee	
6	Growing Trapezoid	Kennedy	Constructing an increasing by less relationship and representing the relationship graphically
7	Growing Trapezoid	Kennedy and Zee	
8	Four Shapes Task	Kennedy	Supporting students' conception of area as a quantity and representing area with a magnitude bar
9	Four Shapes Task	Kennedy and Zee	
10	Growing Triangle/Rectangle	Kennedy	Constructing linear and non-linear relationships and representing each graphically

3.1 Participants, Context, Data Analysis, and Research Question

We conducted our teaching experiment with two sixth-grade students in a public school in the Northeastern U.S. Our primary goal across the teaching experiment was to explore how the 6th-grade students developed graphing meanings with tasks we designed to 1) align with Paoletti et al.'s (2023) framework originally used with advanced 8th-grade students and 2) include added structure, transparency, and accessibility for use by secondary classroom teachers. As such, to design tasks for the teaching experiment, we began by adapting tasks originally described in Paoletti et al. (2023) into the Desmos activity platform. Original tasks had been designed strictly as dynamic applets in GeoGebra, in which the teacher-researcher guided students through exploratory, verbal prompts. By designing tasks as Desmos activities, we could provide students with self-paced, pre-planned, written prompts, share the tasks with teachers via a free and widely used platform, and embed opportunities for teachers' monitoring of students' activities. Because the adaptation to Desmos activities involved fundamental changes to how students would experience the tasks, our teaching experiment enabled us to study students' graphing meanings within this new environment and develop responsive new tasks and prompts to further support their graphical reasoning.

The first author facilitated the teaching experiment as the teacher-researcher (TR). Although we initially selected a pair of students (Kennedy and Zee), Zee was often absent (missing all or most of Sessions 4, 6, 8, and 10; Table 1). Hence, we focus our case on the activity of one participant, Kennedy (age 11; self-identified as female and Dominican, Nova Scotian, and African American). Each teaching episode lasted 35-40 minutes. We video- and audio-recorded the sessions. We digitized records of the students' written work at the end of each session.

Consistent with the teaching experiment methodology (Steffe & Thompson, 2000), the teaching sessions served as an exploratory tool, allowing us to examine and promote Kennedy's mathematical progress over the study. We used both ongoing and retrospective data analysis techniques. In both phases, we conducted conceptual analysis, which is "building models of what students actually know at some specific time and what they comprehend in specific situations" (Thompson, 2008, p. 44). Our goal in conducting such an analysis was to develop and refine models of Kennedy's mathematics that viably explained her activity.

3.1.1. Ongoing analysis

During the teaching experiment, we iteratively built models of Kennedy's mathematics that served as catalysts for designing, refining, or adapting tasks for future sessions. These tasks allowed us to test our models as we predicted how Kennedy might respond to a given task or situation. This ongoing analysis led to our focusing on certain conjectures about Kennedy's meanings which we developed between teaching episodes. This is consistent with how teaching experiments operate (Steffe & Thompson, 2000). Throughout ongoing analysis, the research team met between sessions to debrief, watch videos from the previous session, and discuss Kennedy's reasoning to build preliminary models of her meanings. We documented our hypothesized models made during this analysis as well as instructional decisions that were based on these models.

As the teaching experiment progressed, we conjectured that Kennedy was experiencing complexities related to constructing quantities and bridging her situational and representational meanings. In response to this, we designed the *Four Shapes Task* to explore our conjectures and to support Kennedy's developing meanings. In the results, we report data describing our conjectures regarding Kennedy's reasoning and how our conjectures led to our designing and enacting this new task. We highlight how Paoletti et al.'s (2023) framework allowed us to both analyze a student's reasoning at a fine-grained level and to design a task to support the student's progress towards EGST. We show how this new task provided evidence supporting our conjectured models of Kennedy's mathematics and provided her opportunities to further develop her graphing meanings.

3.1.2. Retrospective analysis

We conducted retrospective analysis after the teaching experiment. During retrospective analysis, we reflected on our preliminary models created throughout the teaching experiment (Steffe & Thompson, 2000). This analysis involved first rewatching the videos to identify instances that provided insights into Kennedy's meanings and reasoning. We transcribed her spoken words, integrating notes about her gestures and computer activity into the transcripts. Keeping our ongoing analysis efforts and the entire data corpus in mind, we then performed conceptual analysis (Thompson, 2008) to generate and test models of Kennedy's meanings and reasoning such that the models provided viable explanations of her activity.

During retrospective analysis, we used Paoletti et al.'s (2023) framework to characterize Kennedy's developing meanings towards EGST. We analyzed the data at the utterance level, highlighting moments in which we inferred Kennedy reasoned about quantities in a situation (M.S.) or quantities in a graphical representation (M.R.). We also noted when Kennedy provided contraindications (Hardison, 2018; Liang & Moore, 2020) of engaging in such reasoning (~M.S. or ~M.R.). Further, we identified instances in which she bridged situational and graphical meanings (M.S.↔M.R.) or when she provided contraindications to bridging such meanings

(~M.S.↔M.R.). We leveraged this analysis to organize an account that addresses our research question, which we share presently.

3.1.3. Research Question

In this report, we address the research question: How can we use Paoletti et al.'s (2023) framework to (a) analyze a student's developing EGST, (b) diagnose complexities in a student's developing EGST, and (c) design tasks that provide opportunities to resolve such complexities?

3.2 Growing Triangle Task

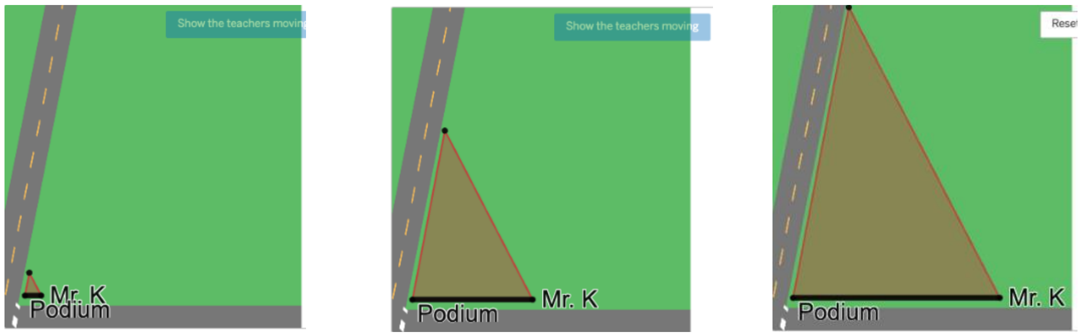
Each task in Table 1 was presented as a series of prompts with associated dynamic applets through which students could watch and/or control the changing of quantities. Before we designed the *Four Shapes Task,* Kennedy engaged in the *Growing Triangle Task*[4]. Intending to provide a context for the growing shape, we presented students with a dynamic scenario in which two teachers (Mr. K and Mrs. B) were walking away from a podium while holding a rope which creates a triangular area where hypothetical students could stand for a presentation (see Figure 4 for several screenshots). Throughout the task, we asked Kennedy to identify and coordinate Mr. K's distance from the podium (i.e., the horizontal base length) and the area of the shape formed (i.e., engage in M.S.). In this scenario, as Mr. K's distance increases, the total area of the shape increases and the AoC of area increases.

Figure 4

Several screenshots of Mr. K. walking away from the podium[5]

[4] The entire activity Kennedy engaged in can be found here: https://tinyurl.com/3djz8cp9.

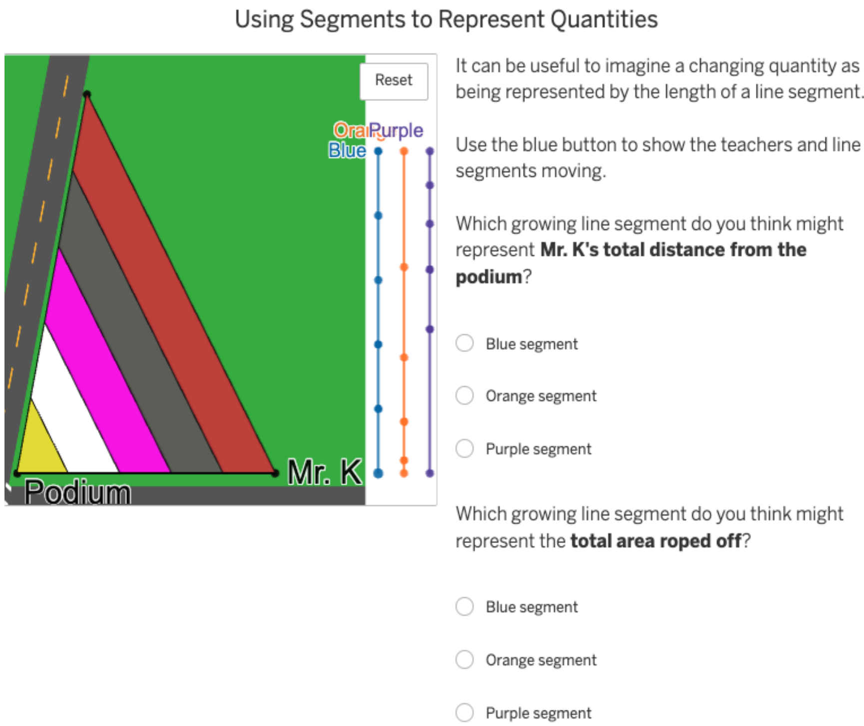
[5] We note we hid Mrs. B's name on the top point of the triangle in Figure 4 after the first screen to minimize the students using Mrs. B's distance from the podium (or location) in their activity.



Our report focuses on Kennedy's activity with two prompts in the *Growing Triangle Task* that we designed to support her in bridging situational and graphical meanings about two quantities (i.e., intending to promote M.S.1↔M.R.1). Prompt 1 (Figure 5) presented Kennedy with three dynamic magnitude bars (blue, orange, and purple) alongside the growing shape. For each increase of Mr. K's distance from the podium of 10 feet, each magnitude bar (hereafter bar) increased while leaving a dot at the prior endpoints of each bar. The blue, orange, and purple bars grew by equal, increasing, and decreasing amounts, respectively, which is reflected in the difference in spacing of the dots along each bar. Similar to the *Which One Task* (Liang & Moore, 2021), the text in Prompt 1 asked Kennedy to select bars that could represent situational quantities (in this case, Mr. K's distance from the podium [blue bar] and the shape's area [orange bar]). Also similar to Liang and Moore's (2021) task, we included a bar that did not represent any situational quantity (purple bar). We conjectured that this bar could support Kennedy in differentiating between various types of change in a bar and related quantity.

Figure 5

A screenshot of Prompt 1, which asks students to connect situational quantities to magnitude bar representations.



Through Prompt 2 we hoped to support Kennedy in bridging her reasoning about two quantities in the situation to represent the two quantities graphically (i.e., intending to promote M.S.2↔M.R.2 and potentially M.S.3↔M.R.3). The screen presented Kennedy with a blank coordinate plane and axes labeled with each focal quantity. The text prompted Kennedy to represent a point in the coordinate system and indicate the direction in which the point would move as the quantities varied.

In the results, we describe how Kennedy's responses to these two prompts and subsequent tasks led us to design a new task, the *Four Shapes Task*, to support her in more explicitly considering how to represent a growing area with a corresponding growing magnitude bar length (M.S.1↔M.R.1). We also highlight how this new task supported Kennedy in reasoning about points as representing the simultaneity of the magnitudes of two changing bars on axes (M.R.3) and working towards EGST (M.E.0).

4. Results

In this section, we describe how we used Paoletti et al.'s (2022) framework to (a) analyze a student's developing EGST, (b) diagnose complexities in the student's developing EGST, and (c) design tasks that provide the student opportunities to resolve such complexities. Although we use a particular student's activity throughout the results, we intend for our presentation to support readers in understanding how to use the framework in their own research, teaching, and curriculum design.

In the results, we first present Kennedy's activity as she addressed the *Growing Triangle Task*. We analyze her reasoning and diagnose complexities in her developing EGST. In particular, she provided us with contraindications that she was bridging her situational and graphical meanings in ways consistent with our intentions (~M.S.↔M.R.). We then describe the *Four Shapes Task,* which we designed to potentially support her in resolving these complexities by providing her opportunities to conceive of varying bars as representing a shape's changing area (M.S.↔M.R.). We provide evidence of Kennedy bridging situational and graphical meanings as she considered how the bars' magnitudes could represent different growing areas. We conclude by highlighting a shift in Kennedy's meanings for graphing.

4.1 Indications of constructing situational quantities (M.S.) with contraindications of representing quantities with graphical objects (~M.R.): Results from Sessions 4-7

In this section, we highlight how we used Paoletti et al.'s (2023) framework to analyze Kennedy's graphing meanings and to diagnose a complexity in her meanings. Across Sessions 4-7, Kennedy provided evidence of constructing several situational quantities in-the-moment (M.S.). However, at times, the precise quantity Kennedy reasoned with was ambiguous to us as researchers (and potentially to herself as well). Therefore, she provided contraindications of conceiving of bar lengths as representing the triangle's area (~M.R.); without such meanings, Kennedy's primary action for graphing points in the plane required moving "over-then-up" (Frank, 2016). We exemplify these claims in the sections that follow.

4.1.1 Indications of constructing situational quantities (M.S.) and contraindications of representing quantities with graphical objects (~M.R.) in Session 4

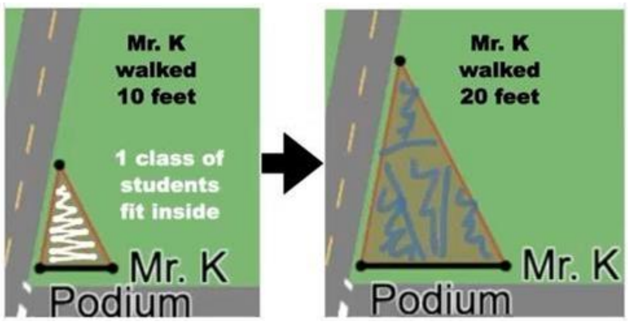
At the beginning of Session 4, the TR introduced the *Growing Triangle Task.* Immediately, Kennedy described that both Mr. K.'s distance from the podium and area were increasing quantities in the situation. Given that one class fit within the small triangle formed when Mr. K walked 10 feet (seen in Figure 6, left), Kennedy drew four triangles (Figure 6, right) to conclude that after Mr. K walked 20 feet, four classes could fit within the resulting area. Additionally, when prompted by a screen to describe "For each jump, how is the added area changing?" and given three options (the amount of change of area is [increasing, decreasing, staying constant] with each jump) Kennedy argued,

> I feel like the amount that he's jumping is um, increasing for each jump. Because as you can see for each jump [*points to yellow triangle in Figure 5*], it gets like bigger. Like the rectangles or whatever [*pointing sequentially from the white trapezoid to the red trapezoid in Figure 5*] gets bigger as he jumps.

Using Paoletti et al.'s (2023) framework, we infer that, in-the-moment, Kennedy had constructed each of Mr. K's distance and area of the triangle (measured in number of classes) as changing quantities in the situation (M.S.1). Further, she conceived that the area increased by an increasing AoC for equal changes in base length (M.S.2 at the stage of directional covariation).

Figure 6

Kennedy's drawing representing four classes fitting in the triangle's area.



However, when we tasked Kennedy with choosing a bar that represented each quantity, she provided a contraindication of bridging her situational meanings to this representation in ways consistent with our intentions (~M.S.1↔M.R.1). Initially, Kennedy guessed the blue bar (growing by equal amounts) and purple bar (growing by less) represented Mr. K's distance from the podium and total area roped off, respectively. She justified her answer by focusing on the timing of the changes of these bars (i.e., she argued that changes to the length of the purple bar and jumps in the area occurred simultaneously). In response to this, the TR asked Kennedy to observe the orange bar and she noted that all three bars changed simultaneously. On the next slide, only showing the bars, Kennedy correctly described that the blue bar grew by equal amounts, the orange bar grew by larger amounts, and the purple bar grew by smaller amounts (M.R.1 with the magnitude bars' lengths as quantities). However, when given a chance to revise her choices for which bar represented which situational quantity, she maintained her original choices, indicating the purple bar, which she described as increasing by a decreasing amount, represented the triangle's area, which she described as increasing by an increasing amount. Hence, we take this as a contraindication of her bridging her meaning for the situational quantity and graphical representation (~M.S.1↔M.R.1).

4.1.2 Kennedy's bridging of situational and graphical meanings but at times in ways unintended by the researcher (M.S.↔M.R.) in Session 4

Over the next five minutes, the TR asked Kennedy questions to shift their discussion from the timing of the changes to how the bars could represent situational quantities. He asked Kennedy to re-describe how the number of classes that fit in the roped-off triangular area changed as the triangle's base length grew by 10-foot increments. In the moment, Kennedy accurately determined total area values (measured in number of classes) at specific values of Mr. K's distance (M.S.2). She first identified that if Mr. K.'s distance was 10 feet, then for the consecutive changes his distance would be 20, 30, 40, and 50 feet. The TR then discussed the blue bar (changing by equal amounts). He indicated that prior to moving, the blue bar represented 0. After hitting play, Kennedy watched the blue bar increase five times, with her verbally saying, "10, 20, 30, 40, 50." In the moment, the TR inferred that Kennedy understood the blue bar represented Mr. K's distance from the podium (M.S.1↔M.R.1).

The TR then turned the conversation with Kennedy to the growing area. He asked Kennedy to again describe how the yellow and white areas in Figure 5 together had a measure of 4 classes. When prompted, Kennedy indicated that 5 classes would fit in the pink area, meaning that 9 classes would fit in the combined areas of the yellow, white, and pink shapes (M.S.1). Connecting this reasoning to the bars, Kennedy provided indications of bridging her meaning for increasing areas to the bars in a way unexpected by the TR (but unnoticed in the moment). Immediately after Kennedy concluded the area of the yellow, white, and pink areas combined was 9, the TR asked her which bar would represent the relationship. Kennedy indicated the orange bar (which increased by increasing amounts) represented area, giving the following justification:

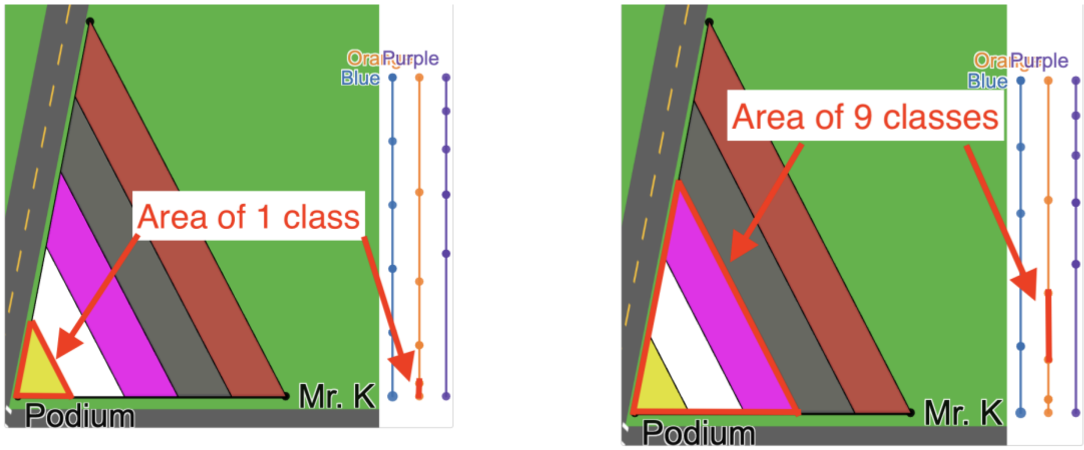
> This starts off as one [*pointing to yellow triangle*] so then this is small [*motioning over the first interval in the orange bar, highlighted in red Figure 7a*]. And then this adds up

> to like four [*motioning over the yellow and white areas*] then this gets bigger [*motioning over the second interval in the orange bar*]. Then all together it adds up as nine [*motioning over yellow, white, and pink areas*] so then this gets bigger [*motions over the third interval in the orange bar, highlighted in red Figure 7b*].

Based on her activity here and later, we identify a complexity (from our perspective) in Kennedy's bridging of her situational and graphical meanings. Kennedy correctly indicated the orange bar reflected the increasing by more relationship she described for the triangle's area. However, Kennedy's reasoning focused on each consecutive interval in the orange bar as representing a new total area rather than an AoC of area. She understood the first orange interval to represent an area of 1 class (i.e., Figure 7a), the second interval to represent an area of 4 classes, and the third interval to represent an area of 9 classes (i.e., Figure 7b). Across her activity, Kennedy described the two changing quantities (Mr. K.'s distance and triangle's area) in terms of numeric values (M.S.2), and we infer that she bridged her situational and graphical meanings to describe a magnitude bar that represented each quantity (M.S.2↔M.R.2). However, her rationale for selecting the orange bar relied on using intervals of the bar to represent different total areas, which was inconsistent with researcher intentions.

Figure 7

A representation of Kennedy's bridging situational total area to a(n unintended) graphical representation of total area when Mr. K has moved (a) 10 feet and (b) 30 feet.



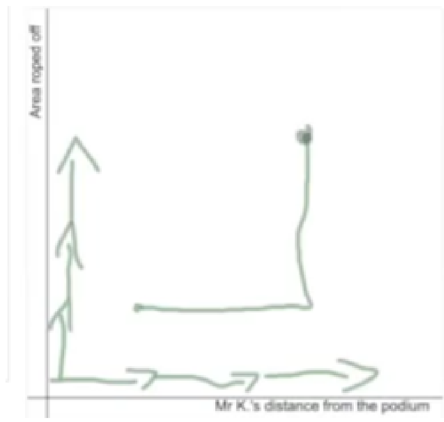
(a) (b)

4.1.3 Kennedy's graphing activity providing contraindications of bridging situational and graphing meanings (~M.S.↔M.R.) at the end of Session 4

In the interactions above from Prompt 1 in Session 4, Kennedy provided evidence of reasoning about two quantities both situationally and representationally, albeit in ways unintended by the TR. However, immediately after this in Session 4, she provided contraindications of conceiving of a point as multiplicative object. A follow-up screen, Prompt 2, asked Kennedy to draw an arrow to show how a point representing Mr. K's distance and the area would move. Kennedy first drew three arrows going toward the right on the horizontal axis and three arrows up the vertical axis (Figure 8). We infer Kennedy was showing directional changes in both quantities (M.S.2↔M.R.2 at the stage of directional covariation). To explore how Kennedy would reason about the movement of a point representing the two quantities, the TR then drew a point on the screen (shown in Figure 8) and asked how this point would move. As a contraindication of simultaneously considering the two changing quantities in the graph (~M.R.3), Kennedy said the point will go "over [to the right] and then straight up." When asked to represent her thinking, Kennedy drew the horizontal line from the point then vertical line from the end of her horizontal line shown in the graph in Figure 8. We take this activity as an indication she was reasoning sequentially, rather than simultaneously about the changes in the bars (and possibly the quantities). Hence, the point did not represent the lengths of simultaneously changing bars on the axes (~M.R.3). Further, if Kennedy was reasoning about the quantities sequentially, she would also be providing a contraindication of having an operative image of covariation of the two quantities (~M.S.3).

Figure 8

The graph Kennedy constructed when asked to represent both quantities changing.



Across Session 4, we note a complexity in Kennedy's graphing activity. Namely, Kennedy initially focused on the timing of the bar movements rather than on the quantities that bars could represent. Kennedy eventually made arguments regarding increasing changes in area and increasing changes in the orange bar's length but did so in ways inconsistent with our intentions. Further, her activity with the bars did not support her point-plotting activity as she focused mainly on directional changes and over-and-up activity.

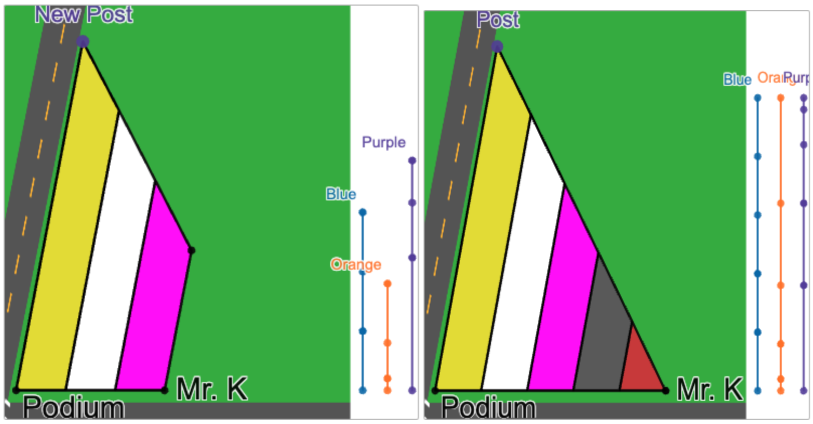
4.1.4 Further contraindications of bridging situational and graphing meanings (~M.S.↔M.R.) in Sessions 6-7

Across Sessions 6-7, Kennedy addressed the *Growing Trapezoid Task* (see Figure 9). We intended to provide additional opportunities for Kennedy to bridge her situational and graphical meanings in ways compatible with Paoletti et al.'s (2023) framework. However, and consistent with her activity in Session 4, Kennedy often experienced complexities as we asked her to connect a growing shape's area with the magnitude bar representation of area (~M.S.1↔M.R.1). Further, and also consistent with her activity in Session 4, Kennedy consistently relied on an 'over-and-up' strategy for marking points in a graph. We conjectured her 'over-and-up' activity was at least partially explainable by her complexities bridging M.S.1 and M.R.1; if Kennedy did not understand how a bar on the vertical axis represented area, it would not be possible for her to understand a point as a multiplicative object that represented an amount of area and base length simultaneously.

Reflecting on these two issues with Paoletti et al.'s (2023) framework in mind, we conjectured that having additional opportunities to represent total area with bars would support Kennedy in constructing area as a quantity and bridging her meanings for area with the bar representation (M.S.1↔M.R.1). Of note regarding our design decisions for the new task for Session 8: In Session 6, Kennedy described each bar are relating to *some* situational quantity when initially asked which bar represented which quantity (Figure 9). For instance, Kennedy correctly connected the blue bar to Mr. K's distance from the podium and the purple bar to the total area. However, she also described that the orange bar represented the distance along the top of the trapezoid (measured from the post). Although the TR did not ask Kennedy to describe her reasoning for the orange bar, we infer she wanted each bar to represent a situational quantity; we have observed similar reasoning by other middle school students in other tasks. Hence, when designing the *Four Shapes Task*, we opted to focus strictly on one quantity (area) using four distinct growing areas resulting in the same shape and four different bars. As such, we hoped for Kennedy to focus on the non-numeric intensity of change represented by both the different areas and bars (Johnson, 2012).

Figure 9

Two screenshots of the *Growing Trapezoid Task*.



4.2 The *Four Shapes Task* and shifts in Kennedy's graphing meanings: Session 8

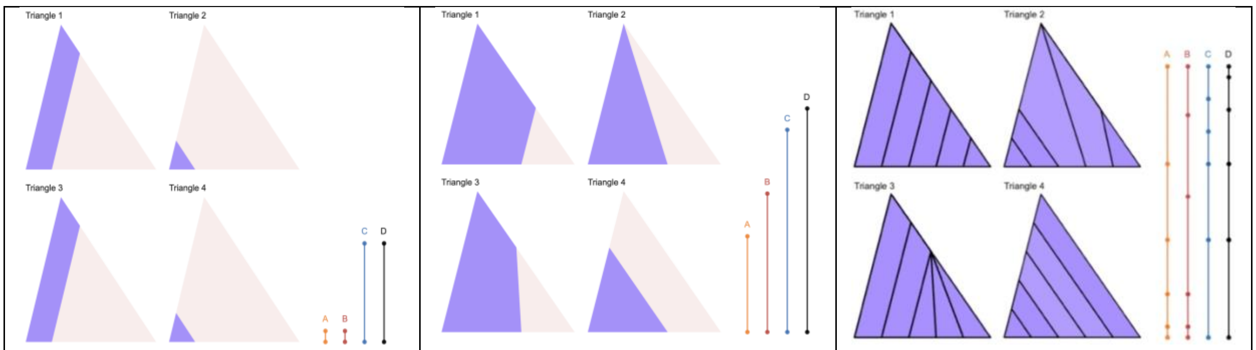
Throughout Sessions 4-7, we had constructed and tested models of Kennedy's mathematics that supported us in diagnosing complexities we observed in her graphing activity. Given our goal to support her developing more sophisticated graphing meanings, we worked to design a task that provided her opportunities to resolve such complexities. Reflecting on Kennedy's meanings in relation to the Paoletti et al. (2023) framework, we designed the four-part *Four Shapes Task*[6] with the goal of supporting her bridging situational and graphical meanings. In Part 1, we provided Kennedy with four different growing shapes that all result in the same final triangle. To encourage her bridging situational meanings for growing areas with graphical representation of areas (i.e., intending to support M.S.1↔M.R.1), we asked her to determine which growing bar corresponded to each shape's changing area (Figure 10). We conjectured that supporting Kennedy to focus on one quantity (area) without dummy segments across a set of several similar growing objects could provide her with an opportunity to consider ways distinct bars could represent different growing areas. Further, as Kennedy was still in the

[6] The Four Shapes Task that Kennedy engaged with can be found here: https://tinyurl.com/ypppr9t9

process of developing stable meanings for how a quantity can be represented via a bar, we conjectured that the removal of dummy segments could be productive in minimizing attempts to make sense of each bar (including dummy bars) in terms of some situational quantity.

Figure 10

(a)-(b) Two screenshots of the Four Shapes Task screen with the shapes and magnitude bars and (c) a screenshot showing five equal changes of base length and the corresponding changes in area for all four shapes.



(a)	(b)	(c)

In Part 2, we asked Kennedy to consider how the quantities of area and base length could be represented with magnitude bars along axes for each of Shapes 1-4 (intending to support M.S.2↔M.R.2). This occurred over several screens in which we removed supports as she progressed through each shape: For Shape 1, we provided the segments along the axes (Figure 12); for Shape 2, Kennedy had to space points along the vertical axis (Figure 13); for Shape 3, Kennedy had to space points along both axes (Figure 14a-b); for Shape 4, Kennedy had to draw all points along each axis (Figure 14c).

Part 3 had a similar structure to Part 2, but we prompted Kennedy to consider points in the plane corresponding to the endpoints of the magnitude bars on each axis (intending to support M.R.3 and possibly M.S.3↔M.R.3; see Figure 15). Finally, Part 4 involved tracing graphs that represented the relationship between area and base length for each shape (intending to support M.E.0 and beyond). In the sections that follow, we provide more detail on Kennedy's activity addressing Parts 1-3 and show how our design was productive in supporting Kennedy's progression towards EGST.

4.2.1 Four Shapes Task - Part 1: Kennedy's activity with magnitude bars

After observing the applet play in Part 1, Kennedy compared the speed at which the different areas filled the shape non-numerically. She noted that Triangles 2 and 4 grew slower than Triangles 1 and 3. While watching the animation a second time, Kennedy noted that Triangle 4 grew the slowest, followed by Triangle 2, then Triangle 3. She confirmed this as she dragged the slider for her third viewing of the growing shapes. Overall, Kennedy provided evidence that she conceived of a growing area for each shape as she used her perceived speed of growth to compare the different growing areas (M.S.1). Such reasoning is compatible with reasoning about the non-numeric intensity of change of area (Johnson, 2012).

The next slide showed the four shapes growing with equal changes in base length and four bars (see Figure 10c for outlines of the five AoC of area and corresponding changes in bar lengths). Each bar (A)-(D) corresponded to exactly one of the total areas of the shapes (Triangle 1, bar D; Triangle 2, bar B; Triangle 3, bar C; Triangle 4, bar A). During this extended interaction, Kennedy experienced a shift in her meanings for the bar lengths as representing area (M.S.1↔M.R.1) that persisted throughout the remainder of the teaching experiment.

With the applet paused after the first increase in base length (Figure 10a), the TR explained that each bar (which he and Kennedy referred to as "segments") represented a different shape's growing total area. He then advanced the animation and paused it after the third increase in base length (Figure 10b; note that the AoCs of area were not outlined). In less than 10 seconds, Kennedy correctly matched each bar to its corresponding shape. She indicated she was attending to:

> the length of this [*motioning over the bars*], the length of the segments and the length of the purple whatever [*waving her hand over the 4 shapes*], the jumps…yeah like the jumps: The shaded-in parts [*motioning the mouse over the purple area in Triangle 1*].

We infer that after viewing four different bar lengths, Kennedy connected these lengths to the four growing shapes' areas ("shaded-in parts"; M.S.1↔M.R.1).

As additional evidence of Kennedy's bridging activity, immediately after the above interaction, the TR had Kennedy go to the next slide, observe the animation, and hit the "Show additional colors" button, resulting in the screen shown in Figure 11. After briefly recapping which bar Kennedy had said represented each growing shape's area, the following interaction ensued:

TR: And how are you deciding that?

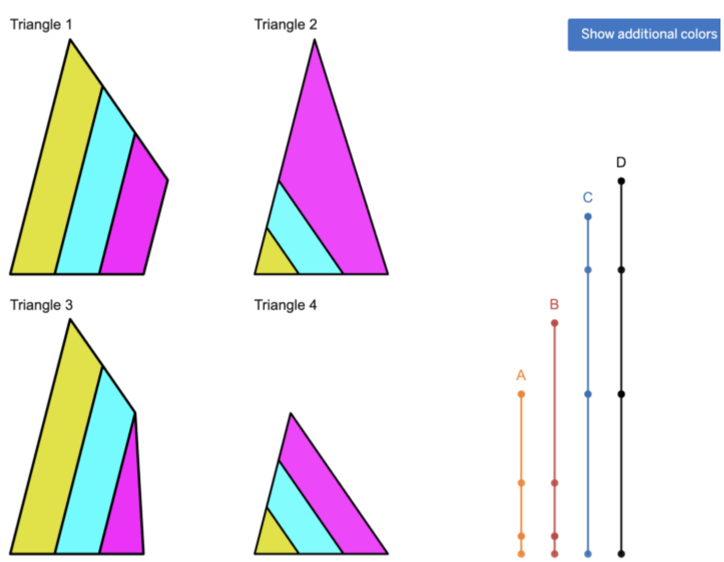
Kennedy: Like how this is small, medium, big I guess [*motioning over yellow, blue, and pink areas in Triangle 4 in Figure 11 as she says each size*]. And then [*moving mouse to Triangle 2*] small, medium, bigger [*motioning over yellow, blue, and pink areas in Triangle 2 as she says each size*]. So then this is small, medium, big I guess [*motioning over the consecutive intervals of bar A*]. And then this is small, medium, and then bigger [*motioning over the consecutive intervals of bar B*].

Immediately after this, Kennedy continued to use similar reasoning as she described how she matched the other two segments with growing shape's area (e.g., arguing the pink area in Triangle 1 is "medium" compared to the pink area in Triangle 3 which is "small").

Throughout this interaction, Kennedy provided numerous instances in which she explicitly connected the AoC of area of each shape to the bar representation. Kennedy repeatedly described the relative AoC of area (e.g., “small”, “medium”, “big”, or “bigger”) and then described how the corresponding bar showed a similar AoC of length. Hence, we infer Kennedy was explicitly connecting the quantity of area to the bar representation (M.S.1↔M.R.1) in ways compatible with our intentions (unlike her activity pictured in Figure 7). Further, we highlight that the task design provided her repeated opportunities to engage in such reasoning as she connected each bar to a unique area.

Figure 11

A screenshot showing the growing shapes after the third increase in base length along with the corresponding magnitude bars.



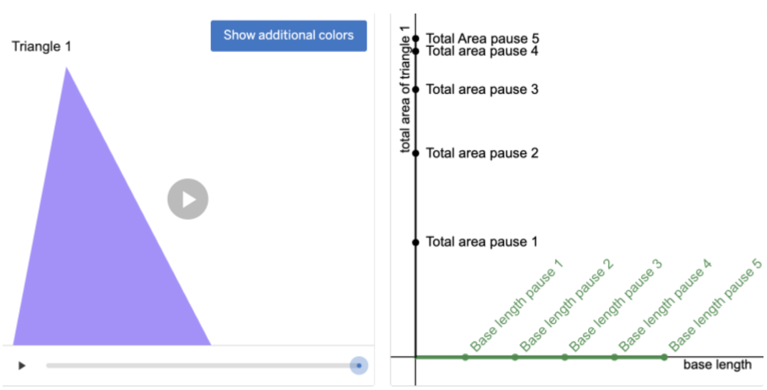
Related to the design of this task and our prior use of dummy bars, we note that immediately after the interaction above, the TR asked, “Is this more helpful than some of the ones we were showing before?” Kennedy responded, “Yeah, I think… ‘cause with the other one it was just one and we had to figure out which one was corresponded to that specific one.” We infer that Kennedy was conveying that in the prior tasks we were asking her about one quantity and tasking her to match which of three magnitude bars represented that quantity (i.e., "just one [quantity] and we had to figure out which one [bar] was corresponded to that specific one [quantity]."). Hence, Kennedy specifically mentioned the affordance of not having dummy bars in this task.

4.2.2 Four Shapes Task - Part 2: Kennedy's activity with quantities represented on axes

Over the next several minutes, Kennedy observed additional screens like the one in Figure 11 but which also included the last two jumps in different colors as well as corresponding colors on each bar. However, when moving to the graphical representations, Kennedy's meanings for bars did not translate to her graphing activity, which is consistent with Sessions 4-7. Instead, when first observing an animation showing a graph, which we intended to represent the base length and total area of Triangle 1 (Figure 12), Kennedy conjectured the bar on the vertical axis was a length in the situation. This provided a contraindication of her bridging her meanings for the shape's total area and the representation of total area via a bar (~M.S.1↔M.R.1) in this new task.

Figure 12

A screenshot of the first slide showing bars on axes in the *Four Shapes Task*.



In the moment, the TR conjectured numeric quantitative reasoning could support Kennedy in connecting her prior bridging of the bars and quantities on the prior slide to her activity in a coordinate system (intending to support M.S.1↔M.R.1). He asked Kennedy to consider specific numeric values of total area and base length in this case. He paused the applet after the first and second increases in base length, describing that the area after each of those 5-unit base length increases was 9 and 16 units respectively. He asked Kennedy what the area would be if the base length increased by 5 more units and the corresponding area increased by 5 units. She quickly responded the new area would be 21 and the corresponding base would be 15 units long. After this, the TR had Kennedy finish the animation to create the full shape and asked:

TR: So one thing I want you to notice is that these points seem to get closer and closer each time [*motioning over the points on the vertical axis*].

Kennedy: Yes

TR: Why is that?

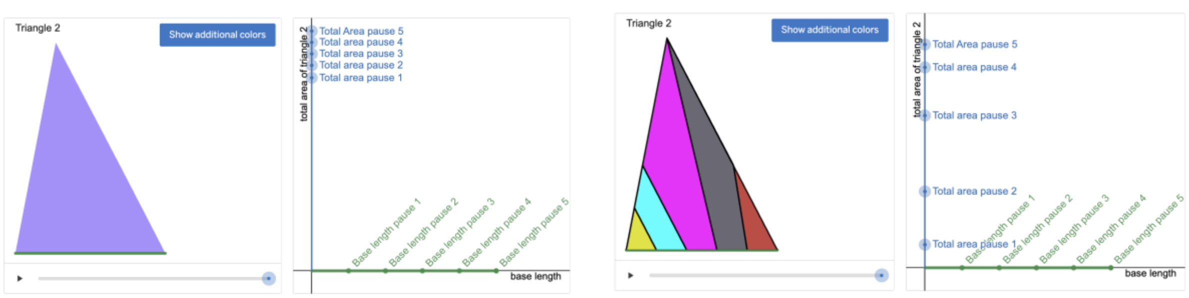
Kennedy: 'cause the colors, they start getting smaller and smaller [*she clicks "Show additional colors" showing an image like Figure 11*]. Like with the colors it starts to get big, small. So it starts off big then gets small, small, smaller, until it's at the smallest [*motioning over consecutive areas in the triangle*] ... And then here it starts off big, bigger, small, smaller, until it's at its smallest [*motioning over consecutive magnitude bar intervals*].

Here, we see evidence Kennedy is beginning to conceive of the bar on the vertical axis as representing total area in ways consistent with researcher intentions (M.S.1↔M.R.1). Further, we note that although Kennedy initially used numbers for area in her reasoning, she shifted to non-numeric reasoning as she connected the areas of the consecutive AoC in the shape to the changes in the bar length. Indicative of a shift in Kennedy's meanings, she consistently described the bar on the vertical axis as representing a shape's growing area after this interaction.

To highlight this shift in Kennedy's meanings and the non-numeric nature of her reasoning about area, we draw on data from the next screen. In this applet, the TR described the new graph, intended to be "broken" as all the points for Triangle 2's area were grouped near the top of the vertical axis (Figure 13a shows this applet at the outset of her activity). The TR asked Kennedy to move the points to match the growing area in Triangle 2. Kennedy turned on the additional colors (shown in Figure 13b). She then spaced the points as shown describing the intervals she created as "small," "bigger," "bigger," "a little…smaller," and "then smaller." The following conversation ensued:

Figure 13

(a) The starting applet and (b) Kennedy's correct spacing of total area points for Triangle 2.



(a) (b)

TR:	Cool. So explain to me how you did that.
Kennedy:	So, I did it because. How I did it was, I looked at the colors, then how here [*points to the yellow area*] the first jump, it starts off small, so put {inaudible} to small to start [*pointing to Total Area Pause 1 point*]. And then this one [*pointing to the teal area*] is medium so I did it [*pointing to Total Area Pause*

2 point] little bit bigger, then this one [*gestures to the Total Area Pause 3 point, then to the pink area, then back to Total Area Pause 3 point*] I did bigger.

[*after a brief aside, Kennedy continues to talk about how she spaced the points on the vertical axis*].

Kennedy: And then I made this space [*motioning over the space between third and fourth Total Area Pause points*] because the area here [*motioning over the gray area*].

TR: The gray?

Kennedy: Yeah, the gray is smaller [*motioning from the space between Total Area Pause 3 and 4 points then to the gray area in the triangle*]. So yeah and then I made this one smaller [*motions over the interval between Total Area Pause points 4 and 5*] because here, red is smaller [*pointing to the red area*].

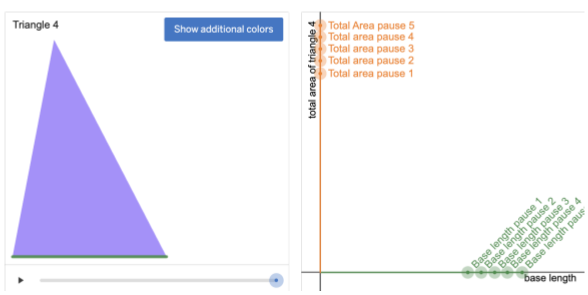
Throughout this activity, Kennedy provides clear and sustained evidence that she consistently bridged her meanings for how AoC of area (represented via different colors) in the situation would be represented by AoC in the bar's magnitude (represented via the space between the "Total Area Pause #" points) along the vertical axis (M.S.1↔M.R.1).

We note two nuances relative to Kennedy's reasoning above in relation to the design of the screen. First, we are unsure the extent to which Kennedy was simultaneously attending to changes in base length as she bridged her situational and graphical meanings for area; if she was also attending to base length, we would infer she was reasoning about covarying quantities in the situation and graph (M.S.2↔M.R.2 at the AoC stage). Second, the extent to which Kennedy was explicitly reasoning about each consecutive non-numeric area as an AoC of area whose accumulation built a total area is vague. This vagueness is due to her using the static screen shown in Figure 13b when describing the connections she made across the situation and the graph. It is possible she was reasoning about each colored area as its own ‘total’ area with the jumps in the segment representing these independent total areas. Regardless, she provided evidence that she understood each change (or “space”) in the bar’s magnitude corresponding to a change in (or particular total) area (e.g., understanding the length between Total Area Pause points 4 and 5 as representing only the pink area). This contrasts with her activity in Figure 7 where she conceived of each change in the bar representing the shape’s total area (e.g., hypothetically conceiving the length between Total Area Pause points 1 and 2 as representing the yellow and blue areas together).

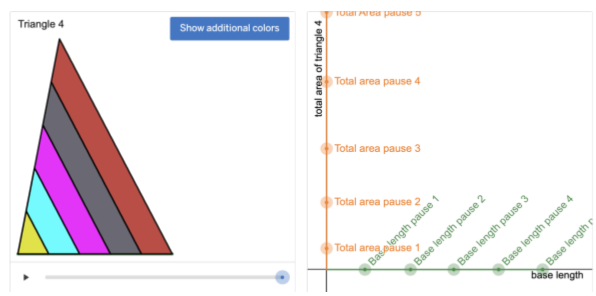
After developing meanings for representing total area with a bar along the vertical axis, Kennedy provided clear evidence that she could coordinate both the total area and base length bars simultaneously while connecting the bars to situational quantities (M.S.2↔M.R.2) in the next few screens. In one screen, Kennedy was tasked with spacing points along both the vertical and horizontal axes for Triangle 4 (Figure 14a). Kennedy immediately moved the base length points along the horizontal axis as shown in Figure 14b. When asked why she placed the points like this, she responded, “because like right here [*gesturing over the base length in the triangle*] you can see the jumps are equal.” (M.S.1↔M.R.1 with respect to base length). She then placed the points along the vertical axis as shown in Figure 14b. When asked what she was trying to represent with her spacing, she indicated “the jumps…how they start off small then get bigger,” indicating that she intended for each consecutive interval to increase (M.S.1↔M.R.1 with respect to area; collectively reflecting M.S.2↔M.R.2 at the AoC stage). On the next screen, Kennedy independently plotted points on each axis while providing explicit attention to representing two situational quantities via bars along two axes (Figure 14c). Hence, Kennedy was bridging her situational meanings for covarying quantities to graphical representations of these quantities (M.S.2↔M.R.2). Further, based on the ease with which she addressed both tasks, we conjecture such reasoning was part of her stable meanings for graphs.

Figure 14

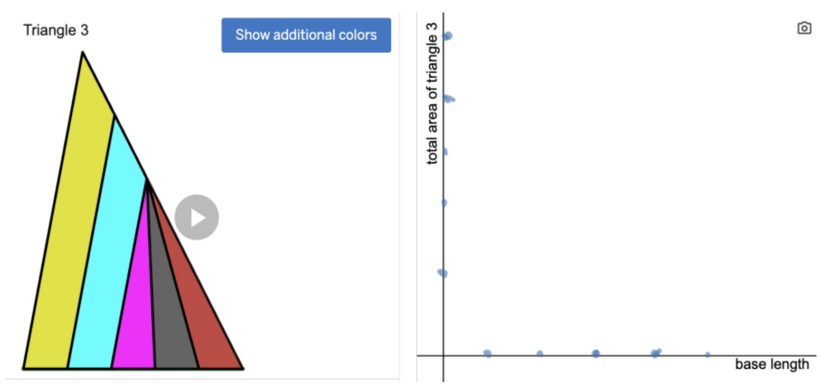
(a) The starting applet, (b) Kennedy's spacing of total area points for Triangle 4, and (c) a screenshot of the points Kennedy drew on the axes to represent the quantities for Triangle 3.



(a)



(b)



(c)

4.2.3 *Four Shapes Task* – Part 3: Points as multiplicative objects.

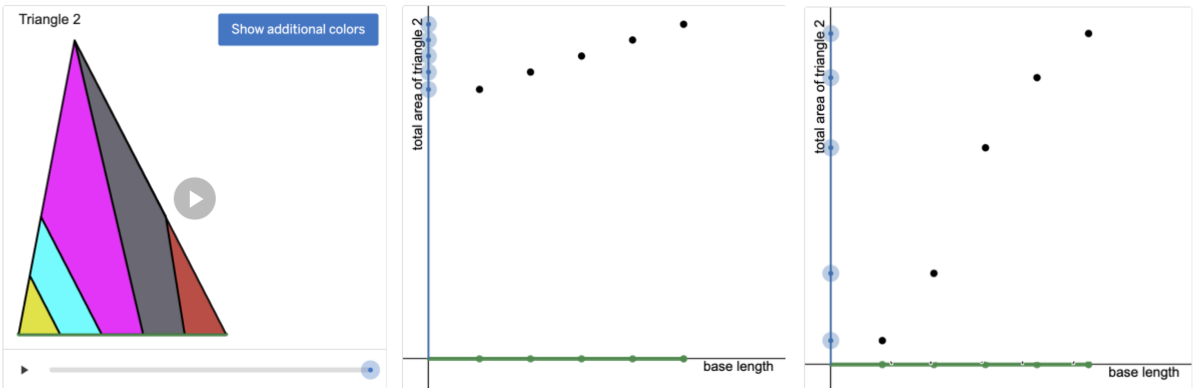
The next four screens had the same functionality as those in Figures 12, 13a, and 14a, but with prompts asking about points in the coordinate plane corresponding to the endpoints of the bars on each axis. For the applet like Figure 12 but with an added point in the coordinate plane, the TR described how the point's location was dictated by the endpoints of each bar on the axes.

The TR then tasked Kennedy to address the next applet (Figure 15a), which had the same prompt as Figure 13a but with additional points in the graph visible. Kennedy initially addressed this task without noticing the new points, moving the endpoints of the bars on the vertical axis like she did in Figure 13b. The TR then prompted Kennedy to look at the points in space as she dragged the endpoints on the vertical axis. Kennedy began to drag the third point on the vertical axis up and down, noticing, “Oh the point! It’s moving.” We infer that the task design did not support Kennedy in spontaneously noticing the movement of the points in Figure 15a, but once prompted she was able to identify the connection between the points and endpoints of the bars on the vertical axis.

The TR then presented Kennedy with an applet like the one in Figure 14a but with added points in the coordinate plane. Kennedy first adjusted the points on the vertical axis, then the points on the horizontal axis. As she dragged the points on the horizontal axis the TR asked, “As you’re dragging those, do you notice what’s happening to the points [in the coordinate plane]?” Kennedy responded, “Oh, they’re moving. Hmm.” In both cases, Kennedy did not spontaneously attend to the location of the points in space, but, when prompted, observed that the points’ locations were dictated by the bar lengths on each axis. Such activity again provided Kennedy opportunities to develop in-the-moment meanings for points as multiplicative objects (M.R.3).

Figure 15

A screenshot of the applet (a) before and (b) after Kennedy adjusted the points.

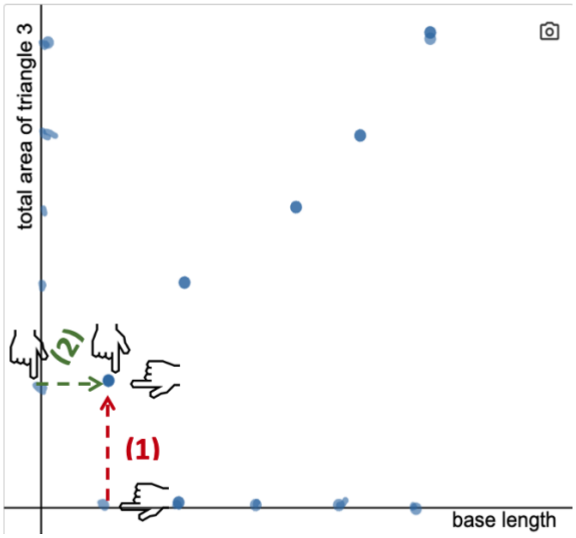


(a) (b)

Kennedy’s activity addressing the last screen provided some evidence she was developing stable meanings for points as representing multiplicative objects (M.R.3). The screen showed Triangle 3 with the points she plotted on the axes in Figure 14c. When the TR asked Kennedy to plot points representing both quantities, she marked the points shown in Figure 16. Describing how she plotted the first point, Kennedy motioned up from the point on the horizontal axis (shown in (1) in Figure 16) and over-to-the-right from the point on the vertical axis (shown in (2) in Figure 16), indicating she intended the first point to align with the two segments on the axes. Although the session ran out of time before the TR had further opportunity to probe Kennedy’s thinking, we note each point in Figure 16 aligned relatively closely with the points she had marked on the horizontal and vertical axes (M.R.3).

Figure 16

A screenshot of Kennedy’s points with annotations of her motions.



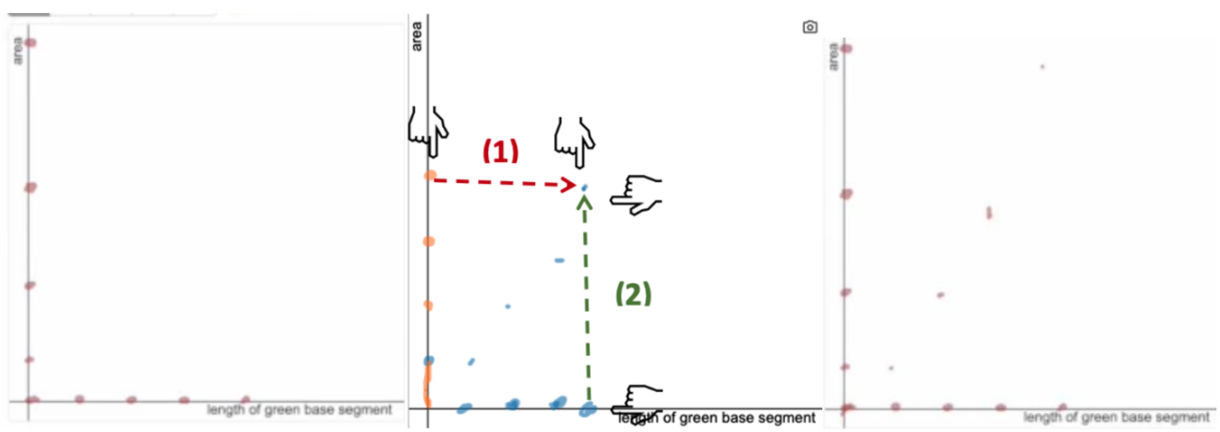
4.4 Triangle/Rectangle Task: Additional Evidence of Kennedy's Stable Meanings for Points as Multiplicative Objects

During Session 10, Kennedy addressed the *Triangle/Rectangle Task*, which entailed two growing shapes (a triangle and rectangle) with equal varying base lengths but different areas. The rectangle grew in one dimension (its width increasing and height remaining constant) while the triangle grew in the same way as Triangle 4 in the *Four Shapes Task*. Kennedy provided initial evidence of the growing stability of her meanings for points as multiplicative objects representing quantities as she addressed this task (M.S.3↔M.R.3). In particular, she described the total area of the growing rectangle and triangle to be quantities that varied as each shape's base length increased (M.S.2 at the stage of directional covariation). Further, she provided additional characterizations of the AoC of area for each shape (i.e., the triangle's area increased by increasing amounts, while the rectangle's area increased by a constant amount; M.S.2 at the AoC stage).

She attended to the situational quantities as she plotted points on the horizontal and vertical axes to represent the quantities' magnitudes. For example, when plotting points to represent the area and base length for the triangle, Kennedy plotted the points on the axes shown in Figure 17a, indicating the points on the vertical axis should get "bigger and bigger" and the points on the horizontal axis would be equally spaced. Kennedy described the spacing of her points in terms of the area and base length of the growing triangle (M.S.2↔M.R.2 at the AoC stage). Similarly, Kennedy plotted the points on the axes shown in Figure 17b to show how the base length and area of the rectangle each increased by equal amounts (M.S.2↔M.R.2 at the AoC stage).

Figure 17

Screenshots of Kennedy's work plotting points (a) on axes representing the area and base length of the growing triangle, (b) in the coordinate plane and on axes representing the area and base length of the growing rectangle with annotations showing her movement, and (c) adding points to her graph in (a) to represent the area and base length of the triangle simultaneously.



(a) (b) (c)

When asked to plot points in the coordinate plane, Kennedy provided clear evidence of understanding points as multiplicative objects (M.R.3). She first drew the points seen in Figure 17b by moving along the horizontal axis to each consecutive point, then moving up. Although this activity is similar to her over-and-up motions from Sessions 4-7, when asked to describe how she located her points, she explicitly described attending to the endpoints of the bars on each axis. For example, when asked how she plotted the top-right-most point, she immediately pointed to the highest point on the vertical axis then moved to the right to the point (indicated by (1) in Figure 17b), saying, “so I went over,” then continued her motions by moving to the right-most point on the horizontal axis and moving up to the point (indicated by (2) in Figure 17b; M.R.3). Similarly, when returning to the triangle situation and asked to plot points, Kennedy’s activity was consistent with understanding points as multiplicative objects (Figure 17c).

We note that Kennedy’s activity plotting points in space did not include specific references to quantities, so we are unable to make claims about whether Kennedy bridged her situational meanings and achieved M.E.0 (the prerequisite for EGST). Regardless, we note substantial progress towards EGST across Session 8 and 10.

7. Discussion

In this report, we used Paoletti et al.’s (2023) framework to (a) analyze a student's developing EGST, (b) diagnose complexities in a student's developing EGST, and (c) design tasks that provide opportunities to resolve such complexities. We discuss how we used the framework to achieve (a)-(c) in the paragraphs that follow.

We provided numerous descriptions of the way Paoletti et al.’s (2023) framework allowed us to analyze Kennedy’s progression toward EGST. In Kennedy’s early activity in Session 4, she provided evidence that she could describe quantities changing in situations (M.S.) and in magnitude bar representations (M.R.) independently (e.g., indicating the area increased by increasing amounts, describing how each of the three magnitude bars grew). As the teaching experiment progressed, we were able to capture ways Kennedy’s meanings developed to bridge her situational and graphical meanings. Particularly, she conceived of bars as representing covarying situational quantities’ magnitudes (M.S.2↔M.R.2) and conceived of points as multiplicative objects representing two situational quantities' magnitudes simultaneously (M.S.3↔M.R.3).

We also showed one way that nuance can be added to the framework when analyzing a student's developing EGST. In particular, we highlighted two different ways students can reason about covarying quantities situationally (M.S.2) and graphically (M.R.2) by differentiating Kennedy's reasoning with respect to directional and amounts of change reasoning (Carlson et al., 2002). Although this distinction was implicit in the eighth-grade students' reasoning in Paoletti et al. (2023), in this report we provided ways to make this distinction explicit for future researchers and teachers.

With respect to diagnosing complexities in a student's developing graphing meanings, in analyzing Kennedy's activity with the Paoletti et al. (2023) framework in mind, we identified both anticipated and unanticipated complexities Kennedy experienced. For example, we were not surprised that bridging her meanings for quantities in the situation (M.S.) with her meanings for varying magnitude bars (M.R.) was non-trivial (i.e., instances of ~M.S.↔M.R.). This finding emphasizes the importance of providing students repeated opportunities to consider magnitude bars as disembedded representations of situational quantities' amounts (Lee et al., 2020).

Kennedy's activity around Figure 7 is an example of an unexpected complexity from our perspective. Whereas prior research (Paoletti, 2015; Paoletti et al., 2019; Moore & Carlson, 2012) has highlighted how students' conceptions of situational quantities can be different than researcher intentions, Kennedy's activity exhibits how a student's meanings for graphical representations of quantities can also differ from researcher intentions. In the moment, the TR did not notice Kennedy's novel way of using the intervals in magnitude bars to represent multiple total areas. He believed she was bridging her situational and graphical meanings in ways compatible with his intentions. It was only during retrospective analysis, as we identified indications and contraindications of reasoning aligned with Paoletti et al.'s (2023) framework, that we noticed such nuances in her meanings for graphical representations. We conjecture other students may experience similar complexities as they attempt to bridge their situational and graphical meanings, and we argue that analysis applying this framework can facilitate future inquiry around this topic.

Paoletti et al.'s (2023) framework also proved productive in providing us with insights into how to support Kennedy in resolving complexities in her graphing activity, notably with respect to task design. By explicitly distinguishing situational and graphical quantitative reasoning, the framework supported us as we designed the *Four Shapes Task*, which provided Kennedy with important opportunities to consider how to use the length of a magnitude bar to represent a (non-length) quantity (i.e., area). Other researchers have suggested initially providing students with "tasks involving 'simpler' attributes, such as height and distance" (Johnson et al., 2017, p. 863) as such quantities are likely easier for students to conceptualize with a magnitude bar. However, Kennedy's activity highlights how providing a sixth-grade student repeated and deliberate opportunities to consider how to represent non-length quantities via magnitude bars can be productive for students as they develop meanings for graphs as representing various types of quantities. In particular, the *Four Shapes Task* supported Kennedy as she disembedded area from the four shapes, projected these areas onto magnitude bars, and maintained a simultaneous awareness of the area within the situation (Lee & Hardison, 2016; Lee et al., 2020; Paoletti et al., 2023).

Moreover, as called for in Paoletti et al.'s (2023) framework, our results emphasize the importance of designing tasks in ways that might provide repeated opportunities for students to engage in situational and graphical reasoning to support them in developing stable meanings related to EGST. In addition to providing Kennedy with repeated prompts to bridge her meanings for area with segment lengths, the task also provided her multiple occasions across four different shapes to consider how quantities can be represented along axes (M.R.2) and how a point simultaneously represents two magnitude bars' lengths (M.R.3). We conjecture that this feature of the task design, motivated by the framework, supported shifts in Kennedy's meanings as she developed stable meanings for representing quantities in a coordinate system.

8. Implications, Limitations, Areas for Future Research, and Concluding Remarks

In this section, we first describe implications, limitations, and areas for future research relevant to students' meanings for graphs, including their numeric and non-numeric quantitative reasoning and differences in Kennedy's progress compared to students' in Paoletti et al. (2023). We then provide implications for task design, including connections Kennedy's activity has for the tasks presented, the use of dummy segments, and Desmos activities more generally. We close with remarks relevant to middle-school students' developing graphing meanings more generally.

8.1 Implications, limitations, and areas for future research: Students' graphing meanings

The current paper adds to the research highlighting how numeric quantitative reasoning can be productive for students' quantitative reasoning (Johnson, 2012; Moore et al., 2019; Paoletti & Vishnubhotla, 2022). Numeric quantitative reasoning was productive for Kennedy as she made sense of both area and AoC of area situationally (i.e., Figure 6). Further, we conjecture this reasoning was critical to her shifts in meanings when first interpreting the points in Figure 12. However, after experiencing this shift, Kennedy was able to reason non-numerically about the intensity of change of area (Johnson, 2012) as she described changing areas and representations of changing areas without specific values. Hence, we conjecture sixth-grade students' numeric quantitative reasoning can support their eventual non-numeric reasoning, with the latter being the essence of quantitative reasoning (Smith & Thompson, 2008). We call for additional research exploring the ways students' numeric quantitative reasoning can form a foundation for their non-numeric reasoning, how to intentionally support and develop both types of reasoning, and to explore the interplay between these types of reasoning for students' quantitative and covariational reasoning more generally.

Also relevant to Kennedy's progression toward EGST, we note Kennedy experienced more and different complexities than the students reported on in Paoletti et al. (2023). We conjecture differences in the students' prior mathematical experience (i.e., currently taking sixth-grade math versus already having finished high school geometry) may largely explain this. We call for researchers to develop theory and empirical findings that describe the prerequisite meanings needed for students to develop the meanings described in Paoletti et al.'s (2023) framework. Such theory and empirical results could support the design of activities that provide sixth-grade students opportunities to develop such prerequisite meanings thereby minimizing the number of complexities students may encounter as they develop EGST.

A limitation of our study is that the teaching experiment ended before we were able to obtain evidence that Kennedy developed all the meanings necessary for EGST. However, the framework provides indications on how to design tasks that could have supported such meanings. If we had more time with Kennedy, we would have designed a task that provided her additional opportunities to bridge M.S.3 and M.R.3 intending to provide her opportunities to develop stable meanings for points as representing two covarying quantities. Such a task may involve asking her to describe how a dynamic point in a graph shows changes in two situational quantities in some new context. We call for future research that explores the productivity of framework-informed tasks to investigate other students' development of EGST that extends beyond what was explored in this report.

8.2 Implications, limitations, and areas for future research: Task design

Although we designed the *Four Shapes Task* with the framework in mind, Kennedy's activity addressing the task provided some areas for improvement of the task itself. First, we note that in her activity around Figure 15, Kennedy did not explicitly attend to the points in space until prompted to by the TR; without prompting, she may not have noticed these points or how they behaved as she moved the points on the axes. We conjecture revising the task to ensure Kennedy's attention was drawn to the points in space rather than the points on axes may have better supported her in developing meanings for points as multiplicative objects (M.R.3) and in bridging her meanings for points and situational quantities (M.S.3↔M.R.3). We would revise the task so that Kennedy needed to drag the points in the coordinate plane to accomplish the familiar goal of creating appropriate spacing for the points along the axes. This suggestion also points to a potentially more general digital task design principle: To direct students' attention to theoretically salient aspects of a representation by making it interactable. We call for future research exploring this specific possibility as well as investigating this more general principle in other tasks and settings.

A second implication of Kennedy's activity relates to the use of dummy magnitude bars. Although other researchers using similar tasks have used dummy bars (e.g., Liang & Moore, 2021; Paoletti et al., 2021; Stevens et al., 2017; Stevens 2023), we have worked with numerous middle school students who, like Kennedy, have tried to ascribe some situational meaning to the dummy bars (e.g., guessing a meaning for the orange segment in Figure 9). We conjecture an affordance of the *Four Shapes Task* was that each segment represented a unique but related situational quantity (i.e., the growing area of different shapes). Kennedy's own reflection suggested not using dummy bars was an affordance of the *Four Shapes Task* compared with the *Growing Triangle Task* (*Four Shapes* had"just one [quantity] and we had to figure out which one [magnitude bar] was corresponded to that specific one [quantity]"). This conjecture is a researchable question. Researchers may be interested in exploring how students at different age levels respond to tasks designed with and without dummy bars. We conjecture for younger students, dummy bars may impede their bridging situational quantities with magnitude bars as they attempt to make sense of each bar in terms of some situational quantity. With older students, using dummy bars may support their becoming reflectively aware of how graphs represent quantities (e.g., Liang & Moore, 2020). We call for additional research exploring these (and other) possibilities.

Finally, we note that, although tasks designed in the Desmos platform provided numerous affordances, especially with their potential for whole-class instruction, they also led to some limitations in our inquiry. Given the need to pre-structure tasks in this environment, the TR had fewer opportunities to spontaneously explore Kennedy's reasoning in-the-moment than in a less structured environment. For example, Kennedy may have been able to reason about area and base length increasing in a continuous fashion (as seen in Figure 10a-b) in the graphing slides without the introduction of the discretely changing areas (as seen in Figure 11). Kennedy may have also spontaneously considered such discrete changes on her own. In both cases, the Desmos activity had discretely changing areas pre-programmed into the activity and this limited our ability to explore such possibilities. However, such a limitation is inherent in achieving our goal of supporting students' reasoning toward EGST while designing tasks that could be easily used by teachers in whole-class settings. Additionally, across our project we found that the Desmos environment provided a number of affordances for digital task design that supports students' developing graphing meanings (e.g., allowing us to provide linked situational and graphical representations easily). We call for future research that attends to the affordances of the digital environment for students' development of EGST.

8.3 Concluding Remarks

In this paper, we described how Paoletti et al.'s (2023) framework was useful for our analyzing a student's graphing meanings, diagnosing complexities in her reasoning, and designing a task that provided her opportunities to resolve these complexities. Collectively, the aforementioned areas for future research could extend the current work by providing additional uses of the framework proposed by Paoletti et al. (2023) for researchers, teachers, and curriculum designers trying to support or make sense of student's graphing meanings. Such research, in conjunction with this report, would add to the growing body of literature exploring ways to promote students' productive development of graphing meanings to avoid the well-documented difficulties students typically experience with graph construction and interpretation (e.g., Glazer, 2011; Thompson et al., 2017). We conjecture this work can lead to tasks that support students' developing initial meanings for graphs which they can leverage in constructing and interpreting graphs in ways that are useful for learning in STEM fields and for engagement as informed citizens (e.g., Gantt et al., 2023b; Glazer, 2011; Potgieter et al., 2008).

Acknowledgements

This material is based upon work supported by National Science Foundation under Grant No. DRL- 2142000. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of National Science Foundation or our respective universities.

Competing Interests

The authors do not have any competing interests.

References

Antonini, S., Baccaglini-Frank, A., & Lisarelli, G. (2020). From experiences in a dynamic environment to written narratives on functions. *Digital Experiences in Mathematics Education*, 6:1–29.

Beckmann, S., & Izsák, A. (2020). A potential foundation for trigonometry and calculus: The variable-parts perspective on proportional relationships and geometric similarity. In Karunakaran, S., Reed, Z., & Higgins, A. (Eds). *Proceedings of the 23rd Annual Conference on Research in Undergraduate Mathematics Education* (pp. 734–741). Boston, MA.

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352–378. https://doi.org/10.2307/4149958

Castillo-Garsow, C. C. (2012). Continuous quantitative reasoning. In R. Mayes, R. Bonillia, L. L. Hatfield, & S. Belbase (Eds.), *Quantitative reasoning: Current state of understanding, WISDOMe Monographs* (Vol. 2, pp. 55–73). University of Wyoming.

Curry, M., Mitchelmore, M. and Outhred, L. July 2006. *Development of children's understanding of length, area, and volume measurement principles*, July, Prague, Czech Republic: Paper presented at the Thirtieth Annual Meeting of the International Group for the Psychology of Mathematics Education.

DiSessa, A. A., Hammer, D., Sherin, B., & Kolpakowski, T. (1991). Inventing graphing: Meta-representational expertise in children. *The Journal of Mathematical Behavior*. *10*(2), 117–160.

Ellis, A. B. (2011). Algebra in the middle school: Developing functional relationships through quantitative reasoning. In J. Cai & E. Knuth (Eds.), *Early Algebraization* (pp. 215–238). Springer Berlin Heidelberg. https://doi.org/10.1007/978-3-642-17735-4_13

Ellis, A. B., Özgür, Z., Kulow, T., Williams, C. C., & Amidon, J. (2015). Quantifying exponential growth: Three conceptual shifts in coordinating multiplicative and additive growth. *The Journal of Mathematical Behavior, 39*, 135–155. https://doi.org/10.1016/j.jmathb.2015.06.004

Frank, K. M. (2016). Plotting points: Implications of “over and up” on students’ covariational reasoning. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), Pro*ceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. The University of Arizona

Gantt, A. L., Paoletti, T., Greenstein, S. (2022). "This one is that": A semiotic lens on quantitative reasoning. In A. Lischka, J. Strayer, J. Lovett, R. S. Jones, & E. Dyer (Eds.) *Proceedings of the 44th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 799–807). https://doi.org/10.51272/pmena.44.2022

Gantt, A. L., Paoletti, T., Acharya, S. & Margolis, C. (2023a). Bridging situational and graphical reasoning to support emergent graphical shape thinking. In T. Lamberg & D. Moss (Eds.), *Proceedings of the 45th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. (Vol. 1, pp. 124-133).

Gantt, A. L., Paoletti, T., & Corven, J. (2023b). Exploring the prevalence of covariational reasoning across mathematics and science using TIMSS 2011 assessment items. *International Journal of Science and Mathematics Education*, 1-25.

Glazer, N. (2011). Challenges with graph interpretation: A review of the literature. *Studies in science education, 47*(2), 183–210. https://doi.org/10.1080/03057267.2011.605307

Goldenberg, E. P., Lewis, P., & O'Keefe, J. (1992). Dynamic representation and the development of a process understanding function. In G. Harel & E. Dubinsky (Eds.), The concept of function (pp. 235–260). Washington, DC: Mathematical Association of America.

Hattikudur, S., Prather, R. W., Asquith, P., Alibali, M. W., Knuth, E. J., & Nathan, M. (2012). Constructing graphical representations: Middle schoolers' intuitions and developing knowledge about slope and y-intercept. *School Science and Mathematics, 112*(4), 230–240. https://doi.org/10.1111/j.1949-8594.2012.00138.x

Hollebrands, McCulloch, Okumus (2021). High school students' use of technology to make sense of functions within the context of geometric transformations. *Digital Experiences in Mathematics Education*, 7:247–275.

Johnson, H. L. (2012). Reasoning about variation in the intensity of change in covarying quantities involved in rate of change. *The Journal of Mathematical Behavior, 31*(3), 313–330. https://doi.org/10.1016/j.jmathb.2012.01.001

Johnson, H. L. (2015). Secondary students' quantification of ratio and rate: A framework for reasoning about change in covarying quantities. *Mathematical Thinking and Learning, 17*(1), 64–90. https://doi.org/10.1080/10986065.2015.981946

Johnson, H. L., McClintock, E., & Hornbein, P. (2017). Ferris wheels and filling bottles: A case of a student's transfer of covariational reasoning across tasks with different backgrounds and features. *ZDM*, 49, 851-864.

Joshua, S., Musgrave, S., Hatfield, N. & Thompson, P. W. (2015) Conceptualizing and reasoning with frames of reference. In Fukawa-Connelly, T., Infante, N.E., Keene, K. & Zandieh, M. (Eds.) *Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education*, pp. 31–44. RUME.

Kamii, C. 2006. Measurement of length: How can we teach it better?. *Teaching children mathematics*, 13(3): 5

Liang, B., & Moore, K. C. (2020). Figurative and operative partitioning activity: students' meanings for amounts of change in covarying quantities. *Mathematical Thinking and Learning*, 1–27. https://doi.org/10.1080/10986065.2020.1789930

Moore, K. C. (2014). Quantitative reasoning and the sine function: The case of Zac. *Journal for Research in Mathematics Education*, *45*(1), 102-138.

Moore, K. C. (2021). Graphical shape thinking and transfer. In C. Hohensee & J. Lobato (Eds.), *Transfer of learning: Progressive perspectives for mathematics education and related fields* (pp. 145—172). Springer. https://doi.org/10.1007/978-3-030-65632-4_7

Lee, H., & Hardison, H. L. (2016). Spatial coordination as a prerequisite for representing quantitative coordination in two dimensions. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North American Chapter of* *the International Group for the Psychology of Mathematics Education (p. 304).* The University of Arizona.

Lee, H. Y., Hardison, H., & Paoletti, T. (2020) Foregrounding the background: Two uses of coordinate systems. *For the Learning of Mathematics,* 40(*1*), 32-37.

Liang, B., & Moore, K. C. (2020, online). Figurative and operative partitioning activity: Students' meanings for amounts of change in covarying quantities. *Mathematical Thinking and Learning*. https://doi.org/10.1080/10986065.2020.1789930

Moore, K. C., Stevens, I. E., Paoletti, T., Hobson, N. L., & Liang, B. (2019). Pre-service teachers' figurative and operative graphing actions. *The Journal of Mathematical Behavior, 56*. https://doi.org/10.1016/j.jmathb.2019.01.008

Moore, K. C., & Thompson, P. W. (2015). Shape thinking and students' graphing activity. In *Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education* (pp. 782-789). RUME.

Nathan, M. J., & Kim, S. (2007). Pattern generalization with graphs and words: A cross-sectional and longitudinal analysis of middle school students' representational fluency. *Mathematical Thinking and Learning, 9*(3), 193–219. https://doi.org/10.1080/10986060701360886

Paoletti, T. (2015) Students' reasoning when constructing quantitatively rich situations. In T. Fukawa-Connolly, N. E. Infante, K. Keene, & M. Zandieh (Eds.), *Proceedings of the Eighteenth Annual Conference on Research in Undergraduate Mathematics Education* (pp. 845-852). Pittsburgh, PA: West Virginia University.

Paoletti, T. (2019). Supporting students' understanding graphs as emergent traces: The faucet task. In M. Graven, H. Vankat, A. A. Essien, & P. Vale (Eds.), *Proceedings of the 43rd*

Conference of the International Group for the Psychology of Mathematics Education (Vol. 3, pp. 185–200). PME.

Paoletti, T., Gantt, A. L. & Corven, J. (2023) A local instructional theory for middle school students' emergent reasoning. *Journal for Research in Mathematics Education*, 54(*3*). https://doi.org/10.5951/jresematheduc-2021-0066

Paoletti, T., Greenstein, S., Vishnubhotla, M., & Mohamed, M. M. (2019) Designing tasks and 3D physical manipulatives to promote students' covariational reasoning. In Graven, M., Venkat, H., Essien, A. & Vale, P. (Eds). (2019). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol 3, pp. 193-200). Pretoria, South Africa: PME.

Paoletti, T., Hardison, H., & Lee, H. Y. (2022) Students' static and emergent graphical shape thinking in spatial and quantitative coordinate systems. *For the Learning of Mathematics*, 42(*2*), 48-49.

Paoletti, T., Lee, H. Y., Rahman, Z., Vishnubhotla, M., & Basu, D. (2020). Comparing graphical representations in mathematics, science, and engineering textbooks and practitioner journals. *International Journal of Mathematical Education in Science and Technology*, 1–20. https://doi.org/10.1080/0020739X.2020.1847336

Paoletti, T., & Moore, K. C. (2017). The parametric nature of two students' covariational reasoning. *The Journal of Mathematical Behavior, 48*, 137–151. https://doi.org/10.1016/j.jmathb.2017.08.003

Paoletti, T. & Vishnubhotla, M. (2022) Leveraging covariational reasoning and emergent shape thinking to distinguish nonlinear and linear relationships. In Karagöz Akar, G., Özgür Zembat, I., Selahattin, A., & Thompson, P. W. (Eds). *Quantitative reasoning in* *mathematics and science education* (pp. 133-168). Springer. https://doi.org/10.1007/978-3-031-14553-7

Piaget, J. (2001). *Studies in reflecting abstraction*. Psychology Press Ltd.

Potgieter, M., Harding, A., & Engelbrecht, J. (2008). Transfer of algebraic and graphical thinking between mathematics and chemistry. *Journal of Research in Science Teaching 45*(2), 197–218. https://doi.org/10.1002/tea.20208

Rolfes, T., Roth, J., & Schnotz, W. (2020). Learning the concept of function with dynamic visualizations. *Frontiers in Psychology, 11* (693). https://doi.org/10.3389/fpsyg.2020.00693

Saldanha, L. A., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensen, K. R. Dawkings, M. Blanton, W. N. Coulombe, J. Kolb, K. Norwood, & L. Stiff (Eds.), *Proceedings of the 20th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 298–303).

Smith, J., & Thompson, P. W. (2008). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the Early Grades* (pp. 95–132). Lawrence Erlbaum Associates.

Steffe, L. P. (1991). Operations that generate quantity. *Journal of Learning and Individual Differences, 3*(1), 61–82. https://doi.org/10.1016/1041-6080(91)90004-K

Steffe, L.P. & Olive, J. (2010) *Children's Fractional Knowledge*. New York: Springer.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. A. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 267–307). Erlbaum.

Stevens, I. E. (2023). Magnitude bars and covariational reasoning. Cook, S., Katz, B. & Moore-Russo D. (Eds.). *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education*. Omaha, NE. (pp. 752–761)

Stevens, I. E., Paoletti, T., Moore, K. C., Liang, B. & Hardison, H. (2017). Principles for designing tasks that promote covariational reasoning. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro & S. Brown (Eds.), *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 928–936). San Diego, CA.

Thompson, P. W. (2002). Didactic objects and didactic models in radical constructivism. In *Symbolizing, modeling and tool use in mathematics education* (pp. 197-220). Springer Netherlands.

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundation of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. SÈpulveda (Eds.), *Proceedings of the Annual Meeting of the International Group for the Psychology of Mathematics Education* (Vol 1, pp. 45-64). PME.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. Chamberlin, L. L. Hatfield, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOM^e* (pp. 33–57).

Thompson, P. W. (2016). Researching mathematical meanings for teaching. In L. D. English & D. Kirchner (Eds.), *Handbook of international research in mathematics education* (pp. 435–461). Taylor & Francis.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation and functions: Foundational ways of mathematical thinking. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421–456). National Council of Teachers of Mathematics.

Thompson, P. W., Hatfield, N. J., Yoon, H., Joshua, S., & Byerley, C. (2017). Covariational reasoning among US and South Korean secondary mathematics teachers. *The Journal of Mathematical Behavior, 48*, 95–111. https://doi.org/10.1016/j.jmathb.2017.08.001

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Falmer Press.

Yeo, S. (2021). Semiotic mediation through technology: The case of fraction reasoning, *The Mathematics Education 60*(10), 1–19.