Non-Gaussian fluctuation dynamics in relativistic fluids

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We consider nonequilibrium evolution of *non-Gaussian* fluctuations within relativistic hydrodynamics relevant for the QCD critical point search in heavy-ion collision experiments. We rely on the hierarchy of relaxation timescales, which emerges in the hydrodynamic regime near the critical point, to focus on the slowest mode such as the fluctuations of specific entropy, whose equilibrium magnitude, non-Gaussianity, and typical relaxation time are increasing as the critical point is approached. We derive evolution equations for the non-Gaussian correlators of this diffusive mode in an arbitrary relativistic hydrodynamic flow. We compare with the simpler case of the stochastic diffusion on a static homogeneous background and identify terms which are specific to the case of the full hydrodynamics with pressure fluctuations and flow.

I. INTRODUCTION

The physics of thermal fluctuations [1] in hydrodynamics [2] has received renewed interest recently in the context of relativistic heavy-ion collision experiments. Hydrodynamics proved to be remarkably successful in describing the data from such collisions [3,4]. One of the major goals of heavy-ion collision experiments is the discovery of the QCD critical point—the end point of the conjectured first-order transition separating hadron gas and the quark-gluon plasma phases [5]. This point is characterized by a certain singular behavior of fluctuations—a universal feature of the critical points [6–8]. Therefore, understanding of fluctuations and, in particular, of their nonequilibrium evolution in the environment of the hydrodynamically expanding QCD fireball is important to enable the experimental search for the QCD critical point [9,10].

The parameter controlling the importance of fluctuations at wavelengths of order ℓ is the ratio of the "correlation volume" of the size of the correlation length ξ to the "homogeneity volume" of order ℓ^3. We shall generically refer to this parameter as ε. The central limit theorem suppresses the magnitude of fluctuations as well as their non-Gaussianity by a power of ε. In a typical condensed matter experiment ε is extremely small

and thus fluctuations often play a negligible role. In heavy-ion collisions, where the system size $[\mathcal{O}(10 \text{ fm})]$ is large compared to the typical correlation length ξ (a fraction of fm), the parameter ε is small but not negligible. Fluctuations are observable in heavy-ion collisions and play important role in understanding the thermodynamic properties of the QCD fireball.

Furthermore, near the critical point, as the correlation length becomes longer, the fluctuations grow and play even more important role. Non-Gaussianity of fluctuations, also controlled by ε, is increasing even faster at the critical point than their magnitude [11], and is expected to show a specific nonmonotonic dependence on the collision energy [12], which in heavy-ion collisions controls the thermodynamic conditions at freeze-out. Therefore, non-Gaussian fluctuation measures have emerged as observables of primary experimental interest in the search for the QCD critical point [13].

Another effect of the critical point is to slow down the relaxation towards local thermodynamic equilibrium. The critical slowing down makes it increasingly important to consider nonequilibrium evolution of fluctuations near the critical point, as earlier estimates and model calculations demonstrate [7,14,15]. To make quantitatively reliable predictions, however, one needs to consider the nonequilibrium behavior of fluctuations in the "first principle" hydrodynamic approach.

There has been considerable recent effort and progress towards understanding non-equilibrium evolution of fluctuations in the context of the hydrodynamics of the heavy-ion collisions with the aim of mapping the QCD phase diagram (see Ref. [10] and references therein). The focus of this paper is on the approach where fluctuations are described in terms of the correlation functions obeying deterministic evolution equations, i.e., the so-called deterministic (also known as hydrokinetic) approach. This approach was introduced and developed recently in the context of relativistic heavy-ion

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collisions [16–18], more generally in [19–23], and much earlier, in the non-relativistic condensed matter context, in [24,25].

Within the deterministic approach most of the work has been done on Gaussian fluctuation measures: two-point correlators, or their Wigner transforms. In this approach, the first step towards the study of the *non-Gaussian* hydrodynamic fluctuations out of equilibrium was made in Ref. [22],[1] where the generalized n-point Wigner transform was introduced and the equations for the corresponding n-point Wigner functions were derived. Reference [22] considered the simplest hydrodynamic system: nonlinear charge diffusion at constant temperature. Such a system is characterized by a single hydrodynamic field: conserved charge density. Full hydrodynamics necessary to describe heavy-ion collisions involves (at least) five hydrodynamic fields: conserved densities of energy, baryon charge, and three-momentum of the fluid. The full theory of hydrodynamic fluctuations is an ambitious goal and in this paper we shall present another step towards it by considering fluctuations of both energy and baryon density in the regime relevant to the critical point search, where the correlation length becomes large compared to typical microscopic scales (while still being smaller than ℓ) and fluctuations of a certain critical mode dominate.

In this regime an important hierarchy of scales emerges. The slowest mode is the diffusive (nonpropagating) mode $m = s/n$, i.e., entropy to baryon number ratio, which we shall refer to as specific entropy. This mode does not mix with propagating sound oscillations, which makes it purely diffusive (unlike fluctuations of energy and baryon densities on their own). Furthermore, unlike the case of other diffusive modes, such as transverse momentum densities, the diffusion constant for m vanishes at the critical point, making m the slowest mode in this regime. Such a hierarchy of scales was exploited in the construction of Hydro+ theory in Ref. [29] as well as in the Hydro++ theory in Ref. [20], where the next-to-slowest momentum diffusion modes were also included. In both cases only two-point correlators were considered. The goal of this work is to extend the Hydro+ formalism to the non-Gaussian fluctuation measures, i.e., n-point correlation functions.

As in Ref. [29] we shall focus on fluctuations of m, which are important for two related reasons. First, as explained above, this mode and its fluctuations are slowest to relax, and therefore are most in need of nonequilibrium description. Second, the magnitude and non-Gaussianity measures of the fluctuations of this mode diverge with the correlation length most strongly (with the largest critical exponent), thus providing the most sensitive observable signatures of the critical point.

Furthermore, in Ref. [22] we considered fluctuations in a stationary fluid without any flow. Here we shall treat the most general relativistic flow in a fully Lorentz covariant formal-

ism necessary for applications to heavy-ion collisions. We review the formalism which was introduced in Ref. [19,20] for two-point correlators with a specific focus on generalizing it to non-Gaussian fluctuations. We use the n-point wave-number dependent correlation functions which we introduced in Ref. [22] by generalizing the well-known Wigner transform and show how the confluent formalism of Refs. [19,20] naturally extends to these objects in Sec. II D.

While the simple diffusion problem in Ref. [22] contains only one fluctuating field variable (charge density n), we find that—even in the regime where the fluctuations of m are the slowest and the fluctuations of the faster hydrodynamic modes, such as pressure p, can be considered as equilibrated—we cannot neglect these fluctuations, as they modify equations for the evolution of fluctuations of m via nonlinearities in the equation of state, or mode coupling. We discuss this nontrivial part of the derivation in Sec. III B.

We should point out again that we set up the formalism in the most general form necessary to tackle the full system of hydrodynamic equations with fluctuations of all hydrodynamic variables. Our focus on the slowest variable m allows us to make the logical first step towards considering the full system. This focus simplifies the calculations and allows us to hone the tools needed to tackle this ambitious goal.

We shall use the approach of Refs. [19,20], i.e., expand the stochastic hydrodynamic equations in powers of fluctuation magnitude. Unlike Refs. [19,20] we shall not consider feedback of fluctuations and assume, based on the analysis of these contributions in Refs. [19,20], that the major effects of these (UV sensitive) contributions have been absorbed into renormalization of hydrodynamic variables, equation of state, and transport coefficients which, therefore, take physical (cutoff independent) values. In the diagrammatic representation we introduced in Ref. [22] such feedback contributions are represented by loop diagrams. As in Ref. [22], we shall consider only "tree-level" terms in the evolution equations for non-Gaussian correlators. Again, this is not a limitation of the approach, but a natural simplifying first step in its development.

In Sec. II we introduce the general formalism for the dynamical evolution equations for the n-point correlation functions in the presence of background hydrodynamic flow. In Sec. III we focus on the specific entropy fluctuations for reasons explained above and, by using the results developed in Sec. II, we derive the evolution equations for the two-, three-, and four-point functions of the specific entropy fluctuations. Thus we arrive at the main results of this paper, which can be found in Sec. III B. We summarize our findings and discuss the outlook for future developments in Sec. IV.

II. GENERAL DETERMINISTIC FORMALISM FOR FIELD FLUCTUATIONS

In this section we extend the formalism for the dynamical evolution equation of relativistic hydrodynamic fluctuations we developed in Refs. [19,20] to n-point correlation functions. In particular, we show that the confluent formalism, which allows us to describe fluctuations in the local rest frame of the fluid in a natural and fully Lorentz covariant way, generalizes naturally to n-point functions. We also use the multipoint

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[1]Evolution of non-Gaussian *cumulants*, i.e., spatial integrals of the correlation functions we consider here, was studied previously in Refs. [15,26] and also within the complementary stochastic approach, e.g., in Ref. [27]. Non-Gaussian correlation functions introduced in Ref. [22] were also studied in the effective theory approach in Ref. [28].

Wigner transform, which we introduced in Ref. [22], to express the n-point correlators in terms of one spatial coordinate of the midpoint and $n - 1$ independent wave vectors. The results of this section are general, i.e., not limited to a particular fluctuating variable of variables.

A. Evolution of correlation functions

Let us first derive the evolution equation for the n-point correlation functions of a set of generic stochastic field variables $\check{\psi}_i$ where the subscript i labels different local hydrodynamic fields (such as entropy per baryon $m \equiv s/n$, pressure p, and fluid velocity u as in Refs. [19,20]). Each of those variables satisfies its own Langevin-type equation that can be generically written in a covariant form,

$$\check{u} \cdot \partial \check{\psi}_i = \check{F}_i + \xi_i, \quad \text{with } \xi_i = \check{H}_{ij}\eta_j, \qquad (2.1)$$

where the symbol "˘" denotes a stochastic quantity. Functions or functionals, such as $\check{u} \equiv u(\check{\psi}_i)$ or $\check{F} \equiv F(\check{\psi})$ inherit the stochastic symbol "˘" from their arguments. The four-velocity obeys $\check{u}^2 = -1$, and should be understood as a four-vector function of ψ_i's (or can also be chosen to be among the variables ψ_i as in Refs. [19,20]). $\check{F}_i$ is the drift "force" and ξ_i is the noise (random "force") for the variable ψ_i, expressed in terms of the canonically normalized local Gaussian noise[2]:

$$\langle \eta_i \rangle = 0, \quad \langle \eta_i(x_1)\eta_j(x_2) \rangle = 2\delta_{ij}\delta^{(4)}(x_1 - x_2). \qquad (2.2)$$

We include multiplicative noise, since $\check{H} = H(\check{\psi})$, and define the product $\check{H}\eta$ in terms of the Itō calculus.[3]

The Onsager matrix (operator) $Q = Q^T$ is the "square" of H: $Q_{ij} \equiv H_{ik}H_{jk}$.

Denoting fluctuation of a given stochastic quantity $\check{X}$ as

$$:\check{X}: \equiv \delta X \equiv \check{X} - \langle \check{X} \rangle \equiv \check{X} - X, \qquad (2.3)$$

and introducing the fluctuation of the variables ψ_i,

$$\phi_i \equiv :\check{\psi}_i: \equiv \check{\psi}_i - \psi_i, \qquad (2.4)$$

we can expand the stochastic equations (2.1) in powers of ϕ_i, using, e.g.,

$$\check{F}_i = \sum_{n=0}^{\infty} \frac{1}{n!} F_{i, j_1...j_n} \phi_{j_1} \cdots \phi_{j_n}$$

$$= F_i + F_{i, j_1}\phi_{j_1} + \frac{1}{2}F_{i, j_1 j_2}\phi_{j_1}\phi_{j_2} + \cdots, \qquad (2.5)$$

where $F \equiv F(\psi)$, etc., and obtain the evolution equation for fluctuation field ϕ_i:

$$u \cdot \partial \phi_i = :\check{u} \cdot \partial \check{\psi}_i - \delta u \cdot \partial \check{\psi}_i :$$

$$= : \sum_{n=1}^{\infty} \frac{1}{n!} L_{i, j_1...j_n} \phi_{j_1} \cdots \phi_{j_n} + \xi_i :, \qquad (2.6)$$

where the term involving δu in Eq. (2.6) must be kept if the fluctuation of velocity is taken into account and

$$L_{i, j_1 j_2...j_n} = F_{i, j_1 j_2...j_n} - u^{\mu}_{, j_1...j_n}(\partial_{\mu}\psi_i)$$

$$- \left[n\delta_{ij_1} u^{\mu}_{, j_2...j_n} \partial^{(j_1)}_{\mu} \right]_{\overline{1...n}}. \qquad (2.7)$$

Here $\partial^{(j_1)}_{\mu}$ acts on ϕ_{j_1} only, and the Einstein summation rule over repeated indices is implied throughout this paper. The subscript $\overline{1 \ldots n}$ denotes the "averaging over permutations", i.e., the sum over all $n!$ permutations of all composite index-position labels $(j_1, x_1), \ldots, (j_n, x_n)$ divided by $n!$,

$$[\ldots]_{\overline{1...n}} = \frac{1}{n!}[\ldots]_{\mathrm{P}_{1...n}}, \qquad (2.8)$$

where we used the notation of Ref. [22], $\mathrm{P}_{1...n}$, to denote the sum over permutations. Furthermore, since Eqs. (2.1) that we consider are *differential* equations, each expansion coefficient such as $L_{i, j_1 j_2...j_n}$ must be treated as a multilinear differential operator. These operators are linear in each of their arguments, which are identified by the matching repeated indices. It is helpful, in this regard, to think of the arguments x_i of the field $\phi_i(x_i)$ as a part of the composite label (i, x_i) denoting position as well as the name or index of the variable.

Having this at hand it is straightforward to write down the evolution equation for the "raw" n-point correlation functions

$$G_{i_1...i_n} \equiv \langle \phi_{i_1}(x_1) \cdots \phi_{i_n}(x_n) \rangle \qquad (2.9)$$

by taking the time derivative in the rest frame of the fluid at midpoint $x \equiv \sum_{i=1}^{n} x_i/n$:

$$u \cdot \partial^{(x)} G_{i_1...i_n}$$

$$\equiv \sum_{i=1}^{n} u \cdot \frac{\partial}{\partial x_i} G_{i_1...i_n} = n\left[u \cdot \frac{\partial}{\partial x_1} G_{i_1...i_n} \right]_{\overline{1...n}}$$

$$= n\Bigg\{ -(y_1 \cdot \partial u) \cdot \frac{\partial}{\partial x_1} G_{i_1...i_n}$$

$$+ \sum_{m=0}^{\infty} \frac{1}{m!} \left[L_{i_1, j_1...j_m}(G_{j_1...j_m i_2...i_n} - G_{j_1...j_m}G_{i_2...i_n}) \right.$$

$$\left. + (n-1)Q_{i_1 i_2, j_1...j_m} G_{j_1...j_m i_3...i_n} \right] \Bigg\}_{\overline{1...n}}, \qquad (2.10)$$

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where we have used[4]

$$u \cdot \frac{\partial}{\partial x_1} G_{i_1 \ldots i_n} = [u(x_1) - (y_1 \cdot \partial u)] \cdot \frac{\partial}{\partial x_1} G_{i_1 \ldots i_n} \qquad (2.11)$$

and

$$
\begin{aligned}
u(x_1) \cdot \frac{\partial}{\partial x_1} G_{i_1 \ldots i_n} &= u(x_1) \cdot \frac{\partial}{\partial x_1} \langle \phi_{i_1}(x_1) \cdots \phi_{i_n}(x_n) \rangle \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} \langle :L_{i_1, j_1 \ldots j_m}(x_1) \phi_{j_1}(x_1) \cdots \phi_{j_m}(x_1): \phi_{i_2}(x_2) \cdots \phi_{i_n}(x_n) \rangle \\
&\quad + \frac{n-1}{m!} \langle Q_{i_1 i_2, j_1 \ldots j_m}(x_1, x_2) \phi_{j_1}(x_1) \cdots \phi_{j_m}(x_1) \phi_{i_3}(x_3) \cdots \phi_{i_n}(x_n) \rangle_{\overline{2 \ldots n}} \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} \Big[L_{i_1, j_1 \ldots j_m} \big(G_{j_1 \ldots j_m i_2 \ldots i_n} - G_{j_1 \ldots j_m} G_{i_2 \ldots i_n} \big) + (n-1) \big(Q_{i_1 i_2, j_1 \ldots j_m} G_{j_1 \ldots j_m i_3 \ldots i_n} \big)_{\overline{2 \ldots n}} \Big]. \qquad (2.12)
\end{aligned}
$$

By the first equality in Eq. (2.10) we defined the differential operator, which can be understood as the derivative with respect to the midpoint x [see also Eq. (2.33)]. The subscripts $\overline{1 \ldots n}$ in Eq. (2.10), and $\overline{2 \ldots n}$ in Eq. (2.11), denote the "averaging over permutations" defined in Eq. (2.8). The factors $n = n!/(n-1)!$ and $n(n-1) = n!/(n-2)!$ in Eq. (2.10) conveniently match the number of the nonidentical terms among the $n!$ permutations. The identical terms arise due to the invariance of the correlators in Eq. (2.9) under the permutation $P_{1 \ldots n}$. The function $Q(x_1, x_2)$ is the integral kernel of the Onsager operator, i.e., $Q(x_1, x_2) \equiv Q \delta^{(3)}(x_1 - x_2)$, where the delta-function support is a line parallel to $u(x)$, i.e., $\delta^{(3)}(y) = 0$ unless $u(x) \cdot y = 0$. The first term on the right-hand side of Eq. (2.10) arises because the velocity at midpoint $u(x)$ on the left-hand side is different from the velocity $u(x_1)$ involved in the equation of motion for $\phi(x_1)$ by the amount proportional to y_1.

B. Power counting and perturbation theory

Evolution equations (2.10) form an infinite system of equations for an infinite set of correlation functions. However, in the regime of applicability of hydrodynamics, there is a natural hierarchy in this system which allows us to systematically truncate and solve these equations. In this section we discuss this hierarchy and the small power counting parameter(s) which control it (see also Refs. [20,22]).

Hydrodynamics is an effective theory that emerges when the thermal state of the system is sufficiently homogeneous. More precisely, hydrodynamic variables (i.e., conserved densities) characterizing the local thermal state of the system must vary on a scale ℓ much longer than the microscopic scale ℓ_{mic}, which typically is of order of the correlation length ξ.[5] Indeed, without such a scale separation one could not describe the state of the system entirely in terms of thermodynamic (conserved) quantities and, e.g., the thermodynamic concept of equation of state would lose meaning. The scale separation $\ell \gg \ell_{\text{mic}}$ is the essential property of the hydrodynamic regime and it gives rise to a small parameter ℓ_{mic}/ℓ (Knudsen number) which controls the gradient expansion of hydrodynamic equations and ensures their locality.

When we consider fluctuations in hydrodynamics the relevant characteristic inhomogeneity scale is the wavelength of the fluctuations, or the inverse of their wave number q, and the corresponding small parameter is $\varepsilon_q \equiv q \ell_{\text{mic}}$.

The small parameter which controls the magnitude and importance of fluctuations is closely related to ε_q. This parameter is the inverse of the typical number of the uncorrelated microscopic cells (of volume ξ^3) in the homogeneity region (of volume ℓ^3, or $1/q^3$): $\varepsilon \equiv (\xi/\ell)^3 \sim (q\xi)^3$. The central limit theorem guarantees that (1) the magnitude of the fluctuations averaged over the scale of $\ell \sim 1/q$ is suppressed by $\sqrt{\varepsilon}$ and (2) the non-Gaussian connected correlators of order $n > 2$ are suppressed by ε^{n-1}.

Both ε_q and ε are small in the hydrodynamic regime, when $q \ell_{\text{mic}} \ll 1$, and $\varepsilon \sim \varepsilon_q^3$. However, for the sake of generality, and to emphasize the difference between the gradient expansion corrections and the fluctuation corrections to hydrodynamics, we shall treat these two parameters as independent.[6]

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[4]Equation (2.11) assumes that the velocity u varies slowly on the characteristic scale of the correlations we study, as is the case in hydrodynamics. This property of hydrodynamics underlines the approach known as "hydrokinetics" [16–20]. Equations (2.10)–(2.12), for $n = 2$ and truncated at $m = 1$, upon Wigner transform reproduce the equations for Gaussian fluctuation correlators known as "hydrokinetic equations." Our equations are more general and include non-Gaussian fluctuations and multiplicative noise.

[5]The vicinity of the critical point is characterized by the correlation length ξ becoming much larger than other microscopic scales. In this interesting case we still require $\ell \gg \xi$, a reasonable condition for the scales relevant for the heavy-ion collisions. Dynamic critical behavior in the regime $\ell \sim \xi$, which is beyond the scope of this work, can be described by a hydrodynamic-like theory such as model-H [30]. In this regime the power counting we use begins to break down.

[6]There are field theories where these two parameters are *parametrically* different, e.g., in in large N theories where $\varepsilon \sim 1/N$, while $\varepsilon_q \sim N^0$.

The corresponding power counting for various quantities involved in our calculation is as follows:

$$G_{i_1 \dots i_n} \sim \varepsilon^{[n/2]}, \quad G^c_{i_1 \dots i_n} \sim \varepsilon^{n-1},$$
$$L_{i_1, i_2 \dots i_n} \sim \varepsilon_q + \mathcal{O}(\varepsilon_q^2), \quad Q_{i_1 i_2, j_1 \dots j_m} \sim \varepsilon_q^2 \varepsilon,$$
$$u \cdot \partial \sim \varepsilon_q^2, \tag{2.13}$$

where $[n/2] = n/2$ for even n and $(n+1)/2$ for odd n, while G^c denote connected correlation functions defined below in Eq. (2.17) or (2.20). The factor $(\sqrt{\varepsilon})^n$ in G is a consequence of the suppression of the magnitude of fluctuations averaged over hydrodynamic scale by a factor $\sqrt{\varepsilon}$. The factor ε^{n-1} in G^c results from the requirement that $n-1$ internal correlations are needed for a fully connected n-point correlator (with each internal two-point correlator being of order ε). The factor ε_q in L reflects the fact that the rate of ideal hydrodynamic evolution for fluctuations is first order in spatial derivatives, while the $\mathcal{O}(\varepsilon_q^2)$ term in L represents dissipative terms in hydrodynamic equations which are second order (as in diffusion). The ε_q term is absent for purely diffusive variables. The factor ε_q^2 in the Onsager operator Q controls the magnitude of the noise and is determined by the fluctuation-dissipation theorem, while the factor ε is due to the locality (short range correlation) of the noise and its suppression on the hydrodynamic scale by the central limit theorem.

The first few equations (with $n = 2, 3, 4$) of Eq. (2.10) are truncated in the double expansion of ε and ε_q, i.e.,

$$u \cdot \partial^{(x)} G_{i_1 i_2}(x_1, x_2) = 2\Bigg[-(y_1 \cdot \partial u) \cdot \frac{\partial}{\partial x_1} G_{i_1 i_2}(x_1, x_2) + L_{i_1, j_1}(x_1) G_{j_1 i_2}(x_1, x_2)$$
$$+ \frac{1}{2} L_{i_1, j_1 j_2}(x_1) G_{j_1 j_2 i_2}(x_1, x_1, x_2) + \frac{1}{6} L_{i_1, j_1 j_2 j_3}(x_1) G_{j_1 j_2 j_3 i_2}(x_1, x_1, x_1, x_2)$$
$$+ Q_{i_1 i_2}(x_1, x_2) + \frac{1}{2} Q_{i_1 i_2, j_1 j_2}(x_1, x_2) G_{j_1 j_2}(x_1, x_1) + \mathcal{O}(\varepsilon_q^2 \varepsilon^3) \Bigg]_{\overline{12}}, \tag{2.14a}$$

$$u \cdot \partial^{(x)} G_{i_1 i_2 i_3}(x_1, x_2, x_3) = 3\Bigg[-(y_1 \cdot \partial u) \cdot \frac{\partial}{\partial x_1} G_{i_1 i_2 i_3}(x_1, x_2, x_3) + L_{i_1, j_1}(x_1) G_{j_1 i_2 i_3}(x_1, x_2, x_3)$$
$$+ \frac{1}{2} L_{i_1, j_1 j_2}(x_1) \big[G_{j_1 j_2 i_2 i_3}(x_1, x_1, x_2, x_3) - G_{j_1 j_2}(x_1, x_1) G_{i_2 i_3}(x_2, x_3) \big]$$
$$+ 2 Q_{i_1 i_2, j_1}(x_1, x_2) G_{j_1 i_3}(x_1, x_3) + \mathcal{O}(\varepsilon_q^2 \varepsilon^3) \Bigg]_{\overline{123}}, \tag{2.14b}$$

$$u \cdot \partial^{(x)} G_{i_1 i_2 i_3 i_4}(x_1, x_2, x_3, x_4) = 4\Bigg[-(y_1 \cdot \partial u) \cdot \frac{\partial}{\partial x_1} G_{i_1 i_2 i_3 i_4}(x_1, x_2, x_3, x_4) + L_{i_1, j_1}(x_1) G_{j_1 i_2 i_3 i_4}(x_1, x_2, x_3, x_4)$$
$$+ \frac{1}{2} L_{i_1, j_1 j_2}(x_1) \big[G_{j_1 j_2 i_2 i_3 i_4}(x_1, x_1, x_2, x_3, x_4) - G_{j_1 j_2}(x_1, x_1) G_{i_2 i_3 i_4}(x_2, x_3, x_4) \big]$$
$$+ \frac{1}{6} L_{i_1, j_1 j_2 j_3}(x_1) G_{j_1 j_2 j_3 i_2 i_3 i_4}(x_1, x_1, x_1, x_2, x_3, x_4)$$
$$+ 3 Q_{i_1 i_2}(x_1, x_2) G_{i_3 i_4}(x_3, x_4) + 3 Q_{i_1 i_2, j_1}(x_1, x_2) G_{j_1 i_3 i_4}(x_1, x_3, x_4)$$
$$+ \frac{3}{2} Q_{i_1 i_2, j_1 j_2}(x_1, x_2) G_{j_1 j_2 i_3 i_4}(x_1, x_1, x_3, x_4) + \mathcal{O}(\varepsilon_q^2 \varepsilon^4) \Bigg]_{\overline{1234}}, \tag{2.14c}$$

and equations for $n \geqslant 5$ can be obtained accordingly. Here we only consider the first-order hydrodynamics. Thus the equations for all multipoint functions are truncated at order ε_q^2.[7] While we keep only the leading order terms in ε in Eqs. (2.14b) and (2.14c), both leading and next-to-leading order terms are kept in Eq. (2.14a), for the purpose of deriving the evolution equation for the four-point connected function, Eq. (2.16c).

Our purpose is to derive the evolution equations for connected correlation functions, which can be directly related to the correlations of particles in experiments [31]. To this end we would need to express the correlation functions G in terms of the connected correlation functions G^c (and vice versa). The equations relevant for our calculation of evolution equations up to $n = 4$ are given by

$$G_{i_1 i_2} = G^c_{i_1 i_2}, \quad G_{i_1 i_2 i_3} = G^c_{i_1 i_2 i_3}, \quad G_{i_1 i_2 i_3 i_4} = \big[G^c_{i_1 i_2 i_3 i_4} + 3 G^c_{i_1 i_2} G^c_{i_3 i_4} \big]_{\overline{1234}}, \quad G_{i_1 i_2 i_3 i_4 i_5} = \big[G^c_{i_1 i_2 i_3 i_4 i_5} + 10 G^c_{i_1 i_2} G^c_{i_3 i_4 i_5} \big]_{\overline{12345}},$$
$$G_{i_1 i_2 i_3 i_4 i_5 i_6} = \big[G^c_{i_1 i_2 i_3 i_4 i_5 i_6} + 15 G^c_{i_1 i_2} G^c_{i_3 i_4 i_5 i_6} + 10 G^c_{i_1 i_2 i_3} G^c_{i_4 i_5 i_6} + 15 G^c_{i_1 i_2} G^c_{i_3 i_4} G^c_{i_5 i_6} \big]_{\overline{123456}}. \tag{2.15}$$

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[7]The first term on the right-hand side of each equation in (2.14), contains the gradient of the background velocity, ∂u, and is thus of order k, the characteristic background wave number, which we treat as being of the same order as $q^2 \sim \varepsilon_q^2$, as in Refs. [19,20].
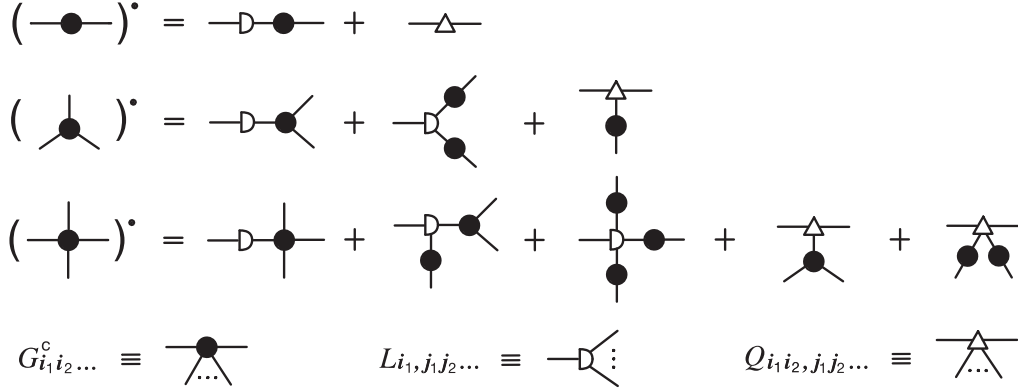
FIG. 1. Diagrammatic representation of the evolution equations (2.16) for multipoint connected correlation functions. The dot on the left-hand side of each diagrammatic equation represents material derivative $u \cdot \partial$. The open half-circle with two legs represents $L_{i_1, j_1} - \delta_{i_1 j_1} (y_1 \cdot \partial u) \cdot (\partial / \partial x_1)$, i.e., it includes the first term on the right-hand side of each equation in (2.16). It is straightforward to generate the evolution equation for G_{n+1}^c recursively by adding in each diagram representing equation for G_n^c either one new leg to filled vertices or one new G_2 correlator to open vertices. This recursive rule can be readily verified using Eq. (2.23); see also Fig. 2. Similar diagrammatic representation applies to evolution equations (3.17) we derive below.

Using Eqs. (2.14) and (2.15) and keeping the leading order terms in ε, we obtain the evolution equations for the "raw" connected functions for $n = 2, 3, 4$:

$$u \cdot \partial^{(x)} G_{i_1 i_2}^c(x_1, x_2) = 2 \left[-(y_1 \cdot \partial u) \cdot \frac{\partial}{\partial x_1} G_{i_1 i_2}^c(x_1, x_2) + L_{i_1, j_1}(x_1) G_{j_1 i_2}^c(x_1, x_2) + Q_{i_1 i_2}(x_1, x_2) + \mathcal{O}(\varepsilon_q^2 \varepsilon^2) \right]_{\overline{12}}, \quad (2.16a)$$

$$u \cdot \partial^{(x)} G_{i_1 i_2 i_3}^c(x_1, x_2, x_3) = 3 \left[-(y_1 \cdot \partial u) \cdot \frac{\partial}{\partial x_1} G_{i_1 i_2 i_3}^c(x_1, x_2, x_3) + L_{i_1, j_1}(x_1) G_{j_1 i_2 i_3}^c(x_1, x_2, x_3) \right.$$

$$\left. + L_{i_1, j_1 j_2}(x_1) G_{j_1 i_2}^c(x_1, x_2) G_{j_2 i_3}^c(x_1, x_3) + 2 Q_{i_1 i_2, j_1}(x_1, x_2) G_{j_1 i_3}^c(x_1, x_3) + \mathcal{O}(\varepsilon_q^2 \varepsilon^3) \right]_{\overline{123}}, \quad (2.16b)$$

$$u \cdot \partial^{(x)} G_{i_1 i_2 i_3 i_4}^c(x_1, x_2, x_3, x_4) = 4 \left[-(y_1 \cdot \partial u) \cdot \frac{\partial}{\partial x_1} G_{i_1 i_2 i_3 i_4}^c(x_1, x_2, x_3, x_4) + L_{i_1, j_1}(x_1) G_{j_1 i_2 i_3 i_4}^c(x_1, x_2, x_3, x_4) \right.$$

$$+ 3 L_{i_1, j_1 j_2}(x_1) G_{j_1 i_2}^c(x_1, x_2) G_{j_2 i_3 i_4}^c(x_1, x_3, x_4) + L_{i_1, j_1 j_2 j_3}(x_1) G_{j_1 i_2}^c(x_1, x_2) G_{j_2 i_3}^c(x_1, x_3) G_{j_3 i_4}^c(x_1, x_4)$$

$$\left. + 3 Q_{i_1 i_2, j_1}(x_1, x_2) G_{j_1 i_3 i_4}^c(x_1, x_3, x_4) + 3 Q_{i_1 i_2, j_1 j_2}(x_1, x_2) G_{j_1 i_3}^c(x_1, x_3) G_{j_2 i_4}^c(x_1, x_4) + \mathcal{O}(\varepsilon_q^2 \varepsilon^4) \right]_{\overline{1234}}. \quad (2.16c)$$

The diagrammatic representation of Eqs. (2.16) is shown in Fig. 1. We find that the leading terms (of order $\varepsilon_q^2 \varepsilon^{n-1}$) are represented by connected tree diagrams.

C. Arbitrary n

Our derivation can be extended beyond $n \leqslant 4$ to higher-order n-point functions, which are also of interest from the experimental point of view [32]. In this subsection we discuss generalization of Eqs. (2.15) and (2.16) to multipoint connected functions of arbitrary order n. In this work we do not need to use such general-n equations, but these equations could facilitate the extension of our results to higher n in future research. They also shed more light on the structure of the $n \leqslant 4$ equations we *do* use.

To start, we need to express correlators G in terms of connected correlators G^c, similarly to Eq. (2.15). The correlator $G_{i_1 \dots i_n}$ is a sum of all possible nonequivalent products of G^c

with indices $i_1 \dots i_n$ divided between the G^c factors in all possible ways. As in Eq. (2.15) we can group these terms into sets within each of which the difference between the terms is a permutation of the indices; e.g., $G_{i_1 i_2} G_{i_3 i_4} + G_{i_1 i_3} G_{i_2 i_4} + G_{i_1 i_4} G_{i_2 i_3}$ is such a set. Using the permutation average notation Eq. (2.8) we can represent each such a set of terms by a single term. The resulting combinatorial factors will simply count the number of terms in each such "equivalent by permutation" set. We thus find the generalization of Eq. (2.15) in the form

$$G_{i_1 \dots i_n} = \sum_{k=1}^{n} \sum_{\{n_1, \dots, n_k\}} \{n_1, \dots, n_k\}! \, G^c_{\underbrace{i_1 \dots i_{n_1}}_{n_1}} G^c_{\underbrace{i_{n_1+1} \dots i_{n_1+n_2}}_{n_2}}$$

$$\cdots G^c_{\underbrace{i_{n-n_k+1} \dots i_n}_{n_k}} \Big|_{\overline{1 \dots n}}, \quad (2.17)$$

where the inner sum is over all ordered sets of integer numbers $\{n_1, \ldots, n_k\} \in \mathbb{N}^k$, such that $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_k$ and $n_1 + \cdots + n_k = n$. Each set describes a partition of the n indices $i_1, \ldots, i_n$ into k groups,

$$\underbrace{i_1, \ldots, i_{n_1}}_{n_1} \,\Big|\, \underbrace{i_{n_1+1}, \ldots, i_{n_1+n_2}}_{n_2} \,\Big|\, \ldots \,\Big|\, \underbrace{i_{n-n_k+1} \ldots i_n}_{n_k}, \quad (2.18)$$

in a way that each term in the sum in Eq. (2.17) is different.

The factor $\{n_1, \ldots, n_k\}!$ is the order of the group of permutations of n indices $i_1, \ldots, i_n$ which leaves the product of G^c's in Eq. (2.17) unchanged due to the symmetry of each G^c with respect to its own indices as well as the commutativity of the product of G^c's themselves. In other words, the factor counts how many identical terms of a given type appear in the sum over *all $n!$* permutations of n indices:

$$\{n_1, \ldots, n_k\}! \equiv \frac{n!}{n_1! \cdots n_k! \, k_1! \cdots k_n!}, \quad (2.19)$$

where k_m is the count of times a given integer m appears in the given set $\{n_1, \ldots, n_k\}$; e.g., for $n = 7$ set $\{2, 2, 3\}$ the counts are $k_1 = 0$, $k_2 = 2$ and $k_3 = 1$. The denominator in Eq. (2.19) can be viewed as a symmetry factor corresponding to a diagram with k vertices (factors of G^c) with $n_1, \ldots, n_k$ equivalent legs (indices) each. In this picture, k_m is the number of equivalent vertices with m legs.

By definition, $k_m = 0$ for $m > n$ and $\sum_m m k_m = n$. For a given k in the outer sum in Eq. (2.17) all terms must obey $k_1 + \cdots + k_n = k$. This ensures that all terms in Eq. (2.17) with the same k are of the same order in ε, i.e., $\mathcal{O}(\varepsilon^{n-k})$, according to Eq. (2.13).

The inverse of Eq. (2.17) gives connected correlator G^c in terms of correlators G's:

$$G^c_{i_1 \ldots i_n} = \sum_{k=1}^{n} \sum_{\{n_1, \ldots, n_k\}} (-1)^{k-1} (k-1)! \{n_1, \ldots, n_k\}! \, G_{\underbrace{i_1 \ldots i_{n_1}}_{n_1}}$$

$$\times \, G_{\underbrace{i_{n_1+1} \ldots i_{n_1+n_2}}_{n_2}} \cdots G_{\underbrace{i_{n-n_k+1} \ldots i_n}_{n_k}} \Bigg|_{1 \ldots n}. \quad (2.20)$$

This relation can be obtained by noting that the connected correlator generating function, $g_c(\mu) = \sum_{n=1}^{\infty} \frac{1}{n!} G^c_n \mu^n$, and the correlator generating function, $g(\mu) = 1 + \sum_{m=1}^{\infty} \frac{1}{m!} G_m \mu^m$, are related by $g_c(\mu) = \ln g(\mu)$; i.e.,

$$\sum_{n=1}^{\infty} \frac{1}{n!} G^c_n \mu^n = \ln\left(1 + \sum_{m=1}^{\infty} \frac{1}{m!} G_m \mu^m\right)$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\sum_{m=1}^{\infty} \frac{1}{m!} G_m \mu^m\right)^k, \quad (2.21)$$

where the logarithmic function is expanded in the last equality. Equating the terms of order μ^n on both sides, and noting that the contributions from $\left(\sum_m \frac{1}{m!} G_m \mu^m\right)^k$ to the terms of order μ^n are simply given by

$$\sum_{\{n_1, \ldots, n_k\}} \frac{k!}{k_1! \ldots k_n!} \frac{1}{n_1! \ldots n_k!} G_{n_1} \cdots G_{n_k} \mu^n, \quad (2.22)$$

where $n_1 + \cdots + n_k = n$ and $k_1 + \cdots + k_n = k$, one obtains Eq. (2.20) immediately. For the sake of notation simplicity, we suppress the multivariable indices here, for instance, $G_m \mu^m \equiv G_{i_1 \ldots i_m} \mu_{i_1} \cdots \mu_{i_m}$.

Using Eqs. (2.10), (2.13), (2.17), and (2.20), we arrive at the generic equation for n-point connected correlation function. At leading order ($\sim \varepsilon_q^2 \varepsilon^{n-1}$) it can be obtained by induction from Eqs. (2.16) that

$$u \cdot \partial^{(x)} G^c_{i_1 \ldots i_n}(x_1, \ldots, x_n)$$

$$= n\Bigg[-(y_1 \cdot \partial u) \cdot \frac{\partial}{\partial x_1} G^c_{i_1 \ldots i_n}(x_1, \ldots, x_n) + \sum_{k=1}^{n-1} \sum_{\substack{\{n_1, \ldots, n_k\} \\ n_1 + \cdots + n_k = n-1}} \{n_1, \ldots, n_k\}! \, L_{i_1, \, j_1 \ldots j_k}(x_1) G^c_{j_1 \underbrace{i_2 \ldots}_{n_1}}(x_1, x_2, \ldots) \cdots G^c_{j_k \underbrace{\ldots i_n}_{n_k}}(x_1, \ldots, x_n)$$

$$+ (n-1) \sum_{k=0}^{n-2} \sum_{\substack{\{n_1, \ldots, n_k\} \\ n_1 + \cdots + n_k = n-2}} \{n_1, \ldots, n_k\}! \, Q_{i_1 i_2, \, j_1 \ldots j_k}(x_1, x_2) G^c_{j_1 \underbrace{i_3 \ldots}_{n_1}}(x_1, x_3, \ldots) \cdots G^c_{j_k \underbrace{\ldots i_n}_{n_k}}(x_1, \ldots, x_n) \Bigg]_{\overline{1 \ldots n}}. \quad (2.23)$$

The diagrammatic representation of Eq. (2.23) is sketched in Fig. 2.

D. Confluent formalism for multipoint equal-time correlators

An important ingredient for our derivation of the fluctuation evolution equations in *relativistic* hydrodynamics is the confluent formalism, which allows us to covariantly describe fluctuations in the local rest frame of the fluid. We follow the approach introduced in Ref. [19] for two-point correlators, but generalize it here to arbitrary n-point correlators. Most of the notations in this section are similar to the ones introduced in Ref. [19] and are summarized in Appendix D for convenience.

The four-velocities u of the flow in two space-time points x and $x + \Delta x$ can be related by a Lorentz boost Λ which we
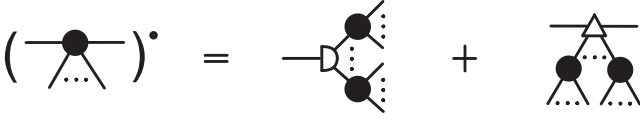
FIG. 2. Diagrammatic representation of the evolution equations for generic multipoint connected correlation functions. Similarly to Fig. 1, the first term on the right-hand side of Eq. (2.23) is included into the diagram with an open semicircle with two legs. The last two terms are drawn in this figure, with all possible combinatorial arrangements at leading order. See Fig. 1 for the meaning of the diagram elements, and for the explicit representation of these diagrams when $n = 2, 3, 4$.

define as[8]

$$\Lambda(\Delta x)u(x + \Delta x) = u(x). \qquad (2.24)$$

For infinitesimal Δx, the four-velocities are different by infinitesimal amount $\Delta u = (\Delta x \cdot \partial)u$ and the boost can be represented by the following matrix close to unity

$$\Lambda(\Delta x) = \mathbb{1} + \Delta\Lambda, \quad \text{where} \quad \Delta\Lambda_\mu^\nu(\Delta x) = -u_\mu \Delta u^\nu + u^\nu \Delta u_\mu. \qquad (2.25)$$

For n-point function the fluctuations of the n variables are evaluated at different points $(x_1, \ldots, x_n)$ with different local velocities $u(x_i)$. Before comparing or correlating these local variables it is natural to boost all of them to the same local rest frame at the midpoint

$$x \equiv \sum_{i=1}^{n} \frac{x_i}{n}. \qquad (2.26)$$

If the variables are components of a four-vector (such as δu^μ), then the boost mixes those components using matrix Λ given explicitly in Eq. (2.25). If the variables are scalars (such as δm or δp) the boost is trivial (identity), or $\Delta\Lambda = 0$. To treat these two cases simultaneously in our formalism, we introduce (as in Ref. [19]) an object u^i whose components equal to u^μ when $i = 0, \ldots, 3$, and zero for all other values of i, corresponding to scalar hydrodynamic variables. Then we can express infinitesimal boost from $x + \Delta x$ to x, as a matrix

$$\Delta\Lambda_i^j(\Delta x) = -\Delta x^\mu \bar{\omega}_{\mu i}^j, \qquad (2.27)$$

where we defined the confluent connection as in Ref. [19],

$$\bar{\omega}_{\mu i}^j = u_i \partial_\mu u^j - u^j \partial_\mu u_i. \qquad (2.28)$$

Boosting all variables into the local rest frame at the midpoint using $\Lambda(x_i - x)$, we can then define the confluent n-point correlator $\bar{G}$ and express it in terms of the "raw"

correlators G:

$$\begin{aligned}
\bar{G}_{i_1 \ldots i_n}&(x_1, \ldots, x_n) \\
&\equiv \Lambda_{i_1}^{j_1}(x_1 - x) \cdots \Lambda_{i_n}^{j_n}(x_n - x) G_{j_1 \ldots j_n}(x_1, \ldots, x_n) \\
&\approx G_{i_1 \ldots i_n}(x_1, \ldots, x_n) - n\big[(x_1 - x)^\mu \bar{\omega}_{\mu i_1}^{j_1} \\
&\quad \times G_{j_1 i_2 \ldots i_n}(x_1, \ldots, x_n)\big]_{\overline{1 \ldots n}},
\end{aligned} \qquad (2.29)$$

where in the second line we have kept only the leading term in the expansion in powers of $x_i - x$. The idea of the confluent correlator is illustrated in Fig. 3.

In order to describe the rate of change of a correlation function such as $\bar{G}_n$ with respect to the midpoint position x, we want to compare the values of $\bar{G}_n$ at two sets of arguments x_i and x_i'. The ordinary derivative would correspond to $x_i' = x_i + \Delta x$ with the same Δx for all i. A more natural measure in the context of hydrodynamics, however, would take into account the change of the four-velocity between the old midpoint x and the new midpoint $x' = x + \Delta x$. Specifically, we want to evaluate the function at a new set of arguments (points x_i'), which are located relative to the new midpoint x' in exactly the same way (in the sense to be precisely defined below) as they were around x *in the rest frame* at midpoint. This is essential if we want to evaluate the rate of change of an *equal-time* correlator.

That rest frame defined by the four-velocity $u(x + \Delta x)$, in general, is different from $u(x)$. The relative position of the points can be described by four-vectors

$$y_i(x) \equiv x_i - x. \qquad (2.30)$$

In order to preserve the relative positions in the rest frame at the midpoint we shall define new relative positions using the same boost as in Eq. (2.24), i.e., $y_i(x + \Delta x) = \Lambda(\Delta x)^{-1} y_i(x)$. This would ensure, in particular, that the time components of the relative four-vectors $y_i(x)$ in the rest frame are preserved: $y_i(x) \cdot u(x) = y_i(x + \Delta x) \cdot u(x + \Delta x)$. Which means that if we define *equal-time* correlator by $y_i(x) \cdot u(x) = 0$, the same relation remain true at the new point: $y_i(x + \Delta x) \cdot u(x + \Delta x) = 0$. In other words, if the points x_i are equal-time in the frame at their midpoint, so are the points x_i'.

Therefore, we define the confluent derivative via the following relation, where Δx is infinitesimal:

$$\begin{aligned}
\Delta x \cdot \bar{\nabla} \bar{G}_{i_1 \ldots i_n} &\equiv \Lambda(\Delta x)_{i_1}^{j_1} \ldots \Lambda(\Delta x)_{i_n}^{j_n} \bar{G}_{j_1 \ldots j_n}(x_1', \ldots, x_n') \\
&\quad - \bar{G}_{i_1 \ldots i_n}(x_1, \ldots, x_n), \\
&\text{where} \quad x_i' = x + \Delta x + \Lambda(\Delta x)^{-1} y_i(x) \\
&\text{and} \quad x_i = x + y_i(x). \qquad (2.31)
\end{aligned}$$

Note that the variables which are being correlated are also boosted accordingly to make sure that only their change with respect to the local rest frame is measured, and not the change of the *components* of these variables due to the change of the rest frame itself. Taking the limit $\Delta x \to 0$, we can write Eq. (2.31) in terms of the partial derivatives of G and the boost
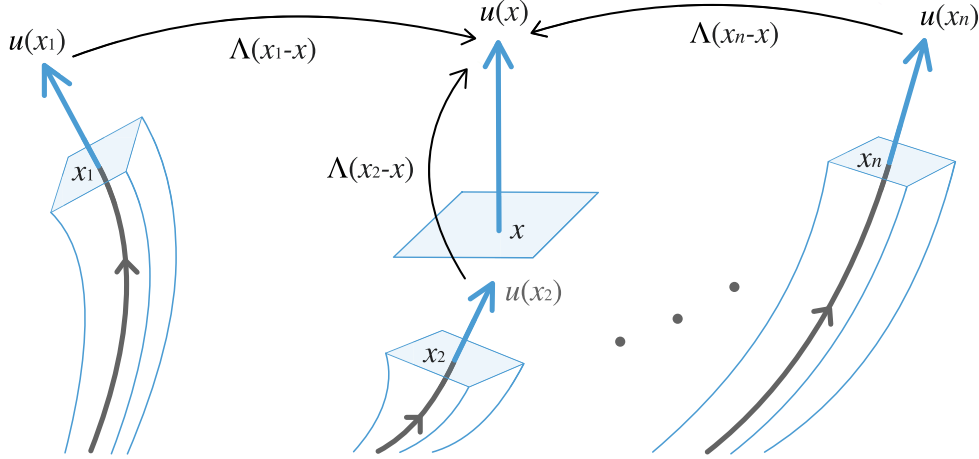
FIG. 3. An illustration, in space-time, of the process involved in constructing the confluent n-point function $\bar{G}$ from the "raw" n-point function G. All n fluctuation fields are boosted from their respective local rest frames at x_i to the same frame at the midpoint x.

connection $\bar{\omega}$:

$$\bar{\nabla}_\mu \bar{G}_{i_1 \ldots i_n} \equiv n \left[\left(\frac{\partial}{\partial x_1^\mu} - \bar{\omega}_{\mu\beta}^\alpha y_1^\beta \frac{\partial}{\partial x_1^\alpha} \right) \bar{G}_{i_1 \ldots i_n} - \bar{\omega}_{\mu i_1}^{j_1} \bar{G}_{j_1 \ldots i_n} \right]_{\overline{1 \ldots n}}. \tag{2.32}$$

At this point it is convenient to introduce the derivative with respect to the midpoint, x which was already defined in Eq. (2.10), as well as the derivatives with respect to separation vectors y_i at fixed midpoint x:

$$\partial^{(x)} \equiv \sum_{i=1}^n \frac{\partial}{\partial x_i}, \quad \partial^{(y_i)} \equiv \frac{\partial}{\partial x_i} - \frac{1}{n} \sum_{j=1}^n \frac{\partial}{\partial x_j} \equiv \frac{\partial}{\partial x_i} - \frac{1}{n} \frac{\partial}{\partial x}. \tag{2.33}$$

Note that y_i variables are not independent, since $\sum_{i=1}^n y_i = 0$, so the derivative $\partial^{(y_i)}$ has unusual, but simple, properties, e.g., $\partial^{(y_i)} x = 0$, $\partial^{(y_i)} y_j = \delta_{ij} - 1/n$. In terms of such derivatives Eq. (2.32) reads

$$\bar{\nabla}_\mu \bar{G}_{i_1 \ldots i_n} \equiv \partial_\mu^{(x)} \bar{G}_{i_1 \ldots i_n} - n \left[\bar{\omega}_{\mu\beta}^\alpha y_1^\beta \partial_\alpha^{(y_1)} \bar{G}_{i_1 \ldots i_n} + \bar{\omega}_{\mu i_1}^{j_1} \bar{G}_{j_1 \ldots i_n} \right]_{\overline{1 \ldots n}}. \tag{2.34}$$

Another derivative which will be even more useful when discussing the Wigner transform is obtained by introducing the local tetrad consisting of vector $u(x)$ and a triad $e_a(x)$, $a = 1, 2, 3$, thus expressing vector y in components: $y(x) = e_a(x) y^a + u(x) y^u$, where $y^u = -u(x) \cdot y$ is the time component in the local rest frame at point x. In this paper we shall only consider equal-time correlators, so we work with $y^u = 0$ [cf. Fig. 5(a)]. We can then define a derivative where y^a and $y^u = 0$ are fixed, i.e.,

$$\Delta x \cdot \mathring{\partial} \bar{G}_{i_1 \ldots i_n} \equiv \bar{G}_{i_1 \ldots i_n}(x_1', \ldots, x_n') - \bar{G}_{i_1 \ldots i_n}(x_1, \ldots, x_n), \quad \text{where } x_i' = x + \Delta x + e_a(x + \Delta x) y_i^a \quad \text{and } x_i = x + e_a(x) y_i^a. \tag{2.35}$$

Taking the limit $\Delta x \to 0$ we find:

$$\mathring{\partial}_\mu \bar{G}_{i_1 \ldots i_n} \equiv n \left[\left\{ \frac{\partial}{\partial x_1^\mu} + \mathring{\omega}_{\mu b}^a y_1^b \frac{\partial}{\partial x_1^a} + (\partial_\mu u_\lambda) y_1^\lambda \left(u \cdot \frac{\partial}{\partial x_1} \right) \right\} \bar{G}_{i_1 \ldots i_n} \right]_{\overline{1 \ldots n}}$$
$$= \partial_\mu^{(x)} \bar{G}_{i_1 \ldots i_n} + n \left[\mathring{\omega}_{\mu b}^a y_1^b \partial_a^{(y_1)} \bar{G}_{i_1 \ldots i_n} - \bar{\omega}_{\mu\lambda}^\nu y_1^\lambda \partial_\nu^{(y_1)} \bar{G}_{i_1 \ldots i_n} \right]_{\overline{1 \ldots n}}, \tag{2.36}$$

where we introduced

$$\mathring{\omega}_{\mu b}^a \equiv e_\nu^a \partial_\mu e_b^\nu \tag{2.37}$$

with $a, b = 1, 2, 3$; the connection associated with the freedom of choice of the local basis triad e_a at each point x. Using this connection we expressed the derivatives of e_a appearing in Eq. (2.36) as

$$\partial_\mu e_a^\nu = \left(e_b^\nu e_\lambda^b - u^\nu u_\lambda \right) \partial_\mu e_a^\lambda = e_b^\nu \mathring{\omega}_{\mu a}^b + e_a^\lambda u^\nu (\partial_\mu u_\lambda) = e_b^\nu \mathring{\omega}_{\mu a}^b - e_a^\lambda \bar{\omega}_{\mu\lambda}^\nu \tag{2.38}$$

where we also used Eq. (2.28) and $u \cdot y_i = 0$. We also defined the derivative

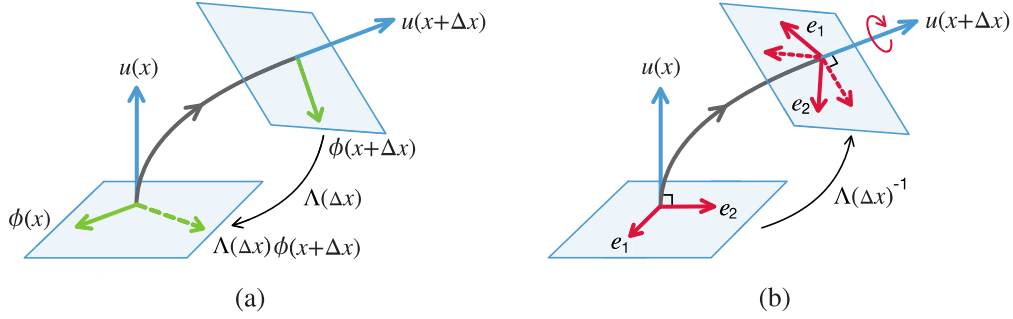$$\partial_a^{(y)} \equiv e_a^\mu \partial_\mu^{(y)}, \tag{2.39}$$

FIG. 4. The origin of two connections $\bar{\omega}$ and $\mathring{\omega}$ in the confluent derivative applying to confluent n-point functions. Panel (a) explains the origin of boost connection $\bar{\omega}$, brought out by boosting the fluctuation variable $\phi(x + \Delta x)$ from the local rest frame of $u(x + \Delta x)$ to the local rest frame of $u(x)$, i.e., $\Lambda(\Delta x)\phi(x + \Delta x)$, using the same boost that takes velocity $u(x + \Delta x)$ to $u(x)$. The action of the boost on ϕ is determined by the tensor rank of ϕ and is illustrated in the figure for the case of a four-vector (e.g., velocity fluctuation). Panel (b) explains the origin of spin connection due to the confluent motion of equal-time surface and the arbitrariness of the local triad e_a choice. Under the boost from the local rest frame of $u(x)$ to $u(x + \Delta x)$ every coordinate vector y in the local equal-time three-hyperplane (represented by the blue rectangle), subject to the constraint $u(x) \cdot y = 0$, must also be boosted simultaneously by $\Lambda(\Delta x)^{-1}$. This includes the basis $e_a(x)$ in that hyperplane (red vectors; only two dimensions $a = 1, 2$ are shown). Upon the boost the basis vectors map onto $\Lambda(\Delta x)^{-1} e_a(x)$ (dashed red vectors) at point $x + \Delta x$. The additional rotation (indicated by the red circular arrow) in the equal-time plane needed to align the boosted basis (dashed red) with the local basis $e_a(x + \Delta x)$ (solid red) gives rise to the $\mathring{\omega}$ connection.

which is essentially the derivative with respect to three-component vector y^a, which, similarly to $\partial_\mu^{(y_i)}$ defined in Eq. (2.33), is preserving the constraint $\sum_{i=1}^n y_i = 0$.

Substituting Eq. (2.36) into Eq. (2.34), we find the relationship between the derivatives we defined:

$$\bar{\nabla}_\mu \bar{G}_{i_1 \ldots i_n} \equiv \mathring{\partial}_\mu \bar{G}_{i_1 \ldots i_n} - n \big[\mathring{\omega}^a_{\mu b} y_1^b \partial_a^{(y_1)} \bar{G}_{i_1 \ldots i_n} + \bar{\omega}^{j_1}_{\mu i_1} \bar{G}_{j_1 \ldots i_n} \big]_{\overline{1 \ldots n}}. \tag{2.40}$$

The confluent derivative in Eq. (2.40) incorporating the two connections defined in Eqs. (2.28) and (2.37) is illustrated in Fig. 4.

Considering now a connected correlation function, $G^{\rm c}(x_1, \ldots, x_n)$ [see Eq. (2.15)], and using the generalized Wigner transform introduced in Ref. [22], we define the Wigner function

$$W_n(x; \boldsymbol{q}_1, \ldots, \boldsymbol{q}_n) = \int \left[\prod_{i=1}^n d^3 y_i^a \, e^{-i q_{ia} y_i^a} \right] \delta^{(3)} \left(\frac{1}{n} \sum_{i=1}^n y_i^a \right) \bar{G}_n^{\rm c}(x + e_a y_1^a, \ldots, x + e_a y_n^a), \tag{2.41}$$

where q^a denote components of three-vector $\boldsymbol{q} = \{q^a\} \in \mathbb{R}^3$ with $a = 1, 2, 3$ which is the wave-number conjugate to vector y^a. To avoid clutter in our notation, we replaced the indices $i_1, \ldots, i_n$, which are the same on $W_{i_1 \ldots i_n}$ and $\bar{G}_{i_1 \ldots i_n}^{\rm c}$, with a single index n. The inverse transformation of Eq. (2.41) is given by

$$\bar{G}_n^{\rm c}\big(x + e_a y_1^a, \ldots, x + e_a y_n^a\big) = \int \left[\prod_{i=1}^n \frac{d^3 \boldsymbol{q}_i}{(2\pi)^3} e^{i q_{ia} y_i^a} \right] \delta^{(3)} \left(\sum_{i=1}^n \frac{\boldsymbol{q}_i}{2\pi} \right) W_n(x; \boldsymbol{q}_1, \ldots, \boldsymbol{q}_n). \tag{2.42}$$

Some of the features of this transformation of variables are illustrated in Fig. 5. In particular we note that, due to the constraint $\sum_i y_i = 0$ implemented by the delta function in Eq. (2.41), the Wigner function is invariant with respect to the shift of all wave numbers $\boldsymbol{q}_i$ by the same vector. That means we can constrain the value of $\sum_i \boldsymbol{q}_i$ without losing any information about the dependence of W on its arguments (of course, this is related to the fact that W_n has one argument more than G_n). The natural choice of the constraint is $\sum_i \boldsymbol{q}_i = 0$ as in Eq. (2.42). An intuitive way to understand this is to think of W_n as an n-point "amplitude" and of $\boldsymbol{q}_i$ as the corresponding "momenta" flowing in. Then the constraint is simply a reflection of "momentum" conservation.

It is easy to see that the derivative of the Wigner function with respect to x at fixed $\boldsymbol{q}_i$'s,

$$\Delta x \cdot \mathring{\partial} W_n(x; \boldsymbol{q}_1, \ldots, \boldsymbol{q}_n)$$
$$= W_n(x + \Delta x; \boldsymbol{q}_1, \ldots, \boldsymbol{q}_n) - W_n(x; \boldsymbol{q}_1, \ldots, \boldsymbol{q}_n), \tag{2.43}$$

is the Wigner transform of $\mathring{\partial} \bar{G}^{\rm c}$; i.e., the derivative $\mathring{\partial}$ commutes with the Wigner transform.

It is also easy to see that the derivative defined in Eqs. (2.39) and (2.33) upon Wigner transform becomes simply the multiplication by the corresponding $i q_a$:

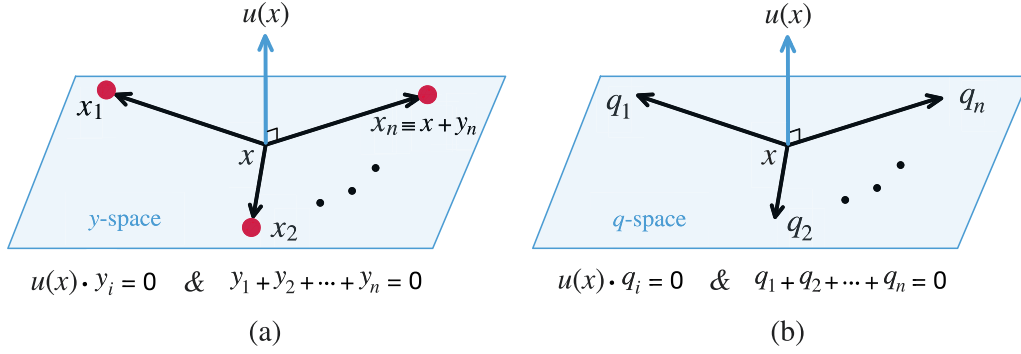$$\partial_a^{(y_i)} \xrightarrow{\text{W.T.}} i q_{ia}. \tag{2.44}$$

FIG. 5. An illustration of the vectors relevant to the Wigner transform of equal-time confluent correlators living in coordinate y-space [shown in panel (a)] to Wigner functions living in wave-number $\boldsymbol{q}$-space [shown in panel (b)]. The wave numbers drawn in the figure are four-vectors $e_a^\mu q^a \equiv q^\mu$. The nature of the constraint $\sum_i \boldsymbol{q}_i = 0$ is discussed in the text.

We shall define $\bar{\nabla}W$ as Wigner transform of $\bar{\nabla}\bar{G}^c$. Using Eq. (2.34) it is then easy to show that

$$\bar{\nabla}_\mu W_{i_1\ldots i_n} = \mathring{\partial}_\mu W_{i_1\ldots i_n} + n\left(\mathring{\omega}_{\mu b}^a q_{1a}\frac{\partial}{\partial q_{1b}}W_{i_1\ldots i_n} - \bar{\omega}_{\mu i_1}^{j_1}W_{j_1 i_2\ldots i_n}\right)_{\overline{1\ldots n}}. \tag{2.45}$$

Now let us turn to the derivation of the evolution equation of connected Wigner functions. Applying the confluent derivative in Eq. (2.45) along $u(x)$ to Eq. (2.41), we express the result in terms of the "raw" connected correlation functions G^c as follows:

$$u \cdot \bar{\nabla}W_{i_1\ldots i_n}(x;\boldsymbol{q}_1,\ldots,\boldsymbol{q}_n)$$

$$= u \cdot \mathring{\partial}W_{i_1\ldots i_n} - nu^\mu\left(\bar{\omega}_{\mu i_1}^{j_1}W_{j_1 i_2\ldots i_n} - \mathring{\omega}_{\mu b}^a q_{1a}\frac{\partial}{\partial q_{1b}}W_{i_1\ldots i_n}\right)_{\overline{1\ldots n}}$$

$$= \int\left[\prod_{i=1}^n d^3 y_i\, e^{-iq_{ia}y_i^a}\right]\delta^{(3)}\left(\frac{1}{n}\sum_{i=1}^n y_i\right)\left\{u \cdot \partial G_{i_1\ldots i_n}^c - n\left[(u^\mu\bar{\omega}_{\mu\lambda}^\nu y_1^\lambda\partial_\nu^{(y_1)}\delta_{i_1}^{j_1} + y_1^\mu\bar{\omega}_{\mu i_1}^{j_1}u \cdot \partial + u^\mu\bar{\omega}_{\mu i_1}^{j_1})G_{j_1 i_2\ldots i_n}^c\right]_{\overline{1\ldots n}}\right\}, \tag{2.46}$$

where we used Eqs. (2.29), (2.34), and (2.36).

We then apply the evolution equations (2.16) [and, generically, Eq. (2.23)] for G_n^c to convert local rest frame time derivatives $u \cdot \partial$ into spatial derivatives and perform the inverse Wigner transform using Eq. (2.42). As a result, we express the confluent local rest frame time derivative of the Wigner function on the left-hand side of Eq. (2.46) in terms of the Wigner functions themselves, thus obtaining a set of local evolution equations.

The explicit form of the evolution equations for Wigner functions will depend on the explicit form of multilinear operators L and Q in Eqs. (2.16) that need to be substituted into Eq. (2.46). Below we shall obtain this explicit form for an important subset of correlators in hydrodynamics.

III. SPECIFIC ENTROPY FLUCTUATIONS IN RELATIVISTIC HYDRODYNAMICS

A. Stochastic equation for specific entropy fluctuations

With the generic formalism established, we now apply it to a more specific hydrodynamic framework governed by the local equations for the conservation of energy, momentum, and charge:

$$\partial_\mu \check{T}^{\mu\nu} = 0, \tag{3.1a}$$

$$\partial_\mu \check{J}^\mu = 0, \tag{3.1b}$$

supplemented by constitutive relations for the stress tensor $T^{\mu\nu}$ and charge current J^μ in the Landau frame for the stochastic relativistic hydrodynamics [33],

$$\check{T}^{\mu\nu} = \check{\epsilon}\check{u}^\mu\check{u}^\nu + \check{p}\check{\Delta}^{\mu\nu} + \check{\Pi}^{\mu\nu} + \check{S}^{\mu\nu}, \tag{3.2a}$$

$$\check{J}^\mu = \check{n}\check{u}^\mu + \check{v}^\mu + \check{I}^\mu, \tag{3.2b}$$

where ϵ and n are the energy density and charge density respectively, measured in the local rest frame; u is the four-velocity already introduced in Eq. (2.1); $\Delta_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ is the transverse projection operator satisfying $\Delta^{\mu\nu}u_\nu = 0$; the pressure p, appearing as the coefficient of $\Delta^{\mu\nu}$ in the ideal part of the stress tensor, is determined by chosen primary hydrodynamic variables (such as ϵ and n) through the equation of state. The explicit forms of the dissipative parts, denoted by $\Pi^{\mu\nu}$ and v^μ for stress tensor and charge current respectively, are determined by applying the second law of thermodynamics [2]:

$$\Pi^{\mu\nu} \equiv -2\eta\sigma^{\mu\nu} - \zeta\theta\Delta^{\mu\nu}, \quad v^\mu \equiv -\lambda\Delta_\nu^\mu\partial^\nu\alpha, \tag{3.3}$$

where $\alpha \equiv \beta\mu \equiv \mu/T$ is the chemical potential to temperature ratio, and gradients of velocity are decomposed into the

traceless and trace parts where

$$\sigma^{\mu\nu} \equiv \theta^{\mu\nu} - \frac{1}{3}\theta\Delta^{\mu\nu}, \quad \theta^{\mu\nu} \equiv \frac{1}{2}\Delta^{\mu\alpha}\Delta^{\nu\beta}(\partial_\alpha u_\beta + \partial_\beta u_\alpha),$$
$$\theta \equiv \theta^\mu_\mu = \partial \cdot u. \tag{3.4}$$

The transport coefficients, denoted by η, ζ, and λ above, are shear viscosity, bulk viscosity, and charge conductivity respectively.

Equations (3.2), accompanied by the constitutive equations (3.2) become stochastic due to the presence of the microscopic scale noises $\check{I}^\mu$ and $\check{S}^{\mu\nu}$. They noises are defined similarly to Eqs. (2.1) and (2.2):

$$\check{I}^\mu = \check{H}^{I}{}_a^\mu \eta_a^I, \quad \text{where}$$
$$\langle \eta_a^I(x_1)\eta_b^I(x_2)\rangle = 2\delta_{ab}\delta^{(4)}(x_1 - x_2) \quad \text{and} \quad H^{I}{}_a^\mu = \sqrt{\lambda}e_a^\mu, \tag{3.5}$$

with $a = 1, 2, 3$ since there are three independent noises (random currents) in the local rest frame, and e_a^μ is an arbitrary spatial basis triad in the local rest frame: $e_a \cdot u = 0$ (as already defined and discussed in Ref. [19] and in the previous section). As a result $(H^{I}{}_a^\mu)H^{I}{}_a^\nu = \lambda\Delta^{\mu\nu}$ (note that the choice of the triad e_a does not matter, since $e_a^\mu e_a^\nu = \Delta^{\mu\nu}$). We define $\check{S}^{\mu\nu}$ accordingly. Since there are six independent noise variables (corresponding to random stress tensor in local rest frame), the components of $\check{S}^{\mu\nu}$ are expressed in terms of a symmetric rank-three random matrix η_{ab}^S, i.e.,

$$\check{S}^{\mu\nu} = \check{H}^{S}{}_{ab}^{\mu\nu} \eta_{ab}^S,$$
$$\text{where} \quad \langle \eta_{ab}^S(x_1)\eta_{cd}^S(x_2)\rangle = (\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})\delta^{(4)}(x_1 - x_2) \quad \text{and}$$

$$H^{S}{}_{ab}^{\mu\nu} = \sqrt{\frac{T\eta}{2}}\left(e_a^\mu e_b^\nu + e_b^\mu e_a^\nu - \frac{2}{3}\Delta^{\mu\nu}\delta_{ab}\right)$$
$$+ \sqrt{\frac{T\zeta}{3}}\Delta^{\mu\nu}\delta_{ab}. \tag{3.6}$$

Similarly to Eqs. (3.5) and (3.6), isotropy requires that the random current and stress noises are statistically independent:

$$\langle \eta_a^I(x_1)\eta_{bc}^S(x_2)\rangle = 0. \tag{3.7}$$

We now turn our focus to the fluctuations of specific entropy (i.e., ratio of entropy density s to charge density n), $m \equiv s/n$, which is parametrically the slowest and also the most significant hydrodynamic mode near a liquid-gas critical point, as we already discussed in Sec. I, Such a focus also underlies the Hydro+ approach in Ref. [29].

Specifically, we are going to derive deterministic equations of motion for the correlators of m. To achieve this, we first obtain the stochastic equation for the evolution of $\check{m}$ starting from the conservation equation (3.1). We find

$$\check{u} \cdot \partial\check{m} = \check{m}_n\partial^\mu\check{\lambda}\check{\Delta}_{\mu\nu}\partial^\nu\check{\alpha} + 2\check{m}_\epsilon\check{\eta}\check{\sigma}^{\mu\nu}\check{\sigma}_{\mu\nu} + \check{m}_\epsilon\check{\zeta}\check{\theta}^2 + \xi_m, \tag{3.8}$$

where

$$m_n \equiv \left(\frac{\partial m}{\partial n}\right)_\epsilon = -\frac{\beta w}{n^2}, \qquad m_\epsilon \equiv \left(\frac{\partial m}{\partial \epsilon}\right)_n = \frac{\beta}{n} \tag{3.9}$$

are expressed in terms of independent thermodynamic derivatives chosen in the set I given by Table I in Appendix A, and

$$\xi_m = -\check{m}_n\partial_\mu\check{I}^\mu + \check{m}_\epsilon\check{u}_\nu\partial_\mu\check{S}^{\mu\nu}. \tag{3.10}$$

Equaiton (3.8) has a form similar to Eq. (2.1) with $\psi = m$. Therefore the formalism of the previous section can be applied. To obtain the equation corresponding to Eq. (2.6), we expand Eq. (3.8) in the fluctuation field, δm, as well as in derivatives applying to fluctuation fields, such as $\partial\delta m = \mathcal{O}(q)$, up to $\mathcal{O}(q^2)$.[9] It is sufficient to truncate the expansion at order δm^3 to obtain equations for the four-point connected functions.[10]

While we are interested in correlators of δm, the evolution of δm also depends on fluctuations of other hydrodynamic variables, in particular on fluctuations of pressure, δp.[11] As a result, the evolution of the correlators of δm will depend on correlators of δp as well.[12] As we shall see below, to the leading order in hydrodynamic (gradient) expansion and in the regime we consider, we only need to include terms linear in δp. Correspondingly, if we only write the terms which will contribute to the correlator evolution equations in the regime we consider [e.g., Eqs. (3.15) or (3.17)], we find

$$u \cdot \partial\delta m =: u \cdot \partial\check{m} :\equiv u \cdot \partial\check{m} - \langle u \cdot \partial\check{m}\rangle$$
$$=: L_{m,m}\delta m + L_{m,p}\delta p + \frac{1}{2}L_{m,mm}\delta m\delta m + L_{m,mp}\delta m\delta p$$
$$+ \frac{1}{6}L_{m,mmm}\delta m\delta m\delta m + \frac{1}{2}L_{m,mmp}\delta m\delta m\delta p + \xi_m : \tag{3.11}$$

---

[9]We keep in mind that fluctuations, such as δm, are characterized by wave number q which is parametrically larger than the wave number k characterizing the background of mean quantities, such as m. Thus $\partial\delta m \sim q \sim \varepsilon_q$, while $\partial m \sim k \sim \varepsilon_q^2$. In the following equation the viscous terms contribute at order higher than ε_q^2 if we focus on the correlators of m only.

[10]To obtain the evolution equation for an n-point connected function at leading order, one needs to expand in fluctuation fields, Eq. (2.6), up to order ϕ^{n-1}, according to Eq. (2.10).

[11]We choose m and p as the independent variables in order to profit from the fact that the fluctuations of m and p are uncorrelated in equilibrium. In addition, m and p represent the basis of normal modes in the ideal hydrodynamics. This significantly simplifies the correlator evolution equations, as we have already observed in Ref. [20]. The simplification is even more significant for non-Gaussian fluctuations.

[12]In the regime we consider, transverse velocity correlators, which ordinarily would mix with the correlators of specific entropy (as in the equation for two-point correlator derived in Ref. [20]), can be considered relaxed (on a parametrically faster timescale) to their equilibrium values. These values are zero by isotropy and, therefore, we need not consider velocity fluctuations in this regime. Correlators involving pressure also relax faster, but their equilibrium values are not zero, as we shall discuss in more detail below.

The multilinear operators $L_{m,m...m}$ serving as expansion coefficients are more explicitly written as[13]

$$L_{m,m}\delta m \equiv L_{m,m}[\delta m] = m_n\partial^\mu\lambda\Delta_{\mu\nu}\partial^\nu\alpha_m\delta m + m_n\partial^\mu\lambda_m\delta m\Delta_{\mu\nu}\partial^\nu\alpha + (m_n)_{,m}\delta m\partial^\mu\lambda\Delta_{\mu\nu}\partial^\nu\alpha$$
$$= m_n\lambda\alpha_m\Delta_{\mu\nu}\partial^\mu\partial^\nu\delta m + \mathcal{O}(kq), \tag{3.12a}$$

$$L_{m,p}\delta p \equiv L_{m,p}[\delta p] = m_n\partial^\mu\lambda\Delta_{\mu\nu}\partial^\nu\alpha_p\delta p + m_n\partial^\mu\lambda_p\delta p\Delta_{\mu\nu}\partial^\nu\alpha + (m_n)_{,p}\delta p\partial^\mu\lambda\Delta_{\mu\nu}\partial^\nu\alpha$$
$$= m_n\lambda\alpha_p\Delta_{\mu\nu}\partial^\mu\partial^\nu\delta p + \mathcal{O}(kq), \tag{3.12b}$$

$$L_{m,mm}\delta m\delta m \equiv L_{m,mm}[\delta m, \delta m]$$
$$= m_n\lambda\alpha_{mm}\Delta_{\mu\nu}\partial^\mu\partial^\nu\delta m^2 + 2m_n\lambda_m\alpha_m\Delta_{\mu\nu}\partial^\mu\delta m\partial^\nu\delta m + 2(m_n)_{,m}\lambda\alpha_m\Delta_{\mu\nu}\delta m\partial^\mu\partial^\nu\delta m + \mathcal{O}(kq), \tag{3.12c}$$

$$L_{m,mp}\delta m\delta p \equiv L_{m,mp}[\delta m, \delta p]$$
$$= m_n\lambda\alpha_{mp}\Delta_{\mu\nu}\partial^\mu\partial^\nu\delta m\delta p + m_n\lambda_m\alpha_p\Delta_{\mu\nu}\partial^\mu\delta m\partial^\nu\delta p + m_n\lambda_p\alpha_m\Delta_{\mu\nu}\partial^\mu\delta p\partial^\nu\delta m$$
$$+ (m_n)_{,m}\lambda\alpha_p\Delta_{\mu\nu}\delta m\partial^\mu\partial^\nu\delta p + (m_n)_{,p}\lambda\alpha_m\Delta_{\mu\nu}\delta p\partial^\mu\partial^\nu\delta m + \mathcal{O}(kq), \tag{3.12d}$$

$$L_{m,mmm}\delta m\delta m\delta m \equiv L_{m,mmm}[\delta m, \delta m, \delta m]$$
$$= m_n\partial^\mu\lambda\alpha_{mmm}\Delta_{\mu\nu}\partial^\nu\delta m^3 + 3m_n\lambda_m\alpha_{mm}\Delta_{\mu\nu}\partial^\mu\delta m\partial^\nu\delta m^2$$
$$+ 3(m_n)_{,m}\lambda\alpha_{mm}\Delta_{\mu\nu}\delta m\partial^\mu\partial^\nu\delta m^2 + 3m_n\lambda_{mm}\alpha_m\Delta_{\mu\nu}\partial^\mu\delta m^2\partial^\nu\delta m + 6(m_n)_{,m}\lambda_m\alpha_m\Delta_{\mu\nu}\delta m\partial^\mu\delta m\partial^\nu\delta m$$
$$+ 3(m_n)_{,mm}\lambda\alpha_m\Delta_{\mu\nu}\delta m^2\partial^\mu\partial^\nu\delta m + \mathcal{O}(kq), \tag{3.12e}$$

where

$$\alpha_{\underbrace{m...m}_{k}\underbrace{p...p}_{\ell}} \equiv \left(\frac{\partial^\ell}{\partial p^\ell}\left(\frac{\partial^{(k)}\alpha}{\partial m^k}\right)_p\right)_m, \quad \lambda_{\underbrace{m...m}_{k}\underbrace{p...p}_{\ell}} \equiv \left(\frac{\partial^{(\ell)}}{\partial p^\ell}\left(\frac{\partial^{(k)}\lambda}{\partial m^k}\right)_p\right)_m. \tag{3.13}$$

In Eqs. (3.12) derivative ∂ applies to all factors to the right of it, and, since we only consider fluctuations of m, i.e., $i, j_1, \ldots, j_n = m$ and $u_{,m...m} = 0$ in Eq. (2.7), we have $L_{m,m...m} = F_{m,m...m}$.

The noise introduced in Eq. (3.10) satisfies

$$Q_{mm}(x_1, x_2) = \tfrac{1}{2}\langle\xi_m(x_1)\xi_m(x_2)\rangle = \tfrac{1}{2}\langle(-\breve{m}_n\partial_\mu\breve{I}^\mu + \breve{m}_\epsilon\breve{u}_\kappa\partial_\lambda\breve{S}^{\lambda\kappa})(x_1)(-\breve{m}_n\partial_\nu\breve{I}^\nu + \breve{m}_\epsilon\breve{u}_\beta\partial_\alpha\breve{S}^{\alpha\beta})(x_2)\rangle$$
$$\approx \tfrac{1}{2}\langle(\breve{m}_n\partial_\mu\breve{I}^\mu)(x_1)(\breve{m}_n\partial_\nu\breve{I}^\nu)(x_2)\rangle = m_n(x_1)m_n(x_2)\partial_\mu^{(x_1)}\partial_\nu^{(x_2)}\lambda(x_1)\Delta^{\mu\nu}(x_1)\delta^{(3)}(x_1 - x_2), \tag{3.14}$$

where in the absence of velocity fluctuation δu, terms involving stress noises $S^{\mu\nu}$ are of a higher order in the gradient expansion, i.e., they are at least of order $kq \sim \varepsilon_q^3$ whereas we truncated Eq. (3.14) at order $q^2 \sim k \sim \varepsilon_q^2$.

B. Evolution of non-Gaussian correlation functions

The evolution equations for $G_{m...m}^c$ can be readily obtained by setting external indices i's $= m$ and internal indices j's $= (m, p)$ in Eq. (2.16):

$$u \cdot \partial G_{mm}^c(x_1, x_2) = 2\left[-(y_1 \cdot \partial u) \cdot \frac{\partial}{\partial x_1}G_{mm}^c(x_1, x_2) + L_{m,m}(x_1)G_{mm}^c(x_1, x_2) + Q_{mm}(x_1, x_2)\right]_{\overline{12}}, \tag{3.15a}$$

$$u \cdot \partial G_{mmm}^c(x_1, x_2, x_3) = 3\left[-(y_1 \cdot \partial u) \cdot \frac{\partial}{\partial x_1}G_{mmm}^c(x_1, x_2, x_3) + L_{m,m}(x_1)G_{mmmm}^c(x_1, x_2, x_3)\right.$$
$$+ L_{m,p}(x_1)G_{pmm}^c(x_1, x_2, x_3) + L_{m,mm}(x_1)G_{mm}^c(x_1, x_2)G_{mm}^c(x_1, x_3)$$
$$\left. + 2Q_{mm,m}(x_1, x_2)G_{mm}^c(x_1, x_3)\right]_{\overline{123}}, \tag{3.15b}$$

$$u \cdot \partial G_{mmmm}^c(x_1, x_2, x_3, x_4) = 4\left[-(y_1 \cdot \partial u) \cdot \frac{\partial}{\partial x_1}G_{mmmm}^c(x_1, x_2, x_3, x_4) + L_{m,m}(x_1)G_{mmmmm}^c(x_1, x_2, x_3, x_4)\right.$$
$$+ L_{m,p}(x_1)G_{pmmmm}^c(x_1, x_2, x_3, x_4) + 3L_{m,mm}(x_1)G_{mm}^c(x_1, x_2)G_{mmm}^c(x_1, x_3, x_4)$$

---

[13]Note that the last two terms on the first line in Eq. (3.12) are of order kq and k^2, respectively, and are neglected on the second line, since they are parametrically smaller than the leading terms of order q^2 being kept.

$$+ 3L_{m,mp}(x_1)G^c_{mm}(x_1, x_2)G^c_{pmm}(x_1, x_3, x_4) + L_{m,mmm}(x_1)G^c_{mm}(x_1, x_2)G^c_{mm}(x_1, x_3)G^c_{mm}(x_1, x_4)$$

$$+ 3Q_{mm,p}(x_1, x_2)G^c_{pmm}(x_1, x_3, x_4) + 3Q_{mm,m}(x_1, x_2)G^c_{mmm}(x_1, x_3, x_4)$$

$$\left. + 3Q_{mm,mm}(x_1, x_2)G^c_{mm}(x_1, x_3)G^c_{mm}(x_1, x_4)\right]_{\overline{1234}}, \tag{3.15c}$$

where $L_{m,m\ldots m}$'s and Q_{mm} are given by Eqs. (3.12) and (3.14) respectively. In deriving Eqs. (3.15), we have used $G^c_{mp} = 0$ following Ref. [20].

Moreover, for ψ_i being *scalars*, Eq. (2.46) is simplified to

$$u \cdot \bar{\nabla} W_{i_1 \ldots i_n}(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n) = \int \left[\prod_{i=1}^n d^3 y_i \, e^{-iq_{ia}y_i^a} \right] \delta^3\left(\frac{1}{n} \sum_{i=1}^n y_i \right) \left\{ u \cdot \partial G^c_{i_1 \ldots i_1} + n\left[(u \cdot \partial u) \cdot y_1 u \cdot \partial^{(y_1)} G^c_{i_1 \ldots i_n} \right]_{\overline{1 \ldots n}} \right\}, \tag{3.16}$$

where we suppressed the argument x of W. Equation (3.16) applies to our situation when $\psi_i = (m, p)$. Following the procedure discussed at the end of Sec. II, we substitute Eqs. (3.15) into Eq. (3.16) and perform the inverse Wigner transform using Eq. (2.42). We then arrive at

$$\mathcal{L}[W_{mm}(\boldsymbol{q}_1, \boldsymbol{q}_2)] = \theta W_{mm}(\boldsymbol{q}_1, \boldsymbol{q}_2) + 2[L_{m,m}(\boldsymbol{q}_1, -\boldsymbol{q}_1)W_{mm}(\boldsymbol{q}_1, \boldsymbol{q}_2) + Q_{mm}(\boldsymbol{q}_1, \boldsymbol{q}_2)]_{\overline{12}}, \tag{3.17a}$$

$$\mathcal{L}[W_{mmm}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3)] = 2\theta W_{mmm}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3) + 3[L_{m,m}(\boldsymbol{q}_1, -\boldsymbol{q}_1)W_{mmm}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3) + L_{m,p}(\boldsymbol{q}_1, -\boldsymbol{q}_1)W_{pmm}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3)$$

$$+ L_{m,mm}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3)W_{mm}(-\boldsymbol{q}_2, \boldsymbol{q}_2)W_{mm}(-\boldsymbol{q}_3, \boldsymbol{q}_3) + 2Q_{mm,m}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3)W_{mm}(-\boldsymbol{q}_3, \boldsymbol{q}_3)]_{\overline{123}}, \tag{3.17b}$$

$$\mathcal{L}[W_{mmmm}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4)] = 3\theta W_{mmmm}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4) + 4[L_{m,m}(\boldsymbol{q}_1, -\boldsymbol{q}_1)W_{mmmm}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4)$$

$$+ L_{m,p}(\boldsymbol{q}_1, -\boldsymbol{q}_1)W_{pmmm}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4) + 3L_{m,mm}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3 + \boldsymbol{q}_4)W_{mm}(-\boldsymbol{q}_2, \boldsymbol{q}_2)$$

$$\times W_{mmm}(-\boldsymbol{q}_3 - \boldsymbol{q}_4, \boldsymbol{q}_3, \boldsymbol{q}_4) + 3L_{m,mp}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3 + \boldsymbol{q}_4)W_{mm}(-\boldsymbol{q}_2, \boldsymbol{q}_2)W_{pmm}(-\boldsymbol{q}_3 - \boldsymbol{q}_4, \boldsymbol{q}_3, \boldsymbol{q}_4)$$

$$+ L_{m,mmm}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4)W_{mm}(-\boldsymbol{q}_2, \boldsymbol{q}_2)W_{mm}(-\boldsymbol{q}_3, \boldsymbol{q}_3)W_{mm}(-\boldsymbol{q}_4, \boldsymbol{q}_4)$$

$$+ 3Q_{mm,m}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3 + \boldsymbol{q}_4)W_{mmm}(-\boldsymbol{q}_3 - \boldsymbol{q}_4, \boldsymbol{q}_3, \boldsymbol{q}_4) + 3Q_{mm,p}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3 + \boldsymbol{q}_4)W_{pmm}(-\boldsymbol{q}_3 - \boldsymbol{q}_4, \boldsymbol{q}_3, \boldsymbol{q}_4)$$

$$+ 3Q_{mm,mm}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4)W_{mm}(-\boldsymbol{q}_3, \boldsymbol{q}_3)W_{mm}(-\boldsymbol{q}_4, \boldsymbol{q}_4)]_{\overline{1234}}, \tag{3.17c}$$

where

$$\mathcal{L}[W] \equiv \left(u \cdot \bar{\nabla} - (\partial_\nu u_\mu)e_a^\mu e_b^\nu q_i^a \frac{\partial}{\partial q_{ib}} \right) W \tag{3.18}$$

is the Liouville-like operator for the scalar Wigner function [19,20], and

$$L_{m,m}(\boldsymbol{q}_1, \boldsymbol{q}_2) = \gamma_{mm}\boldsymbol{q}_1 \cdot \boldsymbol{q}_2, \quad L_{m,p}(\boldsymbol{q}_1, \boldsymbol{q}_2) = \gamma_{mp}\boldsymbol{q}_1 \cdot \boldsymbol{q}_2,$$

$$L_{m,mm}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3) = -(\gamma_{mm})_{,m}\boldsymbol{q}_1^2 + 2(\ln m_n)_{,m}\gamma_{mm}\boldsymbol{q}_2 \cdot \boldsymbol{q}_3,$$

$$L_{m,mp}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3) = (\gamma_{mm})_{,p}\boldsymbol{q}_1 \cdot \boldsymbol{q}_2 + (\gamma_{mp})_{,m}\boldsymbol{q}_1 \cdot \boldsymbol{q}_3 + [(\ln m_n)_{,m}\gamma_{mp} + (\ln m_n)_{,p}\gamma_{mm}]\boldsymbol{q}_2 \cdot \boldsymbol{q}_3,$$

$$L_{m,mmm}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4) = -(\gamma_{mm})_{,mm}\boldsymbol{q}_1^2 + 2((\ln m_n)_{,m}\gamma_{mm})_{,m}(\boldsymbol{q}_2 \cdot \boldsymbol{q}_3 + \boldsymbol{q}_3 \cdot \boldsymbol{q}_4 + \boldsymbol{q}_4 \cdot \boldsymbol{q}_2),$$

$$Q_{mm}(\boldsymbol{q}_1, \boldsymbol{q}_2) = -m_n^2\lambda\boldsymbol{q}_1 \cdot \boldsymbol{q}_2, \quad Q_{mm,m}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3) = -(m_n^2\lambda)_{,m}\boldsymbol{q}_1 \cdot \boldsymbol{q}_2 + (\ln m_n)_{,m}m_n^2\lambda(\boldsymbol{q}_1 + \boldsymbol{q}_2)^2,$$

$$Q_{mm,p}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3) = -(m_n^2\lambda)_{,p}\boldsymbol{q}_1 \cdot \boldsymbol{q}_2 + (\ln m_n)_{,p}m_n^2\lambda(\boldsymbol{q}_1 + \boldsymbol{q}_2)^2,$$

$$Q_{mp,m}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3) = -(m_n p_n\lambda)_{,m}\boldsymbol{q}_1 \cdot \boldsymbol{q}_2 - (\ln p_n)_{,m}m_n p_n\lambda\boldsymbol{q}_1 \cdot \boldsymbol{q}_3 - (\ln m_n)_{,m}m_n p_n\lambda\boldsymbol{q}_2 \cdot \boldsymbol{q}_3,$$

$$Q_{mm,mm}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4) = -(m_n^2\lambda)_{,mm}\boldsymbol{q}_1 \cdot \boldsymbol{q}_2 + ((\ln m_n)_{,m}m_n^2\lambda)_{,m}(\boldsymbol{q}_1 + \boldsymbol{q}_2)^2 - 2((\ln m_n)_{,m})^2 m_n^2\lambda\boldsymbol{q}_3 \cdot \boldsymbol{q}_4, \tag{3.19}$$

where the coefficients in front of $\boldsymbol{q}_i \cdot \boldsymbol{q}_j$ are thermodynamic derivatives of transport coefficients and thermodynamic quantities. In terms of the independent thermodynamic derivatives chosen from set I in Table I, they are given by

$$\gamma_{mm} = m_n\lambda\alpha_m, \quad \gamma_{mp} = m_n\lambda\alpha_p, \quad (\gamma_{mm})_{,m} = \left(\frac{\lambda_m}{\lambda} + \frac{\alpha_{mm}}{\alpha_m} + \frac{\alpha_p}{\beta m_n} - \frac{2 + \alpha_m}{nm_n} \right)\gamma_{mm},$$

$$(\gamma_{mm})_{,p} = \left(\frac{\lambda_p}{\lambda} + \frac{\alpha_{mp}}{\alpha_m} - \frac{\alpha_p}{nm_n} - \frac{\epsilon_p}{w} \right)\gamma_{mm}, \quad (\gamma_{mp})_{,m} = \left(\frac{\lambda_m}{\lambda} + \frac{\alpha_{mp}}{\alpha_m} + \frac{\alpha_p}{\beta m_n} - \frac{2 + \alpha_m}{nm_n} \right)\gamma_{mp},$$

$$(\gamma_{mm})_{,mm} = \left[\frac{\lambda_{mm}}{\lambda} + 2\left(\frac{\alpha_{mm}}{\alpha_m} + \frac{\alpha_p}{\beta m_n} - \frac{2 + \alpha_m}{nm_n} \right)\frac{\lambda_m}{\lambda} + \frac{\alpha_{mmm}}{\alpha_m} + \left(\frac{2\alpha_p}{\beta m_n} - \frac{4 + 3\alpha_m}{nm_n} \right)\frac{\alpha_{mm}}{\alpha_m} + \frac{\alpha_{mp}}{\beta m_n} - \frac{2\alpha_p}{\beta nm_n^2} + \frac{2 + \alpha_m}{n^2 m_n^2} \right]\gamma_{mm},$$

$$(\ln m_n)_{,m} = \frac{\alpha_p}{\beta m_n} - \frac{2 + \alpha_m}{nm_n}, \quad (\ln m_n)_{,p} = -\frac{\alpha_p}{nm_n} - \frac{\epsilon_p}{w},$$
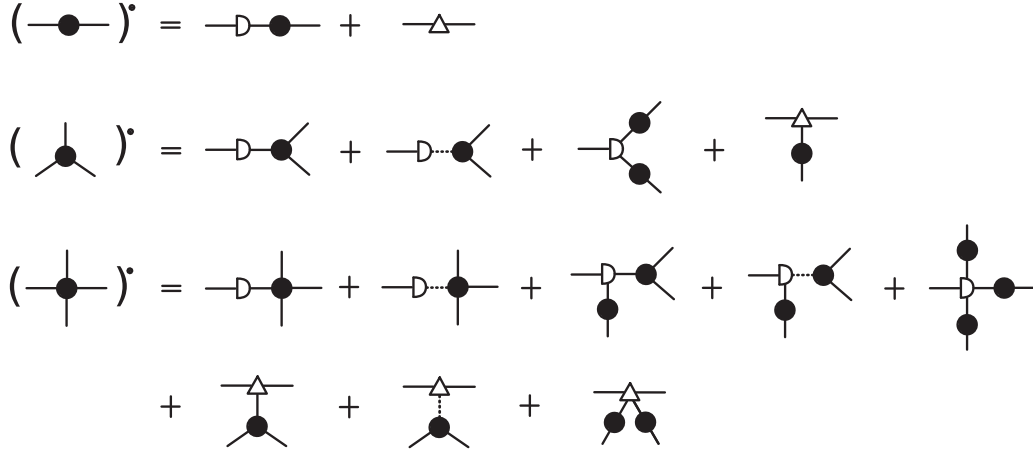
FIG. 6. Diagrammatic representation of the evolution equations (3.17). Similarly to Fig. 1, the solid circles with k legs represent k-point Wigner functions W_k, while the open half-circles represent $L_{m,\dots}$ [the open half-circles with two solid legs in the equation for W_k represent $\sim kL_{m,m} + (k-1)\theta$], and open triangles represent $Q_{mm,\dots}$. The dot on the left-hand side of the diagrammatic equation represents the Liouville operator in Eq. (3.18). A dashed line leg corresponds to index p replacing index m in each of the above objects. In the regime we consider, the Wigner functions involving pressure fluctuations (p index, or dashed line leg) can be replaced by their partial equilibrium values given in Eq. (3.22) and represented diagrammatically in Fig. 7. Note that the diagrams with two-point correlator W_{mp} are not shown, since this correlator vanishes in the regime we consider.

$$(\ln m_n)_{,mm} = \frac{\alpha_{mp}}{\beta m_n} - \frac{\alpha_{mm}}{nm_n} + \frac{\alpha_p}{\beta nm_n^2}\left(2 + 2\alpha_m - \frac{n\alpha_p}{\beta}\right) - \frac{(1+\alpha_m)(2+\alpha_m)}{n^2 m_n^2},$$

$$\left(m_n^2\lambda\right)_{,m} = \left(\frac{\lambda_m}{\lambda} + \frac{2\alpha_p}{\beta m_n} - \frac{4 + 2\alpha_m}{nm_n}\right)m_n^2\lambda, \quad \left(m_n^2\lambda\right)_{,p} = \left(\frac{\lambda_p}{\lambda} - \frac{2\alpha_p}{nm_n} - \frac{2\epsilon_p}{w}\right)m_n^2\lambda,$$

$$\left(m_n p_n\lambda\right)_{,m} = \left[\frac{\lambda_m}{\lambda} + \frac{\alpha_{mp}}{\alpha_p} + \frac{\alpha_p}{\beta m_n\epsilon_p}\left(\frac{w\alpha_{pp}}{\alpha_p} - \frac{2n\alpha_p}{\beta} + 2\epsilon_p + 2\right) - \frac{2}{nm_n}\right]\frac{m_n w\alpha_p}{\beta\epsilon_p}\lambda,$$

$$p_n = \frac{w\alpha_p}{\beta\epsilon_p}, \quad (\ln p_n)_{,m} = \frac{\alpha_{mp}}{\alpha_p} + \frac{\alpha_p}{\beta m_n\epsilon_p}\left(\frac{w\alpha_{pp}}{\alpha_p} - \frac{2n\alpha_p}{\beta} + \epsilon_p + 2\right) + \frac{\alpha_m}{nm_n},$$

$$\left(m_n^2\lambda\right)_{,mm} = \left[\frac{\lambda_{mm}}{\lambda} + 4\left(\frac{\alpha_p}{\beta m_n} - \frac{2+\alpha_m}{nm_n}\right)\frac{\lambda_m}{\lambda} - \frac{2\alpha_{mm}}{nm_n} + \frac{2\alpha_{mp}}{\beta m_n} - \frac{2\alpha_p}{\beta nm_n^2}\left(6 + 2\alpha_m - \frac{n\alpha_p}{\beta}\right) + \frac{2(2+\alpha_m)(3+\alpha_m)}{n^2 m_n^2}\right]m_n^2\lambda.$$

$$(3.20)$$

The corresponding expressions written in terms of independent thermodynamic derivatives from set II given by Table I are given in Eq. (C1). It is worthwhile to mention the following relations:

$$m_n^2\lambda = \frac{\kappa}{n^2}, \quad \alpha_m = -\frac{\beta w}{c_p}, \quad \gamma_{mm} = \frac{\kappa}{c_p}, \qquad (3.21)$$

where κ is the thermal conductivity and $c_p \equiv Tn(\partial m/\partial T)_p$ is the fixed pressure heat capacity per unit volume. Eqs. (3.17) are represented diagrammatically in Fig. 6.

Note that the arguments $\boldsymbol{q}_i$ in each function (W, L, and Q) in Eq. (3.17) sum up to zero, according to the constraint $\sum_i \boldsymbol{q}_i = 0$ discussed in Sec. II D and illustrated in Fig. 5. Diagrammatically, this can be understood as the conservation of "momenta" $\boldsymbol{q}_i$ in each element of the diagram.

In order to close Eqs. (3.17) we need to supply the values of W_{mmp} and W_{mmmp}, i.e., cross-correlators involving fluctuations of pressure. We have not written terms in which the

linear (Gaussian) correlator W_{mp} appears, because this correlator vanishes upon averaging over the timescales longer than the period of sound oscillation, as already observed in Ref. [20]. The vanishing of the linear correlator W_{mp} significantly simplifies equations (3.17). However, we cannot drop the nonlinear correlators W_{mmp} and W_{mmmp}, since they do not average to zero. This can be easily seen by computing [cf. Eqs. (3.23) and (A8)] the equilibrium values of W_{mmp} and W_{mmmp}, which are not zero, unlike the equilibrium value of W_{mp}.

Outside of the regime we consider (i.e., at faster timescales) these nonlinear cross-correlators obey dynamic equations which will be a part of the full system of equations for fluctuation correlators. In this paper we do not intend to derive this full system. Instead, we use the fact that different correlators relax on parametrically different timescales, with correlators of the specific entropy fluctuations being the slowest. We use this hierarchy of scales and observe that pressure fluctuations (upon averaging over the sound oscillations) relax
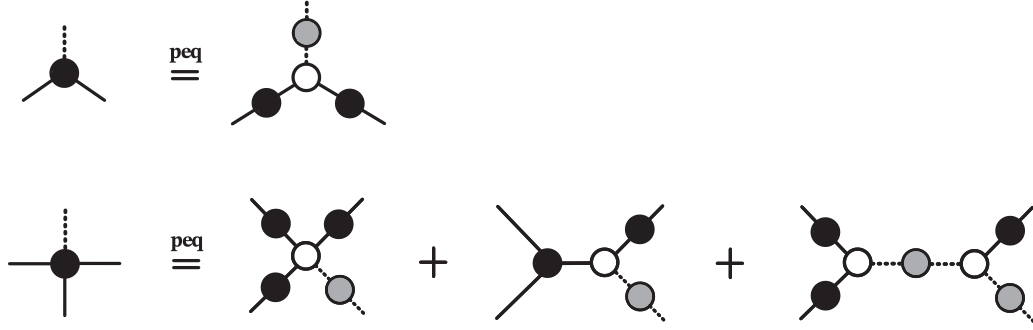
FIG. 7. Diagrammatic representation for the partial equilibrium values of the correlators involving pressure fluctuation, as given by Eqs. (3.22). An open circle with k legs represents the derivative of entropy with respect to k variables, e.g., $S_{,mmp}$ for $k = 3$, with solid and dashed lines representing derivatives with respect to m and p, respectively. The solid gray circle with two dashed line legs represents $-S_{,pp}^{-1} = W_{pp}^{eq}$, the equilibrium value of W_{pp}.

parametrically faster (i.e., on the timescale of sound attenuation) than the timescale of the evolution of the specific entropy fluctuations on which we focus. This parametric separation of scales enables the regime we consider, i.e., Hydro+ regime with only one parametrically slow nonhydrodynamic mode. Effectively, we can consider the fluctuations δm frozen when we determine the fluctuations of δp.

This means that, while in complete equilibrium, by definition, $\delta p = \delta m = 0$, we can also consider *partial equilibrium* where δm is not zero, i.e., not in equilibrium, and determine the "equilibrium" value of δp under these (slowly varying) conditions. Since fluctuations δp relax faster than fluctuations δm, the value of δp becomes a function of δm, and not an independent variable, in the regime we consider. That function is the partial equilibrium value $(\delta p)^{peq}$, which we derive in Appendix B. Consequently, the correlators W_{mmp} and W_{mmmp} can be expressed in terms of the specific entropy correlators as follows:

$$W_{pmm}^{peq}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3) = -S_{,mmp}S_{,pp}^{-1}W_{mm}(-\boldsymbol{q}_2, \boldsymbol{q}_2)W_{mm}(-\boldsymbol{q}_3, \boldsymbol{q}_3),$$

$$W_{pmmm}^{peq}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4) = -3S_{,mmp}S_{,pp}^{-1}W_{mm}(\boldsymbol{q}_2, -\boldsymbol{q}_2)W_{mmm}(-\boldsymbol{q}_3 - \boldsymbol{q}_4, \boldsymbol{q}_3, \boldsymbol{q}_4) - \left(S_{,mmmp}S_{,pp}^{-1} - 3S_{,mmp}S_{,mpp}S_{,pp}^{-2}\right)$$
$$\times W_{mm}(-\boldsymbol{q}_2, \boldsymbol{q}_2)W_{mm}(-\boldsymbol{q}_3, \boldsymbol{q}_3)W_{mm}(-\boldsymbol{q}_4, \boldsymbol{q}_4). \tag{3.22}$$

where $S_{...}$ are derivatives, given by Eqs. (A4), of the entropy functional S that is maximized in equilibrium, as discussed in Appendix A. These equations are represented diagrammatically in Fig. 7. Substituting partial equilibrium values for the cross-correlators W_{pmm} and W_{pmmm} from Eq. (3.22) into Eqs. (3.17), we now obtain a closed system of equations for correlators of the specific entropy.[14]

In order to check the validity of Eqs. (3.17)–(3.20) together with (3.22) we verified that they are solved by the following space-time and wave-number independent values:

$$W_{mm}^{eq} = \frac{m_n}{\alpha_m}, \quad W_{mp}^{eq} = 0, \quad W_{pp}^{eq} = \frac{w}{\beta\epsilon_p}, W_{mmm}^{eq} = -\left(\frac{m_n}{\alpha_m}\right)^2\left(\frac{\alpha_{mm}}{\alpha_m} + \frac{4 + 2\alpha_m}{nm_n} - \frac{3\alpha_p}{\beta m_n}\right), \quad W_{mmp}^{eq} = -\frac{m_n}{\beta\alpha_m},$$

$$W_{mmmm}^{eq} = \left(\frac{m_n}{\alpha_m}\right)^3\left[-\frac{\alpha_{mmm}}{\alpha_m} + 3\left(\frac{\alpha_{mm}}{\alpha_m}\right)^2 + \frac{2\alpha_{mm}}{nm_n\alpha_m}\left(9 + 3\alpha_m - \frac{7n\alpha_p}{\beta}\right) + \frac{6\alpha_{mp}}{\beta m_n} + \frac{3\alpha_m\epsilon_p}{\beta wm_n}\right.$$
$$\left. + \frac{3(2 + \alpha_m)(5 + 2\alpha_m)}{n^2m_n^2} - \frac{\alpha_p}{\beta nm_n^2}\left(44 + 16\alpha_m - 15\frac{n\alpha_p}{\beta}\right)\right],$$

$$W_{mmmp}^{eq} = \frac{2}{\beta}\left(\frac{m_n}{\alpha_m}\right)^2\left(\frac{\alpha_{mm}}{\alpha_m} + \frac{4 + 2\alpha_m}{nm_n} - \frac{3\alpha_p}{\beta m_n}\right), \tag{3.23}$$

which, of course, can be obtained from an independent calculation based on thermodynamics (see Appendix A). This is a highly nontrivial check of the evolution equations, since it involves cancellations of many terms, including the contribution of the pressure fluctuations discussed above.

Eqations (3.23) are expressed in terms of independent thermodynamic derivatives from set I in Table I. This is useful for verifying that equilibrium correlators in Eqs. (3.23) satisfy evolution equations. An alternative representation of Eqs. (3.23) using the independent thermodynamic derivatives from set II is given by Eqs. (C2).

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[14]Interestingly, this substitution generates terms with $W_{m...m}$ correlators similar to the ones already present in Eqs. (3.17). One could say that the effect of the pressure fluctuations is to redefine, or renormalize, the coefficients $L_{m,m...m}$ and $Q_{mm,m...m}$.

The terms involving θ reflect trivial expansion effect on the Wigner function described by the equation $u \cdot \partial W_k = (k-1)\theta W_k$. This effect was already discussed in Ref. [20] for $k = 2$, where it was absorbed by rescaling W_k by a factor of the density n which also obeys a similar equation due to expansion: $u \cdot \partial n = -n\theta$. Generalizing this approach to arbitrary k we can eliminate the terms with θ and simplify the expressions by writing the evolution equations in terms of rescaled Wigner functions N:

$$N_{\underbrace{m...m}_{k}} = n^{k-1}W_{\underbrace{m...m}_{k}}, \quad N_{\underbrace{pm...m}_{k}} = n^{k-1}W_{\underbrace{pm...m}_{k}}. \tag{3.24}$$

In terms of N, Eqs. (3.17) retain exactly the same form, but without θ terms (and with extra factor of n for each $Q_{mm,m...}$, which is understandable recalling that Q_{mm} is a two-point Wigner function), i.e.,

$$\mathcal{L}[N_{mm}(\boldsymbol{q}_1, \boldsymbol{q}_2)] = 2[L_{m,m}(\boldsymbol{q}_1, -\boldsymbol{q}_1)N_{mm}(\boldsymbol{q}_1, \boldsymbol{q}_2) + nQ_{mm}(\boldsymbol{q}_1, \boldsymbol{q}_2)]_{\overline{12}}, \tag{3.25a}$$

$$\mathcal{L}[N_{mmm}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3)] = 3[L_{m,m}(\boldsymbol{q}_1, -\boldsymbol{q}_1)N_{mmm}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3) + L_{m,p}(\boldsymbol{q}_1, -\boldsymbol{q}_1)N_{pmm}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3)$$
$$+ L_{m,mm}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3)N_{mm}(-\boldsymbol{q}_2, \boldsymbol{q}_2)N_{mm}(-\boldsymbol{q}_3, \boldsymbol{q}_3) + 2nQ_{mm,m}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3)N_{mm}(-\boldsymbol{q}_3, \boldsymbol{q}_3)]_{\overline{123}}, \tag{3.25b}$$

$$\mathcal{L}[N_{mmmm}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4)] = 4[L_{m,m}(\boldsymbol{q}_1, -\boldsymbol{q}_1)N_{mmmm}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4) + L_{m,p}(\boldsymbol{q}_1, -\boldsymbol{q}_1)N_{pmmm}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4)$$
$$+ 3L_{m,mm}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3 + \boldsymbol{q}_4)N_{mm}(-\boldsymbol{q}_2, \boldsymbol{q}_2)N_{mmm}(-\boldsymbol{q}_3 - \boldsymbol{q}_4, \boldsymbol{q}_3, \boldsymbol{q}_4)$$
$$+ 3L_{m,mp}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3 + \boldsymbol{q}_4)N_{mm}(-\boldsymbol{q}_2, \boldsymbol{q}_2)N_{pmm}(-\boldsymbol{q}_3 - \boldsymbol{q}_4, \boldsymbol{q}_3, \boldsymbol{q}_4)$$
$$+ L_{m,mmm}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4)N_{mm}(-\boldsymbol{q}_2, \boldsymbol{q}_2)N_{mm}(-\boldsymbol{q}_3, \boldsymbol{q}_3)N_{mm}(-\boldsymbol{q}_4, \boldsymbol{q}_4)$$
$$+ 3nQ_{mm,m}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3 + \boldsymbol{q}_4)N_{mmm}(-\boldsymbol{q}_3 - \boldsymbol{q}_4, \boldsymbol{q}_3, \boldsymbol{q}_4)$$
$$+ 3nQ_{mm,p}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3 + \boldsymbol{q}_4)N_{pmm}(-\boldsymbol{q}_3 - \boldsymbol{q}_4, \boldsymbol{q}_3, \boldsymbol{q}_4)$$
$$+ 3nQ_{mm,mm}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4)N_{mm}(-\boldsymbol{q}_3, \boldsymbol{q}_3)N_{mm}(-\boldsymbol{q}_4, \boldsymbol{q}_4)]_{\overline{1234}}. \tag{3.25c}$$

The equilibrium solutions for Eqs. (3.25) can be obtained accordingly, using Eqs. (3.23) [or Eqs. (C2)] and (3.24).

It is instructive to compare Eqs. (3.17) or (3.25) for fluctuations of m to similar evolution equations for the fluctuations of n at fixed T in the charge diffusion problem derived in Ref. [22]. The following map between the two problems was conjectured in Ref. [22] (this substitution correctly reproduces the equation for the two-point correlator in Eq. (3.17a), already known from Ref. [20]):

$$n \to m, \quad \gamma \to \frac{\kappa}{c_p}, \quad \alpha' \to \frac{n^2}{c_p}. \tag{3.26}$$

While the substitution (3.26) reproduces the terms in Eqs. (3.17) or (3.25), the expressions for the coefficients in these equations given by Eqs. (3.19) contain terms with derivatives of $\ln m_n$ which are not reproduced by the simple map (3.26). These terms reflect the fact that the $\boldsymbol{q} = 0$ mode of n, i.e., $\int d^3x \, n$, is a constant of motion, while $\int d^3x \, m$ is not [see Eq. (3.8)], unless m happens to be a linear function of n, of course, in which case the derivatives of $\ln m_n$ vanish. Also, the contributions of the pressure fluctuations (the terms with $W_{pm...m}$) are not captured by the replacement in Eq. (3.26).[15]

IV. CONCLUSIONS AND OUTLOOK

We have generalized the fully Lorentz covariant deterministic approach to relativistic fluctuating hydrodynamics

to non-Gaussian fluctuations. While the full system of equations involving correlators of all hydrodynamic variables is still a work in progress, here we demonstrate how this approach allows us to derive the relativistically covariant equations for the fluctuations of the slowest hydrodynamic mode, which is specific entropy, or $m = s/n$, in a fluid with arbitrary relativistic flow. Such fluctuations are of special significance near a critical point, in particular, the QCD critical point, for two reasons. First, the equilibrium fluctuations of m are the largest near the critical point: $\langle \delta m^2 \rangle \sim c_p \sim \xi^{2-\eta}$. In that sense, this is the "soft" direction and the source of the most prominent critical point signatures. Second, this mode is also the slowest (not only it is diffusive, but its diffusion coefficient $\gamma_\lambda = \kappa/c_p$ vanishes at the critical point as c_p diverges) and, thus, it is the furthest from equilibrium. Therefore, the nonequilibrium dynamics of this mode of fluctuations are the most consequential from the point of view of predicting the dynamical effects on the signatures of the QCD critical point.

Similar evolution equations for non-Gaussian correlators in a static (nonflowing) fluid at given temperature were derived in Ref. [22]. In this paper we consider diffusion of the slowest diffusive mode in a *flowing* fluid, more relevant for the description of the dynamics of critical fluctuations in heavy-ion collisions.

Comparing evolution equations (3.17) to the results of Ref. [22], we observe many similarities, which were already anticipated in Ref. [22]. In fact, Eqs. (3.17) can be obtained from those in Ref. [22] by a substitution, Eq. (3.26), as conjectured in Ref. [22]. However, the terms containing derivatives of $\ln m_n$ appearing in the coefficients given by Eqs. (3.19)

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[15]One could note, however, that the *leading* critical ($\xi \to \infty$) behavior of the coefficients in Eqs. (3.19) is reproduced correctly by the substitution (3.26).

cannot be obtained that way. Their appearance is related to the qualitative difference between the density n, whose space integral is a conserved quantity, and the specific entropy m, whose space integral (i.e., $q = 0$ mode) is generally *not* conserved. We have also found nontrivial contributions of pressure fluctuations to the evolution of the specific entropy correlators due to nonlinearities of the equation of state (i.e., mode coupling).

As a very nontrivial check of the evolution equations (3.17) and (3.19), we verified that the equilibrium (thermodynamic) correlators in Eq. (3.23) solve the evolution equations. This requires multiple cancellations which cannot be achieved without the terms with derivatives of $\ln m_n$ as well as the contributions of pressure fluctuations.

As we pointed out in the Introduction, our equations include only tree-level contributions (as did the equations derived in Ref. [22]). It could potentially be interesting to extend this analysis to one-loop order and consider "long-time tail" effects on the evolution of fluctuations. The resulting theory would represent the generalization of Hydro+ [29] to non-Gaussian fluctuations.

Although we limited our analysis to the slowest diffusive mode, we presented our derivation in a sufficiently general form to facilitate the extension to fluctuations of faster hydrodynamic modes. The next-to-slowest modes correspond to fluctuations of the transverse velocity of the fluid. The theory describing fluctuations of all *diffusive* hydrodynamic modes (specific entropy and transverse velocity) was referred to as Hydro++ in Ref. [20]. Finally, the full set of hydrodynamic modes includes pressure/longitudinal velocity fluctuations, i.e., they include sound—the fastest hydrodynamic mode. We leave the development of such a full hydrodynamic theory of fluctuations to further work.

Despite the focus on the slowest hydrodynamic mode, we believe the equations presented in this paper are valuable for more realistic simulations of the dynamical evolution of fluctuations in heavy-ion collisions (as a first step one can consider extending exploratory Hydro+ calculations, such as in Refs. [34–36], to non-Gaussian fluctuations). The ingredients required are the same as in the usual hydrodynamic simulation: equation of state, e.g., as in Refs. [37–39], and kinetic coefficients.

APPENDIX A: NON-GAUSSIAN CORRELATORS IN THERMODYNAMICS

In the local rest frame, the total entropy subject to the conservation of charge and energy are given by

$$S(m, p) = \int_x s(m, p) + \bar{\alpha} n(m, p) - \bar{\beta} \epsilon(m, p), \qquad \text{(A1)}$$

where we choose our two independent thermodynamic variables as the specific entropy density m and pressure p associated with conserved quantities n and ϵ, i.e., $m = m(n, \epsilon)$, $p = p(n, \epsilon)$ and the variable associated with momentum (e.g., velocity u) is absent due to the fact that we choose the local rest frame. $\bar{\alpha}$ and $\bar{\beta}$ are the local chemical potential per temperature and inverse of temperature of the heat bath respectively.

For a system with two independent thermodynamic variables (such as m and p), there are $n + 1$ independent nth-order thermodynamic derivatives of entropy. It is convenient for calculations to choose a basis set of independent derivatives for each order to make sure that cancellations are easier to carry out. Table I provides two such basis sets (set I and II) for derivatives up to fourth order. In set I and II we have primarily chosen the derivatives of α and $\ln m_n$, respectively. Of course, one can relate the independent thermodynamic derivatives from set I to those from set II. Taking the second-order derivative as an example, we have

$$
\begin{aligned}
\epsilon_p &= 2 + \alpha_m + n m_n (\ln m_n)_{,m} - w (\ln m_n)_{,p}, \\
\alpha_p &= \beta m_n (\ln m_n)_{,m} + \frac{\beta}{n} (2 + \alpha_m),
\end{aligned}
\qquad \text{(A2)}
$$

while α_m and m_n / α_m are simply functions of each other and the first-order derivative $m_n = -\beta w / n^2$, so we can also treat α_m instead of m_n / α_m as an independent thermodynamic derivative in set II.

As a consequence, all thermodynamic derivatives can be expressed solely in terms of the independent ones such as those listed in Table I. In terms of the independent derivatives from set I given by Table I, useful expressions for some second-, third-, and fourth-order thermodynamic derivatives are presented below:

$$n_m = \frac{1}{m_n}\left(1 - \frac{n\alpha_p}{\beta}\right), \quad n_p = \frac{n\epsilon_p}{w}, \quad \epsilon_m = -\frac{w\alpha_p}{\beta m_n}, \quad \beta_m = \frac{n\alpha_m}{w}, \quad \beta_p = -\frac{\beta}{w}\left(1 - \frac{n\alpha_p}{\beta}\right),$$

$$n_{mm} = \frac{1}{m_n}\left[-\frac{n\alpha_{mp}}{\beta} + \frac{2n\alpha_p^2}{\beta^2 m_n} + \frac{2 + \alpha_m}{nm_n}\left(1 - \frac{2n\alpha_p}{\beta}\right)\right], \quad n_{pp} = -\frac{n}{w^2}(w\epsilon_{pp} - \epsilon_p),$$

TABLE I. Two sets of independent thermodynamic derivatives chosen in this work. The derivative of quantity X with respect to m (or p) while keeping p (or m) fixed is denoted by X_m (or X_p).

Order	Independent derivatives (set I)	Independent derivatives (set II)
2nd	$\epsilon_p,\ \alpha_m,\ \alpha_p$	$\frac{m_n}{\alpha_m},\ (\ln m_n)_{,m},\ (\ln m_n)_{,p}$
3rd	$\epsilon_{pp},\ \alpha_{mm},\ \alpha_{mp},\ \alpha_{pp}$	$\left(\ln \frac{m_n}{\alpha_m}\right)_{,m},\ (\ln m_n)_{,mm},\ (\ln m_n)_{,mp},\ (\ln m_n)_{,pp}$
4th	$\epsilon_{ppp},\ \alpha_{mmm},\ \alpha_{mmp},\ \alpha_{mpp},\ \alpha_{ppp}$	$\left(\ln \frac{m_n}{\alpha_m}\right)_{,mm},\ (\ln m_n)_{,mmm},\ (\ln m_n)_{,mmp},\ (\ln m_n)_{,mpp},\ (\ln m_n)_{,ppp}$

$$\epsilon_{mm} = -\frac{w\alpha_p}{\beta m_n}\left[\frac{\alpha_{mp}}{\alpha_p} + \frac{2}{nm_n}\left(1 + \alpha_m - \frac{n\alpha_p}{\beta}\right)\right], \quad \epsilon_{mp} = -\frac{\alpha_p}{\beta m_n}\left[\frac{w\alpha_{pp}}{\alpha_p} + 2\left(1 + \epsilon_p - \frac{n\alpha_p}{\beta}\right)\right],$$

$$n_{mmm} = \frac{1}{m_n}\left[-\frac{n\alpha_{mmp}}{\beta} + \frac{\alpha_{mm}}{nm_n}\left(1 - \frac{2n\alpha_p}{\beta}\right) - \frac{\alpha_{mp}}{\beta m_n}\left(7 + 4\alpha_m - \frac{6n\alpha_p}{\beta}\right)\right.$$
$$\left. + 6\left(\frac{\alpha_p}{\beta m_n}\right)^2\left(3 + 2\alpha_m - \frac{n\alpha_p}{\beta}\right) + \frac{(2+\alpha_m)(3+2\alpha_m)}{n^2 m_n^2}\left(1 - \frac{3\alpha_p n}{\beta}\right)\right],$$

$$\epsilon_{mmm} = -\frac{w\alpha_p}{\beta m_n}\left[\frac{\alpha_{mmp}}{\alpha_p} + \frac{2\alpha_{mp}}{nm_n\alpha_p}\left(2 + 2\alpha_m - \frac{3n\alpha_p}{\beta}\right) + 6\left(1 + \alpha_m - \frac{\alpha_p}{\beta m_n}\right)^2\right]. \tag{A3}$$

Applying the derivative with respect to m or p to the entropy given by Eq. (A1), we obtain

$$S_{,m} = 0, \quad S_{,p} = 0, \quad S_{,mm} = -\frac{\alpha_m}{m_n}, \quad S_{,mp} = 0, \quad S_{,pp} = -\frac{\beta\epsilon_p}{w}, \quad S_{,mmp} = -\frac{\alpha_m\epsilon_p}{wm_n},$$

$$S_{,mmm} = -\frac{\alpha_m}{m_n}\left(\frac{\alpha_{mm}}{\alpha_m} + \frac{2(2+\alpha_m)}{nm_n} - \frac{3\alpha_p}{\beta m_n}\right), \quad S_{,mpp} = \frac{\alpha_p}{wm_n}\left(\frac{w\alpha_{pp}}{\alpha_p} - \frac{2n\alpha_p}{\beta} + \epsilon_p + 2\right),$$

$$S_{,mmmp} = \frac{\alpha_m}{m_n}\left[\frac{3\alpha_{pp}}{\beta m_n} - \left(\frac{\alpha_{mm}}{\alpha_m} + \frac{2+\alpha_m}{nm_n} - \frac{6\alpha_p}{\beta m_n}\right)\frac{\epsilon_p}{w} + \frac{6\alpha_p}{\beta wm_n}\left(1 - \frac{n\alpha_p}{\beta}\right)\right],$$

$$S_{,mmmm} = -\frac{\alpha_m}{m_n}\left[\frac{\alpha_{mmm}}{\alpha_m} + \frac{2\alpha_{mm}}{nm_n\alpha_m}\left(3 + 3\alpha_m - \frac{2n\alpha_p}{\beta}\right) - \frac{6\alpha_{mp}}{\beta m_n} + \frac{3(2+\alpha_m)(3+2\alpha_m)}{n^2 m_n^2} - \frac{4\alpha_p}{\beta nm_n^2}\left(7 + 5\alpha_m - \frac{3n\alpha_p}{\beta}\right)\right]. \tag{A4}$$

The above expressions for entropy derivatives are evaluated at the maximum of the entropy S (i.e., in equilibrium), which corresponds to $\alpha = \bar{\alpha}$ and $\beta = \bar{\beta}$. As in Ref. [19,20], the choice of m and p as independent thermodynamic variables makes the second-order entropy derivative quadratic form diagonal, i.e., $S_{,mp} = 0$.

The connected n-point correlation functions in thermodynamic equilibrium are given by

$$W_{i_1 i_2}^{\text{eq}} = -S_{,i_1 i_2}^{-1}, \quad W_{i_1 i_2 i_3}^{\text{eq}} = -S_{,i_1 j_1}^{-1} S_{,i_2 j_2}^{-1} S_{,i_3 j_3}^{-1} S_{,j_1 j_2 j_3},$$

$$W_{i_1 i_2 i_3 i_4}^{\text{eq}} = \left[S_{,i_1 j_1}^{-1} S_{,i_2 j_2}^{-1} S_{,i_3 j_3}^{-1} S_{,i_4 j_4}^{-1} S_{,j_1 j_2 j_3 j_4} - 3 S_{,i_1 j_1}^{-1} S_{,i_2 j_2}^{-1} S_{,j_1 j_2 j_3} S_{,j_3 j_4}^{-1} S_{,j_4 j_5 j_6} S_{,j_5 i_3}^{-1} S_{,j_6 i_4}^{-1}\right]_{\overline{i_1 i_2 i_3 i_4}}, \tag{A5}$$

and generically

$$W_{i_1 \ldots i_n}^{\text{eq}} = \left[W_{i_1 j_1}^{\text{eq}} W_{i_2 \ldots i_n, j_1}^{\text{eq}}\right]_{\overline{i_1 \ldots i_n}}, \tag{A6}$$

where indices i_n label the external points while indices j_n label the internal ones. The relation given by Eq. (A6) easily follows from the cumulant generating function $g_c(\mu_1, \ldots, \mu_n)$ such that $W_{i_1 \ldots i_n}^{\text{eq}} = d^n g_c/d\mu_1 \cdots d\mu_n$. Indeed, using the chain rule on the μ derivative in $W_{i_1 \ldots i_n}^{\text{eq}} = [dW_{i_2 \ldots i_n}^{\text{eq}}/d\mu_1]_{\overline{1 \ldots n}}$ and the fact that (susceptibility) $d\psi_j/d\mu_i = W_{ij}^{\text{eq}}$, we obtain Eq. (A6).

In the discussion presented in Sec. III, the indices are chosen as $i = m$ and $j = (m, p)$, thus the connected n-point functions are (see Fig. 8 for diagrammatic representation)

$$W_{mm}^{\text{eq}} = -S_{,mm}^{-1}, \quad W_{mp}^{\text{eq}} = 0, \quad W_{pp}^{\text{eq}} = -S_{,pp}^{-1}, \quad W_{mmm}^{\text{eq}} = -S_{,mm}^{-3} S_{,mmm}, \quad W_{mmp}^{\text{eq}} = -S_{,mm}^{-2} S_{,pp}^{-1} S_{,mmp},$$

$$W_{mmmm}^{\text{eq}} = S_{,mm}^{-4} S_{,mmmm} - 3 S_{,mm}^{-5}(S_{,mmm})^2 - 3 S_{,mm}^{-4} S_{,pp}^{-1}(S_{,mmp})^2,$$

$$W_{mmmp}^{\text{eq}} = S_{,mm}^{-3} S_{,pp}^{-1} S_{,mmmp} - 3 S_{,mm}^{-4} S_{,pp}^{-1} S_{,mmm} S_{,mmp} - 3 S_{,mm}^{-3} S_{,pp}^{-2} S_{,mmp} S_{,mpp}, \tag{A7}$$

where we have used the fact that $S_{,mp} = 0$ given in Eq. (A4), which largely simplifies our calculation. Now substituting Eq. (A4) into Eq. (A7) one immediately obtains Eq. (3.23).
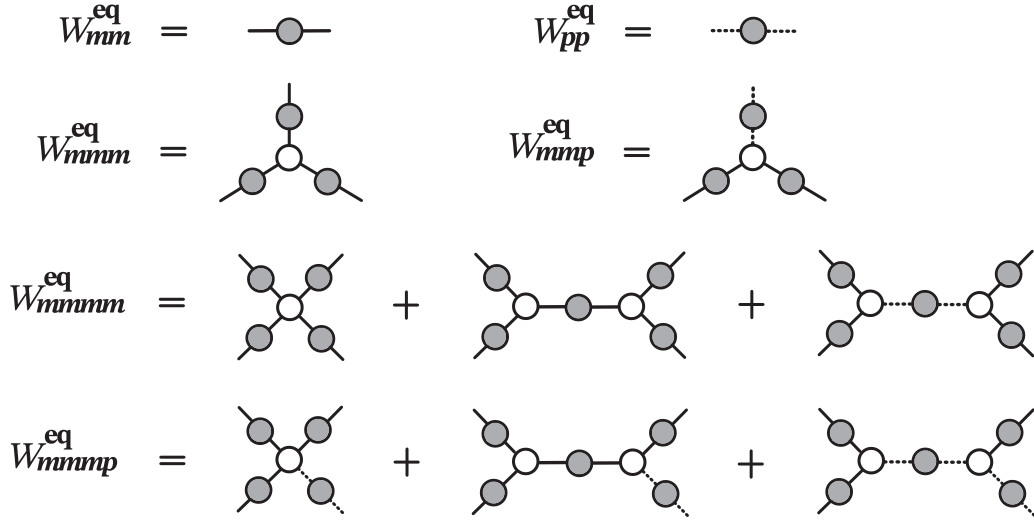
FIG. 8. Diagrammatic representation of the equilibrium correlation functions in Eq. (A7). A solid (dashed) line entering a vertex denotes the m (p) index. Open circles represent entropy derivatives. The solid gray circles with two legs are equilibrium two-point correlators expressed in terms of the second derivatives of the entropy: $W_{mm}^{\mathrm{eq}} = -S_{mm}^{-1}$ and $W_{pp}^{\mathrm{eq}} = -S_{pp}^{-1}$. Note the absence of the cross-correlator $W_{mp}^{\mathrm{eq}} = 0$, which simplifies the calculation.

Interestingly, one can infer the following relations from the above expressions:

$$W_{mmp}^{\mathrm{eq}} = -\frac{1}{\beta} W_{mm}^{\mathrm{eq}}, \quad W_{mmmp}^{\mathrm{eq}} = -\frac{2}{\beta} W_{mmm}^{\mathrm{eq}}. \tag{A8}$$

The above relations, similarly to $W_{mp}^{\mathrm{eq}} = 0$, are consequences of choosing m and p as our independent thermodynamic variables.

APPENDIX B: PARTIAL EQUILIBRIUM FOR PRESSURE FLUCTUATIONS

We shall now use the equilibrium entropy functional given by Eq. (A1) to determine the partial equilibrium value of δp. This value, $(\delta p)^{\mathrm{peq}}$, maximizes the entropy functional S under the condition that δm is fixed. Thus, $(\delta p)^{\mathrm{peq}}$ is determined by solving $S_{,p}(m + \delta m, p + (\delta p)^{\mathrm{peq}}) = 0$, were m and p are full equilibrium values, determined by $S_{,m}(m, p) = S_{,p}(m, p) = 0$. As a result we obtain

$$(\delta p)^{\mathrm{peq}} = -\tfrac{1}{2} S_{,pp}^{-1} S_{,mmp} \delta m^2 - \left(\tfrac{1}{6} S_{,pp}^{-1} S_{,mmmp} - \tfrac{1}{2} S_{,pp}^{-2} S_{,mmp} S_{,mpp} \right) \delta m^3, \tag{B1}$$

where we truncated the solution to the order we need to calculate the third- and fourth-order correlators. Notice that the absence of a term linear in δm is a consequence of the well-known fact that the fluctuations of pressure and specific entropy are not correlated, or $S_{,mp} = 0$. This lack of correlation, however, does not persist beyond the linear order. This observation is important for the non-Gaussian fluctuation equations we derive in this paper.

Using Eq. (B1), we can now calculate partial equilibrium values of the cross-correlators of pressure and specific entropy, such as

$$\begin{aligned}
G_{pmm}^{\mathrm{c,peq}}(x_1, x_2, x_3) &\equiv \langle : (\delta p)^{\mathrm{peq}}(x_1) : \delta m(x_2) \delta m(x_3) \rangle_{\mathrm{c}} = -\tfrac{1}{2} S_{,pp}^{-1} S_{,mmp} \langle : \delta m(x_1)^2 : \delta m(x_2) \delta m(x_3) \rangle_{\mathrm{c}} \\
&= -S_{,pp}^{-1} S_{,mmp} G_{mm}^{\mathrm{c}}(x_1, x_2) G_{mm}^{\mathrm{c}}(x_1, x_3), \\
G_{pmmm}^{\mathrm{c,peq}}(x_1, x_2, x_3, x_4) &\equiv \langle : (\delta p)^{\mathrm{peq}}(x_1) : \delta m(x_2) \delta m(x_3) \delta m(x_4) \rangle_{\mathrm{c}} \\
&= -\tfrac{1}{2} S_{,pp}^{-1} S_{,mmp} \langle : \delta m(x_1)^2 : \delta m(x_2) \delta m(x_3) \delta m(x_4) \rangle_{\mathrm{c}} \\
&\quad - \left(\tfrac{1}{6} S_{,pp}^{-1} S_{,mmmp} - \tfrac{1}{2} S_{,pp}^{-2} S_{,mmp} S_{,mpp} \right) \langle : \delta m(x_1)^3 : \delta m(x_2) \delta m(x_3) \delta m(x_4) \rangle_{\mathrm{c}} \\
&= -3 S_{,pp}^{-1} S_{,mmp} G_{mm}^{\mathrm{c}}(x_1, x_2) G_{mmm}^{\mathrm{c}}(x_1, x_3, x_4) \\
&\quad - \left(S_{,pp}^{-1} S_{,mmmp} - 3 S_{,pp}^{-2} S_{,mmp} S_{,mpp} \right) G_{mm}^{\mathrm{c}}(x_1, x_2) G_{mm}^{\mathrm{c}}(x_1, x_3) G_{mm}^{\mathrm{c}}(x_1, x_4) \big|_{\overline{x_2 x_3 x_4}}, \tag{B2}
\end{aligned}$$

where in the second equality of each above equation we neglected the terms which contribute to higher order in fluctuation expansion parameter ε. Finally, taking the n-point Wigner transform (2.41), we obtain Eqs. (3.22).

APPENDIX C: ALTERNATIVE EXPRESSIONS IN TERMS OF INDEPENDENT THERMODYNAMIC DERIVATIVES

We have expressed Eqs. (3.20) and (3.23) in terms of the independent thermodynamic derivatives chosen from set I given by Table I. There are, of course, numerous different choices for the independent thermodynamic derivatives in terms of which our equations can be formulated. In this section we provide alternative expressions for these equations, which are instead expressed in terms of independent thermodynamic derivatives specified in set II, where all independent derivatives of $\ln m_n$ (with respect to m or p) are chosen as the independent thermodynamic derivatives.

The presence of the derivatives of m_n reflects the fact that the volume integral of m is not a constant of motion, in contrast to the volume integral of n. The choice of the independent thermodynamic derivatives given by set II allows us to quantify such a difference. For example, if m were to be a linear function of n, m_n would be a constant and all thermodynamic derivatives acting on m_n would vanish, significantly simplifying above equations. In this case, there is only one independent nth-order thermodynamic derivative (cf. Table I), as in the charge diffusion problem studied in Ref. [22].

The alternative expressions for Eqs. (3.20), in terms of the independent thermodynamic derivatives from set II, read

$$\gamma_{mm} = m_n \lambda \alpha_m, \quad \gamma_{mp} = \left(\frac{2 + \alpha_m}{nm_n} + (\ln m_n)_{,m} \right) \beta m_n^2 \lambda, \quad (\gamma_{mm})_{,m} = \left((\ln \lambda)_{,m} + 2(\ln m_n)_{,m} - \left(\ln \frac{m_n}{\alpha_m} \right)_{,m} \right) \gamma_{mm},$$

$$(\gamma_{mm})_{,p} = \left[(\ln m_n)_{,mm} - \frac{\alpha_m}{nm_n} \left(\ln \frac{m_n}{\alpha_m} \right)_{,m} + \left(\frac{2 + \alpha_m}{nm_n} + (\ln m_n)_{,m} \right) (\ln m_n)_{,m} \right.$$

$$\left. + \frac{\alpha_m}{\beta m_n} \left[(\ln \lambda)_{,p} + (\ln m_n)_{,p} \right] + \frac{2 + \alpha_m}{n^2 m_n^2} \right] \beta m_n^2 \lambda,$$

$$(\gamma_{mp})_{,m} = \left[(\ln m_n)_{,mm} - \frac{\alpha_m}{nm_n} \left(\ln \frac{m_n}{\alpha_m} \right)_{,m} + \left(\frac{2 + \alpha_m}{nm_n} + (\ln m_n)_{,m} \right) [(\ln \lambda)_{,m} + 2(\ln m_n)_{,m}] + \frac{2 + \alpha_m}{n^2 m_n^2} \right] \beta m_n^2 \lambda,$$

$$(\gamma_{mm})_{,mm} = \left[(\ln \lambda)_{,mm} + 2(\ln m_n)_{,mm} - \left(\ln \frac{m_n}{\alpha_m} \right)_{,mm} + \left((\ln \lambda)_{,m} + 2(\ln m_n)_{,m} - \left(\ln \frac{m_n}{\alpha_m} \right)_{,m} \right)^2 \right] \gamma_{mm},$$

$$\left(m_n^2 \lambda \right)_{,m} = [(\ln \lambda)_{,m} + 2(\ln m_n)_{,m}] m_n^2 \lambda, \quad \left(m_n^2 \lambda \right)_{,p} = [(\ln \lambda)_{,p} + 2(\ln m_n)_{,p}] m_n^2 \lambda,$$

$$(m_n p_n \lambda)_{,m} = [(\ln \lambda)_{,m} + (\ln m_n)_{,m} + (\ln p_n)_{,m}] m_n p_n \lambda, \quad p_n = \frac{w}{n} - \frac{w}{\beta} (\ln m_n)_{,p} \left((\ln m_n)_{,m} + \frac{n(\ln m_n)_{,p}}{\beta} + \frac{2 + \alpha_m}{nm_n} \right)^{-1},$$

$$(\ln p_n)_{,m} = -\frac{2}{nm_n} + \frac{nm_n (\ln m_n)_{,mm} - \alpha_m \left(\ln \frac{m_n}{\alpha_m} \right)_{,m} + \frac{2 + \alpha_m}{nm_n}}{nm_n (\ln m_n)_{,m} + 2 + \alpha_m}$$

$$+ \frac{\alpha_m \left(\ln \frac{m_n}{\alpha_m} \right)_{,m} - nm_n (\ln m_n)_{,mm} + w(\ln m_n)_{,mp} - w(\ln m_n)_{,p} \left((\ln m_n)_{,m} + \frac{2 + \alpha_m}{nm_n} \right) + (\ln m_n)_{,m}}{nm_n (\ln m_n)_{,m} - w(\ln m_n)_{,p} + 2 + \alpha_m},$$

$$\left(m_n^2 \lambda \right)_{,mm} = \{ (\ln \lambda)_{,mm} + 2(\ln m_n)_{,mm} + [(\ln \lambda)_{,m} + 2(\ln m_n)_{,m}]^2 \} m_n^2 \lambda. \tag{C1}$$

Note again that we also treat α_m as an independent second-order thermodynamic derivative as m_n / α_m, since they are different by $m_n = -\beta w / n^2$ which is only a first-order thermodynamic derivative.

The alternative expressions for Eqs. (3.23) likewise read

$$W_{mm}^{\text{eq}} = \frac{m_n}{\alpha_m}, \quad W_{mp}^{\text{eq}} = 0, \quad W_{pp}^{\text{eq}} = -\frac{n}{\beta^2} \left(\frac{2 + \alpha_m}{nm_n} + (\ln m_n)_{,m} - \frac{w}{nm_n} (\ln m_n)_{,p} \right)^{-1},$$

$$W_{mmm}^{\text{eq}} = \left(\frac{m_n}{\alpha_m} \right)^2 \left[\left(\ln \frac{m_n}{\alpha_m} \right)_{,m} + 2(\ln m_n)_{,m} + \frac{2 + \alpha_m}{nm_n} \right], \quad W_{mmp}^{\text{eq}} = -\frac{m_n}{\beta \alpha_m},$$

$$W_{mmmm}^{\text{eq}} = \left(\frac{m_n}{\alpha_m} \right)^3 \left[\left(\ln \frac{m_n}{\alpha_m} \right)_{,mm} + 2 \left(\left(\ln \frac{m_n}{\alpha_m} \right)_{,m} + 5(\ln m_n)_{,m} + \frac{5 + \alpha_m}{nm_n} \right) \left(\ln \frac{m_n}{\alpha_m} \right)_{,m} \right.$$

$$+ 5(\ln m_n)_{,mm} + 9\left((\ln m_n)_{,m} + \frac{2+\alpha_m}{nm_n}\right)(\ln m_n)_{,m} - \frac{3\alpha_m}{\beta m_n}(\ln m_n)_{,p} + \frac{(2+\alpha_m)(7+2\alpha_m)}{n^2 m_n^2}\Bigg],$$

$$W_{mmmp}^{\mathrm{eq}} = -\frac{2}{\beta}\left(\frac{m_n}{\alpha_m}\right)^2\left[\left(\ln\frac{m_n}{\alpha_m}\right)_{,m} + 2(\ln m_n)_{,m} + \frac{2+\alpha_m}{nm_n}\right]. \tag{C2}$$

APPENDIX D: NOTATIONS

This Appendix summarizes the notations introduced in Sec. II D.

$\bar{\nabla}_\mu$: confluent derivative, Eqs. (2.31), (2.45);

$\bar{\omega}_{\lambda\mu}^\nu$: confluent connection, Eq. (2.28);

$\mathring{\omega}_{\mu b}^a$: spin connection for local triad e^a, Eq. (2.37), Fig. 4;

$\mathring{\partial}_\mu G_n(x_1, \ldots, x_n)$: derivative with respect to the midpoint x at fixed y_i^a, Eq. (2.35);

$\bar{\nabla}_\mu \bar{G}_n$: confluent derivative of $\bar{G}_n$, Eq. (2.31);

$G_n \equiv G_{i_1 \ldots i_n}$: "raw" n-point correlator, Eq. (2.9);

$G_n^{\mathrm{c}} \equiv G_{i_1 \ldots i_n}^{\mathrm{c}}$: connected n-point correlator, Eqs. (2.15), (2.17), (2.20);

$\bar{G}_n \equiv \bar{G}_{i_1 \ldots i_n}$: confluent correlator, Eq. (2.29), Fig. 3;

$W_n \equiv W_{i_1 \ldots i_n}$: Wigner transform of connected confluent correlator $\bar{G}_n^{\mathrm{c}}$, Eq. (2.41);

$\mathring{\partial}_\mu W_n(x; \boldsymbol{q}_1, \ldots, \boldsymbol{q}_n)$: partial x derivative at fixed $\boldsymbol{q}_i$'s (Wigner transform of $\mathring{\partial}_\mu \bar{G}_n^{\mathrm{c}}$), Eq. (2.43);

$\bar{\nabla}_\mu W_n$: confluent derivative of W_n (Wigner transform of $\bar{\nabla}_\mu \bar{G}_n^{\mathrm{c}}$), Eq. (2.45);

x: midpoint space-time vector, Eq. (2.26);

$y_i \equiv x_i - x$: separation four-vector, Eq. (2.30);

$y_i^a \equiv e^a \cdot y_i$: components of the separation vector in the local triad basis $e^a(x)$, $a = 1, 2, 3$.

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