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The combinatorics of supertorus sheaf cohomology

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ABSTRACT

Affine superspace $\mathbb{C}^{1|n}$ has a single bosonic coordinate z and n fermionic coordinates $\theta_1, \dots, \theta_n$. Let M be the supertorus obtained by quotienting $\mathbb{C}^{1|n}$ by the abelian group generated by the maps $S : (z, \theta_1, \dots, \theta_n) \mapsto (z+1, \theta_1, \dots, \theta_n)$ and $T : (z, \theta_1, \dots, \theta_n) \mapsto (z+t, \theta_1+\alpha_1, \dots, \theta_n+\alpha_n)$ where $t \in \mathbb{C}$ has positive imaginary part and $\alpha_1, \dots, \alpha_n$ are independent fermionic parameters. We compute the zeroth and first cohomology groups of the structure sheaf $\mathcal{O}$ of M as doubly graded $\mathfrak{S}_n$-modules, exhibiting an instance of Serre duality between these groups. We use skein relations and noncrossing matchings to give a combinatorial presentation of $H^0(M, \mathcal{O})$ in terms of generators and relations.

1. Introduction

In classical algebraic geometry, regular functions on complex varieties X are represented by rational functions f in commuting variables $x_1, x_2, \dots$ whose denominators have no zeros on X. The prototypical model is affine m-space $\mathbb{C}^m$ with coordinate ring $\mathbb{C}[x_1, \dots, x_m]$. In supergeometry, one considers functions of both commuting variables $x_1, x_2, \dots$ representing bosonic coordinates as well as anticommuting variables $\theta_1, \theta_2, \dots$ representing fermionic coordinates. The coordinate ring of affine superspace $\mathbb{C}^{m|n}$ is the tensor product $\mathbb{C}[x_1, \dots, x_m] \otimes \wedge\{\theta_1, \dots, \theta_n\}$ of a polynomial ring of rank m and an exterior algebra of rank n.

Let $\mathbb{C}^{1|n}$ be affine superspace with one bosonic coordinate z and n fermionic coordinates $\theta_1, \dots, \theta_n$. We consider a free exterior algebra $\wedge\{\alpha_1, \dots, \alpha_n\}$ generated by "coefficient" fermionic variables $\alpha_1, \dots, \alpha_n$. In supergeometry, elements of $\wedge\{\alpha_1, \dots, \alpha_n\}$ of odd homogeneous degree may be substituted for the θ_i in any regular functions $f(z, \theta_1, \dots, \theta_n)$ on $\mathbb{C}^{1|n}$; this is the anticommutative analogue of evaluating a polynomial $\mathbb{C}[x_1, \dots, x_m]$ at a point in $\mathbb{C}^m$.

Let $t \in \mathbb{C}$ be a complex number with positive imaginary part. We consider two translation operators $S, T : \mathbb{C}^{1|n} \to \mathbb{C}^{1|n}$ defined in terms of coordinates by

$$S : (z, \theta_1, \dots, \theta_n) \mapsto (z+1, \theta_1, \dots, \theta_n), \tag{1.1}$$

$$T : (z, \theta_1, \dots, \theta_n) \mapsto (z+t, \theta_1+\alpha_1, \dots, \theta_n+\alpha_n). \tag{1.2}$$

We let M be the quotient of $\mathbb{C}^{1|n}$ by the group of operators generated by S and T. If $n = 0$ and no fermionic variables were present, the space M would be a copy of the usual torus $S^1 \times S^1$ with complex structure given by the modulus t. As such, we may regard M as a superspace extension of the torus (technically, a family of supertori over the parameter space $\wedge\{\alpha_1, \dots, \alpha_n\}$).

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Let $\mathcal{O}$ be the structure sheaf of M, so that the zeroth cohomology $H^0(M, \mathcal{O})$ is the algebra of globally defined regular functions on M (see Definition 3.1 for a precise definition of $H^0(M, \mathcal{O})$). If $n = 0$, the space $H^0(M, \mathcal{O})$ is merely a copy of the ground field $\mathbb{C}$; there are no nonconstant regular functions on the torus. On the other hand, we show that $H^0(M, \mathcal{O})$ has rich combinatorial structure in the supertorus case of $n > 0$.

- We show (Corollary 4.8) that $H^0(M, \mathcal{O})$ is generated as a $\mathbb{C}$-algebra by the 'basic invariants' $\alpha_i, \alpha_i\theta_i$, and $\alpha_i\theta_j + \alpha_j\theta_i$ for $1 \le i < j \le n$. Furthermore, we give a combinatorial description (Theorem 4.9) of the relations which hold among these invariants in terms of the classical Ptolemy relation [5,8] which resolves a simple crossing as a sum of its two possible resolutions.
- Considering degree in the coefficient variables α_i as well as the θ_i, the ring $H^0(M, \mathcal{O})$ attains the structure of a bigraded module over the symmetric group $\mathfrak{S}_n$. We compute its bigraded isomorphism type (Theorem 3.3). In particular, the (i, j)-component $H^0(M, \mathcal{O})_{i,j}$ is nonzero if and only if $i \ge j$, in which case it has dimension $\binom{n}{i}\binom{n}{j} - \binom{n}{i+1}\binom{n}{j-1}$. Therefore, the dimensions of the spaces $H^0(M, \mathcal{O})_{i,i}$ are the Narayana numbers $\mathrm{Nar}(n+1, i+1)$ for $i = 1, \ldots, n$ and the sum $\sum_{i=0}^{n} H^0(M, \mathcal{O})_{i,i}$ of these dimensions is the $(n+1)^{st}$ Catalan number $\mathrm{Cat}(n+1)$.

It follows from the second bullet point above that the vector spaces $H^0(M, \mathcal{O})_{i,j}$ and $H^0(M, \mathcal{O})_{n-j,n-i}$ have the same dimension for all $i + j \le n$. We prove (Theorem 3.4) that multiplication by ℓ^{n-i-j} where $\ell = \alpha_1\theta_1 + \cdots + \alpha_n\theta_n$ furnishes a linear isomorphism between these spaces. This fact may be viewed as a bigraded version of the Hard Lefschetz Theorem for the ring $H^0(M, \mathcal{O})$. We also show that the first cohomology group $H^1(M, \mathcal{O})$ (see Definition 3.5) and $H^0(M, \mathcal{O})$ satisfy Serre duality (Theorem 3.8). The $n = 1$ case of these results was proven by the second author [10] in previous work.

We provide a bit more background in supergeometry, although this is not required for the rest of this paper. The $n = 1$ supertorus considered in [10] was in fact a super Riemann surface, with the second group generator above modified to $T : (z, \theta_1) \mapsto (z + t + \theta_1\alpha_1, \theta_1 + \alpha_1)$. In principle, functions in $H^0(M, \mathcal{O})$ or cocycles in $H^1(M, \mathcal{O})$ might depend on z as well as α_1, θ_1. However, one can show by a Fourier series argument [1,4,10] (using the periodicity given by S) that they are in fact independent of z. This implies in particular that the extra even term $\theta_1\alpha_1$ in T does not affect the cohomology and can be ignored. It also makes the generator S trivial and allows us to compute the cohomology from T alone acting on the fermionic variables. In this paper we will compute $H^1(M, \mathcal{O})$ as the group cohomology of the cyclic group generated by T. The equivalence of sheaf cohomology and group cohomology in this instance follows from [9,12].

Serre duality for supermanifolds has been proven in [3] for individual supermanifolds and in [1] for families. The duality we exhibit between $H^1(M, \mathcal{O})$ and $H^0(M, \mathcal{O})$ is an instance of this general structure (technically, as applied to the total space of our family M of supertori).

Our results can be viewed as a first step toward a general theory of Abelian supervarieties, including the Jacobians of supercurves. In such a context the two generators S, T would be extended to a set of $2g$ supertranslations $S_1, \ldots, S_g, T_1, \ldots, T_g$ representing a canonical homology basis of a torus of complex dimension g. The cohomology of the group they generate would be correspondingly more complicated, although still independent of the g even coordinates and hence a purely algebraic problem. The computation of $H^1(M, \mathcal{O})$ is of particular interest in view of its role in classifying topologically trivial line bundles on M [1].

2. Background

We will need a notion of fermionic differentiation. Let $(\omega_1, \ldots, \omega_m)$ be a list of m fermionic variables and let $\wedge\{\omega_1, \ldots, \omega_m\}$ be the exterior algebra over these variables. For $1 \le i \le m$, we define an operator $\partial/\partial\omega_i$ on $\wedge\{\omega_1, \ldots, \omega_m\}$ as the linear extension of

$$\partial/\partial\omega_i(\omega_{j_1}\cdots\omega_{j_r}) := \begin{cases} (-1)^{s-1}\omega_{j_1}\cdots\widehat{\omega_{j_s}}\cdots\omega_{j_r} & \text{if } j_s = i \\ 0 & \text{if } i \neq j_1, \ldots, j_s \end{cases} \tag{2.1}$$

whenever $1 \le j_1, \ldots, j_s \le m$ are distinct indices. Here the hat denotes omission.

Let G be a finite group. The *Grothendieck ring* of G is the free $\mathbb{Z}$-module with basis given by isomorphism classes $[V]$ of irreducible $\mathbb{C}[G]$-modules V. We extend the notation $[-]$ to arbitrary finite-dimensional $\mathbb{C}[G]$-modules by setting $[V] = [W] + [U]$ whenever we have a short exact sequence $0 \to U \to V \to W \to 0$ of finite-dimensional $\mathbb{C}[G]$-modules. The product structure on the Grothendieck ring is given by

$$[V] \cdot [W] := [V \otimes W] \tag{2.2}$$

where G acts on the vector space $V \otimes W$ by $g \cdot (v \otimes w) := (g \cdot v) \otimes (g \cdot w)$ for $g \in G$, $v \in V$, and $w \in W$. We will consider the Grothendieck ring of $G = \mathfrak{S}_n$ in this paper; this is a free $\mathbb{Z}$-algebra with basis indexed by the set of partitions $\lambda \vdash n$. The structure constants for the canonical basis $[V^\lambda]$ of irreducible $\mathfrak{S}_n$-modules are the *Kronecker coefficients*; these are famously difficult to compute.

We shall need a result guaranteeing the invertibility of a certain combinatorial matrix. Recall that the *Boolean poset* B_n has elements given by subsets $S \subseteq \{1, \ldots, n\}$ and order relation $S \le T$ if and only if $S \subseteq T$. Let $B_{n,i}$ denote the family of size

i subsets of $\{1, \ldots, n\}$; these are the rank i elements in B_n. The next result states that the incidence matrix between the complementary ranks $B_{n,i}$ and $B_{n,n-i}$ is invertible.

Theorem 2.1. *For $0 \leq i \leq j \leq n$, let $M_n(i, j)$ be the $\binom{n}{i} \times \binom{n}{j}$ matrix with rows indexed by $B_{n,i}$, columns indexed by $B_{n,j}$, and entries determined by the rule*

$$M_n(i, j)_{S,T} = \begin{cases} 1 & S \subseteq T \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

For all $0 \leq i \leq n/2$, the matrix $M_n(i, n-i)$ is invertible.

The origins of Theorem 2.1 are difficult to trace. In [13], Stanley used his theory of differential posets to calculate the (nonzero) eigenvalues of $M_n(i, n-i)$. Another proof of Theorem 2.1 is due to Hara and Watanabe [2]. In [2] Theorem 2.1 is viewed as a witness for the Hard Lefschetz property of the cohomology ring of the n-fold self product $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ of the Riemann sphere.

3. The bigraded $\mathfrak{S}_n$-structure of $H^0(M, \mathcal{O})$ and $H^1(M, \mathcal{O})$

Let $(\alpha_1, \ldots, \alpha_n)$ and $(\theta_1, \ldots, \theta_n)$ be two lists of fermionic variables and consider the rank $2n$ exterior algebra

$$E_n := \wedge\{\alpha_1, \ldots, \alpha_n, \theta_1, \ldots, \theta_n\} \tag{3.1}$$

on these variables over the ground field $\mathbb{C}$. This algebra is doubly graded, with

$$(E_n)_{i,j} = \wedge^i\{\alpha_1, \ldots, \alpha_n\} \otimes \wedge^j\{\theta_1, \ldots, \theta_n\} \tag{3.2}$$

and carries an action of $\mathfrak{S}_n$

$$w \cdot \theta_i := \theta_{w(i)} \qquad w \cdot \alpha_i := \alpha_{w(i)} \qquad w \in \mathfrak{S}_n, \ 1 \leq i \leq n \tag{3.3}$$

which respects this bigrading.

3.1. The module structure of $H^0(M, \mathcal{O})$

The zeroth cohomology group $H^0(M, \mathcal{O})$ is the set of globally defined regular functions on M, namely the functions on $\mathbb{C}^{1|n}$ that are invariant under the translations S and T of Eqs. (1.1) and (1.2). Since these functions are known to be independent of the even coordinate z [1,4,10], it suffices to check invariance under the action of T on the odd coordinates only.

Thus, we consider the map of algebras

$$T : E_n \to E_n \tag{3.4}$$

defined on generators by

$$T : \theta_i \mapsto \theta_i + \alpha_i \qquad T : \alpha_i \mapsto \alpha_i. \tag{3.5}$$

The map T acts by fermionic translation. The zeroth cohomology group $H^0(M, \mathcal{O})$ is related to T as follows.

Definition 3.1. Let $H^0(M, \mathcal{O}) := \{f \in E_n : T(f) = f\}$ be the subalgebra of E_n fixed by T.

Despite the fact that T is not bigraded, its fixed subspace $H^0(M, \mathcal{O})$ is a bigraded subalgebra of E_n. To show this, we introduce the operator $\tau : E_n \to E_n$ given by

$$\tau(f) := \sum_{i=1}^{n} \alpha_i \cdot (\partial/\partial\theta_i) f \tag{3.6}$$

The operator τ is bihomogeneous of degree $(1, -1)$.

Proposition 3.2. *Let $f \in E_n$. We have $T(f) = f$ if and only if $\tau(f) = 0$. In particular, the subalgebra $H^0(M, \mathcal{O})$ of E_n is bigraded.*

Our proof of Proposition 3.2 proceeds by showing that τ is an 'infinitesimal' version of the translation operator T.

Proof. We claim that we have the equality of operators

$$T = \exp(\tau) := \mathrm{id} + \tau + \frac{1}{2!}\tau^2 + \frac{1}{3!}\tau^3 + \cdots \tag{3.7}$$

on E_n. Since $\tau^{n+1} = 0$ on E_n, the RHS is really a finite sum. Indeed, this equality of linear operators may be easily checked on monomials using the evaluation

$$T : \theta_{i_1} \cdots \theta_{i_r} \cdot \alpha_{j_1} \cdots \alpha_{j_s} \mapsto (\theta_{i_1} + \alpha_{i_1}) \cdots (\theta_{i_r} + \alpha_{i_r}) \cdot \alpha_{j_1} \cdots \alpha_{j_s}. \tag{3.8}$$

If $\tau(f) = 0$, Equation (3.7) immediately gives $T(f) = f$.

On the other hand, if $T(f) = f$, Equation (3.7) gives the relation

$$0 = \tau(f) + \frac{1}{2!}\tau^2(f) + \frac{1}{3!}\tau^3(f) + \cdots \tag{3.9}$$

inside E_n. The T-invariant function f may be written uniquely as a sum $f = \sum_{i,j=0}^n f_{i,j}$ where $f_{i,j} \in (E_n)_{i,j}$. If $f_{i,j} = 0$ for all i, j, then certainly $\tau(f) = 0$ and we are done. Otherwise, choose j_0 minimal such that $f_{i,j_0} \neq 0$ for some $0 \leq i \leq n$ and let $f_{*,j_0} := \sum_{i=0}^n f_{i,j_0}$. Since τ has bidegree $(1, -1)$, Equation (3.9) and our choice of j_0 imply that $\tau(f_{*,j_0}) = 0$. Equation (3.7) shows that $T(f_{*,j_0}) = f_{*,j_0}$, so that $T(f - f_{*,j_0}) = f - f_{*,j_0}$. By induction on bidegree support, we have $\tau(f - f_{*,j_0}) = 0$. We conclude that $\tau(f) = \tau(f_{*,j_0}) + \tau(f - f_{*,j_0}) = 0 + 0 = 0$. □

By Proposition 3.2, we have a direct sum decomposition

$$H^0(M, \mathcal{O}) = \bigoplus_{i,j=0}^{n} H^0(M, \mathcal{O})_{i,j} \tag{3.10}$$

in the category of bigraded rings or bigraded $\mathfrak{S}_n$-modules. The module structure of $H^0(M, \mathcal{O})_{i,j}$ is given as follows.

Theorem 3.3. *We have $H^0(M, \mathcal{O})_{i,j} = 0$ unless $i \geq j$. If $i \geq j$, then*

$$\left[H^0(M, \mathcal{O})_{i,j}\right] = \left[\wedge^i \mathbb{C}^n\right] \cdot \left[\wedge^j \mathbb{C}^n\right] - \left[\wedge^{i+1} \mathbb{C}^n\right] \cdot \left[\wedge^{j-1} \mathbb{C}^n\right] \tag{3.11}$$

$$= [(E_n)_{i,j}] - [(E_n)_{i+1,j-1}] \tag{3.12}$$

within the Grothendieck ring of $\mathfrak{S}_n$ where the action of $\mathfrak{S}_n$ on $\mathbb{C}^n$ is by coordinate permutation.

The bigraded $\mathfrak{S}_n$-structure of $H^0(M, \mathcal{O})$ is very similar to that of the bigraded $\mathfrak{S}_n$-module FDR_n of *fermionic diagonal coinvariants* introduced in [7] and further studied in [6,8]. In [7, Thm. 3.2] it is shown that multiplication by an appropriate power of $\ell := \sum_i \alpha_i \theta_i$ affords a linear isomorphism $\ell^{n-i-j} \times (-) : (E_n)_{i,j} \xrightarrow{\sim} (E_n)_{n-j,n-i}$ whenever $i + j \leq n$. In [7, Thm. 4.2] this isomorphism is used to deduce the bigraded $\mathfrak{S}_n$-structure of $FDR_n := E_n/I$ where $I \subseteq E_n$ is the ideal generated by $\mathfrak{S}_n$-invariants with vanishing constant term. The proof of Theorem 3.3 should be compared to those of [7, Thm. 3.2, Thm. 4.2].

Proof. Let (i, j) be a fixed bidegree. For any $r \geq 0$, we have an operator

$$\tau^r : (E_n)_{i,j} \longrightarrow (E_n)_{i+r,j-r} \tag{3.13}$$

When $i \leq j$ and $r = j - i$, we have a map

$$\tau^{j-i} : (E_n)_{i,j} \longrightarrow (E_n)_{j,i} \tag{3.14}$$

between complementary bidegrees.

We claim that the map τ^{j-i} appearing in (3.14) is bijective. To prove this, we examine its matrix with respect to a strategic choice of bases.

Given subsets $A, B \subseteq \{1, \ldots, n\}$, let $m_{A,B} \in E_n$ be the monomial whose variables consist of α_a for $a \in A$ and θ_b for $b \in B$ written in increasing order with respect to

$$\alpha_1 < \theta_1 < \alpha_2 < \theta_2 < \cdots < \alpha_n < \theta_n. \tag{3.15}$$

For example, if $n = 8$, $A = \{2, 3, 5\}$, and $B = \{1, 3, 4, 6\}$ we have

$$m_{A,B} = \theta_1 \cdot \alpha_2 \cdot \alpha_3\theta_3 \cdot \theta_4 \cdot \alpha_5 \cdot \theta_6.$$

It is not hard to see that

$$\tau(m_{A,B}) = \sum_{\substack{c \notin A \\ c \in B}} m_{A\cup c, B-c}. \tag{3.16}$$

The definition of $m_{A,B}$ was chosen so that Equation (3.16) is free of signs. Iterating Equation (3.16) $j-i$ times, we see that

$$\tau^{j-i}(m_{A,B}) = (j-i)! \cdot \sum_{\substack{|C|=j-i \\ C \subseteq (B-A)}} m_{A\cup C, B-C}. \tag{3.17}$$

The matrix for τ^{j-i} with respect to the $m_{A,B}$ basis therefore breaks up into a direct sum of matrices indexed by subsets $D = A \cup B$ where each direct summand is $(j-i)!$ times a matrix which Theorem 2.1 guarantees is invertible. We conclude that τ^{j-i} is invertible whenever $i \leq j$.

Whenever a composition $f \circ g$ of functions is a bijection, the map f is a surjection and the map g is an injection. We conclude that

$$\tau : (E_n)_{i,j} \longrightarrow (E_n)_{i+1,j-1} \tag{3.18}$$

is injective whenever $i < j$. Proposition 3.2 implies that $H^0(M, \mathcal{O})_{i,j} = 0$ when $i < j$. Similarly, if $i \geq j$, the map in Equation (3.18) is surjective. Since τ is $\mathfrak{S}_n$-equivariant, Proposition 3.2 implies the desired relation in the Groethendieck ring of $\mathfrak{S}_n$. □

As a consequence of Theorem 3.3, we have

$$\dim H^0(M, \mathcal{O})_{i,j} = \begin{cases} \binom{n}{i}\binom{n}{j} - \binom{n}{i+1}\binom{n}{j-1} & i \geq j \\ 0 & i < j \end{cases} \tag{3.19}$$

which implies that

$$\dim H^0(M, \mathcal{O})_{i,j} = \dim H^0(M, \mathcal{O})_{n-j,n-i} \qquad \text{for } i+j \leq n. \tag{3.20}$$

The following result shows that multiplication by the appropriate power of

$$\ell := \alpha_1\theta_1 + \cdots + \alpha_n\theta_n \tag{3.21}$$

yields a linear isomorphism $H^0(M, \mathcal{O})_{i,j} \xrightarrow{\sim} H^0(M, \mathcal{O})_{n-j,n-i}$.

Theorem 3.4. *Suppose $i + j \leq n$. We have a linear isomorphism*

$$(-) \times \ell^{n-i-j} : H^0(M, \mathcal{O})_{i,j} \xrightarrow{\sim} H^0(M, \mathcal{O})_{n-j,n-i} \tag{3.22}$$

where $\ell \in E_n$ is given by (3.21).

Proof. Since ℓ is T-invariant, we have $\ell \in H^0(M, \mathcal{O})$ and the given map is well-defined. By the dimension equality (3.20) it suffices to show that we have a linear isomorphism

$$(-) \times \ell^{n-i-j} : (E_n)_{i,j} \xrightarrow{\sim} (E_n)_{n-j,n-i}. \tag{3.23}$$

The proof that (3.23) is bijective is similar to that of Theorem 3.3. For subsets $A, B \subseteq \{1, \dots, n\}$, let $m'_{A,B} \in E_n$ be the monomial given by

$$m'_{A,B} := \prod_{c \in A\cap B} \alpha_c \cdot \theta_c \times \prod_{a \in A-B} \alpha_a \times \prod_{b \in B-A} \theta_b \tag{3.24}$$

where each product is taken in increasing order. For example, if $A = \{2, 4, 5, 7\}$ and $B = \{3, 4, 7\}$ then $m'_{A,B} = (\alpha_4\theta_4\alpha_7\theta_7) \cdot \alpha_2\alpha_5 \cdot \theta_3$. It is not hard to see that

$$\ell \cdot m'_{A,B} = \sum_{c \notin A\cup B} m'_{A\cup c, B\cup c} \tag{3.25}$$

where the definition of $m'_{A,B}$ guarantees that Equation (3.25) is free of signs. The proof that the map in (3.23) is bijective now follows from the same reasoning as in the proof of Theorem 3.3. The isomorphism (3.23) restricts to give the isomorphism in the statement of the theorem. □

3.2. The module structure of $H^1(M, \mathcal{O})$

The first cohomology group $H^1(M, \mathcal{O})$ of the sheaf $\mathcal{O}$ classifies line bundles over M. In order to compute $H^1(M, \mathcal{O})$, we recall some notions from group cohomology.

Let G be an abelian group and let N be a G-module. A 1-*cocycle* is a function $c : G \to N$ which satisfies

$$c(gh) = h \cdot c(g) + c(h) \qquad \text{for all } g, h \in G. \tag{3.26}$$

A 1-cocycle $c : G \to N$ is determined by its values on a generating set of G. A 1-*coboundary* is a function $c_n : G \to N$ of the form

$$c_n(g) := g \cdot n - n \tag{3.27}$$

for some $n \in N$. It is easily seen that every 1-coboundary is a 1-cocycle. The sets of 1-cocycles and 1-coboundaries both form groups under pointwise addition. The *first cohomology group* is the quotient

$$H^1(G, N) := (\text{1-cocycles})/(\text{1-coboundaries}) \tag{3.28}$$

of the group of 1-cocycles by its subgroup of 1-coboundaries.

From the equivalence between sheaf and group cohomology [9], $H^1(M, \mathcal{O}) \cong H^1(G, N)$, where G is generated by the supertranslations S, T of (1.1), (1.2) acting on the module of functions N on $\mathbb{C}^{1|n}$. Proposition (B.1.3) of [12] gives an exact sequence

$$0 \to H^1((T), N^S) \to H^1(G, N) \to H^1((S), N) \tag{3.29}$$

whose final term vanishes. Thus $H^1(G, N)$ can be computed as the cohomology of the cyclic group generated by T acting on the S-invariant functions in N. Since this cohomology is also known to be independent of z, we can take S to be the identity and simply compute the cohomology of (T) acting on E_n.

Thus, for our purposes, the group $G = \langle g \rangle$ is the infinite cyclic group and the module N is the exterior algebra E_n. The generator g of G acts on E_n by the translation operator T:

$$g \cdot f := T(f) \qquad f \in E_n. \tag{3.30}$$

We may identify the group of 1-cocycles with E_n itself. Indeed, given an element $f \in E_n$, the corresponding 1-cocycle $c_f : G \to M$ determined by $c_f(g) = f$. This motivates the following definition.

Definition 3.5. Let $H^1(M, \mathcal{O})$ be the quotient of E_n by its linear subspace

$$\{T(f) - f \,:\, f \in E_n\} \tag{3.31}$$

of 1-coboundaries.

The denominator in the quotient space of Definition 3.5 may be written in another way using the operator τ. This will allow us to deduce that, like $H^0(M, \mathcal{O})$, the first cohomology $H^1(M, \mathcal{O})$ is bigraded.

Proposition 3.6. *The set $\{T(f) - f \,:\, f \in E_n\}$ of 1-coboundaries equals the image of the map*

$$\tau : E_n \to E_n. \tag{3.32}$$

Thus, the quotient $H^1(M, \mathcal{O}) = \bigoplus_{i,j=0}^{n} H^1(M, \mathcal{O})_{i,j}$ is the cokernel of the bihomogeneous map τ. In particular, the vector space $H^1(M, \mathcal{O})$ is bigraded.

Proof. Equation (3.7) shows that for any $f \in E_n$ we have

$$T(f) - f = \tau(f) + \frac{1}{2!}\tau^2(f) + \frac{1}{3!}\tau^3(f) + \cdots = \tau\left(f + \frac{1}{2!}\tau(f) + \frac{1}{3!}\tau^2(f) + \cdots\right) \tag{3.33}$$

so that every 1-coboundary is in the image of τ.

The reverse containment follows from induction on θ-degree. If $f \in E_n$ has θ-degree 0, then $\tau(f) = 0 = T(0) - 0$ is a 1-coboundary. In general, we have

$$\tau(f) = T(f) - f - \frac{1}{2!}\tau^2(f) - \frac{1}{3!}\tau^3(f) + \cdots = (T(f) - f) - \tau\left(\frac{1}{2!}\tau(f) + \frac{1}{3!}\tau^2(f) + \cdots\right). \tag{3.34}$$

The term $T(f) - f$ is a 1-coboundary by definition and the sum $\frac{1}{2!}\tau(f) + \frac{1}{3!}\tau^2(f) + \cdots$ has strictly lower θ-degree than f, so its image $\tau\left(\frac{1}{2!}\tau(f) + \frac{1}{3!}\tau^2(f) + \cdots\right)$ under τ is a 1-coboundary by induction. $\square$

Since τ is $\mathfrak{S}_n$-equivariant, the bigraded vector space $H^1(M,\mathcal{O}) = \bigoplus_{i,j=0}^n H^1(M,\mathcal{O})_{i,j}$ attains the status of a bigraded $\mathfrak{S}_n$-module. The structure of this module may be determined using the same ideas as in the proof of Theorem 3.3.

Theorem 3.7. *The vector space $H^1(M,\mathcal{O})_{i,j}$ is zero unless $i \le j$. When $i \le j$, we have the equality*

$$\left[H^1(M,\mathcal{O})_{i,j}\right] = \left[\wedge^i \mathbb{C}^n\right] \cdot \left[\wedge^j \mathbb{C}^n\right] - \left[\wedge^{i-1} \mathbb{C}^n\right] \cdot \left[\wedge^{j+1} \mathbb{C}^n\right] \tag{3.35}$$

$$= [(E_n)_{i,j}] - [(E_n)_{i-1,j+1}] \tag{3.36}$$

in the Grothendieck ring of $\mathfrak{S}_n$. Here $\mathfrak{S}_n$ acts on $\mathbb{C}^n$ by coordinate permutation.

Proof. As in the proof of Theorem 3.3, the map

$$\tau : (E_n)_{i,j} \longrightarrow (E_n)_{i+1,j-1} \tag{3.37}$$

is injective whenever $i < j$ and surjective whenever $i \ge j$. Since τ is $\mathfrak{S}_n$-equivariant, the result follows from Proposition 3.6. $\square$

Let V and W be finite-dimensional complex vector spaces and let $\langle -,- \rangle : V \otimes W \to \mathbb{C}$ be a bilinear pairing between them. The pairing $\langle -,- \rangle$ is *perfect* if for all nonzero vectors $v \in V$, there exists a vector $w \in W$ such that $\langle v, w\rangle \neq 0$ and for all nonzero vectors $w \in W$, there exists a vector $v \in V$ such that $\langle v, w\rangle \neq 0$. In particular, if a perfect pairing exists, then $\dim V = \dim W$.

Let X be an n-dimensional smooth complex projective variety with canonical line bundle $\mathcal{K}_X$. If $\mathcal{E}$ is an algebraic vector bundle over X, *Serre duality* furnishes a perfect pairing

$$H^i(X,\mathcal{E}) \otimes H^{n-i}(X, \mathcal{K}_X \otimes \mathcal{E}^*) \longrightarrow \mathbb{C} \tag{3.38}$$

between sheaf cohomology groups of complementary degree. This is an analogue of Poincaré duality for sheaf cohomology.

We describe a supergeometric version of Serre duality which holds between the bigraded rings $H^0(M,\mathcal{O})$ and $H^1(M,\mathcal{O})$. Consider the 'volume form' $\mathrm{vol}_n := \alpha_1 \cdots \alpha_n \cdot \theta_1 \cdots \theta_n$ in E_n and define a bilinear pairing $\langle -,- \rangle$ on E_n by

$$\langle f, g \rangle := \text{coefficient of } \mathrm{vol}_n \text{ in } f \cdot g. \tag{3.39}$$

Theorem 3.8. *The pairing $\langle -,- \rangle$ induces a perfect pairing $H^0(M,\mathcal{O}) \otimes H^1(M,\mathcal{O}) \to \mathbb{C}$.*

Proof. The pairing $\langle -,- \rangle$ is easily seen to be perfect as a map

$$E_n \otimes E_n \longrightarrow \mathbb{C}. \tag{3.40}$$

We claim that the operator τ is self-adjoint with respect to this pairing; that is

$$\langle \tau(f), g \rangle = \langle f, \tau(g) \rangle \qquad \text{for all } f, g \in E_n. \tag{3.41}$$

Indeed, it suffices to check Equation (3.41) in the case where f and g are monomials, and this is a straightforward computation.

Propositions 3.2 and 3.6 together with Equation (3.41) imply that $\langle -,- \rangle$ descends to a well-defined bilinear pairing

$$\langle -,- \rangle : H^0(M,\mathcal{O}) \otimes H^1(M,\mathcal{O}) \longrightarrow \mathbb{C}. \tag{3.42}$$

Indeed, if $f \in H^0(M,\mathcal{O}) = \mathrm{Ker}(\tau)$ and $g \in E_n$ then $\langle f, \tau(g)\rangle = \langle \tau(f), g\rangle = \langle 0, g\rangle = 0$. Since $\dim H^0(M,\mathcal{O})_{i,j} = \dim H^1(M, \mathcal{O})_{n-i,n-j}$, the pairing $\langle -,- \rangle$ remains perfect when regarded as a map $H^0(M,\mathcal{O}) \otimes H^1(M,\mathcal{O}) \to \mathbb{C}$. $\square$

An explicit set of coset representatives for elements in $H^1(M,\mathcal{O})$ can be obtained, if desired, from Theorem 3.8. For any element $f \in E_n$, write $f \odot (-) : E_n \to E_n$ for the differential operator on E_n obtained from f by replacing each instance of θ_i with $\partial/\partial\theta_i$ and each instance of α_i with $\partial/\partial\alpha_i$. If $\mathcal{B}$ is a basis of $H^0(M,\mathcal{O})$, it follows from Theorem 3.8 that $\{f \odot \mathrm{vol}_n : f \in \mathcal{B}\}$ will descend to a basis of $H^1(M,\mathcal{O})$.

For example, when $n = 2$ we have a basis of $H^0(M,\mathcal{O}) = \mathrm{Ker}(\tau)$ given by

$$\mathcal{B} = \{1,\ \alpha_1,\ \alpha_2,\ \alpha_1\theta_1,\ \alpha_2\theta_2,\ \alpha_1\theta_2 + \alpha_2\theta_1,\ \alpha_1\alpha_2\theta_1,\ \alpha_1\alpha_2\theta_2,\ \alpha_1\alpha_2\theta_1\theta_2\}.$$

Recalling that $\mathrm{vol}_2 = \alpha_1\alpha_2\theta_1\theta_2$, the corresponding collection $\{f \odot \mathrm{vol}_2 : f \in \mathcal{B}\}$ of coset representatives of a basis of $H^1(M,\mathcal{O})$ is

$$\{\alpha_1\alpha_2\theta_1\theta_2,\ \alpha_2\theta_1\theta_2,\ -\alpha_1\theta_1\theta_2,\ \alpha_2\theta_2,\ \alpha_1\theta_1,\ -\alpha_2\theta_1 - \alpha_1\theta_2,\ -\theta_2,\ \theta_1,\ 1\}.$$

Remark 3.9. The proofs in this section made heavy use of the operator $\tau = \sum_{i=1}^{n} \alpha_i \cdot (\partial/\partial\theta_i)$ on the exterior algebra E_n. This operator comes from an action of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ on E_n. Let $\sigma : E_n \to E_n$ be the linear operator

$$\sigma(f) := \sum_{i=1}^{n} \theta_i \cdot (\partial/\partial\alpha_i) f \tag{3.43}$$

and let $\eta : E_n \to E_n$ be the linear operator which acts on the subspace $(E_n)_{i,j}$ by the scalar $i - j$. It can be shown that

$$[\tau, \sigma] = \eta \qquad [\eta, \tau] = 2 \cdot \tau \qquad [\eta, \sigma] = -2 \cdot \sigma \tag{3.44}$$

as operators on E_n. Indeed, the relation $[\tau, \sigma] = \eta$ is easily checked on monomials while the relations $[\eta, \tau] = 2 \cdot \tau$ and $[\eta, \sigma] = -2 \cdot \sigma$ follow from bidegree considerations. Since (3.44) are the defining relations of $\mathfrak{sl}_2(\mathbb{C})$, we see that the action of τ, σ, and η endows E_n with the structure of an $\mathfrak{sl}_2(\mathbb{C})$-module.

Similarly, it can be seen that the Lefschetz element $\ell = \sum_i \alpha_i\theta_i$ of bidegree $(1, 1)$ appearing in Theorem 3.4 fits into an $\mathfrak{sl}_2(\mathbb{C})$-action with the operator $\sum_i (\partial/\partial\theta_i)(\partial/\partial\alpha_i)$ of bidegree $(-1, -1)$. Here the Cartan generator of $\mathfrak{sl}_2(\mathbb{C})$ acts on $(E_n)_{i,j}$ by the scalar $i + j - n$.

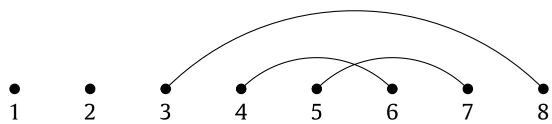
Let $A = \bigoplus_{i=0}^{d} A_i$ be a singly-graded $\mathbb{C}$-algebra equipped with a linear map $L : A \to A$ satisfying $L(A_i) = A_{i+1}$ such that $L^{d-2i} : A_i \to A_{d-i}$ is an isomorphism for all $i \le d/2$. Writing e, f, h for the usual basis of $\mathfrak{sl}_2(\mathbb{C})$, there exists (see e.g. [11, Prop. 3.1.6]) a unique $\mathfrak{sl}_2(\mathbb{C})$-module structure on A such that e acts by L and h acts on A_i by the scalar $2(d - i)$. In our setting we have a doubly-graded algebra E_n with two commuting actions of $\mathfrak{sl}_2(\mathbb{C})$ coming from two Lefschetz elements.

4. Generators and relations for $H^0(M, \mathcal{O})$

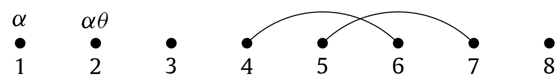
In the last section, we studied the T-invariant subalgebra $H^0(M, \mathcal{O}) \subseteq E_n$ as a doubly-graded $\mathfrak{S}_n$-module. In this section, we focus on its ring structure and give a simple set of generators and combinatorial relations for $H^0(M, \mathcal{O})$.

It can be easily seen that the elements α_i, $\alpha_i\theta_i$, and $\alpha_i\theta_j + \alpha_j\theta_i$ are all translation invariant (we will informally refer to these as "basic invariants"), as are any products of these elements. But there are many dependence relations among products of these elements, and it is not immediately obvious how to pick out a basis from said products. This section will explain how to do so thereby giving a combinatorial basis for the ring of translation invariants.

We will use labelled matchings to index our basis. A *matching of size n* consists of a collection of pairwise disjoint size-two subsets of $\{1, \dots, n\}$. For our purposes we do not require that this collection partition $\{1, \dots, n\}$; there may be unmatched elements. To visualize matchings, we will depict them as n vertices labelled 1 through n, placed on a line, with an arc connecting matched vertices, e.g. when $n = 8$,



Let $\Phi(n)$ consist of all matchings of size n in which each unmatched element is either labelled α, $\alpha\theta$ or unlabelled. To each matching $m \in \Phi(n)$ we associate $F_m \in H^0(M, \mathcal{O})$ as follows. For each unmatched vertex i with an α label, take the product of the α_i's in increasing numerical order. Then, multiply by $\alpha_i\theta_i$ for each vertex i labelled $\alpha\theta$. Finally, multiply by $\alpha_i\theta_j + \alpha_j\theta_i$ for each matched pair $\{i, j\}$. Note that these terms are degree 2 and therefore commute, so there is no ambiguity in the order in which they are multiplied. For example, if $m \in \Phi(8)$ is the labelled matching shown below



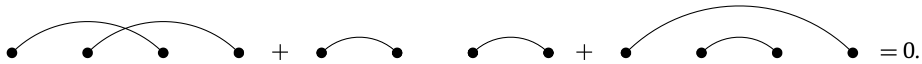
then $F_m \in H^0(M, \mathcal{O})$ is given by $\alpha_1 \cdot \alpha_2\theta_2 \cdot (\alpha_4\theta_6 + \alpha_6\theta_4)(\alpha_5\theta_7 + \alpha_7\theta_5)$.

The benefit of visualizing products of basic invariants as matchings lies in an easier description of the linear dependence relations among these products, as in the following lemmas. The first is the Ptolemy relation (up to sign).

Lemma 4.1. *Let $m \in \Phi(n)$ and suppose that m contains arcs (i, k) and (j, l) with $i < j < k < l$. Let m_0 and m_1 be the matchings obtained by replacing arcs (i, k) and (j, l) with arcs (i, j) and (k, l) or with arcs (i, l) and (j, k) respectively. Then*

$$F_m + F_{m_0} + F_{m_1} = 0.$$

Pictorially:



The second describes a relationship between α-labelled vertices and arcs.

Lemma 4.2. *Let $m \in \Phi(n)$ and suppose that d contains the arc (i, k) and vertex j is labelled by α with $i < j < k$. Let m_0 and m_1 be the diagrams obtained by replacing arc (i, k) with arc (i, j) and vertex k labelled by α or with arc (j, k) and vertex i labelled by α respectively. Then*

$$F_m + F_{m_0} + F_{m_1} = 0.$$

Pictorially:

Lemmas 4.1 and 4.2 are both proven by direct computation; we leave this to the reader. These lemmas allow us to reduce the linearly dependent set of translation invariants

$$\{F_m : m \in \Phi(n)\} \tag{4.1}$$

to a linearly independent subset. Lemma 4.1 allows translation invariants corresponding to matchings to be written as a linear combination of translation invariants corresponding to matchings with fewer crossings or fewer nestings. Lemma 4.2 allows translation invariants corresponding to matchings to be written as a linear combination of translation invariants corresponding to matchings with fewer α labels under arcs. The next two lemmas make this idea explicit.

Let $m \in \Phi(m)$ and let $C(m)$ denote the number of crossings of m, i.e. the number of quadruples $i < j < k < \ell$ such that $\{i, k\}$ and $\{j, \ell\}$ are matched in m. Similarly, let $A(m)$ denote the number of times an α label appears under an arc of m, i.e. the number of triples $i < j < k$ such that j is unmatched and has an α label and $\{i, k\}$ is matched in m.

Lemma 4.3. *If $m \in \Phi(n)$, then m satisfies exactly one of the following*

- $C(m) = A(m) = 0$
- *$C(m) = 0$ and there exists a collection of matchings $m_i \in \Phi(n)$ with $A(m_i) < A(m)$ such that $F_m = \sum_i c_i F_{m_i}$ for some constants c_i.*
- *There exists a collection of matchings $m_i \in \Phi(n)$ with $C(m_i) < C(m)$ with $F_m = \sum_i c_i F_{m_i}$ for some constants c_i.*

Proof. Firstly, suppose $C(m) \neq 0$. So there exists at least one crossing, and we can apply Lemma 4.1 to satisfy the third condition.

Otherwise, suppose $C(m) = 0$ and $A(m) \neq 0$. Then there exists at least one α labelled vertex lying under an arc. Applying Lemma 4.2 to such a vertex and the shortest arc it lies under will allow us to express F_m as a linear combination of F_{m_1} and F_{m_2} for two matchings m_1 and m_2 as defined in Lemma 4.2. Note that since m has no crossings, and we chose the shortest arc, m_1 and m_2 will also have no crossings, and $A(m_1) < A(m)$ and $A(m_2) < A(m)$, so the second condition is satisfied.

Otherwise, the first condition is satisfied. □

Let $NC(n) \subseteq \Phi(n)$ denote the set of noncrossing matchings in which no arc nests an α label. That is, we set

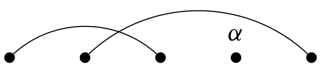
$$NC(n) := \{m \in \Phi(n) : C(m) = A(m) = 0\}. \tag{4.2}$$

Lemma 4.3 has the following corollary.

Corollary 4.4. *For any $m_0 \in \Phi(n)$, the element $F_{m_0} \in H^0(M, \mathcal{O})$ lies in the span of*

$$\{F_m : m \in NC(n)\}. \tag{4.3}$$

Example 4.5. Let $n = 5$ and let m_0 be the labelled matching displayed below



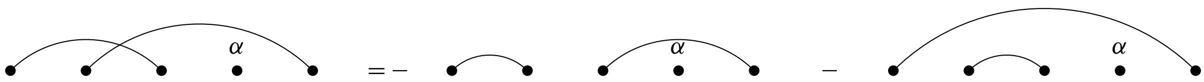
Then $C(m_0) = 1$, $A(m_0) = 1$, and

$$F_{m_0} = \alpha_4(\alpha_1\theta_3 + \alpha_3\theta_1)(\alpha_2\theta_5 + \alpha_5\theta_2). \tag{4.4}$$

We can rewrite F_{m_0} as a sum of F_{m_1} and F_{m_2} for matchings m_1 and m_2 with $C(m_1) = C(m_2) = 0$.

$$F_{m_0} = -\alpha_4(\alpha_1\theta_2 + \alpha_2\theta_1)(\alpha_3\theta_5 + \alpha_5\theta_3) - \alpha_4(\alpha_1\theta_5 + \alpha_5\theta_1)(\alpha_2\theta_3 + \alpha_3\theta_2) \tag{4.5}$$

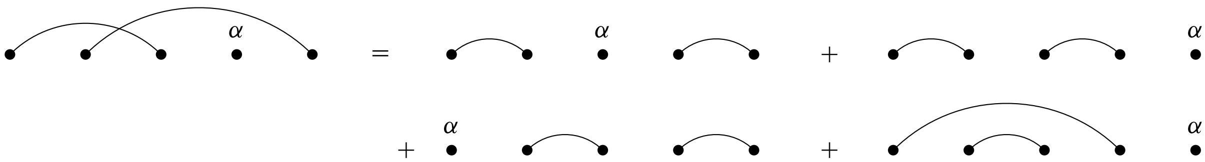
Pictorially:



We can then rewrite these as sums of F_{m_i} for matchings $m_3, \ldots, m_6$ with $A(m_i) = C(m_i) = 0$.

$$\begin{aligned} F_{m_0} &= \alpha_3(\alpha_1\theta_2 + \alpha_2\theta_1)(\alpha_4\theta_5 + \alpha_5\alpha_4) + \alpha_5(\alpha_1\theta_2 + \alpha_2\theta_1)(\alpha_3\theta_4 + \alpha_4\theta_3) \\ &\quad + \alpha_1(\alpha_4\theta_5 + \alpha_5\theta_4)(\alpha_2\theta_3 + \alpha_3\theta_2) + \alpha_5(\alpha_1\theta_4 + \alpha_4\theta_1)(\alpha_2\theta_3 + \alpha_3\theta_2) \end{aligned}$$

Pictorially:



It will turn out that the set in Corollary 4.4 is a basis for $H^0(M, \mathcal{O})$. We will now show that it is linearly independent using a nested induction argument. Let $NC(n,k)$ denote the set of all $m \in NC(n)$ such that F_m is of total degree k.

Proposition 4.6. *The set $\{F_m : m \in NC(n,k)\}$ is linearly independent in $H^0(M, \mathcal{O})$.*

Proof. For any labelled matching m, let $s(m)$ denote the index of the smallest vertex not labelled by α or $\alpha\theta$ in m, or $n+1$ if no such vertex exists.

This proof proceeds via two nested inductions, the first downwards on k and the second on $s(m)$.

When $k = 2n$, there is a unique matching in $NC(n, 2n)$. In this matching, everything is labelled by $\alpha\theta$.

Now assume as an inductive hypothesis that for some $k < 2n$ the set $\{F_m : m \in NC(n, k+1)\}$ is linearly independent, and suppose

$$\sum_{m \in NC(n,k)} c_m F_m = 0. \tag{4.6}$$

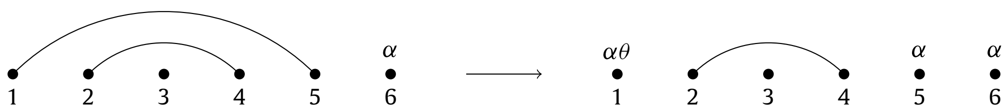
For clarity, we begin with the simpler case where $s(m) = 1$; this is the base case of our induction on $s(m)$. To show that for every matching where $s(m) = 1$ we have $c_m = 0$, multiply the given linear dependence by α_1. We have

$$\sum_{m \in NC(n,k)} c_m \alpha_1 F_m = 0.$$

For any matching m with $s(m) > 1$, vertex 1 is either labelled α or $\alpha\theta$. In either case, $\alpha_1 F_m = 0$. Otherwise, if $s(m) = 1$, then $\alpha_1 F_m = \pm F_{m'}$ where m' is the matching obtained from m by either

(1) Adding an α label to vertex 1 if vertex 1 was not part of any arc.
(2) Removing arc $(1, j)$, then adding an $\alpha\theta$ label to vertex 1 and adding an α label to vertex j, if vertex 1 was part of arc $(1, j)$.

This is because $\alpha_1(\alpha_1\theta_j + \alpha_j\theta_1) = \alpha_1\alpha_j\theta_1$. An example of case 2 is shown in the figure below.



Note that in either case, m' is in $NC(n, k+1)$, since neither vertex 1 nor vertex j can lie under any arc in m'. In case 2, if vertex j did, that arc would have had to cross arc $(1, j)$ in m. By our first inductive hypothesis all such m' are linearly independent, so $c_m = 0$ for any m with $s(m) = 1$.

To complete the inductive step, assume as an inductive hypothesis that $c_m = 0$ for any m with $s(m) < \ell$. Multiply the linear dependence by α_ℓ. We have

$$\sum_{m \in NC(n,k)} c_m \alpha_\ell F_m = 0.$$

For any m with $s(m) > \ell$, $\alpha_\ell F_m = 0$. For any m with $s(m) < \ell$, our inductive hypothesis assumed $c_m = 0$. The analogous argument to the $s(m) = 1$ case therefore shows that $c_m = 0$ for all matchings m with $s(m) = \ell$. Therefore by strong induction $c_m = 0$ for all m, and the set $\{F_m : m \in NC(n,k)\}$ is linearly independent. □

We now know that the set $\{F_m : m \in NC(n)\}$ is linearly independent and spans all products of basic invariants. To show that it is a basis for the entire translation invariant subring, we employ a dimension count. To count the size of our proposed basis, we will give a bijection.

Proposition 4.7. *$NC(n, k)$ is in bijection with ordered pairs (A, B) of subsets of $\{1, \dots, n\}$, where A has size $\lfloor k/2 \rfloor$ and B has size $\lceil k/2 \rceil$.*

Proof. We will show that both sets are in bijection with certain lattice paths in the plane. Let $L(n, k)$ denote the set of all lattice paths in $\mathbb{Z}^2$ which

- Start at the origin.
- End at $(\lfloor \frac{k}{2} \rfloor, \lceil \frac{k}{2} \rceil)$.
- Use exactly n steps.
- Use steps only from $\{(1, 0), (0, 1), (1, 1), (0, 0)\}$.

The bijection to ordered pairs of subsets (A, B) is straightforward, set A records those steps whose x-coordinate is 1, and set B records those steps whose y-coordinate is 1.

For the bijection from $L(n, k)$ to $NC(n, k)$, given a lattice path p, with steps $s_1, \dots, s_n$, associate to it an element $\phi(p) \in NC(n, k)$ defined as follows:

- If $s_i = (0, 0)$, then i is unmatched and unlabelled in $\phi(p)$.
- If $s_i = (1, 1)$, then i is unmatched and labelled $\alpha\theta$ in $\phi(p)$.
- If $s_i = (0, 1)$, draw a ray of slope 1 whose endpoint lies on the midpoint of the step, and which extends to the northeast (a laser). Then i is matched with the smallest $j > i$ for which the ray intersects the lattice path at step s_j. If the ray does not intersect again, then i is unmatched and labelled alpha.
- If $s_i = (1, 0)$, draw a ray of slope 1 whose endpoint lies on the midpoint of the step, and which extends to the southwest (a laser). Then i is matched with the largest $j < i$ for which the ray intersects the lattice path at step s_j. If the ray does not intersect again, then i is unmatched and labelled alpha.

See Fig. 1 for an example. Since all lasers are parallel, we guarantee that $\phi(p)$ is noncrossing. Since each α labelled element of $\phi(p)$ is associated to a laser which never intersects the path again, it cannot lie between a matched pair. So we indeed get an element of $NC(n, k)$. To invert this map, given an $m \in NC(n, k)$ associate to it the lattice path $\phi^{-1}(m) \in L(n, k)$, defined as follows:

- If i is unmatched and unlabelled, then $s_i = (0, 0)$.
- If i is unmatched and labelled $\alpha\theta$, then $s_i = (1, 1)$.
- If i is matched to $j > i$, then $s_i = (0, 1)$.
- If i is matched to $j < i$, then $s_i = (1, 0)$.
- If i is unmatched and labelled α, and among the first half (rounded up) of α-labelled elements, then $s_i = (0, 1)$.
- If i is unmatched and labelled α, and among the second half (rounded down) of α-labelled elements, then $s_i = (1, 0)$.

The composition $\phi^{-1} \circ \phi = \mathrm{id}$ is straightforward to verify. To see that $\phi \circ \phi^{-1}$ is also the identity, note that if i and j are matched in $m \in NC(n, k)$, then the number of $(0, 1)$ steps between s_i and s_j equals the number of $(1, 0)$ steps between s_i and s_j in $\phi^{-1}(m)$, and at any point along travelling from s_i to s_{j-1}, there will have been more $(0, 1)$ steps than $(1, 0)$ steps. So i and j will be matched in $\phi \circ \phi^{-1}(m)$. So these sets are indeed in bijection. □

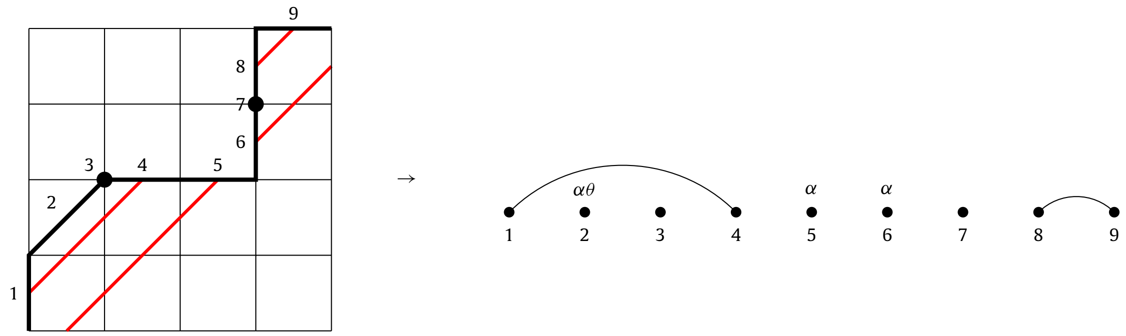


Fig. 1. An example of the bijection described in the proof of Proposition 4.7. Stationary steps are indicated by a filled circle. Rays are cut off at their first intersection for visual clarity.

Proposition 4.7 shows that

$$|NC(n)| = \sum_k |NC(n,k)| = \sum_k \binom{n}{\lfloor k/2 \rfloor}\binom{n}{\lceil k/2 \rceil} = \binom{2n+1}{n}. \tag{4.7}$$

On the other hand, Theorem 3.3 implies that

$$\dim H^0(M,\mathcal{O}) = \sum_{1\le i\le j\le n}\left(\binom{n}{i}\binom{n}{j} - \binom{n}{i-1}\binom{n}{j+1}\right) = \binom{2n+1}{n}. \tag{4.8}$$

Thanks to Proposition 4.6, we have the following corollary.

Corollary 4.8. *The set $\{F_m : m \in NC(n)\}$ is a basis of $H^0(M,\mathcal{O})$.*

We can also give a simple combinatorial presentation for $H^0(M,\mathcal{O})$ as a ring with generators and combinatorial relations. Lemmas 4.1 and 4.2 give combinatorial relations among the basic invariants in $H^0(M,\mathcal{O})$. The remaining relations among these invariants are straightforward.

Theorem 4.9. *The translation invariant ring $H^0(M,\mathcal{O})$ is generated as a $\mathbb{C}$-algebra by the basic invariants $\alpha_i, \alpha_i\theta_i$, and $\alpha_i\theta_j + \alpha_j\theta_i$ subject only to the combinatorial relations in Lemma 4.1 and Lemma 4.2 together with*

$$\begin{cases} \alpha_i^2 = 0 \\ (\alpha_i\theta_i)^2 = 0 \\ (\alpha_i\theta_i)(\alpha_i\theta_j + \alpha_j\theta_i) = 0 \end{cases} \qquad \begin{cases} (\alpha_i\theta_j + \alpha_j\theta_i)^2 = -2(\alpha_i\theta_i)(\alpha_j\theta_j) \\ \alpha_i(\alpha_i\theta_j + \alpha_j\theta_i) = -\alpha_j \cdot (\alpha_i\theta_i) \\ (\alpha_i\theta_i)\cdot(\alpha_i\theta_j + \alpha_j\theta_i) = 0 \end{cases} \tag{4.9}$$

Proof. Corollary 4.8 shows that the basic invariants generate $H^0(M,\mathcal{O})$ as a $\mathbb{C}$-algebra, so we only need to show that any relation among products of basic invariants can be deduced from the given relations. The relations displayed in braces in the statement of the theorem may be used to show that any nonzero product of basic invariants is, up to a scalar, a product F_{m_0} for some $m_0 \in \Phi(n)$. Lemma 4.3 shows that, using only the relations of Lemmas 4.1 and 4.2, the product F_{m_0} can be written as a linear combination of F_m's for various $m \in NC(n)$. Corollary 4.8 says that $\{F_m \; m \in NC(n)\}$ is a basis of $H^0(M,\mathcal{O})$, so every relation among the basic invariants is given in the statement of the theorem. □

5. Conclusion

Section 4 describes the ring $H^0(M,\mathcal{O})$ in terms of generators and relations. The first homology group $H^1(M,\mathcal{O})$ is naturally a module over $H^0(M,\mathcal{O})$. It could be interesting to describe the structure of this module, in terms of a resolution or otherwise.

Throughout this paper, we assumed that the coefficient variables $\alpha_1, \alpha_2, \ldots$ defining the translation T were independent fermionic parameters. However, in supergeometry one often considers more general fermionic translations $T : E_n \to E_n$ of the form $(\theta_1,\ldots,\theta_n) \mapsto (\theta_1+\beta_1,\ldots,\theta_n+\beta_n)$ where the β_i are elements of odd degree in $\alpha_1,\ldots,\alpha_n$, i.e.

$$\beta_1,\ldots,\beta_n \in \bigoplus_{i\ge 0} \wedge^{2i+1}\{\alpha_1,\ldots,\alpha_n\}. \tag{5.1}$$

In particular, the elements $\beta_1,\ldots,\beta_n$ may satisfy nontrivial relations.

Let $R \subseteq E_n$ be the subalgebra of E_n which is invariant under the action of T. Since $T(\alpha_i) = \alpha_i$ for all i, the algebra R has the structure of a module over the exterior algebra $\wedge\{\alpha_1, \dots, \alpha_n\}$.

Problem 5.1. Describe the structure of R as a module over the free exterior algebra $\wedge\{\alpha_1, \dots, \alpha_n\}$.

Theorems 3.3 and 4.9 solve Problem 5.1 when the β_i are independent. The general case is more difficult because relations among the β_i can generate additional invariants in R in ways that are difficult to predict.

Data availability

No data was used for the research described in the article.

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