

RESEARCH ARTICLE

Generic Beauville’s Conjecture

Izzet Coskun¹, Eric Larson² and Isabel Vogt³

¹Department of Mathematics, Statistics, and CS, University of Illinois at Chicago, 851 S. Morgan Street, Chicago, IL 60607, United States; E-mail: icoskun@uic.edu.

²Department of Mathematics, Brown University, 151 Thayer Street, Providence, RI 02912, United States; E-mail: el Larson3@gmail.com.

³Department of Mathematics, Brown University, 151 Thayer Street, Providence, RI 02912, United States; E-mail: ivogt.math@gmail.com.

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Abstract

Let $\alpha: X \rightarrow Y$ be a finite cover of smooth curves. Beauville conjectured that the pushforward of a general vector bundle under α is semistable if the genus of Y is at least 1 and stable if the genus of Y is at least 2. We prove this conjecture if the map α is general in any component of the Hurwitz space of covers of an arbitrary smooth curve Y .

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1. Introduction

Motivated by the study of the theta linear series on the moduli spaces of vector bundles on curves, Beauville in [B00] (see also [B06, Conjecture 6.5]) made the following celebrated conjecture:

Conjecture 1.1 (Beauville). *Let $\alpha: X \rightarrow Y$ be a finite morphism between smooth irreducible projective curves, and let V be a general vector bundle on X . Then α_*V is stable if the genus of Y is at least 2 and semistable if the genus of Y is 1.*

We prove Beauville’s Conjecture when Y is an *arbitrary* smooth irreducible projective curve and X is a general member of *any* component of the Hurwitz space of genus g degree r covers of Y .

Statement of results

Let $\alpha: X \rightarrow Y$ be a finite map of degree r from a smooth irreducible projective curve X of genus g to a smooth irreducible projective curve Y of genus h . We always work over an algebraically closed field of characteristic 0 or greater than r .

For a vector bundle V on a curve X , the slope $\mu(V)$ is defined by $\mu(V) = \frac{\deg(V)}{\text{rk}(V)}$. The bundle V is called (semi)stable if, for every proper subbundle W , we have $\mu(W) \underset{(-)}{<} \mu(V)$. Semistable bundles satisfy nice cohomological and metric properties and form projective moduli spaces. Consequently, determining the stability of naturally defined bundles is an important and fundamental problem.

Let $\mathcal{H}_{r,g}(Y)$ denote the Hurwitz space parameterizing smooth connected degree r genus g covers of Y . In general, $\mathcal{H}_{r,g}(Y)$ is reducible, and when $g > r(h-1) + 1$, the irreducible components correspond to subgroups of the (étale) fundamental group $\pi_1(Y)$ of index dividing r . With this notation, our main theorem is the following.

Theorem 1.2. *Let Y be any smooth irreducible projective curve of genus h . Let $\alpha: X \rightarrow Y$ be a general morphism in any component of $\mathcal{H}_{r,g}(Y)$. Let V be a general vector bundle of any degree and rank on X .*

1. *If $h = 1$, then $\alpha_* V$ is semistable.*
2. *If $h \geq 2$, then $\alpha_* V$ is stable.*

Remark 1.3. It may happen that for special V , the bundle $\alpha_* V$ is not semistable. For example, $\alpha_* \mathcal{O}_X$ has $\mathcal{O}_Y$ as a direct summand. When the map α is ramified, $\mathcal{O}_Y$ destabilizes $\alpha_* \mathcal{O}_X$ (see [CLV22]).

History of the problem

Beauville made Conjecture 1.1 in an unpublished note dating to 2000 [B00]. In the same note, Beauville proved the conjecture if

1. α is étale [B00, Proposition 4.1]; or
2. $r < g(\sqrt{3} + 1) - 1$ [B00, Corollary 3.4]; or
3. V is a line bundle with $|\chi(V)| \leq g + \frac{g^2}{r}$ [B00, Proposition 3.2].

It is an elementary observation that when $h = 0$, the pushforward of a general vector bundle splits as $\bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)$, where $|a_i - a_j| \leq 1$ for every i, j (see [B00, §1]).

Beauville, Narasimhan, and Ramanan [BNR89] earlier proved that a general vector bundle V of degree d and rank r on Y arises as the pushforward of a line bundle from some cover of degree r . Hence, there exists covers of Y for which Beauville's Conjecture is true.

Mehta and Pauly [MP07] proved that if α is the Frobenius morphism, $h \geq 2$, and V is semistable, then $\alpha_* V$ is semistable.

Strategy

Recall that $\alpha: X \rightarrow Y$ is called primitive if the map $\alpha_*: \pi_1(X) \rightarrow \pi_1(Y)$ induced on (étale) fundamental groups is surjective. Every degree r cover $\alpha: X \rightarrow Y$ factors into a primitive map $\alpha^{\text{pr}}: X \rightarrow Y'$ followed by an étale map $\alpha^{\text{ét}}: Y' \rightarrow Y$, where Y' is the étale cover associated with the subgroup $\alpha_* \pi_1(X) \subset \pi_1(Y)$.

We prove Theorem 1.2 by specializing X to a nodal curve. Let $\alpha: X \rightarrow Y$ be a general element in any component of $\mathcal{H}_{r,g}(Y)$. Let $\alpha = \alpha^{\text{pr}} \circ \alpha^{\text{ét}}$ be the primitive-étale factorization of α . Let $\alpha^{\text{ét}}$ and α^{pr} have degrees r' and $s = \frac{r}{r'}$, respectively. Let $\beta_0: X_0 \rightarrow Y'$ be a degree s cyclic étale cover of Y' . The resulting map $\alpha_0: X_0 \rightarrow Y$ is étale and Conjecture 1.1 holds for the map $X_0 \rightarrow Y$ by [B00, Proposition 4.1].

Let p_j and p'_j be points on X_0 contained in a fiber of β_0 such that the cyclic action takes p_j to p'_j . We identify the appropriate number of pairs p_j and p'_j on X_0 to form a nodal curve X_1 of genus g . Let $\nu: X_0 \rightarrow X_1$ be the normalization map. The induced map $\beta_1: X_1 \rightarrow Y'$ is primitive (see Proposition 3.1), and so $\alpha_1 = \alpha^{\text{ét}} \circ \beta_1: X_1 \rightarrow Y$ is in the same irreducible component of $\mathcal{H}_{r,g}(Y)$ as X (see Lemma

2.1). For a general bundle V on X_0 , the pushforward $\alpha_{0*}V = \alpha_{1*}(\nu_*V)$ is stable if $h \geq 2$ and semistable if $h = 1$. Finally, we observe that ν_*V is a limit of vector bundles on nearby deformations of X_1 (see Proposition 3.2). Together with the openness of (semi)stability, this proves Theorem 1.2.

2. Preliminaries

2.1. Basic facts

Let $\alpha: X \rightarrow Y$ be a finite map of degree r from a smooth irreducible projective curve X of genus g to a smooth irreducible projective curve Y of genus h . Since the characteristic is 0 or greater than r , the map α is separable. By the Riemann–Hurwitz formula

$$2g - 2 = r(2h - 2) + b,$$

where b is the degree of the ramification divisor. In particular, $g \geq r(h - 1) + 1$ with equality if and only if α is étale.

If V is a vector bundle of rank s and degree d on X , then $\alpha_*(V)$ is a vector bundle of rank rs on Y . Using the fact that $\chi(V) = \chi(\alpha_*V)$ and the Riemann–Roch formula, we compute the degree d' of $\alpha_*(V)$ via

$$d + s(1 - g) = \chi(V) = \chi(\alpha_*V) = d' + rs(1 - h).$$

We conclude that $d' = d + s(1 - g) - rs(1 - h)$.

2.2. The primitive-étale factorization

Let $\mathcal{H}_{r,g}(Y)$ denote the Hurwitz space parameterizing smooth connected degree r genus g covers of Y . If $g < r(h - 1) + 1$, then $\mathcal{H}_{r,g}(Y)$ is empty. If $g = r(h - 1) + 1$, then the degree r covers of genus g are étale and there are finitely many. In general, the Hurwitz space $\mathcal{H}_{r,g}(Y)$ is not irreducible. The following lemma characterizes the irreducible components.

Lemma 2.1. *Let Y be a smooth and irreducible curve of genus h defined over an algebraically closed field of characteristic 0 or larger than r . Let $g > r(h - 1) + 1$. Then the components of $\mathcal{H}_{r,g}(Y)$ are in bijection with subgroups of $\pi_1(Y)$ of index dividing r .*

Proof. Let $\mathcal{H}$ be an irreducible component of the Hurwitz scheme $\mathcal{H}_{r,g}(Y)$. Given $\alpha: X \rightarrow Y$ in $\mathcal{H}$, the subgroup $\alpha_*\pi_1(X)$ of $\pi_1(Y)$ has index dividing r . Since this is a discrete invariant and is constant in irreducible families, $\alpha_*\pi_1(X)$ is an invariant of $\mathcal{H}$.

Conversely, given a subgroup $G \subset \pi_1(Y)$ of index $r^{\text{ét}}$ dividing r , up to isomorphism there is a unique étale cover $\delta: Y' \rightarrow Y$ corresponding to G of degree $r^{\text{ét}}$ and genus $h' = r^{\text{ét}}(h - 1) + 1$. Let $r^{\text{pr}} = r/r^{\text{ét}}$. Given the inequality

$$g > r(h - 1) + 1 = r^{\text{pr}}(h' - 1) + 1,$$

there exists a genus g primitive cover $\gamma: X \rightarrow Y'$ of degree r^{pr} . For any such cover, we obtain an element of $\mathcal{H}_{r,g}(Y)$ by taking $\alpha = \delta \circ \gamma$. Furthermore, $\alpha_*\pi_1(X) = G$. On the other hand, if $\gamma: X \rightarrow Y'$ is not primitive but $\gamma_*\pi_1(X)$ has index s in $\pi_1(Y')$, then $\alpha_*\pi_1(X)$ has index $sr^{\text{ét}}$ in $\pi_1(Y)$ and cannot be G . We conclude that if $\alpha_*\pi_1(X) = G$, then α must factor as the composition of δ and a primitive cover of Y' . By results of Clebsch [C1872], Fulton [F69], and Gabai–Kazez [GK90] (see [CLV22, Proposition 2.2]), the Hurwitz scheme parameterizing genus g degree r^{pr} primitive covers of Y' is irreducible. We conclude that there is a bijection between irreducible components of $\mathcal{H}_{r,g}$ and subgroups of $\pi_1(Y)$ of index dividing r . $\square$

2.3. The étale case

We briefly recall Beauville's proof of Conjecture 1.1 [B00, Proposition 4.1] in the étale case (see also the proof of [CLV22, Proposition 1.3]).

First, we show that it suffices to consider the case of line bundles. Given a vector bundle V on X of degree d and rank s , let $\delta: Z \rightarrow X$ be an étale cover of degree s . If L is a line bundle of degree d on Z , then $\delta_* L$ is a vector bundle of rank s and degree d on X . Hence, if we prove Conjecture 1.1 for (étale) maps in the case of line bundles, it follows for (étale) maps in higher rank as well.

Let $\alpha: X \rightarrow Y$ be étale and let $\rho: Z \rightarrow Y$ be the Galois cover associated to α with Galois group G . Let Σ be the set of Y -morphisms $\sigma: Z \rightarrow X$. Then

$$W := \rho^* \alpha_* L \cong \bigoplus_{\sigma \in \Sigma} \sigma^* L.$$

The pullback by ρ of any destabilizing subbundle of $\alpha_* L$ would destabilize W . Hence, $\alpha_* L$ is semistable for every line bundle L on X .

If $\alpha_* L$ has a proper subbundle F of the same slope as $\alpha_* L$, then $\rho^* F$ is a G -invariant subbundle of W . Since the category of semistable bundles of a fixed slope is abelian with simple objects stable bundles, $\rho^* F \cong \bigoplus_{\sigma \in \Sigma'} \sigma^* L$ for some $\Sigma' \subset \Sigma$. Since G acts transitively on Σ , it suffices to show that, if $h > 1$ and L is general, the line bundles $\sigma^* L$ are pairwise nonisomorphic as σ varies in Σ .

For any fixed $\sigma \in \Sigma'$, let $H \subset G$ be the subgroup fixing σ . The subvariety $\sigma^* \text{Pic}^d X \subset \text{Pic} Z$ has dimension g , contains $\sigma^* L$, and is invariant under H . On the other hand, if $\sigma^* \text{Pic}^d X$ is invariant under a subgroup H' with $H \subsetneq H' \subset G$, then it would be pulled back from $\text{Pic}(Z/H')$. By the Riemann–Hurwitz formula, the genus of Z/H' is strictly smaller than g . Hence by dimension reasons, this containment is impossible and the $\sigma^* L$ are pairwise distinct. This shows the stability of $\alpha_* L$.

3. Proof of Theorem 1.2

Let Y be a curve of genus h and let g be an integer such that $g > r(h-1) + 1$. Fix an étale cover $\alpha^{\text{ét}}: Y' \rightarrow Y$ of degree $r^{\text{ét}} \mid r$. We first explain how to construct a nodal cover $\alpha_1: X_1 \rightarrow Y$ of arithmetic genus g , whose primitive-étale factorization is

$$X_1 \rightarrow Y' \xrightarrow{\alpha^{\text{ét}}} Y.$$

The first step of our construction is to fix a cyclic étale cover $\beta_0: X_0 \rightarrow Y'$ of degree $r^{\text{pr}} = r/r^{\text{ét}}$. Such covers correspond to points of order r^{pr} in $\text{Jac}(Y')$, which always exist. Write $\tau: X_0 \rightarrow X_0$ for the automorphism corresponding to the generator of $\mathbb{Z}/r^{\text{pr}}\mathbb{Z}$. Let $n := g - r(h-1) - 1$. We then pick general points $p_1, p_2, \dots, p_n \in X_0$, and let $p'_i = \tau(p_i)$. Let X_1 be the curve obtained from X_0 by gluing every p_i to p'_i for $1 \leq i \leq n$ and denote the normalization $\nu: X_0 \rightarrow X_1$. Let $\beta_1: X_1 \rightarrow Y'$ be the induced morphism, and let $\alpha_1 := \alpha^{\text{ét}} \circ \beta_1$.

Proposition 3.1. *The cover $\beta_1: X_1 \rightarrow Y'$ is primitive.*

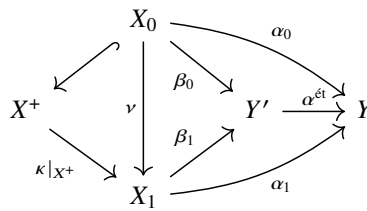
Proof. We must show that the pushforward $\pi_1(X_1) \rightarrow \pi_1(Y')$ is surjective. By [S09, Proposition 5.5.4(2)], this is equivalent to the assertion that for all finite étale connected covers $Y'' \rightarrow Y'$, the fiber product $X_1 \times_{Y'} Y''$ is connected.

Consider the dominant map $\epsilon: X_0 \times_{Y'} Y'' \rightarrow X_1 \times_{Y'} Y''$. By construction, $\mathbb{Z}/r^{\text{pr}}\mathbb{Z}$ acts transitively on the components of $X_0 \times_{Y'} Y''$. Therefore it suffices to show that for any component $Z \subset X_0 \times_{Y'} Y''$, the components $\epsilon(Z)$ and $\epsilon(\tau(Z))$ intersect. Since $Z \rightarrow X_0$ is surjective, Z contains a point of the form (p_1, y'') for some $y'' \in Y''$, and so $\tau(Z)$ contains (p'_1, y'') . Since $\epsilon((p_1, y'')) = \epsilon((p'_1, y''))$, the components $\epsilon(Z)$ and $\epsilon(\tau(Z))$ intersect as desired. $\square$

By the theory of formal patching (see [Li03, Lemma 5.6]), the map β_1 can be smoothed to a map $\beta: X \rightarrow Y'$ with a smooth domain X . Since being primitive is a deformation invariant, the resulting smoothing β is also primitive by Proposition 3.1.

Proposition 3.2. *Given a vector bundle V on X_0 , the pushforward ν_*V to X_1 is a limit of vector bundles of the same rank and slope $\mu(V) + n$ on the smoothing X .*

Proof. Let $\mathcal{X} \rightarrow \Delta$ denote a family of smooth curves specializing to X_1 with smooth total space. Consider the blowup at the nodes of X_1 . In this family, the central fiber is the union of X_0 together with n copies of $\mathbb{P}^1$, where each $\mathbb{P}^1$ is attached at the two preimages of the corresponding node under the normalization map ν . These $\mathbb{P}^1$'s appear with multiplicity 2 in the central fiber. Make a base change of order 2 and normalize the total space to obtain a family $\mathcal{X}^+ \rightarrow \Delta'$. This is a semistable family of smooth curves specializing to the union of X_0 with n copies of $\mathbb{P}^1$, where now the central fiber X^+ is reduced. Write $\kappa: \mathcal{X}^+ \rightarrow \mathcal{X}' := \mathcal{X} \times_{\Delta} \Delta'$. The following diagram illustrates the maps that exist on the central fiber.



Let V^+ denote the vector bundle on X^+ obtained by gluing the vector bundle V on X_0 to $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus \text{rk } V}$ on each $\mathbb{P}^1$ component via any choice of gluing. (In fact the reader may check that any two choices result in isomorphic bundles.) Since V is locally free, $\mathcal{E}xt^i(V^+, V^+) = 0$ for all $i > 0$. Thus, by the local-to-global Ext spectral sequence, we have that $\text{Ext}_{X^+}^2(V^+, V^+) = 0$. By [H, Theorem 7.3 (b)], the obstructions to extending the bundle V^+ to the whole family lie in $\text{Ext}_{X^+}^2(V^+, V^+)$. Consequently, V^+ extends to a vector bundle $\mathcal{V}^+$ on $\mathcal{X}^+$. Observe that $\mathcal{V}^+$ has rank $\text{rk } V$, and by the constancy of the Euler characteristic in flat families, the slope of the restriction of $\mathcal{V}^+$ to the fibers is $\mu(V^+) = \mu(V) + n$.

Now we claim that $\kappa_*\mathcal{V}^+|_{X_1} \simeq \nu_*V$. Once we establish this claim, we obtain that ν_*V is the limit of vector bundles of the same rank and slope $\mu(V) + n$ on the smooth fibers.

Let $(\text{rk}, \chi)(F)$ denote the rank and Euler characteristic of a sheaf F , and write $\mathcal{X}'_t$ and $\mathcal{X}^+_t$ for general fibers of their respective families. We first show that $(\text{rk}, \chi)(\kappa_*\mathcal{V}^+|_{X_1}) = (\text{rk}, \chi)(\nu_*V)$. By the constancy of the rank and the Euler characteristic in flat families, and the fact that κ is an isomorphism away from the central fiber, we have

$$(\text{rk}, \chi)(\kappa_*\mathcal{V}^+|_{X_1}) = (\text{rk}, \chi)(\kappa_*\mathcal{V}^+|_{\mathcal{X}'_t}) = (\text{rk}, \chi)(\mathcal{V}^+|_{\mathcal{X}^+_t}) = (\text{rk}, \chi)(V^+).$$

Furthermore, by considering the exact sequence for restriction to X_0

$$0 \rightarrow \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus \text{rk}(V)} \rightarrow V^+ \rightarrow V^+|_{X_0} = V \rightarrow 0,$$

we see that $(\text{rk}, \chi)(V^+) = (\text{rk}, \chi)(V)$. Finally, $(\text{rk}, \chi)(V) = (\text{rk}, \chi)(\nu_*V)$, which proves that $(\text{rk}, \chi)(\kappa_*\mathcal{V}^+|_{X_1}) = (\text{rk}, \chi)(\nu_*V)$.

Hence, it suffices to construct a surjective map between $\kappa_*\mathcal{V}^+|_{X_1}$ and ν_*V . Consider the exact sequence on $\mathcal{X}^+$

$$0 \rightarrow \mathcal{V}^+(-X_0) \rightarrow \mathcal{V}^+ \rightarrow \mathcal{V}^+|_{X_0} \rightarrow 0.$$

Pushing forward under κ , we obtain

$$0 \rightarrow \kappa_*\mathcal{V}^+(-X_0) \rightarrow \kappa_*\mathcal{V}^+ \rightarrow \kappa_*(\mathcal{V}^+|_{X_0}) \rightarrow R^1\kappa_*(\mathcal{V}^+(-X_0)) \rightarrow \dots$$

Observe that $\kappa_*(\mathcal{V}^+|_{X_0}) \simeq \nu_*V$, and that the map $\kappa_*\mathcal{V}^+ \rightarrow \kappa_*(\mathcal{V}^+|_{X_0}) \simeq \nu_*V$ factors through $(\kappa_*\mathcal{V}^+)|_{X_1}$. Hence it suffices to show that $R^1\kappa_*\mathcal{V}^+(-X_0) = 0$.

Since κ is an isomorphism away from the nodes of the X_1 , the sheaf $R^1\kappa_*\mathcal{V}^+(-X_0)$ is supported on the nodes of X_1 . It therefore suffices to show that its completion at every node p of X_1 vanishes. For this, we use the theorem on formal functions, which states that

$$R^1\kappa_*\mathcal{V}^+(-X_0)_p^\wedge \simeq \varprojlim_n H^1(\mathcal{V}^+(-X_0)|_{n \cdot \mathbb{P}^1}),$$

where $\mathbb{P}^1 = \kappa^{-1}(p)$ is a Cartier divisor on X^+ . It thus suffices to show that $H^1(\mathcal{V}^+(-X_0)|_{n \cdot \mathbb{P}^1}) = 0$ for all n . For this, we use induction on n , with base case $n = 0$, which is clear. For the inductive step, we use the exact sequence for restriction to $n \cdot \mathbb{P}^1$

$$0 \rightarrow \mathcal{V}^+(-X_0 - n \cdot \mathbb{P}^1)|_{\mathbb{P}^1} \rightarrow \mathcal{V}^+(-X_0)|_{(n+1) \cdot \mathbb{P}^1} \rightarrow \mathcal{V}^+(-X_0)|_{n \cdot \mathbb{P}^1} \rightarrow 0.$$

Since $\mathcal{V}^+(-X_0 - n \cdot \mathbb{P}^1)|_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(2n-1)^{\oplus \text{rk } V}$, which has vanishing h^1 , we conclude by induction that the middle term has vanishing h^1 . $\square$

Proof of Theorem 1.2. Let $\mathcal{H}$ be an irreducible component of the Hurwitz space $\mathcal{H}_{r,g}(Y)$. Assume that the corresponding covers have primitive-étale factorization

$$X \rightarrow Y' \xrightarrow{\alpha^{\text{ét}}} Y.$$

Let $\beta_0: X_0 \rightarrow Y'$ be the cyclic étale cover constructed above and let

$$\alpha_1: X_1 \xrightarrow{\beta_1} Y' \xrightarrow{\alpha^{\text{ét}}} Y$$

be the cover constructed above by gluing points in the fibers of $\beta_0: X_0 \rightarrow Y'$.

Let V be a general vector bundle on X_0 of arbitrary degree and rank. By [B00, Proposition 4.1] (see §2.3), the pushforward $\alpha_{0*}V$ is semistable if $h = 1$ and stable if $h \geq 2$. Since

$$\alpha_0 = \alpha^{\text{ét}} \circ \beta_0 = \alpha^{\text{ét}} \circ \beta_1 \circ \nu = \alpha_1 \circ \nu,$$

we conclude that $\alpha_{1*}\nu_*V$ is semistable if $h = 1$ and stable if $h \geq 2$.

By Proposition 3.2, the pushforward bundle ν_*V on X_1 is a limit of vector bundles on a smoothing $X \rightarrow Y' \rightarrow Y$ and these vector bundles can be chosen to have any given degree and rank. The theorem follows by the openness of (semi)stability. $\square$

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