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# Helton-Howe trace, Connes-Chern characters and Toeplitz quantization of Bergman spaces

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A B S T R A C T

We study the Helton-Howe trace and the Connes-Chern character for Toeplitz operators on weighted Bergman spaces via the idea of quantization. We prove a local formula for the large $t$-limit of the Connes-Chern character as the weight goes to infinity. And we show that the Helton-Howe trace of Toeplitz operators is independent of the weight $t$ and obtain a local formula for the Helton-Howe trace for all weighted Bergman spaces using harmonic analysis and quantization.


## 1. Introduction

Toeplitz extensions are fundamental objects in noncommutative geometry. They are natural examples of finite summable Fredholm modules and define elements in the corresponding $K$-homology group. Trace on Toeplitz operators has been well studied with

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many interesting results, cf. [53]. Since the 70s, trace has been employed to extract geometric information of Toeplitz extension. In particular, Connes [14, Sec. 2., Theorem 5] used trace on products of semi-commutators to define the Connes-Chern character of finite summable extensions.

For the unit disk $\mathbb{D}$ in $\mathbb{C}$, let $L^2_a(\mathbb{D})$ be the Bergman space of $L^2$ analytic functions on $\mathbb{D}$. Given $f \in \mathscr{C}^\infty(\overline{\mathbb{D}})$, let $T^{(0)}_f$ be the Toeplitz operator on $L^2_a(\mathbb{D})$ associated to the symbol $f$. The commutator $[T^{(0)}_f, T^{(0)}_g] := T^{(0)}_f T^{(0)}_g - T^{(0)}_g T^{(0)}_f$ for $f, g \in \mathscr{C}^\infty(\overline{\mathbb{D}})$ is a trace class operator. Helton and Howe [30] discovered an interesting formula for the commutator

$$\operatorname{Tr}\big([T^{(0)}_f, T^{(0)}_g]\big) = \frac{1}{2\pi\sqrt{-1}} \int\limits_{\mathbb{D}} \mathrm{d}f \wedge \mathrm{d}g.$$

The above result is deeply connected to the Pincus function for a pair of noncommuting selfadjoint operators, cf. [10,11,42].

Let $\mathbb{B}_n$ be the open unit ball of $\mathbb{C}^n$ and $\mathbb{S}_n = \partial\mathbb{B}_n$ the unit sphere. For $\mathbb{B}_n$, the commutator $[T^{(0)}_f, T^{(0)}_g]$ for two Toeplitz operators with smooth symbols $f, g$ on $L^2_a(\mathbb{B}_n)$ is a Schatten-$p$ class operator for $p > n$. Suppose $f_1, ..., f_{2n} \in \mathscr{C}^\infty(\overline{\mathbb{B}_n})$. Then the product of the commutators

$$[T^{(0)}_{f_1}, T^{(0)}_{f_2}] \cdots [T^{(0)}_{f_{2n-1}}, T^{(0)}_{f_{2n}}]$$

may not be a trace class operator. Helton and Howe [31,33] made a breakthrough by considering the antisymmetric sum of $T^{(0)}_{f_1}, ..., T^{(0)}_{f_{2n}}$ defined by

$$[T^{(0)}_{f_1}, ..., T^{(0)}_{f_{2n}}] := \sum_{\tau \in S_{2n}} \operatorname{sgn}(\tau) T^{(0)}_{f_{\tau(1)}} T^{(0)}_{f_{\tau(2)}} ... T^{(0)}_{f_{\tau(2n)}},$$

where $S_{2n}$ is the permutations group of $2n$ elements and sgn is the sign of the permutation $\tau$. The following is a remarkable generalization of the Helton-Howe trace formula for the commutator of two Toeplitz operators on $L^2_a(\mathbb{D})$.

**Theorem 1.1** *(Helton-Howe). On the Bergman space $L^2_a(\mathbb{B}_n)$ (and the Hardy space $H^2(\mathbb{S}_n)$), the antisymmetric sum $[T^{(0)}_{f_1}, ..., T^{(0)}_{f_{2n}}]$ is a trace class operator, and*

$$\operatorname{Tr}\left([T^{(0)}_{f_1}, ..., T^{(0)}_{f_{2n}}]\right) = \frac{n!}{(2\pi\sqrt{-1})^n} \int\limits_{\mathbb{B}_n} \mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \cdots \wedge \mathrm{d}f_{2n}. \tag{1.1}$$

We observe that by Stoke's theorem, the above integral only depends on the value of $f_1, ..., f_{2n}$ on the unit sphere $\mathbb{S}_n = \partial\overline{\mathbb{B}_n}$, i.e.

$$\frac{n!}{(2\pi\sqrt{-1})^n} \int\limits_{\mathbb{B}_n} \mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \cdots \wedge \mathrm{d}f_{2n} = \frac{n!}{(2\pi\sqrt{-1})^n} \int\limits_{\mathbb{S}_n} f_1 \mathrm{d}f_2 \wedge \cdots \wedge \mathrm{d}f_{2n}.$$

The above idea of Schatten-$p$ commutators was revolutionized by Connes [14] into a fundamental concept in noncommutative geometry as $p$-summable Fredholm modules. And the Helton-Howe trace formula in Theorem 1.1 inspired Connes to his ingenious discovery of cyclic cohomology and the Chern character for $p$-summable Fredholm modules. The building block of the Connes-Chern character is the semi-commutator

$$\sigma_t(f,g) = T_f^{(t)} T_g^{(t)} - T_{fg}^{(t)}.$$

Modulo constants, the Connes-Chern character for the Toeplitz extension is defined to be

$$\tau_t(f_1, \cdots, f_{2p}) := \mathrm{Tr}\left(\sigma_t(f_1,f_2)...\sigma_t(f_{2p-1},f_{2p})\right) - \mathrm{Tr}\left(\sigma_t(f_2,f_3)...\sigma_t(f_{2p},f_1)\right), \quad (1.2)$$

for $p > n$. And the Helton-Howe trace, Equation (1.1), is the top degree component of the above Connes-Chern character. Through out this paper, for the formula of $\tau_t$, we have used $\{f_1, \cdots, f_{2p}\}$ instead of $\{f_0, \cdots, f_{2p-1}\}$, a different one from the classical convention, cf. [14], to make its appearance compatible with the Helton-Howe trace formula.

Connes observed that the above cocycle in general is not local, i.e. the value of $\tau_t(f_1, \cdots, f_{2p})$ can not be expressed by the germ of $f_1 \otimes \cdots \otimes f_{2p}$ on the diagonal in

$$\underbrace{\mathbb{B}_n \times \cdots \times \mathbb{B}_n}_{2p}.$$

Connes [15,16] improved the Chern character, Equation (1.2), by employing the Dixmier trace on the operator ideal $L^{1,\infty}$. In a series of works, Engliš and his coauthors, e.g. [22,23,25,26], studied a generalization of the Helton-Howe trace formula by considering the Dixmier trace on the product

$$[T_{f_1}^{(0)}, T_{f_2}^{(0)}] \cdots [T_{f_{2n-1}}^{(0)}, T_{f_{2n}}^{(0)}].$$

They expressed the Dixmier trace of the above product as an integral of the product of Poisson brackets between $f_{2k-1}$ and $f_{2k}$, $k = 1, \cdots, n$.

In this article, we take a different approach to study the Connes-Chern character (1.2) and the Helton-Howe trace, Theorem 1.1. Our main idea is to put the Bergman space and Hardy space into the family of weighted Bergman spaces $L_a^2(\mathbb{B}_n, \lambda_t) := L_{a,t}^2(\mathbb{B}_n)$ for the measure

$$\mathrm{d}\lambda_t(z) = \frac{(n-1)!}{\pi^n B(n,t+1)} (1-|z|^2)^t \mathrm{d}m(z),$$

where $B(n, t+1)$ is the Beta function. Let $T_f^{(t)}$ be the associated Toeplitz operator on $L_{a,t}^2(\mathbb{B}_n)$ with symbol $f$. We study the large $t$ behavior of the Connes-Chern character and the Helton-Howe trace.

Our first result is about the Helton-Howe trace for $T_{f_1}^{(t)}, \ldots, T_{f_{2n}}^{(t)}$ for functions $f_1, \ldots, f_{2n} \in \mathscr{C}^2(\overline{\mathbb{B}_n})$, which generalizes Theorem 1.1 to all weighted Bergman spaces.

**Theorem 1.2.** *(Theorem 8.1) Suppose* $f_1, f_2, \ldots, f_{2n} \in \mathscr{C}^2(\overline{\mathbb{B}_n})$ *and* $t \geq -1$.

1. $[T_{f_1}^{(t)}, T_{f_2}^{(t)}, \ldots, T_{f_{2n}}^{(t)}]$ *is in the trace class* $\mathcal{S}^1$.
2. $$\mathrm{Tr}[T_{f_1}^{(t)}, T_{f_2}^{(t)}, \ldots, T_{f_{2n}}^{(t)}] = \frac{n!}{(2\pi i)^n} \int\limits_{\mathbb{B}_n} \mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \ldots \wedge \mathrm{d}f_{2n}, \tag{1.3}$$

   *which is independent of* $t$.

Thanks to the use of pseudodifferential calculus and its generalization in the proof, Helton-Howe's original theorem needs to assume that the pseudodifferential operators have smooth symbols. For smooth symbols, Theorem 8.1 might be known to some experts (cf. [22,32]). In this paper, we develop a new approach to study the trace formula using harmonic analysis. As a result, we obtain an improvement of the Helton-Howe trace formula for Toeplitz operators with $\mathscr{C}^2$ symbols.

Our second result is about the Connes-Chern character $\tau_t(f_1, \ldots, f_{2p})$ for $f_1, \ldots, f_{2p} \in \mathscr{C}^2(\overline{\mathbb{B}_n})$. Different from the Helton-Howe trace, $\tau_t$ vanishes as $t$ goes to $\infty$. In the following theorem, we identify the leading term of $\tau_t$ as $t \to \infty$.

**Theorem 1.3.** *(Theorem 8.6) Suppose* $p \geq n+1$ *is an integer and* $f_1, f_2, \ldots, f_{2p} \in \mathscr{C}^2(\overline{\mathbb{B}_n})$. *Set* $f_{2p+1} := f_1$. *Then*

$$\begin{aligned} &\lim_{t\to\infty} t^{p-n} \tau_t(f_1, f_2, \ldots, f_{2p}) \\ =&\frac{n^p}{\pi^n} \int\limits_{\mathbb{B}_n} \Big( \prod_{j=1}^{p} C_1(f_{2j-1}, f_{2j})(z) - \prod_{j=1}^{p} C_1(f_{2j}, f_{2j+1})(z) \Big) \frac{\mathrm{d}m(z)}{(1-|z|^2)^{n+1}}, \end{aligned}$$

*where* $C_1(f, g)$ *is defined as follows,*

$$C_1(f,g)(z) = -\frac{1}{n}(1-|z|^2)\Big[\sum_{i=1}^{n} \partial_i f(z) \bar{\partial}_i g(z) - Rf(z)\bar{R}g(z)\Big],$$

$$R = \sum_{i=1}^{n} z_i \partial_{z_i}, \ \overline{R} = \sum_{i=1}^{n} \bar{z}_i \bar{\partial}_{z_i}. \tag{1.4}$$

Our approach to the above two main theorems is heavily influenced by the idea of quantization, [6,8,9,12,17–20]. Geometrically the defining function $\psi = 1 - |z|^2$ on $\mathbb{B}_n$ defines the Bergman metric in the following way.

$$\begin{aligned}\omega :=& i\frac{-\psi\partial\bar{\partial}\psi+\partial\psi\wedge\bar{\partial}\psi}{\psi^2}\\ =& i\frac{(1-|z|^2)\sum_{j=1}^n \partial z_j\wedge\bar{\partial}\bar{z}_j+(\sum_j \bar{z}_j\partial z_j)\wedge(\sum_{j'} z_{j'}\bar{\partial}\bar{z}_{j'})}{(1-|z|^2)^2}\end{aligned}$$

defines a symplectic form on $\mathbb{B}_n$, cf. [37, Prop. 2.6]. The Toeplitz operator $T_f^{(t)}$ gives a quantization of the symplectic form $i\omega$, e.g. [20, Theorem 3], satisfying

$$||T_f^{(t)}T_g^{(t)}-\sum_{j=0}^k t^{-j}T_{C_j(f,g)}^{(t)}||=O(t^{-k-1}),\ t\to\infty, \tag{1.5}$$

and the $C_j$ are bilinear operators discussed later in Section 6, and $C_1$ is defined in Equation (1.4). The asymptotic expansion formula (1.5) provides the key tool to study the semi-commutator

$$\sigma_t(f,g)=T_f^{(t)}T_g^{(t)}-T_{fg}^{(t)},$$

and therefore also the commutator $[T_f^{(t)},T_g^{(t)}]$, since

$$[T_f^{(t)},T_g^{(t)}]=T_f^{(t)}T_g^{(t)}-T_g^{(t)}T_f^{(t)}=\sigma_t(f,g)-\sigma_t(g,f).$$

The asymptotic expansion formula (1.5) in the literature, e.g. [20], was well studied for estimates on the operator norm. Estimates about the Schatten-$p$ norm in the expansion (1.5) are needed in our applications to the tracial property in Theorem 1.2 and 1.3. We prove these estimates in Theorem 6.3. As we need to study Toeplitz operators with $\mathscr{C}^2$ symbols in Theorem 1.2 and 1.3 and an estimate on Schatten-$p$ norm in Theorem 6.3, we need a new method to develop the asymptotic estimate in Theorem 6.3 different from the classical method via pseudodifferential/Toeplitz operator calculus [7,8,18,20, 22,23,25,26,35], which requires to work with smooth symbols. Our main tool comes from integration formulas in Lemma 2.10 and Lemma 2.15 developed in Section 4 of [47]. Theorem 1.3 follows from the Schatten-$p$ estimate of the semi-commutator $\sigma_t(f,g)$. As a byproduct, our method also provides an explicit algorithm to compute the bilinear differential operator $C_j$ in the asymptotic expansion (1.5), which is in general hard to compute.

A crucial fact used in our estimate is the different behavior of the quantization in complex normal and complex tangential directions (see Remark 6.4 and Corollaries 6.7 and 6.8). Roughly speaking, in the term $I^{e_{i_1,\dots,i_{k+1}},e_{j_1,\dots,j_{k+1}}}(z-w)$ that appears in the quantization formula (6.6) (especially in (6.7)), the Schatten-$p$ membership improves by $\frac{1}{2}$ for each $e_i,e_j$ in the complex tangential direction, whereas improves by 1 for each $e_i,e_j$ in the complex normal direction. Essentially, this allows us to reduce our estimates to the complex tangential direction (see Lemma 2.2 (6) and the proof of Lemma 6.1). In contrast, in pseudodifferential calculous, Helton and Howe [31] considered symbol functions that are homogeneous of order 0 in the $\xi$ variable far away from the zero section of the cotangent bundle, which simplifies the corresponding estimates. The difference between tangential and radial estimates suggests a deep link of our study with the Heisenberg calculus for contact manifolds, e.g. [5,32,43,48].

Instead of a direct computation as in [31], we prove Theorem 1.2 in two steps. Take the $t > -1$ case for example. Suppose $f_1, f_2, \ldots, f_{2n} \in \mathscr{C}^2(\overline{\mathbb{B}_n})$. We prove the following.

1.

$$\mathrm{Tr}[T_{f_1}^{(t)}, T_{f_2}^{(t)}, \ldots, T_{f_{2n}}^{(t)}] = \mathrm{Tr}[T_{f_1}^{(t+1)}, T_{f_2}^{(t+1)}, \ldots, T_{f_{2n}}^{(t+1)}]. \tag{1.6}$$

2.

$$\lim_{s\to\infty} \mathrm{Tr}[T_{f_1}^{(s)}, \ldots, T_{f_{2n}}^{(s)}] = \frac{n!}{(2\pi i)^n} \int \mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_{2n}. \tag{1.7}$$

Again, the proof of Equation (1.6) takes two steps.

(a)

$$\mathrm{Tr}\big\{[T_{f_1}^{(t)}, T_{f_2}^{(t)}, \ldots, T_{f_{2n}}^{(t)}] - [T_{f_1}^{(t+1,t)}, T_{f_2}^{(t+1,t)}, \ldots, T_{f_{2n}}^{(t+1,t)}]\big\} = 0. \tag{1.8}$$

(b)

$$\mathrm{Tr}[T_{f_1}^{(t+1,t)}, \ldots, T_{f_{2n}}^{(t+1,t)}] = \mathrm{Tr}[T_{f_1}^{(t+1)}, \ldots, T_{f_{2n}}^{(t+1)}]. \tag{1.9}$$

Here the operator $T_{f_i}^{(t+1,t)}$ is the restriction of $T_{f_i}^{(t+1)}$ on $L^2_{a,t}(\mathbb{B}_n)$. It serves as a bridge between the weighted spaces $L^2_{a,t}(\mathbb{B}_n)$ and $L^2_{a,t+1}(\mathbb{B}_n)$. The proof of (1.8) is quite complicated and lengthy. We give it in Section 5. Let us briefly sketch the proof. First, we apply the integration formula in Lemma 2.14 and get the decomposition formula $T_{f_i}^{(t)} = T_{f_i}^{(t+1,t)} + B_i$ in Lemmas 5.5 and 5.7. We observe that $T_{f_i}^{(t+1,t)}$ is the "principal part" of $T_{f_i}^{(t)}$ for $f_i \in \mathscr{C}^2(\overline{\mathbb{B}_n})$. We point out that the "minor part" $B_i$ does not live in Schatten class $\mathcal{S}^p$ for $p$ small enough. This fact prevents us from proving (1.8) only using operator-theoretic tools. Then we develop Hypotheses A which handles the operator-theoretic part of the proof of (1.8), and Hypotheses B, where the rest is handled. The proof of (1.9) is an application of Lemma 2.4. Note that $[T_{f_1}^{(t+1,t)}, \ldots, T_{f_{2n}}^{(t+1,t)}]$ is the restriction of $[T_{f_1}^{(t+1)}, T_{f_2}^{(t+1)}, \ldots, T_{f_{2n}}^{(t+1)}]$ on $L^2_{a,t}(\mathbb{B}_n)$. This is done in Section 8, in the proof of Theorem 8.1, after we obtain the trace class membership of $[T_{f_1}^{(t)}, T_{f_2}^{(t)}, \ldots, T_{f_{2n}}^{(t)}]$ and prove (1.7) in Corollary 7.7.

Equation (1.7) is proved in Section 7. The proof relies heavily on the Toeplitz quantization formula and their asymptotic Schatten-norm estimates developed in Section 6. To see the cancellations more clearly (and potentially give new geometric invariants) we introduce first and second partial antisymmetrizations in Section 7. For $f_1, ..., f_n, g_1, ..., g_n \in L^\infty(\mathbb{B}_n)$, define

$$[f_1, g_1, \ldots, f_n, g_n]_t^{\mathrm{fst}} = \sum_{\tau\in S_n} \mathrm{sgn}(\tau)\sigma_t(f_{\tau(1)}, g_1)\ldots\sigma_t(f_{\tau(n)}, g_n),$$

and

$$[f_1, g_1, \ldots, f_n, g_n]_t^{\mathrm{scd}} = \sum_{\tau \in S_n} \mathrm{sgn}(\tau)\sigma_t(f_1, g_{\tau(1)}) \ldots \sigma_t(f_n, g_{\tau(n)}).$$

After a full antisymmetrization they become a constant multiple of $[T_{f_1}^{(t)}, T_{f_2}^{(t)}, \ldots, T_{f_{2n}}^{(t)}]$. We observe that in general the product

$$\sigma_t(f_1, f_2) \ldots \sigma_t(f_{2n-1}, f_{2n})$$

is not a trace class operator (cf. Remark 7.6). Thus the fact that each $[T_{f_1}^{(t)}, T_{f_2}^{(t)}, \ldots, T_{f_{2n}}^{(t)}]$, or the partial antisymmetric sums belong to the trace class is already nontrivial. Besides the quantization this fact also relies on a further antisymmetrization over the complex tangential direction (see Equation (7.9)). This leads to the following.

**Theorem 1.4.** *(Theorem 7.3) Suppose $t \geq -1$ and $f_1, g_1, \ldots, f_n, g_n \in \mathscr{C}^2(\overline{\mathbb{B}_n})$. Then the partial antisymmetrizations $[f_1, g_1, \ldots, f_n, g_n]_t^{\mathrm{fst}}$ and $[f_1, g_1, \ldots, f_n, g_n]_t^{\mathrm{scd}}$ are in the trace class. Moreover,*

$$\begin{aligned}\lim_{t\to\infty} \mathrm{Tr}[f_1, g_1, \ldots, f_n, g_n]_t^{\mathrm{fst}} &= \lim_{t\to\infty} \mathrm{Tr}[f_1, g_1, \ldots, f_n, g_n]_t^{\mathrm{scd}} \\ &= \frac{1}{(2\pi i)^n} \int_{\mathbb{B}_n} \partial f_1 \wedge \bar{\partial} g_1 \wedge \ldots \wedge \partial f_n \wedge \bar{\partial} g_n.\end{aligned}$$

As a corollary of Theorem 1.4, we obtain Equation (1.7) by further antisymmetrization of the first (second) antisymmetric sums. Altogether, Equations (1.6) and (1.7) imply Theorem 1.2.

The proofs of Theorem 1.3 and 1.4 both involve quantization with asymptotic Schatten-norm estimates. In Theorem 1.3, we need to assume that $p$ is greater than or equal to $n+1$ in order for the Connes-Chern character to be well defined. This assumption simplifies the estimates in its proof. In Theorem 1.4, there are only $2n$ functions in $[f_1, g_1, \ldots, f_n, g_n]_t^{\mathrm{fst}}$ and $[f_1, g_1, \ldots, f_n, g_n]_t^{\mathrm{scd}}$. The finiteness of these traces requires a careful proof. To prove Theorem 1.4, we need to consider antisymmetrization over all the complex tangential directions (i.e. Equation (7.9)).

Our proof of the generalized Helton-Howe trace formula through quantization is closely related to the method developed in [9] for a solution to the Atiyah-Weinstein conjecture for quantized contact transform. Such a similarity suggests that our developments can be generalized to strongly pseudoconvex domains, egg domains, Fock spaces, submodules and their quotient modules of $L^2_{a,t}(\mathbb{B}_n)$, e.g. [27], and the Dury-Arveson spaces. More generally, we hope that our study will shed a light on constructing new cyclic cocycles beyond the Helton-Howe traces, which could have applications in noncommutative geometry.

The paper is organized as follows. In Section 2, we recall the definitions of weighted Bergman spaces and Hardy spaces together with their basic properties and properties of Schatten $p$-class operators. Some tools developed in [47] are also reviewed. In Section 3, we develop criteria for integral operators to be bounded between different weighted spaces $L^2_{a,t}(\mathbb{B}_n)$. We develop some useful estimate for integral operators to belong to Schatten-$p$ class in Section 4. In Section 5, we prove Equation (1.8). We develop the asymptotic expansion formula and its Schatten norm estimates in Section 6. We introduce the first and second antisymmetrization and prove Theorem 7.3 and Equation (1.7) in Section 7. The two main theorems, Theorems 1.2 and 1.3, are proved in Section 8.

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## 2. Preliminaries

In this section, we recall some basic definitions and properties about weighted Bergman spaces and Schatten-$p$ class operators.

### *2.1. Spaces on $\mathbb{B}_n$*

Recall that $\mathbb{B}_n$ is the open unit ball of $\mathbb{C}^n$ and $\mathbb{S}_n = \partial\mathbb{B}_n$ is the unit sphere. Let $m$ be the Lebesgue measure and $\sigma$ be the surface measure on $\mathbb{S}_n$. Denote $\sigma_{2n-1} = \sigma(\mathbb{S}_n) = \frac{2\pi^n}{(n-1)!}$.

**Hardy Space:** The Hardy space $H^2(\mathbb{S}_n)$ is the Hilbert space of holomorphic functions on $\mathbb{B}_n$ with the norm

$$\|f\|^2_{H^2(\mathbb{S}_n)} = \sup_{0<r<1} \int_{\mathbb{S}_n} |f(rz)|^2 \frac{d\sigma(z)}{\sigma_{2n-1}}.$$

Equivalently, $H^2(\mathbb{S}_n)$ is the closure of analytic polynomials in $L^2(\mathbb{S}_n) := L^2(\mathbb{S}_n, \frac{d\sigma}{\sigma_{2n-1}})$. The Hardy space is a reproducing kernel Hilbert space on $\mathbb{B}_n$ and the reproducing kernel is

$$K_w^{(-1)}(z) = \frac{1}{(1-\langle z, w\rangle)^n}, \quad \forall w \in \mathbb{B}_n.$$

For any $f \in L^\infty(\mathbb{S}_n)$, the Toeplitz operator on $H^2(\mathbb{S}_n)$ with symbol $f$ is defined to be the compression

$$T_f^{(-1)} = P^{(-1)} M_f|_{H^2(\mathbb{S}_n)},$$

where $M_f$ is the pointwise multiplication on $L^2(\mathbb{S}_n)$, and $P^{(-1)}$ is the orthogonal projection from $L^2(\mathbb{S}_n)$ onto $H^2(\mathbb{S}_n)$. Using the reproducing kernel, we can write $T_f^{(-1)}$ as an integral operator. For $h \in H^2(\mathbb{S}_n)$,

$$T_f^{(-1)} h(z) = \int_{\mathbb{S}_n} f(w)h(w)K_w^{(-1)}(z)\frac{\mathrm{d}\sigma(z)}{\sigma_{2n-1}}, \quad \forall z \in \mathbb{B}_n.$$

Our discussion will also involve Hankel operators. The Hankel operator with symbol $f$ is

$$H_f = (I - P^{(-1)})M_f P^{(-1)}$$

from $H^2(\mathbb{S}_n)$ to $L^2(\mathbb{S}_n)$.

**Weighted Bergman Spaces:** For $t > -1$, define the probability measure on $\mathbb{B}_n$:

$$\mathrm{d}\lambda_t(z) = \frac{(n-1)!}{\pi^n B(n, t+1)}(1 - |z|^2)^t \mathrm{d}m(z).$$

Here $B(n, t+1)$ is the Beta function. The weighted Bergman space $L^2_{a,t}(\mathbb{B}_n)$ is the subspace of $L^2(\mathbb{B}_n, \lambda_t)$ consisting of holomorphic functions on $\mathbb{B}_n$. The reproducing kernel of $L^2_{a,t}(\mathbb{B}_n)$ is

$$K_w^{(t)}(z) = \frac{1}{(1 - \langle z, w\rangle)^{n+1+t}}, \quad \forall w \in \mathbb{B}_n.$$

For any $f \in L^\infty(\mathbb{B}_n)$, the Toeplitz operator $T_f^{(t)}$ is the compression

$$T_f^{(t)} = P^{(t)} M_f^{(t)}|_{L^2_{a,t}(\mathbb{B}_n)},$$

where $P^{(t)}$ is the orthogonal projection from $L^2(\mathbb{B}_n, \lambda_t)$ onto $L^2_{a,t}(\mathbb{B}_n)$, and $M_f^{(t)}$ is the multiplication operator on $L^2(\mathbb{B}_n, \lambda_t)$. The Hankel operator with symbol $f$ is

$$H_f^{(t)} = (I - P^{(t)})M_f^{(t)} P^{(t)}.$$

Using the reproducing kernels, we can write $T_f^{(t)}, H_f^{(t)}$ as integral operators. For $h \in L^2_{a,t}(\mathbb{B}_n)$, we have the following expressions,

$$T_f^{(t)} h(z) = \int_{\mathbb{B}_n} f(w)h(w)K_w^{(t)}(z)\mathrm{d}\lambda_t(w), \quad \forall z \in \mathbb{B}_n,$$

$$H_f^{(t)}h(z) = \int_{\mathbb{B}_n} \big(f(z) - f(w)\big) h(w) K_w^{(t)}(z) \mathrm{d}\lambda_t(w), \quad \forall z \in \mathbb{B}_n.$$

For various reasons including the forms of the reproducing kernels, the Hardy space is often regarded as the $t = -1$ Bergman space. The two spaces can have distinct properties in many ways. But most of the results we prove in this paper will hold for both $t = -1$ and $t > -1$. However in spite of the similarities in these results, the two cases generally require separate (although similar) formulations and proofs. We will generally present them one after the other and note that when we do it.

An important tool on $\mathbb{B}_n$ is the Möbius transform.

**Definition 2.1.** For $z \in \mathbb{B}_n$, $z \neq 0$, the Möbius transform $\varphi_z$ is the biholomorphic mapping on $\mathbb{B}_n$ defined as follows.

$$\varphi_z(w) = \frac{z - P_z(w) - (1 - |z|^2)^{1/2} Q_z(w)}{1 - \langle w, z \rangle}, \quad \forall w \in \overline{\mathbb{B}_n}.$$

Here $P_z$ and $Q_z$ denote the orthogonal projection from $\mathbb{C}^n$ onto $\mathbb{C}z$ and $z^\perp$, respectively. Define

$$\varphi_0(w) = -w, \quad \forall w \in \overline{\mathbb{B}_n}.$$

It is well-known that $\varphi_z$ is an automorphism of $\mathbb{B}_n$ satisfying $\varphi_z \circ \varphi_z = Id$. Also, the two variable function $\rho(z, w) := |\varphi_z(w)| = |\varphi_w(z)|$ defines a metric on $\mathbb{B}_n$. Moreover, $\beta(z, w) := \tanh^{-1} \rho(z, w)$ coincides with the Bergman metric on $\mathbb{B}_n$.

We list some lemmas that serve as basic tools on $\mathbb{B}_n$. Most of the following can be found in [44,52]. Some are proved in our paper [47].

For non-negative values $A, B$, by $A \lesssim B$ we mean that there is a constant $C$ such that $A \leq CB$. Sometimes, to emphasize that the constant $C$ depends on some parameter $a$, we write $A \lesssim_a B$. The notations $\gtrsim, \gtrsim_a, \approx, \approx_a$ are defined similarly.

**Lemma 2.2.** *([47, Lemma 2.2]) Suppose $z, w, \zeta \in \mathbb{B}_n$.*

(1) $\frac{1}{1-\langle \varphi_\zeta(z), \varphi_\zeta(w)\rangle} = \frac{(1-\langle z,\zeta\rangle)(1-\langle \zeta, w\rangle)}{(1-|\zeta|^2)(1-\langle z,w\rangle)}$.
(2) $1 - |\varphi_z(w)|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\langle z,w\rangle|^2}$.
(3) *For any $R > 0$ we have*

$$\frac{1 - |z|^2}{1 - |w|^2} \approx_R 1, \quad \frac{|1 - \langle z, \zeta \rangle|}{|1 - \langle w, \zeta \rangle|} \approx_R 1$$

*whenever $\beta(z, w) < R$ and $\zeta \in \mathbb{B}_n$.*
(4) *The real Jacobian of $\varphi_z$ is $\frac{(1-|z|^2)^{n+1}}{|1-\langle z,\cdot\rangle|^{2n+2}}$ on $\mathbb{B}_n$ and $\frac{(1-|z|^2)^n}{|1-\langle z,\cdot\rangle|^{2n}}$ on $\mathbb{S}_n$.*

(5) *For* $z \in \mathbb{B}_n$,

$$z - \varphi_z(w) = \frac{(1-|z|^2)P_z(w) + (1-|z|^2)^{1/2}Q_z(w)}{1-\langle w,z\rangle} := \frac{A_z w}{1-\langle w,z\rangle},$$

*where* $A_z = [a_z^{ij}]$ *is an* $n \times n$ *matrix depending on* $z$*, and* $w$ *is viewed as a column vector.*

(6) *For any* $z \in \mathbb{B}_n$, $z \neq 0$,

$$|z - P_z(w)| \leq |\varphi_z(w)||1-\langle z,w\rangle|, \quad |Q_z(w)| \lesssim |\varphi_z(w)||1-\langle z,w\rangle|^{1/2}, \tag{2.1}$$

*and*

$$|z-w| \lesssim |\varphi_z(w)||1-\langle z,w\rangle|^{1/2}. \tag{2.2}$$

*If* $n = 1$*, then* $Q_z$ *is identically zero, and the definition of* $\varphi_z(w)$ *directly gives* $|z-w| = |\varphi_z(w)||1-z\bar{w}|$.

(7) $1-|z|^2 \leq 2|1-\langle z,w\rangle|$ *for all* $z, w \in \mathbb{B}_n$.

**Lemma 2.3.** *([47, Lemma 2.4])*

(1) *Suppose* $t > -1$, $c \in \mathbb{R}$. *Then*

$$\int\limits_{\mathbb{B}_n} \frac{(1-|w|^2)^t}{|1-\langle z,w\rangle|^{n+1+t+c}} \mathrm{d}m(w) \lesssim_{t,c} \begin{cases} (1-|z|^2)^{-c}, & c > 0, \\ \ln \frac{1}{1-|z|^2}, & c = 0, \\ 1, & c < 0, \end{cases} \tag{2.3}$$

*and*

$$\int\limits_{\mathbb{S}_n} \frac{1}{|1-\langle z,w\rangle|^{n+c}} \mathrm{d}\sigma(w) \lesssim_{t,c} \begin{cases} (1-|z|^2)^{-c}, & c > 0, \\ \ln \frac{1}{1-|z|^2}, & c = 0, \\ 1, & c < 0, \end{cases} \tag{2.4}$$

*for any* $z \in \mathbb{B}_n$.

(2) *Suppose* $t > -1$, $a, b, c > 0$, $a \geq c, b \geq c$, *and* $a + b < n + 1 + t + c$. *Then for any* $z, \xi \in \mathbb{B}_n$,

$$\int\limits_{\mathbb{B}_n} \frac{(1-|w|^2)^t}{|1-\langle z,w\rangle|^a |1-\langle w,\xi\rangle|^b} \mathrm{d}m(w) \lesssim_{a,b,c,t} \frac{1}{|1-\langle z,\xi\rangle|^c}. \tag{2.5}$$

(3) *Suppose* $\phi : (0,1) \to [0,\infty)$ *is measurable. Suppose* $a > -n$, $b \in \mathbb{R}$, *and*

$$\phi(s) \lesssim s^a(1-s)^b, \quad s \in (0,1).$$

*Then for any $t > -1-b$, $c > -b$ there exists $C > 0$ such that for any $z \in \mathbb{B}_n$,*

$$\int\limits_{\mathbb{B}_n} \phi(|\varphi_z(w)|^2)\frac{(1-|w|^2)^t}{|1-\langle z,w\rangle|^{n+1+t+c}}\mathrm{d}m(w) \le C(1-|z|^2)^{-c}. \tag{2.6}$$

**Lemma 2.4.** *Suppose $s > t \ge -1$ and $T$ is a bounded operator on $L^2_{a,s}(\mathbb{B}_n)$. Suppose $L^2_{a,t}(\mathbb{B}_n)$ is invariant under $T$. Denote $\hat{T}$ its restriction to $L^2_{a,t}(\mathbb{B}_n)$. By the closed graph theorem, $\hat{T}$ is bounded on $L^2_{a,t}(\mathbb{B}_n)$. Assume that $T \in \mathcal{S}^1\big(L^2_{a,s}(\mathbb{B}_n)\big)$ and $\hat{T} \in \mathcal{S}^1\big(L^2_{a,t}(\mathbb{B}_n)\big)$. Then*

$$\mathrm{Tr}T = \mathrm{Tr}\hat{T}.$$

**Proof.** The set $\{z^\alpha\}_{\alpha\in\mathbb{N}_0^n}$ forms an orthogonal basis of both $L^2_{a,s}(\mathbb{B}_n)$ and $L^2_{a,t}(\mathbb{B}_n)$. For $\alpha \in \mathbb{N}_0^n$, we write

$$Tz^\alpha = \hat{T}z^\alpha = \sum_{\beta\in\mathbb{N}_0^n} a_{\alpha,\beta}z^\beta.$$

Compute the traces of $T$ and $\hat{T}$ as follows.

$$\begin{aligned}\mathrm{Tr}T &= \sum_{\alpha\in\mathbb{N}_0^n} \frac{\langle Tz^\alpha, z^\alpha\rangle_{L^2_{a,s}(\mathbb{B}_n)}}{\|z^\alpha\|^2_{L^2_{a,s}(\mathbb{B}_n)}} \\ &= \sum_{\alpha\in\mathbb{N}_0^n} \frac{a_{\alpha,\alpha}\langle z^\alpha, z^\alpha\rangle_{L^2_{a,s}(\mathbb{B}_n)}}{\|z^\alpha\|^2_{L^2_{a,s}(\mathbb{B}_n)}} \\ &= \sum_{\alpha\in\mathbb{N}_0^n} a_{\alpha,\alpha} \\ &= \sum_{\alpha\in\mathbb{N}_0^n} \frac{\langle \hat{T}z^\alpha, z^\alpha\rangle_{L^2_{a,t}(\mathbb{B}_n)}}{\|z^\alpha\|^2_{L^2_{a,t}(\mathbb{B}_n)}} = \mathrm{Tr}\hat{T}.\end{aligned}$$

This completes the proof of Lemma 2.4. □

### *2.2. Schatten class operators*

For $p > 0$, a bounded operator $T$ on a Hilbert space $\mathcal{H}$ is said to be in the Schatten-$p$ class $\mathcal{S}^p$ if $|T|^p$ belongs to the trace class. The Schatten-$p$ class operators $\mathcal{S}^p$ are analogues of $l^p$ spaces in the operator-theoretic setting. Conventionally, $\mathcal{S}^\infty$ denotes the space of compact operators. The following two lemmas will be used constantly.

**Lemma 2.5.** *([46, Theorem 2.8]) Suppose $A$, $B$ are bounded operators on a Hilbert space, and $1 \le p, q, r \le \infty$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. If $A \in \mathcal{S}^p$ and $B \in \mathcal{S}^q$, then*

$$AB \in \mathcal{S}^r, \quad \text{and} \quad \|AB\|_{\mathcal{S}^r} \le \|A\|_{\mathcal{S}^p} \|B\|_{\mathcal{S}^q}.$$

**Lemma 2.6.** *([46, Corollary 3.8]) Suppose $A, B$ are bounded operators on a Hilbert space. If both $AB$ and $BA$ are in the trace class then* $\mathrm{Tr}AB = \mathrm{Tr}BA$ *or equivalently,*

$$\mathrm{Tr}[A, B] = 0.$$

**Remark 2.7.** Suppose $n > 1$ and $X_1, X_2, \dots, X_n$ are bounded operators on a Hilbert space such that

$$X_i \in \mathcal{S}^p, \quad \forall p > n,$$

and there exists $j \in \{1, \dots, n\}$ such that $X_j$ is in $\mathcal{S}^p$ for some $p < n$. Then we can choose $p_1, \dots, p_{j-1}, p_{j+1}, \dots, p_n > n$ and $1 \le p_j < n$ such that

$$X_i \in \mathcal{S}^{p_i}, i = 1, \dots, n, \quad \text{and} \quad \frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_n} = 1.$$

By an inductive application of Lemma 2.5 we have

$$X_1 X_2 \dots X_n \in \mathcal{S}^1.$$

Moreover, by Lemma 2.6, for any $k = 1, \dots, n$,

$$\mathrm{Tr} X_1 X_2 \dots X_n = \mathrm{Tr} X_k X_{k+1} \dots X_n X_1 \dots X_{k-1}.$$

This will be used repeatedly in this paper.

### *2.3. Integration by parts*

Some integral formulas developed in [47] will be used in our proofs. These formulas come from a generalized version of the Bochner-Martinelli formula in several complex variables and are essential to the proof of Equation (1.8) in Section 5 and the Toeplitz quantization formulas in Lemma 6.1. We give a brief review here. Some further remarks about these formulas are given in Remarks 2.11, 2.12 and 2.13. Let us start with introducing some auxiliary functions and operations.

Recall that by Lemma 2.2 (5), for $z \in \mathbb{B}_n$,

$$(1 - |z|^2) P_z(w) + (1 - |z|^2)^{1/2} Q_z(w) = (1 - \langle w, z \rangle)(z - \varphi_z(w)) = A_z w,$$

where $A_z$ is an $n \times n$ matrix depending on $z$, and $w$ is treated as a column vector. In particular, if $z = (z_1, 0, \dots, 0)$ and $w = (w_1, \dots, w_n)$ then

$$A_z w = \left( (1 - |z_1|^2) w_1, (1 - |z_1|^2)^{1/2} w_2, \dots, (1 - |z_1|^2)^{1/2} w_n \right).$$

**Definition 2.8.** For multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ and $\zeta \in \mathbb{C}^n$, denote

$$I^{\alpha,\beta}(\zeta) = \zeta^\alpha \bar{\zeta}^\beta.$$

Suppose $z \in \mathbb{B}_n$, define

$$d_{\alpha,\beta}(z) = \int_{\mathbb{S}_n} I^{\alpha,\beta}\left(A_z\zeta\right) \frac{\mathrm{d}\sigma(\zeta)}{\sigma_{2n-1}}.$$

In particular, $d_{0,0} = 1$, and

$$d_{\alpha,\beta}(z) = \delta_{\alpha,\beta}(1-|z|^2)^{\alpha_1+|\alpha|} \frac{(n-1)!\alpha!}{(n-1+|\alpha|)!}, \quad \text{if } z = (z_1, 0, \dots, 0). \tag{2.7}$$

**Definition 2.9.** For $t \in \mathbb{R}$, denote

$$\phi_t(s) = (1-s)^t.$$

Suppose $\phi : (0,1) \to [0,\infty)$ is a measurable function. For a positive integer $m$ and any $t > -1$, define the operations on $\phi$

$$\mathcal{F}_m^{(t)}\phi(s) = \int_s^1 r^{m-1}\phi(r)(1-r)^t \mathrm{d}r \in [0,\infty], \tag{2.8}$$

and

$$\mathcal{G}_m^{(t)}\phi(s) = \frac{1}{s^m\phi_{t+1}(s)}\mathcal{F}_m^{(t)}\phi(s) = \frac{\int_s^1 r^{m-1}\phi(r)(1-r)^t\mathrm{d}r}{s^m(1-s)^{t+1}} \in [0,\infty]. \tag{2.9}$$

For any $t > -1$, inductively define the functions

$$\Phi_{n,0}^{(t)} \equiv 1, \quad \Phi_{n,k+1}^{(t)} = M_{\phi_1}\left(\mathcal{G}_{n+k}^{(t)}\right)^2 \Phi_{n,k}^{(t)}.$$

Equivalently,

$$\Phi_{n,k}^{(t)} = M_{\phi_1}\left(\mathcal{G}_{n+k-1}^{(t)}\right)^2 \dots M_{\phi_1}\left(\mathcal{G}_n^{(t)}\right)^2 \Phi_{n,0}^{(t)}. \tag{2.10}$$

**Lemma 2.10.** *([47, Lemma 4.2]) Suppose $t > -1$, $\alpha, \beta \in \mathbb{N}_0^n$. Suppose $\phi : (0,1) \to [0,\infty)$ is measurable and $v \in \mathscr{C}^1(\mathbb{B}_n)$. Then the following hold.*

1. *If $|\alpha| \geq |\beta|$ and all integrals converge absolutely, then*

$$\int_{\mathbb{B}_n} \phi(|\varphi_z(w)|^2) I^{\alpha,\beta}(z-w)v(w)K_w^{(t)}(z)\mathrm{d}\lambda_t(w) \tag{2.11}$$

$$= \begin{cases} \frac{d_{\alpha,\beta}(z)}{B(n,t+1)} \cdot \mathcal{F}^{(t)}_{n+|\beta|}\phi(0)v(z) \\ \quad - \sum_{j=1}^{n} \int_{\mathbb{B}_n} \mathcal{G}^{(t)}_{|\beta|+n}\phi(|\varphi_z(w)|^2) I^{\alpha,\beta+e_j}(z-w) S_j(w) K^{(t)}_w(z) \mathrm{d}\lambda_t(w), \\ \hfill v(z) \neq 0, \mathcal{F}^{(t)}_{n+|\beta|}\phi(0) < \infty, \\ \\ - \sum_{j=1}^{n} \int_{\mathbb{B}_n} \mathcal{G}^{(t)}_{|\beta|+n}\phi(|\varphi_z(w)|^2) I^{\alpha,\beta+e_j}(z-w) S_j(w) K^{(t)}_w(z) \mathrm{d}\lambda_t(w), \\ \hfill v(z) = 0, \mathcal{F}^{(t)}_{n+|\beta|}\phi(0) \leq \infty, \end{cases}$$

*where*

$$S_j(w,z) = \frac{(1-|w|^2)\bar{\partial}_{w_j}\left[(1-\langle z,w\rangle)^{|\beta|}v(w)\right]}{(1-\langle w,z\rangle)(1-\langle z,w\rangle)^{|\beta|}}.$$

2. *If* $|\alpha| \leq |\beta|$ *and all integrals converge absolutely, then*

$$\int_{\mathbb{B}_n} \phi(|\varphi_z(w)|^2) I^{\alpha,\beta}(z-w) v(z) K^{(t)}_w(z) \mathrm{d}\lambda_t(z) \tag{2.12}$$

$$= \begin{cases} \frac{d_{\alpha,\beta}(w)}{B(n,t+1)} \cdot \mathcal{F}^{(t)}_{n+|\alpha|}\phi(0)v(w) \\ \quad + \sum_{i=1}^{n} \int_{\mathbb{B}_n} \mathcal{G}^{(t)}_{|\alpha|+n}\phi(|\varphi_z(w)|^2) I^{\alpha+e_i,\beta}(z-w) \widetilde{S}_i(z) K^{(t)}_w(z) \mathrm{d}\lambda_t(z), \\ \hfill v(w) \neq 0, \mathcal{F}^{(t)}_{n+|\alpha|}\phi(0) < \infty, \\ \\ \sum_{i=1}^{n} \int_{\mathbb{B}_n} \mathcal{G}^{(t)}_{|\alpha|+n}\phi(|\varphi_z(w)|^2) I^{\alpha+e_i,\beta}(z-w) \widetilde{S}_i(z) K^{(t)}_w(z) \mathrm{d}\lambda_t(z), \\ \hfill v(w) = 0, \mathcal{F}^{(t)}_{n+|\alpha|}\phi(0) \leq \infty, \end{cases}$$

*where*

$$\widetilde{S}_i(z,w) = \frac{(1-|z|^2)\partial_{z_i}\left[(1-\langle z,w\rangle)^{|\alpha|}v(z)\right]}{(1-\langle w,z\rangle)(1-\langle z,w\rangle)^{|\alpha|}}.$$

**Remark 2.11.** For readers who are familiar with the computation of currents, when $z=0$, Formula (2.11) can be abstracted into an equation of the following form

$$\bar{\partial}\left[I^{\alpha,\beta}(w)\Psi(|w|^2) \wedge \partial|w|^2 \wedge \left(\partial\bar{\partial}|w|^2\right)^{n-1}\right] = c\delta_0 + I^{\alpha,\beta}(w)\Phi(|w|^2)\left(\partial\bar{\partial}|w|^2\right)^n.$$

And the operations $\mathcal{F}^{(t)}, \mathcal{G}^{(t)}$ come from solving $\Psi$ and $c$ from the equation above. Here $\delta_0$ is the point mass at the origin. From this point of view we see Formula (2.11) can be used the same way as Cauchy's formula or the Bochner-Martinelli formula — to solve $\bar{\partial}$-equation of some sort.

**Remark 2.12.** We observe that on the right hand side of Formula (2.11), the function $v$ is differentiated, while the weight is improved by 1 (the term "$(1-|w|^2)$" in $S_j(w,z)$

adds to the weight). For this reason we consider it as integration by parts. Also observe the resemblance of the integrals on both sides of (2.11) and (2.12). This resemblance allows us to iterate them. In fact, this is exactly how we get the quantization formulas in Lemma 6.1 — by iterating the formulas above.

**Remark 2.13.** There is one more benefit of the formulas. Technically, it is easier to work with integral operators with higher weights to study their boundedness, Schatten-class membership, etc.. Applying the integral formula on a Toeplitz operator increases the weight by 1. This improvement offers more flexibility in analyzing these operators. Moreover, since the weight goes up by 1, the integral formula builds a bridge between Toeplitz operators on different weighted Bergman spaces. This idea plays a crucial role in Section 5, where we give the proof of Equation (1.8).

A particular case is to take $\alpha = \beta = 0$, $z = 0$, $\phi = 1$ in (2.11), which gives the following. The estimates (2) and (3) are straightforward to verify.

**Lemma 2.14.** *For any $t > -1$ the following hold.*

(1) *For any $v \in \mathscr{C}^1(\overline{\mathbb{B}_n})$,*

$$\int_{\mathbb{B}_n} v(z)\mathrm{d}\lambda_t(z) = v(0) + \frac{t+1}{n+t+1}\int_{\mathbb{B}_n} \mathcal{G}_n^{(t)}1(|z|^2)\bar{R}v(z)\mathrm{d}\lambda_{t+1}(z). \tag{2.13}$$

(2) *$\mathcal{G}_n^{(t)}1(s) \approx s^{-n}$ in a neighborhood of $0$, and $\lim_{s\to 1^-} \mathcal{G}_n^{(t)}1(s) = \frac{1}{t+1}$.*
(3) *$|\mathcal{G}_n^{(t)}1(s) - \frac{1}{t+1}| \lesssim 1 - s$ for $s$ in a neighborhood of $1$.*

**Lemma 2.15.** *([47, Lemma 4.6]) Suppose $\alpha, \beta \in \mathbb{N}_0^n$, and $v \in \mathscr{C}^1(\overline{\mathbb{B}_n})$. Then the following hold.*

1. *If $|\alpha| \geq |\beta|$, then*

$$\begin{aligned}&\int_{\mathbb{S}_n} I^{\alpha,\beta}(z-w)v(w)K_w^{(-1)}(z)\frac{\mathrm{d}\sigma(w)}{\sigma_{2n-1}} \\ =&d_{\alpha,\beta}(z)v(z) \\ &-\frac{1}{n}\sum_{j=1}^{n}\int_{\mathbb{B}_n}|\varphi_z(w)|^{-2|\beta|-2n}I^{\alpha,\beta+e_j}(z-w)\frac{\bar{\partial}_j\left[(1-\langle z,w\rangle)^{|\beta|}v(w)\right]}{(1-\langle z,w\rangle)^{|\beta|}(1-\langle w,z\rangle)}K_w^{(-1)}(z)\mathrm{d}\lambda_0(w).\end{aligned} \tag{2.14}$$

2. *If $|\alpha| \leq |\beta|$, then*

$$\int_{\mathbb{S}_n} I^{\alpha,\beta}(z-w)v(z)K_w^{(-1)}(z)\frac{\mathrm{d}\sigma(z)}{\sigma_{2n-1}} \tag{2.15}$$

$$=d_{\beta,\alpha}(w)v(w)$$
$$+\frac{1}{n}\sum_{i=1}^{n}\int_{\mathbb{B}_n}|\varphi_z(w)|^{-2|\alpha|-2n}I^{\alpha+e_i,\beta}(z-w)\frac{\partial_i\big[(1-\langle z,w\rangle)^{|\alpha|}v(z)\big]}{(1-\langle z,w\rangle)^{|\alpha|}(1-\langle w,z\rangle)}K_w^{(-1)}(z)\mathrm{d}\lambda_0(z).$$

**Lemma 2.16.** *([47, Lemma 4.3]) Suppose $k$ is a non-negative integer and $\Gamma\subset\mathbb{N}_0^n\times\mathbb{N}_0^n$ is a finite set of multi-indices with $|\alpha|=|\beta|=k$ for every $(\alpha,\beta)\in\Gamma$. Suppose for some $\epsilon>-1-t$, $\{F_{\alpha,\beta}\}_{(\alpha,\beta)\in\Gamma}\subset\mathscr{C}^2(\mathbb{B}_n\times\mathbb{B}_n)$ and*

$$\Big|\sum_{(\alpha,\beta)\in\Gamma}I^{\alpha,\beta}(z-w)F_{\alpha,\beta}(z,w)\Big|\lesssim|\varphi_z(w)|^{2k}|1-\langle z,w\rangle|^{2k+\epsilon},\tag{2.16}$$

$$\Big|\sum_{j=1}^{n}\sum_{(\alpha,\beta)\in\Gamma}I^{\alpha,\beta+e_j}(z-w)\bar\partial_{w_j}F_{\alpha,\beta}(z,w)\Big|\lesssim|\varphi_z(w)|^{2k+1}|1-\langle z,w\rangle|^{2k+\epsilon}.\tag{2.17}$$

*Then*

$$\int_{\mathbb{B}_n^2}\Phi_{n,k}^{(t)}(|\varphi_z(w)|^2)\frac{\sum_{(\alpha,\beta)\in\Gamma}I^{\alpha,\beta}(z-w)F_{\alpha,\beta}(z,w)}{|1-\langle z,w\rangle|^{2k}}K_w^{(t)}(z)\mathrm{d}\lambda_t(w)\mathrm{d}\lambda_t(z)$$
$$=\frac{\mathcal{F}_{n+k}^{(t)}\Phi_{n,k}^{(t)}(0)}{B(n,t+1)}\int_{\mathbb{B}_n}(1-|z|^2)^{-2k}\sum_{(\alpha,\beta)\in\Gamma}d_{\alpha,\beta}(z)F_{\alpha,\beta}(z,z)\mathrm{d}\lambda_t(z)\tag{2.18}$$
$$-\int_{\mathbb{B}_n^2}\Phi_{n,k+1}^{(t)}(|\varphi_z(w)|^2)$$
$$\times\frac{\sum_{i,j=1}^{n}\sum_{(\alpha,\beta)\in\Gamma}I^{\alpha+e_i,\beta+e_j}(z-w)D_{i,j}F_{\alpha,\beta}(z,w)}{|1-\langle z,w\rangle|^{2(k+1)}}K_w^{(t)}(z)\mathrm{d}\lambda_t(z)\mathrm{d}\lambda_t(w).$$

*Here $D_{i,j}$ denotes the operation*

$$D_{i,j}=(1-\langle z,w\rangle)^2\partial_{z_i}\bar\partial_{w_j}.$$

**Lemma 2.17.** *Suppose $k$ is a nonnegative integer. Then there exist $C>c>0$ such that for $t$ large enough,*

$$ct^{-n-k}\le\mathcal{F}_{n+k}^{(t)}\Phi_{n,k}^{(t)}(0)\le Ct^{-n-k},\tag{2.19}$$

*and*

$$\int_0^1\Phi_{n,k}^{(t)}(s)s^{n+k-1}(1-s)^{\frac{t}{4}}\mathrm{d}s\le Ct^{-n-k}.\tag{2.20}$$

**Proof.** The case when $k = 0$ is simply a consequence of [47, Equation (8.9)]. We assume $k > 0$.

From the Sterling's asymptotic formula (cf. [2, Theorem 1.4.1]) and the well-known identity $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ it follows that for fixed $x > -1$,

$$B(x,y) \approx_x y^{-x}$$

for large $y$. By [47, Lemma 8.4], we compute $\mathcal{F}^{(t)}_{n+k}\Phi^{(t)}_{n,k}(0)$ as follows,

$$\begin{aligned}
&\mathcal{F}^{(t)}_{n+k}\Phi^{(t)}_{n,k}(0)\\
=&\mathcal{F}^{(t)}_{n+k}M_{\phi_1}\big(\mathcal{G}^{(t)}_{n+k-1}\big)^2\dots M_{\phi_1}\big(\mathcal{G}^{(t)}_{n}\big)^2 1(0)\\
=&\sum_{j_1,\dots,j_k=0}^{\infty}\frac{B(n+k+j_1+\dots+j_k,t+1)}{1\cdot(1+j_1)(2+j_1)(2+j_1+j_2)\dots(k+j_1+\dots+j_{k-1})(k+j_1+\dots+j_k)}\\
=&\sum_{0\le s_1\le\dots\le s_k<\infty}\frac{B(n+k+s_k,t+1)}{(1+s_1)(2+s_1)\dots(k-1+s_{k-1})(k+s_{k-1})(k+s_k)}\\
\le&\sum_{s_k=0}^{\infty}\sum_{s_1,\dots,s_{k-1}=0}^{\infty}\frac{B(n+k+s_k,t+1)}{(1+s_1)(2+s_1)\dots(k-1+s_{k-1})(k+s_{k-1})(k+s_k)}\\
=&\sum_{s_k=0}^{\infty}\frac{B(n+k+s_k,t+1)}{(k-1)!(k+s_k)}\\
=&\frac{1}{(k-1)!}\int_0^1\Big(\sum_{s_k=0}^{\infty}\frac{x^{n-1+(k+s_k)}}{k+s_k}\Big)(1-x)^t\mathrm{d}x\\
\le&\frac{1}{(k-1)!}\int_0^1\Big(\sum_{s_k=0}^{\infty}x^{n-1}\left(\ln\frac{1}{1-x}\right)(1-x)^t\mathrm{d}x\\
\lesssim&\int_0^1 x^{n-1}(1-x)^{t-1}\mathrm{d}x\\
=&B(n,t)\\
\lesssim&t^{-n-k},
\end{aligned}$$

when $t$ is large. The other inequality $\mathcal{F}^{(t)}_{n+k}\Phi^{(t)}_{n,k}(0) \gtrsim t^{-n-k}$ is also obvious from the equation above. This proves (2.19).

By the expansion

$$\frac{1}{(1-s)^{\frac{3t}{4}}} = \sum_{j=0}^{\infty}\frac{\Gamma(\frac{3t}{4}+j)}{j!\Gamma(\frac{3t}{4})}s^j,$$

we directly compute the following integral,

$$
\begin{aligned}
&\int_0^1 \Phi_{n,k}^{(t)}(s) s^{n+k-1}(1-s)^{\frac{t}{4}} \mathrm{d}s \\
=&\sum_{j=0}^{\infty} \frac{\Gamma(\frac{3t}{4}+j)}{j!\Gamma(\frac{3t}{4})} \int_0^1 \Phi_{n,k}^{(t)}(s) s^{n+k+j-1}(1-s)^{t} \mathrm{d}s \\
=&\sum_{j_0=0}^{\infty} \frac{\Gamma(\frac{3t}{4}+j_0)}{j_0!\Gamma(\frac{3t}{4})} \mathcal{F}_{n+k+j_0}^{(t)} \Phi_{n,k}^{(t)}(0).
\end{aligned}
$$

Similarly as the proof of (2.19), assuming $t$ is large enough, we estimate the above term as follows,

$$
\begin{aligned}
\leq& \sum_{0\leq s_0\leq s_k<\infty} \frac{\Gamma(\frac{3t}{4}+s_0)B(n+k+s_k,t+1)}{(s_0+1)!\Gamma(\frac{3t}{4})(k-1)!(k+s_k)} \\
=&\sum_{a=0}^{\infty}\sum_{b=0}^{\infty} \frac{\Gamma(\frac{3t}{4}+a)B(n+k+a+b,t+1)}{(a+1)!\Gamma(\frac{3t}{4})(k-1)!(k+a+b)} \\
\leq& \sum_{a=0}^{\infty} \frac{\Gamma(\frac{3t}{4}+a)}{(a+1)!\Gamma(\frac{3t}{4})(k-1)!} \int_0^1 \sum_{b=0}^{\infty} \frac{s^{b+1}}{k+a+b} s^{n+k+a-2}(1-s)^t \mathrm{d}s \\
\leq& \sum_{a=0}^{\infty} \frac{\Gamma(\frac{3t}{4}+a)}{(a+1)!\Gamma(\frac{3t}{4})(k-1)!} \int_0^1 \sum_{b=0}^{\infty} \frac{s^{b+1}}{b+1} s^{n+k+a-2}(1-s)^t \mathrm{d}s \\
=&\sum_{a=0}^{\infty} \frac{\Gamma(\frac{3t}{4}+a)}{(a+1)!\Gamma(\frac{3t}{4})(k-1)!} \int_0^1 s^{n+k+a-2}(1-s)^t \ln\frac{1}{1-s} \mathrm{d}s \\
\lesssim& \sum_{a=0}^{\infty} \frac{\Gamma(\frac{3t}{4}+a)}{(a+1)!\Gamma(\frac{3t}{4})(k-1)!} B(n+k+a,t) \\
\leq& \int_0^1 s^{n+k-1}(1-s)^{t-1-\frac{3t}{4}} \mathrm{d}s \\
=&B(n+k,\frac{t}{4}) \\
\lesssim& t^{-n-k}.
\end{aligned}
$$

This completes the proof of Lemma 2.17. □

## 3. Operators between weighted spaces

For our proofs in this paper it is important to obtain criteria for integral operators to be bounded between different weighted spaces. In this section, we introduce some useful criteria to be used in our study. Most of the lemmas established in this section follow from standard techniques from, for example, [44,52]. One thing less standard is that our integral kernels may involve functions of $|\varphi_z(w)|^2$. These functions come from applying the integration by parts formulas in Subsection 2.3 on integral formulas of Toeplitz operators.

**Lemma 3.1.** *Suppose $t > -1, s > -1$, $a > -n$, $b \geq 0$, and $c < t+1+b-\frac{s+1}{2}$. Suppose $F(z,w)$ is measurable on $\mathbb{B}_n \times \mathbb{B}_n$, and*

$$|F(z,w)| \leq \frac{|\varphi_z(w)|^{2a}(1-|\varphi_z(w)|^2)^b}{|1-\langle z,w\rangle|^{n+1+t-c}}, \quad \forall z,w \in \mathbb{B}_n.$$

*Then the integral operator*

$$Th(z) = \int_{\mathbb{B}_n} h(w)F(z,w)\mathrm{d}\lambda_t(w)$$

*is bounded from $L^2(\lambda_{s+2c})$ to $L^2(\lambda_s)$.*

**Proof.** By assumption, we can take $x \in \mathbb{R}$ so that

$$\max\{-1-t-b, -c-s-1-b\} < x < \min\{b-c, t+b-2c-s\}.$$

Then we have the following inequalities,

$$t+x > -1-b, \quad -x-c > -b, \quad x+c+s > -1-b, \quad t-x-2c-s > -b.$$

Take $p(w) = (1-|w|^2)^x$ and $q(z) = (1-|z|^2)^{x+c}$. The integral kernel of $T$ as an operator from $L^2(\lambda_{s+2c})$ to $L^2(\lambda_s)$ is

$$\frac{B(n,s+2c+1)}{B(n,t+1)}F(z,w)(1-|w|^2)^{t-s-2c}.$$

Then by the inequalities above and Lemma 2.3 (3), we have the following estimates of integrals,

$$\int_{\mathbb{B}_n} |F(z,w)|(1-|w|^2)^{t-s-2c}p(w)\mathrm{d}\lambda_{s+2c}(w)$$

$$
\begin{aligned}
&\lesssim \int\limits_{\mathbb{B}_n} |\varphi_z(w)|^{2a}(1-|\varphi_z(w)|^2)^b \frac{(1-|w|^2)^{t+x}}{|1-\langle z,w\rangle|^{n+1+t-c}} \mathrm{d}m(w) \\
&\lesssim (1-|z|^2)^{x+c} = q(z),
\end{aligned}
$$

and

$$
\begin{aligned}
&\int\limits_{\mathbb{B}_n} |F(z,w)|(1-|w|^2)^{t-s-2c} q(z) \mathrm{d}\lambda_s(z) \\
&\lesssim \int\limits_{\mathbb{B}_n} |\varphi_z(w)|^{2a}(1-|\varphi_z(w)|^2)^b \frac{(1-|w|^2)^{t-s-2c}(1-|z|^2)^{x+c+s}}{|1-\langle z,w\rangle|^{n+1+t-c}} \mathrm{d}m(z) \\
&\lesssim (1-|w|^2)^x = p(w).
\end{aligned}
$$

The conclusion follows from Schur's test (cf. [53, Theorem 3.6]). This completes the proof of Lemma 3.1. □

**Lemma 3.2.** *Suppose $t > -1$ and $0 < d < c < t+1$. Suppose $F(z,w)$ is measurable on $\mathbb{B}_n \times \mathbb{B}_n$, and*

$$
|F(z,w)| \le \frac{1}{|1-\langle z,w\rangle|^{n+1+t-c}}, \quad \forall z,w \in \mathbb{B}_n.
$$

*For any $0 < r < 1$, define the integral operator*

$$
T_r h(z) = \int\limits_{\mathbb{B}_n} h(w) F(rz,w) \mathrm{d}\lambda_t(w), \quad z \in \mathbb{S}_n.
$$

*Then each $T_r$ defines a bounded operator from $L^2(\lambda_{-1+2d})$ to $L^2(\mathbb{S}_n)$. Moreover,*

$$
\sup_{0<r<1} \|T_r\|_{L^2(\lambda_{-1+2d}) \to L^2(\mathbb{S}_n)} < \infty.
$$

**Proof.** Take $p(w) = (1-|w|^2)^{-d}$, $q(z) = 1$. The integral kernel of $T_r$ is

$$
T_r(z,w) := \frac{B(n,2d)}{B(n,t+1)} F(rz,w)(1-|w|^2)^{t+1-2d}.
$$

As $t-d > -1$ and $d < c$, by Lemma 2.3, we have the following estimates of integrals,

$$
\int\limits_{\mathbb{B}_n} |T_r(z,w)| p(w) \mathrm{d}\lambda_{-1+2d}(w) \lesssim \int\limits_{\mathbb{B}_n} \frac{(1-|w|^2)^{t-d}}{|1-\langle rz,w\rangle|^{n+1+t-c}} \mathrm{d}m(w) \lesssim 1 = q(z),
$$

and

$$\int_{\mathbb{S}_n} |T_r(z,w)| q(z) \frac{\mathrm{d}\sigma(z)}{\sigma_{2n-1}} \lesssim \int_{\mathbb{S}_n} \frac{(1-|w|^2)^{t+1-2d}}{|1-\langle z, rw\rangle|^{n+1+t-c}} \mathrm{d}\sigma(z)$$
$$\lesssim (1-|w|^2)^{c-2d} \lesssim (1-|w|^2)^{-d} = p(w).$$

The estimates are independent of $r$. Thus the conclusion follows from Schur's test. This completes the proof of Lemma 3.2. □

**Lemma 3.3.** *Suppose $t > -1$ and $0 < d < c < t+1$. Suppose $F(z,w)$ is piecewise continuous on $\mathbb{B}_n \times \mathbb{B}_n$, and*

$$|F(z,w)| \leq \frac{1}{|1-\langle z,w\rangle|^{n+1+t-c}}, \quad \forall z, w \in \mathbb{B}_n.$$

*Assume further that $F(z,w)$ is holomorphic in $z$ for each fixed $w$. Define the operator*

$$Th(z) = \int_{\mathbb{B}_n} h(w) F(z,w) \mathrm{d}\lambda_t(w), \quad z \in \mathbb{B}_n.$$

*Then $T$ defines a bounded operator from $L^2(\lambda_{-1+2d})$ to $H^2(\mathbb{S}_n)$.*

**Proof.** Since $F(z,w)$ is holomorphic in $z$, for every $h$, the function $Th$ is holomorphic in $\mathbb{B}_n$. Denote

$$M = \sup_{0<r<1} \|T_r\|_{L^2(\lambda_{-1+2d}) \to L^2(\mathbb{S}_n)} < \infty.$$

Then for any $h \in L^2(\lambda_{-1+2d})$, we verify the following estimate

$$\sup_{0<r<1} \int_{\mathbb{S}_n} |Th(rz)|^2 \mathrm{d}\sigma(z) = \sup_{0<r<1} \|T_r h\|^2_{L^2(\mathbb{S}_n)} \leq M^2 \|h\|^2_{L^2(\lambda_{-1+2d})}.$$

Therefore $T$ is bounded. This completes the proof of Lemma 3.3. □

The following fact about embedding operators between different weighted spaces is well-known to experts. In our proof, estimates of the operator norms and Schatten norms of these embeddings are needed. We give the calculation for completeness.

**Lemma 3.4.** *Suppose $t \geq -1$ and $c > 0$. Then for any $p > \frac{2n}{c}$ the following hold.*

(1) *The embedding map $E_{t,c}$ from $L^2_{a,t}(\mathbb{B}_n)$ to $L^2_{a,t+c}(\mathbb{B}_n)$ (identifying $L_{a,-1}(\mathbb{B}_n)$ with $H^2(\mathbb{S}_n)$) is in the Schatten $p$ class $\mathcal{S}^p$.*
(2) *There exists $C > 0$, independent of $t$, such that*

$$\|E_{t,c}\| = 1, \quad \|E_{t,c}\|_p \leq C(t+n+1)^{\frac{n}{p}}.$$

**Proof.** For any multi-index $\alpha \in \mathbb{N}_0^n$, we have the norm

$$\|z^\alpha\|^2_{L^2_{a,t}(\mathbb{B}_n)} = \frac{\Gamma(n+t+1)\alpha!}{\Gamma(n+|\alpha|+t+1)}.$$

Thus $E_{t,c}$ is unitarily equivalent to a diagonal operator with entry

$$\frac{\|z^\alpha\|_{L^2_{a,t+c}(\mathbb{B}_n)}}{\|z^\alpha\|_{L^2_{a,t}(\mathbb{B}_n)}} = \sqrt{\frac{\Gamma(n+t+c+1)\Gamma(n+|\alpha|+t+1)}{\Gamma(n+|\alpha|+t+c+1)\Gamma(n+t+1)}} = \sqrt{\frac{B(n+|\alpha|+t+1,c)}{B(n+t+1,c)}}$$

at $\alpha \in \mathbb{N}_0^n$. It follows immediately that $\|E_{t,c}\| = 1$. Thus it suffices to show

$$\begin{aligned}\sum_{\alpha\in\mathbb{N}_0^n}\left(\frac{B(n+|\alpha|+t+1,c)}{B(n+t+1,c)}\right)^{\frac{p}{2}} &= \sum_{d=0}^{\infty}\frac{(d+n-1)!}{d!(n-1)!}\cdot\left(\frac{B(n+d+t+1,c)}{B(n+t+1,c)}\right)^{\frac{p}{2}}\\ &\lesssim (t+n+1)^n.\end{aligned}$$

Since $B(n+d+t+1,c)$ is decreasing in $d$, and $\frac{(d+n-1)!}{d!} \approx d^{n-1}$ for $d \geq 1$, we have the following inequalities,

$$\begin{aligned}&\sum_{d=0}^{\infty}\frac{(d+n-1)!}{d!(n-1)!}\cdot\left(\frac{B(n+d+t+1,c)}{B(n+t+1,c)}\right)^{\frac{p}{2}}\\ \lesssim& 1+\int_0^\infty x^{n-1}\left(\frac{B(n+x+t+1,c)}{B(n+t+1,c)}\right)^{p/2}\mathrm{d}x\\ \lesssim& 1+\int_0^\infty x^{n-1}\left(\frac{n+t+1}{n+x+t+1}\right)^{\frac{pc}{2}}\mathrm{d}x.\end{aligned}$$

If we take the change of variable $y = \frac{x}{n+t+1}$ then the integral above has the following bounds,

$$\lesssim 1+(n+t+1)^n\int_0^\infty y^{n-1}\left(\frac{1}{1+y}\right)^{\frac{pc}{2}}\mathrm{d}y \lesssim (n+t+1)^n$$

when $n < \frac{pc}{2}$, i.e., $p > \frac{2n}{c}$. This completes the proof of Lemma 3.4. $\square$

## 4. Schatten class criteria

In this section, we obtain criteria for integral operators to belong to Schatten class. As explained in the beginning of Section 3, the integral kernels may involve functions of $|\varphi_z(w)|^2$. Thus we need to modify some of the well-known results to fit our case.

A standard way of proving Schatten class membership of integral operators is to use the following lemma. See [53, Lemma 8.26] for a proof when $n = 1$, using interpolation. The exact same proof works for general $n$.

**Lemma 4.1.** *Suppose $t > -1$ and $G(z, w)$ is measurable on $\mathbb{B}_n \times \mathbb{B}_n$. Suppose $p \geq 2$ and*

$$\int_{\mathbb{B}_n^2} |G(z,w)|^p |K_w^{(t)}(z)|^2 \mathrm{d}\lambda_t(w) \mathrm{d}\lambda_t(z) < \infty.$$

*Define the operator*

$$Th(z) = \int_{\mathbb{B}_n} h(w) G(z,w) K_w^{(t)}(z) \mathrm{d}\lambda_t(w).$$

*Then $T$ defines a bounded operator on $L^2(\lambda_t)$ that belongs to $\mathcal{S}^p$.*

**Corollary 4.2.** *Suppose $t > -1$ and $F(z, w)$ is measurable on $\mathbb{B}_n \times \mathbb{B}_n$. Suppose $c > 0$ and*

$$|F(z,w)| \leq \frac{1}{|1 - \langle z, w\rangle|^{n+1+t-c}}, \quad z, w \in \mathbb{B}_n.$$

*Define the integral operator*

$$Th(z) = \int_{\mathbb{B}_n} h(w) F(z,w) \mathrm{d}\lambda_t(w), \quad h \in L^2(\lambda_t).$$

*Then $T$ is bounded on $L^2(\lambda_t)$, and for any $p > \frac{n}{c}$ and $p \geq 2$, $T \in \mathcal{S}^p$.*

**Proof.** Let

$$G(z,w) = \frac{F(z,w)}{K_w^{(t)}(z)}, \quad z, w \in \mathbb{B}_n.$$

Then by assumption, we have the following bound,

$$|G(z,w)| = |F(z,w)| \cdot |1 - \langle z, w\rangle|^{n+1+t} \leq |1 - \langle z, w\rangle|^c.$$

Choose $p$ so that $\frac{n}{c} < p < \frac{n+1+t}{c}$. Then by Lemma 2.3, we compute the following integral.

$$\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} |G(z,w)|^p |K_w^{(t)}(z)|^2 \mathrm{d}\lambda_t(w) \mathrm{d}\lambda_t(z)$$

$$\lesssim \int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{(1-|z|^2)^t(1-|w|^2)^t}{|1-\langle z,w\rangle|^{2(n+1+t)-cp}}\mathrm{d}m(w)\mathrm{d}m(z)$$

$$\lesssim \int_{\mathbb{B}_n} (1-|z|^2)^{t-(n+1+t-cp)}\mathrm{d}m(z)$$

$$= \int_{\mathbb{B}_n} (1-|z|^2)^{-1+(cp-n)}\mathrm{d}m(z) < \infty.$$

Since $G(z,w)$ is bounded, for $p \geq \frac{n+1+t}{c}$ the integral is also finite. Therefore we have the following bound,

$$\int_{\mathbb{B}_n}\int_{\mathbb{B}_n} |G(z,w)|^p |K_w^{(t)}(z)|^2 \mathrm{d}\lambda_t(w)\mathrm{d}\lambda_t(z) < \infty, \quad \forall p > \frac{n}{c}.$$

Finally, by Lemma 4.1, $T \in \mathcal{S}^p$ for any $p > \frac{n}{c}$ and $p \geq 2$. This completes the proof of Corollary 4.2. □

**Corollary 4.3.** *Let $t, c, F, T$ be as in Corollary 4.2. Then for any Lipschitz function $u$ on $\mathbb{B}_n$ and any $p > \frac{n}{c+\frac{1}{2}}$ and $p \geq 2$, the commutator*

$$[T, M_u] \in \mathcal{S}^p.$$

**Proof.** By definition, for $h \in L^2(\lambda_t)$, we have the following expression,

$$[T, M_u]h(z) = \int_{\mathbb{B}_n} \big(u(w) - u(z)\big)F(z,w)h(w)\mathrm{d}\lambda_t(w).$$

Since $u$ is Lipschitz, by Lemma 2.2 (6), we get the following bounds,

$$\big|\big(u(w)-u(z)\big)F(z,w)\big| \lesssim |z-w||F(z,w)| \lesssim \frac{1}{|1-\langle z,w\rangle|^{n+1+t-c-1/2}}.$$

The corollary then follows from Corollary 4.2. This completes the proof of Corollary 4.3. □

It is well-known that Hankel operators with Lipschitz symbols belong to $\mathcal{S}^p$ for any $p > 2n$. In fact, the Schatten-class membership for Hankel operators is completely characterized (see [38], [53, Theorem 8.36]).

**Corollary 4.4.** *Suppose $t > -1$ and $u$ is a Lipschitz function on $\mathbb{B}_n$. Then the Hankel operator $H_u^{(t)} = (I - P^{(t)})M_u P^{(t)}$ belongs to $\mathcal{S}^p$ for any $p > 2n$.*

**Proof.** The corollary follows from Corollary 4.3 and the equation

$$H_u^{(t)} = (I - P^{(t)})M_u P^{(t)} = [M_u, P^{(t)}]P^{(t)}.$$

This completes the proof of Corollary 4.4. □

Lemma 4.1 provides convenient criteria for an integral operator to be in $\mathcal{S}^p$, when $p \geq 2$. However, in this paper we will also need to deal with the case when $1 \leq p < 2$. Moreover, we will need to consider integral operators of the form

$$Th(z) = \int_{\mathbb{B}_n} \phi(|\varphi_z(w)|^2)F(z,w)h(w)\mathrm{d}\lambda_t(w),$$

where $\phi$ is an unbounded function. For such $T$, if we take the double integral as in Lemma 4.1, its integral is very likely to be infinite. In application, it is enough for us to obtain Schatten-class membership of the operators $TP^{(t)}$ or $P^{(t)}T$. An alternative way to obtain such Schatten-class membership result is to take advantage of Lemma 3.4. In particular, the following lemma holds.

**Lemma 4.5.** *Suppose $t > -1$, $a > -n$, $b \geq 0$ and $c > 0$. Suppose $F(z,w)$ is measurable on $\mathbb{B}_n \times \mathbb{B}_n$, $\phi : (0,1) \to [0,\infty)$ is measurable, and*

$$\phi(s) \leq s^a(1-s)^b, \quad s \in (0,1),$$
$$|F(z,w)| \leq \frac{1}{|1-\langle z,w\rangle|^{n+1+t-c}}, \quad \forall z,w \in \mathbb{B}_n.$$

*Define the integral operator on $L^2(\lambda_t)$.*

$$Th(z) = \int_{\mathbb{B}_n} \phi(|\varphi_z(w)|^2)F(z,w)h(w)\mathrm{d}\lambda_t(w).$$

*Then both $P^{(t)}T$ and $TP^{(t)}$ belong to $\mathcal{S}^p$ for any $p > \max\{\frac{n}{c}, \frac{n}{b+\frac{1+t}{2}}\}$.*

**Proof.** Notice that $P^{(t)}T \in \mathcal{S}^p$ if and only if $T^*P^{(t)} \in \mathcal{S}^p$, and that $T^*$ is an integral operator with integral kernel satisfying the same estimate as $T$. Thus it suffices to prove the statement for $TP^{(t)}$. For any $q > \max\{\frac{n}{c}, \frac{n}{b+\frac{1+t}{2}}\}$, let $c' = \frac{n}{q}$. Then $c' < b + \frac{1+t}{2}$. Split the map $TP^{(t)}$ as follows.

$$TP^{(t)} : L^2(\lambda_t) \xrightarrow{P^{(t)}} L^2_{a,t}(\mathbb{B}_n) \xrightarrow{E_{t,2c'}} L^2_{a,t+2c'}(\mathbb{B}_n) \xrightarrow{\hat{T}} L^2(\lambda_t).$$

Here $\hat{T} : L^2_{a,t+2c'}(\mathbb{B}_n) \to L^2(\lambda_t)$ is defined by the same integral formula as $T$. By Lemma 3.4, $E_{t,2c'} \in \mathcal{S}^p$ for any $p > \frac{n}{c'} = q$. Also by Lemma 3.1, $\hat{T}$ is bounded. Since $q$

is any number with $q > \max\{\frac{n}{c}, \frac{n}{b+\frac{1+t}{2}}\}$, we have $TP^{(t)} \in \mathcal{S}^p, \forall p > \max\{\frac{n}{c}, \frac{n}{b+\frac{1+t}{2}}\}$. This completes the proof of Lemma 4.5. □

In the case when $\phi = s^k \Phi_{n,k}^{(t)}$, the following Schatten-norm estimate holds.

**Theorem 4.6.** *Suppose $t > -1$, $c > 0$ and $k$ is a non-negative integer. Suppose $F(z, w)$ is measurable on $\mathbb{B}_n \times \mathbb{B}_n$, and*

$$|F(z,w)| \leq \frac{|\varphi_z(w)|^{2k}}{|1 - \langle z, w\rangle|^{n+1+t-c}}, \quad \forall z, w \in \mathbb{B}_n.$$

*Define the integral operator on $L^2(\mathbb{B}_n, \lambda_t)$.*

$$Th(z) = \int_{\mathbb{B}_n} \Phi_{n,k}^{(t)}(|\varphi_z(w)|^2) F(z,w) h(w) \mathrm{d}\lambda_t(w).$$

*Then both $P^{(t)}T$ and $TP^{(t)}$ belong to $\mathcal{S}^p$ for any $p > \max\{\frac{n}{c}, \frac{n}{k+\frac{1+t}{2}}\}$, $p \geq 1$. Moreover, for $p > \frac{n}{c}$, and $p \geq 1$, and $t$ large enough, we have*

$$\|P^{(t)}T\|_p \lesssim t^{-k+\frac{n}{p}}, \quad \|TP^{(t)}\|_p \lesssim t^{-k+\frac{n}{p}}.$$

As in the proof of Corollary 4.3, Lemma 4.5 and Theorem 4.6 imply the following.

**Corollary 4.7.** *Suppose $t > -1$, $c > 0$, $F(z, w)$ is measurable on $\mathbb{B}_n \times \mathbb{B}_n$, and*

$$|F(z,w)| \leq \frac{1}{|1 - \langle z, w\rangle|^{n+1+t-c}}, \quad z, w \in \mathbb{B}_n.$$

*Suppose $\phi : (0, 1) \to [0, \infty)$ is measurable. Define the integral operator on $L^2(\lambda_t)$,*

$$Th(z) = \int_{\mathbb{B}_n} \phi(|\varphi_z(w)|^2) F(z,w) h(w) \mathrm{d}\lambda_t(w).$$

*Assume that $u$ is a Lipschitz function on $\mathbb{B}_n$.*

1. *Suppose $a > -n$, $b \geq 0$ and $\phi(s) \leq s^a(1 - s)^b$. Then*

$$[T, M_u]P^{(t)}, \quad P^{(t)}[T, M_u] \in \mathcal{S}^p, \quad \forall p > \max\{\frac{n}{c+\frac{1}{2}}, \frac{n}{b+\frac{1+t}{2}}\}.$$

2. *If $\phi(s) = s^k \Phi_{n,k}^{(t)}$, then for $t$ large enough and $p > \frac{n}{c+\frac{1}{2}}$,*

$$\big\|[T, M_u]P^{(t)}\big\|_{\mathcal{S}^p} \lesssim t^{-k+\frac{n}{p}}, \quad \big\|P^{(t)}[T, M_u]\big\|_{\mathcal{S}^p} \lesssim t^{-k+\frac{n}{p}}.$$

A trivial application of Theorem 4.6 gives the following.

**Lemma 4.8.** *Suppose $c > 0$, $t > -1$ and $f \in L^\infty(\mathbb{B}_n)$ satisfies*

$$|f(z)| \leq (1-|z|^2)^c, \quad \forall z \in \mathbb{B}_n.$$

*Then for any $p > \max\{\frac{n}{c}, \frac{2n}{1+t}\}, p \geq 1$,*

$$T_f^{(t)} \in \mathcal{S}^p.$$

*For any $p > \frac{n}{c}$, $p \geq 1$ and $t$ large enough,*

$$\|T_f^{(t)}\|_{\mathcal{S}^p} \lesssim_p t^{\frac{n}{p}}.$$

**Proof.** By definition, we have the following expression of $T_f^{(t)}$,

$$T_f^{(t)}h(z) = \int\limits_{\mathbb{B}_n} f(w)h(w)K_w^{(t)}(z)\mathrm{d}\lambda_t(w).$$

By the assumption, we have the following inequalities

$$|f(w)K_w^{(t)}(z)| \leq \frac{(1-|z|^2)^c}{|1-\langle z,w\rangle|^{n+1+t}} \lesssim \frac{1}{|1-\langle z,w\rangle|^{n+1+t-c}}.$$

Since $\Phi_{n,0}^{(t)} = 1$, the conclusion follows directly from Theorem 4.6. This completes the proof of Lemma 4.8. $\square$

To prove Theorem 4.6, we need the following estimate.

**Lemma 4.9.** *Suppose $\phi : (0,1) \to [0,\infty)$ is measurable. Then for any $c, d \in \mathbb{R}$ there exist $C > 0$ and $t_0 > 0$ such that whenever $t > t_0$,*

$$\int\limits_{\mathbb{B}_n} \phi(|\varphi_z(w)|^2)\frac{(1-|w|^2)^{\frac{t}{2}+c}}{|1-\langle z,w\rangle|^{n+1+t+d}}\mathrm{d}m(w) \leq C\int\limits_0^1 \phi(s)s^{n-1}(1-s)^{\frac{t}{4}}\mathrm{d}s \cdot (1-|z|^2)^{-\frac{t}{2}-d+c}, \quad \forall z \in \mathbb{B}_n. \tag{4.1}$$

**Proof.** Make the change of variable $w = \varphi_z(\xi)$. Using Lemma 2.2 we have the following equation,

$$\frac{(1-|w|^2)^t}{|1-\langle z,w\rangle|^{n+1+t}}\mathrm{d}m(w) \xlongequal[\xi=\varphi_z(w)]{w=\varphi_z(\xi)} \frac{(1-|\xi|^2)^t}{|1-\langle z,\xi\rangle|^{n+1+t}}\mathrm{d}m(\xi).$$

Then we have the following estimate of the left side of Equation (4.1),

$$\begin{aligned}
&\int_{\mathbb{B}_n} \phi(|\varphi_z(w)|^2)\frac{(1-|w|^2)^{\frac{t}{2}+c}}{|1-\langle z,w\rangle|^{n+1+t+d}}\mathrm{d}m(w)\\
=&\int_{\mathbb{B}_n} \phi(|\xi|^2)\left(\frac{(1-|z|^2)(1-|\xi|^2)}{|1-\langle z,\xi\rangle|^2}\right)^{c-\frac{t}{2}}\left(\frac{|1-\langle z,\xi\rangle|}{1-|z|^2}\right)^{d}\frac{(1-|\xi|^2)^t}{|1-\langle z,\xi\rangle|^{n+1+t}}\mathrm{d}m(\xi)\\
=&(1-|z|^2)^{c-\frac{t}{2}-d}\int_{\mathbb{B}_n} \phi(|\xi|^2)\frac{(1-|\xi|^2)^{c+\frac{t}{2}}}{|1-\langle z,\xi\rangle|^{n+1+2c-d}}\mathrm{d}m(\xi)\\
=&(1-|z|^2)^{c-\frac{t}{2}-d}\int_0^1 \phi(r^2)(1-r^2)^{c+\frac{t}{2}}r^{2n-1}\left[\int_{\mathbb{S}_n}\frac{1}{|1-\langle rz,\zeta\rangle|^{n+1+2c-d}}\mathrm{d}\sigma(\zeta)\right]\mathrm{d}r\\
\lesssim&(1-|z|^2)^{c-\frac{t}{2}-d}\int_0^1 \phi(r^2)(1-r^2)^{c+\frac{t}{2}}r^{2n-1}(1-|rz|^2)^{-a}\mathrm{d}r\\
\le&(1-|z|^2)^{c-\frac{t}{2}-d}\int_0^1 \phi(r^2)(1-r^2)^{c-a+\frac{t}{2}}r^{2n-1}\mathrm{d}r\\
\xlongequal{s=r^2}&\frac{1}{2}(1-|z|^2)^{c-\frac{t}{2}-d}\int_0^1 \phi(s)(1-s)^{c-a+\frac{t}{2}}s^{n-1}\mathrm{d}s.
\end{aligned}$$

Here $a=2c-d$ if $2c-d>0$; $a=\frac{1}{2}$ if $2c-d=0$; $a=0$ if $2c-d<0$. For $t$ large enough, $c-a+\frac{t}{2}>\frac{t}{4}$. So we have the following inequality,

$$\int_0^1 \phi(s)(1-s)^{c-a+\frac{t}{2}}s^{n-1}\mathrm{d}s\le\int_0^1 \phi(s)(1-s)^{\frac{t}{4}}s^{n-1}\mathrm{d}s.$$

This completes the proof of Lemma 4.9. □

**Proof of Theorem 4.6.** The Schatten class memberships of $TP^{(t)}$ and $P^{(t)}T$ are implied by Lemma 4.5 and [47, Lemma 8.3]. It remains to prove the Schatten norm estimates. We may assume that $t$ is large enough so that $0<c<k+\frac{1+t}{2}$. Also, as in the proof of Lemma 4.5, it suffices to prove the statement for $TP^{(t)}$.

Split the operator $TP^{(t)}$ as

$$L^2(\mathbb{B}_n,\lambda_t)\xrightarrow{P^{(t)}}L^2_{a,t}(\mathbb{B}_n)\xrightarrow{E_{t,2c}}L^2_{a,t+2c}(\mathbb{B}_n)\xrightarrow{\hat{T}}L^2(\mathbb{B}_n,\lambda_t),$$

where $\hat{T}$ is the same integral operator as $T$. By Lemma 3.4, it suffices to show that $\hat{T}$ defines a bounded operator from $L^2_{a,t+2c}(\mathbb{B}_n)$ to $L^2(\mathbb{B}_n, \lambda_t)$ with $\|\hat{T}\|_{L^2_{a,t+2c}(\mathbb{B}_n)\to L^2_{a,t}(\mathbb{B}_n)} \lesssim t^{-k}$ for large $t$.

This is done by Schur's test. Since $c < k + \frac{t+1}{2}$, if we take $x = c + \frac{t+1}{2}$, then $t+1+k > x > 2c-k$. Take $p(w) = (1-|w|^2)^{-x}$ and $q(z) = (1-|z|^2)^{c-x}$. The integral kernel of $\hat{T} : L^2(\mathbb{B}_n, \lambda_{t+2c}) \to L^2(\mathbb{B}_n, \lambda_t)$ is

$$T(z,w) := \frac{B(n,t+2c+1)}{B(n,t+1)} \Phi^{(t)}_{n,k}(|\varphi_z(w)|^2) F(z,w)(1-|w|^2)^{-2c}.$$

Then by (2.6), (4.1), (2.19) and (2.20), we have the following estimates,

$$\begin{aligned}
&\int_{\mathbb{B}_n} |T(z,w)| p(w) \mathrm{d}\lambda_{t+2c}(w) \\
\approx &\frac{1}{B(n,t+1)} \int_{\mathbb{B}_n} \Phi^{(t)}_{n,k}(|\varphi_z(w)|^2) |\varphi_z(w)|^{2k} \frac{(1-|w|^2)^{t-x}}{|1-\langle z,w\rangle|^{n+1+t-c}} \mathrm{d}m(w) \\
\lesssim &\frac{\int_0^1 \Phi^{(t)}_{n,k}(s) s^{n+k-1}(1-s)^{\frac{t}{4}} \mathrm{d}s}{B(n,t+1)} q(z) \lesssim t^{-k} q(z).
\end{aligned}$$

Similarly, we have the following inequality,

$$\int_{\mathbb{B}_n} |T(z,w)| q(z) \mathrm{d}\lambda_t(z) \lesssim t^{-k} p(w).$$

From this we have the following bound for $p > n/c$ and large $t$,

$$\|T\|_{\mathcal{S}^p} \le \|E_{t,2c}\|_{\mathcal{S}^p} \|\hat{T}\| \lesssim t^{-k+n/p}.$$

This completes the proof of Theorem 4.6. □

## 5. Traces on different weighted Bergman spaces

As explained in the introduction, the goal of this section is to prove Equation (1.8). More precisely, we will prove Lemmas 5.2 and 5.3 stated below.

Suppose $t > s \ge -1$. It is well-known that $L^2_{a,s}(\mathbb{B}_n) \subset L^2_{a,t}(\mathbb{B}_n)$, and $L^2_{a,s}(\mathbb{B}_n)$ is dense in $L^2_{a,t}(\mathbb{B}_n)$. For a Toeplitz operator $T^{(t)}_f$ on $L^2_{a,t}(\mathbb{B}_n)$, if the restriction of $T^{(t)}_f$ on $L^2_{a,s}(\mathbb{B}_n)$ defines a bounded operator on $L^2_{a,s}(\mathbb{B}_n)$, then we denote this restriction to be $T^{(t,s)}_f$. It follows from Lemma 3.1 that if $f \in L^\infty(\mathbb{B}_n)$ and $t > s > -1$, then the restricted operator $T^{(t,s)}_f$ is well-defined. On the other hand, if a Toeplitz operator $T^{(s)}_f$ on $L^2_{a,s}(\mathbb{B}_n)$ extends (uniquely) into a bounded operator on $L^2_{a,t}(\mathbb{B}_n)$, then we denote this operator to be $T^{(s,t)}_f$.

**Notation 5.1.** For operators $S, T$, write $S \sim T$ when $S - T$ is a trace class operator with zero trace. We emphasize that it is not required that $S$ or $T$ is in the trace class. In particular $S \sim 0$ means that $S$ is in the trace class with zero trace.

**Lemma 5.2.** *Suppose $f_1, f_2, \ldots, f_{2n} \in \mathscr{C}^2(\overline{\mathbb{B}_n})$ and $t > -1$. Then*

$$[T_{f_1}^{(t)}, T_{f_2}^{(t)}, \ldots, T_{f_{2n}}^{(t)}] - [T_{f_1}^{(t+1,t)}, T_{f_2}^{(t+1,t)}, \ldots, T_{f_{2n}}^{(t+1,t)}] \sim 0.$$

In the case of the Hardy space we show the following. Note that the index goes up by 2 this time. The reason for doing this is explained in Subsection 5.4, after the proof of Lemma 5.2.

**Lemma 5.3.** *Suppose $f_1, f_2, \ldots, f_{2n} \in \mathscr{C}^2(\overline{\mathbb{B}_n})$. Then the operator on $H^2(\mathbb{S}_n)$*

$$[T_{f_1}^{(-1)}, T_{f_2}^{(-1)}, \ldots, T_{f_{2n}}^{(-1)}] - [T_{f_1}^{(1,-1)}, T_{f_2}^{(1,-1)}, \ldots, T_{f_{2n}}^{(1,-1)}] \sim 0.$$

The fact that each $T_{f_i}^{(1,-1)}$ is well-defined is explained in Remark 5.26.

## *5.1. Decomposition of $T_f^{(t)}$*

The proof of Lemma 5.2 (and Lemma 5.3) involves writing $T_f^{(t)}$ (resp. $T_f^{(-1)}$) as a sum of $T_f^{(t+1,t)}$ (resp. $T_f^{(1,-1)}$) and some perturbation operator. In this subsection we introduce these decompositions. The main results of this subsection are Lemma 5.5 (for $t > -1$) and Lemma 5.7 (for $t = -1$).

**Definition 5.4.** For $t > -1$, define

$$X_w^{(t)}(z) = \frac{t+1}{n+t+1}\mathcal{G}_n^{(t)}1(|w|^2)\bar{R}_w K_w^{(t)}(z) - K_w^{(t+1)}(z), \tag{5.1}$$

and

$$Y_w^{(t)}(z) = \frac{t+1}{n+t+1}\mathcal{G}_n^{(t)}1(|w|^2)K_w^{(t)}(z). \tag{5.2}$$

For a symbol function $g$, formally define the integral operators

$$X_g^{(t)}h(z) = \int\limits_{\mathbb{B}_n} g(w)h(w)X_w^{(t)}(z)\mathrm{d}\lambda_{t+1}(w); \quad Y_g^{(t)}h(z) = \int\limits_{\mathbb{B}_n} g(w)h(w)Y_w^{(t)}(z)\mathrm{d}\lambda_{t+1}(w).$$

It will be clear in subsequent proof that for $f \in \mathscr{C}^2(\overline{\mathbb{B}_n})$, $X_f^{(t)}$ and $Y_{\bar{R}f}^{(t)}$ define bounded operators on $L_{a,t}^2(\mathbb{B}_n)$ (see Lemmas 5.14 and 5.21).

**Lemma 5.5.** *Suppose $f \in \mathscr{C}^2(\overline{\mathbb{B}_n})$. Then for any $t > -1$,*

$$T_f^{(t)} = T_f^{(t+1,t)} + X_f^{(t)} + Y_{\bar{R}f}^{(t)} + f(0)E_0, \tag{5.3}$$

*where $E_0 h = h(0)$ is a rank one operator.*

**Proof.** For any $h \in \mathrm{Hol}(\overline{\mathbb{B}_n})$, by Lemma 2.14, we compute $T_f^{(t)}$ as follows,

$$\begin{aligned} T_f^{(t)} h(z) =& \int_{\mathbb{B}_n} f(w)h(w)K_w^{(t)}(z)\mathrm{d}\lambda_t(w) \\ =& f(0)h(0) + \frac{t+1}{n+t+1}\int_{\mathbb{B}_n} \mathcal{G}_n^{(t)}1(|w|^2)\bar{R}f(w)h(w)K_w^{(t)}(z)\mathrm{d}\lambda_{t+1}(w) \\ &+ \frac{t+1}{n+t+1}\int_{\mathbb{B}_n} \mathcal{G}_n^{(t)}1(|w|^2)f(w)h(w)\bar{R}K_w^{(t)}(z)\mathrm{d}\lambda_{t+1}(w) \\ =& f(0)\big(E_0 f\big)(z) + Y_{\bar{R}f}^{(t)}h(z) + X_f^{(t)}h(z) + T_f^{(t+1,t)}h(z). \end{aligned}$$

This completes the proof of Lemma 5.5. □

In the case of the Hardy space, to make our treatment parallel to the (weighted) Bergman space, we apply integration by parts twice and lift the weight by 2 to achieve the appropriate estimate in Lemma 5.24.

**Definition 5.6.** Let $\phi(s) = s^{-n}$. Define

$$\begin{aligned} X_w^{(-1)}(z) &= \frac{1}{n(n+1)}\mathcal{G}_n^{(0)}\phi(|w|^2)\bar{R}_w^2 K_w^{(-1)}(z) - K_w^{(1)}(z), \\ Y_w^{(-1)}(z) &= \frac{2}{n(n+1)}\mathcal{G}_n^{(0)}\phi(|w|^2)\bar{R}_w K_w^{(-1)}(z), \end{aligned}$$

and

$$Z_w^{(-1)}(z) = \frac{1}{n(n+1)}\mathcal{G}_n^{(0)}\phi(|w|^2)K_w^{(-1)}(z).$$

Formally define the symboled integral operators

$$\begin{aligned} X_f^{(-1)}h(z) &= \int_{\mathbb{B}_n} h(w)f(w)X_w^{(-1)}(z)\mathrm{d}\lambda_1(w), \\ Y_f^{(-1)}h(z) &= \int_{\mathbb{B}_n} h(w)f(w)Y_w^{(-1)}(z)\mathrm{d}\lambda_1(w), \end{aligned}$$

and

$$Z_f^{(-1)}h(z) = \int_{\mathbb{B}_n} h(w)f(w)Z_w^{(-1)}(z)\mathrm{d}\lambda_1(w).$$

Again, it will be clear later that for $f \in \mathscr{C}^2(\overline{\mathbb{B}_n})$, $X_f^{(-1)}$, $Y_{\bar{R}f}^{(-1)}$ and $Z_{\bar{R}^2 f}^{(-1)}$ define bounded operators on $H^2(\mathbb{S}_n)$ (see Lemma 5.25).

Now we give the $t = -1$ analogue of Lemma 5.5.

**Lemma 5.7.** *Suppose $f \in \mathscr{C}^2(\overline{\mathbb{B}_n})$. Then*

$$T_f^{(-1)} = T_f^{(1,-1)} + X_f^{(-1)} + Y_{\bar{R}f}^{(-1)} + Z_{\bar{R}^2 f}^{(-1)} + f(0)E_0.$$

Taking $\alpha = \beta = 0$ and $z = 0$ in Lemma 2.15, we get the following.

**Lemma 5.8.** *Suppose $v \in \mathscr{C}^1(\overline{\mathbb{B}_n})$. Then*

$$\int_{\mathbb{S}_n} v(w)\frac{\mathrm{d}\sigma(w)}{\sigma_{2n-1}} = v(0) + \frac{1}{n}\int_{\mathbb{B}_n} |w|^{-2n}\bar{R}v(w)\mathrm{d}\lambda_0(w). \tag{5.4}$$

Also, taking $\alpha = \beta = 0, t = 0, z = 0$ in Lemma 2.10 gives the following.

**Lemma 5.9.** *Suppose $\phi : (0,1) \to [0,\infty)$ is measurable and $v \in \mathscr{C}^1(\mathbb{B}_n)$. Then whenever all integrals converge absolutely, we have*

$$\int_{\mathbb{B}_n} \phi(|w|^2)v(w)\mathrm{d}\lambda_0(w) \tag{5.5}$$

$$= \begin{cases} \frac{\mathcal{F}_n^{(0)}\phi(0)}{B(n,t+1)}v(0) + \frac{1}{n+1}\int_{\mathbb{B}_n} \mathcal{G}_n^{(0)}\phi(|w|^2)\bar{R}v(w)\mathrm{d}\lambda_1(w), & \mathcal{F}_n^{(0)}\phi(0) < \infty \\ \frac{1}{n+1}\int_{\mathbb{B}_n} \mathcal{G}_n^{(0)}\phi(|w|^2)\bar{R}v(w)\mathrm{d}\lambda_1(w), & \mathcal{F}_n^{(0)}\phi(0) \le \infty,\ v(0) = 0 \end{cases}.$$

**Proof of Lemma 5.7.** By Lemmas 5.8 and 5.9, for $h \in \mathrm{Hol}(\overline{\mathbb{B}_n})$, we compute $T_f^{(-1)}$ as follows,

$$\begin{aligned} T_f^{(-1)}h(z) &= \int_{\mathbb{S}_n} h(w)f(w)K_w^{(-1)}(z)\frac{\mathrm{d}\sigma(w)}{\sigma_{2n-1}} \\ &\overset{(5.4)}{=\!=} f(0)h(0) + \frac{1}{n}\int_{\mathbb{B}_n} \phi(|w|^2)\bar{R}\Big[h(w)f(w)K_w^{(-1)}(z)\Big]\mathrm{d}\lambda_0(w) \\ &\overset{(5.5)}{=\!=} f(0)h(0) + \frac{1}{n(n+1)}\int_{\mathbb{B}_n} \mathcal{G}_n^{(0)}\phi(|w|^2)\bar{R}^2\Big[h(w)f(w)K_w^{(-1)}(z)\Big]\mathrm{d}\lambda_1(w) \end{aligned}$$

$$=f(0)h(0)+\frac{1}{n(n+1)}\int\limits_{\mathbb{B}_n}\mathcal{G}_n^{(0)}\phi(|w|^2)h(w)\mathrm{d}\lambda_1(w)$$

$$\left[\bar{R}^2 f(w)K_w^{(-1)}(z)+2\bar{R}f(w)\bar{R}_w K_w^{(-1)}(z)+f(w)\bar{R}_w^2 K_w^{(-1)}(z)\right]$$

$$=f(0)\big(E_0 f\big)(z)+T_f^{(1,-1)}h(z)+X_f^{(-1)}h(z)+Y_{\bar{R}f}^{(-1)}h(z)+Z_{\bar{R}^2 f}^{(-1)}h(z).$$

This completes the proof of Lemma 5.7. □

In view of (5.3), Lemma 5.2 essentially says that the trace

$$\mathrm{Tr}[T_{f_1}^{(t+1,t)},T_{f_2}^{(t+1,t)},\dots,T_{f_{2n}}^{(t+1,t)}]$$

if it exists, is invariant under the perturbations of $X_{f_i}^{(t)}$, $Y_{\bar{R}f_i}^{(t)}$ and $f_i(0)E_0$. Similarly, Lemma 5.3 can also be interpreted as the stability of trace under certain perturbations.

### *5.2. Hypotheses A*

As explained in the introduction, after establishing the decomposition $T_{f_i}^{(t)}=T_{f_i}^{(t+1,t)}+B_i$, the proof of Equation (1.8) amounts to removing the “minor parts” $B_i$ from the antisymmetric sum $[T_{f_1}^{(t)},T_{f_2}^{(t)},\dots,T_{f_{2n}}^{(t)}]$. We accomplish this in two steps. In the first step, in Section 5.2 we dealt with those parts that can be handled at the level of operator theory. In the second step, in Section 5.3 we use the integral formulas of these operators to handle the rest terms. To treat the $t>-1$ case and $t=-1$ case uniformly, we abstract our conditions into Hypotheses A below and Hypotheses B in the next subsection.

**Hypotheses A:** Suppose $A_1,A_2,\dots,A_{2n},B_1,B_2,\dots,B_{2n}$ are bounded linear operators on a Hilbert space $\mathcal{H}$. Denote $C_i=A_i+B_i, i=1,\dots,2n$. The operators satisfy the following properties.

(1) For any $i=1,\dots,2n$,

$$B_i\in\mathcal{S}^p,\quad\forall p>n.$$

(2) For any $i,j=1,\dots,2n$,

$$[A_i,A_j]\in\mathcal{S}^p,\quad\forall p>n.$$

(3) For any $i,j=1,\dots,2n$,

$$[A_i,B_j]\in\mathcal{S}^p,\qquad\text{for some } p<n.$$

The goal of this subsection is to prove the following.

**Proposition 5.10.** *Assume Hypotheses A. Then the operator*

$$[C_1, C_2, \ldots, C_{2n}] - [A_1, A_2, \ldots, A_{2n}]$$

*is in the trace class. Moreover, the operator*

$$[C_1, C_2, \ldots, C_{2n}] - [A_1, A_2, \ldots, A_{2n}] - \sum_{k=1}^{2n} [A_1, \ldots, A_{k-1}, B_k, A_{k+1}, \ldots, A_{2n}] \sim 0.$$

**Lemma 5.11.** *Suppose $\{X_1, X_2, \ldots, X_{2n}\}$ is a subset of $\{A_1, \ldots, A_{2n}, B_1, \ldots, B_{2n}\}$, and at least two of $X_1, X_2, \ldots, X_{2n}$ are in $\{B_1, B_2, \ldots, B_{2n}\}$. Then*

$$[X_1, X_2, \ldots, X_{2n}] \sim 0.$$

**Proof.** The lemma can be restated as follows:

"if $B \in \{B_1, B_2, \ldots, B_{2n}\}$, $\{X_1, \ldots, X_{2n-1}\} \subset \{A_1, \ldots, A_{2n}, B_1, \ldots, B_{2n}\}$, and at least one of $X_1, \ldots, X_{2n-1}$ is in $\{B_1, \ldots, B_{2n}\}$, then

$$[B, X_1, \ldots, X_{2n-1}] \sim 0''.$$

Under the above assumption on $B$ and $X_1, \cdots, X_{2n-1}$, we compute the antisymmetrization $[B, X_1, \ldots, X_{2n-1}]$ as follows.

$$\begin{aligned}
&[B, X_1, \ldots, X_{2n-1}] \\
=&\sum_{k=1}^{n} \Big( \sum_{\tau \in S_{2n-1}} \operatorname{sgn}(\tau) X_{\tau_1} \ldots X_{\tau_{2k-2}} B X_{\tau_{2k-1}} \ldots X_{\tau_{2n-1}} \\
&\qquad - \sum_{\tau \in S_{2n-1}} \operatorname{sgn}(\tau) X_{\tau_1} \ldots X_{\tau_{2k-1}} B X_{\tau_{2k}} \ldots X_{\tau_{2n-1}} \Big) \\
=&\frac{1}{2} \sum_{k=1}^{n} \Big( \sum_{\tau \in S_{2n-1}} \operatorname{sgn}(\tau) [X_{\tau_1}, X_{\tau_2}] \ldots X_{\tau_{2k-2}} B X_{\tau_{2k-1}} \ldots X_{\tau_{2n-1}} \\
&\qquad - \sum_{\tau \in S_{2n-1}} \operatorname{sgn}(\tau) [X_{\tau_1}, X_{\tau_2}] \ldots X_{\tau_{2k-1}} B X_{\tau_{2k}} \ldots X_{\tau_{2n-1}} \Big) \\
&\ldots \\
=&2^{-n+1} \sum_{k=1}^{n} \Big( \sum_{\tau \in S_{2n-1}} \operatorname{sgn}(\tau) [X_{\tau_1}, X_{\tau_2}] \ldots [X_{\tau_{2k-3}}, X_{\tau_{2k-2}}] B X_{\tau_{2k-1}} [X_{\tau_{2k}}, X_{\tau_{2k+1}}] \ldots [X_{\tau_{2n-2}}, X_{\tau_{2n-1}}] \\
&\quad - \sum_{\tau \in S_{2n-1}} \operatorname{sgn}(\tau) [X_{\tau_1}, X_{\tau_2}] \ldots [X_{\tau_{2k-3}}, X_{\tau_{2k-2}}] X_{\tau_{2k-1}} B [X_{\tau_{2k}}, X_{\tau_{2k+1}}] \ldots [X_{\tau_{2n-2}}, X_{\tau_{2n-1}}] \Big) \\
=&2^{-n+1} \sum_{k=1}^{n} \sum_{\tau \in S_{2n-1}} \operatorname{sgn}(\tau) \Big( [X_{\tau_1}, X_{\tau_2}] \ldots [X_{\tau_{2k-3}}, X_{\tau_{2k-2}}] B X_{\tau_{2k-1}} [X_{\tau_{2k}}, X_{\tau_{2k+1}}] \ldots [X_{\tau_{2n-2}}, X_{\tau_{2n-1}}] \\
&\quad - [X_{\tau_1}, X_{\tau_2}] \ldots [X_{\tau_{2k-3}}, X_{\tau_{2k-2}}] X_{\tau_{2k-1}} B [X_{\tau_{2k}}, X_{\tau_{2k+1}}] \ldots [X_{\tau_{2n-2}}, X_{\tau_{2n-1}}] \Big).
\end{aligned}$$

For each $k = 1, \dots, n$ and $\tau \in S_{2n-1}$, we **claim** that

$$\begin{aligned}&[X_{\tau_1}, X_{\tau_2}] \dots [X_{\tau_{2k-3}}, X_{\tau_{2k-2}}] B X_{\tau_{2k-1}} [X_{\tau_{2k}}, X_{\tau_{2k+1}}] \dots [X_{\tau_{2n-2}}, X_{\tau_{2n-1}}] \\ \sim & B X_{\tau_{2k-1}} [X_{\tau_{2k}}, X_{\tau_{2k+1}}] \dots [X_{\tau_{2n-2}}, X_{\tau_{2n-1}}][X_{\tau_1}, X_{\tau_2}] \dots [X_{\tau_{2k-3}}, X_{\tau_{2k-2}}],\end{aligned}$$

and

$$\begin{aligned}&[X_{\tau_1}, X_{\tau_2}] \dots [X_{\tau_{2k-3}}, X_{\tau_{2k-2}}] X_{\tau_{2k-1}} B [X_{\tau_{2k}}, X_{\tau_{2k+1}}] \dots [X_{\tau_{2n-2}}, X_{\tau_{2n-1}}] \\ \sim & B [X_{\tau_{2k}}, X_{\tau_{2k+1}}] \dots [X_{\tau_{2n-2}}, X_{\tau_{2n-1}}][X_{\tau_1}, X_{\tau_2}] \dots [X_{\tau_{2k-3}}, X_{\tau_{2k-2}}] X_{\tau_{2k-1}}.\end{aligned}$$

If $X_{\tau_{2k-1}} \in \{B_1, B_2, \dots, B_{2n}\}$, then we have

$$B, X_{\tau_{2k-1}} \in \mathcal{S}^p, \forall p > n, \quad [X_{\tau_i}, X_{\tau_j}] \in \mathcal{S}^p, \forall p > n, \forall i, j = 1, \dots, 2n-1.$$

By Remark 2.7 it is easy to see that the claim holds. If $X_{\tau_{2k-1}} \in \{A_1, \dots, A_{2n}\}$, then at least one of $X_{\tau_1}, \dots, X_{\tau_{2k-2}}, X_{\tau_{2k}}, \dots, X_{\tau_{2n-1}}$ is in $\{B_1, B_2, \dots, B_{2n}\}$. Thus at least one of the commutators

$$[X_{\tau_1}, X_{\tau_2}], \dots, [X_{\tau_{2k-3}}, X_{\tau_{2k-2}}], [X_{\tau_{2k}}, X_{\tau_{2k+1}}], \dots, [X_{\tau_{2n-2}}, X_{\tau_{2n-1}}]$$

is in $\mathcal{S}^p$ for some $p < n$. Again, by Remark 2.7, the claim also follows. Thus in both cases the claim holds. By the claim, we compute $[B, X_1, \dots, X_{2n-1}]$.

$$\begin{aligned}&[B, X_1, \dots, X_{2n-1}] \\ =&2^{-n+1} \sum_{k=1}^{n} \sum_{\tau \in S_{2n-1}} \operatorname{sgn}(\tau) \Big( [X_{\tau_1}, X_{\tau_2}] \dots [X_{\tau_{2k-3}}, X_{\tau_{2k-2}}] B X_{\tau_{2k-1}} [X_{\tau_{2k}}, X_{\tau_{2k+1}}] \dots [X_{\tau_{2n-2}}, X_{\tau_{2n-1}}] \\ &\quad - [X_{\tau_1}, X_{\tau_2}] \dots [X_{\tau_{2k-3}}, X_{\tau_{2k-2}}] X_{\tau_{2k-1}} B [X_{\tau_{2k}}, X_{\tau_{2k+1}}] \dots [X_{\tau_{2n-2}}, X_{\tau_{2n-1}}] \Big) \\ \sim&2^{-n+1} \sum_{k=1}^{n} \sum_{\tau \in S_{2n-1}} \operatorname{sgn}(\tau) B X_{\tau_{2k-1}} [X_{\tau_{2k}}, X_{\tau_{2k+1}}] \dots [X_{\tau_{2n-2}}, X_{\tau_{2n-1}}][X_{\tau_1}, X_{\tau_2}] \dots [X_{\tau_{2k-3}}, X_{\tau_{2k-2}}] \\ &- 2^{-n+1} \sum_{k=1}^{n} \sum_{\tau \in S_{2n-1}} \operatorname{sgn}(\tau) B [X_{\tau_{2k}}, X_{\tau_{2k+1}}] \dots [X_{\tau_{2n-2}}, X_{\tau_{2n-1}}][X_{\tau_1}, X_{\tau_2}] \dots [X_{\tau_{2k-3}}, X_{\tau_{2k-2}}] X_{\tau_{2k-1}} \\ =&2^{-n+1} \sum_{k=1}^{n} \sum_{\tau \in S_{2n-1}} \operatorname{sgn}(\tau) B X_{\tau_1} [X_{\tau_2}, X_{\tau_3}] \dots [X_{\tau_{2n-2}}, X_{\tau_{2n-1}}] \\ &- 2^{-n+1} \sum_{k=1}^{n} \sum_{\tau \in S_{2n-1}} \operatorname{sgn}(\tau) B [X_{\tau_1}, X_{\tau_2}] \dots [X_{\tau_{2n-3}}, X_{\tau_{2n-2}}] X_{\tau_{2n-1}} \\ =&\sum_{k=1}^{n} \sum_{\tau \in S_{2n-1}} \operatorname{sgn}(\tau) B X_{\tau_1} \dots X_{\tau_{2n-1}} - \sum_{k=1}^{n} \sum_{\tau \in S_{2n-1}} \operatorname{sgn}(\tau) B X_{\tau_1} \dots X_{\tau_{2n-1}} \\ =&0.\end{aligned}$$

Here the third-to-last equality is because the antisymmetric sums $\sum_{\tau \in S_{2n-1}} \operatorname{sgn}(\tau) \cdots$ are invariant under any even permutation. This completes the proof of Lemma 5.11. □

A verbatim repetition of the proof of Lemma 5.11 also proves the following.

**Corollary 5.12.** *Assume in addition to Hypotheses A that*

$$B_i \in \mathcal{S}^p, \qquad \text{for some } p < n.$$

*Suppose* $\{X_1, X_2, \ldots, X_{2n}\} \subset \{A_1, \ldots, A_{2n}, B_1, \ldots, B_{2n}\}$, *and at least one of* $X_1, X_2, \ldots, X_{2n} \in \{B_1, \ldots, B_{2n}\}$. *Then*

$$[X_1, X_2, \ldots, X_{2n}] \sim 0.$$

*Consequently,*

$$[C_1, C_2, \ldots, C_{2n}] - [A_1, A_2, \ldots, A_{2n}] \sim 0.$$

**Remark 5.13.** Notice that the assumption of Corollary 5.12 is equivalent to the following.

1. For $i, j = 1, \ldots, 2n$,

$$[A_i, A_j] \in \mathcal{S}^p, \quad \forall p > n.$$

2. For $i = 1, \ldots, 2n$,

$$B_i \in \mathcal{S}^p, \qquad \text{for some } p < n.$$

Thus Corollary 5.12 essentially says that the trace

$$\operatorname{Tr}[A_1, A_2, \ldots, A_{2n}]$$

is stable under any perturbation that belongs to $\mathcal{S}^p$ for some $p < n$.

**Proof of Proposition 5.10.** By definition, we have the following equation

$$[C_1, C_2, \ldots, C_{2n}] - [A_1, A_2, \ldots, A_{2n}] = \sum [X_1, X_2, \ldots, X_{2n}],$$

where the sum is taken over all tuples $(X_1, \ldots, X_{2n})$ such that each $X_i$ belongs to $\{A_i, B_i\}$, and at least one of $X_1, \ldots, X_{2n}$ belongs to $\{B_1, B_2, \ldots, B_{2n}\}$. For this tuple $(X_1, X_2, \ldots, X_{2n})$, we have the following expansion

$$
\begin{aligned}
&[X_1, X_2, \dots, X_{2n}] \\
=&\sum_{\tau\in S_{2n}} \operatorname{sgn}(\tau) X_{\tau_1} X_{\tau_2} \dots X_{\tau_{2n}} \\
=&2^{-n} \sum_{\tau\in S_{2n}} \operatorname{sgn}(\tau)[X_{\tau_1}, X_{\tau_2}][X_{\tau_3}, X_{\tau_4}]\dots[X_{\tau_{2n-1}}, X_{\tau_{2n}}].
\end{aligned}
$$

By Hypotheses A, each commutator $[X_i, X_j]$ is at least in $\mathcal{S}^p$ for any $p > n$, and when $X_i \in \{B_1, B_2, \dots, B_{2n}\}$, $[X_i, X_j] \in \mathcal{S}^p$ for some $p < n$, $\forall j = 1, \dots, 2n$. Thus by Lemma 2.5, each product

$$
[X_{\tau_1}, X_{\tau_2}][X_{\tau_3}, X_{\tau_4}]\dots[X_{\tau_{2n-1}}, X_{\tau_{2n}}]
$$

is in the trace class. So we obtain the following estimate

$$
[C_1, C_2, \dots, C_{2n}] - [A_1, A_2, \dots, A_{2n}] \in \mathcal{S}^1.
$$

On the other hand, we have the following equation

$$
\begin{aligned}
&[C_1, C_2, \dots, C_{2n}] - [A_1, A_2, \dots, A_{2n}] - \sum_{k=1}^{2n}[A_1, \dots, A_{k-1}, B_k, A_{k+1}, \dots, A_{2n}] \\
=&\sum [X_1, X_2, \dots, X_{2n}],
\end{aligned}
$$

where each $X_i \in \{A_i, B_i\}$, and at least two of $X_1, X_2, \dots, X_{2n}$ are in $\{B_1, B_2, \dots, B_{2n}\}$. By Lemma 5.11, each $[X_1, X_2, \dots, X_{2n}] \sim 0$. Thus we conclude that

$$
[C_1, C_2, \dots, C_{2n}] - [A_1, A_2, \dots, A_{2n}] - \sum_{k=1}^{2n}[A_1, \dots, A_{k-1}, B_k, A_{k+1}, \dots, A_{2n}] \sim 0.
$$

This completes the proof of Proposition 5.10. □

*5.3. Hypotheses B*

As explained in the beginning of Subsection 5.2. The goal of this subsection is to handle the rest of the perturbations from the level of their integral formulas.

Temporarily fix the notations

$$
A_i = \begin{cases} T_{f_i}^{(t+1,t)}, & \text{if } t > -1 \\ T_{f_i}^{(1,-1)}, & \text{if } t = -1 \end{cases}, \quad B_i = \begin{cases} X_{f_i}^{(t)} + Y_{\bar{R}f_i}^{(t)} + f_i(0)E_0, & \text{if } t > -1 \\ X_{f_i}^{(-1)} + Y_{\bar{R}f_i}^{(-1)} + Z_{\bar{R}^2 f_i}^{(-1)} + f_i(0)E_0, & \text{if } t = -1 \end{cases},
$$

and $C_i = A_i + B_i$. Then by Lemmas 5.5 and 5.7,

$$C_i = \begin{cases} T_{f_i}^{(t)}, & \text{if } t > -1, \\ T_{f_i}^{(-1)}, & \text{if } t = -1. \end{cases}$$

Thus the results of Lemmas 5.2 and 5.3 are equivalent to the property that

$$[C_1, C_2, \dots, C_{2n}] - [A_1, A_2, \dots, A_{2n}] \sim 0.$$

Suppose $X_w(z), Y_w(z)$ are measurable functions on $\mathbb{B}_n \times \mathbb{B}_n$. We introduce the following hypotheses.

**Hypotheses B:** For $t > -1$, we say that $X_w(z), Y_w(z)$ satisfy Hypotheses B at $t$, if there are $\epsilon > 0$ and $C > 0$ such that the following hold.

1. For each $w \in \mathbb{B}_n$, $X_w(z), Y_w(z)$ is holomorphic in $z$,
2. $|X_w(z)| \le C|w|^{-2n+\epsilon}|K_w^{(t)}(z)|$,
3. $|Y_w(z)| \le C|w|^{-2n-1+\epsilon}|K_w^{(t)}(z)|$.

For a function $f$ on $\mathbb{B}_n$, formally define the symboled integral operators

$$X_f h(z) = \int_{\mathbb{B}_n} h(w)f(w)X_w(z)\mathrm{d}\lambda_{t+1}(w), \quad Y_f h(z) = \int_{\mathbb{B}_n} h(w)f(w)Y_w(z)\mathrm{d}\lambda_{t+1}(w). \quad (5.6)$$

**Lemma 5.14.** *Assume Hypotheses B. Then for any $f \in \mathscr{C}^2(\overline{\mathbb{B}_n})$, the integral operators $X_f$ and $Y_{\bar{R}f}$ define bounded operators on $L^2_{a,t}(\mathbb{B}_n)$ that belong to $\mathcal{S}^p$ for any $p > n$.*

**Proof.** For $f \in \mathscr{C}^2(\overline{\mathbb{B}_n})$, we have

$$|f(w)| \lesssim 1, \quad |\bar{R}f(w)| \lesssim |w|.$$

Thus by Hypotheses B, we obtain the following estimates,

$$|f(w)X_w(z)| \lesssim |w|^{-2n+\epsilon}|K_w^{(t)}(z)|, \quad |\bar{R}f(w)Y_w(z)| \lesssim |w|^{-2n+\epsilon}|K_w^{(t)}(z)|.$$

Thus for any $h \in \mathrm{Hol}(\overline{\mathbb{B}_n})$, $X_f h(z), Y_{\bar{R}f}h(z)$ are defined pointwise. If $f$ has compact support contained in $\mathbb{B}_n$, then it is easy to see that $X_f, Y_{\bar{R}f}$ belong to $\mathcal{S}^p$ for any $p$. In general, we can always write $f = f_1 + f_2$, where $f_1$ has compact support in $\mathbb{B}_n$, and the support of $f_2$ is away from the origin. We might as well assume that $f$ itself has support away from zero. In this case, we have the following estimates

$$|f(w)X_w(z)| \lesssim |K_w^{(t)}(z)|, \quad |\bar{R}f(w)Y_w(z)| \lesssim |K_w^{(t)}(z)|.$$

Denote $G_w(z)$ to be either $f(w)X_w(z)$ or $\bar{R}f(w)Y_w(z)$. Then $G_w(z)$ is holomorphic in $z$, and

$$|G_w(z)| \lesssim |K_w^{(t)}(z)|.$$

Define

$$T_G h(z) = \int_{\mathbb{B}_n} h(w) G_w(z) \mathrm{d}\lambda_{t+1}(w).$$

Then $T_G$ equals either $X_f$ or $Y_{\bar{R}f}$. Split the map as follows.

$$T_G : L^2_{a,t}(\mathbb{B}_n) \xrightarrow{E_{t,2}} L^2_{a,t+2}(\mathbb{B}_n) \xrightarrow{\hat{T}_G} L^2_{a,t}(\mathbb{B}_n),$$

where $\hat{T}_G$ is defined by the same integral formula as $T_G$. By Lemma 3.1, $\hat{T}_G$ is bounded. By Lemma 3.4, $E_{t,2} \in \mathcal{S}^p$ for all $p > n$. Therefore $T_G \in \mathcal{S}^p$ for all $p > n$. This completes the proof of Lemma 5.14. $\square$

The main result of this subsection is the following.

**Proposition 5.15.** *Suppose $t > -1$ and $X_w(z), Y_w(z)$ satisfy Hypotheses B at $t$. Suppose that $f_1, f_2, \ldots, f_{2n} \in \mathscr{C}^2(\overline{\mathbb{B}_n})$, and $K_1, K_2, \ldots, K_{2n}$ are bounded operators on $L^2_{a,t}(\mathbb{B}_n)$ that belong to $\mathcal{S}^p$ for some $p < n$. Denote*

$$\hat{A}_i = T_{f_i}^{(t+1,t)}, \quad \hat{B}_i = X_{f_i} + Y_{\bar{R}f_i} + K_i, \quad \hat{C}_i = \hat{A}_i + \hat{B}_i, \quad i = 1, \ldots, 2n.$$

*Then the operator on $L^2_{a,t}(\mathbb{B}_n)$*

$$[\hat{C}_1, \hat{C}_2, \ldots, \hat{C}_{2n}] - [\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_{2n}] \sim 0.$$

Some preparations are needed before proving Proposition 5.15.

**Lemma 5.16.** *Suppose $t > -1$ and $f$, $g$ are Lipschitz functions on $\mathbb{B}_n$. Then*

$$[T_f^{(t+1,t)}, T_g^{(t+1,t)}] \in \mathcal{S}^p, \quad \forall p > n.$$

**Proof.** Denote $P^{(t+1,t)}$ to be the restriction of $P^{(t+1)}$ to $L^2(\lambda_t)$. By Lemma 3.1 it is easy to see that $P^{(t+1,t)}$ is bounded. Obviously, $P^{(t+1,t)}$ satisfies the following identities.

$$P^{(t)} P^{(t+1,t)} = P^{(t+1,t)}, \quad P^{(t+1,t)} P^{(t)} = P^{(t)}, \quad \left(P^{t+1,t}\right)^2 = P^{(t+1,t)}.$$

So we compute the commutator as follows.

$$\begin{aligned}
&[T_f^{(t+1,t)}, T_g^{(t+1,t)}] \\
=&[P^{(t+1,t)} M_f P^{(t)}, P^{(t+1,t)} M_g P^{(t)}] \\
=&P^{(t+1,t)} M_f P^{(t+1,t)} M_g P^{(t)} - P^{(t+1,t)} M_g P^{(t+1,t)} M_f P^{(t)}
\end{aligned}$$

$$=P^{(t+1,t)}M_g\big(I-P^{(t+1,t)}\big)M_fP^{(t)}-P^{(t+1,t)}M_f\big(I-P^{(t+1,t)}\big)M_gP^{(t)}$$
$$=P^{(t+1,t)}[M_g,P^{(t+1,t)}][M_f,P^{t+1,t}]P^{(t)}-P^{(t+1,t)}[M_f,P^{(t+1,t)}][M_g,P^{t+1,t}]P^{(t)}.$$

By Corollary 4.3, we arrive at the following property,

$$[M_g,P^{(t+1,t)}],\quad [M_f,P^{(t+1,t)}]\in\mathcal{S}^p,\ \forall p>2n.$$

The Schatten-$p$ membership of the commutator $[T_f^{(t+1,t)},T_g^{(t+1,t)}]$ follows from the above property easily. This completes the proof of Lemma 5.16. □

**Lemma 5.17.** *Assume Hypotheses B. Then for any $f,g\in\mathscr{C}^2(\overline{\mathbb{B}_n})$, the commutators*

$$[X_f,T_g^{(t+1,t)}]\ \textit{and}\ [Y_{\bar{R}f},T_g^{(t+1,t)}]$$

*belong to $\mathcal{S}^p$ for some $p<n$.*

**Proof.** As in the proof of Lemma 5.14, we may assume the support of $f$ does not contain the origin. For any $h\in L^2_{a,t}(\mathbb{B}_n)$, since $X_w(z),Y_w(z)$ are holomorphic in $z$, it is easy to verify the following integral expression,

$$\Big(T_g^{(t+1,t)}X_f-X_{gf}\Big)h(\xi)=\int\limits_{\mathbb{B}_n^2}\big[g(z)-g(w)\big]f(w)h(w)K_z^{(t+1)}(\xi)X_w(z)\mathrm{d}\lambda_{t+1}(w)\mathrm{d}\lambda_{t+1}(z),$$

$$\Big(X_fT_g^{(t+1,t)}-X_{gf}\Big)h(\xi)=\int\limits_{\mathbb{B}_n^2}f(z)\big[g(w)-g(z)\big]h(w)X_z(\xi)K_w^{(t+1)}(z)\mathrm{d}\lambda_{t+1}(w)\mathrm{d}\lambda_{t+1}(z),$$

and similarly,

$$\Big(T_g^{(t+1,t)}Y_{\bar{R}f}-Y_{g\bar{R}f}\Big)h(\xi)=\int\limits_{\mathbb{B}_n^2}\big[g(z)-g(w)\big]\bar{R}f(w)h(w)K_z^{(t+1)}(\xi)Y_w(z)\mathrm{d}\lambda_{t+1}(w)\mathrm{d}\lambda_{t+1}(z),$$

$$\Big(Y_{\bar{R}f}T_g^{(t+1,t)}-Y_{g\bar{R}f}\Big)h(\xi)=\int\limits_{\mathbb{B}_n^2}\bar{R}f(z)\big[g(w)-g(z)\big]h(w)Y_z(\xi)K_w^{(t+1)}(z)\mathrm{d}\lambda_{t+1}(w)\mathrm{d}\lambda_{t+1}(z).$$

As in the proof of Lemma 5.14, let $G_w(z)$ be either $f(w)X_w(z)$ or $\bar{R}f(w)Y_w(z)$. Write

$$Th(z)=\int\limits_{\mathbb{B}_n}[g(z)-g(w)]h(w)G_w(z)\mathrm{d}\lambda_{t+1}(w),$$

$$Sh(z)=\int\limits_{\mathbb{B}_n}\big[g(w)-g(z)\big]h(w)K_w^{(t+1)}(z)\mathrm{d}\lambda_{t+1}(w),$$

and

$$Wh(\xi) = \int_{\mathbb{B}_n} h(z) G_z(\xi) \mathrm{d}\lambda_{t+1}(z).$$

Then by the above, we have

$$T_g^{(t+1,t)} X_f - X_{gf}, \text{ or } T_g^{(t+1,t)} Y_{\bar{R}f} - Y_{g\bar{R}f} = P^{(t+1)} T,$$
$$X_f T_g^{(t+1,t)} - X_{gf}, \text{ or } Y_{\bar{R}f} T_g^{(t+1,t)} - Y_{g\bar{R}f} = WS.$$

Take $\epsilon > 0$ small enough. Split $P^{(t+1)}T$ and $WS$ as the composition of the following operators.

$$P^{(t+1)}T : L^2_{a,t}(\mathbb{B}_n) \xrightarrow{E_{t,2+\epsilon}} L^2_{a,t+2+\epsilon} \xrightarrow{T} L^2(\lambda_t) \xrightarrow{P^{(t+1)}} L^2_{a,t}(\mathbb{B}_n),$$
$$WS : L^2_{a,t}(\mathbb{B}_n) \xrightarrow{E_{t,2+\epsilon}} L^2_{a,t+2+\epsilon} \xrightarrow{S} L^2(\lambda_{t+2}) \xrightarrow{W} L^2(\lambda_t).$$

By Lemma 2.2, we have the following inequalities,

$$|g(z) - g(w)| \lesssim |z - w| \lesssim |1 - \langle z, w\rangle|^{1/2}.$$

By Lemma 3.1, $T, P^{(t+1)}, S, W$ define bounded operators between the spaces indicated above. Again, by Lemma 3.4, $E_{t,2+\epsilon} \in \mathcal{S}^p$ for some $p < n$. Thus altogether, the operators

$$\left(T_g^{(t+1,t)} X_f - X_{gf}\right), \left(T_g^{(t+1,t)} Y_{\bar{R}f} - Y_{g\bar{R}f}\right), \left(X_f T_g^{(t+1,t)} - X_{gf}\right), \left(Y_{\bar{R}f} T_g^{(t+1,t)} - Y_{g\bar{R}f}\right)$$

are in $\mathcal{S}^p$ for some $p < n$. So are the commutators

$$[T_g^{(t+1,t)}, X_f] = \left(T_g^{(t+1,t)} X_f - X_{gf}\right) - \left(X_f T_g^{(t+1,t)} - X_{gf}\right),$$

and

$$[T_g^{(t+1,t)}, Y_{\bar{R}f}] = \left(T_g^{(t+1,t)} Y_{\bar{R}f} - Y_{g\bar{R}f}\right) - \left(Y_{\bar{R}f} T_g^{(t+1,t)} - Y_{g\bar{R}f}\right).$$

This completes the proof of Lemma 5.17. □

From Lemmas 5.14, 5.16 and 5.17 it follows that if we set

$$A_i = T_{f_i}^{(t+1,t)}, \quad B_i = X_{f_i} + Y_{\bar{R}f_i} + K_i, \quad i = 1, 2, \ldots, 2n,$$

then the operators $\{A_i, B_i\}$ satisfy Hypotheses A defined Subsection 5.2. Thus by Proposition 5.10, the proof of Proposition 5.15 reduces to the proof of

$$\mathrm{Tr}\bigg(\sum_{k=1}^{2n}[A_1,\ldots,A_{k-1},B_k,A_{k+1},\ldots,A_{2n}]\bigg)=0.$$

Each $B_k$ splits into the sum of $X_{f_k}$, $Y_{\bar{R}f_k}$ and $K_k$. The part with $K_k$ can be handled by Corollary 5.12 as $K_k$ is assumed to belong to $\mathcal{S}^p$ for some $p<n$. Thus it remains to prove

$$\mathrm{Tr}\bigg(\sum_{k=1}^{2n}[A_1,\ldots,A_{k-1},X_{f_k}+Y_{\bar{R}f_k},A_{k+1},\ldots,A_{2n}]\bigg)=0. \tag{5.7}$$

The operators $X_{f_k}+Y_{\bar{R}f_k}$ generally do not belong to $\mathcal{S}^p$ for $p<n$. Thus Corollary 5.12 do not apply and it is hard to handle this trace at operator-theoretic level. Instead, we need to treat them as integral operators. Let us explain the idea of the proof. The operator in (5.7) is a sum of compositions of $2n$ integral operators. By [47, Lemma 2.5] and the definitions of $A_i, X_{f_k}, Y_{\bar{R}f_k}$, we can write the trace in (5.7) into a $(2n+1)$-fold integral of the form

$$\int_{\mathbb{B}_n}\bigg[\int_{\mathbb{B}_n^{2n}}K_\xi^{(t)}(z_1)G(z_1,z_2,\ldots,z_{2n},\xi)\mathrm{d}\lambda_{t+1}(z_1)\ldots\mathrm{d}\lambda_{t+1}(z_{2n})\bigg]\mathrm{d}\lambda_t(\xi),$$

where $G(z_1,z_2,\ldots,z_{2n},\xi)$ is holomorphic in $\xi$. If the $(2n+1)$-fold integral converges absolutely, then we can apply Fubini's Theorem and get that the above integral is equal to

$$\int_{\mathbb{B}_n^{2n}}G(z_1,z_2,\ldots,z_{2n},\xi)\big|_{\xi=z_1}\mathrm{d}\lambda_{t+1}(z_1)\ldots\mathrm{d}\lambda_{t+1}(z_{2n}).$$

The anti-symmetrization will then tell us that the above equals zero as $\xi=z_1$. Thus the proof of Proposition 5.15 reduces to reorganize the parts in (5.7) so that each part converges absolutely.

The following lemma helps us recognizing absolutely integrable terms in a multi-fold integral.

**Lemma 5.18.** *Suppose $k$ is a positive integer and $t>-1$. Then*

$$\int_{\mathbb{B}_n^k}\Big|K_{z_1}^{(t)}(z_2)\ldots K_{z_{k-1}}^{(t)}(z_k)\Big|\mathrm{d}\lambda_t(z_1)\ldots\mathrm{d}\lambda_t(z_k)<\infty; \tag{5.8}$$

*and for any $\epsilon>0$,*

$$\int_{\mathbb{B}_n^{k+1}}\Big(\sum_{i,j}|1-\langle z_i,z_j\rangle|\Big)^{n+\epsilon}\Big|K_{z_0}^{(t)}(z_1)K_{z_1}^{(t)}(z_2)\ldots K_{z_{k-1}}^{(t)}(z_k)K_{z_k}^{(t)}(z_0)\Big|\mathrm{d}\lambda_t(z_0)\ldots\mathrm{d}\lambda_t(z_k)$$
$$<\infty. \tag{5.9}$$

**Proof.** First, we notice that (5.8) is a special case of (5.9): take $\epsilon = t+1$, then

$$\begin{aligned}&\int_{\mathbb{B}_n^k}\left|K_{z_1}^{(t)}(z_2)\dots K_{z_{k-1}}^{(t)}(z_k)\right|\mathrm{d}\lambda_t(z_1)\dots\mathrm{d}\lambda_t(z_k)\\&=\int_{\mathbb{B}_n^k}|1-\langle z_1,z_k\rangle|^{n+t+1}\left|K_{z_1}^{(t)}(z_2)\dots K_{z_{k-1}}^{(t)}(z_k)K_{z_k}^{(t)}(z_1)\right|\mathrm{d}\lambda_t(z_1)\dots\mathrm{d}\lambda_t(z_k)\\&\le\int_{\mathbb{B}_n^k}\left(\sum_{i,j}|1-\langle z_i,z_j\rangle|\right)^{n+t+1}\left|K_{z_1}^{(t)}(z_2)\dots K_{z_{k-1}}^{(t)}(z_k)K_{z_k}^{(t)}(z_1)\right|\mathrm{d}\lambda_t(z_1)\dots\mathrm{d}\lambda_t(z_k).\end{aligned}$$

Thus it suffices to prove (5.9). It is well-known that all $l^p$ norms on a finite set are equivalent. So by [44, Proposition 5.1.2] and Lemma 2.2 (7), we obtain the following estimates,

$$|1-\langle z_i,z_j\rangle|=\left(|1-\langle z_i,z_j\rangle|^{1/2}\right)^2\lesssim\left(\sum_{s=0}^{k-1}|1-\langle z_s,z_{s+1}\rangle|^{1/2}\right)^2\lesssim\sum_{s=0}^{k-1}|1-\langle z_s,z_{s+1}\rangle|,\forall i,j,$$

and

$$\left(\sum_{i,j}|1-\langle z_i,z_j\rangle|\right)^{n+\epsilon}\lesssim\left(\sum_{s=0}^{k-1}|1-\langle z_s,z_{s+1}\rangle|\right)^{n+\epsilon}\lesssim\sum_{s=0}^{k-1}|1-\langle z_s,z_{s+1}\rangle|^{n+\epsilon}.$$

Also, for any $s\ge 1$, take the rotation in variables $z_s\mapsto z_0, z_{s+1}\mapsto z_1,\dots,z_k\mapsto z_{k-s}, z_0\mapsto z_{k-s+1}, z_1\mapsto z_{k-s+2},\dots,z_{s-1}\mapsto z_k$, then we have the following equation,

$$\begin{aligned}&\int_{\mathbb{B}_n^{k+1}}|1-\langle z_s,z_{s+1}\rangle|^{n+\epsilon}\left|K_{z_0}^{(t)}(z_1)K_{z_1}^{(t)}(z_2)\dots K_{z_{k-1}}^{(t)}(z_k)K_{z_k}^{(t)}(z_0)\right|\mathrm{d}\lambda_t(z_0)\dots\mathrm{d}\lambda_t(z_k)\\&=\int_{\mathbb{B}_n^{k+1}}|1-\langle z_0,z_1\rangle|^{n+\epsilon}\left|K_{z_0}^{(t)}(z_1)K_{z_1}^{(t)}(z_2)\dots K_{z_{k-1}}^{(t)}(z_k)K_{z_k}^{(t)}(z_0)\right|\mathrm{d}\lambda_t(z_0)\dots\mathrm{d}\lambda_t(z_k).\end{aligned}$$

Notice that the part of the reproducing kernels is invariant under the above rotation of variables. Thus it suffices to prove the integral above is finite. Without loss of generality, assume $0<\epsilon<1+t$. We prove it by induction. For $k=1$, by Lemma 2.3 (1), we compute the integral as follows.

$$\begin{aligned}&\int_{\mathbb{B}_n^2}|1-\langle z_0,z_1\rangle|^{n+\epsilon}\left|K_{z_0}^{(t)}(z_1)K_{z_1}^{(t)}(z_0)\right|\mathrm{d}\lambda_t(z_0)\mathrm{d}\lambda_t(z_1)\\&=\int_{\mathbb{B}_n^2}\frac{1}{|1-\langle z_0,z_1\rangle|^{2(n+1+t)-n-\epsilon}}\mathrm{d}\lambda_t(z_0)\mathrm{d}\lambda_t(z_1)\end{aligned}$$

$$
\begin{aligned}
&\lesssim \int_{\mathbb{B}_n} (1-|z_1|^2)^{\epsilon-(1+t)} \mathrm{d}\lambda_t(z_1) \\
&\lesssim \int_{\mathbb{B}_n} (1-|z_1|^2)^{\epsilon-1} \mathrm{d}m(z_1) \\
&<\infty.
\end{aligned}
$$

Thus (5.9) holds for $k=1$. Suppose (5.9) holds for $k-1$. Then by Lemma 2.3 (2), we have the following estimates by the induction step.

$$
\begin{aligned}
&\int_{\mathbb{B}_n^{k+1}} |1-\langle z_0, z_1\rangle|^{n+\epsilon} \left| K_{z_0}^{(t)}(z_1) K_{z_1}^{(t)}(z_2) \dots K_{z_{k-1}}^{(t)}(z_k) K_{z_k}^{(t)}(z_0) \right| \mathrm{d}\lambda_t(z_0) \dots \mathrm{d}\lambda_t(z_k) \\
&= \int_{\mathbb{B}_n^k} \left( \int_{\mathbb{B}_n} \frac{1}{|1-\langle z_0, z_1\rangle|^{1+t-\epsilon} |1-\langle z_0, z_k\rangle|^{n+1+t}} \mathrm{d}\lambda_t(z_0) \right) \\
&\qquad \cdot \left| K_{z_1}^{(t)}(z_2) \dots K_{z_{k-1}}^{(t)}(z_k) \right| \mathrm{d}\lambda_t(z_1) \dots \mathrm{d}\lambda_t(z_k) \\
&\lesssim \int_{\mathbb{B}_n^k} \frac{1}{|1-\langle z_1, z_k\rangle|^{1+t-\epsilon/2}} \left| K_{z_1}^{(t)}(z_2) \dots K_{z_{k-1}}^{(t)}(z_k) \right| \mathrm{d}\lambda_t(z_1) \dots \mathrm{d}\lambda_t(z_k) \\
&= \int_{\mathbb{B}_n^k} |1-\langle z_1, z_k\rangle|^{n+\epsilon/2} \left| K_{z_1}^{(t)}(z_2) \dots K_{z_{k-1}}^{(t)}(z_k) K_{z_k}^{(t)}(z_1) \right| \mathrm{d}\lambda_t(z_1) \dots \mathrm{d}\lambda_t(z_k) \\
&<\infty.
\end{aligned}
$$

This completes the proof of Lemma 5.18. □

**Proof of Proposition 5.15.** For simplicity of notation, in this proof let us write $A_i$ for $\hat{A}_i$, $B_i$ for $\hat{B}_i$, and $C_i$ for $\hat{C}_i$. The fact that $\{A_i, B_i\}_{i=1}^{2n}$ satisfy Hypotheses A follows from Lemmas 5.14 and 5.17. Thus by Proposition 5.10, we have the following property, i.e.

$$
[C_1, C_2, \dots, C_{2n}] - [A_1, A_2, \dots, A_{2n}] \in \mathcal{S}^1,
$$

and

$$
[C_1, C_2, \dots, C_{2n}] - [A_1, A_2, \dots, A_{2n}] - \sum_{l=1}^{2n} [A_1, \dots, A_{l-1}, B_l, A_{l+1}, \dots, A_{2n}] \sim 0.
$$

Recall that

$$
B_i = X_{f_i} + Y_{\bar{R} f_i} + K_i, \quad i = 1, 2, \dots, 2n.
$$

By Corollary 5.12, each operator

$$[A_1, \ldots, A_{l-1}, K_l, A_{l+1}, \ldots, A_{2n}] \sim 0.$$

Denote

$$X = \sum_{l=1}^{2n} [A_1, \ldots, A_{l-1}, X_{f_l}, A_{l+1}, \ldots, A_{2n}],$$

and

$$Y = \sum_{l=1}^{2n} [A_1, \ldots, A_{l-1}, Y_{\bar{R}f_l}, A_{l+1}, \ldots, A_{2n}].$$

Then it follows from Proposition 5.10 that $X$ and $Y$ are trace class operators on $L^2_{a,t}(\mathbb{B}_n)$. It remains to show that

$$X \sim 0, \quad Y \sim 0.$$

**Notation 5.19.** For two functions $H(\xi), G(\xi)$ on $\mathbb{B}_n$, write $H \simeq G$ if $\int_{\mathbb{B}_n} (H(\xi) - G(\xi)) \mathrm{d}\lambda_t(\xi) = 0$.

By [47, Lemma 2.5], it suffices to show

$$\langle XK_\xi^{(t)}, K_\xi^{(t)} \rangle \simeq 0, \quad \text{and} \quad \langle YK_\xi^{(t)}, K_\xi^{(t)} \rangle \simeq 0. \tag{5.10}$$

A moment of reflection shows that

$$X = \sum_{k=1}^{2n} \sum_{\tau \in S_{2n}} \operatorname{sgn}(\tau) A_{\tau_1} \ldots A_{\tau_{k-1}} X_{f_{\tau_k}} A_{\tau_{k+1}} \ldots A_{\tau_{2n}},$$

and

$$Y = \sum_{k=1}^{2n} \sum_{\tau \in S_{2n}} \operatorname{sgn}(\tau) A_{\tau_1} \ldots A_{\tau_{k-1}} Y_{\bar{R}f_{\tau_k}} A_{\tau_{k+1}} \ldots A_{\tau_{2n}}.$$

For $k = 1, \ldots, 2n$, denote

$$X_k = \sum_{\tau \in S_{2n}} \operatorname{sgn}(\tau) A_{\tau_1} \ldots A_{\tau_{k-1}} X_{f_{\tau_k}} A_{\tau_{k+1}} \ldots A_{\tau_{2n}},$$

and

$$Y_k = \sum_{\tau \in S_{2n}} \operatorname{sgn}(\tau) A_{\tau_1} \ldots A_{\tau_{k-1}} Y_{\bar{R}f_{\tau_k}} A_{\tau_{k+1}} \ldots A_{\tau_{2n}}.$$

Then we have the following expressions for $X$ and $Y$,

$$X = \sum_{k=1}^{2n} X_k, \quad Y = \sum_{k=1}^{2n} Y_k.$$

Define

$$X'_k = \sum_{\tau \in S_{2n}} \operatorname{sgn}(\tau) A_{\tau_{2n}} A_{\tau_1} \dots A_{\tau_{k-1}} X_{f_{\tau_k}} A_{\tau_{k+1}} \dots A_{\tau_{2n-1}}, \quad k = 1, \dots, 2n-1,$$

$$X'_{2n} = \sum_{\tau \in S_{2n}} \operatorname{sgn}(\tau) X_{f_{\tau_{2n}}} A_{\tau_1} \dots A_{\tau_{2n-1}},$$

and

$$Y'_k = \sum_{\tau \in S_{2n}} \operatorname{sgn}(\tau) A_{\tau_{2n}} A_{\tau_1} \dots A_{\tau_{k-1}} Y_{\bar{R} f_{\tau_k}} A_{\tau_{k+1}} \dots A_{\tau_{2n-1}}, \quad k = 1, \dots, 2n-1,$$

$$Y'_{2n} = \sum_{\tau \in S_{2n}} \operatorname{sgn}(\tau) Y_{\bar{R} f_{\tau_{2n}}} A_{\tau_1} \dots A_{\tau_{2n-1}}.$$

In other words, $X'_k, Y'_k$ are obtained from $X_k, Y_k$ by moving each rightmost operator to the leftmost. Define

$$X' = \sum_{k=1}^{2n} X'_k, \quad Y' = \sum_{k=1}^{2n} Y'_k.$$

The anti-symmetrization leads to

$$X' = -X, \quad Y' = -Y. \tag{5.11}$$

Below we show the following.

**Lemma 5.20.**

$$\langle X K_\xi^{(t)}, K_\xi^{(t)} \rangle \simeq \langle X' K_\xi^{(t)}, K_\xi^{(t)} \rangle, \quad \langle Y K_\xi^{(t)}, K_\xi^{(t)} \rangle \simeq \langle Y' K_\xi^{(t)}, K_\xi^{(t)} \rangle.$$

**Proof.** Each $f_i$ has a decomposition $f_i = g_i + h_i$, where $g_i$ has compact support, and $h_i$ has support away from 0. Since $X_{g_i}, Y_{\bar{R} g_i}$ are perturbations that belong to the trace class, we might as well assume that each $f_i$ have support away from the origin.

For $k, i = 1, \dots, 2n$, write

$$X_{k,i}(w, z) = \begin{cases} X_w(z), & \text{if } i = k, \\ K_w^{(t+1)}(z), & \text{otherwise.} \end{cases}$$

By definition, for any $k = 1, \dots, 2n$, we compute $\langle X_k K_\xi^{(t)}, K_\xi^{(t)} \rangle$ as follows.

$$
\begin{aligned}
&\langle X_k K_\xi^{(t)}, K_\xi^{(t)}\rangle \\
=&\Big[\sum_{\tau\in S_{2n}} \operatorname{sgn}(\tau) T^{(t+1,t)}_{f_{\tau_1}} \dots T^{(t+1,t)}_{f_{\tau_{k-1}}} X_{f_{\tau_k}} T^{(t+1,t)}_{f_{\tau_{k+1}}} \dots T^{(t+1,t)}_{f_{\tau_{2n}}} K_\xi^{(t)}\Big](\xi) \\
=&\sum_{\tau\in S_{2n}} \operatorname{sgn}(\tau) \int\limits_{\mathbb{B}_n^{2n}} f_{\tau_1}(z_1)\dots f_{\tau_{2n}}(z_{2n}) K_\xi^{(t)}(z_{2n}) \\
&\quad \cdot X_{k,2n}(z_{2n}, z_{2n-1}) X_{k,2n-1}(z_{2n-1}, z_{2n-2}) \dots X_{k,2}(z_2,z_1) X_{k,1}(z_1,\xi) \mathrm{d}\lambda_{t+1}(z_{2n}) \dots \mathrm{d}\lambda_{t+1}(z_1) \\
=&\int\limits_{\mathbb{B}_n^{2n}} \det\big[f_i(z_j)\big] K_\xi^{(t)}(z_{2n}) \Big(\prod_{i=2}^{2n} X_{k,i}(z_i, z_{i-1})\Big) X_{k,1}(z_1,\xi) \mathrm{d}\lambda_{t+1}(z_{2n}) \dots \mathrm{d}\lambda_{t+1}(z_1).
\end{aligned} \tag{5.12}
$$

Denote $F$ the column vector of $2n$ functions,

$$F(z) = [f_1(z)\ f_2(z)\ \dots\ f_{2n}(z)]^T.$$

Then by Lemma 2.2, we estimate the determinant function,

$$
\begin{aligned}
\Big|\det\big[f_i(z_j)\big]\Big| &= \Big|\det\big[F(z_1)\ \dots\ F(z_{2n})\big]\Big| \\
&= \Big|\det\big[F(z_1)\ F(z_2)-F(z_1)\ \dots\ F(z_{2n})-F(z_1)\big]\Big| \\
&\lesssim \Big(\sum_{i,j} |z_i - z_j|\Big)^{2n-1} \\
&\lesssim \Big(\sum_{i,j} |1-\langle z_i, z_j\rangle|\Big)^{n-1/2}.
\end{aligned}
$$

Also, we obtain the following estimate

$$
|X_{k,i}(w,z)| \lesssim \begin{cases} |w|^{-2n+\epsilon} |K_w^{(t)}(z)|, & i = k, \\ |K_w^{(t+1)}(z)| = \frac{1}{|1-\langle z,w\rangle|} \cdot |K_w^{(t)}(z)|, & i \neq k. \end{cases}
$$

Since we assume that each $f_i$ is supported away from the origin, we continue computing the inner product $\langle X_k K_\xi^{(t)}, K_\xi^{(t)}\rangle$ using Lemma 2.2 and the above estimate.

$$
\begin{aligned}
&\int\limits_{\mathbb{B}_n^{2n+1}} \Big|\det[f_i(z_j)] K_\xi^{(t)}(z_{2n}) \Big(\prod_{i=2}^{2n} X_{k,i}(z_i,z_{i-1})\Big) X_{k,1}(z_1,\xi)\Big| \mathrm{d}\lambda_{t+1}(z_{2n}) \dots \mathrm{d}\lambda_{t+1}(z_1) \mathrm{d}\lambda_t(\xi) \\
\lesssim &\int\limits_{\mathbb{B}_n^{2n+1}} \Big|\det[f_i(z_j)] K_\xi^{(t)}(z_{2n}) \Big(\prod_{i\neq k,1} \frac{1-|z_i|^2}{|1-\langle z_i, z_{i-1}\rangle|}\Big)(1-|z_k|^2) \frac{1-|z_1|^2}{|1-\langle z_1,\xi\rangle|} \\
&\qquad \cdot \Big(\prod_{i=2}^{2n} K_{z_i}^{(t)}(z_{i-1})\Big) K_{z_1}^{(t)}(\xi)\Big| \mathrm{d}\lambda_t(z_{2n}) \dots \mathrm{d}\lambda_t(z_1) \mathrm{d}\lambda_t(\xi)
\end{aligned}
$$

$$\lesssim \int_{\mathbb{B}_n^{2n+1}} \Big(\sum_{i,j} |1-\langle z_i, z_j\rangle|\Big)^{n+\frac{1}{2}} \Big| K_\xi^{(t)}(z_{2n}) \Big(\prod_{i=2}^{2n} K_{z_i}^{(t)}(z_{i-1})\Big) K_{z_1}^{(t)}(\xi)\Big| \mathrm{d}\lambda_t(z_{2n})\ldots \mathrm{d}\lambda_t(z_1)\mathrm{d}\lambda_t(\xi)$$

$$<\infty.$$

Here the last inequality follows from Lemma 5.18. Thus the $(2n+1)$-fold integral obtained by plugging (5.12) into $\int_{\mathbb{B}_n} \langle X_k K_\xi^{(t)}, K_\xi^{(t)}\rangle \mathrm{d}\lambda_t(\xi)$ converges absolutely. By Fubini's Theorem, the variable $\xi$ can be integrated first, and, since each $X_{k,i}(w,z)$ is holomorphic in $z$, we compute the following integral

$$\begin{aligned}
&\int_{\mathbb{B}_n} \langle X_k K_\xi^{(t)}, K_\xi^{(t)}\rangle \mathrm{d}\lambda_t(\xi) \\
&= \int_{\mathbb{B}_n} \Big\{ \int_{\mathbb{B}_n^{2n}} \det\big[f_i(z_j)\big] K_\xi^{(t)}(z_{2n}) \\
&\qquad \Big(\prod_{i=2}^{2n} X_{k,i}(z_i, z_{i-1})\Big) X_{k,1}(z_1,\xi)\mathrm{d}\lambda_{t+1}(z_{2n})\ldots \mathrm{d}\lambda_{t+1}(z_1)\Big\} \mathrm{d}\lambda_t(\xi) \\
&= \int_{\mathbb{B}_n^{2n}} \det\big[f_i(z_j)\big] \Big(\prod_{i=2}^{2n} X_{k,i}(z_i, z_{i-1})\Big) X_{k,1}(z_1, z_{2n})\mathrm{d}\lambda_{t+1}(z_{2n})\ldots \mathrm{d}\lambda_{t+1}(z_1).
\end{aligned}$$

By a similar proof, we have a similar expression for the integral of $\langle X_k' K_\xi^{(t)}, K_\xi^{(t)}\rangle$.

$$\begin{aligned}
&\int_{\mathbb{B}_n} \langle X_k' K_\xi^{(t)}, K_\xi^{(t)}\rangle \mathrm{d}\lambda_t(\xi) \\
&= \int_{\mathbb{B}_n} \Big\{ \int_{\mathbb{B}_n^{2n}} \det[f_i(z_j)] K_\xi^{(t)}(z_{2n-1}) \\
&\qquad \cdot \Big(\prod_{i=2}^{2n-1} X_{k,i}(z_i, z_{i-1})\Big) X_{k,1}(z_1, z_{2n}) X_{k,2n}(z_{2n},\xi)\mathrm{d}\lambda_{t+1}(z_{2n})\ldots \mathrm{d}\lambda_{t+1}(z_1)\Big\} \mathrm{d}\lambda_t(\xi) \\
&= \int_{\mathbb{B}_n^{2n}} \det[f_i(z_j)] \Big(\prod_{i=2}^{2n} X_{k,i}(z_i, z_{i-1})\Big) X_{k,1}(z_1, z_{2n})\mathrm{d}\lambda_{t+1}(z_{2n})\ldots \mathrm{d}\lambda_{t+1}(z_1) \\
&= \int_{\mathbb{B}_n} \langle X_k K_\xi^{(t)}, K_\xi^{(t)}\rangle \mathrm{d}\lambda_t(\xi).
\end{aligned}$$

In other words, we have shown the following equation

$$\langle X_k K_\xi^{(t)}, K_\xi^{(t)}\rangle \simeq \langle X_k' K_\xi^{(t)}, K_\xi^{(t)}\rangle. \tag{5.13}$$

The situation for $Y$ is more complicated. As in the case of $X$, for $k, i = 1, \dots, 2n$, define

$$Y_{k,i}(w,z) = \begin{cases} Y_w(z), & i = k, \\ K_w^{(t+1)}(z), & \text{otherwise.} \end{cases}$$

Then each $Y_{k,i}(w,z)$ is holomorphic in $z$, and

$$|Y_{k,i}(w,z)| \lesssim \begin{cases} |w|^{-2n-1+\epsilon} |K_w^{(t)}(z)|, & i = k, \\ \frac{1}{|1-\langle w,z\rangle|} \cdot |K_w^{(t)}(z)|, & \text{otherwise.} \end{cases}$$

By definition, we compute

$$\langle Y_k K_\xi^{(t)}, K_\xi^{(t)} \rangle = \int_{\mathbb{B}_n^{2n}} \det \left[ F(z_1) \ \dots \ F(z_{k-1}) \ \bar{R}F(z_k) \ F(z_{k+1}) \ \dots \ F(z_{2n}) \right] \\ K_\xi^{(t)}(z_{2n}) \left( \prod_{i=2}^{2n} Y_{k,i}(z_i, z_{i-1}) \right) Y_{k,1}(z_1, \xi) \mathrm{d}\lambda_{t+1}(z_{2n}) \dots \mathrm{d}\lambda_{t+1}(z_1). \tag{5.14}$$

Here $\bar{R}F(z)$ is the column vector

$$\bar{R}F(z) = [\bar{R}f_1(z) \ \dots \ \bar{R}f_{2n}(z)]^T.$$

Direct computation shows that

$$\begin{aligned} &\det \left[ F(z_1) \dots F(z_{k-1}) \bar{R}F(z_k) F(z_{k+1}) \dots F(z_{2n}) \right] \\ =&\det \left[ F(z_1) - F(z_k) \dots F(z_{k-1}) - F(z_k) \bar{R}F(z_k) F(z_{k+1}) - F(z_k) \dots F(z_{2n}) - F(z_k) \right] \\ &+ \sum_{j \neq k} \det F_{j,k}, \end{aligned}$$

where $F_{j,k}$ is the matrix function obtained by replacing the $j$-th column of

$$\left[ F(z_1) \ \dots \ F(z_{k-1}) \ \bar{R}F(z_k) \ F(z_{k+1}) \ \dots \ F(z_{2n}) \right]$$

into $F(z_k)$. Therefore we can compute its determinant as follows.

$$\begin{aligned} &\det \left[ F(z_1) \dots F(z_{k-1}) \bar{R}F(z_k) F(z_{k+1}) \dots F(z_{2n}) \right] \\ =&\det \left[ F(z_1) - F(z_k) \dots F(z_{k-1}) - F(z_k) \bar{R}F(z_k) F(z_{k+1}) - F(z_k) \dots F(z_{2n}) - F(z_k) \right] \\ &+ \sum_{j \neq k} \sum_{\tau \in S_{2n}} \operatorname{sgn}(\tau) f_{\tau_j}(z_k) \bar{R} f_{\tau_k}(z_k) \prod_{i \neq j,k} f_{\tau_i}(z_i) \\ :=& D_k(z_1, \dots, z_{2n}) + \sum_{j \neq k} E_{k,j}(z_1, \dots, z_{2n}), \end{aligned} \tag{5.15}$$

where

$$D_k(z_1, \dots, z_{2n}) = \\ \det \left[ F(z_1) - F(z_k) \dots F(z_{k-1}) - F(z_k) \bar{R} F(z_k) F(z_{k+1}) - F(z_k) \dots F(z_{2n}) - F(z_k) \right],$$

$$E_{k,j}(z_1, \dots, z_{2n}) = \sum_{\tau \in S_{2n}} \operatorname{sgn}(\tau) f_{\tau_j}(z_k) \bar{R} f_{\tau_k}(z_k) \prod_{i \neq j,k} f_{\tau_i}(z_i).$$

Correspondingly, we write

$$\langle Y_k K_\xi^{(t)}, K_\xi^{(t)} \rangle = I_k(\xi) + \sum_{j \neq k} II_{k,j}(\xi),$$

where

$$I_k(\xi) \\ = \int_{\mathbb{B}_n^{2n}} D_k(z_1, \dots, z_{2n}) K_\xi^{(t)}(z_{2n}) \left( \prod_{i=2}^{2n} Y_{k,i}(z_i, z_{i-1}) \right) Y_{k,1}(z_1, \xi) \mathrm{d}\lambda_{t+1}(z_{2n}) \dots \mathrm{d}\lambda_{t+1}(z_1),$$

and

$$II_{k,j}(\xi) \\ = \int_{\mathbb{B}_n^{2n}} E_{k,j}(z_1, \dots, z_{2n}) K_\xi^{(t)}(z_{2n}) \left( \prod_{i=2}^{2n} Y_{k,i}(z_i, z_{i-1}) \right) Y_{k,1}(z_1, \xi) \mathrm{d}\lambda_{t+1}(z_{2n}) \dots \mathrm{d}\lambda_{t+1}(z_1).$$

Similarly, we have the following expression for $\langle Y_k' K_\xi^{(t)}, K_\xi^{(t)} \rangle$,

$$\langle Y_k' K_\xi^{(t)}, K_\xi^{(t)} \rangle = I_k'(\xi) + \sum_{j \neq k} II_{k,j}'(\xi),$$

where

$$I_k'(\xi) \\ = \int_{\mathbb{B}_n^{2n}} D_k(z_1, \dots, z_{2n}) K_\xi^{(t)}(z_{2n-1}) \left( \prod_{i=2}^{2n-1} Y_{k,i}(z_i, z_{i-1}) \right) Y_{k,1}(z_1, z_{2n}) Y_{k,2n}(z_{2n}, \xi) \mathrm{d}\lambda_{t+1}(z_{2n}) \dots \mathrm{d}\lambda_{t+1}(z_1),$$

and

$$II_{k,j}'(\xi) \\ = \int_{\mathbb{B}_n^{2n}} E_{k,j}(z_1, \dots, z_{2n}) K_\xi^{(t)}(z_{2n-1}) \left( \prod_{i=2}^{2n-1} Y_{k,i}(z_i, z_{i-1}) \right) Y_{k,1}(z_1, z_{2n}) Y_{k,2n}(z_{2n}, \xi) \mathrm{d}\lambda_{t+1}(z_{2n}) \dots \mathrm{d}\lambda_{t+1}(z_1).$$

Since

$$
\begin{aligned}
&|D_k| \\
=&\left| \det \left[F(z_1) - F(z_k) \dots F(z_{k-1}) - F(z_k) \bar{R} F(z_k) F(z_{k+1}) - F(z_k) \dots F(z_{2n}) - F(z_k)\right] \right| \\
\lesssim & |z_k| \cdot \left( \sum_{i,j} |1 - \langle z_i, z_j \rangle| \right)^{n+1/2},
\end{aligned}
$$

we can repeat the proof for (5.13) and get the identity

$$
I_k(\xi) \simeq I'_k(\xi). \tag{5.16}
$$

Define

$$
Z_{k,j} = \begin{cases} \sum_{\tau \in S_{2n}} \operatorname{sgn}(\tau) T^{(t+1,t)}_{f_{\tau_1}} \dots \widehat{T^{(t+1,t)}_{f_{\tau_j}}} \dots T^{(t+1,t)}_{f_{\tau_{k-1}}} Y_{f_{\tau_j} \bar{R} f_{\tau_k}} T^{(t+1,t)}_{f_{\tau_{k+1}}} \dots T^{(t+1,t)}_{f_{\tau_{2n}}}, & j < k \\ \sum_{\tau \in S_{2n}} \operatorname{sgn}(\tau) T^{(t+1,t)}_{f_{\tau_1}} \dots T^{(t+1,t)}_{f_{\tau_{k-1}}} Y_{f_{\tau_j} \bar{R} f_{\tau_k}} T^{(t+1,t)}_{f_{\tau_{k+1}}} \dots \widehat{T^{(t+1,t)}_{f_{\tau_j}}} \dots T^{(t+1,t)}_{f_{\tau_{2n}}}, & j > k, \end{cases}
$$

and $Z'_{k,j}$ the operator obtained from $Z_{k,j}$ by moving each rightmost operator to the leftmost. Here $\widehat{A}$ means that $A$ is removed. Checking by definition we get

$$
II_{k,j}(\xi) = \langle Z_{k,j} K^{(t)}_\xi, K^{(t)}_\xi \rangle, \quad II'_{k,j}(\xi) = \langle Z'_{k,j} K^{(t)}_\xi, K^{(t)}_\xi \rangle.
$$

Using antisymmetrization we see that

$$
Z_{k,j} = \begin{cases} (-1)^{k-j-1} Z_{k,k-1}, & j < k, \\ (-1)^{k-j-1} Z_{k,k+1}, & j > k. \end{cases}
$$

Therefore we have the following calculation.

$$
\begin{aligned}
&\sum_{k=1}^{2n} \sum_{j \neq k} Z_{k,j} = \sum_{k \text{ odd}} Z_{k,k+1} + \sum_{k \text{ even}} Z_{k,k-1} = \sum_{m=1}^{n} \Big( Z_{2m-1,2m} + Z_{2m,2m-1} \Big) \\
&= \sum_{m=1}^{n} \Big\{ \sum_{\tau \in S_{2n}} \operatorname{sgn}(\tau) T^{(t+1,t)}_{f_{\tau_1}} \dots T^{(t+1,t)}_{f_{\tau_{2m-2}}} Y_{f_{\tau_{2m}} \bar{R} f_{\tau_{2m-1}}} T^{(t+1,t)}_{f_{\tau_{2m+1}}} \dots T^{(t+1,t)}_{f_{\tau_{2n}}} \\
&\quad + \sum_{\tau \in S_{2n}} \operatorname{sgn}(\tau) T^{(t+1,t)}_{f_{\tau_1}} \dots T^{(t+1,t)}_{f_{\tau_{2m-2}}} Y_{f_{\tau_{2m-1}} \bar{R} f_{\tau_{2m}}} T^{(t+1,t)}_{f_{\tau_{2m+1}}} \dots T^{(t+1,t)}_{f_{\tau_{2n}}} \Big\} \\
&= \sum_{m=1}^{n} \sum_{\tau \in S_{2n}} \operatorname{sgn}(\tau) T^{(t+1,t)}_{f_{\tau_1}} \dots T^{(t+1,t)}_{f_{\tau_{2m-2}}} Y_{f_{\tau_{2m}} \bar{R} f_{\tau_{2m-1}} + f_{\tau_{2m-1}} \bar{R} f_{\tau_{2m}}} T^{(t+1,t)}_{f_{\tau_{2m+1}}} \dots T^{(t+1,t)}_{f_{\tau_{2n}}} \\
&= \sum_{m=1}^{n} \sum_{\tau \in S_{2n}} \operatorname{sgn}(\tau) T^{(t+1,t)}_{f_{\tau_1}} \dots T^{(t+1,t)}_{f_{\tau_{2m-2}}} Y_{\bar{R}[f_{\tau_{2m}} f_{\tau_{2m-1}}]} T^{(t+1,t)}_{f_{\tau_{2m+1}}} \dots T^{(t+1,t)}_{f_{\tau_{2n}}}
\end{aligned}
$$

$=0.$

Similarly, $\sum_{k=1}^{2n}\sum_{j\neq k} Z'_{k,j}=0$. Thus we can verify the following equations.

$$\sum_{k=1}^{2n}\sum_{j\neq k} II_{k,j}(\xi)=\sum_{k=1}^{2n}\sum_{j\neq k}\langle Z_{k,j}K_\xi^{(t)},K_\xi^{(t)}\rangle=\langle\Big(\sum_{k=1}^{2n}\sum_{j\neq k}Z_{k,j}\Big)K_\xi^{(t)},K_\xi^{(t)}\rangle=0,$$
$$\sum_{k=1}^{2n}\sum_{j\neq k} II'_{k,j}(\xi)=\sum_{k=1}^{2n}\sum_{j\neq k}\langle Z'_{k,j}K_\xi^{(t)},K_\xi^{(t)}\rangle=\langle\Big(\sum_{k=1}^{2n}\sum_{j\neq k}Z'_{k,j}\Big)K_\xi^{(t)},K_\xi^{(t)}\rangle=0.$$

This leads to the following identities

$$\langle YK_\xi^{(t)},K_\xi^{(t)}\rangle=\sum_{k=1}^{2n}\langle Y_kK_\xi^{(t)},K_\xi^{(t)}\rangle=\sum_{k=1}^{2n}I_k(\xi)+\sum_{k=1}^{2n}\sum_{j\neq k}II_{k,j}(\xi)=\sum_{k=1}^{2n}I_k(\xi),$$

and

$$\langle Y'K_\xi^{(t)},K_\xi^{(t)}\rangle=\sum_{k=1}^{2n}\langle Y'_kK_\xi^{(t)},K_\xi^{(t)}\rangle=\sum_{k=1}^{2n}I'_k(\xi)+\sum_{k=1}^{2n}\sum_{j\neq k}II'_{k,j}(\xi)=\sum_{k=1}^{2n}I'_k(\xi).$$

Combining with (5.16), we arrive at the following equation

$$\langle YK_\xi^{(t)},K_\xi^{(t)}\rangle\simeq\langle Y'K_\xi^{(t)},K_\xi^{(t)}\rangle.\quad\square \tag{5.17}$$

Equation (5.11) and Lemma 5.20 implies (5.10), which completes the proof of Proposition 5.15. □

### 5.4. *Proof of Lemmas 5.2 and 5.3*

In this subsection we give the proof of Lemmas 5.2 and 5.3.

**Lemma 5.21.** *For $t>-1$, the integral kernels $X_w^{(t)}(z)$ and $Y_w^{(t)}(z)$ satisfy Hypotheses B at $t$.*

**Proof.** The fact that $X_w^{(t)}(z)$ and $Y_w^{(t)}(z)$ are holomorphic in $z$ is obvious from Definition 5.4. By direct computation, we obtain the following equation

$$\bar R_wK_w^{(t)}(z)=(n+1+t)\frac{\langle z,w\rangle}{(1-\langle z,w\rangle)^{n+2+t}}=(n+1+t)K_w^{(t+1)}(z)-(n+1+t)K_w^{(t)}(z).$$

So we obtain the following estimates,

$$|\bar R_wK_w^{(t)}(z)|\lesssim|w||K_w^{(t+1)}(z)|,\quad\big|\bar R_wK_w^{(t)}(z)-(n+1+t)K_w^{(t+1)}(z)\big|\lesssim|K_w^{(t)}(z)|. \tag{5.18}$$

Also by Lemma 5.9, we get the following inequalities

$$|\mathcal{G}_n^{(t)}1(s)| \lesssim s^{-n}, \quad \left|\mathcal{G}_n^{(t)}1(s) - \frac{1}{t+1}\right| \lesssim s^{-n}(1-s). \tag{5.19}$$

Using (5.18) and (5.19), we find

$$\left|Y_w^{(t)}(z)\right| \lesssim |w|^{-2n}|K_w^{(t)}(z)|,$$

and

$$\begin{aligned}\left|X_w^{(t)}(z)\right| =& \left|\left((t+1)\mathcal{G}_n^{(t)}1(|w|^2) - 1\right) \cdot \frac{\bar{R}_w K_w^{(t)}(z)}{n+t+1} + \left(\frac{\bar{R}_w K_w^{(t)}(z)}{n+t+1} - K_w^{(t+1)}(z)\right)\right| \\ \leq& \left|(t+1)\mathcal{G}_n^{(t)}1(|w|^2) - 1\right| \cdot \left|\frac{\bar{R}_w K_w^{(t)}(z)}{n+t+1}\right| + \left|\frac{\bar{R}_w K_w^{(t)}(z)}{n+t+1} - K_w^{(t+1)}(z)\right| \\ \lesssim& |w|^{-2n}(1-|w|^2) \cdot |w|\left|K_w^{(t+1)}(z)\right| + \left|K_w^{(t)}(z)\right| \\ \lesssim& |w|^{-2n+1}\left|K_w^{(t)}(z)\right|.\end{aligned}$$

This completes the proof of Lemma 5.21. □

**Proof of Lemma 5.2.** Take $K_i = f_i(0)E_0$. Then each $K_i$ is a rank one operator. Take $X_w(z) = X_w^{(t)}(z)$ and $Y_w(z) = Y_w^{(t)}(z)$ and define $\hat{A}_i$, $\hat{B}_i$, $\hat{C}_i$ as in Proposition 5.15. Then clearly they satisfy

$$A_i = \hat{A}_i, \quad B_i = \hat{B}_i, \quad C_i = \hat{C}_i, \quad i = 1, \ldots, 2n.$$

By Lemma 5.21, $X_w(z)$ and $Y_w(z)$ satisfy Hypotheses B at $t$. Thus by Proposition 5.15, we verify

$$\begin{aligned}&[T_{f_1}^{(t)}, T_{f_2}^{(t)}, \ldots, T_{f_{2n}}^{(t)}] - [T_{f_1}^{(t+1,t)}, T_{f_2}^{(t+1,t)}, \ldots, T_{f_{2n}}^{(t+1,t)}] \\ =&[C_1, C_2, \ldots, C_{2n}] - [A_1, A_2, \ldots, A_{2n}] \\ =&[\hat{C}_1, \hat{C}_2, \ldots, \hat{C}_{2n}] - [\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_{2n}] \\ \sim&0.\end{aligned}$$

This completes the proof of Lemma 5.2. □

We take a roundabout approach in handling the Hardy space case. In Lemma 5.7 we lift the weight of $T_f$ by 2. Then in Lemma 5.24 we show that the operator on $L^2_{a,0}(\mathbb{B}_n)$

$$[T_{f_1}^{(-1,0)}, T_{f_2}^{(-1,0)}, \ldots, T_{f_{2n}}^{(-1,0)}] - [T_{f_1}^{(1,0)}, T_{f_2}^{(1,0)}, \ldots, T_{f_{2n}}^{(1,0)}] \sim 0.$$

Finally, using Lemma 2.4, we show that the operator

$$[T_{f_1},\dots,T_{f_{2n}}]-[T^{(1,-1)}_{f_1},\dots,T^{(1,-1)}_{f_{2n}}],$$

which is the restriction of the previous operator on $H^2(\mathbb{S}_n)$, also has zero trace. The main reason for taking this approach is to avoid rebuilding results related to Hypotheses B over the Hardy space.

**Lemma 5.22.** *The integral kernels $X^{(-1)}_w(z), Y^{(-1)}_w(z)$ and $Z^{(-1)}_w(z)$ satisfy the following estimates.*

1. $|X^{(-1)}_w(z)|\lesssim |w|^{-2n+1/2}|K^{(0)}_w(z)|$;
2. $|Y^{(-1)}_w(z)|\lesssim |w|^{-2n+1/2}|K^{(0)}_w(z)|$;
3. $|Z^{(-1)}_w(z)|\lesssim |w|^{-2n-1/2}|K^{(-1)}_w(z)|$.

*In particular, $X^{(-1)}_w(z), Y^{(-1)}_w(z)$ satisfy Hypotheses B at $t=0$.*

**Proof.** We compute the following limit by L'Hospital's rule,

$$\lim_{s\to 1^-}\frac{\mathcal{G}^{(0)}_n\phi(s)-1}{1-s}=\lim_{s\to 1^-}\frac{\int_s^1 r^{-1}\mathrm{d}r-(1-s)}{(1-s)^2}=\lim_{s\to 1^-}\frac{-s^{-1}+1}{-2(1-s)}=\frac{1}{2}.$$

Also by [47, Lemma 8.2] with $a=n+\frac{1}{4}$, $\mathcal{G}^{(0)}_n\phi(s)\lesssim s^{-n-\frac{1}{4}}$. Therefore we obtain the estimates

$$\left|\mathcal{G}^{(0)}_n\phi(s)-1\right|\lesssim s^{-n-\frac{1}{4}}(1-s),\quad \left|\mathcal{G}^{(0)}_n\phi(s)\right|\lesssim s^{-n-\frac{1}{4}}. \tag{5.20}$$

Also, direct computation gives the following identities

$$\begin{aligned}\bar{R}_wK^{(-1)}_w(z)&=\frac{n\langle z,w\rangle}{(1-\langle z,w\rangle)^{n+1}}=nK^{(0)}_w(z)-nK^{(-1)}_w(z),\\ \bar{R}^2_wK^{(-1)}_w(z)&=\frac{n(n+1)\langle z,w\rangle}{(1-\langle z,w\rangle)^{n+2}}-\frac{n^2\langle z,w\rangle}{(1-\langle z,w\rangle)}\\ &=n(n+1)K^{(1)}_w(z)-n(2n+1)K^{(0)}_w(z)+n^2K^{(-1)}_w(z).\end{aligned}$$

So we have the following estimates,

$$\left|\bar{R}^2_wK^{(-1)}_w(z)-n(n+1)K^{(1)}_w(z)\right|\lesssim|K^{(0)}_w(z)|, \tag{5.21}$$

and

$$|\bar{R}^2_wK^{(-1)}_w(z)|\lesssim|w||K^{(1)}_w(z)|,\quad |\bar{R}_wK^{(-1)}_w(z)|\lesssim|w||K^{(0)}_w(z)|. \tag{5.22}$$

By (5.20), (5.21) and (5.22), we estimate $|X^{(-1)}_w(z)|, |Y^{(-1)}_w(z)|, |Z^{(-1)}_w(z)|$ as follows,

$$\begin{aligned}|X_w^{(-1)}(z)| \leq & \frac{1}{n(n+1)}\left|\left(\mathcal{G}_n^{(0)}\phi(|w|^2)-1\right)\bar{R}_w^2K_w^{(-1)}(z)\right| + \left|\frac{1}{n(n+1)}\bar{R}_w^2K_w^{(-1)}(z) - K_w^{(1)}(z)\right| \\ \lesssim & |w|^{-2n-\frac{1}{2}}(1-|w|^2)|w||K_w^{(1)}(z)| + |K_w^{(0)}(z)| \\ \lesssim & |w|^{-2n+\frac{1}{2}}|K_w^{(0)}(z)|, \\ |Y_w^{(-1)}(z)| \lesssim & |w|^{-2n-\frac{1}{2}}|w||K_w^{(0)}(z)| = |w|^{-2n+\frac{1}{2}}|K_w^{(0)}(z)|,\end{aligned}$$

and

$$|Z_w^{(-1)}(z)| \lesssim |w|^{-2n-\frac{1}{2}}|K_w^{(-1)}(z)|.$$

This completes the proof of Lemma 5.22. $\square$

By Lemmas 5.14 and 5.22, for $f \in \mathscr{C}^2(\overline{\mathbb{B}_n})$, the integral formulas of $X_f^{(-1)}$ and $Y_{\bar{R}f}^{(-1)}$ define bounded operators on $L^2_{a,0}(\mathbb{B}_n)$. Denote these operators $X_f^{(-1,0)}$ and $Y_{\bar{R}f}^{(-1,0)}$.

**Lemma 5.23.** *Suppose $f, g \in \mathscr{C}^2(\overline{\mathbb{B}_n})$. Then the integral formula of $Z_{\bar{R}^2 f}^{(-1)}$ defines a bounded operator on $L^2_{a,0}(\mathbb{B}_n)$ that belongs to $\mathcal{S}^p$ for any $p > \frac{2n}{3}$. Denote this operator to be $Z_{\bar{R}^2 f}^{(-1,0)}$.*

**Proof.** Split the map as follows.

$$Z_{\bar{R}^2 f}^{(-1,0)} : L^2_{a,0}(\mathbb{B}_n) \xrightarrow{E_{0,3-2\epsilon}} L^2_{a,3-2\epsilon}(\mathbb{B}_n) \xrightarrow{\hat{Z}_{\bar{R}^2 f}^{(-1,0)}} L^2_a(\mathbb{B}_n).$$

Here $\hat{Z}_{\bar{R}^2 f}^{(-1,0)}$ is defined by the same integral formula as $Z_{\bar{R}^2 f}^{(-1,0)}$, and $\epsilon > 0$ is any sufficiently small number. The boundedness of $\hat{Z}_{\bar{R}^2 f}^{(-1,0)}$ follows from Lemmas 3.1 and 5.22. Finally, by Lemma 3.4, $E_{0,3-2\epsilon} \in \mathcal{S}^p$ for any $p > \frac{n}{\frac{3}{2}-\epsilon}$. Since $\epsilon > 0$ is arbitrary, we have $Z_{\bar{R}^2 f}^{(-1,0)} \in \mathcal{S}^p$ for any $p > \frac{2n}{3}$. This completes the proof of Lemma 5.23. $\square$

By Lemma 5.5, for $f \in \mathscr{C}^2(\overline{\mathbb{B}_n})$, the operator

$$T_f^{-1,0} = T_f^{(1,0)} + X_f^{(-1,0)} + Y_{\bar{R}f}^{(-1,0)} + Z_{\bar{R}^2 f}^{(-1,0)} + f(0)E_0$$

is a well-defined bounded operator on $L^2_{a,0}(\mathbb{B}_n)$. It follows from Lemmas 5.22, 5.23 and Proposition 5.15 that the following holds.

**Lemma 5.24.** *Suppose $f_1, f_2, \ldots, f_{2n} \in \mathscr{C}^2(\overline{\mathbb{B}_n})$. Then the operator on $L^2_{a,0}(\mathbb{B}_n)$,*

$$[T_{f_1}^{(-1,0)}, T_{f_2}^{(-1,0)}, \ldots, T_{f_{2n}}^{(-1,0)}] - [T_{f_1}^{(1,0)}, T_{f_2}^{(1,0)}, \ldots, T_{f_{2n}}^{(1,0)}]$$

*is a trace class operator of zero trace.*

**Lemma 5.25.** *Suppose $f, g \in \mathscr{C}^2(\overline{\mathbb{B}_n})$.*

1. *The operators $X_f^{(-1)}, Y_{\bar{R}f}^{(-1)}$ on $H^2(\mathbb{S}_n)$ belong to $\mathcal{S}^p$ for any $p > n$.*
2. *The operator $Z_{\bar{R}^2 f}^{(-1)}$ on $H^2(\mathbb{S}_n)$ belongs to $\mathcal{S}^p$ for any $p > \frac{n}{2}$.*

**Proof.** If $f$ has compact support in $\mathbb{B}_n$, then $X_f^{(-1)}, Y_{\bar{R}f}^{(-1)}, Z_{\bar{R}^2 f}^{(-1)}$ belong to the trace class. Since we have the decomposition $f = f_1 + f_2$, where $f_1$ has compact support in $\mathbb{B}_n$, and $f_2$ has support away from the origin, we might as well assume that the support of $f$ does not contain the origin. Then by Lemma 5.22, we obtain the following bounds

$$|f(w)X_w^{(-1)}(z)| \lesssim |K_w^{(0)}(z)|, \quad |\bar{R}f(w)Y_w^{(-1)}(z)| \lesssim |K_w^{(0)}(z)|, \quad |\bar{R}^2 Z_w^{(-1)}(z)| \lesssim |K_w^{(-1)}(z)|.$$

Split the maps as follows.

$$X_f^{(-1)} : H^2(\mathbb{S}_n) \xrightarrow{E_{-1,2}} L^2_{a,1}(\mathbb{B}_n) \xrightarrow{\hat{X}_f^{(-1)}} H^2(\mathbb{S}_n),$$

$$Y_{\bar{R}f}^{(-1)} : H^2(\mathbb{S}_n) \xrightarrow{E_{-1,2}} L^2_{a,1}(\mathbb{B}_n) \xrightarrow{\hat{Y}_{\bar{R}f}^{(-1)}} H^2(\mathbb{S}_n),$$

$$Z_{\bar{R}^2 f}^{(-1)} : H^2(\mathbb{S}_n) \xrightarrow{E_{-1,4-2\epsilon}} L_{a,3-2\epsilon} \xrightarrow{\hat{Z}_{\bar{R}^2 f}^{(-1)}} H^2(\mathbb{S}_n).$$

Here the operators with hats are defined by the same integral formulas as the corresponding operators without hats, and $\epsilon > 0$ is any sufficiently small number. The boundedness of the operators with hats follows from the estimates in Lemmas 3.3 and 5.22. Finally, by Lemma 3.4, the Schatten-class memberships of $X_f^{(-1)}, Y_{\bar{R}f}^{(-1)}$ and $Z_{\bar{R}^2 f}^{(-1)}$ follow from those of the embedding operators. This completes the proof of Lemma 5.25. □

**Remark 5.26.** It follows from Lemmas 5.7 and 5.25 that for any $f \in \mathscr{C}^2(\overline{\mathbb{B}_n})$, the operator $T_f^{(1,-1)} = T_f^{(-1)} - X_f^{(-1)} - Y_{\bar{R}f}^{(-1)} - Z_{\bar{R}^2 f}^{(-1)} - f(0)E_0$ is a well-defined bounded operator on $H^2(\mathbb{S}_n)$.

**Lemma 5.27.** *Suppose $f, g \in \mathscr{C}^2(\overline{\mathbb{B}_n})$. Then the commutators*

$$[X_f^{(-1)}, T_g^{(1,-1)}], \quad [Y_{\bar{R}f}^{(-1)}, T_f^{(1,-1)}]$$

*belong to $\mathcal{S}^p$ for any $p > \frac{2n}{3}$.*

**Proof.** As in the proof of Lemma 5.25, we may assume the support of $f$ is away from the origin. For any $h \in H^2(\mathbb{S}_n)$, since $X_w^{(-1)}(z), Y_w^{(-1)}(z)$ are holomorphic in $z$, it is easy to verify the following formulas,

$$\left(T_g^{(1,-1)}X_f^{(-1)} - X_{gf}^{(-1)}\right)h(\xi) = \int_{\mathbb{B}_n^2} [g(z)-g(w)]\, f(w)h(w)K_z^{(1)}(\xi)X_w^{(-1)}(z)\mathrm{d}\lambda_1(w)\mathrm{d}\lambda_1(z),$$

$$\left(X_f^{(-1)}T_g^{(1,-1)} - X_{gf}^{(-1)}\right)h(\xi) = \int\limits_{\mathbb{B}_n^2} f(z)\big[g(w) - g(z)\big]h(w)X_z^{(-1)}(\xi)K_w^{(1)}(z)\mathrm{d}\lambda_1(w)\mathrm{d}\lambda_1(z),$$

and similarly,

$$\left(T_g^{(1,-1)}Y_{\bar{R}f}^{(-1)} - Y_{g\bar{R}f}^{(-1)}\right)h(\xi) = \int\limits_{\mathbb{B}_n^2} \big[g(z) - g(w)\big]\bar{R}f(w)h(w)K_z^{(1)}(\xi)Y_w^{(-1)}(z)\mathrm{d}\lambda_1(w)\mathrm{d}\lambda_1(z),$$

$$\left(Y_{\bar{R}f}^{(-1)}T_g^{(1,-1)} - Y_{g\bar{R}f}^{(-1)}\right)h(\xi) = \int\limits_{\mathbb{B}_n^2} \bar{R}f(z)\big[g(w) - g(z)\big]h(w)Y_z^{(-1)}(\xi)K_w^{(1)}(z)\mathrm{d}\lambda_1(w)\mathrm{d}\lambda_1(z).$$

If we denote $G_w(z)$ to be either $f(w)X_w^{(-1)}(z)$ or $\bar{R}f(w)Y_w^{(-1)}(z)$, then set

$$\left(T_g^{(1,-1)}X_f^{(-1)} - X_{gf}^{(-1)}\right), \text{ or } \left(T_g^{(1,-1)}Y_{\bar{R}f}^{(-1)} - Y_{g\bar{R}f}^{(-1)}\right) = H,$$
$$\left(X_f^{(-1)}T_g^{(1,-1)} - X_{gf}^{(-1)}\right), \text{ or } \left(Y_{\bar{R}f}^{(-1)}T_g^{(1,-1)} - Y_{g\bar{R}f}^{(-1)}\right) = WS.$$

Here

$$Sh(z) = \int\limits_{\mathbb{B}_n} \big[g(w) - g(z)\big]h(w)K_w^{(1)}(z)\mathrm{d}\lambda_1(w),\ Wh(\xi) = \int\limits_{\mathbb{B}_n} h(z)G_z(\xi)\mathrm{d}\lambda_1(z),$$

and

$$Hh(\xi) = \int\limits_{\mathbb{B}_n^2} [g(z) - g(w)]h(w)G_w(z)K_z^{(1)}(\xi)\mathrm{d}\lambda_1(w)\mathrm{d}\lambda_1(z) = \int\limits_{\mathbb{B}_n} h(w)H_w(z)\mathrm{d}\lambda_1(w),$$

where

$$H_w(\xi) = \int\limits_{\mathbb{B}_n} [g(z) - g(w)]G_w(z)K_z^{(1)}(\xi)\mathrm{d}\lambda_1(z).$$

By Lemma 5.22 and since we assume the support of $f$ does not contain the origin, we obtain the following bound

$$|G_w(z)| \lesssim |K_w^{(0)}(z)|.$$

By Lemma 2.3, we have the estimate

$$|H_w(\xi)| \lesssim \int\limits_{\mathbb{B}_n} \frac{1}{|1 - \langle z, w\rangle|^{n+1/2}|1 - \langle z, \xi\rangle|^{n+2}}\mathrm{d}\lambda_1(\xi) \lesssim \frac{1}{|1 - \langle w, \xi\rangle|^{n+1/2}}.$$

For any $\epsilon > 0$ sufficiently small, split the map as follows.

$$H : H^2(\mathbb{S}_n) \xrightarrow{E_{-1,3-2\epsilon}} L_{a,2-2\epsilon} \xrightarrow{\hat{H}} H^2(\mathbb{S}_n),$$
$$WS : H^2(\mathbb{S}_n) \xrightarrow{E_{-1,3-2\epsilon}} L^2_{a,2-2\epsilon}(\mathbb{B}_n) \xrightarrow{\hat{S}} L^2(\lambda_{1-2\epsilon}) \xrightarrow{\hat{W}} H^2(\mathbb{S}_n).$$

By the estimates above, Lemmas 3.1 and 3.3, the operators above with hats are bounded. Thus by Lemma 3.4, $H$ and $WS$ belong to $\mathcal{S}^p$ for any $p > \frac{2n}{3}$. Therefore the commutators have the same Schatten-class membership. This completes the proof of Lemma 5.27. □

**Proof of Lemma 5.3.** By Lemma 5.24, the operator on $L^2_{a,0}(\mathbb{B}_n)$

$$[T^{(-1,0)}_{f_1}, \ldots, T^{(-1,0)}_{f_{2n}}] - [T^{(1,0)}_{f_1}, \ldots, T^{(1,0)}_{f_{2n}}] \sim 0.$$

On the other hand, by Lemmas 5.25, 5.27 and Proposition 5.10, the operator

$$[T^{(-1)}_{f_1}, \ldots, T^{(-1)}_{f_{2n}}] - [T^{(1,-1)}_{f_1}, \ldots, T^{(1,-1)}_{f_{2n}}]$$

is a trace class operator on $H^2(\mathbb{S}_n)$. Clearly, we check the following equation

$$\left([T^{(-1,0)}_{f_1}, \ldots, T^{(-1,0)}_{f_{2n}}] - [T^{(1,0)}_{f_1}, \ldots, T^{(1,0)}_{f_{2n}}]\right)\Big|_{H^2(\mathbb{S}_n)}$$
$$= [T^{(-1)}_{f_1}, \ldots, T^{(-1)}_{f_{2n}}] - [T^{(1,-1)}_{f_1}, \ldots, T^{(1,-1)}_{f_{2n}}].$$

Thus by Lemma 2.4, we obtain the following equation

$$\begin{aligned}&\mathrm{Tr}\left([T^{(-1)}_{f_1}, T^{(-1)}_{f_2}, \ldots, T^{(-1)}_{f_{2n}}] - [T^{(1,-1)}_{f_1}, T^{(1,-1)}_{f_2}, \ldots, T^{(1,-1)}_{f_{2n}}]\right)\\ =&\mathrm{Tr}\left([T^{(-1,0)}_{f_1}, T^{(-1,0)}_{f_2}, \ldots, T^{(-1,0)}_{f_{2n}}] - [T^{(1,0)}_{f_1}, T^{(1,0)}_{f_2}, \ldots, T^{(1,0)}_{f_{2n}}]\right)\\ =&0.\end{aligned}$$

This completes the proof of Lemma 5.3. □

## 6. A quantization formula

Toeplitz quantization, or Berezin-Toeplitz quantization [4], has been studied by many researchers on various types of domains. As an incomplete list, it was studied in [4,12,13] on the Fock space on $\mathbb{C}^n$; in [17,36] for planar domains; in [20] for pseudoconvex domains; in [24,49] on symmetric domains; in [18,39,40,45] for Kähler manifolds. See [1,21,45] for some very well-written surveys on this topic. Also see [21, Section 4.6] for some examples, including the unit ball. Quantization provides some of the basic motivation for the questions considered in this paper. However, the results in this paper are self-contained.

In this section we introduce an explicit algorithm for computing the bilinear operators $C_j(f,g)$ in the quantization formula (1.5), and prove Schatten-class norm estimates of the remainder terms. The proof mainly relies on the integration formulas given in Subsection 2.3 and the tools developed in Section 4. We begin this section with stating the main results, Theorem 6.3 and Corollaries 6.7, 6.8. The final half of the section contains the proofs of these results as well as various auxiliary lemmas. We start with explaining the following quantization formula, which is perhaps known to experts.

Recall that we defined the functions $d_{\alpha,\beta}$ and $I^{\alpha,\beta}$ in Definition 2.8.

For $i=1,\dots,n$, denote $e_i$ the multi-index that equals 1 at the $i$-th entry and 0 elsewhere. For $i_1,i_2,\dots,i_k$, denote

$$e_{i_1,i_2,\dots,i_k}=e_{i_1}+e_{i_2}+\ldots+e_{i_k}.$$

**Lemma 6.1.** *Suppose $t>-1$, $k$ is a non-negative integer and $f,g\in\mathscr{C}^{k+1}(\overline{\mathbb{B}_n})$. Then we have the decomposition*

$$T_f^{(t)}T_g^{(t)}=\sum_{l=0}^{k}c_{l,t}T^{(t)}_{C_l(f,g)}+R^{(t)}_{f,g,k+1}, \tag{6.1}$$

*where*

$$c_{0,t}=1,\quad c_{1,t}=nt^{-1}+O(t^{-2});\quad C_0(f,g)=fg,\quad C_1(f,g)-C_1(g,f)=\frac{-i}{n}\{f,g\}. \tag{6.2}$$

*Here $\{f,g\}$ is the Poisson bracket of $f$ and $g$.*

*Moreover, the explicit formulas for $c_{l,t}$, $C_l(f,g)$ and $R^{(t)}_{f,g,k+1}$ are given as follows. For any $l\geq 0$,*

$$c_{l,t}=\frac{\mathcal{F}^{(t)}_{n+l}\Phi^{(t)}_{n,l}(0)}{B(n,t+1)}\approx t^{-l}, \tag{6.3}$$

$$C_l(f,g)(z)=(-1)^l(1-|z|^2)^{-2l}\sum_{i_1,j_1,\dots,i_l,j_l=1}^{n}d_{e_{i_1,\dots,i_l},e_{j_1,\dots,j_l}}(z)\Big[D_{i_l,j_l}\dots D_{i_1,j_1}(f(z)g(w))\Big]\Big|_{w=z}, \tag{6.4}$$

*where*

$$D_{i,j}=(1-\langle z,w\rangle)^2\partial_{z_i}\bar{\partial}_{w_j}. \tag{6.5}$$

*For any $h\in L^2_{a,t}(\mathbb{B}_n)$ and $\xi\in\mathbb{B}_n$,*

$$R^{(t)}_{f,g,k+1}h(\xi)=\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\Phi^{(t)}_{n,k+1}(|\varphi_z(w)|^2)S_{f,g,k+1}(z,w)h(w)K_z^{(t)}(\xi)K_w^{(t)}(z)\mathrm{d}\lambda_t(w)\mathrm{d}\lambda_t(z), \tag{6.6}$$

*where*

$$S_{f,g,k+1}(z,w) = \frac{(-1)^{k+1}}{|1-\langle w,z\rangle|^{2(k+1)}} \cdot \sum_{i_1,j_1,\dots,i_{k+1},j_{k+1}=1}^{n} I^{e_{i_1,\dots,i_{k+1}},e_{j_1,\dots,j_{k+1}}}(z-w) D_{i_{k+1},j_{k+1}}\dots D_{i_1,j_1}[f(z)g(w)]. \tag{6.7}$$

Formula (6.7) leads to the following norm estimate, which is well studied in the theory of Toeplitz quantization. For example, in [20], Engliš gave such estimates under a more general setting.

**Corollary 6.2.** *Under the same assumption of Lemma 6.1,*

$$\|R^{(t)}_{f,g,k+1}\| \lesssim_k t^{-k-1}. \tag{6.8}$$

The novelty of this section is to consider Schatten membership and Schatten norm estimates of remainder terms. The following theorem is crucial to the proof of Theorem 7.3, which is key to our main theorems.

**Theorem 6.3.** *We have the following Schatten class membership and Schatten norm estimates for the remainder terms.*

(1) *On the disk, if $k \geq 0$ and $f,g \in \mathscr{C}^{k+1}(\overline{\mathbb{D}})$, then for any $p \geq 1$ and $t > -1$, $R^{(t)}_{f,g,k+1} \in \mathcal{S}^1$. Moreover, for such $p$,*

$$\|R^{(t)}_{f,g,k+1}\|_{\mathcal{S}^p} \lesssim_{k,p} t^{-(k+1)+\frac{n}{p}}.$$

(2) *On the ball in higher dimensions, if $k \geq 0$ and $f,g \in \mathscr{C}^{k+1}(\overline{\mathbb{B}_n})$, then for any $p > n$ and $t > -1$, $R^{(t)}_{f,g,k+1} \in \mathcal{S}^p$. Moreover, for such $p$,*

$$\|R^{(t)}_{f,g,k+1}\|_{\mathcal{S}^p} \lesssim_{k,p} t^{-(k+1)+\frac{n}{p}}.$$

Recall that

$$d_{e_i,e_j}(z) = \int_{\mathbb{S}_n} (A_z\zeta)_i \overline{(A_z\zeta)_j} \frac{\mathrm{d}\sigma(\zeta)}{\sigma_{2n-1}}.$$

Therefore, we have the following formula for $C_1(f,g)$.

$$\begin{aligned} C_1(f,g)(z) &= -\sum_{i,j=1}^{n} d_{e_i,e_j}(z)\partial_i f(z)\bar{\partial}_j g(z) \\ &= -\int_{\mathbb{S}_n} \left(\sum_{i=1}^{n} (A_z\zeta)_i \partial_i f(z)\right)\left(\sum_{j=1}^{n} \overline{(A_z(\zeta))_j}\bar{\partial}_j g(z)\right) \frac{\mathrm{d}\sigma(\zeta)}{\sigma_{2n-1}} \end{aligned}$$

As in the proof of [47, Equation (4.8)], the functions

$$\sum_{i=1}^{n}\left(A_z\zeta\right)_i\partial_i f(z) \quad \text{and} \quad \sum_{j=1}^{n}\overline{\left(A_z(\zeta)\right)_j}\bar{\partial}_j g(z)=\overline{\sum_{j=1}^{n}\left(A_z(\zeta)\right)_j\partial_j\bar{g}(z)}$$

are independent of the choice of basis. Thus $C_1(f,g)$ is independent of the choice of basis. A similar computation shows that $C_l(f,g)$ $(l=1,2,\cdots)$ are all independent of the choice of basis for general $l$.

**Remark 6.4.** At $z\in\mathbb{B}_n$, $z\neq 0$, choose an orthonormal basis $\mathbf{e}_z=\{e_{z,1},e_{z,2},\ldots,e_{z,n}\}$ of $\mathbb{C}^n$ under which $z$ has coordinates $(z_1,0,\ldots,0)$. Then under $\mathbf{e}_z$, by (2.7),

$$C_1(f,g)(z)=C_N(f,g)(z)+C_T(f,g)(z),$$

where

$$C_N(f,g)(z)=-\frac{1}{n}(1-|z|^2)^2\partial_1 f(z)\bar{\partial}_1 g(z), \quad C_T(f,g)(z)=-\frac{1}{n}(1-|z|^2)\sum_{i=2}^{n}\partial_i f(z)\bar{\partial}_i g(z).$$

The functions $C_N(f,g)$ and $C_T(f,g)$ represent parts of $C_1(f,g)$ involving derivatives of $f,g$ in the complex normal and tangential directions, respectively. It is easy to see that the definitions of $C_N(f,g)$ and $C_T(f,g)$ do not depend on the choice of $\mathbf{e}_z$ (as long as $e_{z,1}$ is in $z$-direction). Locally, we can choose $\mathbf{e}_z$ so that the vectors vary smoothly with respect to $z$. If $f,g\in\mathscr{C}^1(\overline{\mathbb{B}_n})$ then $C_N(f,g),C_T(f,g)\in\mathscr{C}(\overline{\mathbb{B}_n}\backslash\{0\})$. Also, it is easy to see from their definitions and the explicit formulas (6.9) and (6.10) that $C_N(f,g)$, $C_T(f,g)$ are bounded. This implies

$$|C_N(f,g)(z)|\lesssim(1-|z|^2)^2, \quad |C_T(f,g)(z)|\lesssim 1-|z|^2.$$

By Lemma 4.8, for $t$ large enough, we obtain

$$T^{(t)}_{C_N(f,g)}\in\mathcal{S}^p, \quad \|T^{(t)}_{C_N(f,g)}\|_{\mathcal{S}^p}\lesssim_p t^{\frac{n}{p}}, \quad \forall p>\frac{n}{2},p\geq 1,$$

and

$$T^{(t)}_{C_T(f,g)}\in\mathcal{S}^p, \quad \|T^{(t)}_{C_T(f,g)}\|_{\mathcal{S}^p}\lesssim_p t^{\frac{n}{p}}, \quad \forall p>n,p\geq 1.$$

**Remark 6.5.** Continuing with Remark 6.4, we can take $e_{z,1}=\frac{z}{|z|}$. Denote $\mathbf{e}=\{e_1,\ldots,e_n\}$ the canonical basis. Denote $\zeta_i=\zeta_i(\xi)$ to be the $i$-th coordinate of $\xi$ under the basis $\mathbf{e}_z$. Then compute

$$\frac{\partial\xi_i}{\partial\zeta_1}=\overline{\langle e_i,e_{z,1}\rangle}=\left(e_{z,1}\right)_i=\frac{z_i}{|z|}.$$

Thus, we compute the following expressions,

$$\begin{aligned}\left[\frac{\partial f(\xi)}{\partial \zeta_1}\frac{\partial g(\xi)}{\partial \bar{\zeta}_1}\right]\Big|_{\xi=z} &= \Big[\sum_{i,j=1}^{n}\frac{\partial f(\xi)}{\partial \xi_i}\frac{\partial \xi_i}{\partial \zeta_1}\frac{\partial g(\xi)}{\partial \bar{\xi}_j}\overline{\Big(\frac{\partial \xi_j}{\partial \zeta_1}\Big)}\Big]\Big|_{\xi=z}\\ &= \Big[\sum_{i,j=1}^{n}\frac{\partial f(\xi)}{\partial \xi_i}\frac{z_i}{|z|}\frac{\partial g(\xi)}{\partial \bar{\xi}_j}\overline{\Big(\frac{z_j}{|z|}\Big)}\Big]\Big|_{\xi=z}\\ &= |z|^{-2}Rf(z)\bar{R}g(z),\end{aligned}$$

and

$$\begin{aligned}\Big[\sum_{i=2}^{n}\frac{\partial f(\xi)}{\partial \zeta_i}\frac{\partial g(\xi)}{\partial \bar{\zeta}_j}\Big]\Big|_{\xi=z} &= \langle \partial f(z), \bar{\partial} g(z)\rangle - |z|^{-2}Rf(z)\bar{R}g(z)\\ &= \sum_{i,j=1}^{n}\big(\delta_{i,j} - \frac{z_i\bar{z}_j}{|z|^2}\big)\partial_i f(z)\bar{\partial}_j g(z).\end{aligned}$$

In other words, we arrive at the following expression

$$C_N(f,g)(z) = -\frac{1}{n}(1-|z|^2)^2|z|^{-2}Rf(z)\bar{R}g(z), \tag{6.9}$$

and

$$C_T(f,g)(z) = -\frac{1}{n}(1-|z|^2)\sum_{i,j=1}^{n}\big(\delta_{i,j} - \frac{z_i\bar{z}_j}{|z|^2}\big)\partial_i f(z)\bar{\partial}_j g(z). \tag{6.10}$$

Adding up the two equations gives the following formula,

$$C_1(f,g)(z) = -\frac{1}{n}(1-|z|^2)\Big[\sum_{i=1}^{n}\partial_i f(z)\bar{\partial}_i g(z) - Rf(z)\bar{R}g(z)\Big].$$

Motivated by Remark 6.4, we further decompose $R^{(t)}_{f,g,k+1}$ according to the normal and tangential derivatives.

**Definition 6.6.** For $z \in \mathbb{B}_n, z \neq 0$, let $\mathbf{e}_z = \{e_{z,1}, e_{z,2}, \dots, e_{z,n}\}$ be as in Remark 6.4. Then $e_{z,1}$ represents the complex normal direction at $z$, and $e_{z,2}, \dots, e_{z,n}$ represents the complex tangential directions at $z$.

Under the basis $\mathbf{e}_z$, by (6.7), $S_{f,g,k+1}$ decomposes into

$$S_{f,g,k+1}(z,w) = \sum_{1\le|\alpha|,|\beta|\le k+1} V^{\alpha,\beta}_{k+1}(z,w)\partial^\alpha f(z)\bar{\partial}^\beta g(w).$$

For integers $0 \le a, b \le k+1$, define

$$S_{f,g,k+1}^{a,b}(z,w)=\sum_{\substack{1\le|\alpha|,|\beta|\le k+1\\ \alpha_1=a,\beta_1=b}} V_{k+1}^{\alpha,\beta}(z,w)\partial^\alpha f(z)\bar{\partial}^\beta g(w).$$

A moment of reflection shows that the function $S_{f,g,k+1}^{a,b}$ does not depend on the choice of $\mathbf{e}_z$. Define the corresponding operator on $L_{a,t}^2(\mathbb{B}_n)$,

$$R_{f,g,k+1}^{(t)a,b}h(\xi)=\int\limits_{\mathbb{B}_n}\int\limits_{\mathbb{B}_n}\Phi_{n,k+1}^{(t)}(|\varphi_z(w)|^2)S_{f,g,k+1}^{a,b}(z,w)h(w)K_z^{(t)}(\xi)K_w^{(t)}(z)\mathrm{d}\lambda_t(w)\mathrm{d}\lambda_t(z).$$

Then we write

$$R_{f,g,k+1}^{(t)}=\sum_{a,b=0,\ldots,k+1}R_{f,g,k+1}^{(t)a,b}.$$

**Corollary 6.7.** *Suppose $f,g\in\mathscr{C}^{k+1}(\overline{\mathbb{B}_n})$. Then for any $0\le a,b\le k+1$ any $p\ge 1$, $p>\max\{\frac{n}{1+\frac{a+b}{2}},\frac{n}{k+1+\frac{t+1}{2}}\}$, $R_{f,g,k+1}^{(t)a,b}\in\mathcal{S}^p$. Moreover, for such $p$, and $t$ large enough,*

$$\|R_{f,g,k+1}^{(t)a,b}\|_{\mathcal{S}^p}\lesssim_{k,p} t^{-k-1+\frac{n}{p}}.$$

*In particular,*

(1) *if one of $f,g$ has the form $\phi(|z|^2)$, where $\phi\in\mathscr{C}^{k+1}([0,1])$, then*

$$R_{f,g,k+1}^{(t)}\in\mathcal{S}^p,\quad \forall p>\max\{\frac{2n}{3},\frac{n}{k+1+\frac{t+1}{2}}\};$$

(2) *if both $f,g$ are of the form $\phi(|z|^2)$, $\phi\in\mathscr{C}^{k+1}([0,1])$, then*

$$R_{f,g,k+1}^{(t)}\in\mathcal{S}^p,\quad \forall p>\max\{\frac{n}{2},\frac{n}{k+1+\frac{t+1}{2}}\}.$$

*For $p>\frac{2n}{3}$ in case (1) and $p>\frac{n}{2}$ in case (2) and $t$ large enough,*

$$\|R_{f,g,k+1}^{(t)}\|_{\mathcal{S}^p}\lesssim_{k,p} t^{-k-1+\frac{n}{p}}.$$

**Corollary 6.8.** *Suppose $x,y$ are positive integers, and $f\in\mathscr{C}^1(\overline{\mathbb{B}_n})$. Then the following hold.*

(1) *For $p>\max\{\frac{n}{x+\frac{1}{2}},\frac{n}{1+\frac{t+1}{2}}\}$, $T_{(1-|z|^2)^x}^{(t)}T_f^{(t)}-T_{(1-|z|^2)^xf}^{(t)}$ and $T_f^{(t)}T_{(1-|z|^2)^x}^{(t)}-T_{(1-|z|^2)^xf}^{(t)}$ are in $\mathcal{S}^p$. For $p>\frac{n}{x+\frac{1}{2}}$ and $t$ large enough,*

$$\|T_{(1-|z|^2)^x}^{(t)}T_f^{(t)}-T_{(1-|z|^2)^xf}^{(t)}\|_{\mathcal{S}^p}\lesssim_p t^{-1+\frac{n}{p}},\quad \|T_f^{(t)}T_{(1-|z|^2)^x}^{(t)}-T_{(1-|z|^2)^xf}^{(t)}\|_{\mathcal{S}^p}\lesssim_p t^{-1+\frac{n}{p}}.$$

(2) *For* $p > \max\{\frac{n}{x+y}, \frac{n}{1+\frac{t+1}{2}}\}$, $T^{(t)}_{(1-|z|^2)^x} T^{(t)}_{(1-|z|^2)^y} - T^{(t)}_{(1-|z|^2)^{x+y}} \in \mathcal{S}^p$. *For* $p > \frac{n}{x+y}$ *and* $t$ *large enough,*

$$\|T^{(t)}_{(1-|z|^2)^x} T^{(t)}_{(1-|z|^2)^y} - T^{(t)}_{(1-|z|^2)^{x+y}}\|_{\mathcal{S}^p} \lesssim_p t^{-1+\frac{n}{p}}.$$

In general, we would expect that $T^{(t)}_f T^{(t)}_g - T^{(t)}_{fg}$ have better Schatten class membership compared to $T^{(t)}_f T^{(t)}_g$: for arbitrary $f, g \in \mathscr{C}^1(\overline{\mathbb{B}_n})$, $T^{(t)}_f T^{(t)}_g$ is only bounded, whereas the semi-commutator $T^{(t)}_f T^{(t)}_g - T^{(t)}_{fg} \in \mathcal{S}^p$ for any $p > n$. Also in case (1) of the corollary above, for a general function $f \in \mathscr{C}^1(\overline{\mathbb{B}_n})$, $T^{(t)}_{(1-|z|^2)^x} T^{(t)}_f$ is in $\mathcal{S}^p, \forall p > \frac{n}{x}$, while the semi-commutator $T^{(t)}_{(1-|z|^2)^x} T^{(t)}_f - T^{(t)}_{(1-|z|^2)^x f}$ is in $\mathcal{S}^p, \forall p > \frac{n}{x+\frac{1}{2}}$. This is no longer true in case (2): both $T^{(t)}_{(1-|z|^2)^x} T^{(t)}_{(1-|z|^2)^y}$ and $T^{(t)}_{(1-|z|^2)^x} T^{(t)}_{(1-|z|^2)^y} - T^{(t)}_{(1-|z|^2)^{x+y}}$ are in $\mathcal{S}^p$ for $p > \frac{n}{x+y}$. Intuitively, this has to do with the fact that functions of the form $(1-|z|^2)^x$ already vanish along the radial direction to some order.

In the rest of this section, we prove the results above.

Functions of the form

$$\sum_{i_1,\dots,i_l,j_1,\dots,j_l=1}^{n} I^{e_{i_1,\dots,i_l}, e_{j_1,\dots,j_l}}(z-w) D_{i_l,j_l} D_{i_{l-1},j_{l-1}} \dots D_{i_1,j_1}[f(z)g(w)]$$

appear in the formula of $S_{f,g,k+1}(z,w)$ in Lemma 6.1. We need to estimate its absolute value. To start with, we write the above sum in terms of the standard derivation. Recall that

$$D_{i,j} = (1-\langle z,w\rangle)^2 \partial_{z_i} \bar{\partial}_{w_j}.$$

**Definition 6.9.** Denote

$$A_{x,y}(z,w) = \sum_{i_1,\dots,i_x=1}^{n} \sum_{j_1,\dots,j_y=1}^{n} I^{e_{i_1,\dots,i_x}, e_{j_1,\dots,j_y}}(z-w) \partial_{i_1} \dots \partial_{i_x} f(z) \bar{\partial}_{j_1} \dots \bar{\partial}_{j_y} g(w);$$

$$B_1(z,w) = \sum_{i=1}^{n} (z_i - w_i) \partial_{z_i}(1-\langle z,w\rangle) = -\sum_{i=1}^{n} (z_i - w_i)\bar{w}_i = \langle w-z, w\rangle;$$

$$B_2(z,w) = \sum_{j=1}^{n} \overline{(z_j - w_j)} \bar{\partial}_{w_j}(1-\langle z,w\rangle) = \langle z, w-z\rangle;$$

$$C(z,w) = \sum_{i,j=1}^{n} I^{e_i,e_j}(z-w) \partial_{z_i} \bar{\partial}_{w_j}(1-\langle z,w\rangle) = -|z-w|^2.$$

In $D_{i_l,j_l} D_{i_{l-1},j_{l-1}} \dots D_{i_2,j_2}$, the partial derivations $\partial_{z_{i_l}}, \bar{\partial}_{w_{j_l}}, \dots, \partial_{z_{i_2}}, \bar{\partial}_{w_{j_2}}$ fall either on $f(z)g(w)$ or a copy of $(1-\langle z,w\rangle)$. Thus the summation

$$\sum_{i_1,\dots,i_l,j_1,\dots,j_l=1}^{n} I^{e_{i_1,\dots,i_l},e_{j_1,\dots,j_l}}(z-w)D_{i_l,j_l}D_{i_{l-1},j_{l-1}}\dots D_{i_1,j_1}[f(z)g(w)]$$

can be reorganized into sums of functions of the form

$$(1-\langle z,w\rangle)^{s_1}\left[C(z,w)\right]^{s_2}\left[B_1(z,w)\right]^{s_3}\left[B_2(z,w)\right]^{s_4}A_{x,y}(z,w).$$

In total, there are $2l$ steps of taking partial derivatives, and there are $2l$ copies of $(1-\langle z,w\rangle)$ in the above. So

$$2s_2+s_3+s_4+x+y=s_1+s_2+s_3+s_4=2l.$$

Also, the partial derivatives of the first operator, $D_{i_1,j_1}$, always apply on $f(z)g(w)$. So

$$x,y\geq 1.$$

From definition and Lemma 2.2, the following estimates are obvious.

**Lemma 6.10.** *Suppose $f,g\in\mathscr{C}^l(\overline{\mathbb{B}_n})$. Then*

(1) $|A_{x,y}(z,w)|\lesssim|\varphi_z(w)|^{x+y}|1-\langle z,w\rangle|^{\frac{x+y}{2}}$;
(2) $|B_1(z,w)|\lesssim|\varphi_z(w)||1-\langle z,w\rangle|$, $B_2(z,w)|\lesssim|\varphi_z(w)||1-\langle z,w\rangle|$;
(3) $|C(z,w)|\lesssim|\varphi_z(w)|^2|1-\langle z,w\rangle|$.

**Lemma 6.11.** *Suppose $k$ is a non-negative integer and $f,g\in\mathscr{C}^{k+1}(\overline{\mathbb{B}_n})$. For any $l=1,\dots,k+1$, set*

$$G_{i_1,\dots,i_l;j_1,\dots,j_l}(z,w)=D_{i_l,j_l}D_{i_{l-1},j_{l-1}}\dots D_{i_1,j_1}[f(z)g(w)].$$

*Then the following estimates hold.*

$$\left|\sum_{i_1,\dots,i_l,j_1,\dots,j_l=1}^{n} I^{e_{i_1,\dots,i_l},e_{j_1,\dots,j_l}}(z-w)G_{i_1,\dots,i_l;j_1,\dots,j_l}(z,w)\right|\lesssim|\varphi_z(w)|^{2l}|1-\langle z,w\rangle|^{2l+1}, \tag{6.11}$$

$$\left|\sum_{i_1,\dots,i_l,j_1,\dots,j_l,j_{l+1}=1}^{n} I^{e_{i_1,\dots,i_l},e_{j_1,\dots,j_l,j_{l+1}}}(z-w)\bar\partial_{w_{j_{l+1}}}G_{i_1,\dots,i_l;j_1,\dots,j_l}(z,w)\right|$$
$$\lesssim|\varphi_z(w)|^{2l+1}|1-\langle z,w\rangle|^{2l+1}. \tag{6.12}$$

**Proof.** As explained in Definition 6.9, the following sum

$$\sum_{i_1,\dots,i_l,j_1,\dots,j_l=1}^{n} I^{e_{i_1,\dots,i_l},e_{j_1,\dots,j_l}}(z-w)G_{i_1,\dots,i_l;j_1,\dots,j_l}(z,w)$$

splits into sums of functions of the form

$$(1-\langle z,w\rangle)^{s_1}\left[C(z,w)\right]^{s_2}\left[B_1(z,w)\right]^{s_3}\left[B_2(z,w)\right]^{s_4}A_{x,y}(z,w),$$

with

$$x,y\geq 1,\quad 2s_2+s_3+s_4+x+y=s_1+s_2+s_3+s_4=2l.$$

Similarly, the sum

$$\sum_{i_1,\dots,i_l,j_1,\dots,j_l,j_{l+1}=1}^{n} I^{e_{i_1,\dots,i_l},e_{j_1,\dots,j_l,j_{l+1}}}(z-w)\bar{\partial}_{w_{j_{l+1}}}G_{i_1,\dots,i_l;j_1,\dots,j_l}(z,w)$$

splits into sums of functions of above form, with

$$x,y\geq 1,\quad 2s_2+s_3+s_4+x+y=2l+1,\quad s_1+s_2+s_3+s_4=2l.$$

By Lemma 6.10, we have the following estimate

$$\begin{aligned}&\left|(1-\langle z,w\rangle)^{s_1}\left[C(z,w)\right]^{s_2}\left[B_1(z,w)\right]^{s_3}\left[B_2(z,w)\right]^{s_4}A_{x,y}(z,w)\right|\\ \lesssim&|\varphi_z(w)|^{2s_2+s_3+s_4+x+y}|1-\langle z,w\rangle|^{s_1+s_2+s_3+s_4+\frac{x+y}{2}}.\end{aligned}$$

Plugging in the equations for $s_i,x,y$ gives the inequalities (6.11) and (6.12). This completes the proof of Lemma 6.11. □

**Proof of Lemma 6.1 and Corollary 6.2.** Let $G_{i_1,\dots,i_l;j_1,\dots,j_l}$ be defined as in Lemma 6.11. Suppose $h\in\mathrm{Hol}(\overline{\mathbb{B}_n})$ and $\xi\in\mathbb{B}_n$. Write $F=f(z)g(w)h(w)K_z^{(t)}(\xi)$ and

$$F_{i_1,\dots,i_l;j_1,\dots,j_l}=D_{i_l,j_l}\dots D_{i_1,j_1}[f(z)g(w)h(w)K_z^{(t)}(\xi)]=G_{i_1,\dots,i_l;j_1,\dots,j_l}h(w)K_z^{(t)}(\xi).$$

Then we compute $T_f^{(t)}T_g^{(t)}$ as follows.

$$\begin{aligned}&T_f^{(t)}T_g^{(t)}h(\xi)\\ =&\int_{\mathbb{B}_n^2} f(z)g(w)h(w)K_z^{(t)}(\xi)K_w^{(t)}(z)\mathrm{d}\lambda_t(w)\mathrm{d}\lambda_t(z)\\ =&\int_{\mathbb{B}_n^2}\Phi_{n,0}^{(t)}F(z,w)K_w^{(t)}(z)\mathrm{d}\lambda_t(w)\mathrm{d}\lambda_t(z)\\ \overset{(2.18)}{=\!=}&\frac{\mathcal{F}_n^{(t)}\Phi_{n,0}^{(t)}(0)}{B(n,t+1)}\int_{\mathbb{B}_n} d_{0,0}(z)F(z,z)\mathrm{d}\lambda_t(z)\end{aligned}$$

$$
\begin{aligned}
&-\int\limits_{\mathbb{B}_n^2} \Phi_{n,1}^{(t)}(|\varphi_z(w)|^2)\frac{\sum_{i,j=1}^n I^{e_i,e_j}(z-w)D_{i,j}F(z,w)}{|1-\langle z,w\rangle|^2}K_w^{(t)}(z)\mathrm{d}\lambda_t(z)\mathrm{d}\lambda_t(w)\\
=&T_{fg}^{(t)}h(\xi)-\int\limits_{\mathbb{B}_n^2} \Phi_{n,1}^{(t)}(|\varphi_z(w)|^2)\frac{\sum_{i,j=1}^n I^{e_i,e_j}(z-w)F_{i,j}(z,w)}{|1-\langle z,w\rangle|^2}K_w^{(t)}(z)\mathrm{d}\lambda_t(z)\mathrm{d}\lambda_t(w)\\
=&T_{fg}^{(t)}h(\xi)+R_{f,g,1}^{(t)}.
\end{aligned}
$$

The condition for applying Lemma 2.16 is verified by Lemma 6.11. In general, we have the following computation for $R_{f,g,l}^{(t)}$.

$$
\begin{aligned}
&R_{f,g,l}^{(t)}h(\xi)\\
=&(-1)^l\int\limits_{\mathbb{B}_n^2} \Phi_{n,l}^{(t)}(|\varphi_z(w)|^2)\frac{\sum_{i_1,\dots,i_l,j_1,\dots,j_l=1}^n I^{e_{i_1,\dots,i_l},e_{j_1,\dots,j_l}}(z-w)F_{i_l,\dots,i_1;j_l,\dots,j_1}}{|1-\langle z,w\rangle|^{2l}}K_w^{(t)}(z)\mathrm{d}\lambda_t(w)\mathrm{d}\lambda_t(z)\\
=&(-1)^l\frac{\mathcal{F}_{n+l}^{(t)}\Phi_{n,l}^{(t)}(0)}{B(n,t+1)}\int\limits_{\mathbb{B}_n}(1-|z|^2)^{-2l}\sum_{i_1,\dots,i_l,j_1,\dots,j_l=1}^n d_{e_{i_1,\dots,i_l},e_{j_1,\dots,j_l}}(z)F_{i_l,\dots,i_1;j_l,\dots,j_1}(z,z)\mathrm{d}\lambda_t(z)\\
&+(-1)^{l+1}\int\limits_{\mathbb{B}_n^2}\mathrm{d}\lambda_t(z)\mathrm{d}\lambda_t(w)\Phi_{n,l+1}^{(t)}(|\varphi_z(w)|^2)K_w^{(t)}(z)\\
&\qquad\cdot\frac{\sum_{i_1,\dots,i_{l+1},j_1,\dots,j_{l+1}=1}^n I^{e_{i_1,\dots,i_{l+1}},e_{j_1,\dots,j_{l+1}}}(z-w)D_{i_{l+1},j_{l+1}}F_{i_l,\dots,i_1;j_l,\dots,j_1}(z,w)}{|1-\langle z,w\rangle|^{2(l+1)}}\\
=&c_{l,t}T_{C_l(f,g)}^{(t)}h(\xi)+R_{f,g,l+1}^{(t)}h(\xi).
\end{aligned}
$$

This proves the formulas in Lemma 6.1 for $h\in\mathrm{Hol}(\overline{\mathbb{B}_n})$. By [47, Lemma 8.4], we have the following identity,

$$
c_{0,t}=\frac{\mathcal{F}_n^{(t)}1(0)}{B(n,t+1)}=1.
$$

The estimate (a) for $c_{l,t}$ follows from Lemma 2.17. The formulas for $C_0(f,g)$ follow from direct computation. We can also see from the formula that $C_l(f,g)$ does not depend on the choice of an orthonormal basis. Using (2.7) we directly verify that $C_1(f,g)-C_1(g,f)=\frac{-i}{n}\{f,g\}$. By Lemma 6.11, we have the following estimates,

$$
\begin{aligned}
&|S_{f,g,k+1}(z,w)|\\
=&|1-\langle w,z\rangle|^{-2(k+1)}\\
&\times\left|\sum_{i_1,\dots,i_{k+1},j_{k+1},\dots,j_l=1}^n I^{e_{i_1,\dots,i_{k+1}},e_{j_1,\dots,j_{k+1}}}(z-w)G_{i_1,\dots,i_{k+1};j_1,\dots,j_{k+1}}(z,w)\right| \qquad (6.13)\\
\lesssim&|\varphi_z(w)|^{2(k+1)}|1-\langle z,w\rangle|.
\end{aligned}
$$

Thus it follows from Theorem 4.6 that $\|R^{(t)}_{f,g,k+1}\| \lesssim_k t^{-k-1}$, which proves Corollary 6.2.

By [47, Lemma 5.3],

$$\mathcal{F}^{(t)}_{n+1}\Phi^{(t)}_{n,1}(0) = n!t^{-n-1} + o(t^{-n-2})$$

as $t$ tends to infinity. Therefore, we arrive at the following estimate

$$c_{1,t} = \frac{\mathcal{F}^{(t)}_{n+1}\Phi^{(t)}_{n,1}(0)}{B(n,t+1)} = \frac{B(n+1,t+1)}{B(n,t+1)} + O(\frac{B(n+2,t+1)}{B(n,t+1)}) = nt^{-1} + O(t^{-2}).$$

This completes the proof of Lemma 6.1. □

**Proof of Theorem 6.3.** In the case when $n = 1$, the estimates in the proof of Lemma 6.10 are improved into

(1) $|A_{x,y}(z,w)| \lesssim |\varphi_z(w)|^{x+y}|1 - \langle z,w\rangle|^{x+y}$;
(2) $|B_1(z,w)| \lesssim |\varphi_z(w)||1 - \langle z,w\rangle|$, $B_2(z,w)| \lesssim |\varphi_z(w)||1 - \langle z,w\rangle|$;
(3) $|C(z,w)| \lesssim |\varphi_z(w)|^2|1 - \langle z,w\rangle|^2$.

This leads to

$$\left|\sum_{i_1,\dots,i_l,j_1,\dots,j_l=1}^{n} I^{e_{i_1,\dots,i_l},e_{j_1,\dots,j_l}}(z-w)G_{i_1,\dots,i_l;j_1,\dots,j_l}(z,w)\right| \lesssim |\varphi_z(w)|^{2l}|1 - \langle z,w\rangle|^{2l+2} \tag{6.14}$$

and then

$$|S_{f,g,k+1}(z,w)| \lesssim |\varphi_z(w)|^{2(k+1)}|1 - \langle z,w\rangle|^2.$$

Then statement (1) follows from Theorem 4.6. With the same argument, statement (2) follows from (6.13) and Theorem 4.6. This completes the proof. □

**Proof of Corollary 6.7.** As in Definition 6.6, for $z \in \mathbb{B}_n$, $z \neq 0$, let $\mathbf{e}_z$ be an orthonormal basis of $\mathbb{C}^n$ so that $e_{z,1} = \frac{z}{|z|}$. Under the basis $\mathbf{e}_z$, for $0 \le a, b \le k+1$, $S^{a,b}_{f,g,k+1}(z,w)$ consists of the part of $S_{f,g,k+1}$ that contains $\partial^\alpha f(z)\bar{\partial}^\beta g(w)$ with $\alpha_1 = a, \beta_1 = b$.

Since $R^{(t)a,b}_{f,g,k+1} = P^{(t)}T^{(t)a,b}_{f,g,k+1}$, where

$$T^{(t)a,b}_{f,g,k+1}h(z) = \int\limits_{\mathbb{B}_n} \Phi^{(t)}_{n,k+1}(|\varphi_z(w)|^2)S^{a,b}_{f,g,k+1}(z,w)h(w)K^{(t)}_w(z)\mathrm{d}\lambda_t(w).$$

To prove the Schatten class membership of the $R^{(t)a,b}_{f,g,k+1}$ operators, it amounts to prove the corresponding estimates for the kernel $S^{a,b}_{f,g,k+1}(z,w)$, and apply Theorem 4.6.

Locally choose the basis $\mathbf{e}_z$ so that it varies smooth with respect to $z$. Under the basis $\mathbf{e}_z$, define

$$A_{\alpha,\beta}(z,w) = I^{\alpha,\beta}(z-w)\partial^\alpha f(z)\bar{\partial}^\beta g(w).$$

Then by Lemma 2.2, we have the following bound,

$$|A_{\alpha,\beta}(z,w)| \lesssim |\varphi_z(w)|^{|\alpha|+|\beta|}|1-\langle z,w\rangle|^{\frac{|\alpha|+|\beta|+\alpha_1+\beta_1}{2}}.$$

Then $S^{(t)a,b}_{f,g,k+1}(z,w)$ is a finite linear combination of terms like:

$$\frac{1}{|1-\langle w,z\rangle|^{2(k+1)}}\cdot(1-\langle z,w\rangle)^{s_1}\left[C(z,w)\right]^{s_2}\left[B_1(z,w)\right]^{s_3}\left[B_2(z,w)\right]^{s_4}A_{\alpha,\beta}(z,w),$$

where $|\alpha|,|\beta|\geq 1, 2s_2+s_3+s_4+|\alpha|+|\beta| = s_1+s_2+s_3+s_4 = 2(k+1)$, and $\alpha_1 = a, \beta_1 = b$. Therefore we obtain the following estimate

$$\begin{aligned}|S^{a,b}_{f,g,k+1}(z,w)| \lesssim &|\varphi_z(w)|^{2s_2+s_3+s_4+|\alpha|+|\beta|}|1-\langle z,w\rangle|^{-2(k+1)+s_1+s_2+s_3+s_4+\frac{|\alpha|+|\beta|+\alpha_1+\beta_1}{2}}\\ =&|\varphi_z(w)|^{2(k+1)}|1-\langle z,w\rangle|^{\frac{2+a+b}{2}}.\end{aligned}$$

Thus by Theorem 4.6, $R^{(t)a,b}_{f,g,k+1} = P^{(t)}T^{(t)a,b}_{f,g,k+1} \in \mathcal{S}^p, \forall p > \max\{\frac{n}{1+\frac{a+b}{2}}, \frac{n}{k+1+\frac{t+1}{2}}\}$, and for such $p$, we have

$$\|R^{(t)a,b}_{f,g,k+1}\|_{\mathcal{S}^p} \lesssim_{k,p} t^{-k-1+\frac{n}{p}}.$$

If $f = \phi(|z|^2)$ for some $\phi$, then $R^{(t)0,b}_{f,g,k+1} = 0$ for any $b$. If both $f,g$ are of such form then $R^{(t)a,0}_{f,g,k+1}, R^{(t)0,b}_{f,g,k+1} = 0$ for any $a,b$. This gives the improved Schatten-class membership in (1) and (2), and proves Corollary 6.7. □

**Proof of Corollary 6.8.** By Lemma 6.1, for $f,g \in \mathscr{C}^1(\overline{\mathbb{B}_n})$, $T^{(t)}_f T^{(t)}_g - T^{(t)}_{fg} = R^{(t)}_{f,g,1}$, where

$$R^{(t)}_{f,g,1}h(z) = \int\limits_{\mathbb{B}_n^2}\Phi^{(t)}_{n,1}(|\varphi_z(w)|^2)S_{f,g,1}(z,w)h(w)K^{(t)}_z(\xi)K^{(t)}_w(z)\mathrm{d}\lambda_t(w)\mathrm{d}\lambda_t(z),$$

with

$$S_{f,g,1}(z,w) = \frac{-(1-\langle z,w\rangle)^2}{|1-\langle z,w\rangle|^2}\langle\partial f(z),\overline{z-w}\rangle\langle\bar{\partial}g(w),z-w\rangle.$$

If $f(z) = (1-|z|^2)^x$ then we have the following estimate

$$\left|\langle\partial f(z),\overline{z-w}\rangle\right| = \left|(x-1)(1-|z|^2)^{x-1}\langle\bar{z},\overline{z-w}\rangle\right| \lesssim (1-|z|^2)^{x-1}|1-\langle z,w\rangle| \lesssim |1-\langle z,w\rangle|^x.$$

Thus we arrive at the estimate,

$$\begin{aligned}&\left|\langle \partial f(z), \overline{z-w}\rangle \langle \bar\partial g(w), z-w\rangle\right|\\ &\lesssim \begin{cases} |1-\langle z,w\rangle|^{x+1/2}, & \text{if one of } f \text{ or } g \text{ equals } (1-|z|^2)^x\\ |1-\langle z,w\rangle|^{x+y}, & \text{if } f=(1-|z|^2)^x, g(z)=(1-|z|^2)^y.\end{cases}\end{aligned}$$

Corollary 6.8 follows from the above inequality and Theorem 4.6. This completes the proof of Corollary 6.8. □

## 7. First and second antisymmetrizations

As explained in the introduction, the goal of this section is to prove the trace class membership of the antisymmetric sum $[T^{(t)}_{f_1}, T^{(t)}_{f_2}, \dots, T^{(t)}_{f_{2n}}]$ as well as the asymptotic trace formula (1.7). To begin with, we define first and second partial antisymmetrizations, which are generalizations of semi-commutators.

**Definition 7.1.** For $f, g \in C(\overline{\mathbb{B}_n})$ and $t \geq -1$, denote the semi-commutator on $L^2_{a,t}(\mathbb{B}_n)$,

$$\sigma_t(f,g) = T^{(t)}_f T^{(t)}_g - T^{(t)}_{fg}. \tag{7.1}$$

For $f_1, \dots, f_n, g_1, \dots, g_n \in C(\overline{\mathbb{B}_n})$ and $t \geq -1$, define the following partial anti-symmetric sums.

$$[f_1, g_1, \dots, f_n, g_n]^{\text{fst}}_t = \sum_{\tau\in S_n} \operatorname{sgn}(\tau)\sigma_t(f_{\tau_1}, g_1)\dots\sigma_t(f_{\tau_n}, g_n), \tag{7.2}$$

and

$$[f_1, g_1, \dots, f_n, g_n]^{\text{scd}}_t = \sum_{\tau\in S_n} \operatorname{sgn}(\tau)\sigma_t(f_1, g_{\tau_1})\dots\sigma_t(f_n, g_{\tau_n}). \tag{7.3}$$

In the case when $n = 1$, the operators above both agree with the semi-commutator $\sigma_t(f, g)$. In higher dimensions, the partial anti-symmetric sum (7.2) and (7.3) generalize the semi-commutator (7.1).

**Remark 7.2.** Here is another way to describe commutators, semicommutators, and their products. Fix $t > -1$. Under the decomposition $L^2(\lambda_t) = L^2_{a,t}(\mathbb{B}_n) \oplus L^2_{a,t}(\mathbb{B}_n)^\perp$, we can write a multiplication operator $M_f$ as a block matrix

$$M_f = \begin{bmatrix} T^{(t)}_f & H^{(t)*}_{\bar f} \\ H^{(t)}_f & * \end{bmatrix},$$

where $T^{(t)}_f$ and $H^{(t)}_f$ are Toeplitz operator and Hankel operator, respectively, associated to $f$. From the equations

$$\sigma_t(f,g) = -H_{\bar{f}}^{(t)*}H_g^{(t)},$$

and

$$[T_f^{(t)}, T_g^{(t)}] = \sigma_t(f,g) - \sigma_t(g,f) = H_{\bar{g}}^{(t)*}H_f^{(t)} - H_{\bar{f}}^{(t)*}H_g^{(t)},$$

we can see that both operators come from products of off-diagonal terms under the above block matrix representation. Under this point of view, the first and second antisymmetrizations defined above are linear combinations of alternating products of off-diagonal terms, with $f_i$ appearing in the top right, and $g_j$ appearing in the bottom left.

**Theorem 7.3.** *Suppose $t \geq -1$ and $f_1, g_1, \ldots, f_n, g_n \in \mathscr{C}^2(\overline{\mathbb{B}_n})$. Then the partial antisymmetrizations $[f_1, g_1, \ldots, f_n, g_n]_t^{\mathrm{fst}}$ and $[f_1, g_1, \ldots, f_n, g_n]_t^{\mathrm{scd}}$ are in the trace class. Moreover,*

$$\begin{aligned}\lim_{t\to\infty} \mathrm{Tr}[f_1, g_1, \ldots, f_n, g_n]_t^{\mathrm{fst}} &= \lim_{t\to\infty} \mathrm{Tr}[f_1, g_1, \ldots, f_n, g_n]_t^{\mathrm{scd}} \\ &= \frac{1}{(2\pi i)^n}\int_{\mathbb{B}_n} \partial f_1 \wedge \bar{\partial} g_1 \wedge \ldots \wedge \partial f_n \wedge \bar{\partial} g_n. \end{aligned} \tag{7.4}$$

As an application of the above theorem, we obtain the following asymptotic estimate for the Schatten-4 norm of Hankel operators at dimension 2, which seems to be new to us.

**Corollary 7.4.** *Suppose $f_1, f_2 \in \mathscr{C}^2(\overline{\mathbb{B}_2})$ are supported inside $\mathbb{B}_2$. Then the infimum limit*

$$\varliminf_{t\to\infty} \|H_{\bar{f}_1}^{(t)}\|_{\mathcal{S}^4}^2 \|H_{\bar{f}_2}^{(t)}\|_{\mathcal{S}^4}^2$$

*satisfies*

$$\varliminf_{t\to\infty} \|H_{\bar{f}_1}^{(t)}\|_{\mathcal{S}^4}^2 \|H_{\bar{f}_2}^{(t)}\|_{\mathcal{S}^4}^2 \geq \frac{1}{4\pi^2}\int_{\mathbb{B}_2} \partial f_1 \wedge \partial f_2 \wedge \overline{\partial f_1 \wedge \partial f_2}.$$

**Remark 7.5.** We list the $n = 1$ case for comparison and contrast. The equation

$$T_f^{(t)}T_{\bar{f}}^{(t)} - T_{|f|^2}^{(t)} = -H_{\bar{f}}^{(t)*}H_{\bar{f}}^{(t)}$$

and [47, Theorem 1.1] implies that for $t > -1$ and $f \in \mathscr{C}^2(\overline{\mathbb{D}})$,

$$\|H_{\bar{f}}^{(t)}\|_{\mathcal{S}^2}^2 = -\frac{1}{2\pi i}\int_{\mathbb{D}} \partial f \wedge \overline{\partial f} - \int_{\mathbb{D}^2} \rho_t\left(|\varphi_z(w)|^2\right)\Delta f(z)\Delta \bar{f}(w)\mathrm{d}m(z,w).$$

where $\rho_t$ is a positive-valued function on $(0,1)$ with explicit expressions. As a consequence,

$$\lim_{t\to\infty} \|H^{(t)}_{\bar{f}}\|^2_{\mathcal{S}^2} = -\frac{1}{2\pi i}\int_{\mathbb{D}} \partial f \wedge \overline{\partial f}.$$

**Proof of Corollary 7.4.** Take $g_1 = \bar{f}_1$, $g_2 = \bar{f}_2$. By definition, we compute the expression of $[f_1, \bar{f}_1, f_2, \bar{f}_2]^{\mathrm{scd}}$,

$$\begin{aligned}[f_1, \bar{f}_1, f_2, \bar{f}_2]^{\mathrm{scd}} =&\sigma_t(f_1, \bar{f}_1)\sigma_t(f_2, \bar{f}_2) - \sigma_t(f_1, \bar{f}_2)\sigma_t(f_2, \bar{f}_1)\\ =&H^{(t)*}_{\bar{f}_1}H^{(t)}_{\bar{f}_1}H^{(t)*}_{\bar{f}_2}H^{(t)}_{\bar{f}_2} - H^{(t)*}_{\bar{f}_1}H^{(t)}_{\bar{f}_2}H^{(t)*}_{\bar{f}_2}H^{(t)}_{\bar{f}_1}\\ =&|H^{(t)}_{\bar{f}_1}|^2|H^{(t)}_{\bar{f}_2}|^2 - \left(H^{(t)*}_{\bar{f}_2}H^{(t)}_{\bar{f}_1}\right)^* H^{(t)*}_{\bar{f}_2}H^{(t)}_{\bar{f}_1}.\end{aligned}$$

Then by Theorem 7.3, we have the following limit,

$$\begin{aligned}\mathrm{Tr}[f_1, \bar{f}_1, f_2, \bar{f}_2]^{\mathrm{scd}} =&\mathrm{Tr}|H^{(t)}_{\bar{f}_1}|^2|H^{(t)}_{\bar{f}_2}|^2 - \mathrm{Tr}\left(H^{(t)*}_{\bar{f}_2}H^{(t)}_{\bar{f}_1}\right)^* H^{(t)*}_{\bar{f}_2}H^{(t)}_{\bar{f}_1}\\ \to&\frac{1}{(2\pi i)^2}\int_{\mathbb{B}_2} \partial f_1 \wedge \bar{\partial}\bar{f}_1 \wedge \partial f_2 \wedge \bar{\partial}\bar{f}_2\\ =&\frac{1}{4\pi^2}\int_{\mathbb{B}_2} \partial f_1 \wedge \partial f_2 \wedge \overline{\partial f_1 \wedge \partial f_2}, \quad t\to\infty.\end{aligned}$$

Notice that $\left(H^{(t)*}_{\bar{f}_2}H^{(t)}_{\bar{f}_1}\right)^* H^{(t)*}_{\bar{f}_2}H^{(t)}_{\bar{f}_1} \geq 0$. The Cauchy-Schwartz inequality gives the following bound,

$$\begin{aligned}\left|\mathrm{Tr}|H^{(t)}_{\bar{f}_1}|^2|H^{(t)}_{\bar{f}_2}|^2\right| &= \left|\langle |H^{(t)}_{\bar{f}_2}|^2, |H^{(t)}_{\bar{f}_1}|^2\rangle_{\mathcal{S}^2}\right| \leq \langle |H^{(t)}_{\bar{f}_2}|^2, |H^{(t)}_{\bar{f}_2}|^2\rangle^{1/2}_{\mathcal{S}^2}\langle |H^{(t)}_{\bar{f}_1}|^2, |H^{(t)}_{\bar{f}_1}|^2\rangle^{1/2}_{\mathcal{S}^2}\\ &= \|H^{(t)}_{\bar{f}_2}\|^2_{\mathcal{S}^4}\|H^{(t)}_{\bar{f}_1}\|^2_{\mathcal{S}^4}.\end{aligned}$$

We have the following bound from the above computation

$$\varliminf_{t\to\infty} \|H^{(t)}_{\bar{f}_2}\|^2_{\mathcal{S}^4}\|H^{(t)}_{\bar{f}_1}\|^2_{\mathcal{S}^4} \geq \frac{1}{4\pi^2}\int_{\mathbb{B}_2} \partial f_1 \wedge \partial f_2 \wedge \overline{\partial f_1 \wedge \partial f_2}.$$

This completes the proof of Corollary 7.4. □

**Remark 7.6.** Both $[f_1, g_1, \dots, f_n, g_n]^{\mathrm{fst}}_t$ and $[f_1, g_1, \dots, f_n, g_n]^{\mathrm{scd}}_t$ are sums of compositions of $n$ semi-commutators, each belonging to $\mathcal{S}^p, \forall p > n$. This proves that they belong to $\mathcal{S}^p$, $\forall p > 1$. However, the trace class membership of these operators relies on some higher order cancellation and is therefore nontrivial. As an example, we can show by direct computation that each $\sigma_t(z_i, \bar{z}_i)$ is the diagonal operator under the basis $\{z^\alpha/\|z^\alpha\|\}_{\alpha\in\mathbb{N}_0^n}$, with entry $-(|\alpha| - \alpha_i + n + t)/[(n + |\alpha| + t)(n + |\alpha| + t + 1)]$ at $\alpha \in \mathbb{N}_0^n$. Therefore

$\sigma_t(z_i, \bar{z}_i) \notin \mathcal{S}^n$. (In fact, in [51] it was proved that for $f \in H^\infty(\mathbb{B}_n)$, $\sigma_0(f, \bar{f}) \in \mathcal{S}^n$ if and only if $f$ is constant.) However, the theorem states that

$$[z_1, \bar{z}_1, \dots, z_n, \bar{z}_n]_t^{\mathrm{fst}} = \sum_{\tau \in S_n} \mathrm{sgn}(\tau)\sigma_t(z_{\tau_1}, \bar{z}_1)\dots\sigma_t(z_{\tau_n}, \bar{z}_n) \in \mathcal{S}^1.$$

The partial anti-symmetrization needs to be carefully chosen so that the higher cancellation works. For example,

$$\sigma_t(z_1, \bar{z}_1)\dots\sigma_t(z_n, \bar{z}_n) - \sigma_t(\bar{z}_1, z_2)\sigma_t(\bar{z}_2, z_3)\dots\sigma_t(\bar{z}_n, z_1) = \sigma_t(z_1, \bar{z}_1)\dots\sigma_t(z_n, \bar{z}_n) \notin \mathcal{S}^1.$$

Therefore simply taking anti-symmetrization over a rotation generally does not give a trace class operator. The above example also shows that the Connes-Chern character (1.2) for the Toeplitz extension is in general not well-defined at $p = n$. At $p = n + 1$, by Theorem 6.3, for $f, g \in \mathscr{C}^2(\overline{\mathbb{B}_n})$,

$$\|\sigma_t(f, g)\|_{\mathcal{S}^{n+1}} = \|R^{(t)}_{f,g,1}\|_{\mathcal{S}^{n+1}} \lesssim t^{-1+\frac{n}{n+1}} = t^{-\frac{1}{n+1}}.$$

Therefore for $f_1, \dots, f_{2n+2} \in \mathscr{C}^2(\overline{\mathbb{B}_n})$, the Connes-Chern character at $p = n + 1$ satisfies

$$|\tau_t(f_1, \dots, f_{2n+2})| \lesssim t^{-1} \to 0, \quad t \to \infty.$$

In particular, its value depends on $t$. In Subsection 8.2, we consider the Connes-Chern character at $p > n$ after multiplying a suitable power of $t$.

Theorem 7.3 immediately leads to the trace class membership and asymptotic trace formula for the full anti-symmetric sum, which consist an important part of the proof of our main result, Theorem 8.1.

**Corollary 7.7.** *Suppose $f_1, f_2, \dots, f_{2n} \in \mathscr{C}^2(\overline{\mathbb{B}_n})$ and $t \geq -1$. Then $[T^{(t)}_{f_1}, T^{(t)}_{f_2}, \dots, T^{(t)}_{f_{2n}}]$ is in the trace class $\mathcal{S}^1$. Moreover,*

$$\lim_{t\to\infty} \mathrm{Tr}[T^{(t)}_{f_1}, T^{(t)}_{f_2}, \dots, T^{(t)}_{f_{2n}}] = \frac{n!}{(2\pi i)^n} \int_{\mathbb{B}_n} \mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \dots \wedge \mathrm{d}f_{2n}. \tag{7.5}$$

**Proof.** By Remark 7.2, it is easy to see the following identity

$$[T^{(t)}_{f_1}, T^{(t)}_{f_2}, \dots, T^{(t)}_{f_{2n}}] = \frac{1}{n!} \sum_{\tau \in S_{2n}} \mathrm{sgn}(\tau)[f_{\tau_1}, f_{\tau_2}, \dots, f_{\tau_{2n}}]^{\mathrm{fst}}.$$

Therefore it follows from Theorem 7.3 that $[T^{(t)}_{f_1}, T^{(t)}_{f_2}, \dots, T^{(t)}_{f_{2n}}] \in \mathcal{S}^1$. This proves the trace class membership.

Let $H$ be the collection of subsets of $\{1,\dots,2n\}$ consisting of $n$ elements. For each $a \in H$, let $[a]$ be the subset of $S_{2n}$ that sends $\{1,3,\dots,2n-1\}$ to $a$. Then $\#[a]=(n!)^2$ for any $a \in H$. In each $[a]$ there is a unique permutation $\tau_a$ that satisfies

$$\tau_a(1)<\tau_a(3)<\dots<\tau_a(2n-1),\quad \tau_a(2)<\tau_a(4)<\dots<\tau_a(2n).$$

Then we compute the limit of the trace of full antisymmetrization as follows,

$$\begin{aligned}
&\lim_{t\to\infty}\mathrm{Tr}[T^{(t)}_{f_1},T^{(t)}_{f_2},\dots,T^{(t)}_{f_{2n}}]\\
=&\frac{1}{n!}\sum_{\tau\in S_{2n}}\mathrm{sgn}(\tau)\frac{1}{(2\pi i)^n}\int_{\mathbb{B}_n}\partial f_{\tau_1}\wedge\bar\partial f_{\tau_2}\wedge\dots\wedge\partial f_{\tau_{2n-1}}\wedge\bar\partial f_{\tau_{2n}}\\
=&\frac{1}{(2\pi i)^n n!}\sum_{a\in H}\sum_{\tau\in[a]}\mathrm{sgn}(\tau)\int_{\mathbb{B}_n}\partial f_{\tau_1}\wedge\bar\partial f_{\tau_2}\wedge\dots\wedge\partial f_{\tau_{2n-1}}\wedge\bar\partial f_{\tau_{2n}}\\
=&\frac{1}{(2\pi i)^n n!}\sum_{a\in H}(n!)^2\mathrm{sgn}(\tau_a)\int_{\mathbb{B}_n}\partial f_{\tau_a(1)}\wedge\bar\partial f_{\tau_a(2)}\wedge\dots\wedge\partial f_{\tau_a(2n-1)}\wedge\bar\partial f_{\tau_a(2n)}\\
=&\frac{n!}{(2\pi i)^n}\int_{\mathbb{B}_n}\mathrm{d}f_1\wedge\mathrm{d}f_2\wedge\dots\wedge\mathrm{d}f_{2n}.
\end{aligned}$$

This proves (7.5). □

*7.1. Proof of Theorem 7.3 ($t>-1$)*

Since $\sigma_t(f,g)=\left[\sigma_t(\bar g,\bar f)\right]^*$, we can verify that

$$[f_1,g_1,\dots,f_n,g_n]_t^{\mathrm{fst}}=\left([\bar g_n,\bar f_n,\dots,\bar g_1,\bar f_1]_t^{\mathrm{scd}}\right)^*.$$

Thus it suffices to prove the results for the odd partial anti-symmetric sums.

**Notation 7.8.** For two operators $A$ and $B$, temporarily denote $A\sim_w B$ if $A-B$ is a trace class operator. If $A_t$ and $B_t$ are parameterized families of operators on $L^2_{a,t}(\mathbb{B}_n)$, temporarily denote $A_t\sim_a B_t$ if $A_t-B_t$ are trace class operators on $L^2_{a,t}(\mathbb{B}_n)$ with trace norm tending to zero as $t\to\infty$. Again, we do not require $A$ or $B$ ($A_t$ or $B_t$) to be in the trace class.

The proof is split into two steps. In Step 1, we prove that $[f_1,g_1,\dots,f_n,g_n]_t^{\mathrm{fst}}$ belongs to the trace class, i.e.,

$$[f_1,g_1,\dots,f_n,g_n]_t^{\mathrm{fst}}\sim_w 0.$$

The proof starts with repeatedly applying Theorem 6.3 until we get

$$\sigma_t(f_1, g_1)\sigma_t(f_2, g_2)\dots\sigma_t(f_n, g_n) \sim_w \Theta^{(t)}_{f_1,g_1,\dots,f_n,g_n},$$

where the operator $\Theta^{(t)}_{f_1,g_1,\dots,f_n,g_n}$ involves only complex tangential derivatives. Then we show that $\Theta^{(t)}_{f_1,g_1,\dots,f_n,g_n}$ vanishes after antisymmetrization. In Part 2, we show the asymptotic trace formula (7.4). The proof follows the same idea but requires a careful track of the leading term.

**Part 1:** We prove

$$[f_1, g_1, \dots, f_n, g_n]^{\mathrm{fst}}_t \sim_w 0, \quad \forall t > -1.$$

By Lemma 6.1 and Corollary 6.7, for $f, g \in \mathscr{C}^2(\overline{\mathbb{B}_n})$, recall

$$\sigma_t(f, g) = R^{(t)}_{f,g,1},$$

and

$$R^{(t)}_{f,g,1} = \sum_{a,b=0,1} R^{(t)a,b}_{f,g,1},$$

where

$$R^{(t)a,b}_{f,g,1} \in \mathcal{S}^p, \forall p > \max\left\{\frac{n}{1+\frac{a+b}{2}}, \frac{n}{1+\frac{t+1}{2}}\right\}.$$

In particular, if one of $a, b$ is non-zero, then $R^{(t)a,b}_{f,g,1} \in \mathcal{S}^p$ for some $p < n$. It follows immediately that for $f_1, g_1, \dots, f_n, g_n \in \mathscr{C}^2(\overline{\mathbb{B}_n})$, $\tau \in S_n$,

$$\sigma_t(f_{\tau_1}, g_1)\sigma_t(f_{\tau_2}, g_2)\dots\sigma_t(f_{\tau_n}, g_n) \sim_w R^{(t)0,0}_{f_{\tau_1},g_1,1} R^{(t)0,0}_{f_{\tau_2},g_2,1} \dots R^{(t)0,0}_{f_{\tau_n},g_n,1}.$$

Thus in order to prove $[f_1, g_1, \dots, f_n, g_n]^{\mathrm{fst}}_t \sim_w 0$, it remains to prove

$$\sum_{\tau\in S_n} \mathrm{sgn}(\tau) R^{(t)0,0}_{f_{\tau_1},g_1,1} R^{(t)0,0}_{f_{\tau_2},g_2,1} \dots R^{(t)0,0}_{f_{\tau_n},g_n,1} \sim_w 0. \tag{7.6}$$

Recall that by definition the integral kernel of $R^{(t)0,0}_{f,g,1}$ consists of the part that involves only complex tangential derivatives of $f$ and $g$. Tracing back in Lemma 6.1 we have the following formula

$$R^{(t)0,0}_{f,g,1} h(\xi) = -\int_{\mathbb{B}_n^2} \Phi^{(t)}_{n,1}(|\varphi_z(w)|^2) V_1(z, w) \langle Q_{\bar z}\partial f(z), \overline{z - w}\rangle \langle Q_z \bar\partial g(w), z - w\rangle \cdot h(w) K^{(t)}_z(\xi) K^{(t)}_w(z) \mathrm{d}\lambda_t(w) \mathrm{d}\lambda_t(z),$$

where $V_1(z,w) = \frac{1-\langle z,w\rangle}{1-\langle w,z\rangle}$ is bounded. For any $j = 1,\dots,n$ and $f, g \in \mathscr{C}^2(\overline{\mathbb{B}_n})$, define

$$F_{\Lambda_{g,1,j}}(z,w) = -V_1(z,w)(z_j - w_j)\langle Q_z\bar{\partial}g(w), z-w\rangle K_w^{(t)}(z), \quad z,w\in\mathbb{B}_n.$$

Define the operator

$$\Lambda_{g,1,j}h(z) = \int_{\mathbb{B}_n} \Phi_{n,1}^{(t)}(|\varphi_z(w)|^2)F_{\Lambda_{g,1,j}}(z,w)h(w)\mathrm{d}\lambda_t(w). \tag{7.7}$$

Let $[Q_{\bar{z}}\partial f(z)]_j$ be the $j$-th entry of the vector $Q_{\bar{z}}\partial f(z)$. Then by the above expressions, we have the following expression

$$R_{f,g,1}^{(t)0,0} = \sum_{j=1}^{n} P^{(t)} M_{[Q_{\bar{z}}\partial f]_j}^{(t)} \Lambda_{g,1,j}.$$

By Lemma 2.2 we have the bound

$$\left|F_{\Lambda_{g,1,i}}(z,w)\right| \lesssim |\varphi_z(w)|^2 \frac{1}{|1-\langle z,w\rangle|^{n+1+t-1}}, \quad z,w\in\mathbb{B}_n.$$

Thus by Theorem 4.6 and Corollary 4.7, for any Lipschitz function $u$ on $\mathbb{B}_n$, $j = 1,\dots,n$, the following hold.

- Both $P^{(t)}\Lambda_{g,1,j}$ and $\Lambda_{g,1,j}P^{(t)}$ are in $\mathcal{S}^p$ for any $p > n$, with Schatten-$p$ norm $\lesssim_p t^{-1+\frac{n}{p}}$.
- For any Lipschitz function $u$, both $P^{(t)}[\Lambda_{g,1,j}, M_u^{(t)}]$ and $[\Lambda_{g,1,j}, M_u^{(t)}]P^{(t)}$ are in $\mathcal{S}^p$ for any $p > \max\{\frac{2n}{3}, \frac{n}{1+\frac{t+1}{2}}\}$, with Schatten-$p$ norm $\lesssim_p t^{-1+\frac{n}{p}}$. In particular, these operators belong to $\mathcal{S}^p$ for some $p < n$.

Also, by Corollary 4.3, we have

- For any Lipschitz function $u$, $[P^{(t)}, M_u^{(t)}] \in \mathcal{S}^p$ for any $p > 2n$.

Denote

$$u_{i,j}(z) = [Q_{\bar{z}}\partial f_i(z)]_j, \quad i,j = 1,\dots,n.$$

Then by the above discussion, we compute the product of semi-commutators as follows.

$$\begin{aligned} &\sigma_t(f_1,g_1)\sigma_t(f_2,g_2)\dots\sigma_t(f_n,g_n) \\ \sim_w &R_{f_1,g_1,1}^{(t)0,0} R_{f_2,g_2,1}^{(t)0,0} \dots R_{f_n,g_n,1}^{(t)0,0} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j_1,\ldots,j_n=1}^{n} P^{(t)} M^{(t)}_{u_{1,j_1}} \Lambda_{g_1,1,j_1} P^{(t)} M^{(t)}_{u_{2,j_2}} \Lambda_{g_2,1,j_2} \ldots P^{(t)} M^{(t)}_{u_{n,j_n}} \Lambda_{g_n,1,i_n} P^{(t)} \\
&= \sum_{j_1,\ldots,j_n=1}^{n} P^{(t)} \Lambda_{g_1,1,j_1} P^{(t)} M^{(t)}_{u_{1,j_1}} M^{(t)}_{u_{2,j_2}} \Lambda_{g_2,1,j_2} \ldots P^{(t)} M^{(t)}_{u_{n,j_n}} \Lambda_{g_n,1,i_n} P^{(t)} \\
&\quad + \sum_{j_1,\ldots,j_n=1}^{n} P^{(t)} [M^{(t)}_{u_{1,j_1}}, \Lambda_{g_1,1,j_1}] P^{(t)} M^{(t)}_{u_{2,j_2}} \Lambda_{g_2,1,j_2} \ldots P^{(t)} M^{(t)}_{u_{n,j_n}} \Lambda_{g_n,1,i_n} P^{(t)} \\
&\quad + \sum_{j_1,\ldots,j_n=1}^{n} P^{(t)} \Lambda_{g_1,1,j_1} [M^{(t)}_{u_{1,j_1}}, P^{(t)}] M^{(t)}_{u_{2,j_2}} \Lambda_{g_2,1,j_2} \ldots P^{(t)} M^{(t)}_{u_{n,j_n}} \Lambda_{g_n,1,i_n} P^{(t)} \\
&\sim_w \sum_{j_1,\ldots,j_n=1}^{n} P^{(t)} \Lambda_{g_1,1,j_1} P^{(t)} M^{(t)}_{u_{1,j_1}} M^{(t)}_{u_{2,j_2}} \Lambda_{g_2,1,j_2} \ldots P^{(t)} M^{(t)}_{u_{n,j_n}} \Lambda_{g_n,1,i_n} P^{(t)}.
\end{aligned}
$$

Continuing like this, we obtain the following expression

$$
\begin{aligned}
&\sigma_t(f_1, g_1)\sigma_t(f_2, g_2) \ldots \sigma_t(f_n, g_n) \\
&\sim_w \sum_{j_1,\ldots,j_n=1}^{n} P^{(t)} \Lambda_{g_1,1,j_1} P^{(t)} \Lambda_{g_2,1,j_2} \ldots P^{(t)} \Lambda_{g_n,1,j_n} M^{(t)}_{u_{1,j_1}} M^{(t)}_{u_{2,j_2}} \ldots M^{(t)}_{u_{n,j_n}} P^{(t)} \qquad (7.8) \\
&:= \Theta^{(t)}_{f_1,g_1,\ldots,f_n,g_n}.
\end{aligned}
$$

Writing the operator above in integral form, for any $h \in L^2_{a,t}(\mathbb{B}_n)$,

$$
\begin{aligned}
&\Theta^{(t)}_{f_1,g_1,\ldots,f_n,g_n} h(\xi) \\
&= \int_{\mathbb{B}_n^{2n}} \left[\prod_{i=1}^{n} \Phi^{(t)}_{n,1}(|\varphi_{z_i}(w_i)|^2)\right] \cdot \left[\prod_{i=1}^{n} \langle Q_{z_i} \bar{\partial} g_i(w_i), z_i - w_i\rangle\right] \cdot \left[\prod_{i=1}^{n} V_1(z_i, w_i)\right] \\
&\quad \cdot \left[\prod_{i=1}^{n} \langle Q_{\bar{w}_n} \partial f_i(w_n), \overline{z_i - w_i}\rangle\right] h(w_n) \\
&\quad \cdot K^{(t)}_{z_1}(\xi) K^{(t)}_{w_1}(z_1) \ldots K^{(t)}_{z_n}(w_{n-1}) K^{(t)}_{w_n}(z_n) \mathrm{d}\lambda_t(w_n) \mathrm{d}\lambda_t(z_n) \ldots \mathrm{d}\lambda_t(w_1) \mathrm{d}\lambda_t(z_1).
\end{aligned}
$$

We claim that $\Theta^{(t)}_{f_1,g_1,\ldots,f_n,g_n}$ vanishes after antisymmetrization over the $f$-symbols. To show this property, it suffices to show that

$$
\sum_{\tau \in S_n} \operatorname{sgn}(\tau) \prod_{i=1}^{n} \langle Q_{\bar{w}_n} \partial f_{\tau_i}(w_n), \overline{z_i - w_i}\rangle = 0. \qquad (7.9)
$$

Each $\langle Q_{\bar{w}_n} \partial f_{\tau_i}(w_n), \overline{z_i - w_i}\rangle$ is independent of the choice of coordinates. Thus we may assume without loss of generality that $w_n = (r, 0, \ldots, 0)$. In this case, we have the expression

$$\langle Q_{\bar{w}_n}\partial f_{\tau_i}(w_n), \overline{z_i - w_i}\rangle = \sum_{j=2}^{n}(z_{i,j} - w_{i,j})\partial_j f_{\tau_i}(w_n).$$

Then the above equals

$$\sum_{j_1,\dots,j_n=2}^{n}(z_{1,j_1} - w_{1,j_1})\dots(z_{n,j_n} - w_{n,j_n})\sum_{\tau\in S_n}\mathrm{sgn}(\tau)\partial_{j_1}f_{\tau_1}(w_n)\partial_{j_2}f_{\tau_2}(w_n)\dots\partial_{j_n}f_{\tau_n}(w_n).$$

Since $j_1,\dots,j_n$ takes value in $\{2,\dots,n\}$, at least two indices are equal. This implies that

$$\sum_{\tau\in S_n}\mathrm{sgn}(\tau)\partial_{j_1}f_{\tau_1}(w_n)\partial_{j_2}f_{\tau_2}(w_n)\dots\partial_{j_n}f_{\tau_n}(w_n) = 0.$$

Therefore (7.9) holds. We conclude from this fact that the anti-symmetric sum

$$\sum_{\tau\in S_n}\mathrm{sgn}(\tau)\Theta^{(t)}_{f_{\tau_1},g_1,\dots,f_{\tau_n},g_n}$$

equals zero which completes the proof of Part 1. Namely we have

$$[f_1, g_1, \dots, f_n, g_n]_t^{\mathrm{fst}} \sim_w 0.$$

**Part 2:** We prove

$$\lim_{t\to\infty}\mathrm{Tr}[f_1, g_1, \dots, f_n, g_n]_t^{\mathrm{fst}} = \frac{1}{(2\pi i)^n}\int_{\mathbb{B}_n}\partial f_1\wedge\bar{\partial}g_1\wedge\dots\wedge\partial f_n\wedge\bar{\partial}g_n.$$

For this part, we assume that $t$ is large enough. We use the quantization formula (6.1) at $k = 1$. By Lemma 6.1, Remark 6.4 and Corollary 6.7, for $f, g \in \mathscr{C}^2(\overline{\mathbb{B}_n})$, we have the following decomposition

$$\sigma_t(f, g) = R^{(t)}_{f,g,1} = c_{1,t}T^{(t)}_{C_1(f,g)} + R^{(t)}_{f,g,2}. \tag{7.10}$$

The following hold.

1. $$c_{1,t} = nt^{-1} + O(t^{-2}),$$

2. $$C_1(f, g) = C_N(f, g) + C_T(f, g),$$

   where $C_N(f, g)$ ($C_T(f, g)$) denotes the part involving complex normal (tangential) derivatives of $f, g$, and

$$T^{(t)}_{C_N(f,g)} \in \mathcal{S}^p, \forall p > \frac{n}{2},\ \text{with}\ \|T^{(t)}_{C_N(f,g)}\|_{\mathcal{S}^p} \lesssim_p t^{\frac{n}{p}},$$
$$T^{(t)}_{C_T(f,g)} \in \mathcal{S}^p, \forall p > n,\ \text{with}\ \|T^{(t)}_{C_T(f,g)}\|_{\mathcal{S}^p} \lesssim_p t^{\frac{n}{p}}.$$

3. $$R^{(t)}_{f,g,1} = \sum_{a,b=0,1} R^{(t)a,b}_{f,g,1}, \quad R^{(t)}_{f,g,2} = \sum_{a,b=0,1,2} R^{(t)a,b}_{f,g,2},$$

where $a, b$ denote the order of derivatives on $f, g$ in the complex normal directions, and for large $t$,

$$R^{(t)a,b}_{f,g,i} \in \mathcal{S}^p, \forall p > \frac{n}{1+\frac{a+b}{2}}, \quad i = 1, 2,$$
$$\|R^{(t)a,b}_{f,g,i}\|_{\mathcal{S}^p} \lesssim_p t^{-i+\frac{n}{p}}.$$

The above will be the main tool for Part 2 and will be repeatedly used without reference. We will prove Part 2 by establishing the following properties.

$$[f_1, g_1, \dots, f_n, g_n]^{\text{fst}}_t \sim_a c^n_{1,t} \sum_{\tau\in S_n} \operatorname{sgn}(\tau) T^{(t)}_{C_1(f_{\tau_1},g_1)} T^{(t)}_{C_1(f_{\tau_2},g_2)} \dots T^{(t)}_{C_1(f_{\tau_n},g_n)}, \tag{7.11}$$

$$c^n_{1,t} \sum_{\tau\in S_n} \operatorname{sgn}(\tau) T^{(t)}_{C_1(f_{\tau_1},g_1)} T^{(t)}_{C_1(f_{\tau_2},g_2)} \dots T^{(t)}_{C_1(f_{\tau_n},g_n)} \sim_a c^n_{1,t} \sum_{\tau\in S_n} \operatorname{sgn}(\tau) T^{(t)}_{C_1(f_{\tau_1},g_1)C_1(f_{\tau_2},g_2)C_1(f_{\tau_n},g_n)}, \tag{7.12}$$

$$\operatorname{Tr}\left[c^n_{1,t} \sum_{\tau\in S_n} \operatorname{sgn}(\tau) T^{(t)}_{C_1(f_{\tau_1},g_1)C_1(f_{\tau_2},g_2)C_1(f_{\tau_n},g_n)}\right] \to \frac{1}{(2\pi i)^n} \int_{\mathbb{B}_n} \partial f_1 \wedge \bar\partial g_1 \wedge \dots \wedge \partial f_n \wedge \bar\partial g_n, t \to \infty. \tag{7.13}$$

**Proof of (7.11).** By (7.10), we compute the product of semi-commutators

$$\begin{aligned}
&\sigma_t(f_1, g_1)\sigma_t(f_2, g_2)\dots\sigma_t(f_n, g_n) - c^n_{1,t} T^{(t)}_{C_1(f_1,g_1)} T^{(t)}_{C_1(f_2,g_2)} \dots T^{(t)}_{C_1(f_n,g_n)} \\
=& R^{(t)}_{f_1,g_1,1} R^{(t)}_{f_2,g_2,1} \dots R^{(t)}_{f_n,g_n,1} - \left(R^{(1)}_{f_1,g_1,1} - R^{(t)}_{f_1,g_1,2}\right)\left(R^{(1)}_{f_2,g_2,1} - R^{(t)}_{f_2,g_2,2}\right)\dots\left(R^{(1)}_{f_n,g_n,1} - R^{(t)}_{f_n,g_n,2}\right) \\
=& \sum_{(i_1,i_2,\dots,i_n)\in X} \pm R^{(t)}_{f_1,g_1,i_1} R^{(t)}_{f_2,g_2,i_2} \dots R^{(t)}_{f_n,g_n,i_n},
\end{aligned}$$

where

$$X = \{(i_1, i_2, \dots, i_n) : i_j = 1, 2, \text{ and at least one } i_j = 2\}.$$

Therefore

$$[f_1, g_1, \dots, f_n, g_n]^{\text{fst}}_t - c^n_{1,t} \sum_{\tau\in S_n} \operatorname{sgn}(\tau) T^{(t)}_{C_1(f_{\tau_1},g_1)} T^{(t)}_{C_1(f_{\tau_2},g_2)} \dots T^{(t)}_{C_1(f_{\tau_n},g_n)}$$

is a linear combination of operators of the form

$$\sum_{\tau\in S_n} \operatorname{sgn}(\tau) R^{(t)}_{f_{\tau_1},g_1,i_1} R^{(t)}_{f_{\tau_2},g_2,i_2} \dots R^{(t)}_{f_{\tau_n},g_n,i_n},$$

where $(i_1, i_2, \dots, i_n) \in X$. For (7.11), it suffices to prove

$$\sum_{\tau\in S_n} \operatorname{sgn}(\tau) R^{(t)}_{f_{\tau_1},g_1,i_1} R^{(t)}_{f_{\tau_2},g_2,i_2} \dots R^{(t)}_{f_{\tau_n},g_n,i_n} \sim_a 0, \quad \forall (i_1, \dots, i_n) \in X. \tag{7.14}$$

We show the case when $(i_1, i_2, \dots, i_n) = (2, 1, \dots, 1)$, i.e.,

$$\sum_{\tau\in S_n} \operatorname{sgn}(\tau) R^{(t)}_{f_{\tau_1},g_1,2} R^{(t)}_{f_{\tau_2},g_2,1} \dots R^{(t)}_{f_{\tau_n},g_n,1} \sim_a 0.$$

First, each $R^{(t)}_{f,g,i}$ decomposes into the sum of $R^{(t)a,b}_{f,g,i}$. The operator $R^{(t)0,0}_{f,g,i} \in \mathcal{S}^p, \forall p > n$, and

$$\|R^{(t)0,0}_{f,g,1}\|_{\mathcal{S}^p} \lesssim t^{-1+\frac{n}{p}}, \quad \|R^{(t)0,0}_{f,g,2}\|_{\mathcal{S}^p} \lesssim t^{-2+\frac{n}{p}}.$$

If $(a, b) \neq (0, 0)$ then the operator belongs to $\mathcal{S}^p$ for some $p < n$, with asymptotic Schatten-norm estimates

$$\|R^{(t)a,b}_{f,g,1}\|_{\mathcal{S}^p} \lesssim t^{-1+\frac{n}{p}}, \quad \|R^{(t)a,b}_{f,g,2}\|_{\mathcal{S}^p} \lesssim t^{-2+\frac{n}{p}}.$$

Therefore for each $\tau \in S_n$, we have the following equation

$$R^{(t)}_{f_{\tau_1},g_1,2} R^{(t)}_{f_{\tau_2},g_2,1} \dots R^{(t)}_{f_{\tau_n},g_n,1} = \sum_{\substack{a_1,b_1=0,1,2\\ a_2,b_2,\dots,a_n,b_n=0,1}} R^{(t)a_1,b_1}_{f_{\tau_1},g_1,2} R^{(t)a_2,b_2}_{f_{\tau_2},g_2,1} \dots R^{(t)a_n,b_n}_{f_{\tau_n},g_n,1}.$$

Suppose that there is some $(a_k, b_k) \neq (0, 0)$. Then we can choose $1 \le p_1, \dots, p_n < \infty$ such that $\sum_j \frac{1}{p_j} = 1$, and $R^{(t)a_j,b_j}_{f_{\tau_j},g_j,i_j} \in \mathcal{S}^{p_i},\ i = 1, \dots, n$. So

$$R^{(t)a_1,b_1}_{f_{\tau_1},g_1,2} R^{(t)a_2,b_2}_{f_{\tau_2},g_2,1} \dots R^{(t)a_n,b_n}_{f_{\tau_n},g_n,1} \sim_w 0,$$

with trace norm

$$\lesssim t^{-2+\frac{n}{p_1}} \cdot t^{-1+\frac{n}{p_2}} \cdot \ldots \cdot t^{-1+\frac{n}{p_n}} = t^{-1}.$$

It follows that $R^{(t)a_1,b_1}_{f_{\tau_1},g_1,2} R^{(t)a_2,b_2}_{f_{\tau_2},g_2,1} \dots R^{(t)a_n,b_n}_{f_{\tau_n},g_n,1} \sim_a 0$. And we conclude

$$R^{(t)}_{f_{\tau_1},g_1,2} R^{(t)}_{f_{\tau_2},g_2,1} \dots R^{(t)}_{f_{\tau_n},g_n,1} \sim_a R^{(t)0,0}_{f_{\tau_1},g_1,2} R^{(t)0,0}_{f_{\tau_2},g_2,1} \dots R^{(t)0,0}_{f_{\tau_n},g_n,1}. \tag{7.15}$$

Next, we show

$$\sum_{\tau\in S_n} \operatorname{sgn}(\tau) R^{(t)0,0}_{f_{\tau_1},g_1,2} R^{(t)0,0}_{f_{\tau_2},g_2,1} \dots R^{(t)0,0}_{f_{\tau_n},g_n,1} \sim_a 0. \tag{7.16}$$

The proof is an almost verbatim repetition of the proof of (7.6). Tracing back the definition, for $f, g \in \mathscr{C}^2(\overline{\mathbb{B}_n})$ and $h \in L^2_{a,t}(\mathbb{B}_n)$, we compute $R^{(t)0,0}_{f,g,2}$,

$$R^{(t)0,0}_{f,g,2} h(\xi) = \int_{\mathbb{B}_n^2} \Phi^{(t)}_{n,2}(|\varphi_z(w)|^2) V_2(z,w) \langle Q_{\bar{z}} \partial f(z), \overline{z-w}\rangle \langle Q_z \bar{\partial} g(w), z-w\rangle \tag{7.17}$$
$$\cdot h(w) K^{(t)}_z(\xi) K^{(t)}_w(z) \mathrm{d}\lambda_t(w) \mathrm{d}\lambda_t(z),$$

with

$$V_2(z,w) = \frac{\sum_{i_2,j_2=1}^n I^{e_{i_2},e_{j_2}}(z-w) \partial_{z_{i_2}} \bar{\partial}_{w_{j_2}} (1-\langle z,w\rangle)^2}{(1-\langle w,z\rangle)^2} \tag{7.18}$$
$$= 2\frac{1-\langle z,w\rangle}{(1-\langle w,z\rangle)^2} \cdot C(z,w) + \frac{B_1(z,w) B_2(z,w)}{(1-\langle w,z\rangle)^2}, \tag{7.19}$$

where $C(z,w), B_1(z,w), B_2(z,w)$ are as in the proof of Lemma 6.11. By the estimates in the proof, we get

$$|V_2(z,w)| \lesssim |\varphi_z(w)|^2. \tag{7.20}$$

Similarly to the proof of (7.6), we write

$$R^{(t)0,0}_{f,g,2} = \sum_{j=1}^n P^{(t)} M^{(t)}_{[Q_z \partial f]_j} \Lambda_{g,2,j},$$

where

$$\Lambda_{g,2,j} h(z) = \int_{\mathbb{B}_n} \Phi^{(t)}_{n,2}(|\varphi_z(w)|^2) F_{\Lambda_{g,2,j}}(z,w) h(w) \mathrm{d}\lambda_t(w),$$

with

$$F_{\Lambda_{g,2,j}}(z,w) = V_2(z,w)(z_j - w_j) \langle Q_z \bar{\partial} g(w), z-w\rangle K^{(t)}_w(z).$$

By (7.20) and Lemma 2.2, we have the following estimate

$$\left|F_{\Lambda_{g,2,j}}(z,w)\right| \lesssim |\varphi_z(w)|^2 \frac{1}{|1-\langle z,w\rangle|^{n+1+t-1}}.$$

Again, by Theorem 4.6 and Corollary 4.7, for any Lipschitz function $u$ on $\mathbb{B}_n$, we have the following properties.

- Both $P^{(t)}\Lambda_{g,2,j}$ and $\Lambda_{g,2,j}P^{(t)}$ are in $\mathcal{S}^p$ for any $p > n$, with Schatten-$p$ norm $\lesssim_p t^{-2+\frac{n}{p}}$.
- Both $P^{(t)}[\Lambda_{g,2,j}, M_u^{(t)}]$ and $[\Lambda_{g,2,j}, M_u^{(t)}]P^{(t)}$ are in $\mathcal{S}^p$ for any $p > \frac{2n}{3}$, with Schatten-$p$ norm $\lesssim_p t^{-2+\frac{n}{p}}$.
- For any Lipschitz function $u$, $[P^{(t)}, M_u^{(t)}] = [P^{(t)}, M_u^{(t)}]P^{(t)} - P^{(t)}[P^{(t)}, M_u^{(t)}] \in \mathcal{S}^p$ for any $p > 2n$, with $\|[P^{(t)}, M_u^{(t)}]\|_{\mathcal{S}^p} \lesssim_p t^{\frac{n}{p}}$.

Then as in the proof of (7.8), we compute the product of $R^{(t)0,0}_{f_{\tau_1},f_1,2}$ and $R^{(t)0,0}_{f_{\tau_i},g_i,1}$ $(i = 2, \cdots, n)$

$$
\begin{aligned}
&R^{(t)0,0}_{f_{\tau_1},g_1,2}R^{(t)0,0}_{f_{\tau_2},g_2,1}\dots R^{(t)0,0}_{f_{\tau_n},g_n,1}\\
=&\sum_{j_1,\dots,j_n=1}^{n} P^{(t)}M^{(t)}_{u_{\tau_1,j_1}}\Lambda_{g_1,2,j_1}P^{(t)}M^{(t)}_{u_{\tau_2,j_2}}\Lambda_{g_2,1,j_2}\dots P^{(t)}M^{(t)}_{u_{\tau_n,j_n}}\Lambda_{g_n,1,j_n}P^{(t)}\\
\sim_a&\sum_{j_1,\dots,j_n=1}^{n} P^{(t)}\Lambda_{g_1,2,j_1}P^{(t)}\Lambda_{g_2,1,j_2}\dots P^{(t)}\Lambda_{g_n,1,j_n}M^{(t)}_{u_{\tau_1,j_1}}M^{(t)}_{u_{\tau_2,j_2}}\dots M^{(t)}_{u_{\tau_n,j_n}}P^{(t)}\\
:=&\Theta'^{(t)}_{f_{\tau_1},g_1,\dots,f_{\tau_n},g_n}.
\end{aligned}
$$

Again, by (7.9), we conclude with the following equation

$$
\sum_{\tau\in S_n}\operatorname{sgn}(\tau)\Theta'^{(t)}_{f_{\tau_1},g_1,\dots,f_{\tau_n},g_n} = 0.
$$

This proves (7.16). And together with (7.15), it proves (7.14) for $(i_1, \dots, i_n) = (2, 1, \dots, 1)$. The proof for general $(i_1, \dots, i_n) \in X$ is an almost verbatim repetition of the above. This finishes the proof of (7.11).

**Proof of (7.12).** Denote

$$
\phi_i(z) = (1 - |z|^2)^i, \quad i = 1, 2, \dots .
$$

Suppose $f, g \in \mathscr{C}^2(\overline{\mathbb{B}_n})$. For $z \neq 0$, let $\mathbf{e}_z = \{e_{z,1}, e_{z,2}, \dots, e_{z,n}\}$ be an orthonormal basis of $\mathbb{C}^n$ such that $e_{z,1} = \frac{z}{|z|}$. Let $\varphi : \mathbb{C}^n \to [0, 1]$ be a smooth function that equals 1 inside $\frac{1}{4}\mathbb{B}_n$ and vanishes outside $\frac{1}{2}\mathbb{B}_n$. Under the basis $\mathbf{e}_z$, let

$$
D_{f,g,1}(z) = -\frac{1}{n}\left[\sum_{i=2}^{n}\partial_i f(z)\bar{\partial}_i g(z)\right](1 - \varphi(z)), \tag{7.21}
$$

and

$$
D_{f,g,2}(z) = -\frac{1}{n}\partial_1 g(z)\bar{\partial}_1 g(z)(1 - \varphi(z)) + \frac{C_1(f, g)(z)\varphi(z)}{(1 - |z|^2)^2}. \tag{7.22}
$$

Then by Remark 6.4 and direct computation, we have the following decomposition

$$C_1(f,g) = \phi_1 D_{f,g,1} + \phi_2 D_{f,g,2}.$$

By Remark 6.5, $D_{f,g,1}, D_{f,g,2} \in \mathscr{C}^1(\overline{\mathbb{B}_n})$. With the decomposition we have the following formula

$$\begin{aligned} &c_{1,t}^n T^{(t)}_{C_1(f_1,g_1)} T^{(t)}_{C_1(f_2,g_2)} \cdots T^{(t)}_{C_1(f_n,g_n)} - T^{(t)}_{c_{1,t}^n C_1(f_1,g_1) C_1(f_2,g_2) \ldots C_1(f_n,g_n)} \\ =&c_{1,t}^n \sum_{i_1,\ldots,i_n=1,2} \left[ T^{(t)}_{\phi_{i_1} D_{f_1,g_1,i_1}} T^{(t)}_{\phi_{i_2} D_{f_2,g_2,i_2}} \cdots T^{(t)}_{\phi_{i_n} D_{f_n,g_n,i_n}} - T^{(t)}_{\phi_{i_1} D_{f_1,g_1,i_1} \phi_{i_2} D_{f_2,g_2,i_2} \cdots \phi_{i_n} D_{f_n,g_n,i_n}} \right]. \end{aligned} \tag{7.23}$$

By Theorem 6.3, Corollary 6.8 and Lemma 4.8 for $t$ large enough, $i, j = 1, 2, \ldots$, and $u, v \in \mathscr{C}^1(\overline{\mathbb{B}_n})$, we obtain the following estimates.

$$\begin{aligned} T^{(t)}_u T^{(t)}_v - T^{(t)}_{uv} \in \mathcal{S}^p,\ \forall p > n, &\quad \text{and } \|T^{(t)}_u T^{(t)}_v - T^{(t)}_{uv}\|_{\mathcal{S}^p} \lesssim_p t^{-1+\frac{n}{p}}, \\ T^{(t)}_{\phi_i} T^{(t)}_u - T^{(t)}_{\phi_i u} \in \mathcal{S}^p,\ \forall p > \frac{n}{i+\frac{1}{2}}, &\quad \text{and } \|T^{(t)}_{\phi_i} T^{(t)}_u - T^{(t)}_{\phi_i u}\|_{\mathcal{S}^p} \lesssim_p t^{-1+\frac{n}{p}}, \\ T^{(t)}_u T^{(t)}_{\phi_i} - T^{(t)}_{\phi_i u} \in \mathcal{S}^p,\ \forall p > \frac{n}{i+\frac{1}{2}}, &\quad \text{and } \|T^{(t)}_u T^{(t)}_{\phi_i} - T^{(t)}_{\phi_i u}\|_{\mathcal{S}^p} \lesssim_p t^{-1+\frac{n}{p}}, \end{aligned}$$

and

$$\begin{aligned} T^{(t)}_{\phi_i} T^{(t)}_{\phi_j} - T^{(t)}_{\phi_i \phi_j} \in \mathcal{S}^p,\ \forall p > \frac{n}{i+j}, &\quad \text{and } \|T^{(t)}_{\phi_i} T^{(t)}_{\phi_j} - T^{(t)}_{\phi_i \phi_j}\|_{\mathcal{S}^p} \lesssim_p t^{-1+\frac{n}{p}}, \\ T^{(t)}_{\phi_i u} \in \mathcal{S}^p,\ \forall p > \frac{n}{i}, &\quad \text{and } \|T^{(t)}_{\phi_i u}\|_{\mathcal{S}^p} \lesssim_p t^{\frac{n}{p}}, \end{aligned}$$

and

$$c_{1,t} \approx t^{-1}.$$

Therefore by the above estimates, we compute the product of $T^{(t)}_{\phi_{i_j} D_{f_j,g_j,i_j}}$.

$$\begin{aligned} &c_{1,t}^n T^{(t)}_{\phi_{i_1} D_{f_1,g_1,i_1}} T^{(t)}_{\phi_{i_2} D_{f_2,g_2,i_2}} \cdots T^{(t)}_{\phi_{i_n} D_{f_n,g_n,i_n}} \\ =&c_{1,t}^n T^{(t)}_{\phi_{i_1}} T^{(t)}_{D_{f_1,g_1,i_1}} T^{(t)}_{\phi_{i_2} D_{f_2,g_2,i_2}} \cdots T^{(t)}_{\phi_{i_n} D_{f_n,g_n,i_n}} \\ &\qquad - c_{1,t}^n \big( T^{(t)}_{\phi_{i_1}} T^{(t)}_{D_{f_1,g_1,i_1}} - T^{(t)}_{\phi_{i_1} D_{f_1,g_1,i_1}} \big) T^{(t)}_{\phi_{i_2} D_{f_2,g_2,i_2}} \cdots T^{(t)}_{\phi_{i_n} D_{f_n,g_n,i_n}} \\ \sim_a& c_{1,t}^n T^{(t)}_{\phi_{i_1}} T^{(t)}_{D_{f_1,g_1,i_1}} T^{(t)}_{\phi_{i_2} D_{f_2,g_2,i_2}} \cdots T^{(t)}_{\phi_{i_n} D_{f_n,g_n,i_n}} \\ &\cdots \\ \sim_a& c_{1,t}^n T^{(t)}_{\phi_{i_1}} T^{(t)}_{D_{f_1,g_1,i_1}} T^{(t)}_{\phi_{i_2}} T^{(t)}_{D_{f_2,g_2,i_2}} \cdots T^{(t)}_{\phi_{i_n}} T^{(t)}_{D_{f_n,g_n,i_n}} \\ =&c_{1,t}^n T^{(t)}_{\phi_{i_1}} T^{(t)}_{\phi_{i_2}} T^{(t)}_{D_{f_1,g_1,i_1}} T^{(t)}_{D_{f_2,g_2,i_2}} \cdots T^{(t)}_{\phi_{i_n}} T^{(t)}_{D_{f_n,g_n,i_n}} \end{aligned}$$

$$
\begin{aligned}
&\qquad + c_{1,t}^n T_{\phi_{i_1}}^{(t)} \left[T_{D_{f_1,g_1,i_1}}^{(t)}, T_{\phi_{i_2}}^{(t)}\right] T_{D_{f_2,g_2,i_2}}^{(t)} \dots T_{\phi_{i_n}}^{(t)} T_{D_{f_n,g_n,i_n}}^{(t)} \\
\sim_a & c_{1,t}^n T_{\phi_{i_1}}^{(t)} T_{\phi_{i_2}}^{(t)} T_{D_{f_1,g_1,i_1}}^{(t)} T_{D_{f_2,g_2,i_2}}^{(t)} \dots T_{\phi_{i_n}}^{(t)} T_{D_{f_n,g_n,i_n}}^{(t)} \\
&\dots \\
\sim_a & c_{1,t}^n T_{\phi_{i_1}}^{(t)} \dots T_{\phi_{i_n}}^{(t)} T_{D_{f_1,g_1,i_1}}^{(t)} T_{D_{f_2,g_2,i_2}}^{(t)} \dots T_{D_{f_n,g_n,i_n}}^{(t)} \\
= & c_{1,t}^n T_{\phi_{i_1}}^{(t)} \dots T_{\phi_{i_n}}^{(t)} T_{D_{f_1,g_1,i_1} D_{f_2,g_2,i_2}}^{(t)} \dots T_{D_{f_n,g_n,i_n}}^{(t)} \\
&\qquad + c_{1,t}^n T_{\phi_{i_1}}^{(t)} \dots T_{\phi_{i_n}}^{(t)} \left(T_{D_{f_1,g_1,i_1}}^{(t)} T_{D_{f_2,g_2,i_2}}^{(t)} - T_{D_{f_1,g_1,i_1} D_{f_2,g_2,i_2}}^{(t)}\right) T_{\phi_{i_3}}^{(t)} \dots T_{D_{f_n,g_n,i_n}}^{(t)} \\
\sim_a & c_{1,t}^n T_{\phi_{i_1}}^{(t)} \dots T_{\phi_{i_n}}^{(t)} T_{D_{f_1,g_1,i_1} D_{f_2,g_2,i_2}}^{(t)} T_{\phi_{i_3}}^{(t)} \dots T_{D_{f_n,g_n,i_n}}^{(t)} \\
&\dots \\
\sim_a & c_{1,t}^n T_{\phi_{i_1}}^{(t)} \dots T_{\phi_{i_n}}^{(t)} T_{D_{f_1,g_1,i_1} D_{f_2,g_2,i_2} \dots D_{f_n,g_n,i_n}}^{(t)}.
\end{aligned}
$$

If some $i_k = 2$ in the above, we continue the above computation as follows,

$$
\begin{aligned}
\sim_a & c_{1,t}^n T_{\phi_{i_1}\phi_{i_2}}^{(t)} T_{\phi_{i_3}}^{(t)} \dots T_{\phi_{i_n}}^{(t)} T_{D_{f_1,g_1,i_1} D_{f_2,g_2,i_2} \dots D_{f_n,g_n,i_n}}^{(t)} \\
&\dots \\
\sim_a & c_{1,t}^n T_{\phi_{i_1}\phi_{i_2}\dots\phi_{i_n}}^{(t)} T_{D_{f_1,g_1,i_1} D_{f_2,g_2,i_2} \dots D_{f_n,g_n,i_n}}^{(t)} \\
\sim_a & c_{1,t}^n T_{\phi_{i_1}\phi_{i_2}\dots\phi_{i_n} D_{f_1,g_1,i_1} D_{f_2,g_2,i_2} \dots D_{f_n,g_n,i_n}}^{(t)}.
\end{aligned}
$$

This proves that if there is some $i_k = 2$, then we have the following equation

$$
c_{1,t}^n T_{\phi_{i_1} D_{f_1,g_1,i_1}}^{(t)} T_{\phi_{i_2} D_{f_2,g_2,i_2}}^{(t)} \dots T_{\phi_{i_n} D_{f_n,g_n,i_n}}^{(t)} \sim_a c_{1,t}^n T_{\phi_{i_1}\phi_{i_2}\dots\phi_{i_n} D_{f_1,g_1,i_1} D_{f_2,g_2,i_2} \dots D_{f_n,g_n,i_n}}^{(t)}. \tag{7.24}
$$

If all $i_k = 1$, in the steps right above (7.24), the difference operator may not belong to the trace class. We need to take the anti-symmetrization into consideration. Similarly to the proof of (7.9), the odd anti-symmetrization is over $n$ symbols, but $Q_z \partial f$ has only $n-1$ entries under the basis $\mathbf{e}_z$. Thus we have the identities

$$
\sum_{\tau \in S_n} \operatorname{sgn}(\tau) D_{f_{\tau_1},g_1,1} D_{f_{\tau_2},g_2,1} \dots D_{f_{\tau_n},g_n,1} = 0, \tag{7.25}
$$

and

$$
\begin{aligned}
& \sum_{\tau \in S_n} \operatorname{sgn}(\tau) c_{1,t}^n T_{\phi_1 D_{f_{\tau_1},g_1,1}}^{(t)} T_{\phi_1 D_{f_{\tau_2},g_2,1}}^{(t)} \dots T_{\phi_1 D_{f_{\tau_n},g_n,1}}^{(t)} \\
\sim_a & c_{1,t}^n \left[T_{\phi_1}^{(t)}\right]^n T_{\sum_{\tau \in S_n} \operatorname{sgn}(\tau) D_{f_{\tau_1},g_1,1} D_{f_{\tau_2},g_2,1} \dots D_{f_{\tau_n},g_n,1}}^{(t)} \qquad (7.26) \\
= & 0.
\end{aligned}
$$

Also by (7.25), we have the following equation

$$\begin{aligned}&\sum_{\tau\in S_n}\operatorname{sgn}(\tau)c_{1,t}^n T^{(t)}_{\phi_1 D_{f_{\tau_1},g_1,1}\phi_1 D_{f_{\tau_2},g_2,1}\ldots\phi_1 D_{f_{\tau_n},g_n,1}}\\ =\;& T^{(t)}_{\phi_1^n\sum_{\tau\in S_n}\operatorname{sgn}(\tau)D_{f_{\tau_1},g_1,1}D_{f_{\tau_2},g_2,1}\ldots D_{f_{\tau_n},g_n,1}}=0.\end{aligned}\tag{7.27}$$

Altogether, (7.23), (7.24), (7.26) and (7.27) imply (7.12).

**Proof of (7.13).**

Denote

$$F(z)=\sum_{\tau\in S_n}\operatorname{sgn}(\tau)C_1(f_{\tau_1},g_1)\ldots C_1(f_{\tau_n},g_n).$$

Under the basis $\mathbf{e}_z$, we compute the following volume form

$$\begin{aligned}&(-1)^n n^n(1-|z|^2)^{-n-1}F\mathrm{d}z_1\wedge\ldots\wedge\mathrm{d}z_n\wedge\mathrm{d}\bar z_1\wedge\ldots\wedge\mathrm{d}\bar z_n\\ =&\sum_{\tau,\varsigma\in S_n}\operatorname{sgn}(\tau)\partial_{\varsigma(1)}f_{\tau_1}\bar\partial_{\varsigma(1)}g_1\ldots\partial_{\varsigma(n)}f_{\tau_n}\bar\partial_{\varsigma(n)}g_n\mathrm{d}z_1\wedge\ldots\wedge\mathrm{d}z_n\wedge\mathrm{d}\bar z_1\wedge\ldots\wedge\mathrm{d}\bar z_n\\ =&\sum_{\tau,\varsigma\in S_n}\operatorname{sgn}(\tau)\big(\partial_{\varsigma(1)}f_{\tau_1}\mathrm{d}z_{\varsigma(1)}\big)\wedge\ldots\wedge\big(\partial_{\varsigma(n)}f_{\tau_n}\mathrm{d}z_{\varsigma(n)}\big)\wedge\big(\bar\partial_{\varsigma(1)}g_1\mathrm{d}\bar z_{\varsigma(1)}\big)\wedge\ldots\wedge\big(\bar\partial_{\varsigma(n)}g_n\mathrm{d}\bar z_{\varsigma(n)}\big)\\ =&\sum_{\tau,\varsigma\in S_n}\big(\partial_{\varsigma\tau^{-1}(1)}f_1\mathrm{d}z_{\varsigma\tau^{-1}(1)}\big)\wedge\ldots\wedge\big(\partial_{\varsigma\tau^{-1}(n)}f_n\mathrm{d}z_{\varsigma\tau^{-1}(n)}\big)\wedge\big(\bar\partial_{\varsigma(1)}g_1\mathrm{d}\bar z_{\varsigma(1)}\big)\wedge\ldots\wedge\big(\bar\partial_{\varsigma(n)}g_n\mathrm{d}\bar z_{\varsigma(n)}\big)\\ =&\sum_{\iota,\varsigma\in S_n}\big(\partial_{\iota(1)}f_1\mathrm{d}z_{\iota(1)}\big)\wedge\ldots\wedge\big(\partial_{\iota(n)}f_n\mathrm{d}z_{\iota(n)}\big)\wedge\big(\bar\partial_{\varsigma(1)}g_1\mathrm{d}\bar z_{\varsigma(1)}\big)\wedge\ldots\wedge\big(\bar\partial_{\varsigma(n)}g_n\mathrm{d}\bar z_{\varsigma(n)}\big)\\ =&\partial f_1\wedge\ldots\wedge\partial f_n\wedge\bar\partial g_1\wedge\ldots\wedge\bar\partial g_n.\end{aligned}$$

Therefore, we obtain the following formula of the trace

$$\begin{aligned}&\operatorname{Tr}T^{(t)}_{\sum_{\tau\in S_n}\operatorname{sgn}(\tau)c_{1,t}^n C_1(f_{\tau_1},g_1)C_1(f_{\tau_2},g_2)C_1(f_{\tau_n},g_n)}\\ =&c_{1,t}^n\operatorname{Tr}T^{(t)}_F\\ =&\frac{(n-1)!c_{1,t}^n}{\pi^n B(n,t+1)}\int_{\mathbb{B}_n}\frac{F(z)}{(1-|z|^2)^{n+1}}\mathrm{d}m(z)\\ =&\frac{(n-1)!c_{1,t}^n}{\pi^n B(n,t+1)}\frac{(-1)^n}{n^n}\frac{1}{(-2i)^n}\int_{\mathbb{B}_n}\partial f_1\wedge\bar\partial g_1\wedge\ldots\wedge\partial f_n\wedge\bar\partial g_n.\end{aligned}$$

Since

$$c_{1,t}=nt^{-1}+O(t^{-2}),\quad B(n,t+1)=(n-1)!t^{-n}+O(t^{-n-1}),$$

we have

$$\frac{c_{1,t}^n}{B(n,t+1)} \to \frac{n^n}{(n-1)!}, \quad t\to\infty.$$

Simplifying the above gives (7.13).

In summary, we have proved (7.11), (7.12) and (7.13). Altogether they finish Part 2. We have completed the proof of Theorem 7.3 for $t>-1$.

*7.2. Proof of Theorem 7.3: $t=-1$*

In this subsection, we prove that $[f_1,g_1,\dots,f_n,g_n]_{-1}^{\mathrm{fst}}$ and $[f_1,g_1,\dots,f_n,g_n]_{-1}^{\mathrm{scd}}$ belong to the trace class of the Hardy space. The proof follows the same idea as the case of $t>-1$ but requires more careful treatment because the Hardy space norm is defined in a more subtle way. Our approach is to move most of our arguments to the Bergman space $L_{a,0}^2(\mathbb{B}_n)$ via the maps described in Diagram (7.35).

To simplify notations, we write

$$\sigma(f,g)=\sigma_{-1}(f,g),$$

and

$$[f_1,g_1,\dots,f_n,g_n]^{\mathrm{fst}}=[f_1,g_1,\dots,f_n,g_n]_{-1}^{\mathrm{fst}},$$
$$[f_1,g_1,\dots,f_n,g_n]^{\mathrm{scd}}=[f_1,g_1,\dots,f_n,g_n]_{-1}^{\mathrm{scd}}.$$

**Lemma 7.9.** *Suppose $g\in\mathscr{C}^1(\overline{\mathbb{B}_n})$. Then for $h\in H^2(\mathbb{S}_n)$,*

$$T_g^{(-1)}h(z)=g(z)h(z)-\frac{1}{n}\int_{\mathbb{B}_n}|\varphi_z(w)|^{-2n}\frac{\langle\bar\partial g(w),z-w\rangle}{(1-\langle w,z\rangle)}h(w)K_w^{(-1)}(z)\mathrm{d}\lambda_0(w),\quad \forall z\in\mathbb{B}_n. \tag{7.28}$$

*Moreover, for $z\in\mathbb{S}_n$ almost everywhere,*

$$T_g^{(-1)}h(z)=g(z)h(z)-\frac{1}{n}\int_{\mathbb{B}_n}\frac{\langle\bar\partial g(w),z-w\rangle}{1-\langle w,z\rangle}h(w)K_w^{(-1)}(z)\mathrm{d}\lambda_0(w). \tag{7.29}$$

**Proof.** Suppose $h\in\mathrm{Hol}(\overline{\mathbb{B}_n})$. For $z\in\mathbb{B}_n$, apply Lemma 2.15 with $\alpha=\beta=0$, $v(w)=g(w)h(w)$. Then we compute $T_g^{(-1)}$ as follows,

$$\begin{aligned}&T_g^{(-1)}h(z)\\&=\int_{\mathbb{S}_n}g(w)h(w)K_w^{(-1)}(z)\frac{\mathrm{d}\sigma(w)}{\sigma_{2n-1}}\\&=d_{0,0}(z)g(z)h(z)-\frac{1}{n}\int_{\mathbb{B}_n}|\varphi_z(w)|^{-2n}\frac{\sum_{j=1}^n\bar\partial_j\left[g(w)h(w)\right]\overline{(z_j-w_j)}}{1-\langle w,z\rangle}K_w^{(-1)}(z)\mathrm{d}\lambda_0(w)\end{aligned}$$

$$=g(z)h(z)-\frac{1}{n}\int\limits_{\mathbb{B}_n}|\varphi_z(w)|^{-2n}\frac{\langle\bar{\partial}g(w),z-w\rangle}{1-\langle w,z\rangle}h(w)K_w^{(-1)}(z)\mathrm{d}\lambda_0(w).$$

This proves (7.28) for $h\in\mathrm{Hol}(\overline{\mathbb{B}_n})$. The equation for general $h\in H^2(\mathbb{S}_n)$ follows from approximation.

For $h\in\mathrm{Hol}(\overline{\mathbb{B}_n})$, $z\in\mathbb{S}_n$ and $\frac{1}{2}<r<1$, we compute $T_g^{(-1)}h(rz)$

$$T_g^{(-1)}h(rz)=g(rz)h(rz)-\frac{1}{n}\int\limits_{\mathbb{B}_n}|\varphi_{rz}(w)|^{-2n}\frac{\langle\bar{\partial}g(w),rz-w\rangle}{1-\langle w,rz\rangle}h(w)K_w^{(-1)}(rz)\mathrm{d}\lambda_0(w).$$

The first term on the right hand side tends to $g(z)h(z)$. In order to prove (7.29), it suffices to prove the following identity

$$\lim_{r\to1^-}\int\limits_{\mathbb{B}_n}|\varphi_{rz}(w)|^{-2n}\frac{\langle\bar{\partial}g(w),rz-w\rangle}{1-\langle w,rz\rangle}h(w)K_w^{(-1)}(rz)\mathrm{d}\lambda_0(w) \tag{7.30}$$
$$=\int\limits_{\mathbb{B}_n}\frac{\langle\bar{\partial}g(w),z-w\rangle}{1-\langle w,z\rangle}h(w)K_w^{(-1)}(z)\mathrm{d}\lambda_0(w).$$

By Lemma 2.2, we have the following estimate

$$\left|\frac{\langle\bar{\partial}g(w),rz-w\rangle}{1-\langle w,rz\rangle}h(w)K_w^{(-1)}(rz)\right|\lesssim\frac{|\varphi_{rz}(w)|}{|1-\langle rz,w\rangle|^{n+1/2}}.$$

Therefore we have the following estimate of the integrals

$$\begin{aligned}&\Bigg|\int\limits_{\mathbb{B}_n}|\varphi_{rz}(w)|^{-2n}\frac{\langle\bar{\partial}g(w),rz-w\rangle}{1-\langle w,rz\rangle}h(w)K_w^{(-1)}(rz)\mathrm{d}\lambda_0(w)\\&\quad-\int\limits_{\mathbb{B}_n}\frac{\langle\bar{\partial}g(w),rz-w\rangle}{1-\langle w,rz\rangle}h(w)K_w^{(-1)}(rz)\mathrm{d}\lambda_0(w)\Bigg|\\&\lesssim\int\limits_{\mathbb{B}_n}\left(|\varphi_{rz}(w)|^{-2n}-1\right)\cdot\frac{|\varphi_{rz}(w)|}{|1-\langle rz,w\rangle|^{n+1/2}}\mathrm{d}\lambda_0(w)\\&\approx\int\limits_{\mathbb{B}_n}|\varphi_{rz}(w)|^{-2n+1}\left(1-|\varphi_{rz}(w)|^2\right)\frac{1}{|1-\langle rz,w\rangle|^{n+1/2}}\mathrm{d}\lambda_0(w)\\&\lesssim(1-|rz|^2)^{1/2}\to0,\quad r\to1^-.\end{aligned}$$

Here the last inequality follows from Lemma 2.3 (3). Therefore, we have the following limit computation,

$$\lim_{r\to 1^-}\int_{\mathbb{B}_n}|\varphi_{rz}(w)|^{-2n}\frac{\langle\bar\partial g(w), rz-w\rangle}{1-\langle w, rz\rangle}h(w)K_w^{(-1)}(rz)\mathrm{d}\lambda_0(w) \tag{7.31}$$
$$=\lim_{r\to 1^-}\int_{\mathbb{B}_n}\frac{\langle\bar\partial g(w), rz-w\rangle}{1-\langle w, rz\rangle}h(w)K_w^{(-1)}(rz)\mathrm{d}\lambda_0(w).$$

For $\frac{1}{2}<r<1$ and $|a|<1$, it is easy to verify that

$$\frac{1}{|1-ra|}\le\frac{2}{|1-a|}.$$

So we have the following bounds

$$\left|\frac{\langle\bar\partial g(w), rz-w\rangle}{1-\langle w, rz\rangle}h(w)K_w^{(-1)}(rz)\right|\lesssim\frac{|\varphi_{rz}(w)|}{|1-\langle rz,w\rangle|^{n+1/2}}\lesssim\frac{1}{|1-\langle z,w\rangle|^{n+1/2}}.$$

Thus by the Dominated Convergence Theorem, we arrive at the following equation

$$\lim_{r\to 1^-}\int_{\mathbb{B}_n}\frac{\langle\bar\partial g(w), rz-w\rangle}{1-\langle w, rz\rangle}h(w)K_w^{(-1)}(rz)\mathrm{d}\lambda_0(w)=\int_{\mathbb{B}_n}\frac{\langle\bar\partial g(w), z-w\rangle}{1-\langle w, z\rangle}h(w)K_w^{(-1)}(z)\mathrm{d}\lambda_0(w). \tag{7.32}$$

Combining (7.31) and (7.32) gives (7.30). Thus (7.29) holds pointwise for $h\in\mathrm{Hol}(\overline{\mathbb{B}_n})$. The general case follows from approximation. This completes the proof of Lemma 7.9. □

**Lemma 7.10.** *Suppose $f,g\in\mathscr{C}^2(\overline{\mathbb{B}_n})$. Then*

$$\sigma(f,g)=R_{f,g,1},$$

*where the operator $R_{f,g,1}$ is defined as follows.*

$$R_{f,g,1}h(\xi)$$
$$=-\frac{1}{n^2}\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}|\varphi_z(w)|^{-2n}\frac{\langle\partial f(z),\overline{z-w}\rangle\langle\bar\partial g(w), z-w\rangle}{(1-\langle w,z\rangle)^2}K_w^{(-1)}(z)K_z^{(-1)}(\xi)\mathrm{d}\lambda_0(w)\mathrm{d}\lambda_0(z).$$

**Proof.** Suppose $h\in\mathrm{Hol}(\overline{\mathbb{B}_n})$. By Lemma 7.9, for $z\in\mathbb{S}_n$, we compute $T_g^{(-1)}h$

$$T_g^{(-1)}h(z)=g(z)h(z)-\frac{1}{n}\int_{\mathbb{B}_n}\frac{\langle\bar\partial g(w), z-w\rangle}{1-\langle w,z\rangle}h(w)K_w^{(-1)}(z)\mathrm{d}\lambda_0(w).$$

Therefore for $\xi\in\mathbb{B}_n$, we have the following computation of $T_f^{(-1)}T_g^{(-1)}$,

$$
\begin{aligned}
&T_f^{(-1)} T_g^{(-1)} h(\xi) \\
=&\int_{\mathbb{S}_n} \left\{ g(z)h(z) - \frac{1}{n} \int_{\mathbb{B}_n} \frac{\langle \bar{\partial} g(w), z-w\rangle}{1-\langle w,z\rangle} h(w) K_w^{(-1)}(z) \mathrm{d}\lambda_0(w) \right\} f(z) K_z^{(-1)}(\xi) \frac{\mathrm{d}\sigma(z)}{\sigma_{2n-1}} \\
=&\int_{\mathbb{S}_n} g(z)f(z)h(z) K_z^{(-1)}(\xi) \frac{\mathrm{d}\sigma(z)}{\sigma_{2n-1}} - \frac{1}{n} \int_{\mathbb{S}_n} \int_{\mathbb{B}_n} \frac{\langle \bar{\partial} g(w), z-w\rangle}{1-\langle w,z\rangle} h(w) K_w^{(-1)}(z) f(z) K_z^{(-1)}(\xi) \mathrm{d}\lambda_0(w) \frac{\mathrm{d}\sigma(z)}{\sigma_{2n-1}} \\
=&T_{fg}^{(-1)} h(\xi) - \frac{1}{n} \int_{\mathbb{S}_n} \int_{\mathbb{B}_n} \frac{\langle \bar{\partial} g(w), z-w\rangle}{1-\langle w,z\rangle} h(w) K_w^{(-1)}(z) f(z) K_z^{(-1)}(\xi) \mathrm{d}\lambda_0(w) \frac{\mathrm{d}\sigma(z)}{\sigma_{2n-1}}.
\end{aligned}
$$

Since $f, g, h$ are bounded, using Lemma 2.3 we see that the double integral on the right converges absolutely. By Fubini's Theorem, we have

$$
\sigma(f,g)h(\xi) = -\frac{1}{n} \int_{\mathbb{B}_n} \left\{ \int_{\mathbb{S}_n} \frac{f(z)\langle \bar{\partial} g(w), z-w\rangle}{1-\langle w,z\rangle} K_w^{(-1)}(z) K_z^{(-1)}(\xi) \frac{\mathrm{d}\sigma(z)}{\sigma_{2n-1}} \right\} h(w) \mathrm{d}\lambda_0(w).
$$

For fixed $w, \xi \in \mathbb{B}_n$, applying (2.15) with $\alpha = \beta = 0$, $v(z) = \frac{f(z)\langle \bar{\partial} g(w), z-w\rangle}{1-\langle w,z\rangle} K_z^{(-1)}(\xi)$ gives

$$
\begin{aligned}
&\int_{\mathbb{S}_n} \frac{f(z)\langle \bar{\partial} g(w), z-w\rangle}{1-\langle w,z\rangle} K_w^{(-1)}(z) K_z^{(-1)}(\xi) \frac{\mathrm{d}\sigma(z)}{\sigma_{2n-1}} \\
=&\frac{1}{n} \int_{\mathbb{B}_n} |\varphi_z(w)|^{-2n} \frac{\langle \partial f(z), \overline{z-w}\rangle \langle \bar{\partial} g(w), z-w\rangle}{(1-\langle w,z\rangle)^2} K_w^{(-1)}(z) K_z^{(-1)}(\xi) \mathrm{d}\lambda_0(z).
\end{aligned}
$$

Therefore we get the following formula for $\sigma(f,g)$

$$
\begin{aligned}
&\sigma(f,g)h(\xi) \\
=&-\frac{1}{n} \int_{\mathbb{B}_n} \left\{ \frac{1}{n} \int_{\mathbb{B}_n} |\varphi_z(w)|^{-2n} \frac{\langle \partial f(z), \overline{z-w}\rangle \langle \bar{\partial} g(w), z-w\rangle}{(1-\langle w,z\rangle)^2} K_w^{(-1)}(z) K_z^{(-1)}(\xi) \mathrm{d}\lambda_0(z) \right\} h(w) \mathrm{d}\lambda_0(w) \\
=&-\frac{1}{n^2} \int_{\mathbb{B}_n^2} |\varphi_z(w)|^{-2n} \frac{\langle \partial f(z), \overline{z-w}\rangle \langle \bar{\partial} g(w), z-w\rangle}{(1-\langle w,z\rangle)^2} K_w^{(-1)}(z) K_z^{(-1)}(\xi) h(w) \mathrm{d}\lambda_0(w) \mathrm{d}\lambda_0(z).
\end{aligned}
$$

This completes the proof of Lemma 7.10. □

**Definition 7.11.** Suppose $f, g \in \mathscr{C}^2(\overline{\mathbb{B}_n})$ and $h \in L^2(\lambda_0)$.

1. Define

$$
Ph(\xi) = \int_{\mathbb{B}_n} h(z) K_z^{(-1)}(\xi) \mathrm{d}\lambda_0(z),
$$

and

$$\Gamma_{f,g}h(z) = -\frac{1}{n^2}\int\limits_{\mathbb{B}_n} |\varphi_z(w)|^{-2n} \frac{\langle \partial f(z), \overline{z-w}\rangle \langle \bar{\partial} g(w), z-w\rangle}{(1-\langle w,z\rangle)^2} K_w^{(-1)}(z)h(w)\mathrm{d}\lambda_0(w).$$

Then

$$R_{f,g,1} = P\Gamma_{f,g}.$$

2. Define

$$A^0 = \langle Q_{\bar{z}}\partial f(z), \overline{(z-w)}\rangle, \quad A^1 = \langle P_{\bar{z}}\partial f(z), \overline{(z-w)}\rangle,$$
$$B^0 = \langle Q_z \bar{\partial} g(w), z-w\rangle, \quad B^1 = \langle P_z \bar{\partial} g(w), z-w\rangle,$$

and

$$C^{a,b}(z,w) = A^a B^b, \quad a,b = 0,1.$$

Then

$$|C^{a,b}(z,w)| \lesssim |1-\langle z,w\rangle|^{1+\frac{a+b}{2}}, \tag{7.33}$$

and

$$\sum_{a,b=0,1} C^{a,b}(z,w) = \langle \partial f(z), \overline{z-w}\rangle \langle \bar{\partial} g(w), z-w\rangle.$$

3. Define

$$\Gamma_{f,g}^{a,b}h(z) = -\frac{1}{n^2}\int\limits_{\mathbb{B}_n} |\varphi_z(w)|^{-2n} \frac{C^{a,b}(z,w)}{(1-\langle w,z\rangle)^2} K_w^{(-1)}(z)h(w)\mathrm{d}\lambda_0(w).$$

Then

$$\Gamma_{f,g} = \sum_{a,b=0,1} \Gamma_{f,g}^{a,b}.$$

4. Define for $j = 1, \dots, n$,

$$\Lambda_{g,j}h(z) = -\frac{1}{n^2}\int\limits_{\mathbb{B}_n} |\varphi_z(w)|^{-2n} (z_j - w_j) \frac{\langle Q_z \bar{\partial} g(w), z-w\rangle}{(1-\langle w,z\rangle)^2} K_w^{(-1)}(z)h(w)\mathrm{d}\lambda_0(w).$$

Then

$$\Gamma_{f,g}^{0,0} = \sum_{j=1}^{n} M_{[Q_{\bar{z}}\partial f]_j} \Lambda_{g,j}.$$

**Lemma 7.12.** *Suppose $n \geq 2$, $f, g \in \mathscr{C}^2(\overline{\mathbb{B}_n})$, $u \in \mathscr{C}^1(\overline{\mathbb{B}_n})$, and $a, b = 0, 1$. Consider $P, \Gamma^{a,b}_{f,g}, \Lambda_{g,j}$ as operators on $L^2(\lambda_0)$. Then the following hold.*

(1) $[P^{(0)}, M_u] \in \mathcal{S}^p$ *for any* $p > 2n$.
(2) $P \in \mathcal{S}^p$ *for any* $p > n$.
(3) $[P, M_u] \in \mathcal{S}^p$ *for any* $p > \frac{2n}{3}$.
(4) *If* $(a, b) \neq (0, 0)$, *then* $\Gamma^{a,b}_{f,g} P^{(0)} \in \mathcal{S}^p$ *for* $p$ *large enough, and* $\Gamma^{a,b}_{f,g} P \in \mathcal{S}^p$ *for some* $p < n$.
(5) *For each* $j$, $\Lambda_{g,j}$ *is bounded.*
(6) *For each* $j$, $[\Lambda_{g,j}, M_u] P^{(0)} \in \mathcal{S}^p$ *for any* $p > 2n$.
(7) *For each* $j$, $[\Lambda_{g,j}, M_u] P \in \mathcal{S}^p$ *for any* $p > \frac{2n}{3}$.

**Proof.** By comparing the integral formula of $P$ and the Bergman projection $P^{(0)}$, it is easy to see that (2) follows from Corollary 4.2, and (1) follows from Corollary 4.3. If $n \geq 3$ then (3) also follows from Corollary 4.3. At $n = 2$, notice that $P$ is self-adjoint and has range in $L^2_{a,0}(\mathbb{B}_n)$. Thus we compute the commutator $[P, M_u]$

$$[P, M_u] = P^{(0)}[P, M_u] + (1 - P^{(0)})[P, M_u] = \Big([M_{\bar{u}}, P] P^{(0)}\Big)^* - H^{(0)}_u P.$$

By (2) and Corollary 4.4, the second term on the right belongs to $\mathcal{S}^p$ for any $p > \frac{2n}{3}$. For any $h \in \mathrm{Hol}(\overline{\mathbb{B}_n})$, apply (2.11) with $t = 0$, $\alpha = \beta = 0$, $\phi = 1$, and $v(w) = \big(\bar{u}(z) - \bar{u}(w)\big) h(w)(1 - \langle z, w\rangle)$. Then we get the following expression

$$\begin{aligned}
&[M_{\bar{u}}, P] h(z) \\
&= \int_{\mathbb{B}_n} \big(\bar{u}(z) - \bar{u}(w)\big) h(w) K^{(-1)}_w(z) \mathrm{d}\lambda_0(w) \\
&= -\int_{\mathbb{B}_n} \mathcal{G}^{(0)}_n 1(|\varphi_z(w)|^2) \frac{(1 - |w|^2)\langle \bar{\partial} v(w), z - w\rangle}{1 - \langle w, z\rangle} K^{(0)}_w(z) \mathrm{d}\lambda_0(w) \qquad (7.34) \\
&= \int_{\mathbb{B}_n} \mathcal{G}^{(0)}_n 1(|\varphi_z(w)|^2) \Big( (1 - \langle z, w\rangle)\langle \bar{\partial}\bar{u}(w), z - w\rangle + \big(\bar{u}(z) - \bar{u}(w)\big)\langle z, z - w\rangle \Big) \\
&\qquad \cdot \frac{1 - |w|^2}{1 - \langle w, z\rangle} h(w) K^{(0)}_w(z) \mathrm{d}\lambda_0(w).
\end{aligned}$$

Take

$$F(z, w) = \frac{2}{n+1} \Big( (1 - \langle z, w\rangle)\langle \bar{\partial}\bar{u}(w), z - w\rangle + \big(\bar{u}(z) - \bar{u}(w)\big)\langle z, z - w\rangle \Big) \frac{1}{1 - \langle w, z\rangle} K^{(0)}_w(z).$$

Substituting $F$ into the integral on the right hand side of Equation (7.34) gives the following formula

$$[M_{\bar{u}}, P]h(z) = \int_{\mathbb{B}_n} \mathcal{G}_n^{(0)} 1(|\varphi_z(w)|^2) F(z,w) h(w) \mathrm{d}\lambda_1(w).$$

By Lemma 5.9, we have the following estimate

$$|\mathcal{G}_n^{(0)} 1(s)| \lesssim s^{-n}.$$

By Lemma 2.2 and the fact that $\bar{u} \in \mathscr{C}^1(\overline{\mathbb{B}_n})$, we obtain the following bound

$$\left|F(z,w)\right| \lesssim |\varphi_z(w)| \frac{1}{|1-\langle z,w\rangle|^{n+1/2}}.$$

For any $\epsilon > 0$, split the map as follows.

$$[M_{\bar{u}}, P]P^{(0)} : L^2_{a,0}(\mathbb{B}_n) \xrightarrow{E_{0,3-\epsilon}} L^2_{a,3-\epsilon}(\mathbb{B}_n) \xrightarrow{T} L^2(\lambda_0),$$

where $Th(z)$ is defined as in the last line of Equation (7.34). Then by the estimates above and Lemma 3.1, $T$ is bounded. Thus by Lemma 3.4, $[M_{\bar{u}}, P]P^{(0)}$ is in $\mathcal{S}^p$ for any $p > \frac{2n}{3}$. This proves (3).

Let

$$F_{\Gamma^{a,b}}(z,w) = -\frac{1}{n^2} \frac{C^{a,b}(z,w)}{(1-\langle w,z\rangle)^2} K_w^{(-1)}(z), \quad z,w \in \mathbb{B}_n,$$

and

$$F_{\Lambda_{g,j}}(z,w) = -\frac{1}{n^2}(z_j - w_j) \frac{\langle Q_z \bar{\partial} g(w), z-w\rangle}{(1-\langle w,z\rangle)^2} K_w^{(-1)}(z), \quad z,w \in \mathbb{B}_n, j = 1,\dots,n.$$

Then write

$$\Gamma^{a,b}_{f,g} h(z) = \int_{\mathbb{B}_n} |\varphi_z(w)|^{-2n} F_{\Gamma^{a,b}}(z,w) h(w) \mathrm{d}\lambda_0(w),$$

and

$$\Lambda_{g,j} h(z) = \int_{\mathbb{B}_n} |\varphi_z(w)|^{-2n} F_{\Lambda_{g,j}}(z,w) h(w) \mathrm{d}\lambda_0(w).$$

By (7.33) and Lemma 2.2, we have the following bound

$$\left|F_{\Gamma^{a,b}}(z,w)\right| \lesssim |\varphi_z(w)|^2 \frac{1}{|1-\langle z,w\rangle|^{n+1-\frac{a+b}{2}}},$$

and

$$\left|F_{\Lambda_{g,j}}(z,w)\right| \lesssim |\varphi_z(w)|^2 \frac{1}{|1-\langle z,w\rangle|^{n+1}}.$$

Thus by Lemma 3.1, $\Lambda_{g,j}$ is bounded. This proves (5). By Lemma 4.5, $\Gamma^{a,b}_{f,g}P^{(0)} \in \mathcal{S}^p$ for $p$ large enough. Since $P \in \mathcal{S}^p$ for any $p > n$, and $\Gamma^{a,b}_{f,g}P = \Gamma^{a,b}_{f,g}P^{(0)}P$, we have $\Gamma^{a,b}_{f,g}P \in \mathcal{S}^p$ for some $p < n$. This proves (4). Also, (6) follows from the above estimates and Corollary 4.7, and (7) follows again from (6) and the equation $P = P^{(0)}P$. This completes the proof. □

**Lemma 7.13.** *Denote $\hat{P}$ the operator from $L^2(\lambda_0)$ to $H^2(\mathbb{S}_n)$ defined by the same integral formula as $P$. Then $\hat{P} = E^*_{-1,1}$.*

**Proof.** For any $f \in L^2(\lambda_0)$, $g \in H^2(\mathbb{S}_n)$, we compute $\langle \hat{P}f, g\rangle_{H^2(\mathbb{S}_n)}$

$$\begin{aligned}
\langle \hat{P}f, g\rangle_{H^2(\mathbb{S}_n)} &= \int_{\mathbb{S}_n}\int_{\mathbb{B}_n} f(z)K_z^{(-1)}(\xi)\mathrm{d}\lambda_0(z)\overline{g(\xi)}\frac{\mathrm{d}\sigma(\xi)}{\sigma_{2n-1}} \\
&= \int_{\mathbb{B}_n}\int_{\mathbb{S}_n} \overline{g(\xi)}K_z^{(-1)}(\xi)\frac{\mathrm{d}\sigma(\xi)}{\sigma_{2n-1}} f(z)\mathrm{d}\lambda_0(z) \\
&= \int_{\mathbb{B}_n} \overline{g(z)}f(z)\mathrm{d}\lambda_0(z) \\
&= \langle f, E_{-1,1}g\rangle_{L^2(\lambda_0)}.
\end{aligned}$$

This completes the proof. □

**Proof of Theorem 7.3 ($t = -1$).** For any $\tau \in S_n$, split the map

$$\sigma(f_{\tau_1}, g_1)\ldots\sigma(f_{\tau_n}, g_n) = P\Gamma_{f_{\tau_1},g_1}P\Gamma_{f_{\tau_2},g_2}\ldots P\Gamma_{f_{\tau_n},g_n}$$

as follows.

$$H^2(\mathbb{S}_n) \xrightarrow{E_{-1,1}} L^2_{a,0}(\mathbb{B}_n) \xrightarrow{\Gamma_{f_{\tau_1},g_1}P\Gamma_{f_{\tau_2},g_2}\ldots P\Gamma_{f_{\tau_n},g_n}P^{(0)}} L^2(\lambda_0) \xrightarrow{\hat{P}} H^2(\mathbb{S}_n). \tag{7.35}$$

By Lemmas 3.4 and 7.13, the operators on the two ends of (7.35) are in $\mathcal{S}^p$ for any $p > 2n$. Thus it suffices to show that

$$\sum_{\tau\in S_n} \operatorname{sgn}(\tau)\Gamma_{f_{\tau_1},g_1}P\Gamma_{f_{\tau_2},g_2}\ldots P\Gamma_{f_{\tau_n},g_n}P^{(0)}$$

defines an operator in $\mathcal{S}^p(L^2(\lambda_0))$ for some $p < \frac{n}{n-1}$.

**Notation 7.14.** For operators $A, B$, temporarily write $A \sim_p B$ when $A-B$ is in $\mathcal{S}^p(L^2(\lambda_0))$ for some $p < \frac{n}{n-1}$. Again, we do not assume $A$ or $B$ to be in any Schatten-class.

Denote

$$u_{i,j}(z) = [Q_{\bar{z}}\partial f_i(z)]_j.$$

Then by Lemma 7.12, we compute an element in the above sum

$$\begin{aligned}
&\Gamma_{f_{\tau_1},g_1} P \Gamma_{f_{\tau_2},g_2} \dots P \Gamma_{f_{\tau_n},g_n} P^{(0)} \\
=&\Big(\sum_{a,b=0,1} \Gamma^{a,b}_{f_{\tau_1},g_1}\Big) P \Big(\sum_{a,b=0,1} \Gamma^{a,b}_{f_{\tau_2},g_2}\Big) \dots P \Big(\sum_{a,b=0,1} \Gamma^{a,b}_{f_{\tau_n},g_n}\Big) P^{(0)} \\
\sim_p& \Gamma^{0,0}_{f_{\tau_1},g_1} P \Gamma^{0,0}_{f_{\tau_2},g_2} \dots P \Gamma^{0,0}_{f_{\tau_n},g_n} P^{(0)} \\
=& \sum_{j_1,\dots,j_n=1}^{n} M_{u_{\tau_1,j_1}} \Lambda_{g_1,j_1} P M_{u_{\tau_2,j_2}} \Lambda_{g_2,j_2} P \dots M_{u_{\tau_n,j_n}} \Lambda_{g_n,j_n} P^{(0)} \\
=& \sum_{j_1,\dots,j_n=1}^{n} \Lambda_{g_1,j_1} M_{u_{\tau_1,j_1}} P M_{u_{\tau_2,j_2}} \Lambda_{g_2,j_2} P \dots M_{u_{\tau_n,j_n}} \Lambda_{g_n,j_n} P^{(0)} \\
&+ \sum_{j_1,\dots,j_n=1}^{n} [M_{u_{\tau_1,j_1}}, \Lambda_{g_1,j_1}] P M_{u_{\tau_2,j_2}} \Lambda_{g_2,j_2} P \dots M_{u_{\tau_n,j_n}} \Lambda_{g_n,j_n} P^{(0)}.
\end{aligned}$$

Each operator in the second term contains $n-2$ copies of $P$ and one $[M_{u_{\tau_1,j_1}}, \Lambda_{g_1,j_1}]P$. By Lemma 7.12 (2) and (7), it belongs to $\mathcal{S}^p$ for some $p < \frac{n}{n-1}$. Thus we compute the following sum,

$$\begin{aligned}
&\sum_{j_1,\dots,j_n=1}^{n} M_{u_{\tau_1,j_1}} \Lambda_{g_1,j_1} P M_{u_{\tau_2,j_2}} \Lambda_{g_2,j_2} P \dots M_{u_{\tau_n,j_n}} \Lambda_{g_n,j_n} P^{(0)} \\
\sim_p& \sum_{j_1,\dots,j_n=1}^{n} \Lambda_{g_1,j_1} M_{u_{\tau_1,j_1}} P M_{u_{\tau_2,j_2}} \Lambda_{g_2,j_2} P \dots M_{u_{\tau_n,j_n}} \Lambda_{g_n,j_n} P^{(0)} \\
=& \sum_{j_1,\dots,j_n=1}^{n} \Lambda_{g_1,j_1} P M_{u_{\tau_1,j_1}} M_{u_{\tau_2,j_2}} \Lambda_{g_2,j_2} P \dots M_{u_{\tau_n,j_n}} \Lambda_{g_n,j_n} P^{(0)} \\
&+ \sum_{j_1,\dots,j_n=1}^{n} \Lambda_{g_1,j_1} [M_{u_{\tau_1,j_1}}, P] M_{u_{\tau_2,j_2}} \Lambda_{g_2,j_2} P \dots M_{u_{\tau_n,j_n}} \Lambda_{g_n,j_n} P^{(0)}.
\end{aligned}$$

By Lemma 7.12 (2) and (3), the last term belongs to $\mathcal{S}^p$ for some $p < \frac{n}{n-1}$. Therefore we have the following equation

$$\begin{aligned}
&\sum_{j_1,\dots,j_n=1}^{n} M_{u_{\tau_1,j_1}} \Lambda_{g_1,j_1} P M_{u_{\tau_2,j_2}} \Lambda_{g_2,j_2} P \dots M_{u_{\tau_n,j_n}} \Lambda_{g_n,j_n} P^{(0)} \\
\sim_p& \sum_{j_1,\dots,j_n=1}^{n} \Lambda_{g_1,j_1} P M_{u_{\tau_1,j_1}} M_{u_{\tau_2,j_2}} \Lambda_{g_2,j_2} P \dots M_{u_{\tau_n,j_n}} \Lambda_{g_n,j_n} P^{(0)}.
\end{aligned}$$

Continuing like this, we obtain the following equation

$$\begin{aligned}&\sum_{j_1,\dots,j_n=1}^{n} M_{u_{\tau_1,j_1}}\Lambda_{g_1,j_1}PM_{u_{\tau_2,j_2}}\Lambda_{g_2,j_2}P\dots M_{u_{\tau_n,j_n}}\Lambda_{g_n,j_n}P^{(0)}\\ \sim_p&\sum_{j_1,\dots,j_n=1}^{n}\Lambda_{g_1,j_1}P\Lambda_{g_2,j_2}P\dots\Lambda_{g_n,j_n}M_{u_{\tau_1,j_1}}M_{u_{\tau_2,j_2}}\dots M_{u_{\tau_n,j_n}}P^{(0)}\\ =:&\Theta_{f_{\tau_1},g_1,\dots,f_{\tau_n},g_n}.\end{aligned}$$

As in the proof of Theorem 7.3, by writing $\Theta_{f_{\tau_1},g_1,\dots,f_{\tau_n},g_n}$ as an integral operator, and by (7.9), we can show that

$$\sum_{\tau\in S_n}\operatorname{sgn}(\tau)\Theta_{f_{\tau_1},g_1,\dots,f_{\tau_n},g_n}=0.$$

Therefore we conclude

$$\sum_{\tau\in S_n}\operatorname{sgn}(\tau)\Gamma_{f_{\tau_1},g_1}P\Gamma_{f_{\tau_2},g_2}\dots P\Gamma_{f_{\tau_n},g_n}P^{(0)}\in\mathcal{S}^p$$

for some $p<\frac{n}{n-1}$. Thus by (7.35), we obtain

$$[f_1,g_1,\dots,f_n,g_n]^{\text{fst}}=P\Bigg(\sum_{\tau\in S_n}\operatorname{sgn}(\tau)\Gamma_{f_{\tau_1},g_1}P\Gamma_{f_{\tau_2},g_2}\dots P\Gamma_{f_{\tau_n},g_n}P^{(0)}\Bigg)E_{-1,1}$$

is in the trace class. Since

$$[f_1,g_1,\dots,f_n,g_n]^{\text{fst}}=\left([\bar{g_n},\bar{f_n},\dots,\bar{g_1},\bar{f_1}]^{\text{scd}}\right)^*,$$

the second anti-symmetric sum is also in the trace class. Finally, as in the proof of Corollary 7.7, we obtain

$$[T_{f_1}^{(-1)},T_{f_2}^{(-1)},\dots,T_{f_{2n}}^{(-1)}]=\frac{1}{n!}\sum_{\tau\in S_{2n}}\operatorname{sgn}(\tau)[f_{\tau_1},f_{\tau_2},\dots,f_{\tau_{2n}}]^{\text{fst}}\in\mathcal{S}^1.$$

This completes the proof of Theorem 7.3 for $t=-1$. □

## 8. Main theorems

In this section, we prove the main results of this article.

*8.1. Helton-Howe cocycle*

We are ready to prove the following main theorem.

**Theorem 8.1.** *Suppose* $f_1, f_2, \ldots, f_{2n} \in \mathscr{C}^2(\overline{\mathbb{B}_n})$ *and* $t \geq -1$*. Then the following hold.*

1. *The antisymmetric sum* $[T_{f_1}^{(t)}, T_{f_2}^{(t)}, \ldots, T_{f_{2n}}^{(t)}]$ *is in the trace class* $\mathcal{S}^1$*.*
2. $$\mathrm{Tr}[T_{f_1}^{(t)}, T_{f_2}^{(t)}, \ldots, T_{f_{2n}}^{(t)}] = \frac{n!}{(2\pi i)^n} \int_{\mathbb{B}_n} \mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \ldots \wedge \mathrm{d}f_{2n}, \tag{8.1}$$

   *which is independent of* $t$*.*

**Proof.** First, we reduce the Hardy space case to that of a weighted Bergman space. By Lemma 5.3, the operator on $H^2(\mathbb{S}_n)$,

$$[T_{f_1}^{(-1)}, T_{f_2}^{(-1)}, \ldots, T_{f_{2n}}^{(-1)}] - [T_{f_1}^{(1,-1)}, T_{f_2}^{(1,-1)}, \ldots, T_{f_{2n}}^{(1,-1)}]$$

is a trace class operator of zero trace. By Corollary 7.7, the operator

$$[T_{f_1}^{(-1)}, T_{f_2}^{(-1)}, \ldots, T_{f_{2n}}^{(-1)}]$$

is itself in the trace class. Thus we have the following of traces

$$\mathrm{Tr}[T_{f_1}^{(-1)}, T_{f_2}^{(-1)}, \ldots, T_{f_{2n}}^{(-1)}] = \mathrm{Tr}[T_{f_1}^{(1,-1)}, T_{f_2}^{(1,-1)}, \ldots, T_{f_{2n}}^{(1,-1)}].$$

On the other hand, by Corollary 7.7, the operator

$$[T_{f_1}^{(1)}, \ldots, T_{f_{2n}}^{(1)}]$$

is a trace class operator on $L^2_{a,1}(\mathbb{B}_n)$. Since

$$[T_{f_1}^{(1)}, \ldots, T_{f_{2n}}^{(1)}]\big|_{H^2(\mathbb{S}_n)} = [T_{f_1}^{(1,-1)}, T_{f_2}^{(1,-1)}, \ldots, T_{f_{2n}}^{(1,-1)}],$$

by Lemma 2.4, we get the following equation of traces

$$\mathrm{Tr}[T_{f_1}^{(1,-1)}, T_{f_2}^{(1,-1)}, \ldots, T_{f_{2n}}^{(1,-1)}] = \mathrm{Tr}[T_{f_1}^{(1)}, \ldots, T_{f_{2n}}^{(1)}].$$

Therefore we arrive at the following equation

$$\mathrm{Tr}[T_{f_1}^{(-1)}, T_{f_2}^{(-1)}, \ldots, T_{f_{2n}}^{(-1)}] = \mathrm{Tr}[T_{f_1}^{(1)}, \ldots, T_{f_{2n}}^{(1)}].$$

Thus the case of the Hardy space reduces to that of the weighted Bergman space.

Suppose $t > -1$, by Lemma 5.2, the operator on $L^2_{a,t}(\mathbb{B}_n)$

$$[T^{(t)}_{f_1}, T^{(t)}_{f_2}, \ldots, T^{(t)}_{f_{2n}}] - [T^{(t+1,t)}_{f_1}, T^{(t+1,t)}_{f_2}, \ldots, T^{(t+1,t)}_{f_{2n}}]$$

is a trace class operator of zero trace. By Corollary 7.7, we know that the antisymmetrization $[T^{(t)}_{f_1}, T^{(t)}_{f_2}, \ldots, T^{(t)}_{f_{2n}}]$ is itself in the trace class. Thus so does $[T^{(t+1,t)}_{f_1}, T^{(t+1,t)}_{f_2}, \ldots, T^{(t+1,t)}_{f_{2n}}]$. Also by Corollary 7.7,

$$[T^{(t+1)}_{f_1}, T^{(t+1)}_{f_2}, \ldots, T^{(t+1)}_{f_{2n}}]$$

is a trace class operator on $L^2_{a,t+1}(\mathbb{B}_n)$. As explained in the beginning of Section 5, the space $L^2_{a,t}(\mathbb{B}_n)$ is invariant under the operations of each $T^{(t+1)}_{f_i}$. And by direct verification we have

$$[T^{(t+1)}_{f_1}, T^{(t+1)}_{f_2}, \ldots, T^{(t+1)}_{f_{2n}}]\big|_{L^2_{a,t}(\mathbb{B}_n)} = [T^{(t+1,t)}_{f_1}, T^{(t+1,t)}_{f_2}, \ldots, T^{(t+1,t)}_{f_{2n}}].$$

Thus by Lemma 2.4, we have the following equation

$$\mathrm{Tr}[T^{(t+1)}_{f_1}, T^{(t+1)}_{f_2}, \ldots, T^{(t+1)}_{f_{2n}}] = \mathrm{Tr}[T^{(t+1,t)}_{f_1}, T^{(t+1,t)}_{f_2}, \ldots, T^{(t+1,t)}_{f_{2n}}].$$

Therefore we conclude with the following identity

$$\begin{aligned} &\mathrm{Tr}[T^{(t)}_{f_1}, T^{(t)}_{f_2}, \ldots, T^{(t)}_{f_{2n}}] \\ =&\mathrm{Tr}[T^{(t+1,t)}_{f_1}, T^{(t+1,t)}_{f_2}, \ldots, T^{(t+1,t)}_{f_{2n}}] \\ =&\mathrm{Tr}[T^{(t+1)}_{f_1}, T^{(t+1)}_{f_2}, \ldots, T^{(t+1)}_{f_{2n}}]. \end{aligned}$$

This holds for any $t > -1$. Thus by Corollary 7.7, we have shown

$$\begin{aligned} \mathrm{Tr}[T^{(t)}_{f_1}, T^{(t)}_{f_2}, \ldots, T^{(t)}_{f_{2n}}] &= \lim_{k\to\infty} \mathrm{Tr}[T^{(t+k)}_{f_1}, T^{(t+k)}_{f_2}, \ldots, T^{(t+k)}_{f_{2n}}] \\ &= \frac{n!}{(2\pi i)^n} \int\limits_{\mathbb{B}_n} \mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \ldots \wedge \mathrm{d}f_{2n}. \end{aligned}$$

This completes the proof of Theorem 8.1. □

### *8.2. The Connes-Chern character*

As mentioned in Remark 7.6, in this subsection, we consider the Connes-Chern character at $p > n$.

**Proposition 8.2.** *Suppose $p \geq n+1$ is an integer and $f_1, g_1, \dots, f_p, g_p \in \mathscr{C}^2(\overline{\mathbb{B}_n})$. Then for any $t > -1$, the product $\sigma_t(f_1, g_1)\sigma_t(f_2, g_2)\dots\sigma_t(f_p, g_p)$ is in the trace class, and*

$$\begin{aligned}&\lim_{t\to\infty} t^{p-n}\mathrm{Tr}\Big(\sigma_t(f_1, g_1)\sigma_t(f_2, g_2)\dots\sigma_t(f_p, g_p)\Big)\\ =&\frac{n^p}{\pi^n}\int\limits_{\mathbb{B}_n}\prod_{j=1}^{p} C_1(f_j, g_j)(z)\frac{\mathrm{d}m(z)}{(1-|z|^2)^{n+1}}\\ =&\frac{(-1)^p}{\pi^n}\int\limits_{\mathbb{B}_n}\prod_{j=1}^{p}\Big[\sum_{i=1}^{n}\partial_i f_j(z)\bar{\partial}_i g_j(z) - Rf_j(z)\bar{R}g_j(z)\Big](1-|z|^2)^{p-n-1}\mathrm{d}m(z).\end{aligned}$$

**Proof.** The proof is similar to Part 2 of Section 7.1. Recall that the following facts were used in Part 2 and follow from Lemma 6.1, Theorem 6.3 and their corollaries. For $f, g \in \mathscr{C}^2(\overline{\mathbb{B}_n})$, we decompose $\sigma_t(f, g)$ as follows

$$\sigma_t(f, g) = R^{(t)}_{f,g,1} = c_{1,t}T^{(t)}_{C_1(f,g)} + R^{(t)}_{f,g,2}, \tag{8.2}$$

and the following hold.

(1) $c_{1,t} = nt^{-1} + O(t^{-2})$.
(2) $C_1(f, g) = \phi_1 D_{f,g,1} + \phi_2 D_{f,g,2}$, where

$$\phi_i(z) = (1-|z|^2)^i, \quad i = 1, 2, \dots,$$

and $D_{f,g,1}, D_{f,g,2} \in \mathscr{C}^1(\overline{\mathbb{B}_n})$ is defined as in (7.21) and (7.22).
(3) $R^{(t)}_{f,g,i} \in \mathcal{S}^p, \quad i = 1, 2$, and for large $t$, $\|R^{(t)}_{f,g,i}\|_{\mathcal{S}^p} \lesssim t^{-i+\frac{n}{p}}, \quad i = 1, 2$.
(4) For $t$ large enough, $i, j = 1, 2, \dots$, and $u, v \in \mathscr{C}^1(\overline{\mathbb{B}_n})$,
   (a) $T^{(t)}_u T^{(t)}_v - T^{(t)}_{uv} \in \mathcal{S}^p$, $\forall p > n$, and $\|T^{(t)}_u T^{(t)}_v - T^{(t)}_{uv}\|_{\mathcal{S}^p} \lesssim_p t^{-1+\frac{n}{p}}$,
   (b) $T^{(t)}_{\phi_i} T^{(t)}_u - T^{(t)}_{\phi_i u} \in \mathcal{S}^p$, $\forall p > \frac{n}{i+\frac{1}{2}}$, and $\|T^{(t)}_{\phi_i} T^{(t)}_u - T^{(t)}_{\phi_i u}\|_{\mathcal{S}^p} \lesssim_p t^{-1+\frac{n}{p}}$,
   (c) $T^{(t)}_u T^{(t)}_{\phi_i} - T^{(t)}_{\phi_i u} \in \mathcal{S}^p$, $\forall p > \frac{n}{i+\frac{1}{2}}$, and $\|T^{(t)}_u T^{(t)}_{\phi_i} - T^{(t)}_{\phi_i u}\|_{\mathcal{S}^p} \lesssim_p t^{-1+\frac{n}{p}}$,
   (d) $T^{(t)}_{\phi_i} T^{(t)}_{\phi_j} - T^{(t)}_{\phi_i\phi_j} \in \mathcal{S}^p$, $\forall p > \frac{n}{i+j}$, and $\|T^{(t)}_{\phi_i} T^{(t)}_{\phi_j} - T^{(t)}_{\phi_i\phi_j}\|_{\mathcal{S}^p} \lesssim_p t^{-1+\frac{n}{p}}$,
   (e) $T^{(t)}_{\phi_i u} \in \mathcal{S}^p$, $\forall p > \frac{n}{i}$, and $\|T^{(t)}_{\phi_i u}\|_{\mathcal{S}^p} \lesssim_p t^{\frac{n}{p}}$.

Iterating Lemma 2.5 as in Remark 2.7, we have that the property

$$\sigma_t(f_1, g_1)\sigma_t(f_2, g_2)\dots\sigma_t(f_p, g_p) \in \mathcal{S}^1.$$

As in the proof of Theorem 7.3, we write $S \sim_a T$ when $S - T$ is in trace class with trace norm converging to 0. Then by (8.2), we compute

$$\sigma_t(f_1, g_1)\sigma_t(f_2, g_2)\dots\sigma_t(f_p, g_p) - c^p_{1,t}T^{(t)}_{C_1(f_1,g_1)}T^{(t)}_{C_1(f_2,g_2)}\dots T^{(t)}_{C_1(f_p,g_p)}$$

$$\begin{aligned}
&= R^{(t)}_{f_1,g_1,1} R^{(t)}_{f_2,g_2,1} \dots R^{(t)}_{f_p,g_p,1} \\
&\quad - \big(R^{(t)}_{f_1,g_1,1} - R^{(t)}_{f_1,g_1,2}\big)\big(R^{(t)}_{f_2,g_2,1} - R^{(t)}_{f_2,g_2,2}\big) \dots \big(R^{(t)}_{f_p,g_p,1} - R^{(t)}_{f_p,g_p,2}\big) \\
&= \sum \pm R^{(t)}_{f_1,g_1,i_1} R^{(t)}_{f_2,g_2,i_2} \dots R^{(t)}_{f_p,g_p,i_p},
\end{aligned}$$

where $i_1, i_2, \dots, i_p \in \{1, 2\}$ and at least one $i_k = 2$. Again, applying Lemma 2.5 inductively as in Remark 2.7 gives the following bounds

$$\begin{aligned}
\|R^{(t)}_{f_1,g_1,i_1} R^{(t)}_{f_2,g_2,i_2} \dots R^{(t)}_{f_p,g_p,i_p}\| \le \|R^{(t)}_{f_1,g_1,i_1}\|_{\mathcal{S}^p} \dots \|R^{(t)}_{f_p,g_p,i_p}\|_{\mathcal{S}^p} &\lesssim t^{-i_1+\frac{n}{p}} \cdot \dots \cdot t^{-i_p+\frac{n}{p}} \\
&\le t^{-p-1+n}.
\end{aligned}$$

We reach the following equation

$$t^{p-n}\sigma_t(f_1,g_1)\sigma_t(f_2,g_2)\dots\sigma_t(f_p,g_p) \sim_a t^{p-n} c^p_{1,t} T^{(t)}_{C_1(f_1,g_1)} T^{(t)}_{C_1(f_2,g_2)} \dots T^{(t)}_{C_1(f_p,g_p)}. \tag{8.3}$$

Also, by (1) and (4)-(e), we have the following equation

$$t^{p-n} c^p_{1,t} T^{(t)}_{C_1(f_1,g_1)} T^{(t)}_{C_1(f_2,g_2)} \dots T^{(t)}_{C_1(f_p,g_p)} \sim_a n^p t^{-n} T^{(t)}_{C_1(f_1,g_1)} T^{(t)}_{C_1(f_2,g_2)} \dots T^{(t)}_{C_1(f_p,g_p)}. \tag{8.4}$$

Write $u_{ij} = D_{f_i,g_i,j}$, $i = 1, \dots, p$, $j = 1, 2$. Then we arrive at the following equation,

$$t^{-n} T^{(t)}_{C_1(f_1,g_1)} T^{(t)}_{C_1(f_2,g_2)} \dots T^{(t)}_{C_1(f_p,g_p)} = \sum_{j_1,\dots,j_p=1,2} t^{-n} T^{(t)}_{\phi_{j_1} u_{1j_1}} T^{(t)}_{\phi_{j_2} u_{2j_2}} \dots T^{(t)}_{\phi_{j_p} u_{pj_p}}.$$

By (4), we have the following estimate

$$\begin{aligned}
&t^{-n} T^{(t)}_{\phi_{j_1} u_{1j_1}} T^{(t)}_{\phi_{j_2} u_{2j_2}} \dots T^{(t)}_{\phi_{j_p} u_{pj_p}} \\
=& t^{-n}\big(T^{(t)}_{\phi_{j_1} u_{1j_1}} - T^{(t)}_{\phi_i} T^{(t)}_{u_{1j_1}}\big) T^{(t)}_{\phi_{j_2} u_{2j_2}} \dots T^{(t)}_{\phi_{j_p} u_{pj_p}} + t^{-n} T^{(t)}_{\phi_{j_1}} T^{(t)}_{u_{1j_1}} T^{(t)}_{\phi_{j_2} u_{2j_2}} \dots T^{(t)}_{\phi_{j_p} u_{pj_p}} \\
\sim_a& t^{-n} T^{(t)}_{\phi_{j_1}} T^{(t)}_{u_{1j_1}} T^{(t)}_{\phi_{j_2} u_{2j_2}} \dots T^{(t)}_{\phi_{j_p} u_{pj_p}} \\
&\dots \\
\sim_a& t^{-n} T^{(t)}_{\phi_{j_1}} T^{(t)}_{u_{1j_1}} T^{(t)}_{\phi_{j_2}} T^{(t)}_{u_{2j_2}} \dots T^{(t)}_{\phi_{j_p}} T^{(t)}_{u_{pj_p}} \\
=& t^{-n} T^{(t)}_{\phi_{j_1}} [T^{(t)}_{u_{1j_1}}, T^{(t)}_{\phi_{j_2}}] T^{(t)}_{u_{2j_2}} \dots T^{(t)}_{\phi_{j_p}} T^{(t)}_{u_{pj_p}} + t^{-n} T^{(t)}_{\phi_{j_1}} T^{(t)}_{\phi_{j_2}} T^{(t)}_{u_{1j_1}} T^{(t)}_{u_{2j_2}} \dots T^{(t)}_{\phi_{j_p}} T^{(t)}_{u_{pj_p}} \\
\sim_a& t^{-n} T^{(t)}_{\phi_{j_1}} T^{(t)}_{\phi_{j_2}} T^{(t)}_{u_{1j_1}} T^{(t)}_{u_{2j_2}} \dots T^{(t)}_{\phi_{j_p}} T^{(t)}_{u_{pj_p}} \\
&\dots \\
\sim_a& t^{-n} T^{(t)}_{\phi_{j_1}} T^{(t)}_{\phi_{j_2}} \dots T^{(t)}_{\phi_{j_p}} T^{(t)}_{u_{1j_1}} T^{(t)}_{u_{2j_2}} \dots T^{(t)}_{u_{pj_p}} \\
=& t^{-n}\big(T^{(t)}_{\phi_{j_1}} T^{(t)}_{\phi_{j_2}} - T^{(t)}_{\phi_{j_1+j_2}}\big) T^{(t)}_{\phi_{j_3}} \dots T^{(t)}_{\phi_{j_p}} T^{(t)}_{u_{1j_1}} T^{(t)}_{u_{2j_2}} \dots T^{(t)}_{u_{pj_p}} \\
&\qquad + t^{-n} T^{(t)}_{\phi_{j_1+j_2}} T^{(t)}_{\phi_{j_3}} \dots T^{(t)}_{\phi_{j_p}} T^{(t)}_{u_{1j_1}} T^{(t)}_{u_{2j_2}} \dots T^{(t)}_{u_{pj_p}}
\end{aligned}$$

$$
\begin{aligned}
&\sim_a t^{-n} T^{(t)}_{\phi_{j_1+j_2}} T^{(t)}_{\phi_{j_3}} \dots T^{(t)}_{\phi_{j_p}} T^{(t)}_{u_{1j_1}} T^{(t)}_{u_{2j_2}} \dots T^{(t)}_{u_{pj_p}} \\
&\quad \dots \\
&\sim_a t^{-n} T^{(t)}_{\phi_{j_1+j_2+\ldots+j_p}} T^{(t)}_{u_{1j_1}} T^{(t)}_{u_{2j_2}} \dots T^{(t)}_{u_{pj_p}} \\
&= t^{-n} T^{(t)}_{\phi_{j_1+j_2+\ldots+j_p}} \big(T^{(t)}_{u_{1j_1}} T^{(t)}_{u_{2j_2}} - T^{(t)}_{u_{1j_1} u_{2j_2}}\big) T^{(t)}_{u_{3j_3}} \dots T^{(t)}_{u_{pj_p}} \\
&\qquad\qquad + t^{-n} T^{(t)}_{\phi_{j_1+j_2+\ldots+j_p}} T^{(t)}_{u_{1j_1} u_{2j_2}} T^{(t)}_{u_{3j_3}} \dots T^{(t)}_{u_{pj_p}} \\
&\sim_a t^{-n} T^{(t)}_{\phi_{j_1+j_2+\ldots+j_p}} T^{(t)}_{u_{1j_1} u_{2j_2}} T^{(t)}_{u_{3j_3}} \dots T^{(t)}_{u_{pj_p}} \\
&\quad \dots \\
&\sim_a t^{-n} T^{(t)}_{\phi_{j_1+j_2+\ldots+j_p}} T^{(t)}_{u_{1j_1} u_{2j_2} \ldots u_{pj_p}} \\
&\sim_a t^{-n} T^{(t)}_{\phi_{j_1+j_2+\ldots+j_p} u_{1j_1} u_{2j_2} \ldots u_{pj_p}}.
\end{aligned}
$$

Adding up over $j_1, j_2, \ldots, j_p = 1, 2$, we get the following equation

$$
t^{-n} T^{(t)}_{C_1(f_1,g_1)} T^{(t)}_{C_1(f_2,g_2)} \dots T^{(t)}_{C_1(f_p,g_p)} \sim_a t^{-n} T^{(t)}_{C_1(f_1,g_1) C_1(f_2,g_2) \ldots C_1(f_p,g_p)}. \tag{8.5}
$$

Combining (8.3) (8.4) and (8.5), we arrive at the following equation

$$
t^{p-n} \sigma_t(f_1,g_1) \sigma_t(f_2,g_2) \dots \sigma_t(f_p,g_p) \sim_a n^p t^{-n} T^{(t)}_{C_1(f_1,g_1) C_1(f_2,g_2) \ldots C_1(f_p,g_p)}. \tag{8.6}
$$

Denote

$$
F = C_1(f_1,g_1) C_1(f_2,g_2) \dots C_1(f_p,g_p).
$$

Then $|F(z)| \lesssim (1-|z|^2)^p$. By [47, Lemma 2.5] and 2.3, we compute $\mathrm{Tr}\big(n^p t^{-n} T^{(t)}_F\big)$

$$
\begin{aligned}
&\mathrm{Tr}\big(n^p t^{-n} T^{(t)}_F\big) \\
=& n^p t^{-n} \int_{\mathbb{B}_n} \langle T^{(t)}_F K^{(t)}_\xi, K^{(t)}_\xi \rangle \mathrm{d}\lambda_t(\xi) \\
=& n^p t^{-n} \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} F(z) K^{(t)}_\xi(z) K^{(t)}_z(\xi) \mathrm{d}\lambda_t(z) \mathrm{d}\lambda_t(\xi) \\
=& n^p t^{-n} \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} F(z) K^{(t)}_\xi(z) K^{(t)}_z(\xi) \mathrm{d}\lambda_t(\xi) \mathrm{d}\lambda_t(z) \\
=& n^p t^{-n} \int_{\mathbb{B}_n} F(z) K^{(t)}_z(z) \mathrm{d}\lambda_t(z) \\
=& n^p t^{-n} \frac{(n-1)!}{\pi^n B(n,t+1)} \int_{\mathbb{B}_n} \frac{F(z)}{(1-|z|^2)^{n+1}} \mathrm{d}m(z)
\end{aligned}
$$

$$\to \frac{n^p}{\pi^n} \int\limits_{\mathbb{B}_n} \frac{F(z)}{(1-|z|^2)^{n+1}} \mathrm{d}m(z), \quad t \to \infty.$$

This gives the first equation. The second equation follows from plugging in the formula of $C_1(f,g)$ in Remark 6.5. This completes the proof of Proposition 8.2. $\square$

Recall that $C_1(f,g) - C_1(g,f) = \frac{-i}{n}\{f,g\}$. Thus Proposition 8.2 implies the following.

**Corollary 8.3.** *Suppose $p \geq n+1$ is an integer and $f_1, g_1, \ldots, f_p, g_p \in \mathscr{C}^2(\overline{\mathbb{B}_n})$. Then for any $t \geq -1$, $[T^{(t)}_{f_1}, T^{(t)}_{g_1}][T^{(t)}_{f_2}, T^{(t)}_{g_2}] \ldots [T^{(t)}_{f_p}, T^{(t)}_{g_p}]$ is in the trace class, and*

$$\lim_{t\to\infty} t^{p-n} \mathrm{Tr}\left( [T^{(t)}_{f_1}, T^{(t)}_{g_1}][T^{(t)}_{f_2}, T^{(t)}_{g_2}] \ldots [T^{(t)}_{f_p}, T^{(t)}_{g_p}] \right) = \frac{(-i)^p}{\pi^n} \int\limits_{\mathbb{B}_n} \prod_{j=1}^{p} \{f_j, g_j\}(z) \frac{\mathrm{d}m(z)}{(1-|z|^2)^{n+1}}.$$

If $p = n$ then the classical trace of such a product of commutators is infinite but the product does have a finite Dixmier trace. In fact, Engliš, Guo and Zhang showed in [22] that the following holds.

$$Tr_\omega [T^{(t)}_{f_1}, T^{(t)}_{g_1}] \ldots [T^{(t)}_{f_n}, T^{(t)}_{g_n}] = \frac{1}{n!} \int\limits_{\mathbb{S}_n} \prod_{j=1}^{n} \{f_j, g_j\}(z) \frac{\mathrm{d}\sigma(z)}{\sigma_{2n-1}}.$$

Also recall the identity

$$\sigma_t(f,g) = -H^{(t)*}_{\bar{f}} H^{(t)}_g.$$

Thus taking $f_i = \bar{g}, g_i = g, i = 1, \ldots, p$ in Proposition 8.2 gives the following asymptotic formula for Schatten-norm of Hankel operators.

**Corollary 8.4.** *Suppose $p \geq n+1$ is an integer, and $g \in \mathscr{C}^2(\overline{\mathbb{B}_n})$. Then*

$$\lim_{t\to\infty} t^{p-n} \|H^{(t)}_g\|^{2p}_{\mathcal{S}^{2p}} = \frac{1}{\pi^n} \int\limits_{\mathbb{B}_n} \left[ |\bar{\partial} g(z)|^2 - |\bar{R} g(z)|^2 \right]^p (1-|z|^2)^{p-n-1} \mathrm{d}m(z).$$

**Remark 8.5.** There are profound study of Schatten-class membership and Schatten norm formulas for Hankel operators. See [3,28,29,41,51] for Schatten-class membership criteria of Hankel operators. For $q = 2, 4, 6$, Janson, Upmeier and Wallstén [34] gave the following identity on the Hardy space of the unit disk.

$$\|H_\phi\|^q_{\mathcal{S}^q} = c_q \int\limits_{\mathbb{T}} \int\limits_{\mathbb{T}} \frac{|\psi(\zeta) - \psi(\tau)|^q}{|\zeta - \tau|^2} \mathrm{d}\sigma(\zeta) \mathrm{d}\sigma(\tau),$$

where $c_q$ are constants, and $\psi = (I-P)\phi$. In fact, it was shown that such identities hold only for $q = 2, 4, 6$. Recently, Xia [50] extended this formula to the open unit ball $\mathbb{B}_n$.

Proposition 8.2 gives an explicit asymptotic formula for each of the two terms in the Connes-Chern character introduced in Equation (1.2). This observation leads us to the following asymptotic formula for the Connes-Chern character at $p > n$.

**Theorem 8.6.** *Suppose $p \geq n+1$ is an integer and $f_1, f_2, \ldots, f_{2p} \in \mathscr{C}^2(\overline{\mathbb{B}_n})$. Set $f_{2p+1} := f_1$. Then*

$$\begin{aligned}&\lim_{t\to\infty} t^{p-n}\tau_t(f_1, f_2, \ldots, f_{2p})\\ =&\frac{n^p}{\pi^n}\int_{\mathbb{B}_n}\left(\prod_{j=1}^{p} C_1(f_{2j-1}, f_{2j})(z) - \prod_{j=1}^{p} C_1(f_{2j}, f_{2j+1})(z)\right)\frac{\mathrm{d}m(z)}{(1-|z|^2)^{n+1}}.\end{aligned}$$

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