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Trace formula of semicommutators

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ABSTRACT

For weighted Bergman spaces on the unit disk, we give trace formulas of semicommutators of Toeplitz operators with $\mathscr{C}^2(\overline{\mathbb{D}})$ symbols. We generalize this formula to weighted Bergman spaces on the unit ball in higher dimensions. Applications and examples on the Hankel operators are also discussed.

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1. Introduction

For $t > -1$, let $L^2_{a,t}(\mathbb{D})$ be the weighted Bergman space on the open unit disk $\mathbb{D}$. For $f \in L^\infty(\mathbb{D})$, denote $T^{(t)}_f$ the Toeplitz operator on $L^2_{a,t}(\mathbb{D})$ with symbol f. Commutators of Toeplitz operators have been objects of interest in the study of analytic function spaces for a long time. Various properties, such as compactness, Schatten class membership, trace formulas, were studied in a numerous amount of works (cf. [1,8,12,13,22,27,33,35,40,41]). Among others, it is well-known (cf. [24,39]) that for relatively nice symbols f and

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g on the unit disk $\mathbb{D}$, the commutator $[T_f^{(0)}, T_g^{(0)}] = T_f^{(0)} T_g^{(0)} - T_g^{(0)} T_f^{(0)}$ on the Bergman space $L_a^2(\mathbb{D}) = L_{a,0}^2(\mathbb{D})$, is in the trace class, and

$$\operatorname{Tr}[T_f^{(0)}, T_g^{(0)}] = \frac{1}{2\pi i} \int_{\mathbb{D}} \mathrm{d}f \wedge \mathrm{d}g. \tag{1.1}$$

This elegant formula is deeply connected to the Pincus function for a pair of noncommuting selfadjoint operators, cf. [9,10,29].

Our study of trace of semi-commutator is inspired by our investigation [32] of the Connes-Chern character for the Toeplitz extension (cf. [15]). Semi-commutator of Toeplitz operators is the building block in Connes construction. On the other hand, the semi-commutator has its own importance. Let $H_f^{(t)}$ be the Hankel operator with symbol f. The following equation

$$T_f^{(t)} T_g^{(t)} - T_{fg}^{(t)} = -H_{\bar{f}}^{(t)*} H_g^{(t)}, \tag{1.2}$$

provides a natural link between the semi-commutators of Toeplitz operators and Hankel operators, which allows to study the Hilbert-Schmidt norm of a Hankel operator by the trace of the associated semi-commutator. We aim in article to establish a generalization of the Helton-Howe trace formula (1.1) to semi-commutators, which has not been explored in literature.

Suppose f and g are two Lipschitz functions on $\mathbb{D}$. It is well-known that for any $t > -1$, the semi-commutator $T_f^{(t)} T_g^{(t)} - T_{fg}^{(t)}$ is in the trace class (cf. [40]). We will establish a trace formula for $T_f^{(t)} T_g^{(t)} - T_{fg}^{(t)}$ when f, g are nice function. We obtain the following trace formula.

Theorem 1.1. *Suppose $t > -1$ and $f, g \in \mathscr{C}^2(\overline{\mathbb{D}})$. Then*

$$\operatorname{Tr}\left(T_f^{(t)} T_g^{(t)} - T_{fg}^{(t)}\right) = \frac{1}{2\pi i} \int_{\mathbb{D}} \partial f \wedge \bar{\partial} g + \int_{\mathbb{D}^2} \varrho_t(|\varphi_z(w)|^2) \Delta f(z) \Delta g(w) \mathrm{d}m(z, w). \tag{1.3}$$

Here ϱ_t is defined as below and is strictly positive on $(0, 1)$.

$$\varrho_t(s) = \frac{t+1}{16\pi^2} \int_s^1 (1-x)^t x^{-1} F(s, x) \mathrm{d}x,$$

where

$$F(s, x) = -\left[x \ln \frac{s}{x} + (1-x) \ln \frac{1-s}{1-x}\right].$$

Moreover,

$$\lim_{t\to\infty} \mathrm{Tr}\left(T_f^{(t)}T_g^{(t)} - T_{fg}^{(t)}\right) = \frac{1}{2\pi i}\int\limits_{\mathbb{D}} \partial f \wedge \bar{\partial} g. \tag{1.4}$$

Let $H_f^{(t)}$ be the Hankel operator with symbol f. By Equation (1.2), we apply the trace formula (1.3) to study the Hilbert-Schmidt norm of Hankel operators with $\mathscr{C}^2(\overline{\mathbb{D}})$ symbols (See Corollary 6.2 and Corollary 6.3). Also we would like to thank Richard Rochberg for pointing out the resemblance between Formula (1.3) and Equation (a) of [2, Proposition 2.5], or Equation (4) on Page 248 of [17].

Next we generalize Theorem 1.1 to higher dimensions. In general, for $n \geq 2$, semicommutators of Toeplitz operators on $L^2_{a,t}(\mathbb{B}_n)$ with Lipschitz symbols only belong to $\mathcal{S}^p$ for $p > n$. In fact, it was shown in [37] that in the case when $f = \bar{g}$ and g is anti-holomorphic, their semi-commutator is in the trace class only when $g = 0$. To make $T_f^{(t)}T_g^{(t)} - T_{fg}^{(t)}$ belong to trace class, one generally requires some further assumptions. We do not aim to give a criterion of when semi-commutators belong to the trace class. Instead, we focus on giving a trace formula for relatively nice symbols.

Recall that the Levi form $L_z f$ of a function f at a point z is the two form

$$L_z f(\xi) = \sum_{i,j=1}^{n} \partial_i \bar{\partial}_j f(z)\xi_i\bar{\xi}_j, \quad \forall \xi \in \mathbb{C}^n.$$

Define $\partial_z f$ and $\bar{\partial}_z f$ as the n-vectors that has $\partial_i f(z)$ and $\bar{\partial}_i f(z)$ in its i-th entry. Then

$$\langle \partial_z f, \overline{z-w}\rangle = \sum_{i=1}^{n} \partial_i f(z)(z_i - w_i), \quad \langle \bar{\partial}_w f, z-w\rangle = \sum_{j=1}^{n} \bar{\partial}_j f(w)\overline{(z_j - w_j)}.$$

We say that f, g satisfy **Condition 1** if $f, g \in \mathscr{C}^1(\overline{\mathbb{B}_n})$ and there exist $C > 0$, $\epsilon > 0$, such that

$$\left|\langle \partial_z f, \overline{z-w}\rangle\langle \bar{\partial}_w g, z-w\rangle\right| \leq C|\varphi_z(w)|^2|1-\langle z,w\rangle|^{n+\epsilon}, \quad \forall z, w \in \mathbb{B}_n. \tag{1.5}$$

We say that f, g satisfy **Condition 2** if f, g satisfy condition 1, and $f, g \in \mathscr{C}^2(\overline{\mathbb{B}_n})$ satisfy the following inequalities. For any $z, w \in \mathbb{B}_n$

$$\left|\langle \partial_z f, \overline{z-w}\rangle L_w g(z-w)\right| \leq C|\varphi_z(w)|^3|1-\langle z,w\rangle|^{n+\epsilon}, \tag{1.6}$$

$$\left|L_z f(z-w)\langle \bar{\partial}_w g, z-w\rangle\right| \leq C|\varphi_z(w)|^3|1-\langle z,w\rangle|^{n+\epsilon}, \tag{1.7}$$

$$\left|L_z f(z-w) L_w g(z-w)\right| \leq C|\varphi_z(w)|^4|1-\langle z,w\rangle|^{n+\epsilon}. \tag{1.8}$$

Here $|\varphi_z(w)|$ is the length of the Möbius transform $\varphi_z(w)$, or the pseudo-hyperbolic distance of z and w. We obtain the following generalization of Theorem 1.1.

Theorem 1.2. *If $t > 2n - 3$, and f, g satisfy Condition 1, then the semicommutator $T_f^{(t)}T_g^{(t)} - T_{fg}^{(t)}$ is in the trace class. If furthermore f, g satisfy Condition 2, then*

$$\begin{aligned}\operatorname{Tr}\left(T_f^{(t)}T_g^{(t)}-T_{fg}^{(t)}\right)=&a_{n,t}\int\limits_{\mathbb{B}_n}\partial f\wedge\bar{\partial}g\wedge\left[\partial\bar{\partial}\log(1-|w|^2)\right]^{n-1}\\&+\int\limits_{\mathbb{B}_n\times\mathbb{B}_n}\rho_{n,t}(|\varphi_z(w)|^2)L_zf(z-w)L_wg(z-w)\frac{\mathrm{d}m(z,w)}{|1-\langle z,w\rangle|^{2n+2}}.\end{aligned}\tag{1.9}$$

Here

$$a_{n,t}=\frac{-\int_0^1(1-s)^{n-1}s^t\ln s\mathrm{d}s}{\left(B(n,t+1)^2\right)n(2\pi i)^n},$$

and

$$\rho_{n,t}(s)=s^{-n-1}\sum_{k=1}^{n}\frac{(n-1)!\Gamma^2(n+t+1)}{(n-k)!\Gamma(t+1+k)\Gamma(t+1)\pi^{2n}}\int\limits_s^1F(s,x)x^{n-k-1}(1-x)^{t+k-1}\mathrm{d}x.\tag{1.10}$$

In particular, $\rho_{n,t}$ is strictly positive on $(0,1)$*, and*

$$\lim_{t\to\infty}t^{1-n}\operatorname{Tr}\left(T_f^{(t)}T_g^{(t)}-T_{fg}^{(t)}\right)=\frac{1}{(n-1)!(2\pi i)^n}\int\limits_{\mathbb{B}_n}\partial f\wedge\bar{\partial}g\wedge\left[\partial\bar{\partial}\log(1-|w|^2)\right]^{n-1}.\tag{1.11}$$

Remark 1.3. One can show that if the functions f and g are in $\mathscr{C}^2(\overline{\mathbb{D}})$ then they satisfy Condition 1 and 2 with $n=1$. Therefore Theorem 1.2 actually implies Theorem 1.1. Because trace formulas at dimension 1 are of independent interest, and because the proof gets significantly more complicated at higher dimensions, we first give a complete proof of Theorem 1.1 in Section 3.

Remark 1.4. In Lemma 5.5 we give an alternative expression of the first term in the right hand side of (1.9), in terms of radial derivatives $R=\sum_{i=1}^n z_i\partial_i$. So (1.9) and (1.11) can be rewritten as

$$\begin{aligned}\operatorname{Tr}\left(T_f^{(t)}T_g^{(t)}-T_{fg}^{(t)}\right)=&-(2i)^n(n-1)!a_{n,t}\int\limits_{\mathbb{B}_n}\frac{\sum_{i=1}^n\partial_if(w)\bar{\partial}_ig(w)-Rf(w)\bar{R}g(w)}{(1-|w|^2)^n}\mathrm{d}m(w)\\&+\int\limits_{\mathbb{B}_n\times\mathbb{B}_n}\rho_{n,t}(|\varphi_z(w)|^2)L_zf(z-w)L_wg(z-w)\frac{\mathrm{d}m(z,w)}{|1-\langle z,w\rangle|^{2n+2}},\end{aligned}$$

and

$$\lim_{t\to\infty}t^{1-n}\operatorname{Tr}\left(T_f^{(t)}T_g^{(t)}-T_{fg}^{(t)}\right)=\frac{-1}{\pi^n}\int\limits_{\mathbb{B}_n}\frac{\sum_{i=1}^n\partial_if(w)\bar{\partial}_ig(w)-Rf(w)\bar{R}g(w)}{(1-|w|^2)^n}\mathrm{d}m(w).$$

Remark 1.5. It is clear that if we have $H_{\bar{f}}^{(t)} \in \mathcal{S}^p$ and $H_g^{(t)} \in \mathcal{S}^q$ for some $\frac{1}{p}+\frac{1}{q}=1$ then the semi-commutator $T_f^{(t)}T_g^{(t)} - T_{fg}^{(t)}$ belongs to the trace class. However, the converse is not true. The point of Condition 1 and Condition 2 is to give combined conditions of f and g, instead of separate conditions. The estimates

$$|1-\langle z,w\rangle| \approx (1-|z|^2)+(1-|w|^2)+|z-w|^2+|\mathrm{Im}\langle z,w\rangle|,$$
$$|\varphi_z(w)|^2 = \frac{|z-P_z(w)|^2+(1-|z|^2)|Q_z(w)|^2}{|1-\langle z,w\rangle|^2}$$

give us some insight into the two conditions. In Lemmas 6.4-6.6 we give some special cases of Theorem 1.2 that are more intuitive and more convenient to work with.

Our proofs involve applying integration by parts formulas on the unit disk and unit ball (see Lemma 3.3 and Lemma 4.2). These formulas essentially come from the Cauchy formula and a Bochner-Martinelli type formula we develop in Appendix I. In Section 4, integration by parts formulas on $\mathbb{B}_n$ are developed. The formulas involve auxiliary functions and operations, which we define and study in Appendix II. In Section 5, we prove Theorem 1.2. Some applications and examples are given in Section 6.

We end the introduction with some explanation on the relationships between this paper and our other paper [32]. Our study was motivated by the exploration of the Helton-Howe trace and Connes-Chern character in [32], which is an important invariant in noncommutative differential geometry. In [32], we study the Helton-Howe trace and the Connes-Chern character for Toeplitz operators on weighted Bergman spaces via the idea of quantization, [5–7,14,18–21]. As a remainder term in the Toeplitz quantization, semi-commutators naturally appears in the proofs. On the other hand, many of the tools developed here are also heavily used in [32]. The proofs in this paper are intended to be self-contained.

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2. Preliminaries

In this section, we recall some basic definitions and properties about weighted Bergman spaces and Schatten-p class operators.

Let $\mathbb{B}_n$ be the open unit ball of $\mathbb{C}^n$ and $\mathbb{S}_n = \partial\mathbb{B}_n$ the unit sphere. Let m be the Lebesgue measure on $\mathbb{B}_n$ and σ be the surface measure on $\mathbb{S}_n$. Denote $\sigma_{2n-1} = \sigma(\mathbb{S}_n) = \frac{2\pi^n}{(n-1)!}$.

For $t > -1$, define the probability measure on $\mathbb{B}_n$:

$$\mathrm{d}\lambda_t(z) = \frac{(n-1)!}{\pi^n B(n,t+1)}(1-|z|^2)^t \mathrm{d}m(z).$$

Here $B(n,t+1)$ is the Beta function. The weighted Bergman space $L^2_{a,t}(\mathbb{B}_n)$ is the subspace of $L^2(\mathbb{B}_n, \lambda_t)$ consisting of holomorphic functions on $\mathbb{B}_n$. The reproducing kernel of $L^2_{a,t}(\mathbb{B}_n)$ is

$$K_w^{(t)}(z) = \frac{1}{(1-\langle z,w\rangle)^{n+1+t}}, \quad \forall w \in \mathbb{B}_n.$$

For any $f \in L^\infty(\mathbb{B}_n)$, the Toeplitz operator $T_f^{(t)}$ is the compression defined by

$$T_f^{(t)} = P^{(t)} M_f^{(t)}|_{L^2_{a,t}(\mathbb{B}_n)},$$

where $P^{(t)}$ is the orthogonal projection from $L^2(\mathbb{B}_n, \lambda_t)$ onto $L^2_{a,t}(\mathbb{B}_n)$, and $M_f^{(t)}$ is the multiplication operator on $L^2(\mathbb{B}_n, \lambda_t)$. The Hankel operator with symbol f is

$$H_f^{(t)} = (I - P^{(t)}) M_f^{(t)} P^{(t)}.$$

Using the reproducing kernels, we can write $T_f^{(t)}, H_f^{(t)}$ as integral operators. For $h \in L^2_{a,t}(\mathbb{B}_n)$, we have the following expressions,

$$T_f^{(t)} h(z) = \int_{\mathbb{B}_n} f(w)h(w)K_w^{(t)}(z)\mathrm{d}\lambda_t(w), \quad \forall z \in \mathbb{B}_n.$$

$$H_f^{(t)} h(z) = \int_{\mathbb{B}_n} \big(f(z)-f(w)\big)h(w)K_w^{(t)}(z)\mathrm{d}\lambda_t(w), \quad \forall z \in \mathbb{B}_n.$$

An important tool on $\mathbb{B}_n$ is the Möbius transform.

Definition 2.1. For $z \in \mathbb{B}_n$, $z \neq 0$, the Möbius transform φ_z is the biholomorphic mapping on $\mathbb{B}_n$ defined as follows.

$$\varphi_z(w) = \frac{z - P_z(w) - (1-|z|^2)^{1/2} Q_z(w)}{1 - \langle w, z\rangle}, \quad \forall w \in \overline{\mathbb{B}_n}.$$

Here P_z and Q_z denote the orthogonal projection from $\mathbb{C}^n$ onto $\mathbb{C}z$ and $z^\perp$, respectively. Define

$$\varphi_0(w) = -w, \quad \forall w \in \overline{\mathbb{B}_n}.$$

It is well-known that φ_z is an automorphism of $\mathbb{B}_n$ satisfying $\varphi_z \circ \varphi_z = Id$. Also, the two variable function $\rho(z,w) := |\varphi_z(w)| = |\varphi_w(z)|$ defines a metric on $\mathbb{B}_n$. Moreover, $\beta(z,w) := \tanh^{-1}\rho(z,w)$ coincides with the Bergman metric on $\mathbb{B}_n$.

We list some lemmas that serve as basic tools for our study of Toeplitz operators on $\mathbb{B}_n$. Most of the following of this section can be found in [30,40]. A proof will be provided when necessary.

For non-negative values A, B, by $A \lesssim B$ we mean that there is a constant C such that $A \le CB$. Sometimes, to emphasize that the constant C depends on some parameter a, we write $A \lesssim_a B$. The notations $\gtrsim, \gtrsim_a, \approx, \approx_a$ are defined similarly.

Lemma 2.2. *Suppose $z, w, \zeta \in \mathbb{B}_n$.*

(1) $\frac{1}{1-\langle \varphi_\zeta(z), \varphi_\zeta(w)\rangle} = \frac{(1-\langle z,\zeta\rangle)(1-\langle \zeta,w\rangle)}{(1-|\zeta|^2)(1-\langle z,w\rangle)}$.
(2) $1-|\varphi_z(w)|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\langle z,w\rangle|^2}$.
(3) *For any $R > 0$ there exists $C > 1$ such that whenever $\beta(z,w) < R$,*

$$\frac{1}{C} \le \frac{1-|z|^2}{1-|w|^2} \le C, \quad \frac{1}{C} \le \frac{|1-\langle z,\zeta\rangle|}{|1-\langle w,\zeta\rangle|} \le C.$$

(4) *The real Jacobian of φ_z is $\frac{(1-|z|^2)^{n+1}}{|1-\langle z,\cdot\rangle|^{2n+2}}$ on $\mathbb{B}_n$ and $\frac{(1-|z|^2)^n}{|1-\langle z,\cdot\rangle|^{2n}}$ on $\mathbb{S}_n$.*
(5) *For $z \in \mathbb{B}_n$,*

$$z - \varphi_z(w) = \frac{(1-|z|^2)P_z(w) + (1-|z|^2)^{1/2}Q_z(w)}{1-\langle w,z\rangle} := \frac{A_z w}{1-\langle w,z\rangle},$$

where $A_z = [a_z^{ij}]$ is an $n\times n$ matrix depending on z, and w is viewed as a column vector.
(6) *There exists $C > 0$ such that for any $z \in \mathbb{B}_n$, $z \ne 0$,*

$$|z - P_z(w)| \le |\varphi_z(w)||1-\langle z,w\rangle|, \quad |Q_z(w)| \le C|\varphi_z(w)||1-\langle z,w\rangle|^{1/2}, \tag{2.1}$$

and

$$|z-w| \le C|\varphi_z(w)||1-\langle z,w\rangle|^{1/2}. \tag{2.2}$$

In contrast, if $n = 1$, then $|z-w| = |\varphi_z(w)||1-z\bar{w}|$.
(7) $1-|z|^2 \le 2|1-\langle z,w\rangle|$.

Proof. Most of the above are either well-known (cf. [30,40]) or straightforward to verify. The only part that requires some clarification is the second estimate in (6), i.e.,

$$|Q_z(w)| \lesssim |\varphi_z(w)||1-\langle z,w\rangle|^{1/2}. \tag{2.3}$$

On the one hand, from the definition of $\varphi_z(w)$, we easily get the follow inequality,

$$|Q_z(w)| \leq |\varphi_z(w)| \frac{|1-\langle z,w\rangle|}{(1-|z|^2)^{1/2}}.$$

If $|\varphi_z(w)| \leq \frac{1}{2}$, then by (3), $|1-\langle z,w\rangle| \approx 1-|z|^2$. From this (2.3) follows.

On the other hand, since

$$2|1-\langle z,w\rangle| \geq 2-2\mathrm{Re}\langle z,w\rangle \geq |z|^2+|w|^2-2\mathrm{Re}\langle z,w\rangle = |z-w|^2,$$

for $|\varphi_z(w)| > \frac{1}{2}$, we obtain the following estimates,

$$|Q_z(w)| = |Q_z(w-z)| \leq |z-w| \lesssim |1-\langle z,w\rangle|^{\frac{1}{2}} < 2|\varphi_z(w)||1-\langle z,w\rangle|^{\frac{1}{2}}.$$

Thus we get (2.3) in both cases. This completes the proof of Lemma 2.2. $\square$

Lemma 2.3. *([30, Proposition 5.1.2]) The two variable function $d(z,w) = |1-\langle z,w\rangle|^{\frac{1}{2}}$ on $\overline{\mathbb{B}_n}$ satisfies the triangle inequality, i.e.,*

$$d(z,w) \leq d(z,\xi) + d(\xi,w), \quad \forall z,w,\xi \in \overline{\mathbb{B}_n}.$$

Lemma 2.4 *(Rudin-Forelli type estimates).*

(1) *Suppose $t > -1$, $c \in \mathbb{R}$. Then there exists $C > 0$ such that for any $z \in \mathbb{B}_n$,*

$$\int\limits_{\mathbb{B}_n} \frac{(1-|w|^2)^t}{|1-\langle z,w\rangle|^{n+1+t+c}} \mathrm{d}m(w) \leq \begin{cases} C(1-|z|^2)^{-c}, & c>0, \\ C\ln\frac{1}{1-|z|^2}, & c=0, \\ C, & c<0, \end{cases} \tag{2.4}$$

$$\int\limits_{\mathbb{S}_n} \frac{1}{|1-\langle z,w\rangle|^{n+c}} \mathrm{d}\sigma(w) \leq \begin{cases} C(1-|z|^2)^{-c}, & c>0, \\ C\ln\frac{1}{1-|z|^2}, & c=0, \\ C, & c<0. \end{cases} \tag{2.5}$$

(2) *Suppose $t > -1$, $a,b,c > 0$, $a \geq c, b \geq c$, and $a+b < n+1+t+c$. Then there exists $C > 0$ such that for any $z,\xi \in \mathbb{B}_n$,*

$$\int\limits_{\mathbb{B}_n} \frac{(1-|w|^2)^t}{|1-\langle z,w\rangle|^a|1-\langle w,\xi\rangle|^b} \mathrm{d}m(w) \leq C\frac{1}{|1-\langle z,\xi\rangle|^c}. \tag{2.6}$$

(3) *Suppose $\phi : (0,1) \to [0,\infty)$ is measurable. Suppose $a > -n$, $b \in \mathbb{R}$, and*

$$\phi(s) \lesssim s^a(1-s)^b, \quad s \in (0,1).$$

Then for any $t > -1 - b$, $c > -b$ there exists $C > 0$ such that for any $z \in \mathbb{B}_n$,

$$\int_{\mathbb{B}_n} \phi(|\varphi_z(w)|^2) \frac{(1-|w|^2)^t}{|1-\langle z, w\rangle|^{n+1+t+c}} \mathrm{d}m(w) \le C(1-|z|^2)^{-c}. \tag{2.7}$$

Proof. The estimates in (1) are standard Rudin-Forelli estimates. See [30, Proposition 1.4.10] for a proof. Let

$$A = \{w \in \mathbb{B}_n : |1-\langle z, w\rangle| \le |1-\langle w, \xi\rangle|\}; \quad B = \{w \in \mathbb{B}_n : |1-\langle z, w\rangle| > |1-\langle w, \xi\rangle|\}.$$

By Lemma 2.3, we have the following equality,

$$|1-\langle z, \xi\rangle|^{1/2} \le |1-\langle z, w\rangle|^{1/2} + |1-\langle w, \xi\rangle|^{1/2}.$$

Then we obtain the following bounds,

$$|1-\langle z, w\rangle| \ge \frac{1}{4}|1-\langle z, \xi\rangle|, \forall w \in B; \quad |1-\langle w, \xi\rangle| \ge \frac{1}{4}|1-\langle z, \xi\rangle|, \forall w \in A.$$

By assumption, $a \ge c$, $b \ge c$, $a+b-c < n+1+t$, therefore by the standard Rudin-Forelli estimate, we compute the integral as follows,

$$\begin{aligned}
&\int_{\mathbb{B}_n} \frac{(1-|w|^2)^t}{|1-\langle z, w\rangle|^a |1-\langle w, \xi\rangle|^b} \mathrm{d}m(w) \\
=&\int_A \frac{(1-|w|^2)^t}{|1-\langle z, w\rangle|^a |1-\langle w, \xi\rangle|^b} \mathrm{d}m(w) + \int_B \frac{(1-|w|^2)^t}{|1-\langle z, w\rangle|^a |1-\langle w, \xi\rangle|^b} \mathrm{d}m(w) \\
\lesssim&\frac{1}{|1-\langle z, \xi\rangle|^c} \int_A \frac{(1-|w|^2)^t}{|1-\langle z, w\rangle|^{a+b-c}} \mathrm{d}m(w) + \frac{1}{|1-\langle z, \xi\rangle|^c} \int_B \frac{(1-|w|^2)^t}{|1-\langle w, \xi\rangle|^{b+a-c}} \mathrm{d}m(w) \\
\lesssim&\frac{1}{|1-\langle z, \xi\rangle|^c}.
\end{aligned}$$

This proves (2).

To prove (3), make the change of variable $w = \varphi_z(\xi)$ in the left hand side of (2.7). We compute the integral as follows,

$$\begin{aligned}
&\int_{\mathbb{B}_n} \phi(|\varphi_z(w)|^2) \frac{(1-|w|^2)^t}{|1-\langle z, w\rangle|^{n+1+t+c}} \mathrm{d}m(w) \\
=&\frac{1}{(1-|z|^2)^c} \int_{\mathbb{B}_n} \phi(|\xi|^2) \frac{(1-|\xi|^2)^t}{|1-\langle z, \xi\rangle|^{n+1+t-c}} \mathrm{d}m(\xi)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(1-|z|^2)^c} \int_0^1 \phi(r^2) r^{2n-1}(1-r^2)^t \int_{\mathbb{S}_n} \frac{1}{|1-\langle z, r\eta\rangle|^{n+1+t-c}} \mathrm{d}\sigma(\eta)\mathrm{d}r \\
&\lesssim \frac{1}{(1-|z|^2)^c} \int_0^1 \phi(r^2) r^{2n-1}(1-r^2)^m \mathrm{d}r,
\end{aligned}
$$

where $m = c-1$ when $1+t-c > 0$, $m = t$ when $1+t-c < 0$, and when $1+t-c = 0$, we take $m = t - \epsilon$ for a sufficiently small $\epsilon > 0$. With our assumption it is easy to see that $m > -b-1$ and therefore the integral above is finite. This proves (3) and completes the proof of Lemma 2.3. □

For $p > 0$, a bounded operator T on a Hilbert space $\mathcal{H}$ is said to be in the Schatten-p class $\mathcal{S}^p$ if $|T|^p$ belongs to the trace class. The Schatten-p class operators $\mathcal{S}^p$ are analogues of l^p spaces in the operator-theoretic setting and satisfy the Hölder's inequality (see [31, Theorem 2.8]).

The following lemma is well-known. See [41, Theorem 6.4] for a proof at $n = 1$. The same proof works for general n.

Lemma 2.5. *Suppose $t > -1$ and T is a trace class operator on $L^2_{a,t}(\mathbb{B}_n)$. Then*

$$
\mathrm{Tr} T = \int_{\mathbb{B}_n} \langle T K_z^{(t)}, K_z^{(t)} \rangle \mathrm{d}\lambda_t(z).
$$

3. Trace formulas on the disk

In this section we give the proof of Theorem 1.1. The main ingredient of its proof is an integral formula coming from the Cauchy formula.

Definition 3.1. Suppose $t > -1$ and $\phi : (0,1) \to [0,\infty)$ is measurable. Define the operations on ϕ:

$$
\mathcal{F}^{(t)}\phi(s) = \int_s^1 \phi(r)(1-r)^t \mathrm{d}r, \quad \mathcal{G}^{(t)}\phi(s) = s^{-1}(1-s)^{-t-1}\mathcal{F}^{(t)}\phi(s).
$$

Lemma 3.2. *Suppose that $t > -1$, and $\phi : (0,1) \mapsto [0,\infty)$ is a measurable function, and $v \in \mathscr{C}^1(\overline{\mathbb{D}})$. Assume that the two integrals*

$$
\int_{\mathbb{D}} \phi(|z|^2) v(z) \mathrm{d}\lambda_t(z), \quad \int_{\mathbb{D}} (1-|z|^2)\mathcal{G}^{(t)}\phi(|z|^2)\bar{z}\bar{\partial}v(z)\mathrm{d}\lambda_t(z)
$$

both converge absolutely. Then

$$\int_{\mathbb{D}} \phi(|z|^2)v(z)\mathrm{d}\lambda_t(z) \tag{3.1}$$

$$= \begin{cases} (t+1)\mathcal{F}^{(t)}\phi(0)\cdot v(0) + \int_{\mathbb{D}}(1-|z|^2)\mathcal{G}^{(t)}\phi(|z|^2)\bar{z}\bar{\partial}v(z)\mathrm{d}\lambda_t(z), & \textit{if } v(0)\neq 0, \mathcal{F}^{(t)}\phi(0)<\infty, \\ \int_{\mathbb{D}}(1-|z|^2)\mathcal{G}^{(t)}\phi(|z|^2)\bar{z}\bar{\partial}v(z)\mathrm{d}\lambda_t(z), & \textit{if } v(0)=0, \mathcal{F}^{(t)}\phi(0)\leq\infty. \end{cases}$$

Proof. Assume first that $v(0) = 0$. For any $0 < r < 1$, denote σ_r the Euclidean surface measure on $r\mathbb{T}$. If we apply the Cauchy Formula to $\Omega = r\mathbb{D} \subset \mathbb{C}$ and $v \in \mathscr{C}^1(\overline{\mathbb{D}})$, then we have the following equation of integrals,

$$\int_{r\mathbb{T}} v(z)\mathrm{d}\sigma_r(z) = 2r\int_{r\mathbb{D}} \frac{\bar{\partial}v(z)}{z}\mathrm{d}m(z).$$

By assumption the left hand side of (3.1) is absolutely integrable, so we compute the integral as follows,

$$\begin{aligned}
\int_{\mathbb{D}} \phi(|z|^2)v(z)\mathrm{d}\lambda_t(z) &= \frac{(t+1)}{\pi}\int_0^1 \phi(r^2)(1-r^2)^t\left\{\int_{r\mathbb{T}} v(z)\mathrm{d}\sigma_r(z)\right\}\mathrm{d}r \\
&= \frac{(t+1)}{\pi}\int_0^1 \phi(r^2)(1-r^2)^t\left\{2r\int_{r\mathbb{D}} \frac{\bar{\partial}v(z)}{z}\mathrm{d}m(z)\right\}\mathrm{d}r \\
&= \frac{(t+1)}{\pi}\int_{\mathbb{D}}\int_{|z|}^1 2r\phi(r^2)(1-r^2)^t\mathrm{d}r\cdot\frac{\bar{\partial}v(z)}{z}\mathrm{d}m(z) \\
&= \frac{(t+1)}{\pi}\int_{\mathbb{D}}\left[\int_{|z|^2}^1 \phi(s)(1-s)^t\mathrm{d}s\right]\frac{\bar{\partial}v(z)}{z}\mathrm{d}m(z) \\
&= \int_{\mathbb{D}}(1-|z|^2)\mathcal{G}^{(t)}\phi(|z|^2)\bar{z}\bar{\partial}v(z)\mathrm{d}\lambda_t(z).
\end{aligned}$$

The absolute convergence of the integral in the right hand side of (3.1) ensures the third equality above. This proves the second case.

Now assume that $v(0) \neq 0$, and $\mathcal{F}^{(t)}\phi(0) < \infty$. We notice that

$$\int_{\mathbb{D}} \phi(|z|^2)\mathrm{d}\lambda_t(z) = 2\pi\cdot\frac{t+1}{\pi}\int_0^1 r\phi(r^2)(1-r^2)^t\mathrm{d}r = (t+1)\mathcal{F}^{(t)}\phi(0).$$

Then we get the following estimate

$$\int_{\mathbb{D}}\left|\phi(|z|^2)\big(v(z)-v(0)\big)\right|\mathrm{d}\lambda_t(z) \leq \int_{\mathbb{D}}\left|\phi(|z|^2)v(z)\right|\mathrm{d}\lambda_t(z) + (t+1)\mathcal{F}^{(t)}\phi(0)|v(0)| < \infty.$$

Applying the second case to $v(z)-v(0)$ and reorganizing the terms give the first case. This completes the proof of Lemma 3.2. $\square$

Lemma 3.3. *Suppose that $t>-1$, and $\phi:(0,1)\to[0,\infty)$ is a measurable function, and $v\in\mathscr{C}^1(\overline{\mathbb{D}})$.*

(1) *Assuming that $z\in\mathbb{D}$, and the integrals*

$$\int_{\mathbb{D}}\phi(|\varphi_z(w)|^2)v(w)K_w^{(t)}(z)\mathrm{d}\lambda_t(w),$$
$$\int_{\mathbb{D}}\mathcal{G}^{(t)}\phi(|\varphi_z(w)|^2)\tfrac{(1-|w|^2)\overline{(z-w)}}{1-w\bar{z}}\bar{\partial}v(w)K_w^{(t)}(z)\mathrm{d}\lambda_t(w)$$

converge absolutely, then

$$\int_{\mathbb{D}}\phi(|\varphi_z(w)|^2)v(w)K_w^{(t)}(z)\mathrm{d}\lambda_t(w) \tag{3.2}$$
$$=\begin{cases}(t+1)\mathcal{F}^{(t)}\phi(0)\cdot v(z) & \textit{if } v(z)\neq 0,\mathcal{F}^{(t)}\phi(0)<\infty,\\ \quad-\int_{\mathbb{D}}\mathcal{G}^{(t)}\phi(|\varphi_z(w)|^2)\tfrac{(1-|w|^2)\overline{(z-w)}}{1-w\bar{z}}\bar{\partial}v(w)K_w^{(t)}(z)\mathrm{d}\lambda_t(w), & \\ -\int_{\mathbb{D}}\mathcal{G}^{(t)}\phi(|\varphi_z(w)|^2)\tfrac{(1-|w|^2)\overline{(z-w)}}{1-w\bar{z}}\bar{\partial}v(w)K_w^{(t)}(z)\mathrm{d}\lambda_t(w), & \textit{if } v(z)=0,\mathcal{F}^{(t)}\phi(0)\leq\infty.\end{cases}$$

(2) *Assuming that $w\in\mathbb{D}$, and the integrals*

$$\int_{\mathbb{D}}\phi(|\varphi_z(w)|^2)v(z)K_w^{(t)}(z)\mathrm{d}\lambda_t(z),$$
$$\int_{\mathbb{D}}\mathcal{G}^{(t)}\phi(|\varphi_z(w)|^2)\frac{(1-|z|^2)(z-w)}{1-w\bar{z}}\partial v(z)K_w^{(t)}(z)\mathrm{d}\lambda_t(z)$$

converge absolutely, then

$$\int_{\mathbb{D}}\phi(|\varphi_z(w)|^2)v(z)K_w^{(t)}(z)\mathrm{d}\lambda_t(z) \tag{3.3}$$
$$=\begin{cases}(t+1)\mathcal{F}^{(t)}\phi(0)\cdot v(w) & \textit{if } v(w)\neq 0,\mathcal{F}^{(t)}\phi(0)<\infty,\\ \quad+\int_{\mathbb{D}}\mathcal{G}^{(t)}\phi(|\varphi_z(w)|^2)\tfrac{(1-|z|^2)(z-w)}{1-w\bar{z}}\partial v(z)K_w^{(t)}(z)\mathrm{d}\lambda_t(z), & \\ \int_{\mathbb{D}}\mathcal{G}^{(t)}\phi(|\varphi_z(w)|^2)\tfrac{(1-|z|^2)(z-w)}{1-w\bar{z}}\partial v(z)K_w^{(t)}(z)\mathrm{d}\lambda_t(z), & \textit{if } v(w)=0,\mathcal{F}^{(t)}\phi(0)\leq\infty.\end{cases}$$

Proof. First, we prove case (1). The formula is obtained from (3.1) by taking Möbius transforms. By Lemma 2.2 (4) it is easy to verify the following equation,

$$K_w^{(t)}(z)\mathrm{d}\lambda_t(w)\ \xlongequal[\xi=\varphi_z(w)]{w=\varphi_z(\xi)}\ K_z^{(t)}(\xi)\mathrm{d}\lambda_t(\xi). \tag{3.4}$$

By Lemma 2.2 (1)(2), Lemma 3.2 and the above equality, if $\mathcal{F}^{(t)}\phi(0) < \infty$, then we compute the integral as follows.

$$
\begin{aligned}
&\int_{\mathbb{D}} \phi(|\varphi_z(w)|^2)v(w)K_w^{(t)}(z)\mathrm{d}\lambda_t(w)\\
\xlongequal{w=\varphi_z(\xi)}&\int_{\mathbb{D}} \phi(|\xi|^2)v\circ\varphi_z(\xi)K_z^{(t)}(\xi)\mathrm{d}\lambda_t(\xi)\\
\xlongequal{(3.1)}&(t+1)\mathcal{F}^{(t)}\phi(0)\cdot v(z)+\int_{\mathbb{D}} \mathcal{G}^{(t)}\phi(|\xi|^2)(1-|\xi|^2)\bar{\xi}\bar{\partial}_\xi\Big(v\circ\varphi_z(\xi)K_z^{(t)}(\xi)\Big)\mathrm{d}\lambda_t(\xi)\\
=&(t+1)\mathcal{F}^{(t)}\phi(0)\cdot v(z)+\int_{\mathbb{D}} \mathcal{G}^{(t)}\phi(|\xi|^2)(1-|\xi|^2)\bar{\xi}\bar{\partial}v(\varphi_z(\xi))\overline{\varphi_z'(\xi)}K_z^{(t)}(\xi)\mathrm{d}\lambda_t(\xi)\\
=&(t+1)\mathcal{F}^{(t)}\phi(0)\cdot v(z)-\int_{\mathbb{D}} \mathcal{G}^{(t)}\phi(|\xi|^2)(1-|\xi|^2)\bar{\xi}\bar{\partial}v(\varphi_z(\xi))\frac{1-|z|^2}{(1-z\bar{\xi})^2}K_z^{(t)}(\xi)\mathrm{d}\lambda_t(\xi)\\
\xlongequal{\xi=\varphi_z(w)}&(t+1)\mathcal{F}^{(t)}\phi(0)\cdot v(z)-\int_{\mathbb{D}} \mathcal{G}^{(t)}\phi(|\varphi_z(w)|^2)\frac{(1-|w|^2)\overline{(z-w)}}{1-w\bar{z}}\bar{\partial}v(w)K_w^{(t)}(z)\mathrm{d}\lambda_t(w).
\end{aligned}
$$

The case when $\mathcal{F}^{(t)}\phi(0)=\infty, v(z)=0$ is proved by the same equations as above, but with the term "$(t+1)\mathcal{F}^{(t)}\phi(0)\cdot v(z)$" removed. This proves (1). To prove (2), apply (3.2) to $\overline{v(z)}$, then swap z and w, then take conjugate over the equation. Note that the equations $|\varphi_z(w)|=|\varphi_w(z)|$ and $\overline{K_z^{(t)}(w)}=K_w^{(t)}(z)$ is used here. This completes the proof of Lemma 3.3. □

Recall that in Theorem 1.1 we defined

$$F(s,x)=-\left[x\ln\frac{s}{x}+(1-x)\ln\frac{1-s}{1-x}\right].$$

Lemma 3.4. *Suppose* $0<s<x<1$. *Then*

$$\iint_{s<s_1<s_2<x} s_1^{-1}(1-s_1)^{-1}\mathrm{d}s_1\mathrm{d}s_2=F(s,x)$$

Proof.

$$
\begin{aligned}
&\iint_{s<s_1<s_2<x} s_1^{-1}(1-s_1)^{-1}\mathrm{d}s_1\mathrm{d}s_2\\
=&\int_s^x\int_{s_1}^x s_1^{-1}(1-s_1)^{-1}\mathrm{d}s_2\mathrm{d}s_1\\
=&\int_s^x \frac{x-s_1}{s_1(1-s_1)}\mathrm{d}s_1
\end{aligned}
$$

$$
\begin{aligned}
&= \int_s^x \frac{x}{s_1} - \frac{1-x}{1-s_1} \mathrm{d}s_1 \\
&= x \ln \frac{x}{s} + (1-x) \ln \frac{1-x}{1-s} \\
&= F(s,x).
\end{aligned}
$$

This completes the proof. $\square$

Lemma 3.5. *For $0 < \epsilon < 1$ there exists $C > 0$ such that*

$$\int_0^x (1-s)^{-\epsilon} F(s,x) \mathrm{d}s \le C x^2, \quad 0 < x < 1.$$

Proof. By definition,

$$
\begin{aligned}
&\int_0^x (1-s)^{-\epsilon} F(s,x) \mathrm{d}s \\
= &-x \int_0^x (1-s)^{-\epsilon} \ln \frac{s}{x} \mathrm{d}s - (1-x) \int_0^x (1-s)^{-\epsilon} \ln(1-s) \mathrm{d}s + (1-x)\ln(1-x) \\
&\int_0^x (1-s)^{-\epsilon} \mathrm{d}s.
\end{aligned}
$$

For $0 < x < 1$, $\ln(1-x) < 0$. Thus the last term in the above is negative. Therefore

$$\int_0^x (1-s)^{-\epsilon} F(s,x) \mathrm{d}s < -x \int_0^x (1-s)^{-\epsilon} \ln \frac{s}{x} \mathrm{d}s - (1-x) \int_0^x (1-s)^{-\epsilon} \ln(1-s) \mathrm{d}s. \quad (3.5)$$

For the first term in the right hand side of (3.5), take the change of variable $r = \frac{s}{x}$. Then

$$-x \int_0^x (1-s)^{-\epsilon} \ln \frac{s}{x} \mathrm{d}s \xlongequal{r=s/x} -x^2 \int_0^1 (1-rx)^{-\epsilon} \ln r \mathrm{d}r < -x^2 \int_0^1 (1-r)^{-\epsilon} \ln r \mathrm{d}r \le x^2. \quad (3.6)$$

For the second term in the right hand side of (3.5), notice that

$$-\ln(1-s) \lesssim s.$$

So

$$-(1-x)\int_0^x (1-s)^{-\epsilon} s\mathrm{d}s \leq (1-x)^{1-\epsilon}\int_0^x s\mathrm{d}s \lesssim x^2. \tag{3.7}$$

Combining (3.5), (3.6) and (3.7) gives the desired result. □

Proof of Theorem 1.1. By Lemma 2.5, we have the following expression of $\mathrm{Tr}\left(T_f^{(t)}T_g^{(t)} - T_{fg}^{(t)}\right)$:

$$\mathrm{Tr}\left(T_f^{(t)}T_g^{(t)} - T_{fg}^{(t)}\right) = \int_{\mathbb{D}} \langle \left(T_f^{(t)}T_g^{(t)} - T_{fg}^{(t)}\right) K_\xi^{(t)}, K_\xi^{(t)}\rangle \mathrm{d}\lambda_t(\xi)$$

$$= \int_{\mathbb{D}} \left\{ \int_{\mathbb{D}^2} (f(z)-f(w))g(w)K_\xi^{(t)}(w)K_w^{(t)}(z)K_z^{(t)}(\xi)\mathrm{d}\lambda_t(w)\mathrm{d}\lambda_t(z)\right\}\mathrm{d}\lambda_t(\xi).$$

The rest of the proof is simply iterating (3.2) and (3.3) on the integral above. For each fixed $\xi \in \mathbb{D}$, we calculate the inner product $\langle \left(T_f^{(t)}T_g^{(t)} - T_{fg}^{(t)}\right) K_\xi^{(t)}, K_\xi^{(t)}\rangle$ as follows:

$$\begin{aligned}
&\langle \left(T_f^{(t)}T_g^{(t)} - T_{fg}^{(t)}\right) K_\xi^{(t)}, K_\xi^{(t)}\rangle \\
&= \int_{\mathbb{D}^2} (f(z)-f(w))g(w)K_\xi^{(t)}(w)K_w^{(t)}(z)K_z^{(t)}(\xi)\mathrm{d}\lambda_t(w)\mathrm{d}\lambda_t(z) \\
&= \int_{\mathbb{D}} \left\{ \int_{\mathbb{D}} \left[(f(z)-f(w))g(w)K_\xi^{(t)}(w)K_z^{(t)}(\xi)\right] K_w^{(t)}(z)\mathrm{d}\lambda_t(z)\right\}\mathrm{d}\lambda_t(w) \\
&\overset{(3.3)}{=\!=} \int_{\mathbb{D}} \left\{ \int_{\mathbb{D}} \mathcal{G}^{(t)}1(|\varphi_z(w)|^2)\frac{(1-|z|^2)(z-w)}{1-w\bar{z}} \left[\partial f(z)g(w)K_\xi^{(t)}(w)K_z^{(t)}(\xi)\right] K_w^{(t)}(z)\mathrm{d}\lambda_t(z)\right\}\mathrm{d}\lambda_t(w).
\end{aligned}$$

In the above, we apply (3.3) with $\phi = 1$ and $v(z) = \left[(f(z)-f(w))g(w)K_\xi^{(t)}(w)K_z^{(t)}(\xi)\right]$. Here, by direct computation, we obtain

$$\left(\mathcal{G}^{(t)}1\right)(s) = \frac{1}{(t+1)s}.$$

Since $\xi \in \mathbb{D}$ is fixed, and f, g are $\mathscr{C}^1$ to the boundary, the term $\left[\partial f(z)g(w)K_\xi^{(t)}(w)K_z^{(t)}(\xi)\right]$ is bounded. By (2.7), the two-fold integral in the above converges absolutely. Applying Fubini's Theorem, and then (3.2) with

$$\phi = \mathcal{G}^{(t)}1, \quad v(w) = \frac{(1-|z|^2)(z-w)}{1-w\bar{z}}\left[\partial f(z)g(w)K_\xi^{(t)}(w)K_z^{(t)}(\xi)\right],$$

the above integral is computed as follows,

$$\int_{\mathbb{D}} \left\{ \int_{\mathbb{D}} \mathcal{G}^{(t)} 1(|\varphi_z(w)|^2) \frac{(1-|z|^2)(z-w)}{1-w\bar{z}} \left[\partial f(z) g(w) K_\xi^{(t)}(w) K_z^{(t)}(\xi)\right] K_w^{(t)}(z) \mathrm{d}\lambda_t(w) \right\} \mathrm{d}\lambda_t(z)$$

$$\overset{(3.2)}{=\!=} - \int_{\mathbb{D}} \left\{ \int_{\mathbb{D}} \left(\mathcal{G}^{(t)}\right)^2 1(|\varphi_z(w)|^2) \frac{(1-|w|^2)\overline{(z-w)}}{1-w\bar{z}} \cdot \frac{(1-|z|^2)(z-w)}{1-w\bar{z}} \right.$$
$$\left. \cdot \left[\partial f(z) \bar{\partial} g(w) K_\xi^{(t)}(w) K_z^{(t)}(\xi)\right] K_w^{(t)}(z) \mathrm{d}\lambda_t(w) \right\} \mathrm{d}\lambda_t(z)$$
$$= - \int_{\mathbb{D}} \left\{ \int_{\mathbb{D}} \left(\mathcal{G}^{(t)}\right)^2 1(|\varphi_z(w)|^2)(1-|\varphi_z(w)|^2)|\varphi_z(w)|^2 (1-z\bar{w})^2 \right.$$
$$\left. \cdot \left[\partial f(z) \bar{\partial} g(w) K_\xi^{(t)}(w) K_z^{(t)}(\xi)\right] K_w^{(t)}(z) \mathrm{d}\lambda_t(w) \right\} \mathrm{d}\lambda_t(z)$$
$$= - \int_{\mathbb{D}^2} \psi_1(|\varphi_z(w)|^2)(1-z\bar{w})^2 \left[\partial f(z) \bar{\partial} g(w) K_\xi^{(t)}(w) K_z^{(t)}(\xi)\right] K_w^{(t)}(z) \mathrm{d}\lambda_t(w) \mathrm{d}\lambda_t(z).$$

Here we have the following bound of ψ_1,

$$\psi_1(s) = \left[\left(\mathcal{G}^{(t)}\right)^2 1(s)\right](1-s)s = \frac{\int_s^1 r^{-1}(1-r)^t \mathrm{d}r}{(t+1)(1-s)^t} \lesssim_t s^{-1/2}(1-s). \tag{3.8}$$

Using (2.6) and (2.7), we can show that the integrand above is absolutely integrable with the measure $\mathrm{d}\lambda_t(\xi)\mathrm{d}\lambda_t(w)\mathrm{d}\lambda_t(z)$. Therefore by Lemma 2.5 and Fubini's Theorem, we compute $\mathrm{Tr}\left(T_f^{(t)} T_g^{(t)} - T_{fg}^{(t)}\right)$ as follows,

$$\mathrm{Tr}\left(T_f^{(t)} T_g^{(t)} - T_{fg}^{(t)}\right)$$
$$= \int_{\mathbb{D}} \left\{ - \int_{\mathbb{D}^2} \psi_1(|\varphi_z(w)|^2)(1-z\bar{w})^2 \left[\partial f(z) \bar{\partial} g(w) K_\xi^{(t)}(w) K_z^{(t)}(\xi)\right] K_w^{(t)}(z) \mathrm{d}\lambda_t(w) \mathrm{d}\lambda_t(z) \right\} \mathrm{d}\lambda_t(\xi)$$
$$= - \int_{\mathbb{D}^2} \psi_1(|\varphi_z(w)|^2)(1-z\bar{w})^2 \partial f(z) \bar{\partial} g(w) |K_w^{(t)}(z)|^2 \mathrm{d}\lambda_t(w) \mathrm{d}\lambda_t(z).$$

To obtain (1.3), we apply (3.3) again, with z, w reversed,

$$\phi = \psi_1, \qquad v(w) = \left[(1-z\bar{w})^2 \partial f(z) \bar{\partial} g(w) K_w^{(t)}(z)\right].$$

We compute the above integral as follows.

$$
\begin{aligned}
&\operatorname{Tr}\left(T_f^{(t)}T_g^{(t)}-T_{fg}^{(t)}\right)\\
&=-\int\limits_{\mathbb{D}}\left\{\int\limits_{\mathbb{D}}\psi_1(|\varphi_z(w)|^2)\big[(1-z\bar{w})^2\partial f(z)\bar{\partial}g(w)K_w^{(t)}(z)\big]K_z^{(t)}(w)\mathrm{d}\lambda_t(w)\right\}\mathrm{d}\lambda_t(z)\\
&\overset{(3.3)}{=\!=}-\int\limits_{\mathbb{D}}\Bigg\{(t+1)\mathcal{F}^{(t)}\psi_1(0)\cdot(1-|z|^2)^2\partial f(z)\bar{\partial}g(z)K_z^{(t)}(z)\\
&\qquad+\int\limits_{\mathbb{D}}\mathcal{G}^{(t)}\psi_1(|\varphi_z(w)|^2)\frac{(1-|w|^2)(w-z)}{1-z\bar{w}}\\
&\qquad\qquad\big[(1-z\bar{w})^2\partial f(z)\partial\bar{\partial}g(w)K_w^{(t)}(z)\big]K_z^{(t)}(w)\mathrm{d}\lambda_t(w)\Bigg\}\mathrm{d}\lambda_t(z)\\
&=\frac{1}{2\pi i}\int\limits_{\mathbb{D}}\partial f\wedge\bar{\partial}g\\
&\quad-\int\limits_{\mathbb{D}^2}\psi_2(|\varphi_z(w)|^2)\big[(1-|w|^2)(w-z)(1-z\bar{w})\partial f(z)\partial\bar{\partial}g(w)K_w^{(t)}(z)\big]K_z^{(t)}(w)\mathrm{d}\lambda_t(w)\mathrm{d}\lambda_t(z).
\end{aligned}
$$

Here we have the following expressions

$$
\mathcal{F}^{(t)}\psi_1(0)=\frac{\int_0^1\int_s^1 r^{-1}(1-r)^t\mathrm{d}r\mathrm{d}s}{t+1}=\frac{\int_0^1\int_0^r r^{-1}(1-r)^t\mathrm{d}s\mathrm{d}r}{t+1}=\frac{\int_0^1(1-r)^t\mathrm{d}r}{t+1}=(t+1)^{-2}, \tag{3.9}
$$

and

$$
\psi_2(s)=\mathcal{G}^{(t)}\psi_1(s)=\frac{\int_s^1\psi_1(r)(1-r)^t\mathrm{d}r}{s(1-s)^{t+1}}=\frac{\int_s^1\int_r^1 x^{-1}(1-x)^t\mathrm{d}x\mathrm{d}r}{(t+1)s(1-s)^{t+1}}\lesssim_t s^{-1/2}(1-s). \tag{3.10}
$$

Again, applying Fubini's Theorem and (3.2), with z,w reversed, $\phi=\psi_2$, $v(z)=\big[(1-|w|^2)(w-z)(1-z\bar{w})\partial f(z)\partial\bar{\partial}g(w)K_w^{(t)}(z)\big]$, we compute the second integral in the above $\operatorname{Tr}\left(T_f^{(t)}T_g^{(t)}-T_{fg}^{(t)}\right)$ as follows.

$$
\begin{aligned}
&-\int\limits_{\mathbb{D}^2}\psi_2(|\varphi_z(w)|^2)\big[(1-|w|^2)(w-z)(1-z\bar{w})\partial f(z)\partial\bar{\partial}g(w)K_w^{(t)}(z)\big]K_z^{(t)}(w)\mathrm{d}\lambda_t(w)\mathrm{d}\lambda_t(z)\\
&=-\int\limits_{\mathbb{D}}\left\{\int\limits_{\mathbb{D}}\psi_2(|\varphi_z(w)|^2)\big[(1-|w|^2)(w-z)(1-z\bar{w})\partial f(z)\partial\bar{\partial}g(w)K_w^{(t)}(z)\big]K_z^{(t)}(w)\mathrm{d}\lambda_t(z)\right\}\mathrm{d}\lambda_t(w)\\
&\overset{(3.2)}{=\!=}\int\limits_{\mathbb{D}}\Bigg\{\int\limits_{\mathbb{D}}\mathcal{G}^{(t)}\psi_2(|\varphi_z(w)|^2)\frac{(1-|z|^2)\overline{(w-z)}}{1-z\bar{w}}\\
&\qquad\cdot\big[(1-|w|^2)(w-z)(1-z\bar{w})\bar{\partial}\partial f(z)\partial\bar{\partial}g(w)K_w^{(t)}(z)\big]K_z^{(t)}(w)\mathrm{d}\lambda_t(z)\Bigg\}\mathrm{d}\lambda_t(w)
\end{aligned}
$$

$$=\frac{(t+1)^2}{16\pi^2}\int\limits_{\mathbb{D}^2}\mathcal{G}^{(t)}\psi_2(|\varphi_z(w)|^2)\frac{(1-|z|^2)^{t+1}(1-|w|^2)^{t+1}|w-z|^2}{|1-z\bar{w}|^{4+2t}}\Delta f(z)\Delta g(w)|\mathrm{d}m(z,w)$$

$$=\int\limits_{\mathbb{D}^2}\varrho_t(|\varphi_z(w)|^2)\Delta f(z)\Delta g(w)\mathrm{d}m(z,w).$$

Here, by Lemma 2.2, we have

$$\begin{aligned}\varrho_t(s)=&\frac{(t+1)^2}{16\pi^2}\mathcal{G}^{(t)}\psi_2(s)(1-s)^{t+1}s=\frac{(t+1)^2}{16\pi^2}\int_s^1\psi_2(s_1)(1-s_1)^t\mathrm{d}s_1\\=&\frac{(t+1)}{16\pi^2}\int_s^1\int_{s_1}^1\int_{s_2}^1 s_1^{-1}(1-s_1)^{-1}s_3^{-1}(1-s_3)^t\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1.\end{aligned}\tag{3.11}$$

Therefore we have reached the following identity

$$\operatorname{Tr}\left(T_f^{(t)}T_g^{(t)}-T_{fg}^{(t)}\right)=\frac{1}{2\pi i}\int\limits_{\mathbb{D}}\partial f\wedge\bar{\partial}g+\int\limits_{\mathbb{D}^2}\varrho_t(|\varphi_z(w)|^2)\Delta f(z)\Delta g(w)\mathrm{d}m(z,w).\tag{3.12}$$

Let us simplify the expression of ϱ_t.

$$\begin{aligned}&\int_s^1\int_{s_1}^1\int_{s_2}^1 s_1^{-1}(1-s_1)^{-1}s_3^{-1}(1-s_3)^t\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1\\=&\iiint\limits_{\{(s_1,s_2,s_3):s<s_1<s_2<s_3<1\}} s_1^{-1}(1-s_1)^{-1}s_3^{-1}(1-s_3)^t\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3\\=&\int_s^1\left\{\iint\limits_{\{(s_1,s_2):s<s_1<s_2<s_3\}} s_1^{-1}(1-s_1)^{-1}\mathrm{d}s_1\mathrm{d}s_2\right\}s_3^{-1}(1-s_3)^t\mathrm{d}s_3\\=&\int_s^1 F(s,s_3)s_3^{-1}(1-s_3)^t\mathrm{d}s_3.\end{aligned}$$

Here the last equality is by Lemma 3.4. This proves the equation for ϱ_t. It also follows from Lemma 3.4 that $F(s,x)$ is strictly positive on $(0,1)$. Therefore ρ_t is strictly positive on $(0,1)$.

It remains to prove (1.4). In other words, the second term of (3.12) tends to zero as t tends to infinity. Clearly the absolute value of the second term has the following bound,

$$\lesssim\int\limits_{\mathbb{D}}\int\limits_{\mathbb{D}}\varrho_t(|\varphi_z(w)|^2)\mathrm{d}m(w)\mathrm{d}m(z)=\int\limits_{\mathbb{D}}\int\limits_{\mathbb{D}}\varrho_t(|\zeta|^2)\frac{(1-|z|^2)^2}{|1-\zeta\bar{z}|^4}\mathrm{d}m(\zeta)\mathrm{d}m(z)$$

$$
\begin{aligned}
&\lesssim \int_{\mathbb{D}} \varrho_t(|\zeta|^2) \ln \frac{1}{1-|\zeta|^2} \mathrm{d}m(\zeta) \lesssim \int_{\mathbb{D}} \varrho_t(|\zeta|)(1-|\zeta|^2)^{-1/2} \mathrm{d}m(\zeta) \\
&\approx \int_0^1 \varrho_t(s)(1-s)^{-1/2} \mathrm{d}s.
\end{aligned}
$$

Plugging in the formula of ϱ_t and applying the Fubini's theorem gives

$$
\begin{aligned}
&\int_0^1 \varrho_t(s)(1-s)^{-1/2} \mathrm{d}s \\
=&\frac{t+1}{16\pi^2} \int_0^1 \int_s^1 (1-s)^{-1/2} F(s,x) x^{-1}(1-x)^t \mathrm{d}x\mathrm{d}s \\
=&\frac{t+1}{16\pi^2} \int_0^1 \left[\int_0^x (1-s)^{-1/2} F(s,x)\mathrm{d}s\right] x^{-1}(1-x)^t \mathrm{d}x.
\end{aligned}
$$

By Lemma 3.5,

$$
0 < \int_0^x (1-s)^{-1/2} F(s,x)\mathrm{d}s \lesssim x^2.
$$

So

$$
\begin{aligned}
\int_0^1 \varrho_t(s)(1-s)^{-1/2} \mathrm{d}s =&\frac{t+1}{16\pi^2} \int_0^1 \left[\int_0^x (1-s)^{-1/2} F(s,x)\mathrm{d}s\right] x^{-1}(1-x)^t \mathrm{d}x \\
\lesssim&(t+1) \int_0^1 x(1-x)^t \mathrm{d}x \\
=&(t+1) B(2, t+1) \\
\approx& t^{-1}.
\end{aligned}
$$

Therefore the second term in (3.12) vanishes as $t \to \infty$. This completes the proof of Theorem 1.1. □

4. Integration by parts

In the remaining sections of the article, we aim to extend Theorem 1.1 to higher dimensions and prove Theorem 1.2. Reviewing the proof of Theorem 1.1, we notice that there are two key ingredients:

(1) the integral formulas in Lemma 3.3;
(2) the auxiliary operations $\mathcal{F}^{(t)}$, $\mathcal{G}^{(t)}$ that record the change in ϕ after each application of the formulas.

With the above tools, we obtain the trace formula in Theorem 1.1 by applying iterations of Lemma 3.3.

The proof of Theorem 1.2 relies on generalizing (1) and (2). The goal of this section is to establish Lemma 4.2, which is an analogue of Lemma 3.3 in higher dimensions. Applying iteration of Lemma 4.2 two times, we get Lemma 4.3. In Appendix II, more general auxiliary operations $\mathcal{F}_m^{(t)}$, $\mathcal{G}_m^{(t)}$ are defined and some basic properties are established. In Section 5, we apply Lemma 4.3 to obtain a formula for the semi-commutator (see Lemma 5.1), and we apply Lemma 4.3 again with z, w reversed, to get the trace formula in Theorem 1.2.

The proof of Lemma 4.2 relies on a Bochner-Martinelli type formula that we establish in Appendix I. Similar integral formulas as in Lemma 4.2 were discovered by Charpentier in [11]. Such integral formulas were used in [28] to study Bergman-Besov spaces and in [16] to study the corona problem on the multiplier algebra of the Drury-Arveson space.

Let us start with a few definitions. Recall that by Lemma 2.2 (5), for $z \in \mathbb{B}_n$,

$$(1-|z|^2)P_z(w) + (1-|z|^2)^{1/2}Q_z(w) = (1-\langle w, z\rangle)(z-\varphi_z(w)) = A_z w,$$

where A_z is an $n \times n$ matrix depending on z, and w is treated as a column vector.

Definition 4.1. For multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ and $z \in \mathbb{B}_n$, define

$$d_{\alpha,\beta}(z) = \int_{\mathbb{S}_n} \left(A_z\zeta\right)^{\alpha} \overline{\left(A_z\zeta\right)}^{\beta} \frac{\mathrm{d}\sigma(\zeta)}{\sigma_{2n-1}}.$$

In particular, $d_{0,0} = 1$, and

$$d_{\alpha,\beta}(z) = \delta_{\alpha,\beta}(1-|z|^2)^{\alpha_1+|\alpha|}\frac{(n-1)!\alpha!}{(n-1+|\alpha|)!}, \quad \text{if } z = (z_1, 0, \ldots, 0). \tag{4.1}$$

For multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ and $\zeta \in \mathbb{C}^n$, denote

$$I^{\alpha,\beta}(\zeta) = \zeta^{\alpha}\bar{\zeta}^{\beta}.$$

Lemma 4.2. *Suppose $t > -1$, $\alpha, \beta \in \mathbb{N}_0^n$. Suppose $\phi : (0,1) \to [0,\infty)$ is measurable and $v \in \mathscr{C}^1(\mathbb{B}_n)$. Then the following hold.*

1. *If $|\alpha| \geq |\beta|$ and all integrals converge absolutely, then*

$$\int_{\mathbb{B}_n} \phi(|\varphi_z(w)|^2) I^{\alpha,\beta}(z-w)v(w)K_w^{(t)}(z)\mathrm{d}\lambda_t(w) \tag{4.2}$$

$$= \begin{cases} \frac{d_{\alpha,\beta}(z)}{B(n,t+1)} \cdot \mathcal{F}^{(t)}_{n+|\beta|}\phi(0)v(z) - \sum_{j=1}^n \int_{\mathbb{B}_n} \mathcal{G}^{(t)}_{|\beta|+n}\phi(|\varphi_z(w)|^2) I^{\alpha,\beta+e_j}(z-w) S_j(w) K^{(t)}_w(z) \mathrm{d}\lambda_t(w), \\ \hfill v(z) \neq 0, \mathcal{F}^{(t)}_{n+|\beta|}\phi(0) < \infty, \\ - \sum_{j=1}^n \int_{\mathbb{B}_n} \mathcal{G}^{(t)}_{|\beta|+n}\phi(|\varphi_z(w)|^2) I^{\alpha,\beta+e_j}(z-w) S_j(w) K^{(t)}_w(z) \mathrm{d}\lambda_t(w), \\ \hfill v(z) = 0, \mathcal{F}^{(t)}_{n+|\beta|}\phi(0) \leq \infty, \end{cases}$$

where

$$S_j(w,z) = \frac{(1-|w|^2)\bar{\partial}_{w_j}\left[(1-\langle z,w\rangle)^{|\beta|} v(w)\right]}{(1-\langle w,z\rangle)(1-\langle z,w\rangle)^{|\beta|}}.$$

2. *If* $|\alpha| \leq |\beta|$ *and all integrals converge absolutely, then*

$$\int_{\mathbb{B}_n} \phi(|\varphi_z(w)|^2) I^{\alpha,\beta}(z-w) v(z) K^{(t)}_w(z) \mathrm{d}\lambda_t(z) \tag{4.3}$$

$$= \begin{cases} \frac{d_{\alpha,\beta}(w)}{B(n,t+1)} \cdot \mathcal{F}^{(t)}_{n+|\alpha|}\phi(0)v(w) + \sum_{i=1}^n \int_{\mathbb{B}_n} \mathcal{G}^{(t)}_{|\alpha|+n}\phi(|\varphi_z(w)|^2) I^{\alpha+e_i,\beta}(z-w) \widetilde{S}_i(z) K^{(t)}_w(z) \mathrm{d}\lambda_t(z), \\ \hfill v(w) \neq 0, \mathcal{F}^{(t)}_{n+|\alpha|}\phi(0) < \infty, \\ \sum_{i=1}^n \int_{\mathbb{B}_n} \mathcal{G}^{(t)}_{|\alpha|+n}\phi(|\varphi_z(w)|^2) I^{\alpha+e_i,\beta}(z-w) \widetilde{S}_i(z) K^{(t)}_w(z) \mathrm{d}\lambda_t(z), \\ \hfill v(w) = 0, \mathcal{F}^{(t)}_{n+|\alpha|}\phi(0) \leq \infty, \end{cases}$$

where

$$\widetilde{S}_i(z,w) = \frac{(1-|z|^2)\partial_{z_i}\left[(1-\langle z,w\rangle)^{|\alpha|} v(z)\right]}{(1-\langle w,z\rangle)(1-\langle z,w\rangle)^{|\alpha|}}.$$

With Lemma 4.2, we show the following.

Lemma 4.3. *Suppose* k *is a non-negative integer and* $\Gamma \subset \mathbb{N}_0^n \times \mathbb{N}_0^n$ *is a finite set of multi-indices with* $|\alpha| = |\beta| = k$ *for every* $(\alpha,\beta) \in \Gamma$. *Suppose for some* $\epsilon > -1-t$, $\{F_{\alpha,\beta}\}_{(\alpha,\beta)\in\Gamma} \subset \mathscr{C}^2(\mathbb{B}_n \times \mathbb{B}_n)$ *and*

$$\left| \sum_{(\alpha,\beta)\in\Gamma} I^{\alpha,\beta}(z-w) F_{\alpha,\beta}(z,w) \right| \lesssim |\varphi_z(w)|^{2k} |1-\langle z,w\rangle|^{2k+\epsilon}, \tag{4.4}$$

$$\left| \sum_{j=1}^n \sum_{(\alpha,\beta)\in\Gamma} I^{\alpha,\beta+e_j}(z-w) \bar{\partial}_{w_j} F_{\alpha,\beta}(z,w) \right| \lesssim |\varphi_z(w)|^{2k+1} |1-\langle z,w\rangle|^{2k+\epsilon}. \tag{4.5}$$

Then

$$\int_{\mathbb{B}_n^2} \Phi^{(t)}_{n,k}(|\varphi_z(w)|^2) \frac{\sum_{(\alpha,\beta)\in\Gamma} I^{\alpha,\beta}(z-w) F_{\alpha,\beta}(z,w)}{|1-\langle z,w\rangle|^{2k}} K^{(t)}_w(z) \mathrm{d}\lambda_t(w) \mathrm{d}\lambda_t(z)$$

$$
\begin{aligned}
&= \frac{\mathcal{F}_{n+k}^{(t)}\Phi_{n,k}^{(t)}(0)}{B(n,t+1)} \int_{\mathbb{B}_n} (1-|z|^2)^{-2k} \sum_{(\alpha,\beta)\in\Gamma} d_{\alpha,\beta}(z) F_{\alpha,\beta}(z,z) \mathrm{d}\lambda_t(z) \\
&\quad - \int_{\mathbb{B}_n^2} \Phi_{n,k+1}^{(t)}(|\varphi_z(w)|^2) \frac{\sum_{i,j=1}^n \sum_{(\alpha,\beta)\in\Gamma} I^{\alpha+e_i,\beta+e_j}(z-w) D_{i,j} F_{\alpha,\beta}(z,w)}{|1-\langle z,w\rangle|^{2(k+1)}} K_w^{(t)}(z) \mathrm{d}\lambda_t(z) \mathrm{d}\lambda_t(w).
\end{aligned} \tag{4.6}
$$

Here $D_{i,j}$ *denotes the operation*

$$
D_{i,j} = (1-\langle z,w\rangle)^2 \partial_{z_i} \bar{\partial}_{w_j}.
$$

Lemma 4.3 will be a key ingredient in the proof of Theorem 1.2. For the rest of this section, we prove the two lemmas. As in Section 3, we start by proving a version of Lemma 4.2 at the point 0.

Lemma 4.4. *Suppose* $t > -1$, k, l *are non-negative integers with* $k \geq l$, *and* $\Gamma \subset \mathbb{N}_0^n \times \mathbb{N}_0^n$ *is a finite set of multi-indices with* $|\kappa| = k, |\gamma| = l$ *for every* $(\kappa, \gamma) \in \Gamma$. *Suppose* $\{c_{\kappa,\gamma}\}_{(\kappa,\gamma)\in\Gamma} \subset \mathbb{C}$, $\phi : (0,1) \to [0,\infty)$ *is measurable and* $v \in \mathscr{C}^1(\mathbb{B}_n)$ *satisfies that both*

$$
\int_{\mathbb{B}_n} \phi(|\zeta|^2) \left[\sum_{(\kappa,\gamma)\in\Gamma} c_{\kappa,\gamma} \zeta^\kappa \bar{\zeta}^\gamma \right] v(\zeta) \mathrm{d}\lambda_t(\zeta)
$$

and

$$
\int_{\mathbb{B}_n} \mathcal{G}_{l+n}^{(t)} \phi(|\zeta|^2)(1-|\zeta|^2) \left[\sum_{(\kappa,\gamma)\in\Gamma} c_{\kappa,\gamma} \zeta^\kappa \bar{\zeta}^\gamma \right] \bar{R} v(\zeta) \mathrm{d}\lambda_t(\zeta)
$$

converge absolutely. Then

$$
\begin{aligned}
&\int_{\mathbb{B}_n} \phi(|\zeta|^2) \left[\sum_{(\kappa,\gamma)\in\Gamma} c_{\kappa,\gamma} \zeta^\kappa \bar{\zeta}^\gamma \right] v(\zeta) \mathrm{d}\lambda_t(\zeta) \\
&= \begin{cases} cv(0) + \int_{\mathbb{B}_n} \mathcal{G}_{l+n}^{(t)} \phi(|\zeta|^2)(1-|\zeta|^2) \left[\sum_{(\kappa,\gamma)\in\Gamma} c_{\kappa,\gamma} \zeta^\kappa \bar{\zeta}^\gamma \right] \bar{R} v(\zeta) \mathrm{d}\lambda_t(\zeta), & v(0) \neq 0, \mathcal{F}_{n+l}^{(t)} \phi(0) < \infty. \\ \int_{\mathbb{B}_n} \mathcal{G}_{l+n}^{(t)} \phi(|\zeta|^2)(1-|\zeta|^2) \left[\sum_{(\kappa,\gamma)\in\Gamma} c_{\kappa,\gamma} \zeta^\kappa \bar{\zeta}^\gamma \right] \bar{R} v(\zeta) \mathrm{d}\lambda_t(\zeta), & v(0) = 0, \mathcal{F}_{n+l}^{(t)} \phi(0) \leq \infty. \end{cases}
\end{aligned} \tag{4.7}
$$

Here

$$
c = \int_{\mathbb{B}_n} \phi(|\zeta|^2) \left[\sum_{(\kappa,\gamma)\in\Gamma} c_{\kappa,\gamma} \zeta^\kappa \bar{\zeta}^\gamma \right] \mathrm{d}\lambda_t(\zeta).
$$

Proof. Assume first that $v(0) = 0$. As in Appendix II we use ϕ_t to stand for the function $(1-s)^t$. By assumption on v, the first line of (4.7) converges absolutely. Therefore

$$\begin{aligned}
&\int_{\mathbb{B}_n} \phi(|\zeta|^2)\Big[\sum_{(\kappa,\gamma)\in\Gamma} c_{\kappa,\gamma}\zeta^\kappa\bar{\zeta}^\gamma\Big]v(\zeta)\mathrm{d}\lambda_t(\zeta)\\
=&\frac{(n-1)!}{\pi^n B(n,t+1)}\int_{\mathbb{B}_n}\phi\phi_t(|\zeta|^2)\Big[\sum_{(\kappa,\gamma)\in\Gamma} c_{\kappa,\gamma}\zeta^\kappa\bar{\zeta}^\gamma\Big]v(\zeta)\mathrm{d}m(\zeta)\\
=&\frac{(n-1)!}{\pi^n B(n,t+1)}\int_0^1\phi\phi_t(r^2)\Big[\sum_{(\kappa,\gamma)\in\Gamma} c_{\kappa,\gamma}\int_{r\mathbb{S}_n}\zeta^\kappa\bar{\zeta}^\gamma v(\zeta)\mathrm{d}\sigma_r(\zeta)\Big]\mathrm{d}r.
\end{aligned}$$

Define $R=\sum_{i=1}^n z_i\partial_{z_i}$ be the radial derivative operator, and $\bar{R}=\sum_{i=1}^n \bar{z}_i\bar{\partial}_{z_i}$. In Appendix I, Lemma 7.1, we show that

$$\int_{r\mathbb{S}_n}\zeta^\kappa\bar{\zeta}^\gamma v(\zeta)\mathrm{d}\sigma_r(\zeta)=2r^{2l+2n-1}\int_{r\mathbb{B}_n}\frac{\zeta^\kappa\bar{\zeta}^\gamma}{|\zeta|^{2l+2n}}\bar{R}v(\zeta)\mathrm{d}m(\zeta).$$

Plugging it back gives

$$\begin{aligned}
&\int_{\mathbb{B}_n} \phi(|\zeta|^2)\Big[\sum_{(\kappa,\gamma)\in\Gamma} c_{\kappa,\gamma}\zeta^\kappa\bar{\zeta}^\gamma\Big]v(\zeta)\mathrm{d}\lambda_t(\zeta)\\
=&\frac{(n-1)!}{\pi^n B(n,t+1)}\int_0^1\phi\phi_t(r^2)\Big[\sum_{(\kappa,\gamma)\in\Gamma} c_{\kappa,\gamma}\cdot 2r^{2l+2n-1}\int_{r\mathbb{B}_n}\frac{\zeta^\kappa\bar{\zeta}^\gamma}{|\zeta|^{2|\gamma|+2n}}\bar{R}v(\zeta)\mathrm{d}m(\zeta)\Big]\mathrm{d}r\\
=&\frac{(n-1)!}{\pi^n B(n,t+1)}\int_0^1\int_{r\mathbb{B}_n}\phi\phi_t(r^2)2r^{2l+2n-1}\Big[\sum_{(\kappa,\gamma)\in\Gamma} c_{\kappa,\gamma}\cdot\zeta^\kappa\bar{\zeta}^\gamma\Big]|\zeta|^{-2l-2n}\bar{R}v(\zeta)\mathrm{d}m(\zeta)\mathrm{d}r\\
=&\frac{(n-1)!}{\pi^n B(n,t+1)}\int_{\mathbb{B}_n}\Big[|\zeta|^{-2l-2n}\int_{|\zeta|}^1\phi\phi_t(r^2)2r^{2l+2n-1}\mathrm{d}r\Big]\Big[\sum_{(\kappa,\gamma)\in\Gamma} c_{\kappa,\gamma}\zeta^\kappa\bar{\zeta}^\gamma\Big]\bar{R}v(\zeta)\mathrm{d}m(\zeta)\\
=&\int_{\mathbb{B}_n}\mathcal{G}_{l+n}^{(t)}\phi(|\zeta|^2)(1-|\zeta|^2)\Big[\sum_{(\kappa,\gamma)\in\Gamma} c_{\kappa,\gamma}\zeta^\kappa\bar{\zeta}^\gamma\Big]\bar{R}v(\zeta)\mathrm{d}\lambda_t(\zeta).
\end{aligned}$$

Here in the last equality we used that by Definition 8.1,

$$\begin{aligned}
|\zeta|^{-2l-2n}\int_{|\zeta|}^1\phi\phi_t(r^2)2r^{2l+2n-1}\mathrm{d}r &\overset{s=r^2}{=\!=\!=} |\zeta|^{-2l-2n}\int_{|\zeta|^2}^1\phi\phi_t(s)s^{l+n-1}\mathrm{d}s\\
&=(1-|\zeta|^2)^{t+1}\mathcal{G}_{l+n}^{(t)}\phi(|\zeta|^2).
\end{aligned}$$

In the second to last equality, Fubini's theorem is applied: since by assumption the last integral converges absolutely, the condition for Fubini's theorem is satisfied. This proves the second case.

Assuming that $v(0) \neq 0$ and $\mathcal{F}_{l+n}^{(t)}\phi(0) < \infty$, then we have

$$
\begin{aligned}
&\int_{\mathbb{B}_n} \left| \phi(|\zeta|^2) \Big[\sum_{(\kappa,\gamma)\in\Gamma} c_{\kappa,\gamma} \zeta^\kappa \bar{\zeta}^\gamma \Big] \big(v(\zeta) - v(0) \big) \right| \mathrm{d}\lambda_t(\zeta) \\
&\leq \int_{\mathbb{B}_n} \left| \phi(|\zeta|^2) \Big[\sum_{(\kappa,\gamma)\in\Gamma} c_{\kappa,\gamma} \zeta^\kappa \bar{\zeta}^\gamma \Big] v(\zeta) \right| \mathrm{d}\lambda_t(\zeta) + |v(0)| \int_{\mathbb{B}_n} \left| \phi(|\zeta|^2) \Big[\sum_{(\kappa,\gamma)\in\Gamma} c_{\kappa,\gamma} \zeta^\kappa \bar{\zeta}^\gamma \Big] \right| \mathrm{d}\lambda_t(\zeta).
\end{aligned}
$$

By assumption, the first integral converges. Also since $|\kappa| = k \geq l = |\gamma|$,

$$
\begin{aligned}
&\int_{\mathbb{B}_n} \left| \phi(|\zeta|^2) \Big[\sum_{(\kappa,\gamma)\in\Gamma} c_{\kappa,\gamma} \zeta^\kappa \bar{\zeta}^\gamma \Big] \right| \mathrm{d}\lambda_t(\zeta) \\
&\lesssim \int_{\mathbb{B}_n} \phi(|\zeta|^2) |\zeta|^{2l} \mathrm{d}\lambda_t(\zeta) \\
&= \frac{(n-1)!\sigma_{2n-1}}{\pi^n B(n,t+1)} \int_0^1 \phi(r^2) r^{2n+2l-1} (1-r^2)^t \mathrm{d}r \\
&\overset{s=r^2}{=\!=\!=} \frac{(n-1)!\sigma_{2n-1}}{2\pi^n B(n,t+1)} \int_0^1 \phi(s) s^{n+l-1} (1-s)^t \mathrm{d}s \\
&= \frac{(n-1)!\sigma_{2n-1}}{2\pi^n B(n,t+1)} \mathcal{F}_{n+l}^{(t)}\phi(0) \\
&< \infty.
\end{aligned}
$$

Therefore we may apply the formula with $v(\zeta) - v(0)$ replacing $v(\zeta)$. This gives the first case. $\square$

Suppose $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_n\}$ are two orthonormal basis of $\mathbb{C}^n$. Suppose

$$
\sum_{i=1}^n \zeta_i e_i = \sum_{i=1}^n \xi_i f_i; \quad \sum_{i=1}^n \lambda_i e_i = \sum_{i=1}^n \omega_i f_i.
$$

Then there is a unitary matrix $U = [u_{ij}]$ such that

$$
\xi = U\zeta, \quad \omega = U\lambda.
$$

Denote $U^* = [u^*_{ij}]$. Therefore

$$\sum_{i=1}^{n}\frac{\partial v}{\partial\bar{\zeta}_i}\bar{\lambda}_i=\sum_{i,j,k=1}^{n}\left(\frac{\partial v}{\partial\bar{\xi}_j}\bar{u}_{ji}\bar{\zeta}_i\right)\overline{(u^*_{ik}\omega_k)}=\sum_{j,k=1}^{n}\frac{\partial v}{\partial\bar{\xi}_j}\delta_{jk}\bar{w}_k=\sum_{j=1}^{n}\frac{\partial v}{\partial\bar{\xi}_j}\bar{w}_j. \tag{4.8}$$

In other words, the function

$$\sum_{i=1}^{n}\bar{\partial}_i v\bar{\lambda}_i=\langle\bar{\partial}v,\lambda\rangle$$

does not depend on the choice of a basis.

Lemma 4.5. *Suppose $v\in\mathscr{C}^1(\mathbb{B}_n)$ and $z\in\mathbb{B}_n$. Then*

$$\langle\bar{\partial}_\zeta v(\zeta),\zeta\rangle=-\frac{\langle\bar{\partial}\big(v\circ\varphi_z\big)(\varphi_z(\zeta)),z-\varphi_z(\zeta)\rangle}{1-\langle z,\zeta\rangle}. \tag{4.9}$$

Proof. By (4.8), both sides of (4.9) do not depend on the choice of basis. Thus we may assume $z=(r,0,\dots,0)$. In this case, we have the following expression

$$\varphi_z(w)=\frac{1}{1-w_1r}\big(r-w_1,\ \ -(1-r^2)^{1/2}w_2,\ \ \dots,\ \ -(1-r^2)^{1/2}w_n\big).$$

Consequently, we compute the Jacobian

$$\left[\frac{\partial(\varphi_z)_i}{\partial w_j}\right]=\begin{bmatrix}-\frac{1-r^2}{(1-w_1r)^2} & 0 & 0 & 0 & \dots & 0\\ -\frac{(1-r^2)^{1/2}w_2r}{(1-w_1r)^2} & -\frac{(1-r^2)^{1/2}}{1-w_1r} & 0 & 0 & \dots & 0\\ -\frac{(1-r^2)^{1/2}w_3r}{(1-w_1r)^2} & 0 & -\frac{(1-r^2)^{1/2}}{1-w_1r} & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ -\frac{(1-r^2)^{1/2}w_nr}{(1-w_1r)^2} & 0 & 0 & 0 & \dots & -\frac{(1-r^2)^{1/2}}{1-w_1r}\end{bmatrix}.$$

Therefore we have the following expression,

$$\langle\bar{\partial}\big(v\circ\varphi_z\big)(w),z-w\rangle=\sum_{i,j=1}^{n}\bar{\partial}_i v\big(\varphi_z(w)\big)\overline{\left(\frac{\partial\big(\varphi_z\big)_i}{\partial w_j}\right)}\overline{(z_j-w_j)}=\langle\bar{\partial}v\big(\varphi_z(w)\big),\xi\rangle, \tag{4.10}$$

where

$$\xi_i=\sum_{j=1}^{n}\frac{\partial\big(\varphi_z\big)_i}{\partial w_j}(z_j-w_j)=\frac{\partial\big(\varphi_z\big)_i}{\partial w_1}(r-w_1)-\sum_{j=2}^{n}\frac{\partial\big(\varphi_z\big)_i}{\partial w_j}w_j.$$

By the above, set

$$\xi_1=-\frac{(1-r^2)(r-w_1)}{(1-w_1r)^2},$$

and for $i = 2, \dots, n$,

$$\xi_i = -\frac{(1-r^2)^{1/2} w_i r(r-w_1)}{(1-w_1 r)^2} + \frac{(1-r^2)^{1/2} w_i}{1-w_1 r} = \frac{(1-r^2)^{1/2}}{(1-w_1 r)^2}\big(w_i - w_i r^2\big) = \frac{(1-r^2)^{3/2}}{(1-w_1 r)^2} w_i.$$

Thus we have

$$\xi = -\frac{1-r^2}{1-w_1 r}\varphi_z(w). \tag{4.11}$$

If we plug in $w = \varphi_z(\zeta)$ then by (4.10), (4.11) and Lemma 2.2, we obtain the equalities

$$\langle \bar\partial\big(v\circ\varphi_z\big)(\varphi_z(\zeta)), z-\varphi_z(\zeta)\rangle = \langle \bar\partial v(\zeta), -\frac{1-|z|^2}{1-\langle\varphi_z(\zeta), z\rangle}\zeta\rangle = -(1-\langle z,\zeta\rangle)\langle\bar\partial v(\zeta),\zeta\rangle.$$

Equivalently, (4.9) holds. This completes the proof of Lemma 4.5. $\square$

Proof of Lemma 4.2. By Lemma 2.2 (4) (5),

$$K_w^{(t)}(z)\mathrm{d}\lambda_t(w) \xlongequal[\zeta=\varphi_z(w)]{w=\varphi_z(\zeta)} K_z^{(t)}(\zeta)\mathrm{d}\lambda_t(\zeta),$$

and

$$z - \varphi_z(\zeta) = \frac{A_z\zeta}{1-\langle\zeta, z\rangle}, \tag{4.12}$$

where $A_z w$ is the linear transformation

$$A_z\zeta = (1-|z|^2)P_z(\zeta) + (1-|z|^2)^{1/2}Q_z(\zeta).$$

Write $|\alpha| = k, |\beta| = l$. Assume that $v(z) \neq 0$ and $\mathcal{F}_{n+l}^{(t)}\phi(0) < \infty$.

$$\begin{aligned}
&\int\limits_{\mathbb{B}_n} \phi(|\varphi_z(w)|^2) I^{\alpha,\beta}(z-w)v(w)K_w^{(t)}(z)\mathrm{d}\lambda_t(w)\\
\xlongequal{w=\varphi_z(\zeta)}&\int\limits_{\mathbb{B}_n} \phi(|\zeta|^2) I^{\alpha,\beta}(z-\varphi_z(\zeta))v\circ\varphi_z(\zeta)K_z^{(t)}(\zeta)\mathrm{d}\lambda_t(\zeta).
\end{aligned}$$

By (4.12),

$$I^{\alpha,\beta}(z-\varphi_z(\zeta)) = \frac{\sum_{|\kappa|=k,|\gamma|=l} c_{\alpha,\beta,\kappa,\gamma,z}\zeta^\kappa\bar\zeta^\gamma}{(1-\langle\zeta,z\rangle)^k(1-\langle z,\zeta\rangle)^l}. \tag{4.13}$$

By the above and Lemma 4.4,

$$
\begin{aligned}
&\int\limits_{\mathbb{B}_n} \phi(|\varphi_z(w)|^2) I^{\alpha,\beta}(z-w)v(w)K_w^{(t)}(z)\mathrm{d}\lambda_t(w)\\
=&\int\limits_{\mathbb{B}_n} \phi(|\zeta|^2)\Big[\sum_{|\kappa|=k,|\gamma|=l} c_{\alpha,\beta,\kappa,\gamma,z}\zeta^\kappa\bar{\zeta}^\gamma\Big]\frac{v\circ\varphi_z(\zeta)}{(1-\langle\zeta,z\rangle)^k(1-\langle z,\zeta\rangle)^l}K_z^{(t)}(\zeta)\mathrm{d}\lambda_t(\zeta)\\
=&cv(z)\\
&\quad+\int\limits_{\mathbb{B}_n}\mathcal{G}_{l+n}^{(t)}\phi(|\zeta|^2)(1-|\zeta|^2)\Big[\sum_{|\kappa|=k,|\gamma|=l} c_{\alpha,\beta,\kappa,\gamma,z}\zeta^\kappa\bar{\zeta}^\gamma\Big]\bar{R}\Big[\frac{v\circ\varphi_z(\zeta)}{(1-\langle\zeta,z\rangle)^k(1-\langle z,\zeta\rangle)^l}K_z^{(t)}(\zeta)\Big]\mathrm{d}\lambda_t(\zeta)\\
=&cv(z)\\
&\quad+\int\limits_{\mathbb{B}_n}\mathcal{G}_{l+n}^{(t)}\phi(|\zeta|^2)(1-|\zeta|^2)\Big[\sum_{|\kappa|=k,|\gamma|=l} c_{\alpha,\beta,\kappa,\gamma,z}\zeta^\kappa\bar{\zeta}^\gamma\Big]\bar{R}\Big[\frac{v\circ\varphi_z(\zeta)}{(1-\langle z,\zeta\rangle)^l}\Big](1-\langle\zeta,z\rangle)^{-k}K_z^{(t)}(\zeta)\mathrm{d}\lambda_t(\zeta)\\
=&cv(z)\\
&\quad+\int\limits_{\mathbb{B}_n}\mathcal{G}_{l+n}^{(t)}\phi(|\zeta|^2)(1-|\zeta|^2)I^{\alpha,\beta}(z-\varphi_z(\zeta))(1-\langle z,\zeta\rangle)^l\bar{R}\Big[\frac{v\circ\varphi_z(\zeta)}{(1-\langle z,\zeta\rangle)^l}\Big]K_z^{(t)}(\zeta)\mathrm{d}\lambda_t(\zeta).
\end{aligned}
$$

Write $h(\zeta)=\frac{v\circ\varphi_z(\zeta)}{(1-\langle z,\zeta\rangle)^l}$. Then

$$
h\circ\varphi_z(w)=\frac{(1-\langle z,w\rangle)^l v(w)}{(1-|z|^2)^l}.
$$

By Lemma 4.5,

$$
\bar{R}\Big[\frac{v\circ\varphi_z(\zeta)}{(1-\langle z,\zeta\rangle)^l}\Big]=\bar{R}h(\zeta)=\langle\bar{\partial}_\zeta h(\zeta),\zeta\rangle=-\frac{\langle\bar{\partial}\big(h\circ\varphi_z\big)(\varphi_z(\zeta)),z-\varphi_z(\zeta)\rangle}{1-\langle z,\zeta\rangle}.
$$

Taking the change of variable $\zeta=\varphi_z(w)$ we get

$$
\begin{aligned}
&\int\limits_{\mathbb{B}_n}\mathcal{G}_{l+n}^{(t)}\phi(|\zeta|^2)(1-|\zeta|^2)I^{\alpha,\beta}(z-\varphi_z(\zeta))(1-\langle z,\zeta\rangle)^l\bar{R}\Big[\frac{v\circ\varphi_z(\zeta)}{(1-\langle z,\zeta\rangle)^l}\Big]K_z^{(t)}(\zeta)\mathrm{d}\lambda_t(\zeta)\\
=&-\int\limits_{\mathbb{B}_n}\mathcal{G}_{l+n}^{(t)}\phi(|\zeta|^2)(1-|\zeta|^2)\\
&\qquad I^{\alpha,\beta}(z-\varphi_z(\zeta))(1-\langle z,\zeta\rangle)^{l-1}\langle\bar{\partial}\big(h\circ\varphi_z\big)(\varphi_z(\zeta)),z-\varphi_z(\zeta)\rangle K_z^{(t)}(\zeta)\mathrm{d}\lambda_t(\zeta)\\
=&-\int\limits_{\mathbb{B}_n}\mathcal{G}_{l+n}^{(t)}\phi(|\varphi_z(w)|^2)(1-|\varphi_z(w)|^2)\\
&\qquad I^{\alpha,\beta}(z-w)\Big[\frac{1-|z|^2}{1-\langle z,w\rangle}\Big]^{l-1}\langle\bar{\partial}\Big[\frac{(1-\langle z,w\rangle)^l v(w)}{(1-|z|^2)^l}\Big](w),z-w\rangle K_w^{(t)}(z)\mathrm{d}\lambda_t(w)
\end{aligned}
$$

$$= -\sum_{j=1}^{n}\int_{\mathbb{B}_n} \mathcal{G}_{l+n}^{(t)}\phi(|\varphi_z(w)|^2)I^{\alpha,\beta+e_j}(z-w)\frac{1-|w|^2}{1-\langle w,z\rangle}\frac{\bar{\partial}_j\left[(1-\langle z,w\rangle)^l v(w)\right]}{(1-\langle z,w\rangle)^l}K_w^{(t)}(z)\mathrm{d}\lambda_t(w)$$

$$= -\sum_{j=1}^{n}\int_{\mathbb{B}_n} \mathcal{G}_{l+n}^{(t)}\phi(|\varphi_z(w)|^2)I^{\alpha,\beta+e_j}(z-w)S_j(w)K_w^{(t)}(z)\mathrm{d}\lambda_t(w).$$

To find the constant c, recall that by Lemma 4.4,

$$c = \int_{\mathbb{B}_n} \phi(|\zeta|^2)\left[\sum_{|\kappa|=l,|\gamma|=l} c_{\alpha,\beta,\kappa,\gamma,z}\zeta^\kappa\bar{\zeta}^\gamma\right]\mathrm{d}\lambda_t(\zeta).$$

Clearly if $k \neq l$ then $c = 0$. Assuming $k = l$, then by (4.13), we compute c as follows,

$$\begin{aligned}
c &= \int_{\mathbb{B}_n} \phi(|\zeta|^2)|1-\langle z,\zeta\rangle|^{2l}I^{\alpha,\beta}(z-\varphi_z(\zeta))\mathrm{d}\lambda_t(\zeta)\\
&= \int_{\mathbb{B}_n} \phi(|\zeta|^2)I^{\alpha,\beta}(A_z\zeta)\mathrm{d}\lambda_t(\zeta)\\
&= \frac{(n-1)!}{\pi^n B(n,t+1)}\int_0^1 \phi(r^2)r^{2n++2l-1}(1-r^2)^t\left[\int_{\mathbb{S}_n} I^{\alpha,\beta}(A_z\zeta)\mathrm{d}\sigma(\zeta)\right]\mathrm{d}r\\
&= \frac{(n-1)!}{\pi^n B(n,t+1)}\cdot\frac{1}{2}\mathcal{F}_{n+l}^{(t)}\phi(0)\cdot\sigma_{2n-1}d_{\alpha,\beta}(z)\\
&= \frac{\mathcal{F}_{n+l}^{(t)}\phi(0)}{B(n,t+1)}d_{\alpha,\beta}(z).
\end{aligned}$$

This proves the first case of (4.2). The second case is proved in the same way.

In (4.2), reverse (z,w), (α,β), and replace v with $\bar{v}$. Then

$$\int_{\mathbb{B}_n} \phi(|\varphi_w(z)|^2)I^{\beta,\alpha}(w-z)\bar{v}(z)K_z^{(t)}(w)\mathrm{d}\lambda_t(z)$$

$$= \begin{cases} \frac{d_{\beta,\alpha}(w)}{B(n,t+1)}\cdot\mathcal{F}_{n+|\alpha|}^{(t)}\phi(0)\bar{v}(w) - \sum_{j=1}^n\int_{\mathbb{B}_n}\mathcal{G}_{|\alpha|+n}^{(t)}\phi(|\varphi_w(z)|^2)I^{\beta,\alpha+e_j}(w-z)S_j(z)K_z^{(t)}(w)\mathrm{d}\lambda_t(z), \\ \qquad\qquad v(w)\neq 0, \mathcal{F}_{n+|\alpha|}^{(t)}\phi(0)<\infty, \\ -\sum_{j=1}^n\int_{\mathbb{B}_n}\mathcal{G}_{|\alpha|+n}^{(t)}\phi(|\varphi_w(z)|^2)I^{\beta,\alpha+e_j}(w-z)S_j(z)K_z^{(t)}(w)\mathrm{d}\lambda_t(z), \\ \qquad\qquad v(w)=0, \mathcal{F}_{n+|\alpha|}^{(t)}\phi(0)\leq\infty, \end{cases}$$

where

$$S_j(z) = \frac{(1-|z|^2)\bar{\partial}_{z_j}\left[(1-\langle w,z\rangle)^{|\alpha|}\bar{v}(z)\right]}{(1-\langle z,w\rangle)(1-\langle w,z\rangle)^{|\alpha|}}.$$

Taking conjugate on both sides, we get Equation ((4.3)) from

$$\overline{d_{\alpha,\beta}(z)} = d_{\beta,\alpha}(z), \quad |\varphi_z(w)| = |\varphi_w(z)|.$$

This completes the proof. $\square$

Proof of Lemma 4.3. By Estimates (8.4), (2.7), and assumption (4.4), we conclude that the left hand side of Equation (4.6) is absolutely integrable. For each $z \in \mathbb{B}_n, (\alpha,\beta) \in \Gamma$, we compute the following integral,

$$\begin{aligned}
&\int_{\mathbb{B}_n} \Phi^{(t)}_{n,k}(|\varphi_z(w)|^2) I^{\alpha,\beta}(z-w) \frac{F_{\alpha,\beta}(z,w)}{|1-\langle z,w\rangle|^{2k}} K^{(t)}_w(z) \mathrm{d}\lambda_t(w) \\
\overset{(4.2)}{=\!=\!=}\, & \frac{d_{\alpha,\beta}(z)}{B(n,t+1)} \mathcal{F}^{(t)}_{n+k} \Phi^{(t)}_{n,k}(0) \frac{F_{\alpha,\beta}(z,z)}{(1-|z|^2)^{2k}} \\
& - \sum_{j=1}^n \int_{\mathbb{B}_n} \mathcal{G}^{(t)}_{n+k} \Phi^{(t)}_{n,k}(|\varphi_z(w)|^2) I^{\alpha,\beta+e_j}(z-w) \frac{(1-|w|^2)\bar{\partial}_{w_j} F_{\alpha,\beta}(z,w)}{(1-\langle w,z\rangle)^{k+1}(1-\langle z,w\rangle)^k} K^{(t)}_w(z) \mathrm{d}\lambda_t(w).
\end{aligned}$$

By (8.5), (2.7) and assumption (4.5), the integral

$$\int_{\mathbb{B}_n^2} \mathcal{G}^{(t)}_{n+k} \Phi^{(t)}_{n,k}(|\varphi_z(w)|^2) \frac{\sum_{j=1}^n \sum_{(\alpha,\beta)\in\Gamma} I^{\alpha,\beta+e_j}(z-w)(1-|w|^2)\bar{\partial}_{w_j} F(z,w)}{(1-\langle w,z\rangle)^{k+1}(1-\langle z,w\rangle)^k} K^{(t)}_w(z) \mathrm{d}\lambda_t(w) \mathrm{d}\lambda_t(z)$$

converges absolutely. Therefore the first line of (4.6) equals

$$\begin{aligned}
&\int_{\mathbb{B}_n} \sum_{(\alpha,\beta)\in\Gamma} \frac{d_{\alpha,\beta}(z)}{B(n,t+1)} \mathcal{F}^{(t)}_{n+k} \Phi^{(t)}_{n,k}(0) \frac{F_{\alpha,\beta}(z,z)}{(1-|z|^2)^{2k}} \mathrm{d}\lambda_t(z) \\
&\quad - \int_{\mathbb{B}_n^2} \mathcal{G}^{(t)}_{n+k} \Phi^{(t)}_{n,k}(|\varphi_z(w)|^2) \\
&\quad\quad \frac{\sum_{j=1}^n \sum_{(\alpha,\beta)\in\Gamma} I^{\alpha,\beta+e_j}(z-w)(1-|w|^2)\bar{\partial}_{w_j} F(z,w)}{(1-\langle w,z\rangle)^{k+1}(1-\langle z,w\rangle)^k} K^{(t)}_w(z) \mathrm{d}\lambda_t(z) \mathrm{d}\lambda_t(w).
\end{aligned}$$

Again, for any $w \in \mathbb{B}_n$, $j = 1, \ldots, n$ and $(\alpha,\beta) \in \Gamma$, we compute the following integral,

$$\int_{\mathbb{B}_n} \mathcal{G}^{(t)}_{n+k} \Phi^{(t)}_{n,k}(|\varphi_z(w)|^2) I^{\alpha,\beta+e_j}(z-w) \frac{(1-|w|^2)\bar{\partial}_{w_j} F(z,w)}{(1-\langle w,z\rangle)^{k+1}(1-\langle z,w\rangle)^k} K^{(t)}_w(z) \mathrm{d}\lambda_t(z)$$

$$\overset{(4.3)}{=\!=}\sum_{i=1}^{n}\int\limits_{\mathbb{B}_n}\left(\mathcal{G}_{n+k}^{(t)}\right)^2\Phi_{n,k}^{(t)}(|\varphi_z(w)|^2)$$

$$I^{\alpha+e_i,\beta+e_j}(z-w)\frac{(1-|z|^2)(1-|w|^2)\partial_{z_i}\bar{\partial}_{w_j}F(z,w)}{(1-\langle w,z\rangle)^{k+2}(1-\langle z,w\rangle)^{k}}K_w^{(t)}(z)\mathrm{d}\lambda_t(z)$$

$$=\sum_{i=1}^{n}\int\limits_{\mathbb{B}_n}M_{\phi_1}\left(\mathcal{G}_{n+k}^{(t)}\right)^2\Phi_{n,k}^{(t)}(|\varphi_z(w)|^2)$$

$$I^{\alpha+e_i,\beta+e_j}(z-w)\frac{\partial_{z_i}\bar{\partial}_{w_j}F(z,w)}{(1-\langle w,z\rangle)^{k+1}(1-\langle z,w\rangle)^{k-1}}K_w^{(t)}(z)\mathrm{d}\lambda_t(z)$$

$$=\sum_{i=1}^{n}\int\limits_{\mathbb{B}_n}\Phi_{n,k+1}^{(t)}(|\varphi_z(w)|^2)I^{\alpha+e_i,\beta+e_j}(z-w)\frac{D_{i,j}F(z,w)}{|1-\langle w,z\rangle|^{2(k+1)}}K_w^{(t)}(z)\mathrm{d}\lambda_t(z).$$

Altogether, the first line of (4.6) equals

$$\int\limits_{\mathbb{B}_n}\sum_{(\alpha,\beta)\in\Gamma}\frac{d_{\alpha,\beta}(z)}{B(n,t+1)}\mathcal{F}_{n+k}^{(t)}\Phi_{n,k}^{(t)}(0)\frac{F_{\alpha,\beta}(z,z)}{(1-|z|^2)^{2k}}\mathrm{d}\lambda_t(z)$$

$$+\int\limits_{\mathbb{B}_n^2}\Phi_{n,k+1}^{(t)}(|\varphi_z(w)|^2)\frac{\sum_{i=1,j}^{n}\sum_{(\alpha,\beta)\in\Gamma}I^{\alpha+e_i,\beta+e_j}(z-w)D_{i,j}F(z,w)}{|1-\langle w,z\rangle|^{2(k+1)}}K_w^{(t)}(z)\mathrm{d}\lambda_t(z)\mathrm{d}\lambda_t(w).$$

This completes the proof of Lemma 4.3. $\square$

Using the same proof of Lemma 4.2, one can show the following.

Lemma 4.6. *Suppose $\alpha,\beta\in\mathbb{N}_0^n$, and $v\in\mathscr{C}^1(\overline{\mathbb{B}_n})$. Then the following hold.*

1. *If $|\alpha|\geq|\beta|$, then*

$$\begin{aligned}&\int\limits_{\mathbb{S}_n}I^{\alpha,\beta}(z-w)v(w)K_w(z)\frac{\mathrm{d}\sigma(w)}{\sigma_{2n-1}}\\&=d_{\alpha,\beta}(z)v(z)-\frac{1}{n}\sum_{j=1}^{n}\int\limits_{\mathbb{B}_n}|\varphi_z(w)|^{-2|\beta|-2n}I^{\alpha,\beta+e_j}(z-w)\frac{\bar{\partial}_j\left[(1-\langle z,w\rangle)^{|\beta|}v(w)\right]}{(1-\langle z,w\rangle)^{|\beta|}(1-\langle w,z\rangle)}K_w(z)\mathrm{d}\lambda_0(w),\end{aligned}\tag{4.14}$$

2. *If $|\alpha|\leq|\beta|$, then*

$$\begin{aligned}&\int\limits_{\mathbb{S}_n}I^{\alpha,\beta}(z-w)v(z)K_w(z)\frac{\mathrm{d}\sigma(z)}{\sigma_{2n-1}}\\&=d_{\beta,\alpha}(w)v(w)+\frac{1}{n}\sum_{i=1}^{n}\int\limits_{\mathbb{B}_n}|\varphi_z(w)|^{-2|\alpha|-2n}I^{\alpha+e_i,\beta}(z-w)\frac{\partial_i\left[(1-\langle z,w\rangle)^{|\alpha|}v(z)\right]}{(1-\langle z,w\rangle)^{|\alpha|}(1-\langle w,z\rangle)}K_w(z)\mathrm{d}\lambda_0(z),\end{aligned}\tag{4.15}$$

5. The higher dimensions

The goal of this section is to prove Theorem 1.2. To start with, we apply Lemma 4.3 to get the following.

Lemma 5.1. *Suppose $t > -1$ and $f, g \in \mathscr{C}^1(\mathbb{B}_n)$. Suppose $f, g, \partial f, \bar{\partial} g$ are bounded on $\mathbb{B}_n$. Then*

$$T_f^{(t)} T_g^{(t)} - T_{fg}^{(t)} = P^{(t)} R,$$

where $R : L^2_{a,t}(\mathbb{B}_n) \to L^2(\lambda_t)$ is defined by

$$Rh(z) = -\int_{\mathbb{B}_n} \Phi_{n,1}^{(t)}(|\varphi_z(w)|^2) \frac{1 - \langle z, w\rangle}{1 - \langle w, z\rangle} \langle \partial_z f, \overline{z - w}\rangle \langle \bar{\partial}_w g, z - w\rangle h(w) K_w^{(t)}(z) \mathrm{d}\lambda_t(w).$$

Proof. By definition, for $h \in H^\infty(\mathbb{B}_n)$,

$$\left(T_f^{(t)} T_g^{(t)} - T_{fg}^{(t)}\right) h(\xi) = \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \big(f(z)g(w) - f(w)g(w)\big) h(w) K_w^{(t)}(z) K_z^{(t)}(\xi) \mathrm{d}\lambda_t(w) \mathrm{d}\lambda_t(z).$$

Denote $F_\xi(z, w) = \big(f(z)g(w) - f(w)g(w)\big) h(w) K_z^{(t)}(\xi)$. Then

$$F(z, z) = 0.$$

For fixed $\xi \in \mathbb{B}_n$, $F_\xi(z, w)$ is bounded, and by Lemma 2.2,

$$\left| \sum_{j=1}^n I^{0,e_j}(z - w) \bar{\partial}_{w_j} F(z, w) \right| \lesssim |z - w| \lesssim |\varphi_z(w)||1 - \langle z, w\rangle|^{1/2}.$$

Then the assumption of Lemma 4.3 is satisfied when we take $\Gamma = \{(0, 0)\}$, $k = 0$, $\epsilon = 0$ and $F_{0,0} = F_\xi$. Applying the lemma, we obtain the following computation,

$$\begin{aligned}
&\left(T_f^{(t)} T_g^{(t)} - T_{fg}^{(t)}\right) h(\xi) \\
&= \int_{\mathbb{B}_n \times \mathbb{B}_n} \Phi_{n,0}^{(t)}(|\varphi_z(w)|^2) F(z, w) K_w^{(t)}(z) \mathrm{d}\lambda_t(w) \mathrm{d}\lambda_t(z) \\
&= -\int_{\mathbb{B}_n \times \mathbb{B}_n} \Phi_{n,1}^{(t)}(|\varphi_z(w)|^2) \frac{\sum_{i,j=1}^n I^{e_i,e_j}(z - w) D_{i,j} F(z, w)}{|1 - \langle z, w\rangle|^2} K_w^{(t)}(z) \mathrm{d}\lambda_t(z) \mathrm{d}\lambda_t(w) \\
&= -\int_{\mathbb{B}_n \times \mathbb{B}_n} \Phi_{n,1}^{(t)}(|\varphi_z(w)|^2) \frac{1 - \langle z, w\rangle}{1 - \langle w, z\rangle}
\end{aligned}$$

$$\sum_{i,j=1}^{n} I^{e_i,e_j}(z-w)\partial_i f(z)\bar{\partial}_j g(w)h(w)K_z^{(t)}(\xi)K_w^{(t)}(z)\mathrm{d}\lambda_t(z)\mathrm{d}\lambda_t(w)$$

$$=P^{(t)}Rh(\xi).$$

This completes the proof. $\square$

Lemma 5.2. *Suppose $t > 2n-3$ and f, g satisfy Condition 1. Then the semicommutator $T_f^{(t)}T_g^{(t)} - T_{fg}^{(t)}$ belongs to the trace class.*

Proof. Divide $T_f^{(t)}T_g^{(t)} - T_{fg}^{(t)} = P^{(t)}R$ as in Lemma 5.1. Take $\epsilon > 0$ so that $t > 2n-3+2\epsilon$. Let $c = n+\epsilon$ and denote $\hat{R} : L^2_{a,t+2c}(\mathbb{B}_n) \to L^2(\lambda_t)$ the integral operator with the same integral formula as R, i.e.,

$$\hat{R}h(z) = -\int_{\mathbb{B}_n} \Phi_{n,1}^{(t)}(|\varphi_z(w)|^2)\frac{1-\langle z,w\rangle}{1-\langle w,z\rangle}\langle \partial_z f, \overline{z-w}\rangle\langle \bar{\partial}_w g, z-w\rangle h(w)K_w^{(t)}(z)\mathrm{d}\lambda_t(w).$$

Let $E : L^2_{a,t}(\mathbb{B}_n) \to L^2_{a,t+2c}(\mathbb{B}_n)$ be the embedding map. Split $T_f^{(t)}T_g^{(t)} - T_{fg}^{(t)}$ as

$$T_f^{(t)}T_g^{(t)} - T_{fg}^{(t)} : L^2_{a,t}(\mathbb{B}_n) \xrightarrow{E} L^2_{a,t+2c}(\mathbb{B}_n) \xrightarrow{\hat{R}} L^2(\lambda_t) \xrightarrow{P^{(t)}} L^2_{a,t}(\mathbb{B}_n).$$

It is well-known that E is in the trace class [32]. It remains to show that $\hat{R}$ is bounded. By definition, the operator $\hat{R}$ has integral kernel

$$\hat{F}(z,w) = C\Phi_{n,1}^{(t)}(|\varphi_z(w)|^2)\frac{1-\langle z,w\rangle}{1-\langle w,z\rangle}\langle \partial_z f, \overline{z-w}\rangle\langle \bar{\partial}_w g, z-w\rangle K_w^{(t)}(z)(1-|w|^2)^{-2c},$$

where C is a constant. By assumption,

$$|\hat{F}(z,w)| \lesssim \frac{\Phi_{n,1}^{(t)}(|\varphi_z(w)|^2)|\varphi_z(w)|^2}{|1-\langle z,w\rangle|^{1+t-\epsilon}(1-|w|^2)^{2c}}.$$

By Lemma 8.3,

$$\Phi_{n,1}^{(t)}(s) \lesssim s^{-n-1/2}(1-s).$$

Take x so that

$$-2(n+\epsilon)+1 > x > -2-t.$$

Let $y = n+\epsilon+x$, $p(w) = (1-|w|^2)^x$ and $q(z) = (1-|z|^2)^y$. Then by Lemma 2.4 (3),

$$\int_{\mathbb{B}_n} |\hat{F}(z,w)|p(w)\mathrm{d}\lambda_{t+2c}(w) \lesssim \int_{\mathbb{B}_n} \Phi_{n,1}^{(t)}(|\varphi_z(w)|^2)|\varphi_z(w)|^2\frac{(1-|w|^2)^{t+x}}{|1-\langle z,w\rangle|^{1+t-\epsilon}}\mathrm{d}m(w) \lesssim q(z),$$

$$\int_{\mathbb{B}_n} |\hat{F}(z,w)|q(z)\mathrm{d}\lambda_t(z) \lesssim (1-|w|^2)^{-2c} \int_{\mathbb{B}_n} \Phi_{n,1}^{(t)}(|\varphi_z(w)|^2)|\varphi_z(w)|^2 \frac{(1-|z|^2)^{t+y}}{|1-\langle z,w\rangle|^{1+t-\epsilon}} \mathrm{d}m(z)$$
$$\lesssim p(w).$$

By Schur's test, $\hat{R}$ is bounded. Therefore the semicommutator is in the trace class. This completes the proof. $\square$

Recall that the operations $\mathcal{F}_m^{(t)}, \mathcal{G}_m^{(t)}$ and the functions $\Phi_{n,k}^{(t)}$ are defined in Appendix II.

Lemma 5.3. *We have*

$$\mathcal{F}_{n+1}^{(t)}\Phi_{n,1}^{(t)}(0) = \sum_{j=0}^{\infty} \frac{B(n+1+j,t+1)}{1+j} = -\int_0^1 (1-s)^{n-1}s^t \ln s\mathrm{d}s.$$

Consequently, as t tends to infinity,

$$\mathcal{F}_{n+1}^{(t)}\Phi_{n,1}^{(t)}(0) = n!t^{-n-1} + o(t^{-n-2}).$$

Proof. First, by Lemma 8.4,

$$\mathcal{F}_{n+1}^{(t)}\Phi_{n,1}^{(t)}(0) = \mathcal{F}_{n+1}^{(t)}M_{\phi_1}\left(\mathcal{G}_n^{(t)}\right)^2 1(0) = \mathcal{F}_{n+1}^{(t)}\mathcal{G}_n^{(t)}1(0)$$
$$= \sum_{j=0}^{\infty} \frac{1}{1+j}\mathcal{F}_{n+1+j}^{(t)}1(0) = \sum_{j=0}^{\infty} \frac{1}{1+j}B(n+1+j,t+1).$$

By definition, the above equals

$$\sum_{j=0}^{\infty} \frac{1}{1+j} \int_0^1 (1-s)^{n+j}s^t\mathrm{d}s = -\int_0^1 (1-s)^{n-1}s^t \ln s\mathrm{d}s.$$

This proves the first line of equations. The second line of equation follows from the estimate

$$|-\ln s - 1 + s| \lesssim (1-s)^2 s^{-1}.$$

This completes the proof. $\square$

Lemma 5.4. *We have*

$$\Phi_{n,2}^{(t)}(s) = (1-s)^{-t}s^{-n-1} \sum_{k=1}^{n} \frac{(n-1)!\Gamma(t+1)}{(n-k)!\Gamma(t+1+k)} \int_s^1 F(s,x)x^{n-k-1}(1-x)^{t+k-1}\mathrm{d}x.$$

Proof. By Definition 8.1, we compute $\Phi^{(t)}_{n,2}(s)$ as follows,

$$\begin{aligned}
&\Phi^{(t)}_{n,2}(s)\\
=&(1-s)\big(\mathcal{G}^{(t)}_{n+1}\big)^2 M_{\phi_1}\big(\mathcal{G}^{(t)}_n\big)^2 1(s)\\
=&(1-s)^{-t}s^{-n-1}\int_s^1 s_1^n(1-s_1)^t\mathcal{G}^{(t)}_{n+1}M_{\phi_1}\big(\mathcal{G}^{(t)}_n\big)^2 1(s_1)\mathrm{d}s_1\\
=&(1-s)^{-t}s^{-n-1}\int_s^1 s_1^{-1}(1-s_1)^{-1}\int_{s_1}^1 s_2^n(1-s_2)^{t+1}\big(\mathcal{G}^{(t)}_n\big)^2 1(s_2)\mathrm{d}s_2\mathrm{d}s_1\\
=&(1-s)^{-t}s^{-n-1}\int_s^1 s_1^{-1}(1-s_1)^{-1}\int_{s_1}^1\int_{s_2}^1 s_3^{n-1}(1-s_3)^t\mathcal{G}^{(t)}_n 1(s_3)\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1\\
=&(1-s)^{-t}s^{-n-1}\int_s^1 s_1^{-1}(1-s_1)^{-1}\int_{s_1}^1\int_{s_2}^1 s_3^{-1}(1-s_3)^{-1}\int_{s_3}^1 s_4^{n-1}(1-s_4)^t\mathrm{d}s_4\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1\\
=&(1-s)^{-t}s^{-n-1}\iiiint\limits_{s<s_1<s_2<s_3<s_4<1} s_1^{-1}(1-s_1)^{-1}s_3^{-1}(1-s_3)^{-1}s_4^{n-1}(1-s_4)^t\mathrm{d}s_4\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1\\
=&(1-s)^{-t}s^{-n-1}\int_s^1\Bigg\{\iint\limits_{s<s_1<s_2<s_3} s_1^{-1}(1-s_1)^{-1}\mathrm{d}s_1\mathrm{d}s_2\Bigg\}\\
&\qquad\cdot\Bigg\{\int_{s_3}^1 s_4^{n-1}(1-s_4)^t\mathrm{d}s_4\Bigg\}s_3^{-1}(1-s_3)^{-1}\mathrm{d}s_3.
\end{aligned}$$

By Lemma 3.4, we have the following integral,

$$\iint\limits_{s<s_1<s_2<s_3} s_1^{-1}(1-s_1)^{-1}\mathrm{d}s_1\mathrm{d}s_2 = F(s,s_3).$$

For a positive integer m, and $x>-1$, temporarily denote $I(m,x)=\int_{s_3}^1 s_4^{m-1}(1-s_4)^x\mathrm{d}s_4$. Then, we obtain the following relations,

$$I(1,x)=\frac{(1-s_3)^{x+1}}{x+1},$$

and

$$I(m+1,x)=\int_{s_3}^1 s_4^m(1-s_4)^x\mathrm{d}s_4$$

$$
\begin{aligned}
&= - (x+1)^{-1} \int_{s_3}^{1} s_4^m \mathrm{d}(1-s_4)^{x+1} \\
&= \frac{1}{x+1} s_3^m (1-s_3)^{x+1} + \frac{m}{x+1} \int_{s_3}^{1} s_4^{m-1} (1-s_4)^{x+1} \mathrm{d}s_4 \\
&= \frac{1}{x+1} s_3^m (1-s_3)^{x+1} + \frac{m}{x+1} I(m, x+1).
\end{aligned}
$$

Thus by induction, we obtain the following formula for $I(n,t)$,

$$
\int_{s_3}^{1} s_4^{n-1}(1-s_4)^t \mathrm{d}s_4 = I(n,t) = \sum_{k=1}^{n} \frac{(n-1)!\Gamma(t+1)}{(n-k)!\Gamma(t+1+k)} s_3^{n-k}(1-s_3)^{t+k}.
$$

Therefore, we conclude with the following formula for $\Phi_{n,2}^{(t)}$,

$$
\Phi_{n,2}^{(t)}(s) = (1-s)^{-t} s^{-n-1} \sum_{k=1}^{n} \frac{(n-1)!\Gamma(t+1)}{(n-k)!\Gamma(t+1+k)} \int_{s}^{1} F(s,s_3) s_3^{n-k-1} (1-s_3)^{t+k-1} \mathrm{d}s_3.
$$

This completes the proof. $\square$

The following lemma helps us study the first term of $\mathrm{Tr}\left(T_f^{(t)} T_g^{(t)} - T_{fg}^{(t)}\right)$ after iteration.

Lemma 5.5. *For $f, g \in \mathscr{C}^1(\mathbb{B}_n)$, the following are equal whenever the integrals converge.*

$$
\begin{aligned}
&\int_{\mathbb{B}_n} \frac{\sum_{i,j=1}^n d_{e_i,e_j}(w) \partial_i f(w) \bar{\partial}_j g(w)}{(1-|w|^2)^{n+1}} \mathrm{d}m(w) \\
&= \frac{-1}{(2i)^n n!} \int_{\mathbb{B}_n} \partial f \wedge \bar{\partial} g \wedge \left[\partial\bar{\partial} \log(1-|w|^2)\right]^{n-1} \\
&= \frac{1}{n} \int_{\mathbb{B}_n} \frac{\sum_{i=1}^n \partial_i f(w) \bar{\partial}_i g(w) - Rf(w)\bar{R}g(w)}{(1-|w|^2)^n} \mathrm{d}m(w).
\end{aligned}
\tag{5.1}
$$

Proof. By Definition 4.1, we consider

$$
\sum_{i,j=1}^n d_{e_i,e_j}(w) \partial_i f(w) \bar{\partial}_j g(w) = \int_{\mathbb{S}_n} \left[\sum_{i=1}^n \left(A_w\zeta\right)_i \partial_i f(w)\right] \left[\sum_{j=1}^n \overline{\left(A_w\zeta\right)_j} \bar{\partial}_j g(w)\right] \frac{\mathrm{d}\sigma(\zeta)}{\sigma_{2n-1}}.
$$

By an argument similar as in the proof of (4.8), the sum in each big bracket is independent of the choice of an orthonormal basis of $\mathbb{C}^n$. Thus the integrand in the left hand side of (5.1) does not depend on the choice of a basis. At each $w \in \mathbb{B}_n$, $w \neq 0$, choose a basis under which $w = (w_1, 0, \ldots, 0)$. By (4.1),

$$\frac{\sum_{i,j=1}^n d_{e_i,e_j}(w)\partial_i f(w)\bar{\partial}_j g(w)}{(1-|w|^2)^{n+1}} = \frac{1}{n}\left[\frac{\partial_1 f(w)\bar{\partial}_1 g(w)}{(1-|w|^2)^{n-1}} + \frac{\sum_{i=2}^n \partial_i f(w)\bar{\partial}_j g(w)}{(1-|w|^2)^n}\right]. \tag{5.2}$$

On the other hand, we compute

$$\partial\bar{\partial}\log(1-|w|^2) = -\partial\left[\frac{\sum_{j=1}^n w_j \mathrm{d}\bar{w}_j}{1-|w|^2}\right] = -\frac{\sum_{i,j=1}^n \bar{w}_i w_j \mathrm{d}w_i \wedge \mathrm{d}\bar{w}_j}{(1-|w|^2)^2} - \frac{\sum_{j=1}^n \mathrm{d}w_j \wedge \mathrm{d}\bar{w}_j}{1-|w|^2}.$$

At $w = (w_1, 0, \ldots, 0)$ the above equals

$$-\frac{|w|^2 \mathrm{d}w_1 \wedge \mathrm{d}\bar{w}_1}{(1-|w|^2)^2} - \sum_{j=1}^n \frac{\mathrm{d}w_j \wedge \mathrm{d}\bar{w}_j}{1-|w|^2} = -\frac{\mathrm{d}w_1 \wedge \mathrm{d}\bar{w}_1}{(1-|w|^2)^2} - \sum_{j=2}^n \frac{\mathrm{d}w_j \wedge \mathrm{d}\bar{w}_j}{1-|w|^2}.$$

Thus, we have

$$\begin{aligned}
&\left[\partial\bar{\partial}\log(1-|w|^2)\right]^{n-1}\Big|_{w=(w_1,0,\ldots,0)} \\
&\qquad = (-1)^{n-1}(n-1)!\left[\frac{\bigwedge_{j=2}^n \left(\mathrm{d}w_j \wedge \mathrm{d}\bar{w}_j\right)}{(1-|w|^2)^{n-1}} + \sum_{i=2}^n \frac{\bigwedge_{j\neq i}\left(\mathrm{d}w_j \wedge \mathrm{d}\bar{w}_j\right)}{(1-|w|^2)^n}\right].
\end{aligned}$$

Therefore at $w = (w_1, 0, \ldots, 0)$, we have

$$\begin{aligned}
&\partial f \wedge \bar{\partial} g \wedge \left[\partial\bar{\partial}\log(1-|w|^2)\right]^{n-1} \\
=&(-1)^{n-1}(n-1)!\left[\frac{\partial_1 f(w)\bar{\partial}_1 g(w)}{(1-|w|^2)^{n-1}} + \frac{\sum_{i=2}^n \partial_i f(w)\bar{\partial}_i g(w)}{(1-|w|^2)^n}\right]\bigwedge_{j=1}^n \left(\mathrm{d}w_j \wedge \mathrm{d}\bar{w}_j\right) \\
=&-(2i)^n(n-1)!\left[\frac{\partial_1 f(w)\bar{\partial}_1 g(w)}{(1-|w|^2)^{n-1}} + \frac{\sum_{i=2}^n \partial_i f(w)\bar{\partial}_i g(w)}{(1-|w|^2)^n}\right]\mathrm{d}m(w).
\end{aligned} \tag{5.3}$$

Comparing (5.2) and (5.3), we conclude that at $w = (w_1, 0, \ldots, 0)$,

$$\partial f \wedge \bar{\partial} g \wedge \left[\partial\bar{\partial}\log(1-|w|^2)\right]^{n-1} = -(2i)^n n! \frac{\sum_{i,j=1}^n d_{e_i,e_j}(w)\partial_i f(w)\bar{\partial}_j g(w)}{(1-|w|^2)^{n+1}}\mathrm{d}m(w).$$

Since both sides are independent of the choice of basis, the equation holds for general w. This proves the first equality.

Also, it is easy to see that $\sum_{i=1}^n \partial_i f(w)\bar{\partial}_i g(w) - Rf(w)\bar{R}g(w)$ is invariant of the choice of a basis. Again, if one chooses a basis so that $w = (w_1, 0, \ldots, 0)$, then

$$\sum_{i=1}^n \partial_i f(w)\bar{\partial}_i g(w) - Rf(w)\bar{R}g(w) = (1-|w|^2)\partial_1 f(w)\bar{\partial}_1 g(w) + \sum_{i=2}^n \partial_i f(w)\bar{\partial}_i g(w).$$

Comparing the above and (5.2) gives

$$\sum_{i=1}^n \partial_i f(w)\bar{\partial}_i g(w) - Rf(w)\bar{R}g(w) = n\frac{\sum_{i,j=1}^n d_{e_i,e_j}(w)\partial_i f(w)\bar{\partial}_j g(w)}{1-|w|^2}. \tag{5.4}$$

Since both sides are independent of the choice of a basis the equation holds for general w. Plugging (5.4) into the first equality gives the second equality. This completes the proof. □

Proof of Theorem 1.2. The fact that $T_f^{(t)}T_g^{(t)} - T_{fg}^{(t)}$ belongs to the trace class is proved in Lemma 5.2. By Lemma 2.5 and Lemma 5.1, we compute the trace of the semi-commutator as follows,

$$\begin{aligned}
&\operatorname{Tr}\left(T_f^{(t)}T_g^{(t)} - T_{fg}^{(t)}\right)\\
&= \int_{\mathbb{B}_n} \langle \left(T_f^{(t)}T_g^{(t)} - T_{fg}^{(t)}\right)K_\xi^{(t)}, K_\xi^{(t)}\rangle\\
&= -\int_{\mathbb{B}_n}\left\{\int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \Phi_{n,1}^{(t)}(|\varphi_z(w)|^2)\frac{1-\langle z,w\rangle}{1-\langle w,z\rangle}\langle \partial_z f, \overline{z-w}\rangle\langle \bar{\partial}_w g, z-w\rangle\right.\\
&\qquad\qquad \left. K_\xi^{(t)}(w)K_w^{(t)}(z)K_z^{(t)}(\xi)\mathrm{d}\lambda_t(w)\mathrm{d}\lambda_t(z)\right\}\mathrm{d}\lambda_t(\xi).
\end{aligned}$$

It follows from our assumption that f, g satisfy condition 1 and Lemma 2.4 that the integral converges absolutely. Applying Fubini's theorem, we continue the above computation,

$$\begin{aligned}
&-\int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \Phi_{n,1}^{(t)}(|\varphi_z(w)|^2)\frac{1-\langle z,w\rangle}{1-\langle w,z\rangle}\langle \partial_z f, \overline{z-w}\rangle\langle \bar{\partial}_w g, z-w\rangle\\
&\qquad\qquad \left\{\int_{\mathbb{B}_n} K_\xi^{(t)}(w)K_z^{(t)}(\xi)\mathrm{d}\lambda_t(\xi)\right\}K_w^{(t)}(z)\mathrm{d}\lambda_t(w)\mathrm{d}\lambda_t(z)\\
&= -\int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \Phi_{n,1}^{(t)}(|\varphi_z(w)|^2)\frac{1-\langle z,w\rangle}{1-\langle w,z\rangle}\langle \partial_z f, \overline{z-w}\rangle\langle \bar{\partial}_w g, z-w\rangle|K_w^{(t)}(z)|^2\mathrm{d}\lambda_t(w)\mathrm{d}\lambda_t(z)
\end{aligned}$$

$$=-\int\limits_{\mathbb{B}_n}\int\limits_{\mathbb{B}_n} \Phi_{n,1}^{(t)}(|\varphi_z(w)|^2)\frac{(1-\langle w,z\rangle)^2\sum_{i,j=1}^n I^{e_i,e_j}(w-z)\partial_i\bar{g}(w)\bar{\partial}_j\bar{f}(z)}{|1-\langle z,w\rangle|^2}|K_w^{(t)}(z)|^2\mathrm{d}\lambda_t(w)\mathrm{d}\lambda_t(z).$$

Applying Lemma 4.3 with $\Gamma=\{(e_i,e_j):i,j=1,\dots,n\}$, $k=1$, we obtain

$$F_{e_i,e_j}(z,w)=(1-\langle w,z\rangle)^2\partial_i\bar{g}(w)\bar{\partial}_j\bar{f}(z)K_z^{(t)}(w)$$

and also using Lemma 5.5 we get

$$\begin{aligned}
&\int\limits_{\mathbb{B}_n}\int\limits_{\mathbb{B}_n} \Phi_{n,1}^{(t)}(|\varphi_z(w)|^2)\frac{(1-\langle w,z\rangle)^2\sum_{i,j=1}^n I^{e_i,e_j}(w-z)\partial_i\bar{g}(w)\bar{\partial}_j\bar{f}(z)}{|1-\langle z,w\rangle|^2}|K_w^{(t)}(z)|^2\mathrm{d}\lambda_t(w)\mathrm{d}\lambda_t(z)\\
=&\int\limits_{\mathbb{B}_n}\int\limits_{\mathbb{B}_n} \Phi_{n,1}^{(t)}(|\varphi_z(w)|^2)\frac{\sum_{i,j=1}^n F_{e_i,e_j}(z,w)}{|1-\langle z,w\rangle|^2}K_w^{(t)}(z)\mathrm{d}\lambda_t(w)\mathrm{d}\lambda_t(z)\\
=&\frac{\mathcal{F}_{n+1}^{(t)}\Phi_{n,1}^{(t)}(0)}{B(n,t+1)}\int\limits_{\mathbb{B}_n}(1-|z|^2)^{-2}\sum_{i,j=1}^n d_{e_i,e_j}(z)F_{e_i,e_j}(z,z)\mathrm{d}\lambda_t(z)\\
&-\int\limits_{\mathbb{B}_n\times\mathbb{B}_n}\Phi_{n,2}^{(t)}(|\varphi_z(w)|^2)\frac{\sum_{i,j,k,l=1}^n I^{e_i+e_k,e_j+e_l}(z-w)(1-\langle z,w\rangle)^2\partial_{z_k}\bar{\partial}_{w_l}F_{e_i,e_j}(z,w)}{|1-\langle z,w\rangle|^4}\\
&\qquad K_w^{(t)}(z)\mathrm{d}\lambda_t(z)\mathrm{d}\lambda_t(w)\\
=&\frac{\mathcal{F}_{n+1}^{(t)}\Phi_{n,1}^{(t)}(0)}{B(n,t+1)}\int\limits_{\mathbb{B}_n}\frac{\sum_{i,j=1}^n d_{e_i,e_j}(z)\partial_i\bar{g}(z)\bar{\partial}_j\bar{f}(z)}{(1-|z|^2)^{n+1+t}}\mathrm{d}\lambda_t(z)\\
&-\int\limits_{\mathbb{B}_n\times\mathbb{B}_n}\Phi_{n,2}^{(t)}(|\varphi_z(w)|^2)\sum_{i,j,k,l=1}^n I^{e_i+e_k,e_j+e_l}(z-w)\bar{\partial}_l\partial_i\bar{g}(w)\partial_k\bar{\partial}_j\bar{f}(z)|K_w^{(t)}(z)|^2\mathrm{d}\lambda_t(z)\mathrm{d}\lambda_t(w)\\
=&\frac{\mathcal{F}_{n+1}^{(t)}\Phi_{n,1}^{(t)}(0)}{B(n,t+1)}\cdot\frac{(n-1)!}{B(n,t+1)\pi^n}\cdot\frac{-1}{(2i)^n n!}\int\limits_{\mathbb{B}_n}\partial\bar{g}\wedge\bar{\partial}\bar{f}\wedge\left[\partial\bar{\partial}\log(1-|w|^2)\right]^{n-1}\\
&-\int\limits_{\mathbb{B}_n\times\mathbb{B}_n}\Phi_{n,2}^{(t)}(|\varphi_z(w)|^2)L_z\bar{f}(z-w)L_w\bar{g}(z-w)|K_w^{(t)}(z)|^2\mathrm{d}\lambda_t(z)\mathrm{d}\lambda_t(w).
\end{aligned}$$

Therefore, we continue the computation of the semi-commutator using the above calculation,

$$\begin{aligned}
&\mathrm{Tr}\left(T_f^{(t)}T_g^{(t)}-T_{fg}^{(t)}\right)\\
=&-\int\limits_{\mathbb{B}_n}\int\limits_{\mathbb{B}_n} \Phi_{n,1}^{(t)}(|\varphi_z(w)|^2)\frac{(1-\langle w,z\rangle)^2\sum_{i,j=1}^n I^{e_i,e_j}(w-z)\partial_i\bar{g}(w)\bar{\partial}_j\bar{f}(z)}{|1-\langle z,w\rangle|^2}|K_w^{(t)}(z)|^2\mathrm{d}\lambda_t(w)\mathrm{d}\lambda_t(z)
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\mathcal{F}_{n+1}^{(t)}\Phi_{n,1}^{(t)}(0)}{B(n,t+1)} \cdot \frac{(n-1)!}{B(n,t+1)\pi^n} \cdot \frac{-1}{(-2i)^n n!} \int\limits_{\mathbb{B}_n} \bar{\partial} g \wedge \partial f \wedge \Big[-\partial\bar{\partial}\log(1-|w|^2)\Big]^{n-1} \\
&\quad + \int\limits_{\mathbb{B}_n\times\mathbb{B}_n} \Phi_{n,2}^{(t)}(|\varphi_z(w)|^2) L_z f(z-w) L_w g(z-w) |K_w^{(t)}(z)|^2 \mathrm{d}\lambda_t(z)\mathrm{d}\lambda_t(w) \\
&= a_{n,t} \int\limits_{\mathbb{B}_n} \partial f \wedge \bar{\partial} g \wedge \Big[\partial\bar{\partial}\log(1-|w|^2)\Big]^{n-1} \\
&\quad + \int\limits_{\mathbb{B}_n\times\mathbb{B}_n} \rho_{n,t}(|\varphi_z(w)|^2) L_z f(z-w) L_w g(z-w) \frac{\mathrm{d}m(z,w)}{|1-\langle z,w\rangle|^{2n+2}},
\end{aligned}
$$

where

$$
a_{n,t} = \frac{\mathcal{F}_{n+1}^{(t)}\Phi_{n,1}^{(t)}(0)}{\big(B(n,t+1)^2\big) n (2\pi i)^n} \quad \text{and} \quad \rho_{n,t}(s) = \left(\frac{(n-1)!}{\pi^n B(n,t+1)}\right)^2 (1-s)^t \Phi_{n,2}^{(t)}(s).
$$

By Lemma 5.3, we have the following estimate,

$$
a_{n,t} = \frac{-\int_0^1 (1-s)^{n-1} s^t \ln s \mathrm{d}s}{\big(B(n,t+1)^2\big) n (2\pi i)^n} = \frac{n! t^{-n-1} + o(t^{-n-2})}{\big(B(n,t+1)^2\big) n (2\pi i)^n} = \frac{t^{n-1}}{(n-1)!(2\pi i)^n} + o(t^{n-2}). \tag{5.5}
$$

By Lemma 5.4, we have the following formula,

$$
\rho_{n,t}(s) = s^{-n-1} \sum_{k=1}^{n} \frac{(n-1)!\Gamma^2(n+t+1)}{(n-k)!\Gamma(t+1+k)\Gamma(t+1)\pi^{2n}} \int\limits_s^1 F(s,x) x^{n-k-1}(1-x)^{t+k-1} \mathrm{d}x. \tag{5.6}
$$

This proves Equation (1.9). It remains to prove (1.11). By our assumption on Condition 2, Lemmas 2.2, 2.4 and 3.5, we have the following estimates,

$$
\begin{aligned}
& t^{1-n} \Bigg| \int\limits_{\mathbb{B}_n\times\mathbb{B}_n} \rho_{n,t}(|\varphi_z(w)|^2) L_z f(z-w) L_w g(z-w) \frac{\mathrm{d}m(z,w)}{|1-\langle z,w\rangle|^{2n+2}} \Bigg| \\
\lesssim & t^{1-n} \int\limits_{\mathbb{B}_n}\int\limits_{\mathbb{B}_n} \frac{\rho_{n,t}(|\varphi_z(w)|^2)|\varphi_z(w)|^4}{|1-\langle z,w\rangle|^{n+2-\epsilon}} \mathrm{d}m(w)\mathrm{d}m(z) \\
\xlongequal{\zeta=\varphi_z(w)} & t^{1-n} \int\limits_{\mathbb{B}_n}\int\limits_{\mathbb{B}_n} \rho_{n,t}(|\zeta|^2)|\zeta|^4 \frac{|1-\langle z,\zeta\rangle|^{n+2-\epsilon}}{(1-|z|^2)^{n+2-\epsilon}} \cdot \frac{(1-|z|^2)^{n+1}}{|1-\langle z,\zeta\rangle|^{2n+2}} \mathrm{d}m(\zeta)\mathrm{d}m(z) \\
= & t^{1-n} \int\limits_{\mathbb{B}_n} \rho_{n,t}(|\zeta|^2)|\zeta|^4 \int\limits_{\mathbb{B}_n} \frac{(1-|z|^2)^{-1+\epsilon}}{|1-\langle z,\zeta\rangle|^{n+\epsilon}} \mathrm{d}m(z)\mathrm{d}m(\zeta)
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim t^{1-n}\int\limits_{\mathbb{B}_n}\rho_{n,t}(|\zeta|^2)|\zeta|^4\ln\frac{1}{1-|\zeta|^2}\mathrm{d}m(\zeta)\\
&\approx t^{1-n}\int\limits_0^1 s^{n+1}\rho_{n,t}(s)\ln\frac{1}{1-s}\mathrm{d}s\\
&\lesssim t^{1-n}\int\limits_0^1 s^{n+1}\rho_{n,t}(s)(1-s)^{-1/2}\mathrm{d}s\\
&=t^{1-n}\int\limits_0^1 s^{n+1}\Bigg\{s^{-n-1}\sum_{k=1}^{n}\frac{(n-1)!\Gamma^2(n+t+1)}{(n-k)!\Gamma(t+1+k)\Gamma(t+1)\pi^{2n}}\\
&\qquad\qquad\int\limits_s^1 F(s,x)x^{n-k-1}(1-x)^{t+k-1}\mathrm{d}x\Bigg\}(1-s)^{-1/2}\mathrm{d}s\\
&=t^{1-n}\sum_{k=1}^{n}\frac{(n-1)!\Gamma^2(n+t+1)}{(n-k)!\Gamma(t+1+k)\Gamma(t+1)\pi^{2n}}\\
&\qquad\qquad\int\limits_0^1\int\limits_s^1 F(s,x)x^{n-k-1}(1-x)^{t+k-1}\mathrm{d}x(1-s)^{-1/2}\mathrm{d}s\\
&=t^{1-n}\sum_{k=1}^{n}\frac{(n-1)!\Gamma^2(n+t+1)}{(n-k)!\Gamma(t+1+k)\Gamma(t+1)\pi^{2n}}\\
&\qquad\qquad\int\limits_0^1\Bigg\{\int\limits_0^x F(s,x)(1-s)^{-1/2}\mathrm{d}s\Bigg\}x^{n-k-1}(1-x)^{t+k-1}\mathrm{d}x\\
&\lesssim t^{1-n}\sum_{k=1}^{n}\frac{(n-1)!\Gamma^2(n+t+1)}{(n-k)!\Gamma(t+1+k)\Gamma(t+1)\pi^{2n}}\int\limits_0^1 x^{n-k+1}(1-x)^{t+k-1}\mathrm{d}x\\
&=t^{1-n}\sum_{k=1}^{n}\frac{(n-1)!\Gamma^2(n+t+1)}{(n-k)!\Gamma(t+1+k)\Gamma(t+1)\pi^{2n}}B(n-k+2,t+k)\\
&=o(t^{-1})\\
&\quad\to 0,
\end{aligned}
$$

as t tends to infinity. Combining the above, (5.5), and (1.9) we obtain (1.11). This completes the proof. □

6. Applications and examples

We start this section with some applications of Theorem 1.1. Since $|\varphi_z(w)| = |\varphi_w(z)|$, it follows immediately that the second term in (1.3) is symmetric in the symbols f and g. As a consequence, the following trace formula for commutators of Toeplitz operators holds.

Corollary 6.1. *Suppose $t > -1$ and $f, g \in \mathscr{C}^2(\overline{\mathbb{D}})$. Then*

$$\operatorname{Tr}[T_f^{(t)}, T_g^{(t)}] = \frac{1}{2\pi i} \int_{\mathbb{D}} \mathrm{d}f \wedge \mathrm{d}g. \tag{6.1}$$

For the case when $t = 0$, this result is well-known (cf. [23,39]).

We can apply Theorem 1.1 to study Hankel operators. Recall that the Hankel operator with symbol g is defined on $L^2(\lambda_t)$ by

$$H_g^{(t)} = (I - P^{(t)}) M_g P^{(t)},$$

where $P^{(t)}$ is the Bergman projection. By the identity

$$T_f^{(t)} T_g^{(t)} - T_{fg}^{(t)} = -H_{\bar{f}}^{(t)*} H_g^{(t)},$$

we have

$$\operatorname{Tr}(T_f^{(t)} T_g^{(t)} - T_{fg}^{(t)}) = -\operatorname{Tr}(H_{\bar{f}}^{(t)*} H_g^{(t)}) = -\langle H_g^{(t)}, H_{\bar{f}}^{(t)} \rangle_{\mathcal{S}^2}.$$

Thus (1.3) leads to a formula for the inner product of Hankel operators in the Hilbert-Schmidt class. In particular, it leads to a formula for the Hilbert-Schmidt norm of Hankel operators.

Corollary 6.2. *Suppose $t > -1$ and $g \in \mathscr{C}^2(\overline{\mathbb{D}})$. Then*

$$\|H_g^{(t)}\|_{\mathcal{S}^2}^2 = \frac{1}{\pi} \int_{\mathbb{D}} |\bar{\partial} g|^2 \mathrm{d}m - \int_{\mathbb{D}^2} \varrho_t(|\varphi_z(w)|^2) \Delta \bar{g}(z) \Delta g(w) \mathrm{d}m(z, w).$$

where ϱ_t is defined as in Theorem 1.1. In particular,

$$\lim_{t \to \infty} \|H_g^{(t)}\|_{\mathcal{S}^2}^2 = \frac{1}{\pi} \int_{\mathbb{D}} |\bar{\partial} g|^2 \mathrm{d}m.$$

For Hankel operators with real subharmonic symbols, the second term is non-negative. So the following holds.

Corollary 6.3. *Suppose $t > -1$ and $g \in \mathscr{C}^2(\overline{\mathbb{D}})$ is real-valued and subharmonic in $\mathbb{D}$. Then*

$$\|H_g^{(t)}\|_{\mathcal{S}^2}^2 \leq \frac{1}{\pi} \int\limits_{\mathbb{D}} |\bar{\partial} g|^2 \mathrm{d}m, \tag{6.2}$$

with equality holds if and only if g is harmonic in $\mathbb{D}$.

As explained in the introduction, in this paper we focus more on the trace formula (1.9) and asymptotic trace formula (1.11) of semi-commutators with relatively nice symbols. Nonetheless, the following lemma and the examples that follow show that Condition 1 is a natural condition to work with.

Lemma 6.4. *Suppose $n \geq 2$ and f, g satisfy Condition 1. Then there exists a constant $C > 0$ such that for any $z \in \mathbb{B}_n \backslash \{0\}$,*

$$\left|\langle \partial_z f, \bar{\zeta}\rangle \langle \bar{\partial}_z g, \zeta\rangle\right| \leq C\Big(|P_z(\zeta)|^2 + (1-|z|^2)|Q_z(\zeta)|^2\Big)(1-|z|^2)^{n-2+\epsilon}, \quad \forall \zeta \in \mathbb{C}^n. \tag{6.3}$$

(1) *In the special case when $f = \bar{g}$, (6.3) becomes*

$$\left|\langle \bar{\partial}_z g, \zeta\rangle\right| \leq C_1\Big(|P_z(\zeta)| + (1-|z|^2)^{1/2}|Q_z(\zeta)|\Big)(1-|z|^2)^{\frac{n-2+\epsilon}{2}}, \tag{6.4}$$

which is equivalent to Condition 1. Here C_1 is another constant.

(2) *If there are $a, b \geq 0$, $a + b > n - 2$ such that*

$$\left|\langle \partial_z f, \bar{\zeta}\rangle\right| \leq C_2\Big(|P_z(\zeta)| + (1-|z|^2)^{1/2}|Q_z(\zeta)|\Big)(1-|z|^2)^{a}, \tag{6.5}$$

$$\left|\langle \bar{\partial}_z g, \zeta\rangle\right| \leq C_2\Big(|P_z(\zeta)| + (1-|z|^2)^{1/2}|Q_z(\zeta)|\Big)(1-|z|^2)^{b} \tag{6.6}$$

with some constant C_2, then f, g satisfy Condition 1.

Proof. Note that

$$|1 - \langle z, w\rangle| \approx (1-|z|^2) + (1-|w|^2) + |z-w|^2 + |\mathrm{Im}\langle z, w\rangle|,$$

$$|\varphi_z(w)|^2 = \frac{|z - P_z(w)|^2 + (1-|z|^2)|Q_z(w)|^2}{|1-\langle z, w\rangle|^2}. \tag{6.7}$$

Take $w = z + \lambda\zeta$ where $\lambda \in \mathbb{C}$ is sufficiently small. Then by definition,

$$|\varphi_z(w)|^2 = |\lambda|^2 \frac{|P_z(\zeta)|^2 + (1-|z|^2)|Q_z(\zeta)|^2}{|1-\langle z, w\rangle|^2}.$$

Condition 1 implies

$$\left|\langle \partial_z f, \bar{\zeta}\rangle \langle \bar{\partial}_w g, \zeta\rangle\right| |\lambda|^2 \le C|\lambda|^2 \frac{|P_z(\zeta)|^2 + (1-|z|^2)|Q_z(\zeta)|^2}{|1-\langle z,w\rangle|^2} |1-\langle z,w\rangle|^{n+\epsilon}.$$

Canceling out $|\lambda|^2$ and letting $\lambda \to 0$ we obtain the first inequality in (6.3). The second inequality is proved similarly.

On the other hand, suppose $f = \bar{g}$ and (6.4) holds. Then

$$\begin{aligned}\left|\langle \bar{\partial}_z g, z-w\rangle\right|^2 &\lesssim \Big(|z-P_z(w)|^2 + (1-|z|^2)|Q_z(w)|^2\Big)(1-|z|^2)^{n-2+\epsilon} \\ &\lesssim \Big(|z-P_z(w)|^2 + (1-|z|^2)|Q_z(w)|^2\Big)|1-\langle z,w\rangle|^{n-2+\epsilon} \\ &= |\varphi_z(w)|^2 |1-\langle z,w\rangle|^{n+\epsilon}.\end{aligned}$$

Equivalently,

$$\left|\langle \bar{\partial}_w g, z-w\rangle\right|^2 \lesssim |\varphi_z(w)|^2 |1-\langle z,w\rangle|^{n+\epsilon}$$

and also

$$\left|\langle \partial_z \bar{g}, \overline{z-w}\rangle\right|^2 \lesssim |\varphi_z(w)|^2 |1-\langle z,w\rangle|^{n+\epsilon}.$$

Multiplying the two inequalities and taking square root gives Condition 1 for $f = \bar{g}$. This proves (1). Statement (2) is proved in the same way as (1). We omit the details. □

Similarly, one may give sufficient conditions for Condition 2 in terms of growth rates of second order derivatives. Taking the case $f = \bar{g}$ for example, we have the following.

Lemma 6.5. *Suppose $n \ge 2$ and $g \in \mathscr{C}^2(\mathbb{B}_n)$. If g satisfies (6.4) and for some constant $C > 0$ and $a \ge \max\{0, \frac{n}{2} - 2\}$,*

$$|L_z g(\zeta)| \le C\Big(|P_z(\zeta)|^2 + (1-|z|^2)|Q_z(\zeta)|^2\Big)(1-|z|^2)^a,$$

then $f = \bar{g}$ and g satisfy Condition 1 and 2.

The Schatten class criterion of Hankel operators is throughly studied, cf. [4,3,26,36, 38]. There are also some results on the Schatten norms of Hankel operator with anti-holomorphic symbols, cf. [25][34]. In [26, Theorem 3.1], Li and Luecking gave a criterion for Hankel operators to be in $\mathcal{S}^p$. Our condition in (1) of Lemma 6.4 is consistent with that of f_2 in Li and Luecking's Theorem 3.1 when $p = 2$. One can also check using [26, Theorem 3.1] that when (2) of Lemma 6.4 holds, $H_{\bar{f}}^{(t)} \in \mathcal{S}^p$ and $H_g^{(t)} \in \mathcal{S}^q$ for some

$\frac{1}{p}+\frac{1}{q}=1$. So the trace class membership of $T_f^{(t)}T_g^{(t)}-T_{fg}^{(t)}$ follows from the identity $T_f^{(t)}T_g^{(t)}-T_{fg}^{(t)}=-H_{\bar{f}}^{(t)*}H_g^{(t)}$. The converse, however, is not true: there are symbols f,g such that $H_{\bar{f}}^{(t)}$ and $H_g^{(t)}$ belong only to bigger Schatten classes but their product belongs to the trace class. The following lemma gives us a clue.

Lemma 6.6. *Suppose $f,g\in\mathscr{C}^2(\mathbb{B}_n)$ are bounded and have bounded first and second order derivatives. If* $\operatorname{supp} fg$ *is a compact subset in $\mathbb{B}_n$ then f,g satisfy Conditions 1 and 2 in Theorem 1.2. In particular (1.9) and (1.11) hold.*

Proof. By (6.7), in this case $|1-\langle z,w\rangle|$ is bounded away from 0 for $(z,w)\in\operatorname{supp} f\times\operatorname{supp} g$. From this it is easy to verify Conditions 1 and 2. $\square$

In the case when $\operatorname{supp} f$ and $\operatorname{supp} g$ do meet on the boundary, Condition 1 gives us an idea of how much decay is needed when they meet. See the following example.

Example 6.7. Let $\epsilon>0$ and ψ be a $\mathscr{C}^1$ function on $\mathbb{R}$ such that

$$\psi'(s)=0 \text{ for } s<0, \text{ and } |\psi'(s)|\lesssim s^{1+\epsilon} \text{ for } s\geq 0.$$

Let $n=2$ and

$$f(z)=\psi(|z_1|^2-|z_2|^2),\quad g(z)=\psi(|z_2|^2-|z_1|^2).$$

Then we compute

$$|\partial_z f|\begin{cases}=0, & \text{if } |z_1|<|z_2|\\ \lesssim(|z_1|^2-|z_2|^2)^{1+\epsilon}, & \text{if } |z_1|\geq|z_2|\end{cases},$$

$$|\bar{\partial}_w g|\begin{cases}=0, & \text{if } |w_2|<|w_1|\\ \lesssim(|w_2|^2-|w_1|^2)^{1+\epsilon}, & \text{if } |w_2|\geq|w_1|\end{cases}$$

Whenever $|\partial_z f||\bar{\partial}_w g|$ is non-zero we have $|z_1|>|z_2|$ and $|w_2|>|w_1|$, in which case

$$|z-w|\approx|z_1-w_1|+|z_2-w_2|\geq|z_1|-|w_1|+|w_2|-|z_2|=(|z_1|-|z_2|)+(|w_2|-|w_1|).$$

So we have the following estimate

$$|\partial_z f||\bar{\partial}_w g|\lesssim|z-w|^{2+2\epsilon}.$$

Using the above inequality, we reach the following estimate,

$$|\langle\partial_z f,\overline{z-w}\rangle||\langle\bar{\partial}_w g,z-w\rangle|\lesssim|z-w|^2|\partial_z f||\bar{\partial}_w g|\lesssim|z-w|^{4+2\epsilon}\lesssim|\varphi_z(w)|^2|1-\langle z,w\rangle|^{2+\epsilon}.$$

So Condition 1 is satisfied and by Theorem 1.2, $T_f^{(t)}T_g^{(t)}-T_{fg}^{(t)}$ is in the trace class.

7. Appendix I: a formula of Bochner-Martinelli type

Recall that $R = \sum_{i=1}^{n} z_i \partial_{z_i}$ is the radial derivative operator, and $\bar{R} = \sum_{i=1}^{n} \bar{z}_i \bar{\partial}_{z_i}$. In this appendix we prove the following lemma.

Lemma 7.1. *Suppose $r > 0$, $\alpha, \beta \in \mathbb{N}_0^n$, $|\alpha| \geq |\beta|$ and $v \in \mathscr{C}^1(\overline{r\mathbb{B}_n})$. Then*

$$\begin{aligned}&\int\limits_{r\mathbb{S}_n} z^\alpha \bar{z}^\beta v(z) \mathrm{d}\sigma_r(z) \\ =& a_{\alpha,\beta} \sigma_{2n-1} r^{2|\beta|+2n-1} v(0) + 2r^{2|\beta|+2n-1} \int\limits_{r\mathbb{B}_n} \frac{z^\alpha \bar{z}^\beta}{|z|^{2|\beta|+2n}} \bar{R} v(z) \mathrm{d}m(z).\end{aligned} \tag{7.1}$$

Here

$$a_{\alpha,\beta} = \delta_{\alpha,\beta} \frac{(n-1)!\alpha!}{(n-1+|\alpha|)!}.$$

Lemma 7.1 can be verified directly on $v(z) = z^\gamma \bar{z}^\iota$, and then using approximation of v by polynomials. For future reference, we show in the rest of this appendix that it can be viewed as a special case of a Bochner-Martinelli type formula (see Proposition 7.5 below).

For a (p, q)-form $u = \sum_{|I|=p,|J|=q} u_{I,J} \mathrm{d}z_I \wedge \mathrm{d}z_J$,

$$\partial u = \sum_{k=1}^{n} \sum_{|I|=p,|J|=q} \partial_k u_{I,J} \mathrm{d}z_k \wedge \mathrm{d}z_I \wedge \mathrm{d}z_J, \quad \bar{\partial} u = \sum_{k=1}^{n} \sum_{|I|=p,|J|=q} \bar{\partial}_k u_{I,J} \mathrm{d}\bar{z}_k \wedge \mathrm{d}z_I \wedge \mathrm{d}z_J.$$

Then $\mathrm{d} = \partial + \bar{\partial}$ is the exterior derivative.

In some of the estimates, we may abuse notations and use ∂f, $\bar{\partial} f$ to denote holomorphic, and anti-holomorphic gradient of a $\mathscr{C}^1$ function f, i.e.,

$$\partial f(z) = (\partial_1 f(z), \partial_2 f(z), \ldots, \partial_n f(z)), \quad \bar{\partial} f(z) = (\bar{\partial}_1 f(z), \bar{\partial}_2 f(z), \ldots, \bar{\partial}_n f(z)),$$

considered as column vectors.

Cauchy formula. Let $\Omega \subset \mathbb{C}$ be a bounded open set with $\mathscr{C}^1$ boundary and $0 \in \Omega$. Then for every $v \in \mathscr{C}^1(\overline{\Omega})$,

$$\frac{1}{2\pi i} \int\limits_{\partial\Omega} \frac{v(z)}{z} \mathrm{d}z = v(0) - \frac{1}{2\pi i} \int\limits_{\Omega} \frac{\bar{\partial} v(z)}{z} \mathrm{d}z \wedge \mathrm{d}\bar{z}.$$

Its generalization to higher dimensions is the Bochner-Martinelli Formula.

Bochner-Martinelli formula. Let $\Omega \subset \mathbb{C}^n$ be a bounded open set with $\mathscr{C}^1$ boundary and $0 \in \Omega$. Then for every $v \in \mathscr{C}^1(\overline{\mathbb{B}_n})$,

$$\int_{\partial\Omega} v(z) k_{BM}(z) = v(0) + \int_{\Omega} \bar{\partial} v(z) \wedge k_{BM}(z). \tag{7.2}$$

Here k_{BM} is the Bochner-Martinelli kernel, defined by

$$k_{BM}(z) = \frac{(-1)^{n-1}}{(2\pi i)^n} |z|^{-2n} \partial |z|^2 \wedge \left(\partial\bar{\partial}|z|^2\right)^{n-1}. \tag{7.3}$$

In this section, we prove a generalization of the above Bochner-Martinelli Formula (7.2). First, we review the definition of currents on $\mathbb{C}^n$.

Definition 7.2. For $p, q = 0, \dots, n$, denote $\mathscr{D}^{p,q}$ the locally convex space of smooth (p,q)-forms on $\mathbb{C}^n$ with compact support. The topology of $\mathscr{D}^{p,q}$ is defined by the collection of semi-norms

$$\|u\|_{\mathscr{D}^{p,q}(K),N} := \sum_{|I|=p,|J|=q} \sup_{|\alpha|+|\beta|\le N} \sup_{z\in K} |\partial^\alpha \bar{\partial}^\beta u_{I,J}(z)|, \quad u = \sum_{|I|=p,|J|=q} u_{I,J}(z) \mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J,$$

where N ranges over all positive integers, and K is any compact subset in $\mathbb{C}^n$. The space of currents of bidegree (p,q), denoted by $\mathscr{D}'^{p,q}$, is the dual space of $\mathscr{D}^{n-p,n-q}$, endowed with the weak* topology. The currents in $\mathscr{D}'^{p,q}$ can be viewed as (p,q)-forms with distribution coefficients. In particular, any (p,q)-form with locally integrable coefficients is a current of bidegree (p,q). With the identification of Lebesgue measure $\mathrm{d}m(z)$ with the Euclidean volume form

$$\mathrm{d}v := \frac{1}{(-2i)^n} \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge \ldots \wedge \mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n,$$

a distribution T on $\mathbb{C}^n$ can be viewed as either a current of bidegree $(0,0)$ or (n,n): for h on $\mathbb{C}^n$ smooth and compactly supported,

$$\langle T, h\rangle = \langle T, h\mathrm{d}v\rangle.$$

Differential operators act on currents by duality and are continuous with respect to the weak* topology.

Definition 7.3. For multi-indices $\alpha, \beta \in \mathbb{N}_0^n$, define

$$H_{\alpha,\beta}(z) = \lim_{\epsilon\to 0^+} \left(\frac{z^\alpha \bar{z}^\beta}{|z|^{2|\beta|}} k_{BM}(z) \right)\Big|_{\mathbb{C}^n \setminus \epsilon\mathbb{B}_n}, \tag{7.4}$$

in the sense of current.

In the case when $|\beta| \leq |\alpha|$, $H_{\alpha,\beta}$ has locally integrable distributions. So we can simply write

$$H_{\alpha,\beta}(z) = \frac{z^\alpha \bar{z}^\beta}{|z|^{2|\beta|}} k_{BM}(z).$$

Only this case will be used in this paper. For completeness and future reference, we include the case when $|\beta| > |\alpha|$. In this case, the current $H_{\alpha,\beta}$ has coefficient distributions which are not locally integrable, and we need to define it in the style of a principal value. For any $1 \leq j \leq n$ and $\mathscr{C}^\infty$, compactly supported function h on $\mathbb{C}^n$, take the Taylor expansion

$$h(z) = \sum_{|\gamma_1|+|\gamma_2| \leq |\beta|-|\alpha|} \frac{\partial^{\gamma_1} \bar{\partial}^{\gamma_2} h(0)}{\gamma_1! \gamma_2!} z^{\gamma_1} \bar{z}^{\gamma_2} + O(|z|^{|\beta|-|\alpha|+1}).$$

For each $\epsilon > 0$, the current inside the limit sign of (7.4) vanishes on every term except for $O(|z|^{|\beta|-|\alpha|+1})$. Thus the current $H_{\alpha,\beta}$ for $|\beta| > |\alpha|$ is well-defined.

The standard Bochner-Martinelli formula follows from Stoke's Theorem and the following identity.

$$\bar{\partial} k_{BM} = \delta_0, \tag{7.5}$$

where δ_0 is the point mass at 0. Standard arguments show that the following holds.

Lemma 7.4. *We have*

$$\bar{\partial} H_{\alpha,\beta} = (-1)^{l-k} a_{\alpha,\beta} \partial^{\beta-\alpha} \delta_0. \tag{7.6}$$

Here

$$a_{\alpha,\beta} = \begin{cases} \frac{(n-1)!\alpha!}{(n-1+|\beta|)!}, & \text{if } \alpha \leq \beta, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 7.5. *Let $\Omega \subset \mathbb{C}^n$ be a bounded open set with $\mathscr{C}^1$ boundary and $0 \in \Omega$. Then for multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ and $v \in \mathscr{C}^1(\overline{\Omega})$,*

$$\int_{\partial\Omega} v(z) H_{\alpha,\beta}(z) = a_{\alpha,\beta} \partial^{\beta-\alpha} v(0) + \int_{\Omega} \bar{\partial} v(z) \wedge H_{\alpha,\beta}(z). \tag{7.7}$$

Proof of Proposition 7.5. By Lemma 7.4 and Stokes' Theorem for currents, we have the following equations

$$\int_{\partial\Omega} v(z)H_{\alpha,\beta}(z) = \int_{\Omega} \mathrm{d}\Big(v(z)H_{\alpha,\beta}(z)\Big) = \int_{\Omega} \bar\partial\Big(v(z)H_{\alpha,\beta}(z)\Big)$$
$$= \int_{\Omega} v(z)\bar\partial H_{\alpha,\beta}(z) + \int_{\Omega} \bar\partial v(z) \wedge H_{\alpha,\beta}(z) = a_{\alpha,\beta}\partial^{\beta-\alpha}v(0) + \int_{\Omega} \bar\partial v(z) \wedge H_{\alpha,\beta}(z).$$

This completes the proof of Proposition 7.5. $\square$

Taking $\Omega = r\mathbb{B}_n$ and assuming $|\alpha| \geq |\beta|$ in Proposition 7.5 gives Lemma 7.1.

8. Appendix II: auxiliary functions and operations

In Section 3, the integral operations $\mathcal{F}^{(t)}, \mathcal{G}^{(t)}$ simplify our computation. To work in higher dimensions, it is necessary to extend those integral operations and establish some basic properties. This is the goal of the current section.

Definition 8.1. For $t \in \mathbb{R}$, denote

$$\phi_t(s) = (1-s)^t.$$

Suppose $\phi : (0,1) \to [0,\infty)$ is a measurable function. For a positive integer m and any $t > -1$, define the operations on ϕ

$$\mathcal{F}_m^{(t)}\phi(s) = \int_s^1 r^{m-1}\phi(r)(1-r)^t \mathrm{d}r \in [0,\infty], \tag{8.1}$$

and

$$\mathcal{G}_m^{(t)}\phi(s) = \frac{1}{s^m\phi_{t+1}(s)}\mathcal{F}_m^{(t)}\phi(s) = \frac{\int_s^1 r^{m-1}\phi(r)(1-r)^t \mathrm{d}r}{s^m(1-s)^{t+1}} \in [0,\infty]. \tag{8.2}$$

For any $t > -1$, inductively define the functions

$$\Phi_{n,0}^{(t)} \equiv 1, \quad \Phi_{n,k+1}^{(t)} = M_{\phi_1}\big(\mathcal{G}_{n+k}^{(t)}\big)^2\Phi_{n,k}^{(t)}.$$

Equivalently,

$$\Phi_{n,k}^{(t)} = M_{\phi_1}\big(\mathcal{G}_{n+k-1}^{(t)}\big)^2 \dots M_{\phi_1}\big(\mathcal{G}_n^{(t)}\big)^2 1. \tag{8.3}$$

It is straightforward to verify that the following estimates hold.

Lemma 8.2. *Suppose $a > m$ is not an integer and $b \geq 0$. Suppose $\phi : (0,1) \to [0,\infty)$ is measurable and*

$$\phi(s) \lesssim s^{-a}(1-s)^b.$$

Then

$$\mathcal{G}_m^{(t)}\phi(s) \lesssim s^{-a}(1-s)^b.$$

As a consequence, the following hold.

Lemma 8.3. *For any $t > -1$ and integers $n > 0, k \geq 0$,*

$$\Phi_{n,k}^{(t)}(s) \lesssim s^{-n-k+\frac{1}{2}}(1-s)^k, \tag{8.4}$$

and

$$\mathcal{G}_{n+k}^{(t)}\Phi_{n,k}^{(t)}(s) \lesssim s^{-n-k-\frac{1}{2}}(1-s)^k. \tag{8.5}$$

Lemma 8.4. *For any $t > -1$ and positive integers m, k and $\phi : (0,1) \to [0,\infty)$ we have*

$$\mathcal{F}_{m+k}^{(t)} M_{\phi_1} \mathcal{G}_m^{(t)}\phi(0) = \frac{1}{k}\mathcal{F}_{m+k}^{(t)}\phi(0), \tag{8.6}$$

$$\mathcal{F}_{m+k}^{(t)}\mathcal{G}_m^{(t)}\phi(0) = \sum_{j=0}^{\infty}\frac{1}{k+j}\mathcal{F}_{m+k+j}^{(t)}\phi(0), \tag{8.7}$$

$$\mathcal{F}_m^{(t)}\phi = \mathcal{F}_m^{(t)} M_{\phi_1}\phi(0) + \mathcal{F}_{m+1}^{(t)}\phi, \tag{8.8}$$

and

$$\mathcal{F}_m^{(t)} 1(0) = B(m, t+1). \tag{8.9}$$

Proof. The proof is a simple application of Fubini's Theorem. By definition, we have the following computation for $\mathcal{F}_{m+k}^{(t)} M_{\phi_1}\mathcal{G}_m^{(t)}\phi(0)$,

$$\begin{aligned}
&\mathcal{F}_{m+k}^{(t)} M_{\phi_1}\mathcal{G}_m^{(t)}\phi(0)\\
&= \int_0^1 r^{m+k-1}(1-r)^{t+1}\mathcal{G}_m^{(t)}\phi(r)\mathrm{d}r\\
&= \int_0^1 r^{k-1}\int_r^1 s^{m-1}(1-s)^t\phi(s)\mathrm{d}s\mathrm{d}r\\
&= \int_0^1\int_0^s r^{k-1}\mathrm{d}r s^{m-1}(1-s)^t\phi(s)\mathrm{d}s
\end{aligned}$$

$$=\frac{1}{k}\int_0^1 s^{m+k-1}(1-s)^t\phi(s)\mathrm{d}s$$
$$=\frac{1}{k}\mathcal{F}^{(t)}_{m+k}\phi(0).$$

This proves (8.6). We prove (8.7) by Lemma 8.3 using the expansion

$$\frac{1}{1-s}=\sum_{j=0}^{\infty}s^j.$$

Finally, we arrive at the following equation,

$$\mathcal{F}^{(t)}_m 1(0)=\int_0^1 r^{m-1}(1-r)^t\mathrm{d}r=B(m,t+1).$$

This completes the proof of Lemma 8.4. □

Data availability

No data was used for the research described in the article.

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