

On the Connes–Kasparov isomorphism, I

The reduced C^* -algebra of a real reductive group and the K -theory of the tempered dual

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Abstract. This is the first of two papers dedicated to the detailed determination of the reduced C^* -algebra of a connected, linear, real reductive group up to Morita equivalence, and a new and very explicit proof of the Connes–Kasparov conjecture for these groups using representation theory. In this part we shall give details of the C^* -algebraic Morita equivalence and then explain how the Connes–Kasparov morphism in operator K -theory may be computed using what we call the Matching Theorem, which is a purely representation-theoretic result. We shall prove our Matching Theorem in the sequel, and indeed go further by giving a simple, direct construction of the components

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of the tempered dual that have non-trivial K -theory using David Vogan's approach to the classification of the tempered dual.

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1. Introduction

If π is a unitary representation of a locally compact group G on a Hilbert space H , then the formula

$$(1.1) \quad \pi(f) = \int_G f(g)\pi(g) dg \quad (f \in C_c^\infty(G))$$

defines a representation of the group C^* -algebra $C^*(G)$ as bounded operators on H . In this way the category of unitary representations of G becomes equivalent to the category of (non-degenerate) representations of $C^*(G)$. See [Dix77, Chapter 13].

The C^* -algebra point of view equips the unitary dual of G with a topology whose closed sets are in bijection with two-sided ideals $J \triangleleft C^*(G)$: the closed set determined by J is the set of all irreducible unitary representations that vanish on J . The *reduced dual* of G is by definition the closed subset of the unitary dual that is associated to the kernel of the left regular representation

$$(1.2) \quad \lambda: C^*(G) \longrightarrow \mathfrak{B}(L^2(G)).$$

If G is a real reductive group, then the representations in the reduced dual are precisely Harish-Chandra's *tempered* irreducible unitary representations [HC66, Section 25]. See for instance [Kna86, Theorem 12.23] and [CHH88, Theorems 1 and 2], together with [CHH88, Remark (b), p. 103] for a proof of this. To a first approximation, the goals of this paper are to determine the tempered dual of a real reductive group as a topological space, and to compute its K -theory.

The tempered dual may be identified with the (topological) space of irreducible representations of the *reduced C^* -algebra* $C_r^*(G)$, which is the quotient of the full group C^* -algebra by the kernel of the regular representation (1.2). Our precise goals are to determine $C_r^*(G)$ up to Morita equivalence, and to compute its K -theory.

Studying the tempered dual as a topological space (and at the same time studying the reduced C^* -algebra) is rewarded in spectacular fashion by a beautiful isomorphism statement in K -theory that was conjectured

by A. Connes [Ros84, Conjecture 4.1] and G. Kasparov [Kas84, Section 5, Conjecture 1]. This *Connes–Kasparov isomorphism* is now viewed as part of the more general *Baum–Connes conjecture* about the reduced group C^* -algebra of any locally compact group (including any discrete group). See [BCH94, §4] or [GAJV19] for a recent survey of the status of the conjecture.

In this work we shall present the full details of a representation-theoretic proof of the Connes–Kasparov isomorphism for connected, linear, real reductive groups. Such a proof was announced in outline form only by A. Wassermann in [Was87], following pioneering work by M. Penington and R. Plymen [PP83] and A. Valette [Val84], [Val85]. Subsequently, V. Lafforgue gave an entirely new proof using his work on the Baum–Connes conjecture [Laf02b]. In some places we shall follow Wassermann’s outline, but elsewhere we shall follow a quite different route.

Lafforgue explains in [Laf02a, §2] that if G is of equal rank and has compact center, so that it possesses discrete series representations, then one can recover Harish-Chandra’s classification of the discrete series in terms of Harish-Chandra parameters as a consequence of the Connes–Kasparov isomorphism for G . The starting point is the observation that each discrete series is isolated in the tempered dual, and so is detectable in K -theory. Our approach will in effect use Harish-Chandra’s classification, rather than provide an independent verification. But in place of that we shall give an explicit answer to a natural question that arises from the classification. Harish-Chandra’s parameters are weights (for a chosen maximal torus of a maximal compact subgroup K of G) that satisfy a non-singularity condition. The same parameters determine Dirac-type operators on the symmetric space G/K , and as R. Parthasarathy [Par72] and Atiyah–Schmid [AS77] explained, the associated discrete series can be constructed as the space of harmonic spinors for this operator. If Harish-Chandra’s non-singularity condition is dropped, there is *still* an associated Dirac operator. We shall show that this Dirac operator determines not a single representation of G but a single *connected component* of the tempered dual, and we shall describe this component in full detail, thereby in some sense completing Harish-Chandra’s parametrization.

We shall reach this goal in the second paper. Our starting point here is an earlier paper [CCH16] that gave a detailed account of the structure of $C_r^*(G)$, in the form of a Paley–Wiener type theorem for the reduced group C^* -algebra. Here we shall go one step further and give a computation of the K -theory of the reduced C^* -algebra using a dichotomy, first observed by Wassermann in [Was87], that invokes the Knapp–Stein theory of intertwining operators [KS71], [KS80] to separate the components of the tempered dual with trivial K -theory from those with non-trivial

K -theory. The result is that the K -theory of the tempered dual is carried by those components that are *essential*, which means that every Knapp–Stein intertwiner for the component belongs to the Knapp–Stein R -group. See Definition 4.1. Moreover each essential component contributes one free generator to K -theory.

On the other hand, Connes and Kasparov conjectured that the K -theory is freely generated by the indices of indecomposable Dirac-type operators on the symmetric space G/K , where K is a maximal compact subgroup in G . These Dirac operators are easily parametrized using more or less just the representation theory of K . In contrast the set of essential components of the tempered dual is far more mysterious.

Nevertheless, it is a remarkable fact that the K -theory generators associated to the essential components and the K -theory generators associated to indecomposable Dirac operators are, up to sign, *exactly the same*. We shall conclude the present paper by precisely formulating a slightly weaker correspondence in what we call the *Matching Theorem*; see Theorem 6.3. Finally, we shall explain how the Connes–Kasparov isomorphism follows quickly from the Matching Theorem. We shall present two arguments—one that relies on fundamental results by Kasparov in KK -theory, and one that is purely representation theoretic.

In the second paper we shall complete our account of the K -theory of the tempered dual and the Connes–Kasparov isomorphism by proving the Matching Theorem, and more. In contrast to earlier works on the Connes–Kasparov isomorphism, we shall use David Vogan’s approach to the construction and classification of the tempered dual [Vog81]. We shall show that Vogan’s theory leads to a simple construction of all the essential components of the tempered dual, and only those components, all at once. Although Dirac operators are not used in our construction, the data used to construct an essential component turns out to be exactly the same as the data used to construct an indecomposable Dirac operator. From the point of view of representation theory, this is the fundamental result underlying the Connes–Kasparov isomorphism.

Notes on Terminology

Throughout this paper and the sequel, by a *real reductive group*, we shall always mean the group G of real points in a connected complex reductive linear algebraic group that is defined over \mathbb{R} . See for instance [Mil17, Chapter 19]. The main reason for this assumption is to guarantee that the theory developed in Vogan’s monograph [Vog81], which will be crucial in the sequel, will apply to G . But *we shall also assume that G is itself connected*, which will considerably simplify both the statements of theorems

and their proofs. In this paper we shall often refer to the text [Kna86], so let us note here that our groups are the same (up to isomorphism) as the linear connected reductive groups in [Kna86, §I.1].

We shall use fraktur letters such as \mathfrak{g} , etc, to refer to the Lie algebras of Lie groups such as G , etc, and not to the complexifications of these Lie algebras. This is because we shall have little use for the complexified Lie algebras in this first paper. But in the second paper we shall use the complexifications extensively and we shall follow a different convention.

When discussing Dirac operators we shall follow conventions appropriate to index theory on manifolds. These are a bit different from the conventions in representation theory, where so-called Dirac cohomology is studied. But in the second paper we shall switch and follow the Dirac operator conventions that are used in representation theory.

2. Parabolic induction and the reduced group C^* -algebra

In this section we shall review the description of the reduced C^* -algebra of a real reductive group that was obtained in [CCH16] using results in tempered representation theory due to Harish-Chandra, R. Langlands and others. Then, following Wassermann [Was87], we shall refine that description so as to determine the reduced group C^* -algebra up to Morita equivalence.

We shall fix, once and for all in this paper, a maximal compact subgroup $K \subseteq G$ and Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. We shall also fix a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{s}$ and a compatible Iwasawa decomposition $G = KAN$. The associated minimal parabolic subgroup is $P_{\min} = MAN$, where M is the centralizer of A in K . See [Kna86, Chapter V].

Since we shall be working with convolution algebras we shall also fix a Haar measure on G , as well as a normalized Haar measure on K .

Parabolic induction

We begin by reviewing some essential points about parabolic induction [CCH16]. A standard parabolic subgroup of G is any closed subgroup P of G that includes P_{\min} . It decomposes as a semi-direct product $P = L_P N_P$ of a Levi component L_P that is mapped to itself by the Cartan involution and the unipotent radical N_P . Furthermore, the Levi component L_P is the product of the closed subgroup M_P that is generated by all compact subgroups of L_P and the split component A_P of L_P . This leads to a Langlands decomposition $P = M_P A_P N_P$. See [Kna86, §V.5] or [Kna02, Chapter VII].

If π is a unitary representation of L_P , then we may form the (*unitarily parabolically induced representation*) $\text{Ind}_P^G \pi$, which is the unitary action of G by left translation on the Hilbert space completion of the vector space of smooth functions

$$(2.1) \quad \{f: G \longrightarrow H_\pi : f(gman) = e^{-\rho(\log a)} \pi(ma)^{-1} f(g)\}.$$

The completion is taken with respect to the inner product

$$(2.2) \quad \langle f_1, f_2 \rangle = \int_K \langle f_1(k), f_2(k) \rangle dk,$$

and $\rho \in \mathfrak{a}_P^*$ is defined by

$$\rho(X) = \frac{1}{2} \text{Trace}(\text{ad}_X: \mathfrak{n}_P \longrightarrow \mathfrak{n}_P).$$

See [Kna86, §VII.1].

2.3 Definition. Let P be a standard parabolic subgroup and let σ be an irreducible, discrete series representation of M_P (that is, an irreducible unitary representation of M_P , all of whose matrix coefficient functions are square-integrable) and let $\varphi \in \mathfrak{a}_P^*$. The formula

$$\sigma \otimes e^{i\varphi}: ma \longmapsto e^{i\varphi(\log a)} \sigma(m)$$

defines a unitary representation of $L_P = M_P A_P$ on the Hilbert space of the representation σ . The associated (P, σ) -principal series representation of G is the unitary representation $\pi_{\sigma, \varphi}$ that is obtained from $\sigma \otimes e^{i\varphi}$ by parabolic induction. We shall denote by $\text{Ind}_P^G H_\sigma \otimes \mathbb{C}_{i\varphi}$ the Hilbert space on which it acts.

2.4 Remark. The Langlands decomposition $P = MAN$, which is used in the definition above, will be important at a number of places below. It supplies each principal series of representations with a natural base point, where $\varphi = 0$. This base point is not always available in other contexts, including for instance that of p -adic groups, and the paper [AA22] examines some of the difficulties that can arise as a result.

If we restrict the functions in (2.1) to K , then all the representations in the (P, σ) -principal series can be regarded as acting on the following common Hilbert space (and we shall mostly do so from now on):

2.5 Definition. We shall denote by $\text{Ind}_P^G H_\sigma$ the Hilbert space completion of the space of smooth functions

$$(2.6) \quad \{f: K \longrightarrow H_\sigma : f(kh) = \sigma(h)^{-1} f(k) \ \forall h \in K \cap L, \forall k \in K\}$$

in the inner product (2.2).

Intertwining operators

The following concept gives us a first, large-scale view of the tempered dual:

2.7 Definition. *Let $P_1 = M_1 A_1 N_1$ and $P_2 = M_2 A_2 N_2$ be standard parabolic subgroups and let σ_1, σ_2 be irreducible square-integrable representations of M_1 and M_2 respectively. The pairs (P_1, σ_1) and (P_2, σ_2) are associate if there exists an element $k \in K$ such that*

$$\text{Ad}_k[L_{P_1}] = L_{P_2} \quad \text{and} \quad \text{Ad}_k^* \sigma_1 \simeq \sigma_2.$$

We shall call an equivalence class of pairs under this relation an associate class and use the notation $[P, \sigma]$.

The theorem below summarizes some of the important results of Harish-Chandra, Langlands and others, and it is the foundation for our study of the tempered dual.

2.8 Theorem. *The tempered dual admits a disjoint union decomposition*

$$\widehat{G}_{\text{temp}} = \bigsqcup_{[P, \sigma]} \widehat{G}_{P, \sigma}$$

as a topological space, where $\widehat{G}_{P, \sigma}$ consists of the irreducible components of the (P, σ) -principal series representations and the union is indexed by associate classes. Each part $\widehat{G}_{P, \sigma}$ is a connected and open subset of $\widehat{G}_{\text{temp}}$ (and it follows that each part is also closed).

As for the proof, it is explained by Lipsman in [Lip70] that for semisimple groups with finite center, the topology on each minimal principal series component of the tempered dual coincides with the “natural topology” inherited from the space of continuous parameters of the principal series, and in particular each principal series is connected. The general case of Lipsman’s result (covering all reductive groups and all cuspidal principal series) may be proved in the same way. But in any case, a complete proof of this and all the other assertions in Theorem 2.8 is given in [CCH16]. Theorem 6.8 in [CCH16], which is reproduced below as Theorem 2.19, gives a nearly complete description of the reduced C^* -algebra of G , using which the spectrum of the reduced C^* -algebra is easily determined. But that spectrum is precisely the tempered dual, as we noted in the Introduction. So for instance, the fact that each principal series component is open and closed in the tempered dual follows from the direct sum decomposition in Theorem 6.8 of [CCH16].

To probe the equivalences within a single principal series family, as well as the possible reducibility of the representations within that family, one studies intertwining operators.

2.9 Definition. *The intertwining group associated with a pair (P, σ) as above is the finite group*

$$W_\sigma = \{w \in N_K(L_P) : \text{Ad}_w^* \sigma \simeq \sigma\} / K \cap L_P.$$

The theory of intertwining operators, due to Knapp and Stein [KS71], [KS80], associates to each $w \in W_\sigma$, and each $\varphi \in \mathfrak{a}_P^*$, a unitary operator

$$(2.10) \quad U_{w,\varphi} : \text{Ind}_P^G H_\sigma \longrightarrow \text{Ind}_P^G H_\sigma,$$

that intertwines the principal series representations $\pi_{\sigma,\varphi}$ and $\pi_{\sigma,w(\varphi)}$, so that if $g \in G$, then

$$(2.11) \quad U_{w,\varphi} \pi_{\sigma,\varphi}(g) = \pi_{\sigma,w(\varphi)}(g) U_{w,\varphi}.$$

Here $w(\varphi)(X) = \varphi(\text{Ad}_{w^{-1}}(X))$. The operators $U_{w,\varphi}$ vary strongly-continuously with $\varphi \in \mathfrak{a}_P^*$.

The construction of $U_{w,\varphi}$ involves a choice of unitary equivalence of representations $\text{Ad}_w^* \sigma \simeq \sigma$. By Schur's lemma the choice is unique up to a multiplicative scalar of modulus one, but it is not absolutely unique. This leads to a cocycle relation

$$(2.12) \quad U_{w_1, w_2(\varphi)} U_{w_2, \varphi} = c(w_1, w_2) U_{w_1 w_2, \varphi} \quad \forall \varphi \in \mathfrak{a}_P^*$$

with $|c(w_1, w_2)| = 1$.

In fact the equivalences $\text{Ad}_w^* \sigma \simeq \sigma$ may be chosen so that $c(w_1, w_2) = 1$ for all $w_1, w_2 \in W_\sigma$. This is not trivial, but it is important for what follows, so let us describe the method.

2.13 Definition ([KS80, §13]). *Denote by $W'_\sigma \triangleleft W_\sigma$ the normal subgroup*

$$W'_\sigma = \{w' \in W_\sigma : \text{The intertwiner } U_{w',0} \text{ acts as a scalar on } \text{Ind}_P^G H_\sigma\}.$$

Denote by R_σ the quotient group W_σ / W'_σ .

For each $w' \in W'_\sigma$ we can choose an equivalence $\text{Ad}_{w'}^* \sigma \simeq \sigma$ so that in fact each $U_{w',0}$ acts as the identity operator on $\text{Ind}_P^G H_\sigma$, and then having done so we obtain $c(w'_1, w'_2) = 1$ for all $w'_1, w'_2 \in W'_\sigma$.

2.14 Theorem ([KS80, Theorem 13.4]). *The quotient group homomorphism from W_σ to R_σ splits, and the intertwining group W_σ therefore admits a semi-direct product decomposition $W_\sigma = W'_\sigma \rtimes R_\sigma$.*

2.15 Theorem ([KS80, §13 and §15] and [Kna82, §6]). *The R -group is abelian, and indeed a finite product of groups of order two. Moreover there is a splitting, as in the previous theorem, so that the associated intertwiners $U_{w,0}$ ($w \in R_\sigma$) also pairwise commute.*

Returning to the cocycle issue, fix a splitting as in Theorem 2.15. Since R_σ is a direct product of groups of order two, and since the associated Knapp–Stein intertwiners commute, we can certainly choose equivalences $\text{Ad}_w^* \sigma \simeq \sigma$ for $w \in R_\sigma$ so that $c(w_1, w_2) = 1$ in (2.12), for all $w_1, w_2 \in R_\sigma$. If we now make further adjustments by scalars so that $U_{w,0}U_{w',0} = U_{ww',0}$ for all $w \in R_\sigma$ and all $w' \in W'_\sigma$, then we shall obtain $c(w_1, w_2) \equiv 1$ for all $w_1, w_2 \in W_\sigma$, as required. We shall use these adjusted Knapp–Stein intertwining operators from now on.

The Knapp–Stein intertwining operators completely account for the decomposition of principal series representations into irreducible representations, and for equivalences among these irreducible summands. We refer to [CCH16, §6] for a summary that is adapted to our purposes; the same information will be encoded in the description of the reduced C*-algebra in Theorem 2.19 below.

The reduced group C-algebra*

Let $P = M_P A_P N_P$ be a parabolic subgroup of G , and let σ be an irreducible square-integrable representation of M_P . The (P, σ) -principal series representations are tempered, and they therefore determine representations $\pi_{\sigma, \varphi}$ of the reduced group C*-algebra using formula (1.1). We now introduce the C*-algebra

$$(2.16) \quad C_0(\mathfrak{a}_P^*, \mathfrak{K}(\text{Ind}_P^G H_\sigma))$$

of norm-continuous functions, vanishing in norm at infinity, from the locally compact space \mathfrak{a}_P^* to the C*-algebra of compact operators on the Hilbert space $\text{Ind}_P^G H_\sigma$.

2.17 Proposition ([CCH16, Corollary 4.12]). *There is a (unique) C*-algebra homomorphism*

$$\pi_\sigma: C_r^*(G) \longrightarrow C_0(\mathfrak{a}_P^*, \mathfrak{K}(\text{Ind}_P^G H_\sigma))$$

such that $\pi_\sigma(f)(\varphi) = \pi_{\sigma, \varphi}(f)$ for every $f \in C_c^\infty(G)$ and every $\varphi \in \mathfrak{a}_P^*$.

2.18 Remark. In [CCH16], the right-hand side is described in terms of functions on \widehat{A}_P , which identifies with \mathfrak{a}_P^* through the exponential map as in Definition 2.3.

There is an action of the intertwining group W_σ on the C*-algebra (2.16) that is characterized by the formula

$$w(f)(w(\varphi)) = U_{w, \varphi} f(\varphi) U_{w, \varphi}^*$$

for all $w \in W_\sigma$ and all $\varphi \in \mathfrak{a}_P^*$. It follows from the intertwining property (2.11) that the image of the morphism π_σ in Proposition 2.17 is fixed pointwise by this action of W_σ . The description of the reduced C^* -algebra given in [CCH16] is as follows:

2.19 Theorem ([CCH16, Theorem 6.8]). *The morphisms in Proposition 2.17 combine to give an isomorphism of C^* -algebras*

$$C_r^*(G) \xrightarrow{\cong} \bigoplus_{[P,\sigma]} C_0(\mathfrak{a}_P^*, \mathfrak{K}(\text{Ind}_P^G H_\sigma))^{W_\sigma}.$$

The direct sum is the C_0 -direct sum of C^ -algebras over a choice of representatives of the associate classes $[P,\sigma]$.*

The principal series as an equivariant bundle

In this section we shall show, following Wassermann [Was87], that the W_σ -action that is used to define the fixed-point algebra Theorem 2.19 can be replaced by a much simpler action in a way that does not change those fixed-point algebra, up to $*$ -isomorphism.

Form the trivial bundle of Hilbert spaces with fiber $\text{Ind}_P^G H_\sigma$ over the locally compact space \mathfrak{a}_P^* . The Knapp–Stein intertwiners determine an action on this bundle,

$$(2.20) \quad W_\sigma \times (\mathfrak{a}_P^* \times \text{Ind}_P^G H_\sigma) \longrightarrow (\mathfrak{a}_P^* \times \text{Ind}_P^G H_\sigma),$$

via the formula

$$(2.21) \quad w \cdot (\varphi, v) = (w(\varphi), U_{w,\varphi} v).$$

We shall now give a simpler description, up to isomorphism, of this W_σ -equivariant Hilbert space bundle.

2.22 Remark. The bundle $\mathfrak{a}_P^* \times \text{Ind}_P^G H_\sigma$ is infinite-dimensional, but it decomposes canonically as the orthogonal Hilbert direct sum of its finite-dimensional K -isotypic components, which are finite-dimensional W_σ -equivariant bundles in their own right. One could, if one preferred, work with these finite-dimensional bundles.

Define a second W_σ -action on the bundle $\mathfrak{a}_P^* \times \text{Ind}_P^G H_\sigma$ by the formula

$$(2.23) \quad w \cdot (\varphi, v) = (w(\varphi), U_{w,0} v).$$

2.24 Proposition (cf. [Was87, Corollary 5]). *The two W_σ -equivariant bundle structures on $\mathfrak{a}_P^* \times \text{Ind}_P^G H_\sigma$ defined by the two actions (2.21) and (2.23) are unitarily equivalent.*

Proof. The single-point subset $\{0\} \subseteq \mathfrak{a}_P^*$ is a W_σ -equivariant deformation retract. It therefore follows from elementary vector bundle theory that any two W_σ -equivariant bundles over \mathfrak{a}_P^* whose fibers over 0 are unitarily equivalent as representations of W_σ are in fact unitarily equivariantly isomorphic as bundles. \square

2.25 Corollary. *The W_σ -actions on the C^* -algebra $C_0(\mathfrak{a}_P^*, \mathfrak{K}(\text{Ind}_P^G H_\sigma))$ defined by the formulas (2.21) and (2.23) are conjugate by a C^* -algebra automorphism. In particular, the corresponding fixed-point C^* -subalgebras are isomorphic.*

The reduced C^ -algebra up to Morita equivalence*

In this section we shall construct a Morita equivalence between each summand in the decomposition of Theorem 2.19 and a still more elementary C^* -algebra. We shall continue to follow Wassermann [Was87] closely.

A complete treatment of the Morita equivalence has appeared very recently in [AA22], where many situations involving p -adic groups are also considered. For the sake of completeness we shall nonetheless present the argument below, although in view of the existence of [AA22] we shall take the liberty of omitting some details.

We shall use Corollary 2.25, and throughout this subsection we shall work with the action of W_σ on the C^* -algebra $C_0(\mathfrak{a}_P^*, \mathfrak{K}(\text{Ind}_P^G H_\sigma))$ that is derived from (2.23). We shall determine the fixed-point C^* -subalgebra up to Morita equivalence.

The group R_σ acts on the C^* -algebra $C_0(\mathfrak{a}_P^*/W'_\sigma, \mathfrak{K}(\text{Ind}_P^G H_\sigma))$ via the formula

$$(r \cdot f)([\varphi]) = U_{w,0} f([w^{-1}(\varphi)]) U_{w,0}^*,$$

where w is any preimage in W_σ of $r \in R_\sigma$. The morphism

$$(2.26) \quad C_0(\mathfrak{a}_P^*, \mathfrak{K}(\text{Ind}_P^G H_\sigma))^{W_\sigma} \xrightarrow{\cong} C_0(\mathfrak{a}_P^*/W'_\sigma, \mathfrak{K}(\text{Ind}_P^G H_\sigma))^{R_\sigma}$$

defined by the formula

$$f \longmapsto [[\varphi] \longmapsto f(\varphi)]$$

is an isomorphism of C^* -algebras. We shall therefore concentrate on the R_σ -fixed point C^* -algebra.

Now form the space $\mathfrak{K}(\text{Ind}_P^G H_\sigma, \ell^2 R_\sigma)$ of compact Hilbert space operators from $\text{Ind}_P^G H_\sigma$ into the finite-dimensional Hilbert space $\ell^2 R_\sigma$. Use the action of R_σ on $\text{Ind}_P^G H$, along with the left-translation action of R_σ on $\ell^2 R_\sigma$, to define an R_σ -action on $\mathfrak{K}(\text{Ind}_P^G H_\sigma, \ell^2 R_\sigma)$.

Finally, form the Banach space

$$(2.27) \quad C_0(\mathfrak{a}_P^*/W'_\sigma, \mathfrak{K}(\text{Ind}_P^G H_\sigma, \ell^2 R_\sigma))^{R_\sigma}.$$

It carries commuting actions of the C^* -algebra $C_0(\mathfrak{a}_P^*/W'_\sigma, \mathfrak{K}(\text{Ind}_P^G H_\sigma))^{R_\sigma}$ on the right, by pointwise composition, and of $C_0(\mathfrak{a}_P^*/W'_\sigma, \mathfrak{K}(\ell^2 R_\sigma))^{R_\sigma}$ on the left.

2.28 Theorem (cf. [Was87, Corollary 7]). *For each associate class $[P, \sigma]$, the bimodule*

$$C_0(\mathfrak{a}_P^*/W'_\sigma, \mathfrak{K}(\text{Ind}_P^G H_\sigma, \ell^2 R_\sigma))^{R_\sigma}$$

implements a strong Morita equivalence

$$C_0(\mathfrak{a}_P^*/W'_\sigma, \mathfrak{K}(\ell^2 R_\sigma))^{R_\sigma} \underset{\text{Morita}}{\simeq} C_0(\mathfrak{a}_P^*/W'_\sigma, \mathfrak{K}(\text{Ind}_P^G H_\sigma))^{R_\sigma}.$$

Let us quickly recall the C^* -algebraic concept of Morita equivalence, which includes analytic requirements that are obviously absent from the purely algebraic theory (among other things, they help extending the reach of the theory to non-unital C^* -algebras). A succinct formulation is as follows: an equivalence A - B -bimodule must have the form pCp^\perp , where

- (i) C is a C^* -algebra and p is a projection in the multiplier algebra of C [Ped79, §3.12];
- (ii) C^* -algebra isomorphisms are provided between pCp and A , and between $p^\perp C p^\perp$ and B (where $p^\perp = 1 - p$); and
- (iii) $pCp^\perp C p$ and $p^\perp C p C p^\perp$ are dense in pCp and $p^\perp C p^\perp$, respectively.

See [RW98]. In the present case, C will be the C^* -algebra of R_σ -fixed functions of class C_0 from $\mathfrak{a}_P^*/W'_\sigma$ to the C^* -algebra of compact operators on the direct sum Hilbert space $\text{Ind}_P^G H_\sigma \oplus \ell^2 R_\sigma$, and p will be the projection onto the second Hilbert space direct summand, and then pCp^\perp will be (2.27).

2.29 Lemma. *Let Γ be a finite group acting properly on a locally compact Hausdorff space X , and let H_1 and H_2 be Hilbert spaces equipped with unitary representations of Γ . If for every $x \in X$, H_1 and H_2 are weakly equivalent representations of the stabilizer subgroup Γ_x (that is, each is contained in a multiple of the other), then the bimodule*

$$C_0(X, \mathfrak{K}(H_2, H_1))^\Gamma$$

implements a Morita equivalence of C^* -algebras

$$C_0(X, \mathfrak{K}(H_1))^\Gamma \underset{\text{Morita}}{\simeq} C_0(X, \mathfrak{K}(H_2))^\Gamma.$$

Proof. Denote the two C^* -algebras in the statement of the proposition by A and B , and the A - B -bimodule by E . We need to show that the sets

$$\{fg^* : f, g \in E\} \quad \text{and} \quad \{f^*g : f, g \in E\}$$

span dense ideals in A and B , respectively.

If an ideal in a C^* -algebra is *not* dense, then there is an irreducible representation of the C^* -algebra that vanishes on the ideal [Dix77, Chapter 2]. So to prove density in A we need only show that for every irreducible representation of A there is some element $f \in E$ such that the representation is non-zero on $ff^* \in A$.

Each irreducible representation of A must factor through evaluation of functions A in at some point $x \in X$, since by Schur's lemma all functions in $C_0(X)^\Gamma$ must act on the representation space as scalar multiples of the identity. But the image of A under evaluation at $x \in X$ is $\mathfrak{K}(H_1)^{\Gamma_x}$, and its irreducible representations are precisely the non-zero Γ_x -isotypical subspaces H_1^ρ of H_1 ($\rho \in \widehat{\Gamma}_x$).

By hypothesis, the isotypical subspace H_2^ρ is non-zero, and hence there is a non-zero Γ_x -equivariant compact operator $T : H_2 \rightarrow H_1$ whose range lies in H_1^ρ , and there is a function $f \in E$ whose value at x is T . But now the value of $ff^* \in A$ at x is equal to TT^* , which is non-zero. \square

We shall need to combine the simple computation above with the following more substantial result from the Knapp–Stein theory:

2.30 Theorem ([Kna86, Theorem 14.43]). *The intertwining operators*

$$U_{w,0} : \text{Ind}_P^G H_\sigma \longrightarrow \text{Ind}_P^G H_\sigma \quad (w \in R_\sigma)$$

are linearly independent of one another.

2.31 Corollary. *The representation of R_σ on $\text{Ind}_P^G H_\sigma$ includes a copy of every irreducible representation of R_σ .*

Proof. The representation of R_σ on $\text{Ind}_P^G H_\sigma$ determines a representation of the complex group algebra of R_σ , and Theorem 2.30 implies that this algebra representation is faithful. That is, every element of the group algebra acts as a non-zero operator. So each of the isotypical projections associated to the irreducible representations of R_σ acts as a non-zero operator on $\text{Ind}_P^G H_\sigma$. \square

Proof of Theorem 2.28. It follows from the corollary above that $\text{Ind}_P^G H_\sigma$ includes a copy of every irreducible representation of every subgroup of R_σ , and certainly the same is true of $\ell^2 R_\sigma$. So Lemma 2.29 applies with $X = \mathfrak{a}_P^*/W'_\sigma$ and $\Gamma = R_\sigma$. \square

We shall conclude this section by showing how the statement of Theorem 2.28 can be streamlined using some standard C^* -algebra language (although we shall not use this language in what follows).

Let Γ be a finite group. Denote by λ and ρ the actions of Γ on $\mathfrak{K}(\ell^2\Gamma)$ associated with the left and right regular representations, respectively. In addition, if $\gamma \in \Gamma$, then denote by e_γ the rank-one projection onto the functions in $\ell^2\Gamma$ that are supported on γ . If A is any C^* -algebra with a Γ -action, then we denote by $A \rtimes \Gamma$ the crossed product C^* -algebra.

2.32 Lemma ([Rie80, Proposition 4.3]). *The linear map*

$$A \rtimes \Gamma \longrightarrow (A \otimes \mathfrak{K}(\ell^2\Gamma))^{\Gamma, \lambda}$$

defined by

$$a \longmapsto \sum_{\gamma \in \Gamma} \gamma(a) \otimes e_\gamma \quad \text{and} \quad \gamma \longmapsto 1 \otimes \rho(\gamma),$$

is an isomorphism of C^* -algebras.

Combining Lemma 2.32 with Theorem 2.28, we obtain for any component $[P, \sigma]$ of the tempered dual a Morita equivalence

$$(2.33) \quad C_0(\mathfrak{a}^*, \mathfrak{K}(\text{Ind}_P^G H_\sigma))^{W_\sigma} \underset{\text{Morita}}{\simeq} C_0(\mathfrak{a}_P^*/W'_\sigma) \rtimes R_\sigma.$$

Assembling the summands using the isomorphism of Theorem 2.19, we obtain the following picture of the reduced C^* -algebra up to Morita equivalence, due to Wassermann [Was87].

2.34 Theorem ([Was87, Theorem 8]). *There is a Morita equivalence of C^* -algebras*

$$C_r^*(G) \underset{\text{Morita}}{\simeq} \bigoplus_{[P, \sigma]} C_0(\mathfrak{a}_P^*/W'_\sigma) \rtimes R_\sigma,$$

where the sum is over representatives of the associate classes $[P, \sigma]$.

3. Further information about the Knapp–Stein intertwining groups

The results in the preceding section give an account of the structure of $C_r^*(G)$ up to Morita equivalence in terms of the intertwining groups W_σ and their semi-direct product decompositions $W_\sigma = W'_\sigma \rtimes R_\sigma$. In this section we shall summarize the additional facts about these decompositions that we shall need to complete the computations in this paper.

The W' -group

We defined W'_σ using the action of the Knapp–Stein intertwining operators on the representations $\pi_{\sigma,0}$. An important result is that W'_σ is also the Weyl group of a root system:

3.1 Theorem ([KS80, §13 and §15] and [Kna82, §6]). *The subgroup $W'_\sigma \triangleleft W_\sigma$ is the Weyl group of a (possibly non-reduced) root system Δ'_σ spanning a subspace¹ of \mathfrak{a}_P^* . The action of the group W_σ on \mathfrak{a}_P^* permutes the roots in Δ'_σ .*

3.2 Definition. *We shall denote by*

$$\mathfrak{a}_{\sigma,+}^* \subseteq \mathfrak{a}_P^*$$

the (closed) dominant Weyl chamber in \mathfrak{a}_P^ associated to some fixed system of positive roots $\Delta'_{\sigma,+} \subseteq \Delta'_\sigma$.*

See [Kna86, Chapter XIV, Section 9] for the definition of the root system Δ'_σ . One important consequence of Theorem 3.1 for us will be that the quotient $\mathfrak{a}_P^*/W'_\sigma$ may be identified with the dominant chamber $\mathfrak{a}_{\sigma,+}^*$ (in more detail, we shall use the fact that the projection map from the closed dominant chamber to $\mathfrak{a}_P^*/W'_\sigma$ is a homeomorphism).

The R -group

Using the system of positive roots, Knapp and Stein define R_σ as a subgroup of W_σ , as follows:

3.3 Definition. *The Knapp–Stein R -group $R_\sigma \subseteq W_\sigma$ is the subgroup consisting of those elements that permute the positive roots $\Delta'_{\sigma,+} \subseteq \Delta'_\sigma$ among themselves.*

This is consistent with our previous terminology: the subgroup R_σ normalizes W'_σ , and since W_σ acts by permutations on the Weyl chambers in \mathfrak{a}_P^* for the root system Δ'_σ , while W'_σ acts on the chambers simply-transitively, there is a semi-direct product decomposition $W_\sigma = W'_\sigma \rtimes R_\sigma$.

Since R_σ permutes the positive roots among themselves, the action of R_σ on \mathfrak{a}_P^* restricts to an action

$$R_\sigma \times \mathfrak{a}_{\sigma,+}^* \longrightarrow \mathfrak{a}_{\sigma,+}^*.$$

¹ To be precise, there is an isomorphism from W'_σ to the Weyl group of a root system Δ'_σ spanning a subspace, and the isomorphism gives the action of W'_σ on that subspace. There is a complementary subspace on which the action of W'_σ is trivial.

We shall use this action in the next section.

By a *reflection* of \mathfrak{a}_P^* we shall mean an isometric involution of \mathfrak{a}_P^* with a one-dimensional (-1) -eigenspace. Two reflections are *orthogonal* if their (-1) -eigenspaces are orthogonal.

3.4 Theorem ([KS80, §13 and §15] and [Kna82, §6]). *The R -group associated to every associate class $[P, \sigma]$ is a finite product of groups of order two that act by pairwise orthogonal reflections on \mathfrak{a}_P^* .*

Finally, we shall need the size of the group R_σ in a crucial special case.

3.5 Definition. Denote by \mathfrak{a}_{\max} the split part of a maximally compact Cartan subalgebra of $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. Thus \mathfrak{a}_{\max} is the fixed part in \mathfrak{s} of the action of a maximal torus in K .

The space \mathfrak{a}_{\max} is unique up to conjugation by elements of K , and so its dimension $\dim(\mathfrak{a}_{\max})$ is independent of any choices.

3.6 Lemma. $\dim(\mathfrak{a}_{\max}) \equiv \dim(G/K) \pmod{2}$.

Proof. The action of the maximal torus associated to \mathfrak{a}_{\max} on $\mathfrak{s} \ominus \mathfrak{a}_{\max}$ has no non-zero fixed vectors. Since every non-trivial irreducible representation of the torus has dimension 2, it follows that $\mathfrak{s} \ominus \mathfrak{a}_{\max}$ is even-dimensional. \square

3.7 Theorem. *If $[P, \sigma]$ is an associate class, and if $W'_\sigma = \{e\}$, then the group R_σ is generated by $\dim(\mathfrak{a}_P) - \dim(\mathfrak{a}_{\max})$ pairwise orthogonal reflections on \mathfrak{a} .*

We refer the reader to [CHS22] for a proof using an alternative approach to the R -group due to Vogan [Vog81, Section 4.3] (which seems to be much better suited to the problem of computing R_σ in the essential case).

4. K-Theory of the reduced C^* -algebra

In this section we shall compute the K -theory [RLL00] of $C_r^*(G)$ as an abstract abelian group. Since it is a basic feature of K -theory that for any family of C^* -algebras $\{A_\alpha\}$ the natural map

$$\bigoplus_{\alpha} K_*(A_\alpha) \longrightarrow K_*(\bigoplus_{\alpha} A_\alpha)$$

is an isomorphism, we can and shall focus on the individual fixed-point algebras

$$C_0(\mathfrak{a}_P^*, \mathfrak{K}(\text{Ind}_P^G H_\sigma))^{W_\sigma}$$

that make up the reduced group C^* -algebra. We have seen that these are Morita equivalent to the C^* -algebras

$$C_0(\mathfrak{a}_P^*/W'_\sigma, \mathfrak{K}(\ell^2 R_\sigma))^{R_\sigma}.$$

Since K -theory is a Morita invariant it suffices to study the latter.

The computations below are very simple from a K -theoretic point of view, but they require the difficult results about the R -group that we surveyed in the last section.

Essential and inessential components

The results in this section are due to Wassermann [Was87]. We start from the following partition of the set of associate classes $[P, \sigma]$.

4.1 Definition. *An associate class $[P, \sigma]$ is called **essential** if the normal subgroup $W'_\sigma \triangleleft W_\sigma$ is trivial. Otherwise $[P, \sigma]$ is called **inessential**.*

4.2 Theorem. *If $[P, \sigma]$ is inessential, then $K_*(C_0(\mathfrak{a}_P^*/W'_\sigma, \mathfrak{K}(\ell^2 R_\sigma))^{R_\sigma}) = 0$.*

Proof. Identify the quotient $\mathfrak{a}_P^*/W'_\sigma$ with the dominant Weyl chamber $\mathfrak{a}_{\sigma,+}^* \subseteq \mathfrak{a}_P^*$. The half-sum of the positive roots is a non-zero vector ρ in the chamber that is fixed under the action of R_σ . The translations by non-negative multiples of ρ map $\mathfrak{a}_{\sigma,+}^*$ into itself and give an R_σ -equivariant homotopy between the identity morphism on the C^* -algebra $C_0(\mathfrak{a}_{\sigma,+}^*, \mathfrak{K}(\ell^2 R_\sigma))$ and the zero morphism. So the R_σ -fixed-point algebra is homotopy equivalent to zero. \square

The essential components have non-zero K -theory, and their treatment requires more of the R -group results from Sect. 3.

4.3 Theorem. *If $[P, \sigma]$ is essential, then $K_*(C_0(\mathfrak{a}_P^*, \mathfrak{K}(\ell^2 R_\sigma))^{R_\sigma})$ is a free abelian group on one generator, which lies in degree $\dim(G/K) \pmod{2}$.*

Actually, for the sake of a later calculation we shall make a more precise statement directly in terms of the K -theory of $C_0(\mathfrak{a}_P^*, \mathfrak{K}(\text{Ind}_P^G H_\sigma))$. The assumption that $[P, \sigma]$ is essential implies that the group R_σ decomposes as a direct product

$$R_\sigma \cong \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{q \text{ times}},$$

where $q = \dim(\mathfrak{a}_P) - \dim(\mathfrak{a}_{\max})$; see Theorem 3.7. The generators of the factors act on \mathfrak{a}_P^* as pairwise orthogonal reflections and the fixed subspace

$$\mathfrak{a}_P^{*, R_\sigma} \subseteq \mathfrak{a}_P^*$$

for the action of R_σ has dimension $d = \dim(\mathfrak{a}_{\max})$.

It follows from Theorem 2.30 and Proposition 2.24 that for $\varphi \in \mathfrak{a}_P^{*,R_\sigma}$ the elements of the group R_σ act linearly independently on the representation space $\text{Ind}_P^G H_\sigma \otimes \mathbb{C}_{i\varphi}$, which therefore decomposes into a direct sum of $|R_\sigma|$ distinct irreducible subrepresentations,

$$(4.4) \quad \text{Ind}_P^G H_\sigma \otimes \mathbb{C}_{i\varphi} = \bigoplus_{\mu} X_{\mu,\varphi},$$

on each of which R_σ acts as multiples of a distinct character. So we can write

$$(4.5) \quad C_0(\mathfrak{a}_P^{*,R_\sigma}, \mathfrak{K}(\text{Ind}_P^G H_\sigma))^{R_\sigma} = \bigoplus_{\mu} C_0(\mathfrak{a}_P^{*,R_\sigma}, \mathfrak{K}(\mathbf{X}_\mu)),$$

where \mathbf{X}_μ is the bundle of Hilbert spaces with fibers $X_{\mu,\varphi}$. We can therefore form the C^* -algebra morphism

$$(4.6) \quad C_0(\mathfrak{a}_P^*, \mathfrak{K}(\text{Ind}_P^G H_\sigma))^{R_\sigma} \longrightarrow C_0(\mathfrak{a}_P^{*,R_\sigma}, \mathfrak{K}(\text{Ind}_P^G H_\sigma))^{R_\sigma} \longrightarrow C_0(\mathfrak{a}_P^{*,R_\sigma}, \mathfrak{K}(\mathbf{X}_\mu))$$

in which the first map is restriction to $\mathfrak{a}_P^{*,R_\sigma} \subseteq \mathfrak{a}_P^*$ and the second is projection to a single summand in (4.5). The target C^* -algebra is Morita equivalent to $C_0(\mathfrak{a}_P^{*,R_\sigma})$ via the bimodule $C_0(\mathfrak{a}_P^{*,R_\sigma}, \mathbf{X}_\mu)$ and so we can formulate a more precise version of Theorem 4.3 as follows:

4.7 Theorem. *If $[P, \sigma]$ is essential, then for every μ the restriction-projection morphism*

$$C_0(\mathfrak{a}_P^*, \mathfrak{K}(\text{Ind}_P^G H_\sigma))^{R_\sigma} \longrightarrow C_0(\mathfrak{a}_P^{*,R_\sigma}, \mathfrak{K}(\mathbf{X}_\mu))$$

in (4.6) induces an isomorphism

$$K_*(C_0(\mathfrak{a}_P^*, \mathfrak{K}(\text{Ind}_P^G H_\sigma))^{R_\sigma}) \xrightarrow{\cong} K_*(C_0(\mathfrak{a}_P^{*,R_\sigma})).$$

Proof. We shall prove the Morita equivalent version of the theorem that uses $\ell^2 R_\sigma$. The C^* -algebra $C_0(\mathfrak{a}_P^*, \mathfrak{K}(\ell^2 R_\sigma))^{R_\sigma}$ admits a tensor product decomposition

$$(4.8) \quad \begin{aligned} C_0(\mathfrak{a}_P^*, \mathfrak{K}(\ell^2 R_\sigma))^{R_\sigma} &\cong C_0(\mathbb{R}^d) \otimes C_0(\mathbb{R}, \mathfrak{K}(\ell^2 \mathbb{Z}_2))^{\mathbb{Z}_2} \otimes \cdots \\ &\quad \otimes C_0(\mathbb{R}, \mathfrak{K}(\ell^2 \mathbb{Z}_2))^{\mathbb{Z}_2}, \end{aligned}$$

where d is the dimension of the subspace of \mathfrak{a}_P^* fixed by the R_σ -action, and where there are as many factors of $C_0(\mathbb{R}, \mathfrak{K}(\ell^2 \mathbb{Z}_2))^{\mathbb{Z}_2}$ as there are factors of \mathbb{Z}_2 in the group R_σ . Now each of the fixed point algebras in the factorization above fits in an extension

$$0 \longrightarrow \mathfrak{J} \longrightarrow C_0(\mathbb{R}, \mathfrak{K}(\ell^2 \mathbb{Z}_2))^{\mathbb{Z}_2} \xrightarrow{\pi} \mathbb{C} \longrightarrow 0,$$

where the quotient map π is evaluation at $0 \in \mathbb{R}$, followed by compression to the subspace of constant functions in $\ell^2\mathbb{Z}_2$. The kernel \mathfrak{J} is Morita equivalent to the contractible C^* -algebra of C_0 -functions on $[0, \infty)$. It follows that the tensor product of the morphisms π above gives an isomorphism in K -theory from the right-hand side in (4.8) to $C_0(\mathbb{R}^d)$, and since the tensor product of the morphisms π is the same as (4.6), the theorem follows. \square

Let us summarize:

4.9 Theorem. *The group $K_{\dim(G/K)}(C_r^*(G))$ is a free abelian group on the set of essential associate classes, while the group $K_{\dim(G/K)+1}(C_r^*(G))$ is zero. More precisely, the K -theory of each essential summand of $C_r^*(G)$ is free abelian in one generator in degree $\dim(G/K)$, while the K -theory of each inessential summand of $C_r^*(G)$ is zero.*

5. The Connes–Kasparov index homomorphism

In this section we shall review the construction of Dirac operators on the symmetric space $K \backslash G$ of right K -cosets in G and the definition of the Connes–Kasparov index homomorphism.

We shall fix for the rest of the paper a G -invariant symmetric bilinear form

$$(5.1) \quad B: \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$$

that is positive-definite on \mathfrak{s} and negative-definite on \mathfrak{k} (where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ is the Cartan decomposition that was fixed at the beginning of Sect. 2).

Spin modules

The definitions and results in this section may be applied to any orthogonal action of a compact Lie group K on a finite-dimensional Euclidean vector space \mathfrak{s} . But soon the only example of interest will be the adjoint action of the maximal compact subgroup $K \subseteq G$ on the space \mathfrak{s} in the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. With this in mind we shall denote the K -action by

$$(k, X) \longmapsto \text{Ad}_k(X)$$

for $k \in K$ and $X \in \mathfrak{s}$ (even though this is a slight abuse of notation in general).

Form the Clifford algebra $\text{Cliff}(\mathfrak{s})$ using the convention that the square of any element from \mathfrak{s} is *minus* the norm-squared of that element, for the

inner product \mathfrak{s} .² If $X \in \mathfrak{s}$, then we shall denote by $c(X)$ the corresponding element in the Clifford algebra, so that the convention reads:

$$c(X)^2 = -\|X\|^2 \cdot 1.$$

5.2 Definition. *If \mathfrak{s} is even-dimensional, then a spin module for the pair (K, \mathfrak{s}) is a finite-dimensional, \mathbb{Z}_2 -graded, complex Hilbert space S that is equipped with*

- (i) *a representation of $\text{Cliff}(\mathfrak{s})$, written*

$$(X, s) \longmapsto X \cdot s$$

for $X \in \mathfrak{s}$ and $s \in S$, in which each X acts as a grading-degree one, skew-adjoint operator, and

- (ii) *a grading-degree zero, unitary representation of K that is compatible with the representation of $\text{Cliff}(\mathfrak{s})$ in the sense that*

$$k \cdot (X \cdot s) = \text{Ad}_k(X) \cdot (k \cdot s)$$

for every $k \in K$, every $X \in \mathfrak{s}$, and every $s \in S$.

If \mathfrak{s} is odd-dimensional, then a spin module for (K, \mathfrak{s}) is a spin module for $(K, \mathfrak{s} \oplus \mathbb{R})$, where \mathbb{R} is equipped with the trivial action of K . A spin module for (K, \mathfrak{s}) is irreducible if it cannot be decomposed into a direct sum of two spin submodules.

5.3 Definition. *We shall denote by $R_{\text{spin}}(K, \mathfrak{s})$ the abelian group generated by isomorphism classes of spin modules subject to the relations*

$$[S_1] + [S_2] = [S_1 \oplus S_2] \quad \text{and} \quad [S] + [S^{\text{opp}}] = 0,$$

where S^{opp} is obtained from S by reversing the \mathbb{Z}_2 -grading.

This group may be analyzed using the following standard construction from Clifford algebras and Lie theory (compare [HP06, Section 2.3] or [Mei13, Section 2.2.10]):

² This convention agrees with [HP06], which is one of the main references for the material here, but it disagrees with [Mei13], which is the other main reference.

5.4 Definition. *The fundamental morphism*

$$\alpha: \mathfrak{k} \longrightarrow \text{Cliff}(\mathfrak{s})$$

is defined by the formula

$$\alpha(Z) = -\frac{1}{4} \sum c(\text{ad}_Z(X_a)) \cdot c(X_a),$$

where the sum is over any orthonormal basis $\{X_a\}$ of \mathfrak{s} (the sum is independent of the choice).

The fundamental morphism is a Lie algebra morphism (for the commutator bracket on the Clifford algebra) and moreover

$$c(\text{ad}_Z(X)) = [\alpha(Z), c(X)]$$

for all $X \in \mathfrak{s}$ and all $Z \in \mathfrak{k}$. If S is a spin module, then by composing the Clifford algebra action on S with α we obtain a representation of \mathfrak{k} on S .

Suppose for a moment that \mathfrak{s} is even-dimensional. Fix an irreducible representation S_{irr} of the Clifford algebra on a finite-dimensional \mathbb{Z}_2 -graded Hilbert space (S_{irr} is unique up to a possibly grading-reversing unitary equivalence). The fundamental morphism endows S_{irr} with a \mathfrak{k} -action, and so to any spin module S we can associate the \mathbb{Z}_2 -graded \mathfrak{k} -module

$$(5.5) \quad \text{mod}(S) = \text{Hom}_{\text{Cliff}(\mathfrak{s})}(S, S_{\text{irr}}),$$

(the morphisms in $\text{mod}(S)$ need not be grading-preserving). We can reconstruct S from $\text{mod}(S)$ via the canonical isomorphism

$$(5.6) \quad S_{\text{irr}} \widehat{\otimes} \text{mod}(S)^* \xrightarrow{\cong} S,$$

where the tensor product is given the diagonal \mathfrak{k} -action. Note that as a result, if S is irreducible in the sense of Definition 5.2, then $\text{mod}(S)$ is an irreducible \mathfrak{k} -module.

If \mathfrak{s} is odd-dimensional, then we repeat the above with $\text{Cliff}(\mathfrak{s} \oplus \mathbb{R})$ in place of $\text{Cliff}(\mathfrak{s})$. In either case, the \mathfrak{k} -module $\text{mod}(S)$ does not necessarily integrate to a representation of K . However if we define a compact group \tilde{K} by means of the pullback diagram

$$\begin{array}{ccc} \tilde{K} & \longrightarrow & \text{Spin}(\mathfrak{s}) \\ \downarrow & & \downarrow \\ K & \longrightarrow & SO(\mathfrak{s}) \end{array}$$

in which the bottom morphism comes from the adjoint action of K on \mathfrak{s} , then $\text{mod}(S)$ integrates to a representation of \tilde{K} .

The pullback group \tilde{K} may or may not be connected, but in any case the kernel of the morphism $\tilde{K} \rightarrow K$ is a two-element group, and there is a unique morphism from \tilde{K} into the group of invertible elements in $\text{Cliff}(\mathfrak{s})$ whose associated Lie algebra morphism is α , and which maps the non-trivial element of the kernel to minus the identity.

5.7 Definition. *We shall say that a representation of \tilde{K} is genuine if the non-trivial element in the kernel of $\tilde{K} \rightarrow K$ acts as $-I$ in the representation.*

5.8 Theorem (See for instance [EP09, Theorem 0.1]). *The abelian group $R_{\text{spin}}(K, \mathfrak{s})$ is isomorphic via the correspondence $S \mapsto \text{mod}(S)$ to the free abelian group on the set of equivalence classes of irreducible and genuine representations of \tilde{K} .*

The Dirac operator and its square

For the rest of this section K will be the given maximal compact subgroup of our real reductive group G , and \mathfrak{s} will be the complementary subspace to \mathfrak{k} in the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. We shall equip \mathfrak{s} with the inner product coming from (5.1).

Given a spin module S , form the space $[C_c^\infty(G) \otimes S]^K$, where K acts diagonally, and where the K -action on $C_c^\infty(G)$ is by left translation.

5.9 Definition. *The Dirac operator associated to a spin module S is the linear operator*

$$\not{D}_S: [C_c^\infty(G) \otimes S]^K \longrightarrow [C_c^\infty(G) \otimes S]^K$$

given by the formula

$$\not{D}_S = \sum X_a \otimes c(X_a),$$

in which the sum is over any orthonormal basis $\{X_a\}$ for \mathfrak{s} , as in Definition 5.4, and X_a acts on $C_c^\infty(G)$ via the left-translation action of G on $C_c^\infty(G)$. Compare [Par72], [AS77].

5.10 Definition. *The Casimir element for G is the element*

$$\Omega_G = \sum Y^a Y_a$$

in the enveloping algebra³ $\mathcal{U}(\mathfrak{g})$, where the sum is over any basis $\{Y_a\}$ for \mathfrak{g} with dual basis $\{Y^a\}$ for the invariant form B (so that $B(Y^a, Y_b) = \delta_b^a$). Similarly the Casimir element for K is the element

$$\Omega_K = \sum Z^b Z_b \in \mathcal{U}(\mathfrak{k}),$$

where the sum is over any basis $\{Z_b\}$ for \mathfrak{k} and dual basis $\{Z^b\}$ for the invariant form B , restricted to K .

5.11 Definition. *The diagonal morphism*

$$\Delta: \mathcal{U}(\mathfrak{k}) \longrightarrow \mathcal{U}(\mathfrak{g}) \otimes \text{Cliff}(\mathfrak{s})$$

is the morphism of associative algebras for which

$$\Delta(Z) = Z \otimes I + I \otimes \alpha(Z)$$

for all $Z \in \mathfrak{k}$, where α is the fundamental morphism from Definition 5.4.

In the next result, it is convenient to view the Dirac operator algebraically, as an element of $\mathcal{U}(\mathfrak{g}) \otimes \text{Cliff}(\mathfrak{s})$; the choice of spin module S is therefore no longer immediately relevant. The expression for the square of the Dirac operator in Theorem 5.13 below is essentially due to Parthasarathy [Par72]; see [HP06, Proposition 3.1.6] for a modern account.

The bilinear form B in (5.1) may be extended to a non-degenerate symmetric complex-bilinear form

$$(5.12) \quad B: \mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \longrightarrow \mathbb{C}$$

on the complexification of \mathfrak{g} . This restricts to a non-degenerate bilinear form on each Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$, and also on each Cartan subalgebra of $\mathfrak{k}_{\mathbb{C}}$. These restrictions may be used to identify the Cartan subalgebras with their complex vector space duals in the usual way, and so we obtain non-degenerate complex-bilinear forms B^* on these dual spaces.

5.13 Theorem. *Let ρ_K and ρ_G be the half-sums of the positive roots for $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{C}}$ (formed using any choices of Cartan subalgebras in the complexified Lie algebras and any systems of positive roots). The square of the Dirac operator in $\mathcal{U}(\mathfrak{g}) \otimes \text{Cliff}(\mathfrak{s})$ is given by the formula*

$$D^2 = \Delta(\Omega_K + B^*(\rho_K, \rho_K)) - (\Omega_G + B^*(\rho_G, \rho_G)).$$

³ Generally we shall work with the complexification of the enveloping algebra of the real Lie algebra \mathfrak{g} , or equivalently the enveloping algebra of the complexification, but here it is not necessary to do so.

The scalars $B^*(\rho_K, \rho_K)$ and $B^*(\rho_G, \rho_G)$ are independent of the choices of Cartan subalgebras and systems of positive roots. Moreover they are real-valued, and indeed non-negative. This may be seen by selecting a (complexification of a) θ -stable Cartan subalgebra $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a} \subseteq \mathfrak{g}$ with \mathfrak{t} maximal abelian in \mathfrak{k} and \mathfrak{a} orthogonal to \mathfrak{k} , and observing that ρ_K and ρ_G are imaginary-valued on \mathfrak{t} , where B is positive definite, while ρ_G is real-valued on \mathfrak{a} , where B is real-valued.

In order to reflect this non-negativity, it will be convenient to change the notation in Theorem 5.13. The formula

$$(5.14) \quad \langle X, Y \rangle = -B(X, \theta(Y)) \quad (X, Y \in \mathfrak{g})$$

defines a positive-definite inner product on the real Lie algebra \mathfrak{g} . Let us extend it to a complex-sesquilinear inner product on the complexification $\mathfrak{g}_{\mathbb{C}}$. Of course, this extension restricts to an inner product on any Cartan subalgebra of the complexification, and the restriction induces an inner product on the vector space dual of the Cartan subalgebra. The same goes for any Cartan subalgebra of the Lie algebra of K . With this notation, Theorem 5.13 may be restated as

$$(5.15) \quad \not{D}^2 = \Delta(\Omega_K + \|\rho_K\|^2) - (\Omega_G + \|\rho_G\|^2).$$

Let us now bring the spin module S back into the picture and compute the operator

$$\Delta(\Omega_K + \|\rho_K\|^2): [C_c^\infty(G) \otimes S]^K \longrightarrow [C_c^\infty(G) \otimes S]^K$$

arising from Theorem 5.13.

5.16 Definition. *If S is an irreducible spin module (Definition 5.2), then we define*

$$\|S\| = \|\mu + \rho_K\|,$$

where μ is the highest weight of the irreducible \mathfrak{k} -module $\text{mod}(S)$ (both μ and ρ_K depend on a choice of Cartan subalgebra of $\mathfrak{k}_{\mathbb{C}}$ and system of positive roots, but the norm does not).

5.17 Lemma. *If S is an irreducible spin module, then the operator*

$$\Delta(\Omega_K + \|\rho_K\|^2): [C_c^\infty(G) \otimes S]^K \longrightarrow [C_c^\infty(G) \otimes S]^K$$

is $\|S\|^2$ times the identity operator.

Proof. Write $S \cong S_{\text{irr}} \otimes \text{mod}(S)^*$ as in (5.6). Under this isomorphism, the Clifford algebra acts on S_{irr} , but not on $\text{mod}(S)^*$. So the diagonal morphism Δ gives the action

$$\Delta(Z) = Z \otimes 1 \otimes 1 + 1 \otimes \alpha(Z) \otimes 1$$

of the Lie algebra \mathfrak{k} on

$$C_c^\infty(G) \otimes S \cong C_c^\infty(G) \otimes S_{\text{irr}} \otimes \text{mod}(S)^*.$$

In contrast, the K -fixed part of this space is computed using the actions of K (or, strictly speaking \tilde{K}) on all three factors, so that

$$[C_c^\infty(G) \otimes S]^K \cong \text{Hom}_K(\mathbb{C}, C_c^\infty(G) \otimes S) \cong \text{Hom}_{\tilde{K}}(\text{mod}(S), C_c^\infty(G) \otimes S_{\text{irr}}).$$

We can therefore compute the action of $\Delta(\Omega_K)$ using either the action of \mathfrak{k} on $C_c^\infty(G) \otimes S_{\text{irr}}$ or the action of \mathfrak{k} on $\text{mod}(S)$. Using the latter it is well known that we obtain $\|\mu + \rho_K\|^2 - \|\rho_K\|^2$; See [KV95, Proposition 4.120] for instance. \square

With this, we can simplify the formula for the square of the Dirac operator:

5.18 Theorem. *The square of the Dirac operator*

$$\not{D}_S: [C_c^\infty(G) \otimes S]^K \longrightarrow [C_c^\infty(G) \otimes S]^K$$

associated to an irreducible spin module is given by the formula

$$\not{D}_S^2 = \|S\|^2 - (\Omega_G + \|\rho_G\|^2).$$

Infinitesimal characters

Let us quickly review some basic topics in representation theory. For a further discussion of all these concepts and results, see for instance [Kna86, Chapter VIII].

If π is any continuous representation of G on a Hilbert space H_π by bounded, invertible operators, then we shall denote by $H_{\pi,\text{fin}}$ the space of K -finite vectors in H_π . The representation π is said to be *admissible* if each K -isotypical subspace in $H_{\pi,\text{fin}}$ is finite-dimensional. According to a theorem of Harish-Chandra, if π is unitary and irreducible, then it is admissible.

If π is admissible (but not necessarily unitary), then $H_{\pi,\text{fin}}$ is included within the smooth vectors in H_π and so it carries a representation of the complexified Lie algebra $\mathfrak{g}_\mathbb{C}$. One says that π is *quasi-simple* if $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$, the center of the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$, acts as multiples of the identity operator.

Let \mathfrak{h} be any Cartan subalgebra of \mathfrak{g} . Harish-Chandra defined an isomorphism

$$(5.19) \quad HC: \mathcal{Z}(\mathfrak{g}_\mathbb{C}) \xrightarrow{\cong} \mathcal{S}(\mathfrak{h}_\mathbb{C})^W,$$

where W is the Weyl group associated to $(\mathfrak{g}, \mathfrak{h})$ and $\mathcal{S}(\mathfrak{h}_{\mathbb{C}})^W$ is the W -invariant part of the symmetric algebra of $\mathfrak{h}_{\mathbb{C}}$, the complexification of the real Lie algebra \mathfrak{h} . We shall identify the range in (5.19) with the algebra of W -invariant complex polynomial functions on the complex vector space $\text{Hom}_{\mathbb{R}}(\mathfrak{h}, \mathbb{C})$.

If π is an admissible and quasi-simple representation of G , then the *infinitesimal character* of π is the algebra homomorphism

$$\text{inf. ch.}(\pi): \mathcal{L}(\mathfrak{g}_{\mathbb{C}}) \longrightarrow \mathbb{C}$$

that gives the action of the center of the enveloping algebra on $H_{\pi, \text{fin}}$. Using the Harish-Chandra isomorphism we can and will view the infinitesimal character as (any representative of) a W -orbit in $\text{Hom}_{\mathbb{R}}(\mathfrak{h}, \mathbb{C})$.

5.20 Definition. Let $\mathfrak{h} = \mathfrak{t}_{\mathfrak{h}} \oplus \mathfrak{a}_{\mathfrak{h}}$ be a θ -stable Cartan subalgebra of \mathfrak{g} , with $\mathfrak{t}_{\mathfrak{h}} \subseteq \mathfrak{k}$ and $\mathfrak{a}_{\mathfrak{h}} \subseteq \mathfrak{s}$. An element of $\text{Hom}_{\mathbb{R}}(\mathfrak{h}, \mathbb{C})$ is said to be *real* if it belongs to

$$\text{Hom}_{\mathbb{R}}(\mathfrak{t}_{\mathfrak{h}}, i\mathbb{R}) \oplus \text{Hom}_{\mathbb{R}}(\mathfrak{a}_{\mathfrak{h}}, \mathbb{R})$$

and it is said to be *imaginary* if it belongs to

$$\text{Hom}_{\mathbb{R}}(\mathfrak{t}_{\mathfrak{h}}, \mathbb{R}) \oplus \text{Hom}_{\mathbb{R}}(\mathfrak{a}_{\mathfrak{h}}, i\mathbb{R}).$$

The same terminology may be applied to infinitesimal characters, using the Harish-Chandra isomorphism. Whether or not an infinitesimal character is real or imaginary does not depend on the choice of representative of $\text{inf. ch.}(\pi)$ within its W -orbit. Nor does it depend on the choice of Cartan subalgebra (as long as the Cartan subalgebra is stable under the Cartan involution). See [Kna86, p. 535].

For the next result, recall that the complexification process outlined in the discussion preceding (5.15) endows $\text{Hom}_{\mathbb{R}}(\mathfrak{h}, \mathbb{C})$ with an inner product. Every element of $\text{Hom}_{\mathbb{R}}(\mathfrak{h}, \mathbb{C})$ decomposes as a sum of real and imaginary parts, and we shall use the standard notation for these.

5.21 Lemma. Let π be a unitary, admissible and quasi-simple representation of G on a Hilbert space H_{π} . If the real and imaginary parts of the infinitesimal character of π are orthogonal, then the operator $\pi(\Omega_G) + \|\rho_G\|^2$ acts on $H_{\pi, \text{fin}}$ as the scalar

$$\|\text{Re}(\text{inf. ch.}(\pi))\|^2 - \|\text{Im}(\text{inf. ch.}(\pi))\|^2.$$

Proof. The formula in the statement of the lemma is a special case of the following standard identity for the Harish-Chandra homomorphism (5.19):

$$HC(\Omega_G)(\lambda) + B^*(\rho_G, \rho_G) = B^*(\lambda, \lambda) \quad \forall \lambda \in \mathfrak{h}_{\mathbb{C}}^*$$

(we introduced the form B^* in the discussion prior to the statement of Theorem 5.13). For a proof see for instance [KV95, Proposition 4.120]; the relation between the form $\langle \cdot, \cdot \rangle$ that appears there and our B^* is explained on p.299 of [KV95]. \square

Let us now apply this to the unitary principal series representations of G . Let $[P, \sigma]$ be an associate class, and let $P = M_P A_P N_P$ be the Langlands decomposition of P , so that σ is an irreducible square-integrable representation of M_P . Harish-Chandra showed that:

5.22 Theorem. *Whenever M_P carries an irreducible square-integrable representation, the Lie algebra \mathfrak{t}_P of any maximal torus in $K \cap M_P$ is a Cartan subalgebra of \mathfrak{m}_P .*

5.23 Theorem. *Every irreducible, square-integrable representation of M has real infinitesimal character. Moreover, for every $N > 0$ the set of equivalence classes of irreducible, square-integrable representations of M with $\|\inf. \text{ch.}(\sigma)\| < N$ is finite.*

For an exposition of these results, see for instance [Kna86]. We shall use the second statement in the second theorem in the next subsection. As a result of the first theorem, the Lie algebra $\mathfrak{t}_P \oplus \mathfrak{a}_P$ is a Cartan subalgebra of \mathfrak{g} , and we may compute the infinitesimal characters for the (P, σ) -principal series as follows:

5.24 Lemma (See for example [Kna86, Proposition 8.22]). *The infinitesimal character of the unitary (P, σ) -principal series representation $\pi_{\sigma, \varphi}$ is*

$$\inf. \text{ch.}(\sigma) \oplus \varphi \in \text{Hom}_{\mathbb{R}}(\mathfrak{t}_P, i\mathbb{R}) \oplus \text{Hom}_{\mathbb{R}}(\mathfrak{a}_P, i\mathbb{R}).$$

Note that the two summands in the infinitesimal character above are its real and imaginary parts, respectively.

Dirac operator from the representation theory point of view

Now form the Hilbert space $\text{Ind}_P^G H_\sigma$ as in Definition 2.5, and given a spin module S , form the fixed space

$$[\text{Ind}_P^G H_\sigma \otimes S]^K.$$

The same space is obtained if we replace $\text{Ind}_P^G H_\sigma$ by its subspace of K -finite vectors, and as a result $[\text{Ind}_P^G H_\sigma \otimes S]^K$ carries an action of \mathfrak{g} . So if we regard $\text{Ind}_P^G H_\sigma$ as carrying the principal series representation $\pi_{\sigma, \varphi}$, then we may form the operator

(5.25)

$$\mathcal{D}_{\sigma, \varphi, S} = \sum \pi_{\sigma, \varphi}(X_a) \otimes c(X_a) : [\text{Ind}_P^G H_\sigma \otimes S]^K \longrightarrow [\text{Ind}_P^G H_\sigma \otimes S]^K.$$

The following is an immediate consequence of Lemmas 5.21 and 5.24:

5.26 Lemma. *The operator $\pi_{\sigma,\varphi}(\Omega_G) + \|\rho_G\|^2$ acts on $[\text{Ind}_P^G H_\sigma \otimes S]^K$ as the scalar $\|\text{inf. ch.}(\sigma)\|^2 - \|\varphi\|^2$.*

Putting this together with Theorems 5.13 and 5.18, we arrive at the following result (compare [Was87, p. 562]):

5.27 Theorem. *If $\pi_{\sigma,\varphi}$ is any (P, σ) -principal series representation, and if S is any irreducible spin module, then*

$$\not{D}_{\sigma,\varphi,S}^2 = \|S\|^2 - \|\text{inf. ch.}(\sigma)\|^2 + \|\varphi\|^2.$$

5.28 Remark. Strictly speaking, to reach this conclusion we need the formula

$$\Delta(\Omega_K + \|\rho_K\|^2) = \|S\|^2 : [\text{Ind}_P^G H_\sigma \otimes S]^K \longrightarrow [\text{Ind}_P^G H_\sigma \otimes S]^K,$$

which is a version of Lemma 5.17 with $C_c^\infty(G)$ replaced by the K -finite vectors in $\text{Ind}_P^G H_\sigma$. This follows by a verbatim repetition of the proof of Lemma 5.17.

5.29 Corollary. *For every spin module S , the space $[\text{Ind}_P^G H_\sigma \otimes S]^K$ is zero for all but finitely many associate classes $[P, \sigma]$.*

Proof. The formula

$$\not{D}_{\sigma,0,S}^2 = \|S\|^2 - \|\text{inf. ch.}(\sigma)\|^2$$

shows that $\not{D}_{\sigma,0,S}^2$ will be negative whenever $\|\text{inf. ch.}(\sigma)\|^2 > \|S\|^2$, assuming that $[\text{Ind}_P^G H_\sigma \otimes S]^K$ is non-zero. But the Dirac operator is self-adjoint, so its square is positive semidefinite. So necessarily $[\text{Ind}_P^G H_\sigma \otimes S]^K$ is zero in these cases. The corollary now follows from Theorem 5.23. \square

Dirac operator from a functional analytic point of view

The Dirac operator \not{D}_S can be viewed as an unbounded operator on the Hilbert space $[L^2(G) \otimes S]^K$ with domain $[C_c^\infty(G) \otimes S]^K$. The Dirac operator so viewed is essentially self-adjoint (see [Che73]), and there is therefore an associated one-parameter group of unitary operators $\exp(it\not{D}_S)$. These restrict to operators

$$(5.30) \quad \exp(it\not{D}_S) : [C_c^\infty(G) \otimes S]^K \longrightarrow [C_c^\infty(G) \otimes S]^K$$

(see [Che73] again); this is the finite propagation property of the Dirac operator.

But we are more interested in viewing \mathcal{D}_S as an unbounded operator on the space $[C_r^*(G) \otimes S]^K$, which becomes a *Hilbert C*-module* over $C_r^*(G)$ when equipped with the right action of $C_r^*(G)$ on the first factor and the $C_r^*(G)$ -valued inner product

$$\langle f_1 \otimes s_1, f_2 \otimes s_2 \rangle = f_1^* f_2 \langle s_1, s_2 \rangle.$$

See [Lan95] for general information about Hilbert C*-modules. The operators $\exp(it\mathcal{D}_S)$ in (5.30) extend to a one-parameter group of unitary operators

$$\exp(it\mathcal{D}_S): [C_r^*(G) \otimes S]^K \longrightarrow [C_r^*(G) \otimes S]^K,$$

and the generator of this one-parameter group is a regular and self-adjoint operator on the Hilbert module $[C_r^*(G) \otimes S]^K$ in the sense of [Lan95, Chapter 9]. We shall use the same notation \mathcal{D}_S for the extension.

5.31 Definition. *The bounded transform of \mathcal{D}_S is the operator*

$$\mathcal{F}_S = \mathcal{D}_S(I + \mathcal{D}_S^2)^{-1/2}: [C_r^*(G) \otimes S]^K \longrightarrow [C_r^*(G) \otimes S]^K$$

that is defined using the functional calculus for regular self-adjoint Hilbert module operators.

As in the previous section, given a (P, σ) -principal series representation

$$\pi_{\sigma, \varphi}: G \longrightarrow U(\text{Ind}_P^G H_\sigma)$$

we may form the operator

$$\mathcal{D}_{\sigma, \varphi, S} = \sum \pi_{\sigma, \varphi}(X_a) \otimes c(X_a): [\text{Ind}_P^G H_\sigma \otimes S]^K \longrightarrow [\text{Ind}_P^G H_\sigma \otimes S]^K,$$

with $\{X_a\}$ an orthonormal basis for \mathfrak{s} , as usual, and then its bounded transform⁴

$$\mathcal{F}_{\sigma, \varphi, S} = \mathcal{D}_{\sigma, \varphi, S}(I + \mathcal{D}_{\sigma, \varphi, S}^2)^{-1/2}: [\text{Ind}_P^G H_\sigma \otimes S]^K \longrightarrow [\text{Ind}_P^G H_\sigma \otimes S]^K.$$

5.32 Lemma. *Under isomorphism of Hilbert modules*

$$[C_r^*(G) \otimes S]^K \cong \bigoplus_{[P, \sigma]} [C_0(\mathfrak{a}_P^*, \mathfrak{K}(\text{Ind}_P^G H_\sigma))^{W_\sigma} \otimes S]^K$$

associated with the C-algebra isomorphism of Theorem 2.19, the operator*

$$\mathcal{F}_S: [C_r^*(G) \otimes S]^K \longrightarrow [C_r^*(G) \otimes S]^K$$

⁴ Of course, the operator $\mathcal{D}_{\sigma, \varphi, S}$ is acting on a finite-dimensional Hilbert space, and is therefore already bounded itself.

acts as

$$f \otimes s \longmapsto [\varphi \mapsto \mathbb{F}_{\sigma, \varphi, S} \cdot (f(\varphi) \otimes s)]$$

for all $f \in C_0(\mathfrak{a}_P^*, \mathfrak{K}(\text{Ind}_P^G H_\sigma))^{W_\sigma}$ and all $s \in S$, where the product \cdot on the right hand side is composition of linear operators on the finite-dimensional space $[\text{Ind}_P^G H_\sigma \otimes S]^K$.

Proof. The analogous result for $\exp(it\mathbb{D}_S)$ is readily verified on $[C_c^\infty(G) \otimes S]^K$, and the stated result follows from this. \square

See [Lan95, Chapter 1] for the meaning of *compact* in following fundamental result:

5.33 Theorem. *The operator*

$$I - \mathbb{F}_S^2 = (I + \mathbb{D}_S^2)^{-1}$$

is a compact operator on the Hilbert module $[C_r^(G) \otimes S]^K$.*

We shall prove this using the representation theory calculations from the previous section, since those results are at hand. The original proof, due to [Kas83], uses the basic elliptic estimates for the Dirac operator and the Rellich lemma. See [BCH94] for a general account of these matters.

Proof. The formula for \mathbb{F}_S in Lemma 5.32 and the formula for $\mathbb{D}_{S, \sigma, \varphi}$ in Theorem 5.27 combine to give a formula for $1 - \mathbb{F}_S^2$ as an operator on

$$[C_r^*(G) \otimes S]^K \cong \bigoplus_{[P, \sigma]} [C_0(\mathfrak{a}_P^*, \mathfrak{K}(\text{Ind}_P^G H_\sigma))^{W_\sigma} \otimes S]^K.$$

The direct sum here is actually a finite direct sum, in view of Corollary 5.29, and in each summand $1 - \mathbb{F}_S^2$ acts as multiplication by a C_0 -scalar-valued function. Each such operator is compact, thanks to the finite-dimensionality of the spaces $[\text{Ind}_P^G H_\sigma \otimes S]^K$. \square

Now the compact operators on any Hilbert C^* -module form an ideal in the C^* -algebra of all bounded, adjointable operators, and by definition a bounded adjointable operator is *Fredholm* if it is invertible modulo this ideal. In the present case, we see from the theorem above that \mathbb{F}_S is its own inverse, modulo compact operators. Therefore:

5.34 Corollary. *The operator*

$$\mathbb{F}_S: [C_r^*(G) \otimes S]^K \longrightarrow [C_r^*(G) \otimes S]^K$$

is a bounded, self-adjoint, odd-graded, Fredholm operator on the \mathbb{Z}_2 -graded Hilbert $C_r^(G)$ -module $[C_r^*(G) \otimes S]^K$.*

The Connes–Kasparov index homomorphism

In order to define the index homomorphism it is convenient to use Kasparov’s approach of C^* -algebra K -theory [Kas81] (see [Hig90, Section 3] for an exposition).

Kasparov *defines* the K_0 -group of a C^* -algebra A as the group of homotopy classes of bounded, self-adjoint, odd-graded, Fredholm operators F on \mathbb{Z}_2 -graded Hilbert A -modules. In addition, he defines the K_1 -group in the same way, except that the Hilbert A -module \mathcal{E} on which F acts is required to carry an additional odd-graded skew-symmetry

$$(5.35) \quad \begin{aligned} \gamma: \mathcal{E} &\longrightarrow \mathcal{E}, \\ \gamma^* &= -\gamma, \quad \gamma^2 = -1 \end{aligned}$$

that anti-commutes with F .

5.36 Remark. We note for later use that, as a consequence of the way homotopy is defined, it is an elementary property of K -theory that if a Fredholm operator is actually invertible (not merely invertible modulo compact operators), then it determines the zero class in K -theory.

Kasparov’s definitions are made with Dirac operators in mind, and it follows immediately from the definitions and the results we have summarized above that if S is any spin module, then the Fredholm operator \mathbb{F}_S determines a class

$$\text{Index}(\mathbb{F}_S) \in K_{\dim(G/K)}(C_r^*(G))$$

(in the case where $\dim(G/K)$ is odd, the skew-symmetry γ is Clifford multiplication by the generator in $\text{Cliff}(\mathfrak{s} \oplus \mathbb{R})$ associated to the \mathbb{R} -summand).

5.37 Definition. *The Connes–Kasparov index homomorphism is the homomorphism of abelian groups*

$$R_{\text{spin}}(K, \mathfrak{s}) \longrightarrow K_{\dim(G/K)}(C_r^*(G))$$

that maps the class of a spin module S to the index of \mathbb{F}_S in K -theory.

Our aim is to prove that:

5.38 Theorem (Connes–Kasparov Isomorphism). *If G is a connected, linear real reductive Lie group, then Connes–Kasparov index map*

$$R_{\text{spin}}(K, \mathfrak{s}) \longrightarrow K_{\dim(G/K)}(C_r^*(G))$$

is an isomorphism of abelian groups. Moreover $K_{\dim(G/K)+1}(C_r^(G)) = 0$.*

5.39 Remarks. There is a Connes–Kasparov index homomorphism for any almost-connected Lie group (that is, any Lie group with finitely many connected components) and moreover it is an isomorphism in this generality [CEN03]. The definition of the index homomorphism for connected Lie groups is essentially the same as the one we have presented. But beyond connected groups, and even within the realm of real reductive groups, the definition of the index homomorphism needs to be adjusted [EP09]. Among other things it is possible that both K -theory groups for $C_r^*(G)$ might be non-zero at the same time, as is the case for $GL(2, \mathbb{R})$, for instance.

6. The Matching Theorem

In this section we shall state a purely representation-theoretic result that will lead quickly (in the next section) to a proof that the Connes–Kasparov index homomorphism is an isomorphism.

Statement of the Matching Theorem

6.1 Definition. *We shall say that an irreducible spin module S for (K, \mathfrak{s}) and an associate class $[P, \sigma]$ are matched if*

- (i) *the space $[\text{Ind}_P^G H_\sigma \otimes S]^K$ is non-zero, and*
- (ii) *the Dirac operator $\mathcal{D}_{\sigma, 0, S}$ vanishes on $[\text{Ind}_P^G H_\sigma \otimes S]^K$.*

6.2 Remark. It follows from Theorem 5.27 and the fact that the Dirac operator is self-adjoint that the second condition in the definition above is equivalent to the identity $\|S\| = \|\text{inf. ch.}(\sigma)\|$.

The result that we shall use to establish the Connes–Kasparov isomorphism is as follows:

6.3 Theorem (Matching Theorem). *Let G be a connected linear real reductive group.*

- (i) *For every essential associate class $[P, \sigma]$ there is a unique irreducible spin module S to which $[P, \sigma]$ is matched.*
- (ii) *For every irreducible spin module S there is a unique essential associate class $[P, \sigma]$ to which S is matched.*

We shall prove this in a separate article [CHS22] using a number of important (and quite difficult) results of Vogan from [Vog81]. But let us give some examples.

6.4 Example. If σ is an irreducible square-integrable representation of G , then it is an essential component all by itself, labelled by the associate class $[G, \sigma]$. The irreducible spin module matched to $[G, \sigma]$ is the unique one, up to not-necessarily-grading-preserving isomorphism, for which

$$[H \otimes S]^K \neq 0.$$

Moreover if μ is the highest weight of the irreducible and genuine representation $\text{mod}(S)$ of \tilde{K} , then $\mu + \rho_K$ is the so-called *Harish-Chandra parameter* of σ . Compare [AS77, Theorem 9.3] or [Laf02a, §2].

6.5 Example. The reduced C^* -algebra for $G=SL(2, \mathbb{R})$ was mostly described in [CCH16, Example 6.10] (and in several earlier works). The decomposition of the intertwining groups W_σ as semidirect products $W'_\sigma \rtimes R_\sigma$ was not discussed there, but one may determine by direct computation that the only inessential component in the tempered dual is the spherical principal series component. The essential associate classes are therefore of two types: the discrete series and the odd principal series.

If $[P, \sigma]$ is a discrete series component with Harish-Chandra parameter $n \in \mathbb{Z}$, $n \neq 0$, then the matching spin module S_n is

$$S_n = S_{\text{irr}} \otimes \mathbb{C}_n,$$

where \mathbb{C}_n denotes the weight- n irreducible representation of $SO(2)$, viewed as a genuine representation of $\tilde{K} \cong K \times \mathbb{Z}_2$. This may be computed directly, but the result is in line with Example 6.4 above. If $[P, \sigma]$ is the odd principal series component of the tempered dual, then the matching spin module is $S_0 = S_{\text{irr}} \otimes \mathbb{C}_0$; the matching conditions in Definition 6.1 may again be checked by direct computation.

6.6 Example. If G is a complex reductive group, then all essential associate classes are attached to the minimal parabolic $P_{\min} = MAN$, for which M is a maximal torus in a maximal compact subgroup of G . The Connes–Kasparov isomorphism was established in [PP83]. A bit more generally, the Matching Theorem was established for semi-simple Lie groups having only one conjugacy class of Cartan subgroups by Valette in [Val85] (see Theorem 3.12). In the complex case the correspondence provided by the Matching Theorem is as follows: the spin module $S_{\text{irr}} \otimes V_\mu$, where V_μ is irreducible with highest weight μ , is matched to the associate class $[P_{\min}, \sigma]$, where the differential of $\sigma \in \hat{M}$ is $\mu + \rho_K$.

6.7 Example. A more complicated and more interesting example is that of the real symplectic group $G=Sp(4, \mathbb{R})$.

There are three components of the tempered dual associated with the minimal parabolic subgroup $P_{\min} = MAN$ of $Sp(4, \mathbb{R})$, given by three

characters σ_0 , σ_1 and σ_2 of the finite group M (there are four characters altogether, but two lead to the same associate class). None of the principal series components is essential.

There are two other associate classes of proper parabolic subgroups, with Levi factors

$$L_1 \cong GL(2, \mathbb{R}) \quad \text{and} \quad L_2 \cong GL(1, \mathbb{R}) \times SL(2, \mathbb{R}).$$

The compactly generated subgroups $M_1 \subseteq L_1$ and $M_2 \subseteq L_2$ both carry discrete series. The group M_1 is isomorphic to $SL_{\pm}(2, \mathbb{R})$, and its discrete series are parametrized by positive integers; let us write these representations as $\sigma_{1,k}$ ($k > 0$). The group M_2 is isomorphic to $O(1) \times SL(2, \mathbb{R})$, and its discrete series are parametrized by pairs (ℓ, k) where $\ell \in \mathbb{Z}_2$ and $k \in \mathbb{Z}$, $k \neq 0$; let us write these representations as $\sigma_{2,\ell,k}$. Altogether, the discrete series

$$\sigma_{1,k}, \quad (k \in \mathbb{Z}, \quad k > 0) \quad \text{and} \quad \sigma_{2,\ell,k}, \quad (\ell \in \mathbb{Z}_2, \quad k \in \mathbb{Z}, \quad k \neq 0)$$

label the components in the tempered dual of G that are associated to the intermediate parabolic subgroups, and the above representations label these components without repetition. The essential components associated to the intermediate parabolic subgroups are

$$\sigma_{1,k}, \quad (k \in 2\mathbb{Z}, \quad k > 0) \quad \text{and} \quad \sigma_{2,0,k}, \quad (k \in \mathbb{Z}, \quad k \neq 0).$$

Finally, there are the discrete series of $Sp(4, \mathbb{R})$. The maximal compact subgroup of $Sp(4, \mathbb{R})$ is $K \cong U(2)$. Using the diagonal maximal torus in $U(2)$, the irreducible representations of K may be identified, via highest weights, with pairs of integers (m, n) such that $m \geq n$, and the Harish-Chandra parameters of the discrete series may be identified with pairs (m, n) with

$$m > n, \quad m \neq 0, \quad n \neq 0 \quad \text{and} \quad m \neq -n.$$

To describe the Matching Theorem, it is convenient to associate to the spin module $S_{\text{irr}} \otimes V_{\mu}$ (where V_{μ} is irreducible with highest weight μ) the parameter $\mu + \rho_K$. Compare Example 6.4. The parameters $\mu + \rho_K$ range over all pairs of integers (m, n) with $m > n$, and using the $\mu + \rho_K$ parametrization, the Matching Theorem is illustrated in Fig. 1.

The computations involved in checking the Matching Theorem are greatly simplified by David Vogan's theory of minimal K -types, and we shall say more about this in the second paper of the series [CHS22, Theorem 8.4].

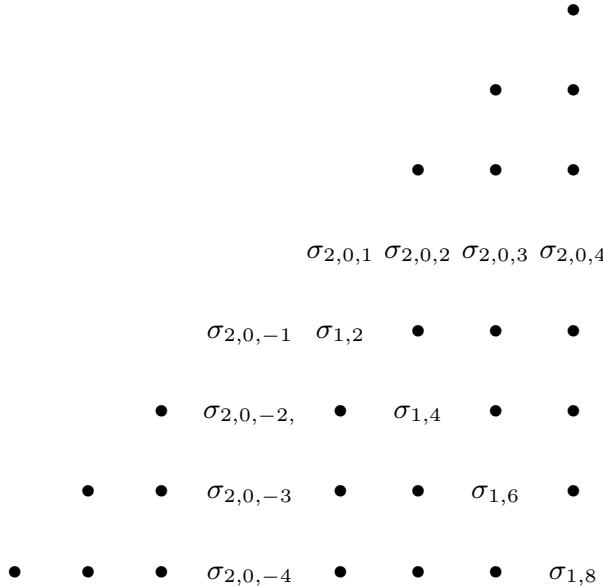


Fig. 1. The Matching Theorem for $Sp(4, \mathbb{R})$. The nodes in this diagram are the integer lattice points (m, n) in the plane with $m > n$; the entry $\sigma_{2,2}$ appears in position $(1, -1)$. The σ -labels are some of the discrete series attached to intermediate Levi subgroups of $Sp(4, \mathbb{R})$, and together with the discrete series of $Sp(4, \mathbb{R})$, they give the full list of essential components in the tempered dual of $Sp(4, \mathbb{R})$. Each essential component is placed at the location $(m, n) = \mu + \rho_K$, where μ is the highest weight of the genuine irreducible representation of \tilde{K} to which it is matched. The bullet points represent the discrete series for $Sp(4, \mathbb{R})$; their locations are also their Harish-Chandra parameters

7. First proof of the Connes–Kasparov isomorphism

In this section we shall use the Matching Theorem formulated in the previous section to prove that the Connes–Kasparov index homomorphism is an isomorphism. We shall follow the shortest route to do so, which uses the fact, proved by Kasparov, that the index homomorphism is a split-injective homomorphism of abelian groups (that is, the index homomorphism has a left-inverse). While this is certainly a significant new ingredient in the proof, injectivity is a considerably simpler and more accessible result than surjectivity. (In any case, in the next section we shall take a different approach to the proof of the Connes–Kasparov isomorphism that avoids Kasparov’s result.)

Kasparov proved split injectivity in a much broader context than the one we are considering here—Involving both continuous and discrete groups—in the course of proving groundbreaking results on the Novikov conjecture in differential topology. But let us record his result as it applies in our case:

Theorem ([Kas88]). *The Connes–Kasparov index morphism*

$$R_{\text{spin}}(K, \mathfrak{s}) \longrightarrow K_{\dim(G/K)}(C_r^*(G))$$

is a split injection of abelian groups.

Proof of the Connes–Kasparov isomorphism theorem using split injectivity

To begin with, Theorem 4.9 shows in particular that

$$K_{\dim(G/K)+1}(C_r^*(G)) = 0,$$

which is one of the assertions in Theorem 5.38. The main task is to show that the index homomorphism

$$R_{\text{spin}}(K, \mathfrak{s}) \longrightarrow K_{\dim(G/K)}(C_r^*(G))$$

is an isomorphism of abelian groups.

The C^* -algebra isomorphism in Theorem 2.19 determines a K -theory direct sum decomposition

$$(7.1) \quad K_{\dim(G/K)}(C_r^*(G)) \cong \bigoplus_{[P,\sigma]} K_{\dim(G/K)}(C_0(\mathfrak{a}_P^*, \mathfrak{K}(\text{Ind}_P^G H_\sigma))^{W_\sigma}).$$

If S is a spin module for (K, \mathfrak{s}) , then we shall denote by

$$\text{Index}_{[P,\sigma]}(\not{D}_S) \in K_{\dim(G/K)}(C_0(\mathfrak{a}_P^*, \mathfrak{K}(\text{Ind}_P^G H_\sigma))^{W_\sigma})$$

the $[P, \sigma]$ -component in (7.1) of the image of S under the Connes–Kasparov index homomorphism.

7.2 Lemma. *Let S be an irreducible spin module for (K, \mathfrak{s}) . If $[P, \sigma]$ and S are unmatched, then $\text{Index}_{[P,\sigma]}(\not{D}_S) = 0$.*

Proof. If $[\text{Ind}_P^G H_\sigma \otimes S]^K = 0$, then certainly $\text{Index}_{[P,\sigma]}(\not{D}_S) = 0$. If $[\text{Ind}_P^G H_\sigma \otimes S]^K$ is non-zero but $[P, \sigma]$ and S are unmatched, then the operator $\not{D}_{\sigma, 0, S}$ is non-zero on $[\text{Ind}_P^G H_\sigma \otimes S]^K$. Since the Dirac operator is self-adjoint, the square is also non-zero on $[\text{Ind}_P^G H_\sigma \otimes S]^K$, and therefore, by Theorem 5.27,

$$\|S\|^2 - \|\text{inf. ch.}(\sigma)\|^2 > 0.$$

But Theorem 5.27 asserts more generally that

$$\not{D}_{\sigma, \varphi, S}^2 = \|S\|^2 - \|\text{inf. ch.}(\sigma)\|^2 + \|\varphi\|^2,$$

and therefore \not{D}_S^2 is uniformly bounded below over the $[P, \sigma]$ -component of $C_r^*(G)$. The bounded operator \not{F}_S is therefore invertible there, and hence the index is zero. \square

Proof of Theorem 5.38. We shall use the Matching Theorem. Let S be an irreducible spin module. Lemma 7.2 implies that the image of S under the index homomorphism is concentrated in the summand in (7.1) associated to the unique $[P, \sigma]$ to which S is matched. Since the index homomorphism is injective, the image there must be non-zero. In fact, because the index homomorphism is split injective, while the summand is isomorphic to \mathbb{Z} , the image must be a generator. That is, the index homomorphism maps the basis of $R_{\text{spin}}(K, \mathfrak{s})$ determined by the irreducible spin modules to the basis determined up to signs by Theorem 4.3. \square

8. Second proof of the Connes–Kasparov isomorphism

In this final section we shall study the Dirac operator \not{D}_S in more detail, and by doing so give a proof of the Connes–Kasparov isomorphism that is independent of Kasparov’s split-injectivity result. This is probably more in line with the approach that Wassermann intended to take, as sketched in the note [Was87].

K-theoretic preliminaries

As we have seen, the Connes–Kasparov index homomorphism carries the natural basis for $R_{\text{spin}}(K, \mathfrak{s})$ to the natural basis⁵ for the K -theory of $C_r^*(G)$ (labeled by the essential components of the tempered dual; see Theorem 4.9). A striking feature of the Connes–Kasparov index is that in fact it carries natural basis elements to natural basis elements *at the level of cycles*, and not merely at the level of K -theory classes. In this section we shall describe those cycles.

8.1 Definition. Let V be a finite-dimensional Euclidean vector space of dimension d . A *Bott element* for V consists of a finite-dimensional $\mathbb{Z}/2$ -graded Hilbert space S with

$$\dim(S) = \begin{cases} 2^{d/2} & d: \text{even}, \\ 2^{(d-1)/2} & d: \text{odd} \end{cases}$$

and an \mathbb{R} -linear map $v \mapsto D_v$ from V into the odd-graded, self-adjoint operators on S such that $D_v^2 = \|v\|^2$ for all $v \in V$. When the dimension of V is odd, we also require that S be equipped with a symmetry γ as in (5.35) that anti-commutes with all D_v .

⁵ To be accurate, both bases are defined up to choices of signs.

It follows from the elementary theory of Clifford algebras that Bott elements are unique up to isomorphism. Each Bott element may be regarded as a Fredholm operator on the Hilbert $C_0(V)$ -module $C_0(V, S)$ of the sort considered by Kasparov (D is unbounded, but one can take the bounded transform to obtain a bounded Fredholm operator F if preferred). There is therefore an index

$$\text{Index}(D) \in K_d(C_0(V)).$$

Here is one form of the Bott periodicity theorem (see [Kas81, Theorem 7 on p. 547]):

8.2 Theorem. *Let V be a finite-dimensional Euclidean vector space of dimension d . The K_d -group of $C_0(V)$ is freely generated by the index of any Bott element, and the K_{d+1} -group is zero.*

Representation-theoretic preliminaries

Now let $[P, \sigma]$ be an essential associate class. As noted earlier, there is a decomposition of the parabolically induced representation $\pi_{\sigma, 0}$ into finitely many irreducible subrepresentations,

$$(8.3) \quad \text{Ind}_P^G H_\sigma = \bigoplus_{\mu} X_\mu,$$

and the index set in the direct sum is the set \widehat{R}_σ of characters of the finite abelian group R_σ . But we can index the sum in a different way using Vogan's theory of minimal K -types [Vog81], and it will be very useful to do so in what follows.

It will not be important to present the precise definition of minimal K -type here. It will suffice to recall that the K -types of a representation π of G are the irreducible representations of K that occur upon restriction of π from G to K , and that every representation has a finite number of *minimal* K -types among these, which depend only on the set of all K -types in π .

The deeper properties of minimal K -types that we shall use below are as follows:

8.4 Theorem. *Let $[P, \sigma]$ be an essential associate class, and let S be the irreducible spin module to which it is matched.*

- (i) *Each minimal K -type of $\text{Ind}_P^G H_\sigma$ has multiplicity one⁶, and each irreducible direct summand X_μ of $\text{Ind}_P^G H_\sigma$, as in (8.3), includes precisely one of these minimal K -types.*

⁶ That is, the underlying irreducible representation of K occurs precisely once in any decomposition of $\text{Ind}_P^G H_\sigma$ into irreducible representations of K .

(ii) If X_μ is any irreducible summand of $\text{Ind}_P^G H_\sigma$, then

$$\dim[X_\mu \otimes S]^K = 2^{[(\dim(\mathfrak{a}_{\max})+1)/2]},$$

where the brackets $[]$ in the exponent denote the integer part.

(iii) If X_μ is any irreducible summand of $\text{Ind}_P^G H_\sigma$, and if $V_\mu \subseteq X_\mu$ is its minimal K -type, then the inclusion

$$[V_\mu \otimes S]^K \longrightarrow [X_\mu \otimes S]^K$$

is a vector space isomorphism.

We shall prove this theorem in [CHS22, Section 8] (mostly by collecting results from elsewhere in the representation theory literature).

It follows from parts (i) and (iii) of the theorem, together with the direct sum decomposition (8.3), that if $\{V_\mu\}$ is the set of minimal K -types in $\text{Ind}_P^G H_\sigma$, then the inclusion

$$(8.5) \quad \bigoplus_{\mu} [V_\mu \otimes S]^K \longrightarrow [\text{Ind}_P^G H_\sigma \otimes S]^K$$

is a vector space isomorphism. This gives a very concrete and convenient description of the space $[\text{Ind}_P^G H_\sigma \otimes S]^K$. The following lemmas examine the Dirac operators that act on this space.

8.6 Lemma. *Let $[P, \sigma]$ be an essential associate class and let S be the irreducible spin module to which it is matched. The operators*

$$\not{D}_{\sigma, \varphi, S} : [V_\mu \otimes S]^K \longrightarrow [V_\mu \otimes S]^K$$

are linear functions of $\varphi \in \mathfrak{a}_P^$.*

Proof. The action of \mathfrak{g} on the smooth vectors in any principal series representation space such as $\text{Ind}_P^G H_\sigma \otimes \mathbb{C}_{i\varphi}$ is affine-linear in φ (compare [KV95, Proposition 11.47]), and so $\not{D}_{\sigma, \varphi, S}$ is affine-linear in φ . But since S is matched to (P, σ) , the operator $\not{D}_{\sigma, 0, S}$ is zero. So $\not{D}_{\sigma, \varphi, S}$ is actually linear in φ . \square

8.7 Lemma. *Let $[P, \sigma]$ be an essential associate class and let S be the irreducible spin module to which it is matched. Denote by $\mathfrak{a}_P^{*, R_\sigma} \subseteq \mathfrak{a}_P^*$ the subspace that is fixed under the action of the group R_σ . The image of each direct summand in (8.5) is invariant under the Dirac operators*

$$\not{D}_{\sigma, \varphi, S} : [\text{Ind}_P^G H_\sigma \otimes S]^K \longrightarrow [\text{Ind}_P^G H_\sigma \otimes S]^K$$

for all $\varphi \in \mathfrak{a}_P^{, R_\sigma}$.*

Proof. The representations $X_{\mu,\varphi}$ that appear in the direct sum decomposition

$$\mathrm{Ind}_P^G H_\sigma \otimes \mathbb{C}_{i\varphi} = \bigoplus_{\mu} X_{\mu,\varphi},$$

compare (4.4), have the same K -isotypic decompositions as the representations X_μ . Therefore for every $\varphi \in \mathfrak{a}_P^{*,R_\sigma}$ the K -type V_μ appears in $X_{\mu,\varphi}$ as a minimal K -type, and the inclusion

$$[V_\mu \otimes S]^K \longrightarrow [X_{\mu,\varphi} \otimes S]^K$$

is a vector space isomorphism, since $[X_{\mu,\varphi} \otimes S]^K$ depends only on the K -structure of $X_{\mu,\varphi}$, and not on φ . The Dirac operator $D_{\sigma,\varphi,S}$ certainly maps $[X_{\mu,\varphi} \otimes S]^K$ to itself, and so it maps $[V_\mu \otimes S]^K$ to itself, as claimed. \square

8.8 Theorem. *If $[P,\sigma]$ is any essential associate class, and if S is the irreducible spin module to which it is matched, then for any μ the family of Dirac operators*

$$\mathcal{D}_{\sigma,\varphi,S} : [X_\mu \otimes S]^K \longrightarrow [X_\mu \otimes S]^K \quad (\varphi \in \mathfrak{a}_P^{*,R_\sigma})$$

is a Bott element for $\mathfrak{a}_P^{,R_\sigma}$.*

Proof. This follows from the preceding two lemmas and Theorem 5.27. \square

Completion of the second proof of the Connes–Kasparov isomorphism

The Matching Theorem and Lemma 7.2 show that the Connes–Kasparov index morphism maps an irreducible spin module for (K,\mathfrak{s}) to the index of the family of Dirac operators

$$(8.9) \quad \mathcal{D}_{\sigma,\varphi,S} : [\mathrm{Ind}_P^G H_\sigma \otimes S]^K \longrightarrow [\mathrm{Ind}_P^G H_\sigma \otimes S]^K,$$

in $K_{\dim(G/K)}(C_0(\mathfrak{a}_P^*, \mathfrak{K}(\mathrm{Ind}_P^G H_\sigma))^{W_\sigma})$, where $[P,\sigma]$ is the unique essential associate class matched to S . To prove the Connes–Kasparov isomorphism, it remains to show that the homotopy class of the family (8.9) is a generator for the K -theory group.

By Theorem 8.8, for any summand X_μ of $\mathrm{Ind}_P^G H_\sigma$ the family of Dirac operators

$$\mathcal{D}_{\sigma,\varphi,S} : [X_\mu \otimes S]^K \longrightarrow [X_\mu \otimes S]^K \quad (\varphi \in \mathfrak{a}_P^{*,R_\sigma})$$

is a Bott element for $\mathfrak{a}_P^{*,R_\sigma}$ and therefore its index is a generator of the K -theory group $K_{\dim(G/K)}(C_0(\mathfrak{a}_P^{*,R_\sigma}))$. But (8.9) is precisely the image of the cycle that defines the Connes–Kasparov index

$$\mathrm{Index}_{[P,\sigma]}(\mathcal{D}_S) \in K_{\dim(G/K)}(C_0(\mathfrak{a}_P^*, \mathfrak{K}(\mathrm{Ind}_P^G H_\sigma))^{W_\sigma})$$

under the K -theory isomorphism in Theorem 4.7. The proof is complete.

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